Topological invariants
for discontinuous mappings
Differential topology in Sobolev spaces

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Kolokwium MIM, 7 listopada 2019
Why do we need
• non-differentiable
• discontinuous
mappings in real-life applications?
Why do we need non-differentiable discontinuous mappings in real-life applications?

Two reasons:

- natural: folding, breaking, changes of state;
- technical: necessary because of the available mathematical tools.
Variational principles

- Fermat’s principle (Hero of Alexandria, Pierre de Fermat): Light travels through media along paths of shortest time.
- Extremal action principle (Euler, Maupertuis): Bodies travel along paths locally minimizing the reduced action (integral of the momentum).

Other examples: Dirichlet’s principle for harmonic maps, Einstein-Hilbert action in general relativity etc.
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Minima of accumulative, i.e, integral quantities ⇒ we measure distance to minimizer with integral expressions.
Harmonic maps and $W^{1,2}$

Canonical example: **harmonic maps**

A map $f : \mathbb{B}^n \to \mathbb{R}^m$ is *minimizing harmonic* if it is a minimum of the Dirichlet energy

$$\mathcal{E}(f) = \int_{\mathbb{B}^n} |Df|^2$$

among all maps with the same boundary values as $f$. 

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Topological invariants for discontinuous mappings

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among all maps with the same boundary values as $f$. What all maps?
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$C^1(B^n)$, with distance $d(f, g) = |\mathcal{E}(f) - \mathcal{E}(g)|$?
Canonical example: \textit{harmonic maps}

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\( C^1(\mathbb{B}^n) \), with distance \( d(f, g) = |\mathcal{E}(f) - \mathcal{E}(g)| \)?

Problems:

- \( d \) is not a metric (remedy: add \( \|f - g\|_{L^2} \) to \( d(f, g) \))
- space is not complete \( \Rightarrow \) existence of minima is not guaranteed.
Harmonic functions and $W^{1,2}$

Right choice: Sobolev space $W^{1,2}(\mathbb{B}^n)$ – the completion of $C^1$ (or $C^\infty$) maps, in the norm

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But if $n \geq 2$, then this space contains discontinuous functions:

$$\log(\log(e + \frac{1}{|x|})) \in W^{1,2}(\mathbb{B}^n).$$
Definition

A function $g \in L^1_{loc}$ is the weak derivative in the $i$-th direction of $f \in L^1_{loc}(U)$, if

$$\int_U f \frac{\partial}{\partial x^i} \phi = - \int_U g \phi \quad \text{for any } \phi \in C_0^\infty(U).$$

We write $g = D^i f$. If $f \in C^1(U)$, then $D^i f = \frac{\partial}{\partial x^i} f$. 
Introduction

Weak derivative and Sobolev spaces

Definition

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We write \( g = D^i f \). If \( f \in C^1(U) \), then \( D^i f = \frac{\partial}{\partial x_i} f \).

Definition (Sobolev space)

\[
W^{1,p}(U) = \{ f \in L^p(U) : D^i f \in L^p(U) \text{ for } i = 1, \ldots, n \}
\]

If \( p \leq n = \dim U \) and \( n \geq 2 \), then \( W^{1,p}(U) \) contains discontinuous functions.
Advantages of Sobolev spaces:
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* Banach spaces with uniformly convex norm $\Rightarrow$ reflexive $\Rightarrow$ weak compactness of bounded subsets.

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Sobolev and Orlicz-Sobolev spaces

Advantages of Sobolev spaces:
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• nice embedding properties,
* Banach spaces with uniformly convex norm $\Rightarrow$ reflexive $\Rightarrow$ weak compactness of bounded subsets.
* means that sufficiently nice functionals attain their minima
(minimizing sequences have weakly convergent subsequences).

