Approximation in the calculus of variations

Iwona Chlebicka

MIMUW @ University of Warsaw, Poland

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Variational functionals

Consider a functional

\[ u \mapsto \mathcal{F}[u] = \int_{\Omega} F(x, u, Du) \, dx , \quad \Omega \subset \mathbb{R}^n, \quad u : \Omega \to \mathbb{R} . \]
Variational functionals

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The calculus of variation asks if

\[ \inf \mathcal{F}[u] \]

is attained and what are the properties of the minimizer (e.g. \( C^{0,\alpha}, C^{1,\alpha} \) regularity, partial regularity, higher integrability).
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**Typical regularity method**

- consider nice enough \( \mathcal{F}_\varepsilon \) and find a minimizer \( u_\varepsilon \) (for every \( \varepsilon \))
- show that \( u_\varepsilon \to u \) and \( \mathcal{F}_\varepsilon[u_\varepsilon] \to \mathcal{F}[u] \) well enough
- show that the limit function \( u \) shares regularity with each \( u_\varepsilon \)
Natural space to minimize a functional $1/2$

We note that

$$\inf_{\text{all } u} \mathcal{F}[u] \leq \inf_{\text{regular } u} \mathcal{F}[u],$$

where we think about all functions that make the functional finite.
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Lavrentiev, 1926, Mania, 1934

First considerations on when \( \inf_{\text{all } u} \mathcal{F}[u] < \inf_{\text{regular } u} \mathcal{F}[u] \).
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$$\mathcal{F}[u] = \int_{0}^{1} (x - u^3)^2 (u')^6 \, dx,$$
for $$u(0) = 0, \quad u(1) = 1.$$
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Then for \( w \in C^1([0,1]), \mathcal{F}[w] > c > 0, \)
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but there is an absolutely continuous function making $$\mathcal{F}[u_{\min}] = 0.$$
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Then for \( w \in C^1([0, 1]) \), \( \mathcal{F}[w] > c > 0 \),
but there is an absolutely continuous function making \( \mathcal{F}[u_{\text{min}}] = 0 \).

\[ (u_{\text{min}}(x) = x^{\frac{1}{3}}) \]
Natural space to minimize a functional 2/2

Denoting

\( W(\Omega) := \{ \text{all functions with finite natural energy for minimizers of } \mathcal{F}[u] \} \)

\( H(\Omega) := \{ \text{regular functions from } W(\Omega) \} \)
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we always have
\[ \inf_{u \in u_0 + W} \mathcal{F}[u] \leq \inf_{u \in u_0 + H} \mathcal{F}[u]. \]
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\[ \inf_{u \in u_0 + W} F[u] \leq \inf_{u \in u_0 + H} F[u]. \]

When the inequality above is $<$, we call it **Lavrentiev’s gap**.
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Previous example showed the functional with the gap between absolutely continuous functions \( AC([0, 1]) \) and \( C^1([0, 1]) \).
No gap in the classical case

**Dirichlet principle.** The scalar Euler–Lagrange equation

$$-\Delta u = 0 \quad \text{in } \Omega$$

is associated to the energy functional

$$u \mapsto \mathcal{F}[u] = \frac{1}{2} \int_\Omega |Du|^2 \, dx.$$
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is associated to the energy functional

\[ u \mapsto F[u] = \frac{1}{2} \int_{\Omega} |Du|^2 \, dx. \]

It does not matter if we minimize over

\[ W(\Omega) := \left\{ \text{all functions with } Du \text{ such that } \int_{\Omega} |Du|^2 \, dx < \infty \right\} \]

\[ H(\Omega) := \overline{C^\infty_0(\Omega)}^W \]

because

\[ \inf_{u \in u_0 + H} F[u] = \inf_{u \in u_0 + W} F[u]. \]
No gap in the power case

Consider a functional

\[ u \mapsto \mathcal{F}[u] = \int_{\Omega} F(x, u, Du) \, dx, \quad \Omega \subset \mathbb{R}^n \]

with the growth of \( F \) governed by a power function for \( 1 < p < \infty \):

\[ \nu |\xi|^p \leq F(x, s, \xi) \leq L(|\xi|^p + 1). \]
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Then the Meyers–Serrin theorem ’64 \((H = W)\) yields that

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\[ \inf_{u \in u_0 + H} \mathcal{F}[u] = \inf_{u \in u_0 + W} \mathcal{F}[u]. \]

This property may fail if \( F \) is governed by not regular enough inhomogeneous function, e.g. \( |\xi|^{p(x)} \) or \( |\xi|^p + a(x)|\xi|^q \).
1. Calculus of variations
Is there a Lavrentiev’s phenomenon?

