The Weak Ramsey Property

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Motivation
Motivation

Theorem (Kechris, Pestov, Todorcevic 2005)

Let $\mathcal{F}$ be a relational Fraïssé class with the Fraïssé limit $U$ and let $G = \text{Aut}(U)$ be endowed with the pointwise convergence topology. TFAE:

(a) $G$ is extremely amenable.
(b) $\mathcal{F}$ has the Ramsey property and the ordering property.

$G \leq S_{\infty}$

closed
F consists of finite structures with relations

- F is hereditary
- F has the joint embedding property, the amalgamation property, and ctly many types.

Homogeneity:

\[ \sim \]

\[ \text{AE}_F \]

\[ \text{Aut}(U) \uparrow \text{Emb}(A, U) \]

\[ g, gf, f \]
### Categories

Categories will be denoted by letters $\mathcal{G}$, $\mathcal{C}$, $\mathcal{L}$, etc. Given a category $\mathcal{C}$ and two objects $A, B \in \text{Obj}(\mathcal{C})$ the set of all $\mathcal{C}$-arrows from $A$ to $B$ will be denoted by $\mathcal{C}(A, B)$.
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The setup

We shall work with a pair $\langle \mathcal{G}, \mathcal{L} \rangle$, where $\mathcal{L}$ is a category and $\mathcal{G}$ is its full subcategory. We shall assume that $\text{Obj}(\mathcal{L})$ consists of all colimits of chains (sequences) in $\mathcal{G}$ and all $\mathcal{L}$-arrows are monic.

We assume $\mathcal{G}$ is directed, i.e., for every $a, b \in \text{Obj}(\mathcal{G})$ there is $c \in \text{Obj}(\mathcal{G})$ with

$$\mathcal{G}(a, c) \neq \emptyset \neq \mathcal{G}(b, c).$$
Definition

Fix $\alpha : a \rightarrow a'$ in $\mathcal{G}$. We say that $\mathcal{G}$ has the weak amalgamation property (WAP) at $\alpha$ if for every $\mathcal{G}$-arrows $f : a' \rightarrow x$, $g : a' \rightarrow y$ there are $\mathcal{G}$-arrows $f' : x \rightarrow w$ and $g' : y \rightarrow w$ satisfying

$$f' \circ f \circ \alpha = g' \circ g \circ \alpha.$$
Example (Pouzet)

Let $\mathcal{F}$ be the class of all finite linearly ordered sets, where the linear ordering $<$ is replaced by the following ternary relation:

$$R(x, y, z) \iff x < y \& x < z \& y \neq z.$$
Weak Fraïssé category:
- directed
- with the WAP
- countably dominated
Weak Fraïssé sequences

Definition

A normalized weak Fraïssé sequence in $\mathcal{G}$ is a sequence $\tilde{u}: \omega \rightarrow \mathcal{G}$ satisfying the following condition.

(W) For every $n \in \omega$, for every $\mathcal{G}$-arrow $f: u_{n+1} \rightarrow y$ there exist $m > n$ and an $\mathcal{G}$-arrow $g: y \rightarrow u_m$ such that

$$g \circ f \circ u_{n+1} = u_m.$$
The framework

We assume $U = \lim \bar{u}$, where $\bar{u} : \omega \to S$ is a normalized weak Fraïssé sequence.

Furthermore:

(F) For every $a \in \text{Obj}(S)$, for every $f : a \to \lim \bar{u}$ there are $n \in \omega$ and $f' : a \to u_n$ such that $f = u_n \circ f'$.
The topology on $G := \text{Aut}(U)$

**Definition**

A basic neighborhood of $\text{id}_U \in G$ is defined to be any set of the form

$$V_m = \{g \in G : g \circ u_m^\infty = u_m^\infty\},$$

where $m \in \omega$. 

\[ \begin{array}{c}
\text{\underline{a}_0 \xrightarrow{\text{\underline{u}_0}} \text{\underline{a}_1} \xrightarrow{\text{\underline{u}_1}} \ldots \xrightarrow{\text{\underline{u}_m}} \text{\underline{u}_m}} \\
\text{\underline{u}_0 \xrightarrow{\text{id}_{\underline{u}_m}} \text{\underline{a}_0}} \\
\end{array} \]
The topology on $G := \text{Aut}(U)$

**Definition**

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where $m \in \omega$.

