Maximal regularity in Partial Differential Equations

Tomasz Piasecki

MIM Faculty Colloquium

Warsaw, November 17, 2022

Outline

1. Derivation of basic equations
2. Notion of maximal regularity
3. Simple examples of maximal regularity
4. Application to Compressible Navier-Stokes system

Introduction

- Partial Differential Equations originates from applications; we don't want to forget about it so we will start with some physical motivation
- If we want to describe some phenomenon possibly accurately, the resulting PDE is very often impossible to solve with explicit formula;
- Then the first step in the analysis is to know if a solution exists. We are even more happy if the solution is **unique.** In this talk we will concentrate on these aspects.

Gauss Divergence Theorem and corollaries

Assume $\Omega \subset \mathbb{R}^k$ sufficiently regular and $\mathbf{v}$ is a continuously differentiable vector field defined on $\bar{\Omega}$. Then

$$\int_\Omega \operatorname{div} \mathbf{v} \, \mathrm{dx} = \int_{\partial\Omega} \mathbf{v} \cdot \mathbf{n} \, \mathrm{dS},$$

where $\mathbf{n} = (n^1, \ldots, n^k)$ is the outer unit normal to $\partial\Omega$.

Corollary: integration by parts:

- $\int_\Omega f_{x_i} = \int_{\partial\Omega} f n^i \, \mathrm{dS}$
- $\int_\Omega f g_{x_i} = -\int_\Omega g_{x_i} f \; \mathrm{dx} + \int_{\partial\Omega} f g n^i \; \mathrm{dS}$
- $\int_\Omega f \Delta g \, \mathrm{dx} = -\int_\Omega \nabla g \cdot \nabla f \; \mathrm{dx} + \int_{\partial\Omega} f \frac{\partial g}{\partial \mathbf{n}} \, \mathrm{dS}$

Laplace equation and heat equation (1)

Assume u is a density of some diffusing substance, We want to describe the evolution of total mass of the substance in given domain Ω. Then

$$\underbrace{\frac{\partial}{\partial t}\int_\Omega u(x)\,\mathrm{dx}}_{\text{change of mass}} = \underbrace{-\int_{\partial\Omega} F(u)\cdot\mathbf{n}\,\mathrm{dS}}_{\text{diffusion}} + \underbrace{\int_\Omega f\,\mathrm{dx}}_{\text{production}},$$

where $\mathbf{n}$ is the outer unit normal vector to the boundary of Ω and $F(u)$ is the flux through the boundary. In a simplest case we can assume that the flux is proportional to ∇u and diffusion is in the direction of lower density. Then

$$F(u) = -c\nabla u,$$

therefore by the divergence theorem

$$-\int_{\partial\Omega} F(u)\cdot\mathbf{n}\,\mathrm{dS} = c\int_{\partial\Omega}\nabla u\cdot\mathbf{n}\,\mathrm{dS} = c\int_\Omega \Delta u\,\mathrm{dx}$$

and we obtain

$$\frac{\partial}{\partial t}\int_\Omega u\,\mathrm{dx} - c\int_\Omega \Delta u\,\mathrm{dx} = \int_\Omega f\,\mathrm{dx}$$

Laplace equation and heat equation (2)

Assuming we can pass with time derivative under the integral we obtain

$$\int_\Omega (u_t - c\Delta u)\,\mathrm{dx} = \int_\Omega f\,\mathrm{dx},$$

therefore, since Ω is arbitrary, we obtain the **heat equation**

$$u_t - \Delta u = f.$$

Assuming the total mass is conserved we arrive at the **Laplace equation**

$$-\Delta u = f$$

These are model examples of, correspondingly, **parabolic** and **elliptic** problems.

Elliptic and parabolic equations

General elliptic equation: $Au(x) = f(x)$

General parabolic equation: $\partial_t u(t,x) + Au(t,x) = f(t,x)$

where

$$Au = \sum (a_{ij} u_{x_i})_{x_j}$$

and the coefficients a_{ij} satisfy **ellipticity** (or **coercivity**) condition:

$$\exists \theta > 0 : \quad \sum_{i,j} a_{ij} y_i y_j \geq \theta |y|^2$$

Originate from many applications: fluid mechanics, macroscale description of collective dynamics, biological models, ...

