Property A and duality in linear programming

Andrzej Nagórko
joint work with G. C. Bell (UNCG)

University of Warsaw
1. What is ... property A?

2. Property A as a linear problem and the dual problem

3. Examples: primal and dual solutions

4. Primal relaxation: Cheeger constant

5. Applications: hypercubes and graphs with large girth
What is ... property A?
Property A was introduced in 2000 and turns out to be of great importance in many areas of mathematics [1]. Perhaps the most striking example is the following implication that follows from results in [4].

*If group $G$ has Property A then the Novikov conjecture is true for all closed manifolds with fundamental group $G$.*

The Novikov conjecture asserts homotopy invariance of higher signatures of smooth manifolds. Is is one of most important unsolved problems in topology.

Graph as a metric space

Path-length metric on a graph.

- Graphs are oriented \((E \subset V \times V)\).
- We allow infinite distance.
- If \(ij \in E\) is used to denote an edge, then \(i\) is the source and \(j\) is the target vertex of \(ij\).
Optimization Problem I. Let $G = (V, E)$ be a graph and let $S \geq 0$. Find minimal $\epsilon = \epsilon_{S,G}$ (variation) and a family $\{\xi_i : V \to \mathbb{R}\}_{i \in V}$ of functionals (probability measures) such that

1. Each $\xi_i$ is a probability measure, i.e.

   $$\|\xi_i\|_1 = 1 \text{ and } \xi_i \geq 0 \text{ for each } i \in V;$$

2. Variation on edge $ij$ does not exceed $\epsilon$, i.e.

   $$\|\xi_i - \xi_j\|_1 \leq \epsilon \text{ for each } ij \in E;$$

3. Each $\xi_i$ is supported by $B(i, S)$, i.e.

   $$\text{supp } \xi_i = \{j \in V : \xi_i(j) > 0\} \subset B(i, S) \text{ for each } i \in V,$$

   where $B(i, S)$ is ball of radius $S$ centered at $i$. 
Optimization Problem I. Find minimal $\varepsilon_{S,G} > 0$ with functionals $\xi_i$ on $G$ satisfying

1. $\|\xi_i\|_1 = 1$ and $\xi_i \geq 0$ for each $i \in V$;
2. $\|\xi_i - \xi_j\|_1 \leq \varepsilon$ for each $ij \in E$;
3. $\text{supp} \xi_i = \{j \in V: \xi_i(j) > 0\} \subset B(i, S)$ for each $i \in V$.

This is optimal solution with objective $\varepsilon = \frac{2}{3}$.

Optimality of the solution is not trivial to show.
Optimization Problem I. Find minimal $\varepsilon_{S,G} > 0$ with functionals $\xi_i$ on $G$ satisfying

1. $\|\xi_i\|_1 = 1$ and $\xi_i \geq 0$ for each $i \in V$;
2. $\|\xi_i - \xi_j\|_1 \leq \varepsilon$ for each $ij \in E$;
3. $\text{supp} \xi_i = \{j \in V : \xi_i(j) > 0\} \subset B(i,S)$ for each $i \in V$.

Property A

Let $G$ be a graph and for each $S \geq 0$ let $\varepsilon_{S,G}$ be the minimal variation of probability measures on $G$ at scale $S$, i.e. the solution of Optimization Problem I at scale $S$. We say that $G$ has property A iff

$$\lim_{S \to \infty} \varepsilon_{S,G} = 0.$$
Property A

Let $G$ be a graph and for each $S \geq 0$ let $\varepsilon_{S,G}$ be the minimal variation of probability measures on $G$ at scale $S$, i.e. the solution of Optimization Problem I at scale $S$. We say that $G$ has property A iff

$$\lim_{S \to \infty} \varepsilon_{S,G} = 0.$$
Property A

Let $G$ be a graph and for each $S \geq 0$ let $\varepsilon_{S,G}$ be the minimal variation of probability measures on $G$ at scale $S$, i.e. the solution of Optimization Problem $I$ at scale $S$. We say that $G$ has property A iff

$$\lim_{S \to \infty} \varepsilon_{S,G} = 0.$$  

• To prove property A for $G$ it is enough to find upper bounds $\varepsilon_{S,G} \leq \hat{\varepsilon}_{S,G}$ such that

$$\lim_{S \to \infty} \hat{\varepsilon}_{S,G} = 0.$$
Property A

Let $G$ be a graph and for each $S \geq 0$ let $\varepsilon_{S,G}$ be the minimal variation of probability measures on $G$ at scale $S$, i.e. the solution of Optimization Problem I at scale $S$. We say that $G$ has **property A** iff