**Claim**

$\text{Aut}(U)$ is a completely metrizable non-archimedean group.

Each $V_m$ is an open subgroup.
The weak big Ramsey property

**Theorem**

Assume \( G = \text{Aut}(U) \) is extremely amenable. Then for every \( a \in \text{Obj}(\mathcal{G}) \) there exists an \( \mathcal{G} \)-arrow \( \alpha : a \to a' \) satisfying:

\[(wB) \quad \text{For every } k \in \omega, \text{ for every finite } F \subseteq \mathcal{L}(a', U), \text{ for every } \varphi : \mathcal{L}(a', U) \circ \alpha \to k \text{ there is } g \in G \text{ such that } \varphi \text{ is constant on } g \circ F \circ \alpha.\]

\[a \xrightarrow{\alpha} a' \]

\[G \not\supset \mathcal{L}(a', U) \circ \alpha \text{ transitively} \]

\[G \not\supset k \mathcal{L}(a', U) \circ \alpha \]

\[G \supset \text{compact} \]

\[U \]

\[\sim \]

\[g \]

\[U \]

W. Kubiš (http://www.math.cas.cz/kubis/)

The Weak Ramsey Property
If \( a \xrightarrow{\mathcal{L}} a' \) is amalgamable then

\[
h \circ f_1 \circ \alpha = f_2 \circ \alpha
\]
The weak Ramsey property

**Definition**

We say $\mathcal{G}$ has the **weak Ramsey property** if for every $a \in \text{Obj}(\mathcal{G})$ there is an $\mathcal{G}$-arrow $\alpha : a \rightarrow a'$ satisfying

$$(wR) \quad \text{For every } b \in \text{Obj}(\mathcal{G}), \text{ for every } k \in \omega, \text{ for every finite } F \subseteq \mathcal{G}(a', b) \text{ there is } v \in \text{Obj}(\mathcal{G}) \text{ such that for every } \varphi : \mathcal{G}(a', v) \circ \alpha \rightarrow k \text{ there exists } e : b \rightarrow v \text{ such that } \varphi \upharpoonright e \circ F \circ \alpha \text{ is constant.}$$
Proposition

The weak Ramsey property implies WAP.
Proof.

Fix $a \in \text{Obj}(\mathcal{G})$, let $k = 2$, and let $\alpha: a \to a'$ be as in (wR). Fix $f_0, f_1 \in \mathcal{G}$ with $\text{dom}(f_0) = a' = \text{dom}(f_1)$. Using directedness, choose $b \in \text{Obj}(\mathcal{G})$ and $g_0, g_1 \in \mathcal{G}$ such that $g_i \circ f_i \in \mathcal{G}(a', b)$ for $i = 0, 1$. Let $F = \{g_0 \circ f_0, g_1 \circ f_1\}$.

Find $v \in \text{Obj}(\mathcal{G})$ from the weak Ramsey property applied to $F$. Define $\varphi: \mathcal{G}(a', v) \circ \alpha \to 2$ by setting $\varphi(g) = 1$ if and only if $g = g' \circ f_1 \circ \alpha$ for some $g' \in \mathcal{G}$. The weak Ramsey property says there exists $e: b \to v$ such that $\varphi$ is constant on $e \circ F \circ \alpha$. Note that $\varphi(e \circ (g_1 \circ f_1) \circ \alpha) = 1$, for obvious reasons. Thus also $\varphi(e \circ (g_0 \circ f_0) \circ \alpha) = 1$, which means that there exists $h$ such that

$$e \circ g_0 \circ f_0 \circ \alpha = h \circ f_1 \circ \alpha.$$ 

We are done, because $e \circ g_0$ and $h$ witness the weak amalgamation.
Theorem

Assume $G = \text{Aut}(U)$ is extremely amenable. Then $G$ has the weak Ramsey property.
Theorem

Assume $\mathcal{G}$ has the weak Ramsey property and $U$ is as above. Then $\text{Aut}(U)$ is extremely amenable.
References


Go back to Pouset example...

\[ \text{Ant}(U) = \text{Ant}(Q, \prec) \uparrow \text{extremely amenable} \]
$k$-AP \iff WAP

4-AP

\textit{xournalpp}