Lebesque and Bochner spaces

- For $\Omega \subset \mathbb{R}^n$ and $1 \leq p \leq \infty$ we define

$$\|f\|_{L_p(\Omega)} = \begin{cases} \left(\int_\Omega |f|^p \,\mathrm{dx}\right)^{1/p}, & 1 \leq p < \infty \\ \operatorname{ess\,sup}_{x\in\Omega}|f|, & p = \infty \end{cases}$$

  Then

$$L_p(\Omega) = \{f : \Omega \to \mathbb{R},\ \|f\|_{L_p(\Omega)} < +\infty\}$$

- Let $I \subset \mathbb{R}$ an interval and X a Banach space. Then for $f : I \to X$ we define

$$\|f\|_{L_p(I;X)} = \begin{cases} \left(\int_I \|f(t)\|_X^p\right)^{1/p}, & 1 \leq p < \infty \\ \operatorname{ess\,sup}_{t\in I}\|f(t)\|_X, & p = \infty \end{cases}$$

  Then we define

$$L_p(I;X) = \{f : I \to X,\ \|f\|_{L_p(I;X)} < \infty\}$$

Weak derivatives

Corollary from integration by parts: if $\phi \in C_c^\infty(\Omega)$ then

$$\int_\Omega f\phi_{x_i} = \int_\Omega \phi_{x_i} f \,\mathrm{dx}$$

Assume $\Omega \subset \mathbb{R}^k$ open and $f, g \in L^1_{loc}(\Omega)$. We say that $g = \partial_{x_i} f$ in the weak sense, if

$$\int_\Omega f\partial_{x_i}\phi \,\mathrm{dx} = -\int_\Omega g\phi \,\mathrm{dx} \quad \forall \phi \in C_c^\infty(\Omega)$$

Multiindex notation:

$$\alpha = (\alpha_1, \dots \alpha_m),\ \alpha_i \geq 0,\ |\alpha| := \sum_{i=1}^m \alpha_i,$$

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \dots \partial x_m^{\alpha_m}}$$

Assume $\Omega \subset \mathbb{R}^k$ open and $f, g \in L^1_{loc}(\Omega)$. We say that $g = D^\alpha f$ in the weak sense, if

$$\int_\Omega f\partial_\alpha\phi \,\mathrm{dx} = (-1)^{|\alpha|}\int_\Omega g\phi \,\mathrm{dx} \quad \forall \phi \in C_c^\infty(\Omega).$$

Sobolev spaces

For $k \in \mathbb{N}$ and $p \in [1, \infty)$ we define

$$W_p^k(\Omega) = \{f \in L_p(\Omega) : \ D^\alpha f \in L_p(\Omega) \text{ for } |\alpha| \leq p\}$$

with the norm

$$\|f\|_{W_p^k(\Omega)} = \sum_{|\alpha| \leq k} \|D^\alpha f\|_{L_p(\Omega)}$$

- Sobolev spaces are Banach spaces
- We now describe briefly two other important properties: Trace Theorem and Imbedding Theorem

Boundary traces

- **Trace theorems** state that functions from Sobolev spaces can be defined on a boundary (functions only from L_p are defined almost everywhere; so in general have no meaning on the boundary);
- **basic Trace Theorem:** There exists $T : W_p^1(\Omega) \to L_p(\Omega)$ such that $Tu = u$ for $u \in W_p^1(\Omega) \cap C^1(\bar{\Omega})$ and $\|Tu\|_{L_p(\Omega)} \leq C\|u\|_{W_p^1(\Omega)}$
- This theorem is not optimal; indeed the trace is more regular: it is in the fractional space $W_p^{1-1/p}(\partial\Omega)$ which is somehow 'between' L_p and W_p^1. Then also inverse holds: any function from $W_p^{1-1/p}(\Omega)$ admits and extension $Eu \in W_p^1(\Omega)$.
- Functions with zero trace ('vanishing' on the boundary):

$$W_0^{1,p}(\Omega) = \{u \in W^{1,p}(\Omega) : \exists u \in C_c^\infty(\Omega), u_n \to u \text{ in } W^{1,p}(\Omega)\}.$$

- Poincaré inequality (very important for us):

$$\|u\|_{L_p(\Omega)} \leq C\|\nabla u\|_{L_p(\Omega)} \text{ for } u \in W_0^{1,p}(\Omega)$$

Sobolev imbedding

Gagliardo-Nirenberg-Sobolev inequality: for $p < n$ and compactly supported $\phi \in C^1(\mathbb{R}^n)$:

$$\|\phi\|_{L_p^*} \leq C\|\nabla\phi\|_{L_p},\ C \text{ independent of } \phi\,(!)$$

where $p^* = \frac{np}{n-p}$ is the Sobolev conjugate exponent to p.