$$\lim_{S \to \infty} \varepsilon_{S,G} = 0.$$

- To prove property A for $G$ it is enough to find upper bounds $\varepsilon_{S,G} \leq \hat{\varepsilon}_{S,G}$ such that

$$\lim_{S \to \infty} \hat{\varepsilon}_{S,G} = 0.$$

- To prove that $G$ does not have property A we have to show that

$$\limsup_{S \to \infty} \varepsilon_{S,G} > 0.$$ so we have to consider optimal solutions $\varepsilon_{G,S}$. 

A. Nagórko
• If $S \geq \text{diam } G$, then $\varepsilon_{S,G} = 0$, so property A is trivial for finite graphs.
• If \( S \geq \text{diam} \, G \), then \( \varepsilon_{S,G} = 0 \), so property A is trivial for finite graphs.

**Theorem**

Let \( G \) be a graph and assume that \( G = \bigcup_{n \in \mathbb{N}} G_n \), with \( G_1 \subset G_2 \subset G_3 \subset \cdots \) an ascending sequence of convex subgraphs of \( G \). Let \( \varepsilon_{S,G_n} \) be the minimal variation of probability measures at scale \( S \) for graph \( G_n \). Graph \( G \) has property A iff

\[
\lim_{S \to \infty} \lim_{n \to \infty} \varepsilon_{S,G_n} = 0.
\]
• If $S \geq \text{diam } G$, then $\varepsilon_{S,G} = 0$, so property A is trivial for finite graphs.

**Theorem**

Let $G$ be a graph and assume that $G = \bigcup_{n \in \mathbb{N}} G_n$, with $G_1 \subset G_2 \subset G_3 \subset \cdots$ an ascending sequence of convex subgraphs of $G$. Let $\varepsilon_{S,G_n}$ be the minimal variation of probability measures at scale $S$ for graph $G_n$. Graph $G$ has property A iff

$$\lim_{S \to \infty} \lim_{n \to \infty} \varepsilon_{S,G_n} = 0.$$ 

Important case: disjoint sum of finite subgraphs.
Graph $G$ has property A iff
\[ \lim_{S \to \infty} \lim_{n \to \infty} \epsilon_{S,G_n} = 0. \]

To check that
\[ \lim_{S \to \infty} \lim_{n \to \infty} \epsilon_{S,G_n} \neq 0 \]
we need a lower bound on $\epsilon_{S,G_n}$. 
Property A as a linear problem and the dual problem
Optimization Problem I. Find minimal $\varepsilon_{S,G} > 0$ with functionals $\xi_i$ on $G$ satisfying

1. $\|\xi_i\|_1 = 1$ and $\xi_i \geq 0$ for each $i \in V$;
2. $\|\xi_i - \xi_j\|_1 \leq \varepsilon$ for each $ij \in E$;
3. $\text{supp} \xi_i = \{j \in V : \xi_i(j) > 0\} \subset B(i, S)$ for each $i \in V$.

Primal problem.

minimize $e$

subject to

$\sum_{j \in V} x_{i,j} = 1$ for each $i \in V$,

$x_{i,j} = 0$ for each $i \in V, j \in V \setminus B(i, S)$,

$x_{j,k} - x_{i,k} \leq e_{ij,k}$ for each $ij \in E, k \in V$,

$x_{i,k} - x_{j,k} \leq e_{ij,k}$ for each $ij \in E, k \in V$,

$\sum_{k \in V} e_{ij,k} \leq e$ for each $ij \in E$,

$x_{i,j}, e_{ij,k}, e \geq 0$
The dual problem

Dual problem.

maximize \[ \sum_{i \in V} \eta_i \]
subject to
\[ \sum_{ij \in E} \kappa_{ij} \leq 1 \]
\[ \varphi_{k,ij} \leq \kappa_{ij} \text{ for each } ij \in E, k \in V, \]
\[ -\varphi_{k,ij} \leq \kappa_{ij} \text{ for each } ij \in E, k \in V, \]
\[ \sum_{mi \in E, m \in V} \varphi_{k,mi} - \sum_{im \in E, m \in V} \varphi_{k,im} \geq \eta_i \text{ for each } k \in V, i \in B(k, S), \]
\[ \eta_i, \varphi_{k,ij}, \kappa_{ij} \in \mathbb{R}, \kappa_{ij} \geq 0 \]
Primal vs dual

