Probabilistic programming semantics for name generation

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A syntax for "probabilistic programming" (actually "name generation"

The following is the grammar for \( \nu \)-calculus

\[
M = x \mid \lambda x . M \mid MM \quad \text{application of } x \rightarrow M
\]

\[
1 \text{ true } \mid \text{ false } \mid \text{ if } M \text{ then } M \text{ else } M
\]

\[
1 \text{ } M = M \mid \nu n M \quad \text{generate "random" name } n
\]

\[\begin{array}{c}
\text{types} \\
\text{ground} \\
\text{boolean} \\
\text{names}
\end{array}
\]

\[\begin{array}{c}
B \\
N \\
\sigma \rightarrow \sigma
\end{array}\]

1st order types

\[
G_1 \rightarrow (G_2 \rightarrow \ldots G_n)
\]

2nd order

\[
(B \rightarrow B) \rightarrow N
\]

Examples

1) \( \nu n \nu m \ n = m \quad B \)
2) \( \nu n \lambda x . x = n \quad N \rightarrow B \)
3) \( \lambda x. \text{false} \quad N \rightarrow B \)

Operational semantics expresses iterated evaluation of terms

\[
M \Downarrow C \quad \text{stands for }
\]

\[
M \text{ evaluates to } C
\]
Def (observational equivalence)
If $M_1$ and $M_2$ are $v$-terms, of the same type,
$$M_1 \simeq M_2 \text{ if for every context } P[-] \text{ we have }$$
$$P[M_1] \cup b \iff P[M_2] \cup b$$
whenever $P[M_i]$ are well-formed expressions.

For 1st order types there is a format system called logical relations that determines $\simeq$
$$M_1 \simeq M_2 \iff M_1 \vdash M_2$$
defined inductively and resembles a proof system.

Examples
1) $\forall n \forall m \ n = m \ \land \ false$
2) $\forall n \forall m \ n = m \ \land \ false$
(thesis is called the privacy equation)

Semantics
In semantics of programming languages, we often need function spaces:

to each type $\sigma$, we associate a set $\mathbf{X}_\sigma$.
E.g. $\sigma = \mathbb{B}$, $\mathbf{X}_\sigma = \{0, 1\}$. 
\( \sigma = N, \quad X_\sigma = \mathbb{R} \)

for type \( \sigma \rightarrow \tau \) we want to have \( X_\tau \)

In probabilistic programming we want to have \( \mathbb{R} \) as the space associated to type \( N \).

we want to treat it as a space with a \( \sigma \)-algebra (measurable space).

The problem appears at the construction of function spaces.

**Theorem (Aumann 1969)** There is no \( \sigma \)-algebra on \( 2^\mathbb{R} \) such

\[
\begin{align*}
\mathbb{R} \times 2^\mathbb{R} & \rightarrow \mathbb{R} \\
(\xi, f) & \mapsto f(\xi)
\end{align*}
\]

is measurable.

**Def** (Hennec- Kammar; Aatao - Yang 1977)

A Quasi-Borel space is a set \( X \) together with

\( M_X \) of functions \( f : \mathbb{R} \rightarrow X \)

satisfying

1) all constant functions are in \( M_X \)

2) if \( f \in M_X, \alpha : \mathbb{R} \rightarrow \mathbb{R} \) is Borel

\[ f \circ \alpha \in M_X \]

3) if \( IR = \bigcup B_n \) Borel

\[ f_n \in M_X \] then \( \bigcup f_n \bigcup B_n \in M_X \)

This forms a category.

**Remark** 1) Every standard Borel space \( X \) is a quasi-

Borel space \( (M_X - all \ Borel \ maps) \)
2) The category allows for function spaces and products and \( x \mapsto \mathcal{G}(x) \)

If \( X, Y \in \text{QBS} \)

\( Y^X \in \text{QBS} \) with these

\( f: \mathbb{R} \rightarrow Y^X \)

s.t. \( f': \mathbb{R} \times X \rightarrow Y \) is measurable

\( f'(r, x) = f(r)(x) \)

On \( \mathcal{Z}^\mathbb{R} \) (this is the set of Borel subsets of \( \mathbb{R} \) )

Fact: Every \( f: \mathbb{R} \rightarrow \mathcal{Z}^\mathbb{R} \) measurable in QBS

\( \implies \) (Borel-on-Borel) - measurable
Interpretation

we can interpreted \( r \)-calculus in QBS

\( N \sim \mathbb{R} \quad B \sim 2: \mathcal{O} / \mathcal{B} \)

\[
\llbracket \lambda x. M \rrbracket \quad \text{"} e^{x_\sigma} " = \delta_x
\]

\[
\llbracket \forall x. M \rrbracket = \int_{\mathbb{R}} \llbracket M \rrbracket \, dv
\]

Examples

1) \( \llbracket \lambda x. \text{false} \rrbracket = \delta_\emptyset \)

2) \( \llbracket \forall x. \lambda x. x = n \rrbracket = \int_{\mathbb{R}} \delta_{x=n} \, dv \)

both are measures on Borel-on-Borel set

\[
\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}
\]

\[
\int_{\mathbb{R}} \delta_{x=n} \, dv = \int_{\mathbb{R}} \left\{ \begin{array}{ll} 1 & \text{if } x = n \\ 0 & \text{if } x \neq n \end{array} \right\} \, dv(n)
\]

Theorem (SSSW) if \( M_1 \) and \( M_2 \) are

1st order \( r \)-terms then

\( M_1 \simeq M_2 \quad \text{iff} \quad \llbracket M_1 \rrbracket = \llbracket M_2 \rrbracket \)
we will look at the special case of

- $M_1 = \lambda x \text{ false}$
- $M_2 = \forall x \lambda x \ x = n$

**Lemma** If $F \in \text{Borel-on-Borel}$, then

$$x \in \mathbb{N} \iff \{ x \in \mathbb{R} : \exists y \in F \} \text{ is co-countable}$$

**PT sketch**

$s.t. \ \phi \in \mathbb{N} \text{ but } \exists x \in \mathbb{R} : \exists y \neq F \text{ is uncountable}$

WLOG $\phi 

Recall Becker's Theorem

WF, UB are Borel inseparable

ie. there exist $B$ Borel set $B \subseteq \mathbb{R} \times \mathbb{R}$

s.t. WF: $\exists x \in \mathbb{R} : |B_x| = 0$

UB: $\exists x \in \mathbb{R} : |B_x| = 1$ are Borel inseparable

If $\{ x : B_x \in F \}$ is Borel by $F$ was Borel-on-Borel

but it separates WF from UB.

This lemma implies that

$$\delta_\chi (F) = \int \delta_{\chi(n)} (F) d\nu(n)$$

for every $F \text{ Borel-on-Borel.}$
In general, we introduce a "normal form" \( M \mapsto \langle M \rangle \)

such that \( M_1 \equiv M_2 \) if \( \langle M_1 \rangle = \langle M_2 \rangle \)