This I will NOT discuss.
FAQ

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Is there a Lavrentiev’s phenomenon?
= Can we approximate functions minimizing variational functionals?

Other questions:
how to relax dependence of the integrand for the
theory to make sense?
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2. Functional analysis
Can any function from an unconventional space be approximated?

3. PDEs
What existence results one can provide thanks to the density
properties?
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   Can any function from an unconventional space be approximated?
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Calculus of variations motivation

minimization of \( u \mapsto \int_{\Omega} F(x, u, Du) \, dx \) for \( u : \Omega \to \mathbb{R} \)

Gap or no gap for minimizers

- Lavrentiev '26, Mania '34
- Gossez '82, Zhikov 80’-10’
- Buttazzo & Mizel '95, Belloni & Buttazzo '92
- Fonseca, Malý, Mingione '04, Esposito, Leonetti, Mingione '04, Balci, Diening, Surnachev '20
- Esposito, Leonetti, Petricca '19, Leonetti, De Filippis '22, Koch '22
- Bousquet '23
- 2018+ via density: Ahmida, Alberico, Borowski, Bulíček, Chlebicka (Skrzypczak), Cianchi, Gwiazda, Skrzeczkowski, Świerczewska-Gwiazda, Wróblewska-Kamińska, Youssfi, Zatorska-Golstein
Real-world motivation
Real-world motivation
Real-world motivation

cheese issue

Energy density: \[
\int_\Omega |Du|^p + a(x)|Du|^q \, dx
\]
with nasty weight \( a \)

Figure: Inhomogeneous medium

- not enough approximation properties of the function space
- no regularity of minimizers or solutions
Examples of spaces with functions that cannot be approximated 1/2

Variable exponent spaces

\[ W := \{ f \in W^{1,1}_{loc} : |Df|^{p(x)} \in L^1 \} \]
when the exponent \( p \) is \textbf{not} log-Hölder continuous \cite{Zhikov1986}
Examples of spaces with functions that cannot be approximated $1/2$

**Variable exponent spaces**

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when the exponent $p$ is not log-Hölder continuous [Zhikov1986]

- Checkerboard on a 2d-plane
  - $u_0$ – nice trace

- **Nasty** exponent: $p > 2$ in $V$
  - and $p < 2$ outside $V$

- **Bad** $u_* \in W$ has $Du_* \equiv 0$ in $V$
  - (but it jumps)

Then

$$\inf_{u \in u_0 + W} \mathcal{F}[u] \leq \mathcal{F}[u_*] < \inf_{u \in u_0 + H} \mathcal{F}[u]$$
Examples of spaces with functions that cannot be approximated 2/2

Double phase spaces

\[ W := \{ f \in W_{loc}^{1,1} : |Df|^p + a(x)|Df|^q \in L^1 \} \]

with \( a : \Omega \to [0, \infty) \), \( a \in C^{0,\alpha} \),

when powers do not satisfy \( p < n < n + \alpha < q \)

see [Zhikov1995], [Esposito, Leonetti, & Mingione, JDE2004]
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- Checkerboard on a 2d-plane, extended to n-d; \( u_0 \) – nice trace
- **Nasty** weight \( a \in C^{0,\alpha} \) with \( \text{supp} \ a \subset V. \)
- Bad \( u_* \in W \) with \( Du_* \equiv 0 \) in \( B_1 \setminus V \) (but it jumps).
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Use of fractals to get rid of the dimensional threshold
[Balci, Diening, & Surnachev, CalcVar2020]
Consequences for minimizers

Double phase spaces

[Fonseca, Malý, & Mingione, ARMA2004]

For a functional \[ \int_{\Omega} |Du|^p + a(x)|Du|^q \, dx \] with \( a \in C^{0,\alpha} \),

if \( p < n < n + \alpha < q \)