Primal

\[
\begin{align*}
\min & \quad e \\
\text{s.t.} & \quad \sum_{j \in V} x_{i,j} = 1, \\
& \quad x_{i,j} = 0, \\
& \quad x_{j,k} - x_{i,k} \leq e_{ij,k}, \\
& \quad x_{i,k} - x_{j,k} \leq e_{ij,k}, \\
& \quad \sum_{k \in V} e_{ij,k} \leq e, \\
& \quad x_{i,j}, e_{ij,k}, e \geq 0
\end{align*}
\]

Dual

\[
\begin{align*}
\max & \quad \sum_{i \in V} \eta_i \\
\text{s.t.} & \quad \sum_{ij \in E} \kappa_{ij} \leq 1, \\
& \quad \varphi_{k,ij} \leq \kappa_{ij}, \\
& \quad -\varphi_{k,ij} \leq \kappa_{ij}, \\
& \quad \sum_{mi \in E, m \in V} \varphi_{k,mi} - \sum_{im \in E, m \in V} \varphi_{k,im} \geq \eta_i, \\
& \quad \eta_i, \varphi_{k,ij} \in \mathbb{R}, \kappa_{ij} \geq 0
\end{align*}
\]
Examples: primal and dual solutions
\[ \|\xi_1 - \xi_0\|_1 = |\xi_1(0) - \xi_0(0)| + |\xi_1(1) - \xi_0(1)| + |\xi_1(2) - \xi_0(2)| + |\xi_1(3) - \xi_0(3)| \leq \varepsilon \]

\[ \Downarrow \]

\[ -\frac{1}{12} (\xi_1(0) - \xi_0(0)) + \frac{1}{12} (\xi_1(1) - \xi_0(1)) + \frac{1}{4} (\xi_1(2) - 0) - \frac{1}{4} (0 - \xi_0(3)) \leq \frac{1}{4} \varepsilon \]
Theorem

Dual problem at scale $S$ is dual to Primal problem at scale $S$. In particular, for each admissible solution of each problem, we have

$$\sum_{i \in V} \eta_i \leq e$$

and the optimal solutions are equal.
Square graph - solution by hand
Cube graph, $S = 2$, primal solution
Cube graph, $S = 2$, dual solution
Primal relaxation: Cheeger constant
Capacity and supply is implicit if we know flows

Note that the capacity $\kappa$ and supply $\eta$ is implicit in the solution of the dual problem as the optimal values for chosen pseudo-flows $\varphi_i$ can be easily computed.

Dual Problem

$$\text{maximize} \quad \sum_{i \in V} \eta_i$$

subject to

$$\sum_{ij \in E} \kappa_{ij} \leq 1,$$

$$\varphi_{k,ij} \leq \kappa_{ij} \quad \text{for each } ij \in E, k \in V,$$

$$-\varphi_{k,ij} \leq \kappa_{ij} \quad \text{for each } ij \in E, k \in V,$$

$$\sum_{mi \in E, m \in V} \varphi_{k,mi} - \sum_{im \in E, m \in V} \varphi_{k,im} \geq \eta_i \quad \text{for each } k \in V, i \in B(k, S),$$

$$\eta_i, \varphi_{k,ij} \in \mathbb{R}, \quad \kappa_{ij} \geq 0$$
Averaged solutions

Theorem

Let $G$ be a graph. Let $\Gamma$ be a group that acts on $G$ by automorphisms. If $\Gamma$ acts transitively both on edges and on vertices of $G$, then there exists an optimal solution of the dual problem such that $\eta_i = \eta_j$ for each $i, j \in V$ and $\varepsilon_{ij} = \frac{1}{|E|}$ for each $ij \in E$. 

This is not always the case.
Averaged solutions

Theorem

Let $G$ be a graph. Let $\Gamma$ be a group that acts on $G$ by automorphisms. If $\Gamma$ acts transitively both on edges and on vertices of $G$, then there exists an optimal solution of the dual problem such that $\eta_i = \eta_j$ for each $i, j \in V$ and $\varepsilon_{ij} = \frac{1}{|E|}$ for each $ij \in E$.