Morrey inequality: for $p > n$ and $\phi \in C^1(\mathbb{R}^n)$

$$\|\phi\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C\|\phi\|_{W^{1,p}(\mathbb{R}^n)}, \quad \alpha = 1 - \frac{n}{p}$$

Sobolev imbedding theorem: For $\Omega \subset \mathbb{R}^n$ bounded we have

- $p < n \Rightarrow W_p^1(\Omega) \subset L_q(\Omega)$ for $q \leq p^*$
- $p > n \Rightarrow W_p^1(\Omega) \subset C^{0,\alpha}(\Omega),\ \alpha = 1 - \frac{n}{p}$

Important corollary: Functions from sufficiently high Sobolev spaces are classically differentiable, possibly many times.

Weak solutions

- Weak solutions are functions from some Sobolev-type spaces which satisfy certain integral identities related to the original formulation.
- **Example:** A weak solution to the Laplace equation

$$\Delta u = f \in \Omega, \quad u = 0 \text{ on } \partial\Omega$$

  is a function $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{dx} = \int_\Omega f v \, \mathrm{dx} \quad \forall\, v \in W_0^{1,2}(\Omega)$$

- Sometimes it is the only thing we can get (it is not so bad: integral formulation may be natural, recall derivation of the Laplace Equation)
- Sometimes it is a first step, then we can show that weak solutions are more regular

What is maximal regularity?

Assume Au is a second order differential operator. Then we say that A has L_p **maximal regularity** property if a solution u to the problem

$$u_t + Au = f, \quad u|_{t=0} = 0$$

satisfies

$$\|u_t\|_{L_p(0,T;X)} + \|Au\|_{L_p(0,T;X)} \leq C\|f\|_{L_p(0,T;X)}, \tag{1}$$

Where X is some Banach space.
If we also have **elliptic regularity** $\|D^2u\|_X \leq C\|Au\|_X$ then (1) implies

$$\|u_t\|_{L_p(0,T;X)} + \|D^2u\|_{L_p(0,T;X)} \leq C\|f\|_{L_p(0,T;X)}, \tag{2}$$

why is it important?

- If X is sufficiently high Sobolev space then, by Sobolev Imbedding, maximal regularity implies that the solution, although we work in L_p setting, is indeed classical.
- It is also powerful tool in solving nonlinear partial differential equations (we will see an example in the end of the talk)

Simple examples in the whole space via Fourier Transform

For $u \in L_1(\mathbb{R}^n)$ we define the Fourier transform:

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) \, \mathrm{dx}$$

Basic properties:

- $\hat{u}_{x_i} = i\xi_i \hat{u}$
- Plancherel Theorem:
  $u \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n) \Rightarrow \|u\|_{L_2(\mathbb{R}^n)} = \|\hat{u}\|_{L_2(\mathbb{R}^n)}$

Plancherel Theorem allows to extend Fourier transform to $L_2(\mathbb{R}^n)$ by density argument: if $u \in L_2(\mathbb{R}^n)$ then there exists $u_n \in L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^n)$ such that $u_n \to u$ in $L_2(\mathbb{R}^n)$. Then in particular $\{u_n\}$ is a Cauchy sequence in $L_2(\mathbb{R}^n)$, so by Plancherel Theorem $\{\hat{u}_n\}$ is a Cauchy sequence in $L_2(\mathbb{R}^n)$, therefore

$$\hat{u}_n \to g \text{ in } L_2(\mathbb{R}^n)$$

for some $g \in L_2(\mathbb{R}^n)$ and we define $\hat{u} := g$.

Laplace equation in $\mathbb{R}^n$

Consider

$$-\Delta u = f \text{ in } \mathbb{R}^n$$

with $f \in L_2(\mathbb{R}^n)$. Applying Fourier Transform to the equation we get

$$|\xi|^2 \hat{u} = \hat{f} \text{ in } \mathbb{R}^n,$$

therefore by Plancherel Thm

$$\|u_{x_i x_j}\|_{L_2(\mathbb{R}^n)} = \|\widehat{u_{x_i x_j}}\|_{L_2(\mathbb{R}^n)} = \|\xi_i \xi_j \hat{u}\|_{L_2(\mathbb{R}^n)}$$

$$\leq \||\xi|^2 \hat{u}\|_{L_2(\mathbb{R}^n)} = \|\hat{f}\|_{L_2(\mathbb{R}^n)} = \|f\|_{L_2(\mathbb{R}^n)}$$