Of interest are also Orlicz-Sobolev functions

$$W^{1,P}(U) = \{ f \in L^1(U) : P(|D^i f|) \in L^1(U) \text{ for } i = 1, \ldots, n \},$$

e.g., for $P(t) = t^n / \log(e + t)$ and $p < n$, $W^{1,p} \subset W^{1,P} \subset W^{1,n}$. 
Topology encoded in the derivative

**invertibility:**
Assume $X, Y$ are Banach, $F: X \rightarrow Y$ is $C^1$. If

$$DF(x): X \rightarrow Y \text{ is a linear isomorphism} \quad (1)$$

then $F$ is a local homeomorphism at $x$.
If (1) holds for all $x \in X$, then $F$ is an open mapping.
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local orientation:
If $F: \Omega \subset \mathbb{R}^n \rightarrow F(\Omega) \subset \mathbb{R}^n$ is $C^1$ and

$$J_F(x) = \det DF(x) > 0,$$

then $F$ preserves local orientation at $x$. 
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**Brouwer degree:** Let $M, N$ be closed manifolds of the same dimension.

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Other topological invariants (Hopf index, characteristic classes) of a mapping can be determined if the derivative is known.
For weakly differentiable mappings, we can calculate the values of ‘invariants’, substituting the weak derivative where the strong one should go.

Two (closely related) questions:

• If two weakly differentiable mappings are suff. close in some Sobolev norm, do their ‘invariants’ match?

• If a **continuous**, non-differentiable mapping is weakly differentiable, does this value of the ‘invariant’ match the true topological invariant?

And are these ‘invariants’ of any use?
The formula for the Brouwer degree:

$$\deg F = \frac{1}{\text{vol}(N^n)} \int_{M^n} \det DF(x) \, dx$$

defines a map $\deg : C^1(M, N) \to \mathbb{Z}$, continuous in $C^1$ norm.
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defines a map \( \text{deg} : C^1(M, N) \rightarrow \mathbb{Z} \), continuous in \( C^1 \) norm. It is also continuous in \( C^1(M, N) \) with \( W^{1,n} \) norm.
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It is also continuous in \( C^1(M, N) \) with \( W^{1,n} \) norm.

We extend \( \deg \) by density to arbitrary \( W^{1,n}(M, N) \) maps, possibly discontinuous.

**Theorem**

Assume \( f_m : M \to N, m = 1, 2, \ldots \), is a sequence of \( C^1 \) maps convergent in \( W^{1,n}(M, N) \). Then the sequence \( \deg f_m \) is constant for suff. large \( m \).
Take any other invariant of a map $f$, e.g., homotopy class $[f]$.

**Question**

Assume $f_m : M \to N$ is a sequence of smooth / $C^1$ / continuous maps convergent in $W^{1,p}(M,N)$. Does the value of the invariant ($[f_m]$) become constant for large $m$?

If so, then the invariant (e.g., homotopy classes) can be extended, by density, to all $W^{1,p}(M,N)$. 
**Problem**

*Find a non-trivial (non-constant) harmonic function* $h : \mathbb{B} \to \mathbb{R}$.

**Solution:** minimize Dirichlet’s energy $\mathcal{E}$ among all $W^{1,2}(\mathbb{B})$ functions with prescribed, non-trivial values on $\partial\mathbb{B}$.

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*Problem* $S^n$ has no boundary! $\quad$ *Solution:* minimize Dirichlet’s energy $\mathcal{E}$ among all $W^{1,2}(S^2)$ functions with prescribed, non-zero homotopy class. *(White, Eells&Ferreira)*
Problem

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Problem

Find a non-trivial harmonic function $h : \mathbb{S}^2 \rightarrow \mathbb{S}^2$. 
Introduction

Motivation

**Problem**

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\text{Find a non-trivial (non-constant) harmonic function } h : \mathbb{B} \rightarrow \mathbb{R}.
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**Problem**

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(White, Eells&Ferreira)
Definition

Assume $N$ is isometrically embedded in $\mathbb{R}^k$.

$$W^{1,p}(M, N) = \{ f \in W^{1,p}(M, \mathbb{R}^k) : f(x) \in N \text{ for a.e. } x \in M \}$$

Smooth mappings need not be dense in $W^{1,p}(M, N)$ for $p < n$; $H^{1,p}(M, N) :=$ the closure of $C^\infty$ in $W^{1,p}$.

