Closed groups generated by
generic measure preserving transformations

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Introduction
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Introduction

Basics

**Polish space** = a separable completely metrizable space

**Polish group** = a topological group whose group topology is Polish

**Standard Borel space** = a set $X$ equipped with the $\sigma$-algebra of sets Borel with respect to a Polish topology on $X$

**Borel measure** = a measure defined on the $\sigma$-algebra of a standard Borel space

All atomless Borel probability measures are isomorphic to each other, so we can think of such a measure as Lebesgue measure $\lambda$ on $[0, 1]$. 
The group of measure preserving transformations

\((X, \gamma) = \text{a standard Borel space with an atomless Borel probability measure}\)

\(\text{Aut} = \text{the Polish group of all measure preserving transformations of } (X, \gamma)\)

Measure preserving transformations are identified if they coincide on a set of full measure.

\(\text{Aut} \text{ is taken with composition and is topologized so that}\)

\[T_n \to T \text{ iff } \gamma(T_n(A) \Delta T(A)) \to 0, \text{ for each Borel } A \subseteq X.\]
Genericity

$X$ a Polish space, $P$ a property

A **generic** $x \in X$ **has** $P$ if $\{x \in X \mid x \text{ does not have } P\}$ is meager.
The subject matter of the talk

For a Polish group $G$ and $g \in G$, let

$$\langle g \rangle_c = \text{closure}(\{g^n \mid n \in \mathbb{Z}\}).$$

We study closed subgroups of $\text{Aut}$ generated by generic elements of $\text{Aut}$, that is, groups of the form

$$\langle T \rangle_c,$$

for a generic measure preserving transformation $T$. 
**Boolean actions**

$G$ a Polish group

A **boolean action of $G$ on $(X, \gamma)$** is a continuous homomorphism $\zeta: G \to \text{Aut}$.

The word action is justified by viewing $G$ as acting on the boolean algebra of measure classes of measurable subsets of $(X, \gamma)$ by

$$gB = \zeta(g)(B).$$

For example, the Polish group $\langle T \rangle_c$, for $T \in \text{Aut}$, has a natural boolean action being a subgroup of $\text{Aut}$. 
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Two more groups
The group of measurable functions

\( \mathbb{T} = \) the group of all complex numbers of unit length taken with multiplication

\( \lambda = \) Lebesgue measure on \([0, 1]\)

\( L^0(\lambda, \mathbb{T}) = \) the Polish group of all measurable functions from \([0, 1]\) to \( \mathbb{T} \)

\( L^0(\lambda, \mathbb{T}) \) is taken with pointwise multiplication and the topology of convergence in measure.

**Notation:** We write \( L^0 \) for \( L^0(\lambda, \mathbb{T}) \).
$L^0(\lambda, \mathbb{R})$ = the Polish linear space of all measurable functions from $[0, 1]$ to $\mathbb{R}$

There is a continuous surjective homomorphism

$$L^0(\lambda, \mathbb{R}) \to L^0,$$

namely,

$$f \to \exp(if).$$
The unitary group

\( H = \) the separable, infinite dimensional, complex Hilbert space

\( \mathcal{U} = \) the Polish group of unitary transformations of \( H \)

\( \mathcal{U} \) is taken with composition and the strong operator topology.
The question and the theorem
Recall, for $T \in \text{Aut}$,

$$\langle T \rangle_c = \text{closure}(\{ T^n \mid n \in \mathbb{Z} \}).$$

**Glasner–Weiss:** Is it the case that for a generic $T \in \text{Aut}$, $\langle T \rangle_c$ is isomorphic to $L^0$?
Motivation for the question

Qualifications

**Glasner**: $L^0$ is monothetic.

Analogy

**Melleray–Tsankov**: $\langle U \rangle_c$ is isomorphic to $L^0$ for a generic $U \in \mathcal{U}$. 
Structure

**Ageev:** For a generic $T \in \text{Aut}$, each finite abelian group embeds into $\langle T \rangle_c$.

**S.:** For a generic $T \in \text{Aut}$, there is a Polish linear space $L_T$ and a continuous surjective homomorphism $L_T \rightarrow \langle T \rangle_c$.

Dynamics

**Glasner–Weiss:** For a generic $T \in \text{Aut}$, the natural boolean action of $\langle T \rangle_c$ on $(X, \gamma)$ is whirly.
Theorem (S., 2021)

For a generic transformation $T \in \text{Aut}$, the group $\langle T \rangle_c$ is not isomorphic to $L^0$. 
A rough outline of the proof

Prove the following two points.

1. If $L^0 \cong \langle T \rangle_c < \text{Aut}$, for a generic $T \in \text{Aut}$, then some ergodic boolean action of $L^0$ has \textbf{spectral properties} similar to spectral properties of a generic $T \in \text{Aut}$.