(\( = \) the closeness condition on the weights is not satisfied),

then minimizers are almost as bad as any Sobolev functions.
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It is the fault of \( a \).
Real-world motivation
Real-world motivation

thick soup case

Energy density: \[ \int_{\Omega} |Du|^p + a(x)|Du|^q \, dx \text{ with nice weight } a \]

Figure: Inhomogeneous medium

- good approximation properties of the function space
- possible for study regularity of minimizers or solutions
Regularity of minimizers

thick soup case

For a minimizer to the problem

\[ u \mapsto \int_{\Omega} |Du|^p + a(x)|Du|^q \, dx \]

with nice weight \( a \in C^{0,\alpha} \),
Regularity of minimizers
thick soup case

For a minimizer to the problem

\[ u \mapsto \int_{\Omega} |Du|^p + a(x)|Du|^q \, dx \quad \text{with nice weight } a \in C^{0,\alpha}, \]

if any of those holds true

- \( \frac{q}{p} < 1 + \frac{\alpha}{n}, \)
- a priori \( u \in L^\infty \) and \( q < p + \alpha, \)
- a priori \( u \in C^{0,\gamma} \) and \( q < p + \frac{\alpha}{1-\gamma}, \)

(\( = \text{decay of } a \text{ says how close have to be powers} \))
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then \( Du \in C^{0,\beta} \) for some \( \beta. \)
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series of works by Baroni, Colombo, Mingione 2015-18
continued by Harjulehto, Hästö, Byun, and their collaborators
PDE motivation
heating up a thick soup

Goal
General theory for nonlinear diffusion equations in inhomogeneous media.
PDE motivation

heating up a thick soup

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General theory for nonlinear diffusion equations in inhomogeneous media. Well-posedness of problems like

$$\partial_t u - \text{div} A(t, x, Du) = f(t, x)$$

with $A(t, x, \xi)$ of growth given by $M : [0, T] \times \Omega \times \mathbb{R}^n \to [0, \infty)$. 
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In nonreflexive space needed density result $\approx$ no gap

If $M$ is regular enough, then for any $u \in W$ there exists $\{u_k\} \subset C_0^\infty$: 
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General theory for nonlinear diffusion equations in inhomogeneous media.
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$$\partial_t u - \text{div} \mathcal{A}(t, x, Du) = f(t, x)$$

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In nonreflexive space needed density result $\approx$ no gap
If $M$ is regular enough, then for any $u \in W$ there exists $\{u_k\} \subset C^\infty_0$:

$$u_k \to u \text{ in } L^1 \quad \text{and} \quad \exists \lambda > 0 \quad \int_{\Omega_T} M \left( t, x, \frac{D u_k - D u}{\lambda} \right) \, dx \, dt \xrightarrow{k \to \infty} 0.$$
PDE motivation

heating up a thick soup

**Goal**

General theory for nonlinear diffusion equations in **inhomogeneous** media. Well-posedness of problems like

\[
\partial_t u - \text{div} A(t, x, Du) = f(t, x)
\]

with \( A(t, x, \xi) \) of growth given by \( M : [0, T] \times \Omega \times \mathbb{R}^n \rightarrow [0, \infty) \).

**In nonreflexive space needed density result \( \approx \) no gap**

If \( M \) is regular enough, then for any \( u \in W \) there exists \( \{u_k\} \subset C_0^\infty : \)

\[
u_k \rightarrow u \text{ in } L^1 \quad \text{and} \quad \exists \lambda > 0 \quad \int_{\Omega_T} M \left( t, x, \frac{Du_k - Du}{\lambda} \right) d x d t \xrightarrow{k \rightarrow \infty} 0 .
\]

In the classical case of \( W^{1,p} \) we have \( M(t, x, \xi) \equiv |\xi|^p \).