Orlicz-Sobolev maps $W^{1,p}$ between manifolds are defined in the same way; we are interested in those that are only slightly larger than $W^{1,n}$ (e.g., $P(t) = t^n / \log(e + t)$).
Homotopy classes

Theorem (White, 1986)

If $M, N$ are compact, oriented Riemannian manifolds, $\dim M = n$, then the homotopy classes are well defined in $W^{1,n}(M, N)$.

Theorem (G., Hajłasz, 2012)

The homotopy classes are well defined in $H^{1,p}(M, N)$, $n - 1 \leq p < n$, if $\pi_n(N) = 0$, and they cannot be well defined in $H^{1,p}(S^n, N)$, if $\pi_n(N) = 0$. An analogous result holds for a certain class of Orlicz-Sobolev spaces, slightly larger than $W^{1,n}(M, N)$. 
Theorem (G., Hajłasz, 2012)

If $M, N$ are compact, oriented $n$-dim. Riemannian manifolds w/out boundary, $n - 1 \leq p < n$. Then the degree is well defined in $H^{1,p}(M,N)$ (and in the aforementioned class of Orlicz-Sobolev spaces) if and only if the universal cover of $N$ is not a rational homology sphere (RHS).

A *rational homology $n$-sphere*: an $n$-dimensional smooth manifold with the same deRham cohomology as $S^n$. 
**Fact**

If \( f, g \in C^1(\mathbb{B}^n, \mathbb{R}^n) \cap C(\overline{\mathbb{B}}^n, \mathbb{R}^n) \) and \( f = g \) on \( \partial \mathbb{B} \), then

\[
\int_{\mathbb{B}^n} J_f = \int_{\mathbb{B}^n} J_g.
\]

Because \( \mathbb{R}^n \) is contractible. It holds for \( W^{1,n} \) maps, as well. What if the target is not contractible?
Results

Importance of the degree

Lemma

If $M$ is a smooth, compact, connected, oriented, $n$-dim. manifold w/out boundary, $n \geq 2$, then there is a smooth mapping $f : S^n \to M$ of non-zero degree if and only if the universal cover of $M$ is a RHS.

Corollary

If $f, g \in C^1(B^n, N) \cap C(\overline{B^n}, N)$, $f = g$ on $\partial B^n$, and the universal cover of $N$ is not a RHS, then

$$\int_{B^n} J_f = \int_{B^n} J_g.$$
Continuity of mappings with positive Jacobian

Theorem (Gol’dšteĩn & Vodop’yanov, 1976)

If $U \subset \mathbb{R}^n$ is open and $f \in W^{1,n}(U, \mathbb{R}^n)$ has positive Jacobian a.e., then $f$ is continuous.

The same holds for our Orlicz-Sobolev maps $W^{1,P}$. How about mappings between manifolds?
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Theorem (G., Hajłasz, Pakzad, 2019)

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The same holds in our Orlicz-Sobolev spaces $W^{1,P}$, if the universal cover of $N$ is not a RHS. However, if the universal cover of $N$ is $S^n$, then there exist discontinuous $W^{1,P}$ maps of positive Jacobian.
Other invariants, other directions...

Question

Assume $f$ is a weakly or approximately differentiable, continuous map. How much of its topology can be read from the ‘invariants’ calculated using its weak/approximate derivative (which we can study and estimate by analytic means)?

- local degree/orientation $\rightarrow$ study of homeomorphisms with Jacobian changing sign (2017, 2019),
- Hopf invariant and its generalizations $\rightarrow$ study of $C^1$ maps with degenerate derivative, topologically non-trivial examples showing sharpness of assumptions to the Sard theorem (2018, 2019).
Thank you for your attention