so we obtain

$$\|D^2 u\|_{L_2(\mathbb{R}^n)} \leq \|f\|_{L_2(\mathbb{R}^n)}$$

General elliptic equation in $\mathbb{R}^n$

$$-\sum_{i,j} a_{ij} u_{x_i x_j} = f \text{ in } \mathbb{R}^n$$

with $f \in L_2(\mathbb{R}^n)$. Applying Fourier Transform to the equation we get

$$\sum a_{ij} \xi_i \xi_j \hat{u} = \hat{f},$$

but by ellipticity condition

$$\sum a_{ij} \xi_i \xi_j \geq \theta |\xi|^2,$$

therefore

$$\theta \| |\xi|^2 \hat{u} \|_{L_2(\mathbb{R}^n)} \leq \frac{1}{\theta} \| \hat{f} \|_{L_2(\mathbb{R}^n)}$$

and, similarly as before

$$\| u_{x_i x_j} \|_{L_2(\mathbb{R}^n)} \leq \| |\xi|^2 \hat{u} \|_{L_2(\mathbb{R}^n)} \leq \frac{1}{\theta} \| \hat{f} \|_{L_2(\mathbb{R}^n)} \leq \frac{1}{\theta} \| f \|_{L_2(\mathbb{R}^n)}.$$

Heat equation in $\mathbb{R}^n$ (1)

$$\begin{aligned} & u_t - \Delta u = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+ \\ & u|_{t=0} = g(x) \end{aligned} \tag{3}$$

Applying Fourier transform in space variable we obtain

$$\hat{u}_t + |\xi|^2 \hat{u} = 0, \quad \hat{u}|_{t=0} = \hat{g},$$

therefore

$$\hat{u}(t,\xi) = e^{-t|\xi|^2} \hat{g},$$

and so

$$\|u_{x_i x_j}(t,\cdot)\|_{L_2} = \|\widehat{u_{x_i x_j}}(t,\cdot)\|_{L_2} = \|\xi_i \xi_j \hat{u}\|_{L_2(\mathbb{R}^n)}$$

$$\leq \||\xi|^2 \hat{u}(t,\cdot)\|_{L_2(\mathbb{R}^n)} = \||\xi|^2 e^{-t|\xi|^2} \hat{g}\|_{L_2(\mathbb{R}^n)} \leq C\|g\|_{L_2(R^n)}$$

Remarks to heat equation in the whole space:

- Inverting the Fourier transform we obtain explicit integral formula for the solution;
- the formula $\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{g}$ implies

$$\||\xi|^n \hat{u}(t,\xi)\|_{L_2(\mathbb{R}^n)} \leq C\|\hat{g}\|_{L_2(\mathbb{R}^n)}$$

So, at least formally, the derivatives of u of arbitrary order are square integrable. This can be justified rigorously due to Sobolev imbedding theorem. We conclude that solutions are smooth in the space variable for any positive time, even if initial data is only in L_2.

General parabolic equation in $\mathbb{R}^n$

$$\begin{aligned} & u_t + \sum a_{ij} u_{x_i x_j} = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}_+ \\ & u|_{t=0} = g(x) \end{aligned} \tag{4}$$

Applying Fourier transform in space variable we obtain

$$\hat{u}_t + \sum a_{ij}\xi_j\xi_j\hat{u} = 0, \quad \hat{u}|_{t=0} = \hat{g},$$

therefore

$$\hat{u}(t,\xi) = e^{-t\sum a_{ij}\xi_i\xi_j}\hat{g} \leq e^{-\theta t|\xi|^2}\hat{g},$$

so, similarly as before,

$$\|D^2 u(t,\cdot)\|_{L_2} \leq C\|g\|_{L_2(\mathbb{R}^n)}.$$

The remark on higher regularity also remains valid.

