A universal coregular countable second-countable space

Taras Banakh

Lviv & Kielce

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For a topological vector space $X$ over a field $F$ and a natural number $k$ let $Gr_k(X)$ be the space of $k$-dimensional linear subspaces of $X$.

The space $Gr_k(X)$ is called the $k$-th Grassmannian of $X$.

We shall be interested in the simplest case of 1-Grassmannians.

In this case $Gr_1(X)$ is the space of lines in $X$, or else the projective space of $X$.

It is well-known and well-studied space.

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Topologically, $\text{Gr}_1(X)$ is the quotient space $X^*/F^*$ of the space $X \setminus \{0\}$ by the action of the multiplicative group $F^* = F \setminus \{0\}$.

So, $\text{Gr}_1(X)$ carries the quotient topology with respect to the orbit map $X^* \to \text{Gr}_1(X)$, which is open (but not necessarily closed).

General topologists know that quotient topologies are dangerous and can provide many surprises.

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Let us consider the simplest surprising case.
In the countable power $\mathbb{Q}^\omega$ of the fields of rationals $\mathbb{Q}$, consider the countable linear subspace

$$\mathbb{Q}^{<\omega} = \{(x_i)_{i \in \omega} \in \mathbb{Q}^\omega : |\{i \in \omega : x_i \neq 0\}| < \omega\}$$

consisting of all eventually zero sequences of rational numbers.

The space $\mathbb{Q}^{<\omega}$ carries the Tychonoff product topology inherited from $\mathbb{Q}^\omega$. This is the topology of simple convergence.

It is clear that $X = \mathbb{Q}^{<\omega}$ is a countable metrizable space without isolated points, so is homeomorphic to $\mathbb{Q}$ according to the classical

Theorem (Sierpiński)

A topological space $X$ is homeomorphic to $\mathbb{Q}$ if and only if $X$ is countable metrizable and without isolated points.
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The metrizability in this theorem can be weakened to the second countability (= existence of a countable base of the topology) according to another classical

**Theorem (Urysohn)**

A topological space \( X \) is metrizable and separable if and only if \( X \) is regular and second-countable.

Those two theorems imply

**Corollary (Sierpiński–Urysohn)**

A topological space \( X \) is homeomorphic to \( \mathbb{Q} \) if and only if \( X \) is countable, regular, second-countable and has no isolated points.
### Theorem (Sierpiński)

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### Corollary (Sierpiński–Urysohn)

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Topological characterizations of $\mathbb{Q}$

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Theorem (Sierpiński–Urysohn)

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Let us recall that a topological space $X$ is regular if for any open set $U \subset X$ and point $x \in U$ there exists an open set $V$ such that

$$x \in V \subseteq \overline{V} \subseteq U.$$
Theorem (Sierpiński–Urysohn)

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Let us return to our linear topological space $X = \mathbb{Q}^<\omega$ and its projective space

$$\mathbb{Q}P^\infty = X^*/\mathbb{Q}^*.$$ 

It is clear that the space $\mathbb{Q}P^\infty$ is countable, second-countable, and has no isolated points.

What about the regularity of $\mathbb{Q}P^\infty$?

Surprise (first noticed by Gelfand and Fuks in 1967)

The space $\mathbb{Q}P^\infty$ is not regular.

Moreover, it is countable and connected!

How this is possible?
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How this is possible?
Take any non-empty open set $U \subseteq \mathbb{QP}^\infty$ and let $q^{-1}[U]$ be its preimage under the quotient map $q : \mathbb{Q}^{<\omega} \setminus \{0\} \to \mathbb{QP}^\infty$.

The set $q^{-1}[U]$ is open and $\mathbb{Q}^*$-conical, i.e., $\mathbb{Q}^* \cdot q^{-1}[U] = q^{-1}[U]$.

Since $q^{-1}[U]$ is open in the Tychonoff product topology, it contains an open set of form $V \times \mathbb{Q}^{\omega \setminus n}$ for some $n = \{0, \ldots, n-1\} \in \omega$ and some open set $V \subseteq \mathbb{Q}^n \setminus \{0\}$.

