Ellipticity in geometric variational problems

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\[ \Phi : \{\text{geometric objects}\} \rightarrow [0, \infty] \]

**Goal:** Study critical points (in particular minima) of \( \Phi \).
Geometric variational problems

\[ \Phi : \{ \text{geometric objects} \} \to [0, \infty] \]

**Goal:** Study critical points (in particular minima) of \( \Phi \).

**Examples:**

1. **The Plateau problem**
   - geometric objects \( \rightarrow \) surfaces with a fixed boundary
   - \( \Phi \) \( \rightarrow \) measure of the surface

2. **The isoperimetric problem**
   - geometric objects \( \rightarrow \) open sets with fixed volume
   - \( \Phi \) \( \rightarrow \) measure of the boundary
1930s  parameterised surfaces
J. Douglas (Fields medal) and T. Radó

1960s  rise of GMT: rectifiable currents and varifolds, Caccioppoli sets

2010s  new models and new solutions, more abstract setting
G. David, J. Harrison, C. De Lellis, F. Maggi, G. De Philippis et al.
How to find a minimiser?

$$\Phi : \{\text{geometric objects}\} \rightarrow [0, \infty]$$

$$S_i \in \text{dmn } \Phi, \quad \Phi(S_i) \xrightarrow{i \to \infty} \inf \text{ im } \Phi$$

- Is there a convergent sub-sequence of \(\{S_1, S_2, \ldots\}\)?
- If so, is the limit in the domain of \(\Phi\)?
Definition

Let $X \subseteq \mathbb{R}^N$. Define $\mathcal{H}(X)$ to be the space of those continuous functions $f : X \to \mathbb{R}$ which satisfy

$$\text{Clos}\{x \in X : f(x) \neq 0\} \text{ is compact .}$$

$\mathcal{H}(X)$ is a locally convex topological vector space.

Definition

*Radon measure* over $X$ is any continuous linear functional $\mu : \mathcal{H}(X) \to \mathbb{R}$ such that $\mu(f) \geq 0$ whenever $f \geq 0$.

Definition

Radon measures $\mu_1, \mu_2, \ldots$ over $X$ converge weakly to $\mu$ if

$$\lim_{i \to \infty} \mu_i(f) = \mu(f) \quad \text{for all } f \in \mathcal{H}(X).$$
Examples

Let $M$ be a $C^1$-smooth submanifold of $\mathbb{R}^n$ of dimension $k$ and such that $M \cap K$ has finite measure whenever $K \subseteq \mathbb{R}^n$ is compact.

1. One can associate a Radon measure $\mu_M$ to $M$ by setting

$$\mu_M(f) = \int f \, d\text{vol}_M \quad \text{for } f \in \mathcal{K}(\mathbb{R}^n).$$
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2. Varifold associated to $M$ is the Radon measure $\nu_k(M)$ over $\mathbb{R}^n \times G(n,k)$

$$\nu_k(M)(f) = \int f(x, \text{Tan}(M, x)) \, d\text{vol}_M(x)$$

for $f \in \mathcal{K}(\mathbb{R}^n \times G(n,k))$, where

$$G(n,k) = \{ T : T \text{ a linear subspace of } \mathbb{R}^n, \dim T = k \}$$

$$\simeq \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \cap \{ T : T \circ T = T, T^* = T, \text{trace } T = k \}.$$
**Definition**

A $k$-dimensional *varifold* in an open set $U \subseteq \mathbb{R}^n$ is a Radon measure over $U \times \mathcal{G}(n,k)$.

**Definition**

A varifold $V$ is called *rectifiable* if for $i = 1, 2, \ldots$ there are positive numbers $a_i$ and $C^1$-manifolds $M_i$ such that

$$V = \sum_{i=1}^{\infty} a_i v_k(M_i).$$

The *weight measure* of $V$ is the Radon measure $\|V\|$ over $U$ s.t.

$$\|V\|(A) = V(A \times \mathcal{G}(n,k)) \quad \text{for } A \subseteq U.$$

The *$k$-density* of $V$ at $x$ is defined (if the limit exists) by

$$\Theta_k(\|V\|, x) = \lim_{r \downarrow 0} \frac{\|V\| B(x, r)}{\alpha(k)r^k}.$$
Convergence example

\[ \mu_{M_1} \quad M_1 \quad \nu_k(M_1) \]
\[ \mu_{M_2} \quad M_2 \quad \nu_k(M_2) \]
\[ \mu_{M_3} \quad M_3 \quad \nu_k(M_3) \]
\[ \vdots \]
\[ \lambda / \mu_{M_\infty} \quad M_\infty \]

Not a rectifiable varifold

\[ V(f) = \lambda \int \int f(x,T) \, d\nu(T) \, d\text{vol}_{M_\infty} \]

for some \( \lambda \in \mathbb{R} \) and \( \nu \in \text{Prob}(\mathcal{G}(m,\lambda)) \).
Competitors: family $C$ of varifolds associated to properly embedded $C^1$-submanifolds of $U \subseteq \mathbb{R}^n$ with a given boundary.