L_2 maximal regularity in a bounded domain: energy methods

Formal energy estimate (1)

$$u_t - \Delta u = f \text{ in } \Omega \times (0,T), \quad u|_{\partial\Omega} = 0, \; u|_{t=0} = 0; \tag{5}$$

We multiply the equation by u and integrate by parts; we get

$$\int_\Omega u_t u \,\mathrm{dx} + \int_\Omega |\nabla u|^2 \,\mathrm{dx} = \int_\Omega f u \,\mathrm{dx},$$

which is equivalent to

$$\frac{1}{2}\partial_t \|u(t)\|^2_{L_2(\Omega)} + \|\nabla u\|^2_{L_2(\Omega)} = \int_\Omega f u \,\mathrm{dx}.$$

By the Poincaré and Young inequalities

$$\Big| \int_\Omega f u \,\mathrm{dx} \Big| \le \epsilon \|u\|^2_{L_2(\Omega)} + C(\epsilon)\|f\|^2_{L_2(\Omega)} \le \epsilon\, C_P \|\nabla u\|^2_{L_2(\Omega)} + C(\epsilon)\|f\|^2_{L_2(\Omega)}$$

where C_P is the constant from Poincaré inequality. Therefore

$$\frac{1}{2}\partial_t \|u(t)\|^2_{L_2(\Omega)} + (1 - \epsilon C_P)\|\nabla u\|^2_{L_2(\Omega)} \le C\|f\|^2_{L_2(\Omega)}$$

Integrating this inequality in time we get

$$\sup_{t\in(0,T)} \|u(t)\|_{L_2(\Omega)} + \|\nabla u(t)\|_{L_2(0,T;L_2(\Omega))} \le C\|f\|_{L_2(0,T;L_2(\Omega))}$$

Formal energy estimate (2)

$$u_t - \Delta u = f \text{ in } \Omega \times (0,T), \quad u|_{t=0} = 0; \tag{6}$$

We multiply the equation by $-\Delta u$ and integrate by parts. Notice that

$$-\int_\Omega u_t u_{x_i x_i} = \int_\Omega u_{t,x_i} u_{x_i}\, dx = \int_\Omega (u_{x_i})_t u_{x_i}\, dx = \frac{1}{2}\frac{\partial}{\partial t}\int_\Omega |u_{x_i}|^2\, \mathrm{dx},$$

therefore we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\|\nabla u\|^2_{L_2(\Omega)} + \|\Delta u\|^2_{L_2(\Omega)} = \int_\Omega f \Delta u\, \mathrm{dx} \le \frac{1}{2}\|f\|_{L_2(\Omega)} + \frac{1}{2}\|\Delta u\|^2_{L_2(\Omega)}$$

Integrating this inequality in time we get

$$\sup_{t\in(0,T)}\|\nabla u(t)\|_{L_2(\Omega)} + \|\Delta u(t)\|_{L_2(0,T;L_2(\Omega))} \le C\|f\|_{L_2(0,T;L_2(\Omega))}.$$

Formal energy estimate (3)

To have a full maximal regularity we need:

$$\|D^2 u\|_{L_2(\Omega)} \le C\|\Delta u\|_{L_2(\Omega)}, \tag{7}$$

Assuming u is sufficiently regular, integrating twice by parts we get

$$\int_\Omega u^2_{x_i x_j}\,\mathrm{dx} = -\int_\Omega (u_{x_i x_j})_{x_i} u_{x_j}\,\mathrm{dx} = \int_\Omega u_{x_i x_i} u_{x_j x_j}\,\mathrm{dx},$$

therefore

$$\int_\Omega u^2_{x_i x_j}\,\mathrm{dx} \le \frac{1}{2}\int_\Omega \left(u^2_{x_i x_i} + u^2_{x_j x_j}\right)\mathrm{dx}.$$

Summing over i, j we obtain (7). Alltogether

$$\|u_t\|_{L_2(0,T;L_2(\Omega))} + \|D^2 u\|_{L_2(0,T;L_2(\Omega))} + \sup_{t\in(0,T)} \|u\|_{W^{1,2}(\Omega)} \le C\|f\|_{L_2(0,T;L_2(\Omega))}$$

Higher regularity

Differentiating the equation (assuming that it is allowed) we obtain in a similar way for $k \in \mathbb{N}$.

$$\|u_t\|_{L_2(0,T;W^k(\Omega))} + \|D^2 u\|_{L_2(0,T;W^k(\Omega))} + \sup_{t\in(0,T)} \|u\|_{W^{1,k+1}(\Omega)}$$

$$\leq C\|f\|_{L_2(0,T;W^k(\Omega))}$$

- By Sobolev imbedding, we can conclude that the solution is smooth provided f is smooth (!)
- We have seen just formal estimates, but this approach can be made rigorous in the language of weak solutions: we obtain the estimates chosing appropriate test functions in the weak formulation