Being $\mathbb{Q}^*$-conical, the set $q^{-1}[U]$ contains the $\mathbb{Q}^*$-cone

$$\mathbb{Q}^* \cdot (V \times \mathbb{Q}^{\omega \setminus n}) = (\mathbb{Q}^* \cdot V) \times \mathbb{Q}^{\omega \setminus n}$$

and then its closure

$$\overline{q^{-1}[U]} \supset \overline{\mathbb{Q}^* \cdot V \times \mathbb{Q}^{\omega \setminus n}} = \{0\}^n \times \mathbb{Q}^{\omega \setminus n}$$

contains the linear subspace of finite codimension.

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Therefore, for any nonempty open set $U \subseteq \mathbb{Q}P^\infty$ the closure $\overline{U}$ contains the image $q[\{0\}^n \times \mathbb{Q}^\omega \setminus n]$ for some $n \in \omega$.

Consequently, for any nonempty open sets $U_1, \cdots, U_k \subseteq \mathbb{Q}P^\infty$ there exists $n \in \omega$ such that

$$\overline{U}_1 \cap \cdots \cap \overline{U}_k \supset q[\{0\}^n \times \mathbb{Q}^\omega \setminus n] \neq \emptyset.$$ 

So, $\mathbb{Q}P^\infty$ is connected and moreover, $\mathbb{Q}P^\infty$ is superconnected!

**Definition**

A topological space $X$ is called *superconnected* if for any nonempty open sets $U_1, \ldots, U_k$ the intersection $\overline{U}_1 \cap \cdots \cap \overline{U}_k$ is not empty.

**Remark**

*Each superconnected space $X$ is connected:* assuming that $X$ is disconnected, we could write $X$ as the union $X = U_1 \cup U_2$ of two non-empty disjoint open sets and then $\overline{U}_1 \cap \overline{U}_2 = U_1 \cap U_2 = \emptyset$.  

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Therefore the countable second-countable space $\mathbb{Q}P^\infty$ is superconnected and not regular (otherwise it would be metrizable and disconnected).

But it is not regular to a very small extent.

**Observation**

For any nonempty open sets $U_1, \ldots, U_k \subseteq \mathbb{Q}P^\infty$ the complement $\mathbb{Q}P^\infty \setminus (\overline{U_1} \cap \cdots \cap \overline{U_k})$ is a regular space!

Because $\mathbb{Q}P^\infty \setminus (\overline{U_1} \cap \cdots \cap \overline{U_k}) \supseteq q[(\mathbb{Q}^n \setminus \{0\}) \times \mathbb{Q}^\omega \setminus n]$.  

**Definition**

A topological space $X$ is coregular if $X$ is Hausdorff and for any nonempty open sets $U_1, \ldots, U_k \subseteq X$ the complement $X \setminus (\overline{U_1} \cap \cdots \cap \overline{U_k})$ is a regular space.
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Unified Definition

A Hausdorff topological space $X$ is superconnected and coregular if for any nonempty open sets $U_1, \ldots, U_k \subseteq X$ the intersection $\overline{U_1} \cap \cdots \cap \overline{U_k}$ is not empty and its complement $X \setminus (\overline{U_1} \cap \cdots \cap \overline{U_k})$ is a regular space.

If $\{U_n\}_{n \in \omega}$ is a countable base of the topology in a superconnected coregular Hausdorff space, then for every $n \in \omega$ the set

$$X_n = \overline{U_1} \cap \cdots \cap \overline{U_n}$$

is non-empty and its complement $X \setminus X_n$ is a regular topological space. Moreover the sequence $(X_n)_{n \in \omega}$ is decreasing and has empty intersection $\bigcap_{n \in \omega} X_n = \emptyset$. 
A Hausdorff topological space $X$ is superconnected and coregular if for any nonempty open sets $U_1, \ldots, U_k \subseteq X$ the intersection $\overline{U}_1 \cap \cdots \cap \overline{U}_k$ is not empty and its complement $X \setminus (\overline{U}_1 \cap \cdots \cap \overline{U}_k)$ is a regular space.

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is non-empty and its complement $X \setminus X_n$ is a regular topological space. Moreover the sequence $(X_n)_{n \in \omega}$ is decreasing and has empty intersection $\bigcap_{n \in \omega} X_n = \emptyset$. 
Is there any topological characterization of the space $\mathbb{Q}\mathbb{P}^\infty$, analogical to the topological characterization of the space $\mathbb{Q}$?