Functional: $\Phi : C \to \mathbb{R}$ defined by $\Phi(V) = V(1)$, i.e., the total mass of $V \in C$.

1. choose a minimising sequence $V_i \in C$, i.e., $\lim_{i \to \infty} \Phi(V_i) = \inf \text{im } \Phi$

2. use the Banach-Alaoglu theorem to find a weakly convergent sub-sequence

3. pass to the limit $V_i \xrightarrow{i \to \infty} V$

Questions. Is $V$ rectifiable? Is it associated to a single manifold?
How to solve the Plateau problem?

**Competitors:** family $\mathcal{C}$ of varifolds associated to properly embedded $C^1$-submanifolds of $U \subseteq \mathbb{R}^n$ with a given boundary.

**Functional:** $\Phi : \mathcal{C} \to \overline{\mathbb{R}}$ defined by $\Phi(V) = V(1)$, i.e., the total mass of $V \in \mathcal{C}$.

1. choose a minimising sequence $V_i \in \mathcal{C}$, i.e.,
   \[ \lim_{i \to \infty} \Phi(V_i) = \inf \text{im } \Phi \]
2. use the Banach-Alaoglu theorem to find a weakly convergent sub-sequence
3. pass to the limit $V_i \xrightarrow{i \to \infty} V$

**Questions.** Is $V$ rectifiable? Is it associated to a single manifold?

**Remark.** One can consider $\Phi_F(V) = V(F)$, where $F : U \times G(n, k) \to \mathbb{R}$ is continuous and s.t.

\[ 0 < \inf \text{im } F < \sup \text{im } F < \infty. \]
Theorem (Almgren, Ann. of Math, 1968)

Assume $C$ is a class of rectifiable varifolds “spanning” a given boundary, $F : U \times G(n, k) \to \mathbb{R}$ is of class $C^k$ and s.t. $0 < \inf \text{im } F < \sup \text{im } F < \infty$, $\Phi_F(V) = V(F)$ for $V \in C$, $V$ is a minimiser of $\Phi_F$, and $F$ is uniformly elliptic. Then $V$ is regular (of class $C^{k-1}$) almost everywhere, i.e., there exists a manifold $M$ of class $C^{k-1}$, which is dense in $\text{spt } V$ and such that $\|V\| (\text{spt } V \sim M) = 0$.

Theorem (Almgren’s big regularity paper, 2000)

If $F \equiv 1$, then the singular set has Hausdorff dimension $\leq k - 2$.

Theorem (Federer, Bull. AMS, 1970)

If $F \equiv 1$ and $k = n - 1$, then the singular set has Hausdorff dimension $\leq k - 7$. 
Assume

1. $\mu$ is a Borel regular measure over $\mathbb{R}^n$ (i.e. open sets are $\mu$-measurable and each set is a subset of a Borel set with the same measure),
2. $\mu$ is translation invariant,
3. $\mu$ is finite on any bounded subset of any $T \in G(n,k)$,
4. $\mu$ is continuous w.r.t. $C^1$-deformations, i.e., given $f_k \xrightarrow{k \to \infty} 0$ in $C^1(\mathbb{R}^n, \mathbb{R}^n)$ and a bounded set $A \subseteq \mathbb{R}^n$ with $\mu(A) < \infty$ there holds

$$
\mu((\text{id}_{\mathbb{R}^n} + f_k)[A]) \xrightarrow{k \to \infty} \mu(A).
$$

Then there exists $F : G(n,k) \to [0, \infty)$ s.t.