Applications to nonlinear PDE

Consider the Navier-Stokes system describing flow of compressible, viscous fluid

$$\begin{aligned}
&\varrho_t + \operatorname{div}(\varrho \mathbf{u}) = 0 \\
&\varrho \mathbf{u}_t + \varrho \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - \nu \nabla \operatorname{div} \mathbf{u} + \nabla \pi(\varrho) = \varrho \mathbf{f}, \\
&(\varrho, \mathbf{u})|_{t=0} = (\varrho_0, \mathbf{u}_0), \quad \mathbf{u}|_{\partial\Omega} = 0.
\end{aligned}$$

- ϱ is the density, $\mathbf{u}$ the velocity, $\pi(\varrho)$ the pressure given by some constitutive relation
- First equation is the conservation of mass, second describes the balance of momentum

Fixed point approach to nonlinear PDEs

How to solve a nonlinear PDE:

$$Lu = f(u),$$

where L is a linear differential operator and f is nonlinear?
One of possible approaches:

- Linearize: $Lu = f(w)$
- Solve the corresponding linear problem $Lu = f$
- Define a solution operator $u = Tw \iff Lu = f(w)$, then the fixed point of T is a solution to our nonlinear problem
- Use some fixed point theorem to prove the existence of the fixed point

Let's come back to the Navier-Stokes system

$$
\begin{aligned}
&\varrho_t + \varrho \cdot \nabla \mathbf{u} + \varrho \mathrm{div}\, \mathbf{u} = 0 \\
&\varrho \mathbf{u}_t + \varrho \mathbf{u} \cdot \nabla \mathbf{u} - \mu \Delta \mathbf{u} - \nu \nabla \mathrm{div}\, \mathbf{u} + \nabla \pi(\varrho) = \varrho \mathbf{f}, \\
&(\varrho, \mathbf{u})|_{t=0} = (\varrho_0, \mathbf{u}_0), \quad \mathbf{u}|_{\partial\Omega} = 0.
\end{aligned}
$$

- The blue term is an elliptic linear operator under reasonable assumptions on the coefficients μ, ν. So the second equation has some nice properties discussed before.
- In the first equation there is no regularization;
- On of common approaches is application of so called **Lagrangian coordinates.** Then

$$
\partial_t + \mathbf{u} \cdot \nabla \longrightarrow \partial_t
$$

Lagrangian coordinates

Lagrangian coordinates y is a solution to the system of ordinary differential equations:

$$\partial_t X(t,y) = \mathbf{u}(t, X(t,y)), \quad X(0,y) = y.$$

Denoting $\tilde{f}(t,y) = f(t, X(t,y))$ we have

$$\partial_t \tilde{f} = \partial_t f + \partial_t X \cdot \nabla_x f = \partial_t f + \mathbf{u} \cdot \nabla_x f$$

The continuity equation in Lagrangian coordinates becomes

$$\partial_t \tilde{\varrho} = -\tilde{\varrho} \operatorname{div}_x \mathbf{u}, \quad \tilde{\varrho}(0,y) = \varrho_0(y)$$

solving this ordinary differential equation we get

$$\tilde{\varrho}(t,y) = \exp\Big(- \int_0^t \operatorname{div}_x \mathbf{u}(s,y) ds \Big),$$

provided $\nabla \mathbf{u} \in L_1(0,T; L_\infty(\Omega))$

Maximal regularity for the momentum equation

Linearization of the momentum equation reads

$$\varrho^* \partial_t \mathbf{u} - \mu \Delta \mathbf{u} - \nu \nabla \operatorname{div} \mathbf{u} = F$$

for some constant ϱ^* and given function F.

- The blue term is elliptic under reasonable assumptions on the coefficients μ and ν.
- Using energy methods similar to described before we can show maximal regularity:

$$\|\partial_t \mathbf{u}\|_{L_2(0,T;L_2)} + \|\mathbf{u}\|_{L_2(0,T;W^{2,2})} \leq C\left[\|F\|_{L_2(0,T;L_2)} + \|\mathbf{u}_0\|_{W^{1,2}}\right]$$

- But recall that Sobolev imbedding does not give $\nabla \mathbf{u} \in L_\infty$ for $u \in W^{2,2}$. Therefore we have problems to define Lagrangian coordinates (and also other difficulities without going into details).

How to get $\nabla \mathbf{u}$ bounded: 1st approach

- Work in L_2 but with more regular solutions; then we need more regular data and we can show

$$\|\partial_t \mathbf{u}\|_{L_2(0,T;W^{1,2})} + \|\mathbf{u}\|_{L_2(0,T;W^{3,2})} \leq C\left[\|F\|_{L_2(0,T;W^{1,2})} + \|\mathbf{u}_0\|_{W^{2,2}}\right]$$

  and now $W^{3,2} \subset W^{1,\infty}$ in space dimension ≤ 3.