2. \textbf{No} ergodic boolean actions of $L^0$ has \textbf{spectral properties} similar to spectral properties of a generic $T \in \text{Aut}$.
Spectral behavior
Spectral behavior of a generic $T \in \text{Aut}$

$\nu(T) = \text{maximal spectral type of } T \in \text{Aut}.$

Building on earlier work of Choksi–Nadkarni, Katok, and Stepin, del Junco–Lemańczyk proved the following theorem.

**Theorem (del Junco–Lemańczyk, 1992)**

For a generic $T \in \text{Aut}$ and $\ell_1, \ldots, \ell_p, \ell'_1, \ldots, \ell'_{p'} \in \mathbb{N}$, if $(\ell_1, \ldots, \ell_p)$ and $(\ell'_1, \ldots, \ell'_{p'})$ are not rearrangements, then

$$\nu(T^{\ell_1}) \ast \cdots \ast \nu(T^{\ell_p}) \perp \nu(T^{\ell'_1}) \ast \cdots \ast \nu(T^{\ell'_{p'}}).$$

Call the condition above the **del Junco–Lemańczyk condition**.
Spectral behavior of $L^0$

A unitary representation of $L^0$ can be constructed as follows. Given $\phi \in L^0$, let

$$L^2(\lambda) \ni f \rightarrow \phi \cdot f \in L^2(\lambda).$$

This is a unitary representation in $\mathcal{U}(L^2(\lambda))$. 
**Spectral behavior of $L^0$**

A unitary representation of $L^0$ can be constructed as follows. Given $\phi \in L^0$, let

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Spectral behavior of $L^0$

A unitary representation of $L^0$ can be constructed as follows. Given $\phi \in L^0$, let

$$L^2(\mu) \ni f \mapsto \phi^k \cdot f \in L^2(\mu),$$

for $\mu \leq \lambda$ and $k \in \mathbb{Z}$. This is a unitary representation in $\mathcal{U}(L^2(\mu))$. 
$\mathbb{Z}^\times = \mathbb{Z} \setminus \{0\}$

$\mathbb{N}[\mathbb{Z}^\times]$ consists of all finite functions $x$ such that

$\emptyset \neq \text{dom}(x) \subseteq \mathbb{Z}^\times$ and $\text{rng}(x) \subseteq \mathbb{N}$
For $x \in \mathbb{N}[\mathbb{Z}^\times]$, let

$$D(x) = \{(k, i) \mid k \in \text{dom}(x), \ i \leq x(k)\},$$

and

$$C_x = [0, 1]^{D(x)}.$$

For $(k, i) \in D(x)$, let

$$\pi_{k,i} : C_x \rightarrow [0, 1] = \text{projection onto coordinate } (k, i).$$

A permutation $\delta$ of $D(x)$ is **good** if, for each $(k, i) \in D(x)$, $\delta(k, i) = (k, j)$ for some $j$. 
A finite Borel measure $\mu$ on $C_x$ is **compatible with** $x \in \mathbb{N}[\mathbb{Z}^\times]$ if

— the marginals of $\mu$ given by $\pi_{k,i}$, for $(k, i) \in D(x)$, are absolutely continuous with respect to $\lambda$;
— $\mu$ is invariant under good permutations of $D(x)$;
— all diagonals of $C_x$ have measure zero with respect to $\mu$. 
\( \mu \) compatible with \( \times \)

\( \tilde{L}^2(\mu) = \) the closed subspace of \( L^2(\mu) \) consisting of all functions invariant under good permutations
For $x \in \mathbb{N}[\mathbb{Z}^\times]$ and $\phi \in L^0$, 

$$R_x(\phi) = \prod_{(k,i) \in D(x)} (\phi \circ \pi_{k,i})^k.$$ 

If $\mu$ is compatible with $x$, then 

$$f \in \tilde{L}^2(\mu) \Rightarrow R_x(\phi) \cdot f \in \tilde{L}^2(\mu).$$
Fix $\xi : L^0 \rightarrow \mathcal{U}$ a unitary representation without non-zero fixed points.

**Theorem (S., 2014)**

$\xi$ is determined by finite Borel measures $(\mu_x)_{x \in \mathbb{N}[\mathbb{Z}^\times]}$ with $\mu_x$ compatible with $x$.

$\xi$ is isomorphic to the $\ell^2$-sum over $x \in \mathbb{N}[\mathbb{Z}^\times]$ of the representations

$$L^0 \times \tilde{L}^2(\mu_x) \ni (\phi, f) \rightarrow R_x(\phi) \cdot f \in \tilde{L}^2(\mu_x).$$

The sequence $(\mu_x)_{x \in \mathbb{N}[\mathbb{Z}^\times]}$ is unique up to mutual absolute continuity of its entries.