Well, let us list what we know about the space $\mathbb{Q}\mathbb{P}^\infty$:

- countable,
- second-countable,
- Hausdorff;
- superconnected;
- coregular;
- locally metrizable.

Do these properties uniquely identify the topology of $\mathbb{Q}\mathbb{P}^\infty$?

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Main Theorem

**Theorem**

A topological space $X$ is homeomorphic to the space $\mathbb{Q}P^\infty$ if and only if $X$ is countable, second-countable and possesses a decreasing sequence of non-empty closed sets $(X_n)_{n \in \omega}$ such that

- $X_0 = X$, $\bigcap_{n \in \omega} X_n = \emptyset$, and $X_{n+1} \subseteq X_n$ for all $n$;
- for every $n \in \omega$ the complement $X \setminus X_n$ is a regular topological space;
- for every $n \in \omega$ and a nonempty relatively open set $U \subseteq X_n$ the closure $\overline{U}$ contains some $X_m$.

The sequence $(X_n)_{n \in \omega}$ with the above properties is called a superskeleton of $X$. If every set $X_{n+1}$ is nowhere dense in $X_n$, then the superskeleton is called canonical.

A canonical superskeleton in $\mathbb{Q}P^\infty$ is the sequence $(X_n)_{n \in \omega}$ of closed subsets $X_n = q[\{0\}^n \times \mathbb{Q}^\omega \setminus n]$. 

T. Banakh

Rational projective space
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The proof is technically very difficult and exploits the classical back-and-forth method of Cantor.

Given a canonical supersekeleton \((X_n)_{n \in \omega}\) in a coregular superconnected space \(X\), we construct inductively two sequences \((x_i)_{i \in \omega}\) in \(X\) and \((y_i)_{i \in \omega}\) in \(\mathbb{Q}P^\infty\) so that the correspondence \(h : x_n \to y_n\) determines a homeomorphism between \(X\) and \(\mathbb{Q}P^\infty\) mapping the sets \(X_n\) of the supersekeleton in \(X\) to the corresponding sets in the canonical superskeleton in the space \(\mathbb{Q}P^\infty\).

The construction of the sequences \((x_i)_{i \in \omega}\) and \((y_i)_{i \in \omega}\) is inductive with many conditions. Besides the points \(x_i\) and \(y_i\) we also construct their basic neighborhoods \(U_{i,j}\) and \(V_{i,j}\) in order to guarantee that the bijection \(h : x_n \to y_n\) will be a homeomorphism. The induction is done over the set \(\Gamma = \omega \cup (\omega \times \omega)\), ordered by a suitable well-order.
Proof of the Main Theorem

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The construction of the sequences \((x_i)_{i \in \omega}\) and \((y_i)_{i \in \omega}\) is inductive with many conditions. Besides the points \(x_i\) and \(y_i\) we also construct their basic neighborhoods \(U_{i,j}\) and \(V_{i,j}\) in order to guarantee that the bijection \(h : x_n \to y_n\) will be a homeomorphism. The induction is done over the set \(\Gamma = \omega \cup (\omega \times \omega)\), ordered by a suitable well-order.
Inductively we shall construct sequences of points \( \{x_n\}_{n \in \omega} \subseteq X \), \( \{y_n\}_{n \in \omega} \subseteq Y \), a double sequences of open sets \( \{U_{n,k}\}_{n,k \in \omega} \subseteq \tau_X \), \( \{V_{n,k}\}_{k,n \in \omega} \subseteq \tau_Y \), and a function \( \ell : \Gamma \to \omega \) such that for any \( \gamma \in \Gamma \) the following conditions are satisfied:

1. If \( \gamma = n \) for some number \( n \in \omega \), then
   1a. \( \ell(\gamma) = \ell_X(x_n) = \ell_Y(y_n) \);
   1b. \( x_n \notin \{x_k\}_{k \in \downarrow \gamma} \) and \( y_n \notin \{y_k\}_{k \in \downarrow \gamma} \);
   1c. \( \{(i,j) \in \downarrow \gamma : x_n \in U_{i,j}\} = \{(i,j) \in \downarrow \gamma : y_n \in V_{i,j}\} \);
   1d. \( \{(i,j) \in \downarrow \gamma : x_n \in \overline{U}_{i,j}\} = \{(i,j) \in \downarrow \gamma : y_n \in \overline{V}_{i,j}\} \);
   1e. If \( n \in \Omega \), then \( x_n = \xi(n) \) and \( y_n = f(x_n) \);
   1f. If \( n \notin \Omega \), then \( x_n = \min(X' \setminus \{x_k\}_{k \in \downarrow \gamma}) \) and \( y_n \notin \overline{B} \);
   1g. If \( n \notin \Omega \), then \( y_n = \min(Y' \setminus \{y_k\}_{k \in \downarrow \gamma}) \) and \( x_n \notin A \).

2. If \( \gamma = (n,k) \) for some \( n,k \in \omega \), then
   2a. \( \ell(\gamma) \geq 2 + \max\{\ell(\alpha) : \alpha \in \downarrow \gamma\} \);
   2b. for any \( m \in \omega \cap \downarrow \gamma \) with \( m \neq n \), we have \( x_m \notin U_{n,k} \) and \( y_m \notin V_{n,k} \);
   2c. \( x_n \in U_{n,k} \subseteq O_k^X(x_n) \subseteq X \setminus X_{1+\ell(n)} \) and \( y_n \in V_{n,k} \subseteq O_k^Y(x_n) \subseteq Y \setminus Y_{1+\ell(n)} \);
   2d. \( \{(i,j) \in \downarrow \gamma : U_{n,k} \subseteq U_{i,j}\} = \{(i,j) \in \downarrow \gamma : x_n \in U_{i,j}\} \) and
      \( \{(i,j) \in \downarrow \gamma : V_{n,k} \subseteq V_{i,j}\} = \{(i,j) \in \downarrow \gamma : y_n \in V_{i,j}\} \);
   2e. \( \{(i,j) \in \downarrow \gamma : U_{n,k} \cap \overline{U}_{i,j} = \emptyset\} = \{(i,j) \in \downarrow \gamma : x_n \notin \overline{U}_{i,j}\} \) and
      \( \{(i,j) \in \downarrow \gamma : V_{n,k} \cap \overline{V}_{i,j} = \emptyset\} = \{(i,j) \in \downarrow \gamma : y_n \notin \overline{V}_{i,j}\} \);
   2f. \( X_{\ell(\gamma)} = \partial U_{n,k} \) and \( Y_{\ell(\gamma)} = \partial V_{n,k} \subseteq V_{n,k} \cap Y_{\ell(n)} \);
   2g. if \( n \in \Omega \) then \( f(U_{n,k} \cap A) = V_{n,k} \cap \overline{B} \);
   2h. if \( n \notin \Omega \) then \( U_{n,k} \cap A = \emptyset = V_{n,k} \cap \overline{B} \);
   2i. If \( \Omega \neq \emptyset \), then \( X_{\ell(\gamma)} = \partial U_{n,k} \subseteq \overline{U}_{n,k} \cap X_{\ell(n)} \).
We recall that a topological space $X$ is *coregular* if it is Hausdorff and for any nonempty open sets $U_1, \ldots, U_n$ the complement $X \setminus (\overline{U_1} \cap \cdots \cap \overline{U_n})$ is a regular topological space.

So, every regular topological space $X$ is coregular.

The coregular space $\mathbb{QP}^\infty$ has the following universal property.

**Theorem**

*Every countable second-countable coregular topological space is homeomorphic to a subspace of $\mathbb{QP}^\infty$.***
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\textit{Every countable second-countable coregular topological space is homeomorphic to a subspace of $\mathbb{Q}P^\infty$.}
A subset of a topological space is called *regular open* if it is equal to the interior of its closure.

A topological space is called *semiregular* if it has a base of the topology consisting of regular open sets.

**Proposition**

*Every coregular space is semiregular.*

In particular

**Corollary**

*The superconnected countable space $\mathbb{Q}P^\infty$ is semiregular.*
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\textit{The superconnected countable space $\mathbb{Q} P^\infty$ is semiregular.}
It is easy to see that for any lines $\ell, \ell'$ in the ltp $\mathbb{Q}^{<\omega}$ there exists a linear homeomorphism $H$ of $\mathbb{Q}^{<\omega}$ such that $H(\ell) = \ell'$.