This is not always the case.
Forced equal capacities and supplies

\[
\sigma_{k,i} = \sum_{m_i \in E, m \in V} \varphi_{k,m_i} - \sum_{i \in E, m \in V} \varphi_{k,im}
\]

<table>
<thead>
<tr>
<th></th>
<th>\sigma_{0,i}</th>
<th>\sigma_{1,i}</th>
<th>\sigma_{2,i}</th>
<th>\sigma_{3,i}</th>
<th>\sigma_{4,i}</th>
<th>\sigma_{5,i}</th>
<th>\sigma_{6,i}</th>
<th>\sigma_{7,i}</th>
<th>\min_k \sigma_{k,i}</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>\eta_0 = 1/28</td>
</tr>
<tr>
<td>1</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>\eta_1 = 1/28</td>
</tr>
<tr>
<td>2</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>\eta_2 = 1/28</td>
</tr>
<tr>
<td>3</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>\eta_3 = 1/28</td>
</tr>
<tr>
<td>4</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>\eta_4 = 1/28</td>
</tr>
<tr>
<td>5</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>\eta_5 = 1/28</td>
</tr>
<tr>
<td>6</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>\eta_6 = 1/28</td>
</tr>
<tr>
<td>7</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>1/28</td>
<td>\eta_7 = 1/28</td>
</tr>
</tbody>
</table>
\[
\sum = \frac{2}{7}
\]
The dual problem with extra constraints

Uniform flows.

maximize $|V| \cdot \eta$

subject to

$\varepsilon_{ij,k} \leq \frac{1}{|E|}$ for each $ij \in E, k \in V,$

$-\varepsilon_{ij,k} \leq \frac{1}{|E|}$ for each $ij \in E, k \in V,$

$\sum_{j \in V, ji \in E} \varepsilon_{ji,k} - \sum_{j \in V, ij \in E} \varepsilon_{ij,k} \geq \eta$ for each $k \in V, i \in B(k, S),$

$\eta, \varepsilon_{ij,k} \in \mathbb{R}$
3 × 3 grid, Uniform Flows problem at scale $S = 1$
Let \( S \subset V \). The **edge boundary of** \( S \) is \( \partial S = E[S, V \setminus S] \). For \( S \neq \emptyset \) we let
\[
\varphi(S) = \frac{|\partial S|}{|S|}
\]
be the **isoperimetric number of** \( S \).

Let \( S \geq 0 \) be a scale on a graph \( G \). We let
\[
\gamma(G, S) = \min_{T \subset B(i, S), i \in V, T \neq \emptyset} \varphi(T)
\]
be the **Cheeger constant** of \( G \) at scale \( S \).
Theorem

The optimal solution of Uniform Flow problem at scale $S$ is equal to

$$\frac{|V|}{|E|} \gamma(G, S),$$

the Cheeger constant of $G$ at scale $S$ multiplied by $\frac{|V|}{|E|}$. 


Minimal isoperimetric number over $B(k, S)$

Minimal isoperimetric number

maximize \( \eta \)

subject to

\( \varepsilon_{ij} \leq 1 \) for each \( ij \in E \),

\( -\varepsilon_{ij} \leq 1 \) for each \( ij \in E \),

\[
\sum_{j \in V,i \in E} \varepsilon_{ji} - \sum_{j \in V,i \in E} \varepsilon_{ij} \geq \eta \text{ for each } i \in B(k, S),
\]

\( \eta, \varepsilon_{ij} \in \mathbb{R} \)
Minimal isoperimetric number - the dual

minimize \ \ \ \ \sum_{ij \in E} |a_i - a_j|

subject to

\sum_{i \in S} a_i = 1,$

\begin{align*}
a_i &= 0 \text{ for each } i \in V \setminus B(k, S), \\
a_i &\geq 0 \text{ for each } i \in B(k, S)
\end{align*}

**Theorem**

*There exists an optimal solution of the above problem with all non-zero values equal.*
Minimal isoperimetric number over $B(k, S)$

**Minimal isoperimetric number - the dual**

\[
\text{minimize} \quad \sum_{ij \in E} |a_i - a_j|
\]

subject to

\[
\sum_{i \in S} a_i = 1,
\]

\[
a_i = 0 \quad \text{for each } i \in V \setminus B(k, S),
\]

\[
a_i \geq 0 \quad \text{for each } i \in B(k, S)
\]

For such solution, if we take $T = \{i : a_i > 0\}$, then the value of the objective function is $\frac{|\partial T|}{|T|} = \eta(T)$. But this is the isoperimetric number of $T$ - and it is the minimal one.
Therefore the minimal isoperimetric number dual is equivalent to:

**Minimal isoperimetric number - the dual reinterpreted**

Maximize $\eta$ such that for each $T \subset B(k, S)$ we have

$$\eta \leq \frac{|\partial T|}{|T|}.$$ 

This is the Cheeger constant. Remember that we rescaled the original problem by $\frac{|E|}{|V|}$. 
The dual problem to Uniform flows

Uniform flows - Cheeger constant times $\frac{|V|}{|E|}$.

maximize $|V| \cdot \eta$

subject to

$$\varepsilon_{ij,k} \leq \frac{1}{|E|} \text{ for each } ij \in E, k \in V,$$

$$-\varepsilon_{ij,k} \leq \frac{1}{|E|} \text{ for each } ij \in E, k \in V,$$

$$\sum_{j \in V, ji \in E} \varepsilon_{ji,k} - \sum_{j \in V, ij \in E} \varepsilon_{ij,k} \geq \eta \text{ for each } k \in V, i \in B(k, S),$$

$$\eta, \varepsilon_{ij,k} \in \mathbb{R}$$
For each $S > 0$ find $\epsilon_{S,G}$ and a family of functionals $\{\psi_i\}$ on $G$ satisfying

1. $\{\psi_i\}$ has norm 1 on average, i.e.
   \[
   \frac{1}{|V|} \left\| \sum_{i \in V} \psi_i \right\|_1 = 1,
   \]
   and $\psi_i \geq 0$ for each $i \in V$.

2. $\{\psi_i\}$ has $\epsilon$-variation on average, i.e.
   \[
   \frac{1}{|E|} \sum_{ij \in E} \sum_{k \in V} |\psi_i(k) - \psi_j(k)| = \frac{1}{|E|} \sum_{ij \in E} \|\psi_i - \psi_j\|_1 \leq \epsilon.
   \]

3. $\text{supp } \psi_i \subset B(i, S)$ for each $i \in V$.

If $\lim_{S \to \infty} \epsilon_{S,G} = 0$, then $G$ has mean property A.
Applications: hypercubes and graphs with large girth
Theorem

Let $Q_n$ be the $n$-dimensional hypercube graph. The minimal variation of probability measures for $Q_n$ at scale $S$ is

$$\varepsilon_{S,Q_n} = \frac{2 \binom{n-1}{S}}{\sum_{k=0}^{S} \binom{n}{k}}.$$
Theorem

Let $Q_n$ be the $n$-dimensional hypercube graph. The minimal variation of probability measures for $Q_n$ at scale $S$ is

$$
\varepsilon_{S,Q_n} = \frac{2^{n-1}}{\sum_{k=0}^{S} \binom{n}{k}}.
$$

For $n = 3, S = 2$ we have

$$
\varepsilon = \frac{2 \cdot \binom{2}{2}}{\binom{3}{0} + \binom{3}{1} + \binom{3}{2}} = \frac{2}{7}.
$$

(We found this number before.)
Corollary (P. Nowak, [2])

The disjoint union

$$\bigsqcup_{n \in \mathbb{N}} \{0, 1\}^n$$

with $\ell_1$ metric does not have property A.

Proof.
The proof follows from the observation that for each $S \geq 0$ we have

$$\lim_{n \to \infty} \frac{2^n - 1}{\sum_{k=0}^S \binom{n}{k}} = 2.$$
Some experimental results and a quiz

Connected simple graphs with 12 edges (29503 graphs)

Top 12 solutions, $S = 1$, computation time 154.6743s (3 cores).
Some experimental results and a quiz

Connected simple graphs with 12 edges (29503 graphs)

Top 12 solutions, $S = 2$, computation time 207.7706s (3 cores).
Theorem
Let $G(d, c)$ be a $d$-regular graph with girth $c$. Let $2S + 1 < c$. The minimal variation of probability measures for $G(d, c)$ at scale $S$ is

$$\frac{2(d - 1)^S(2 - d)}{2 - d(d - 1)^S}.$$

Corollary (R. Willett, [3])
Suppose $d_i$ is a bounded sequence of integers with $d_i \geq 3$ and suppose $c_i$ is a sequence of integers going to infinity. Then, the disjoint union of the graphs $G(d_i, c_i)$ fails to have property A.

Proof.
The proof follows from the observation that

$$\lim_{S \to \infty} \frac{2(d - 1)^S(2 - d)}{2 - d(d - 1)^S} = 2 - \frac{4}{d}.$$
Thank you for your attention!

P. Nowak and G. Yu.
What is . . . property A?

P. W. Nowak.
Coarsely embeddable metric spaces without Property A.

R. Willett.
Property A and graphs with large girth.

G. Yu.
The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space.