The above is true modulo multiplicity.
Theorem (S., 2014)

\( \xi \) is determined by a sequence of finite Borel measures 
\( (\mu^j_x)_{x \in \mathbb{N}[\mathbb{Z}^\times], j \in \mathbb{N}} \) such that, for each \( j \),

\[
\mu^j_x \text{ is compatible with } x, \text{ and } \mu^{j+1}_x \preceq \mu^j_x.
\]

\( \xi \) is isomorphic to the \( \ell^2 \)-sum over \( x \in \mathbb{N}[\mathbb{Z}^\times] \) and \( j \in \mathbb{N} \) of the representations

\[
L^0 \times \tilde{L}^2(\mu^j_x) \ni (\phi, f) \to R_x(\phi) \cdot f \in \tilde{L}^2(\mu^j_x).
\]

The sequence \( (\mu^j_x)_{x \in \mathbb{N}[\mathbb{Z}^\times], j \in \mathbb{N}} \) is unique up to mutual absolute continuity of its entries.
The del Junco–Lemańczyk condition for $L^0$

$\mathbb{N}[\mathbb{Z}^\times]$ comes equipped with coordinatewise addition

$$x \oplus y.$$

Given: $\mu$ on $C_x$ compatible with $x$, and $\nu$ on $C_y$ compatible with $y$. Since $C_{x\oplus y} \sim C_x \times C_y$, we can define the “symmetried” product

$$\mu \otimes \nu$$

on $C_{x\oplus y}$ compatible with $x \oplus y$. 
Recall: we have $\xi : L^0 \to \mathcal{U}$, a unitary representation without non-zero fixed points, with $H$ separable.
Theorem (Etedadialiabadi, 2016/20)

**Assume:** for a generic $\phi \in L^0$ and $\ell_1, \ldots, \ell_p, \ell'_1, \ldots, \ell'_p \in \mathbb{N}$ such that $(\ell_1, \ldots, \ell_p)$ and $(\ell'_1, \ldots, \ell'_p)$ are not rearrangements, we have

$$\nu(\phi^{\ell_1}) \ast \cdots \ast \nu(\phi^{\ell_p}) \perp \nu(\phi^{\ell'_1}) \ast \cdots \ast \nu(\phi^{\ell'_p}),$$

where $\nu(\psi) = \text{maximal spectral type of } \xi(\psi)$.

**Then:** for $x_1, \ldots, x_p \in \mathbb{N}[\mathbb{Z}^\times]$ with $p > 1$, we have

$$\mu_{x_1} \otimes \cdots \otimes \mu_{x_p} \perp \mu_{x_1 \oplus \cdots \oplus x_p}.$$
Theorem on Koopman representations of $L^0$
Given a boolean action $\zeta : G \to \text{Aut}$, the **Koopman representation associated with** $\zeta$ is given by

$$G \ni g \to U_g \in \mathcal{U}(L^2(\gamma)),$$

where, for $f \in L^2(\gamma)$,

$$U_g(f) = f \circ (\zeta(g))^{-1}.$$
The proposition below gives a connection of Etedadialabiabadi’s theorem with the Glasner–Weiss question.

**Proposition (S., 2021)**

*Assume that there is a non-meager set of $T \in \text{Aut}$ such that $\langle T \rangle_c$ is isomorphic to $L^0$. There exists an ergodic boolean action of $L^0$ on $(X, \gamma)$, whose Koopman representation is such that*

$$\mu_x \otimes \mu_x \perp \mu_{x \oplus x}, \text{ for all } x \in \mathbb{N}[\mathbb{Z}^\times].$$
Theorem (S. 2021)

\( \xi = \text{the Koopman representation associated with an ergodic boolean action of } L^0. \)

Then, for \( x_1, \ldots, x_p \in \mathbb{N}[\mathbb{Z}^\times], \) we have

\[ \mu_{x_1} \otimes \cdots \otimes \mu_{x_p} \preceq \mu_{x_1 \oplus \cdots \oplus x_p}. \]
In particular, for ergodic **Koopman** representations of $L^0$

$$\mu_x \otimes \mu_x \preceq \mu_{x \oplus x}, \text{ for all } x.$$ 

Contrast the above statement with

$$\mu_x \otimes \mu_x \perp \mu_{x \oplus x}, \text{ for all } x,$$

for the ergodic Koopman representation of $L^0$ found in the proposition.
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Questions
Is there a Polish group $G$ such that $\langle T \rangle_c$ is isomorphic to $G$, for a generic $T \in \text{Aut}$?

**Glasner–Weiss:** Is the group $\langle T \rangle_c$ a Lévy group for a generic $T \in \text{Aut}$?