$$
\mu(M) = \int F(\text{Tan}(M, x)) \, d\text{vol}_M(x) \quad \text{for any } C^1\text{-manifold } M \subseteq \mathbb{R}^n.
$$
Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a (non-Euclidean) norm. The $k$-dimensional Hausdorff measure is

$$H^k_\phi (A) = \liminf_{\delta \downarrow 0} \left\{ \frac{\alpha(k)}{2^k} \sum_{E \in \mathcal{F}} \text{diam}_\phi(E)^k : \begin{array}{l} \mathcal{F} \subseteq 2^{\mathbb{R}^n} \text{ countable}, \\ A \subseteq \bigcup \mathcal{F}, \\ \text{diam}_\phi(E) < \delta \text{ for } E \in \mathcal{F} \end{array} \right\}$$

**Lemma (Busemann, Ann. of Math, 1947)**

For any $C^1$-manifold $M \subseteq \mathbb{R}^n$ there holds

$$H^k_\phi (M) = \int_{\mathbb{R}^n} F^{BH}(\text{Tan}(M, x)) \, d\text{vol}_M(x),$$

where

$$F^{BH}(T) = \frac{\alpha(k)}{H^k(B_\phi(0, 1) \cap T)}.$$

**Remark.** $H^n = \mathcal{L}^n$ over $\mathbb{R}^n$. 
Why consider $\Phi_F$

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\[ \mu(D_1) \neq \mu(D_2) \]
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external anisotropic force (e.g. gravity)
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Assume $V$ is a $k$-varifold in $U \subseteq \mathbb{R}^n$ s.t. for any choice of $g : U \to \mathbb{R}^n$ a $C^1$-vectorfield with compact support, defining

$$f_t(x) = x + tg(x) \quad \text{for } x \in U \text{ and } t \in (-\varepsilon, \varepsilon),$$

there holds

$$\frac{d}{dt} \bigg|_{t=0} \Phi_F(f_t\#V) = 0.$$ 

**Question.** What can we say about the regularity of $V$?
For simplicity for the rest of my talk assume $F(x, T) = F(T)$.

$$g : U \to \mathbb{R}^n, \quad f_t(x) = x + tg(x),$$

$$\delta_F V(g) = \frac{d}{dt} \bigg|_{t=0} \Phi_F(f_t#V)$$

$$= \int \text{trace} \left( P_F(T) \circ Dg(x) \right) F(T) \, dV(x, T),$$

where $P_F(T) \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ is a (non-orthogonal) projection onto $T$.

**Definition**

We say that $V$ is $F$-stationary if $\delta_F V = 0$.

**Remark.** In case $F \equiv 1$ and $M \subseteq \mathbb{R}^3$ is a minimal surface, then $\delta_F V = 0$. 


**Definition**

\((S, D)\) is a *test pair* if \(S\) is a \(k\)-rectifiable set, \(D\) is a flat \(k\)-disc, and \(S\) cannot be retracted onto \(\partial D\).

**Definition**

\[ F \in \text{AE} \] iff. for any test pair \((S, D)\) with \(\mathcal{H}^k(S) > \mathcal{H}^k(D)\)

\[ \Phi_F(S) > \Phi_F(D). \]

\[ F \in \text{UAE} \] iff. there is \(c > 0\) s.t. for any test pair \((S, D)\)

\[ \Phi_F(S) - \Phi_F(D) \geq c(\mathcal{H}^k(S) - \mathcal{H}^k(D)). \]
More ellipticity conditions

\[ F \in \text{BC} \iff \text{for any } T \in \mathbf{G}(n,k) \text{ and } \mu \text{ a probability measure over } \mathbf{G}(n,k) \]

\[ \delta_F(\mathcal{H}^k \triangleleft T \times \mu) = 0 \implies \mu = \text{Dirac}(T), \]

i.e., any stationary varifold supported in some \( T \in \mathbf{G}(n,k) \) which is translation invariant in \( T \) is rectifiable.

Let

\[ \mathcal{G}_F = \{ P_F(T) : T \in \mathbf{G}(n,k) \} \subseteq \text{Hom}(\mathbb{R}^n, \mathbb{R}^n). \]

\[ F \in \text{AC} \iff \mathcal{G}_F \text{ is the set of extreme points of } \text{conv } \mathcal{G}_F \text{ and} \]

\[ \text{conv } \mathcal{G}_F \cap \{ A : \dim \text{ im } A \leq k \} = \mathcal{G}_F. \]
Theorem (De Philippis, De Rosa, Ghiraldin, CPAM, 2018)

Assume $F \in \text{AC}$, $c > 0$, $V$ is a $k$-varifolds such that $\Theta^k(\|V\|, x) > c$ for $\|V\|$ almost all $x$ and $\delta_F V$ is representable by integration. Then $V$ is rectifiable.