This implies that the projective space $\mathbb{QP}^{\infty}$ is topologically homogeneous: for any points $x, y \in \mathbb{QP}^{\infty}$ there exists a homeomorphism $h$ of $\mathbb{QP}^{\infty}$ such that $h(x) = y$.

In fact, the space $\mathbb{QP}^{\infty}$ is homogeneous in a much stronger sense.
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In fact, the space $\mathbb{Q}\mathbb{P}^\infty$ is homogeneous in a much stronger sense.
Deep and shallow subsets

A subset $A$ of a topological space $X$ is called

- **deep** if for any non-empty open sets $U_1, \ldots, U_n \subseteq X$ the set $A \setminus (U_1 \cap \cdots \cap U_n)$ is finite.

- **shallow** if there exist non-empty open sets $U_1, \ldots, U_n \subseteq X$ such that $A \cap (U_1 \cap \cdots \cap U_n) = \emptyset$.

**Fact 1:** For any deep (shallow) set $A$ in a topological space $X$ and any homeomorphism $h : X \to X$ the set $h(A)$ is deep (shallow).

**Fact 2:** Any infinite set in a second-countable space contains an infinite subset which is either deep or shallow.

**Fact 3:** Any finite set in a Hausdorff space is shallow.

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**Theorem (Dichotomous Homogeneity of $\mathbb{QP}^\infty$)**

Let $A, B$ be two closed discrete subsets of $\mathbb{QP}^\infty$. If the sets $A, B$ are either both deep or both shallow, then any bijection $f : A \to B$ extends to a homeomorphism $h$ of $\mathbb{QP}^\infty$ such that $h(A) = B$. 
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Let $A, B$ be two closed discrete subsets of $\mathbb{QP}^\infty$. If the sets $A, B$ are either both deep or both shallow, then any bijection $f : A \to B$ extends to a homeomorphism $h$ of $\mathbb{QP}^\infty$ such that $h(A) = B$. 
A subset $A$ of a topological space $X$ is called

- **deep** if for any non-empty open sets $U_1, \ldots, U_n \subseteq X$ the set $A \setminus (\overline{U_1} \cap \cdots \cap \overline{U_n})$ is finite.
- **shallow** if there exist non-empty open sets $U_1, \ldots, U_n \subseteq X$ such that $A \cap (\overline{U_1} \cap \cdots \cap \overline{U_n}) = \emptyset$.

**Fact 1:** For any deep (shallow) set $A$ in a topological space $X$ and any homeomorphism $h : X \to X$ the set $h(A)$ is deep (shallow).

**Fact 2:** Any infinite set in a second-countable space contains an infinite subset which is either deep or shallow.

**Fact 3:** Any finite set in a Hausdorff space is shallow.

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Since finite subsets are shallow, we have

**Corollary (Finite homogeneity of $\mathbb{QP}^\infty$)**

Any bijection $h : A \to B$ between finite subsets of $\mathbb{QP}^\infty$ extends to a homeomorphism of $\mathbb{QP}^\infty$.

**Theorem (Discrete homogeneity of $\mathbb{Q}$)**

Any bijection $h : A \to B$ between closed discrete subspaces $A, B \subset \mathbb{Q}$ extends to a homeomorphism of $\mathbb{Q}$.

How about $\mathbb{QP}^\infty$?

**Example**

$\mathbb{QP}^\infty$ contains two closed discrete subsets $A, B$ (one shallow and other deep) such that no homeomorphism of $\mathbb{QP}^\infty$ sends $A$ onto $B$. 
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**Corollary (Finite homogeneity of $\mathbb{Q}P^\infty$)**

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How about \(\mathbb{Q}P^\infty\)?

**Example**

\(\mathbb{Q}P^\infty\) contains two closed discrete subsets \(A, B\) (one shallow and other deep) such that no homeomorphism of \(\mathbb{Q}P^\infty\) sends \(A\) onto \(B\).
The space $\mathbb{Q}P^{\infty}$ is an orbit space of the action of the multiplicative group $\mathbb{Q}^*$ on $\mathbb{Q}^{<\omega} \setminus \{0\}$, so it is natural to look for topological copies of the space $\mathbb{Q}P^{\infty}$ among orbit spaces of group actions.