Then the above density is in norm. It can be obtained by mollification.
PDE motivation

heating up a thick soup

Existence result

Well-posedness of problems like

$$\partial_t u - \text{div}\ A(t, x, Du) = f(t, x)$$

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heating up a thick soup

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Special cases of $\text{div} A$ are:

$$\Delta u = \text{div} Du$$
PDE motivation
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Special cases of $\text{div} A$ are:

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and $\Delta_p u = \text{div}(|Du|^{p-2}Du), \ 1 < p < \infty,$
PDE motivation

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Special cases of $\text{div} A$ are:

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PDE motivation

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general growth (Orlicz) $\Delta_A$,
PDE motivation
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and $\Delta_p u = \text{div } (|Du|^{p-2} Du)$, $1 < p < \infty$,

but also their counterparts that are inhomogeneous $\Delta_p(x)$,
general growth (Orlicz) $\Delta_A$, anisotropic $\Delta_{\bar{p}}$ and more...
Sufficient conditions on $M$ for $H = W$

thick soup case

By the many efforts including [Ahmida, C, Gwiazda, Youssfi, JFA 2018], [Hästo, Harjulehto, Springer Lecture Notes 2019], [C, Gwiazda, Świerczewska-Gwiazda, Wróblewska-Kamińska, Springer Monographs in Mathematics 2021], [Borowski & C, JFA 2022], [Buliček, Gwiazda, Skrzeczkowski, ARMA 2022] we know that

Balance condition ($B$)
Sufficient conditions on $M$ for $H = W$

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Balance condition $(B)$
For $M : \Omega \times \mathbb{R}^n \rightarrow [0, \infty)$ there exists a constant $C_M > 1$ such that

$$\sup_{y \in B(x)} M(y, \xi) \leq M(x, C_M \xi)$$

for prescribed ranges of $x$ and $\xi$ ← we fight for these ranges
Sufficient conditions on $M$ for $H = W$

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implies needed *modular* density of $H$ in $W$. 

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Basic ideas

thick soup case

1) Approximation is based on convolution
2) \( \Omega \) is decomposed to star-shaped sub-domains \( \Omega_i \)
3) For \( \Omega_i \) star-shaped with respect to \( B_i(0, r) \), \( \xi \in \mathcal{L}_M(\Omega_i) \) with \( \text{supp} \xi \subset \Omega \) and \( \kappa_\delta < 1 \), we define

\[
S_\delta \xi(x) = \int_{\Omega_i} \rho_\delta(x - y) \xi(y/\kappa_\delta) \, dy.
\]

4) we need a kind of Jensen inequality to take convolution \( S_\delta \) with respect to \( x \) from inside of \( M \) to get

\[
M \left( x, \frac{D S_\delta \varphi(x)}{\lambda} \right) \lesssim S_\delta M \left( \cdot, \frac{D \varphi(\cdot)}{\lambda} \right) (x) + 1
\]

but \( M \) depends on \( x \) (our space is defined via \( \int_\Omega M(x, \xi) \, dx \))

this step essentially requires a balance condition \((B)\).
Fight for the range of exponents

Functional

$$\int_{\Omega} |Du|^p + a(x)|Du|^q \, dx$$

is typically considered with $a \in C^{0,\alpha}(\Omega)$ and $1 < p \leq q$. 

There are various regimes treated as natural

- $q/p \leq 1 + \alpha/n$ is the scope when the maximal function is continuous,
- $q \leq p + \alpha$ is the scope for the absence of Lavrentiev's gap.

Is it possible to introduce a scale for $a$ ensuring good properties of a Sobolev-type space for $q$ and $p$ further apart?

Answer YES is coming soon.
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Borowski, C, Miasojedow, De Filippis, Absence and presence of Lavrentiev’s phenomenon in double phase functionals for every choice of exponents.
Inhomogeneous media

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Summary 1/2

Inhomogeneous media

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**Figure:** Nasty $M = \text{bad medium}$

**Figure:** Nice $M = \text{good medium}$
Summary 2/2

Having the *modular density* result of *smooth functions* we can study
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- regularity and other properties of minimizers to

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\int_{\Omega} F(x, u, Du) - f(x)u \, dx,
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for broad class of \(F(x, z, \xi)\) with growth controlled by \(M(x, \xi)\),
Having the **modular density** result of **smooth functions** we can study

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for broad class of \( F(x, z, \xi) \) with growth controlled by \( M(x, \xi) \),

- well-posedness of

\[
\partial_t u - \text{div} A(t, x, Du) = f(t, x),
\]

where \( A(t, x, \xi) \) has growth given by \( M(t, x, \xi) \); then theory of PDEs (local and global qualitative properties of solutions like uniqueness, multiplicity, symmetry, local regularity, optimal transfer regularity from data to solutions, asymptotic behaviour...)


Thank you for your attention!