Remark. In particular, any $F$-stationary varifold with $k$-density bounded away from zero is rectifiable.

Remark. $F \in \text{AC}$ is also a necessary condition for the above implication.

Theorem (De Rosa, K., CPAM, 2020)

$\text{AC} = \text{BC} \subseteq \text{AE}$
Regularity of critical graphs

\[ F \in \text{mUSAC} \iff \text{there is } c > 0 \text{ s.t. for each } T \in G(n, k) \text{ there exists } N \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \text{ s.t.} \]

\[ N \perp \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \cap \{ A : T \subseteq \text{im } A \} \]

and \[ N \cdot P_F(S) \geq c\|S - T\|^2. \]

**Theorem (De Rosa, Tione, Invent. math., 2022)**

Assume \( p > k, F \in \text{mUSAC}, \Omega \subseteq \mathbb{R}^k, V \) is associated to a graph of a Lipschitz function \( f : \Omega \to \mathbb{R}^{n-k} \), and \( \delta_F V \) representable by integration against some \( L_p(\|V\|, \mathbb{R}^n) \) function. Then there exists an open set \( \Omega_0 \subseteq \Omega \) of full measure s.t. \( f|\Omega_0 \) is of class \( C^{1,\alpha} \), where \( \alpha = 1 - k/p. \)
\( F \in \text{mUSAC} \) iff. there is \( c > 0 \) s.t. for each \( T \in \mathbb{G}(n,k) \) there exists \( N \in \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \) s.t.

\[
N \perp \text{Hom}(\mathbb{R}^n, \mathbb{R}^n) \cap \{ A : T \subseteq \text{im} A \},
\]

and

\[
N \cdot P_F(S) \geq c \| S - T \|^2 \quad \text{for } S \in \mathbb{G}(n,k).
\]
Regularity of critical points of the area integrand

**Theorem (Allard, Ann. of Math., 1972)**

Assume $p > k$, $F \equiv 1$, $V$ is a $k$-varifol with density bounded away from zero, $\delta_F V$ representable by integration against some $L_p(\|V\|, \mathbb{R}^n)$ function. Then the set of regular points is open and dense in $\text{spt} \|V\|$.

**Theorem (Wickramasekera, Ann. of Math, 2014)**

Assume $k = n - 1$, $F \equiv 1$, $V$ is an integral $k$-varifol which is stationary and stable. Then the singular set consists of points which have a neighbourhood in which $V$ is made of at least three $C^{1,\alpha}$ hypersurfaces meeting along their common boundary and the rest which has Hausdorff dimension at most $k - 7$. 
If \( k = n - 1 \), then \( S^k \xrightarrow{\pi} \mathbb{RP}(k) \simeq G(n, k) \).

**Fact.** \( F \in \text{AC} \) iff. there exists a strictly convex norm \( G \) s.t. \( F(\pi(\nu)) = G(\nu) \) for \( \nu \in S^k \).

**Fact.** \( F \in \text{UAE} \) iff. there exists a uniformly convex norm \( G \) s.t. \( F(\pi(\nu)) = G(\nu) \) for \( \nu \in S^k \).

**Remark.** In higher co-dimension no particular examples of \( F \in \text{AC} \) are known!
Summary

- $F \in \text{UAE}$ gives partial regularity of minimisers
- $F \in \text{AC} = \text{BC} \subseteq \text{AE}$ gives rectifiability of critical points
- $F \in \text{mUSAC}$ gives partial regularity for critical graphs
- $F \equiv 1$
  - singular set of a critical point is open and dense in the support
  - singular set of a minimiser is of dimension $\leq k - 2$
  - if $n = k + 1$, singular set of a minimiser is of dimension $\leq k - 7$
  - if $n = k + 1$, singularities of a stable critical point are classified into two categories; the dimension of one of them is $\leq k - 7$, the other has dimensions $k - 1$ is unavoidable and is well understood.
Open questions

1. $AE = AC$?
2. $mUSAC \subseteq UE$?
3. $F_{BH} \in AC$?
4. How to construct examples of $F \in AC$?
5. Does any analogue of the Allard regularity theorem holds for $F \in mUSAC$?
6. Is the singular set of measure zero? (not known even for $F \equiv 1$)
7. What is the precise dimension of the singular set?
Thank you for listening.