By a group act we understand a topological space $X$ endowed with an action $\alpha : G \times X \to X$ a group $G$. The action $\alpha$ satisfies the following axioms:

- for every $g \in G$ the map $\alpha(g, \cdot) : X \to X$, $\alpha(g, \cdot) : x \mapsto gx := \alpha(g, x)$, is a homeomorphism of $X$;
- for the identity $1_G$ of the group $G$ and every $x \in X$ we have $1_Gx = x$;
- $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$.

In this case we also say that $X$ is a $G$-space.
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We say that a G-space \( X \) has \textit{closed orbits} if for any point \( x \in X \) its \textit{orbit} \( Gx = \{gx : g \in G\} \) is a closed subset of \( X \).

A subset \( A \subseteq X \) is called \textit{G-invariant} if it coincides with its \( G \)-saturation \( GA = \bigcup_{x \in A} Gx \).

The action of \( G \) on \( X \) induces the equivalence relation

\[
E = \{(x, gx) : x \in X, \ g \in G\}.
\]

The quotient space \( X/E \) by this equivalence relation is called the \textit{orbit space} of the \( G \)-space and is denoted by \( X/G \).
We say that a $G$-space $X$ has *closed orbits* if for any point $x \in X$ its *orbit* $Gx = \{gx : g \in G\}$ is a closed subset of $X$.

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Some properties of $G$-spaces

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The quotient space $X/E$ by this equivalence relation is called the \textit{orbit space} of the $G$-space and is denoted by $X/G$. 
Let $X$ be a $G$-space with closed $G$-orbits, possessing a vanishing sequence $(X_n)_{n \in \omega}$ of nonempty $G$-invariant closed subsets such that

1. for any $n \in \omega$ and nonempty open $G$-invariant set $U \subseteq X_n$, the closure $\overline{U}$ contains some set $X_m$;

2. for any $n \in \omega$, point $x \in X \setminus X_n$, and open $G$-invariant neighborhood $U \subseteq X$ of $x \in U$, there exists an open $G$-invariant neighborhood $V \subseteq X$ of $x$ such that $\overline{V} \subseteq U \cup X_n$.

Then the orbit space $X/G$ has a superskeleton.

If $X$ is first-countable and $X/G$ is countable, then the space $X/G$ is homeomorphic to $\mathbb{Q}P^\infty$. 
A topological space \( X \) endowed with a continuous action \( \alpha : G \times X \to X \) of a Hausdorff topological group \( G \) is called **singular** if it has the following properties:

(i) the topological space \( X \) is regular and infinite;
(ii) the set \( \text{Fix}_G(X) = \{ x \in X : Gx = \{ x \} \} \) is a singleton;
(iii) for every \( x \in X \setminus \text{Fix}_G(X) \) the map \( \alpha_x : G \to X, \alpha_x : g \mapsto gx = \alpha(g, x) \), is injective and open;
(iv) the orbit \( Gx \) of every point \( x \in X \setminus \text{Fix}_G(X) \) contains the singleton \( \text{Fix}_G(X) \) in its closure \( \overline{Gx} \);
(v) for any points \( x \in X \setminus \text{Fix}_G(X) \) and \( y \in X \), there exists a neighborhood \( U \subseteq X \) of \( y \) such that for any neighborhood \( W \subseteq X \) of the singleton \( \text{Fix}_G(X) \), there exists a neighborhood \( V \subseteq X \) of \( \text{Fix}_G(X) \) such that \( \alpha_u(\alpha_x^{-1}(V)) \subseteq W \) for every \( u \in U \).
(v) for any points $x \in X \setminus \operatorname{Fix}_G(X)$ and $y \in X$, there exists a neighborhood $U \subseteq X$ of $y$ such that for any neighborhood $W \subseteq X$ of the singleton $0 = \operatorname{Fix}_G(X)$, there exists a neighborhood $V \subseteq X$ of $\operatorname{Fix}_G(X)$ such that $\alpha_u(\alpha^{-1}_x(V)) \subseteq W$ for every $u \in U$. 