- First results on regular solutions to the compressible Navier-Stokes equations were proved to exist with this approach: Matsumura-Nishida, Valli, Valli & Zajączkowski '80s

How to get $\nabla \mathbf{u}$ bounded: 2nd approach

- Work in L_p where p is higher then the space dimension; Sobolev imbedding then gives $W^{2,p} \subset W^{1,\infty}$. So we get $\nabla \mathbf{u} \in L_\infty$ provided we have $L_p - L_q$ maximal regularity:

$$\|\partial_t \mathbf{u}\|_{L_p(0,T;L_q)} + \|\mathbf{u}\|_{L_p(0,T;W^{2,q})} \leq C \left[\|F\|_{L_p(0,T;L_q)} + \|\mathbf{u}_0\|_X \right]$$

  where X is some Banach space, usually of Besov type (we shall not define it now)

- This approach to compressible Navier-Stokes started to develop in the 90's and continued in 2000's: Mucha, Zajączkowski, Choe et al., Shibata et al., Hieber et al. ...

$L_p - L_q$ maximal regularity for Compressible Navier-Stokes:

Enomoto, Shibata (2013)

Ideas

- Lagrangian coordinates: $\partial_t + \mathbf{u}\cdot\nabla \to \partial_t$; we obtain

$$
\begin{aligned}
&\varrho_t + \varrho_0 \mathrm{div}\,\mathbf{u} = g(\mathbf{u}, w)\\
&\varrho_0\mathbf{u}_t - \mu\Delta\mathbf{u} - \nu\nabla\mathrm{div}\,\mathbf{u} + \pi'(\varrho_0)\nabla\varrho = \mathbf{f}(\mathbf{u}, w),\\
&(\varrho, \mathbf{u})|_{t=0} = (\varrho_0, \mathbf{u}_0), \quad \mathbf{u}|_{\partial\Omega} = 0.
\end{aligned}
$$

- Solution operator: $(\mathbf{u}, \varrho) = T(\mathbf{w}, \eta) \iff$

$$
\begin{aligned}
&\varrho_t + \varrho_0 \mathrm{div}\,\mathbf{u} = g(\mathbf{w}, \eta)\\
&\varrho_0\mathbf{u}_t - \mu\Delta\mathbf{u} - \nu\nabla\mathrm{div}\,\mathbf{u} + \pi'(\varrho_0)\nabla\varrho = \mathbf{f}(\mathbf{w}, \eta),\\
&(\varrho, \mathbf{u})|_{t=0} = (\varrho_0, \mathbf{u}_0), \quad \mathbf{u}|_{\partial\Omega} = 0.
\end{aligned}
$$

- Fixed point of T is a solution to the nonlinear problem.

Linear problem

$$
\begin{aligned}
&\varrho_t + \varrho_0 \mathrm{div}\,\mathbf{u} = G \\
&\varrho_0 \mathbf{u}_t - \mu\Delta\mathbf{u} - \nu\nabla\mathrm{div}\,\mathbf{u} + \pi'(\varrho_0)\nabla\varrho = \mathbf{F}, \\
&(\varrho, \mathbf{u})|_{t=0} = (\varrho_0, \mathbf{u}_0), \quad \mathbf{u}|_{\partial\Omega} = 0.
\end{aligned}
$$

Maximal regularity:

$$
\begin{aligned}
&\|\partial_t \mathbf{u}\|_{L_p(0,T;L_q)} + \|\mathbf{u}\|_{L_p(0,T;W^{2,q})} + \|\varrho, \varrho_t\|_{L_p(0,T;L_q)} \\
&\leq C\big[\|\mathbf{F}\|_{L_p(0,T;L_q)} + \|G\|_{L_p(0,T;W^{1,q})} + \|\mathbf{u}_0\|_X\big]
\end{aligned}
$$

together with nonlinear estimate (here embedding $W^1_p \subset L_\infty$ is also necessary):

$$
\|g(\mathbf{w}, \eta)\|_{L_p(0,T;W^{1,q})} + \|\mathbf{f}(\mathbf{w}, \eta)\|_{L_p(0,T;L_q)} \leq C(w, \eta)
$$

allow to prove existence of a unique solution to the nonlinear problem via Banach fixed point theorem.