\begin{center}
\begin{tikzpicture}
\draw (0,0) -- (3,3);
\draw (0,0) circle (0.5cm);
\draw (3,3) circle (0.5cm);
\draw (0,0) node {$0$};
\draw (3,3) node {$\operatorname{Fix}_G(X)$};
\draw (0,0) node {$V$};
\draw (3,3) node {$U$};
\draw (0,0) node {$W$};
\draw (0,0) node {$x$};
\draw (3,3) node {$y$};
\end{tikzpicture}
\end{center}
Examples of singular $G$-spaces:

1. The complex plane $\mathbb{C}$ endowed with the action of the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers.
2. Any subfield $F \subseteq \mathbb{C}$ endowed with the action of the multiplicative group $F^* = F \setminus \{0\}$.
3. The real line $\mathbb{R}$ endowed with the action of the multiplicative group $\mathbb{R}_+$ of positive real numbers.
4. The closed half-line $\overline{\mathbb{R}}_+ = [0, \infty)$ endowed with the action of the multiplicative group $\mathbb{R}_+$.
5. The space $\mathbb{Q}$ of rationals, endowed with the action of the multiplicative group $\mathbb{Q}_+$ of positive rational numbers.
6. The one-point compactification $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$ of the discrete space $\mathbb{Z}$ endowed with the natural action of the additive group $\mathbb{Z}$ of integer numbers.
7. The one-point compactification of any non-compact locally compact topological group $G$, endowed with the natural action of the topological group $G$. 
Examples of singular $G$-spaces:

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Examples of singular $G$-spaces:

1. The complex plane $\mathbb{C}$ endowed with the action of the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers.
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7. The one-point compactification of any non-compact locally compact topological group $G$, endowed with the natural action of the topological group $G$. 
Given a singular $G$-space $X$, consider the $G$-space $X^\omega$ endowed with the Tychonoff product topology and the coordinatewise action of the group $G$.

Let $s$ be the unique point of the singleton $\text{Fix}(X; G)$.

Consider the subspaces of $X^\omega$:

$X^{<\omega} := \{ x \in X^\omega : |\{ n \in \omega : x(n) \neq s \}| < \omega \}$ and $X^{<\omega}_0 := X^{<\omega}\backslash \{s\}^\omega$.

The orbit space $X^{<\omega}_0/G$ is called the \textit{infinite projective space} of the singular $G$-space $X$ and is denoted by $XP^\infty$.

If $X = \mathbb{F}$ is a non-discrete topological field endowed with the action of its multiplicative group $\mathbb{F}^*$, then $\mathbb{F}^{<\omega}$ is a topological vector space over the field $\mathbb{F}$ and $\mathbb{F}P^\infty$ is the projective space of $\mathbb{F}^{<\omega}$ in the standard sense. In particular, $\mathbb{Q}P^\infty$ is the projective space of the tvp $\mathbb{Q}^{<\omega}$ over the topological field $\mathbb{Q}$ of rational numbers.
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Theorem

The infinite projective space $XP^\infty$ of any singular $G$-space $X$ possesses a canonical superskeleton.

If the singular $G$ space $X$ is countable and metrizable, then its infinite projective space $XP^\infty$ is homeomorphic to the space $\mathbb{Q}P^\infty$. 
Let $\mathbb{F}$ be a topological field. Three elements $\mathbb{F}^*x, \mathbb{F}^*y, \mathbb{F}^*z$ of the projective space $\mathbb{F}P^\infty$ are called *collinear* if the union $\mathbb{F}^*x \cup \mathbb{F}^*y \cup \mathbb{F}^*z$ is contained in some 2-dimensional vector subspace of $\mathbb{F}^{<\omega}$.

For two topological fields $\mathbb{F}_1, \mathbb{F}_2$ a map $f : \mathbb{F}_1P^\infty \to \mathbb{F}_2P^\infty$ is called *affine* if for any collinear elements $\mathbb{F}_1^*x, \mathbb{F}_1^*y, \mathbb{F}_1^*z \in \mathbb{F}_1P^\infty$, the elements $f(\mathbb{F}_1^*x), f(\mathbb{F}_1^*y), f(\mathbb{F}_1^*z)$ are collinear in the projective space $\mathbb{F}_2^*P^\infty$.