Ideas of the proof of $L_p - L_q$ maximal regularity

Resolvent problem

Maximal regularity is shown by analysing the corresponding resolvent problem:

$$
\begin{aligned}
&\lambda\varrho_\lambda + \varrho_0 \mathrm{div}\,\mathbf{u}_\lambda = g \\
&\varrho_0\lambda\mathbf{u}_\lambda - \mu\Delta\mathbf{u}_\lambda - \nu\nabla\mathrm{div}\,\mathbf{u}_\lambda + \pi'(\varrho_0)\nabla\varrho_\lambda = \mathbf{f}, \\
&\mathbf{u}_\lambda|_{\partial\Omega} = 0.
\end{aligned}
$$

which is obtained by application of Laplace transform in time:

$$
\mathcal{L}[f](\lambda) = \int_{\mathbb{R}} e^{-\lambda t} f(t)
$$

$\mathcal{R}$ - boundedness and maximal regularity

Definition

Let X and Y be two Banach spaces. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called **$\mathcal{R}$-bounded** *on $\mathcal{L}(X,Y)$ if there exist constants $C > 0$ and $p \in [1,\infty)$ such that for any $n \in$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$ and $\{f_j\}_{j=1}^n \subset X$, the inequality*

$$\int_0^1 \| \sum_{j=1}^n r_j(u) T_j f_j \|_Y^p \, du \leq C \int_0^1 \| \sum_{j=1}^n r_j(u) f_j \|_X^p \, du,$$

where $r_j : [0,1] \to \{-1,1\}$, $j \in$, are the Rademacher functions given by $r_j(t) = \mathrm{sign}(\sin(2^j \pi t))$. **The smallest such C is called $\mathcal{R}$-bound** *of $\mathcal{T}$ on $\mathcal{L}(X,Y)$ which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}\mathcal{T}$.*

Weis Theorem

Theorem (Weis (2001))

X, Y - UMD spaces, $1 < p < \infty$. Then $\mathcal{R}$-boundedness for the resolvent problem in $\mathcal{L}(X, Y)$ implies maximal regularity for the original problem in $L_p(X) - L_p(Y)$.

Here we have $X = W^1_q(\Omega) \times L_q(\Omega)$, $Y = W^1_q(\Omega) \times W^2_q(\Omega)$.

Corollary

$\mathcal{R}$-boundedness for the resolvent problem implies maximal regularity for the time-dependent problem.

$\mathcal{R}$-boundedness for linearized problem - general idea:

Continuity Equation: $\varrho_\lambda = \lambda^{-1}(g - \varrho_0 \mathrm{div}\, \mathbf{u}_\lambda) \Longrightarrow$

$$\varrho_0 \lambda \mathbf{u}_\lambda - \mu \Delta \mathbf{u}_\lambda - (\nu + \lambda^{-1}\gamma\varrho_0)\nabla \mathrm{div}\, \mathbf{u}_\lambda = \mathbf{f} - \lambda^{-1}\mathrm{div}\, \mathbf{u} \nabla \varrho_0, \quad (ME)$$

where $\gamma = \pi'(\varrho_0) \geq 0$.

- The problem is reduced to showing $\mathcal{R}$ - boundedness for (ME).
- Regularity for continuity equation is for free!

proving $\mathcal{R}$-boundedness ...

... is a long and technical story - no details here.

- [Denk, Hieber, Prüss (2003)]: for heat equation
- [Shibata, Shimizu (2001,2008)]: Stokes
- [Enomoto, Shibata (2013)]: compressible Stokes
- Generalizations: Free boundary problems, fluid - structure etc. works of Y. Shibata group in Waseda and M. Hieber group in Darmstadt

$\mathcal{R}$-boundedness for Momentum Equation - scheme:

- Problem with fixed coefficients in $\mathbb{R}^n$: theory based on explicit solution formula Fourier transform
- Problem with fixed coefficients in $\mathbb{R}^n_+$: solution formula for the Fourier transform of solution is also available
- Problem in a perturbed half-space using regularity of the coefficients of the linear problem
- Partition of unity to treat the general domain

Remarks on the range of p,q:

- Maximal regularity for the linear problem holds for $1 < p, q < \infty$;
- Local well-posedness and global well-posedness for small data for nonlinear problem holds for $1 < p < \infty$, $q > d$ where d is the space dimension (crucial imbedding $W^1_q \subset L_\infty$).

THANK YOU !