A bijective map $f : \mathbb{F}_1P^\infty \to \mathbb{F}_2P^\infty$ is called an *affine isomorphism* if both maps $f$ and $f^{-1}$ are affine.

If an affine isomorphism $f : \mathbb{F}_1P^\infty \to \mathbb{F}_2P^\infty$ is also a homeomorphism, then $f$ is called an *affine topological isomorphism*.

The projective spaces $\mathbb{F}_1P^\infty, \mathbb{F}_2P^\infty$ are called *affinely isomorphic* (resp. *affinely homeomorphic*) if there exists an affine topological isomorphism $f : \mathbb{F}_1P^\infty \to \mathbb{F}_2P^\infty$. 
Let $F$ be a topological field. Three elements $F^*x, F^*y, F^*z$ of the projective space $FP^\infty$ are called *collinear* if the union $F^*x \cup F^*y \cup F^*z$ is contained in some 2-dimensional vector subspace of $F^{<\omega}$.

For two topological filed $F_1, F_2$ a map $f : F_1P^\infty \to F_2P^\infty$ is called *affine* if for any collinear elements $F_1^*x, F_1^*y, F_1^*z \in F_1P^\infty$, the elements $f(F_1^*x), f(F_1^*y), f(F_1^*z)$ are collinear in the projective space $F_2^*P^\infty$.

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The projective spaces $\mathbb{F}_1P^\infty, \mathbb{F}_2P^\infty$ are called \textit{affinely isomorphic} (resp. \textit{affinely homeomorphic}) if there exists an affine topological isomorphism $f : \mathbb{F}_1P^\infty \to \mathbb{F}_2P^\infty$. 
In spite of the fact that for any countable subfields $\mathbb{F}_1, \mathbb{F}_2 \subseteq \mathbb{C}$, the infinite projective spaces $\mathbb{F}_1 \mathbb{P}^\infty$ and $\mathbb{F}_2 \mathbb{P}^\infty$ are homeomorphic (to $\mathbb{Q} \mathbb{P}^\infty$), we have the following rigidity result for affine isomorphisms between infinite projective spaces.

**Theorem**

Two (topological) fields $\mathbb{F}_1, \mathbb{F}_2$ are (topologically) isomorphic iff their infinite projective spaces $\mathbb{F}_1 \mathbb{P}^\infty$, $\mathbb{F}_2 \mathbb{P}^\infty$ are affinely isomorphic (affinely homeomorphic).
In spite of the fact that for any countable subfields $F_1, F_2 \subseteq \mathbb{C}$, the infinite projective spaces $F_1\mathbb{P}^\infty$ and $F_2\mathbb{P}^\infty$ are homeomorphic (to $\mathbb{Q}\mathbb{P}^\infty$), we have the following rigidity result for affine isomorphisms between infinite projective spaces.

**Theorem**

Two (topological) fields $F_1, F_2$ are (topologically) isomorphic iff their infinite projective spaces $F_1\mathbb{P}^\infty$, $F_2\mathbb{P}^\infty$ are affinely isomorphic (affinely homeomorphic).
The spaces $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_+$ endowed with suitable group actions are singular $G$-spaces.

By a preceding theorem, the infinite projective spaces $\mathbb{C}P^\infty$, $\mathbb{R}P^\infty$, $\mathbb{R}_+P^\infty$ possess (canonical) superskeleta.

Each of these spaces has a countable base of the topology consisting of sets, homeomorphic to the space $\mathbb{R}^{<\omega}$, so is a (non-metrizable) $\mathbb{R}^{<\omega}$-manifold.

It can be shown that the $\mathbb{R}^{<\omega}$-manifolds $\mathbb{C}P^\infty$, $\mathbb{R}P^\infty$, $\mathbb{R}_+P^\infty$ are pairwise non-homeomorphic (because of different homotopical properties of complements $Y_0 \setminus Y_n$ of their canonical skeleta).

The distinguishing topological property of the space $\mathbb{R}_+P^\infty$ is possessing a superskeleton $(Y_n)_{n \in \omega}$ such that for every $n < m$ in $\omega$ the complement $Y_n \setminus Y_m$ is contractible.
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Each of these spaces has a countable base of the topology consisting of sets, homeomorphic to the space $\mathbb{R}^{<\omega}$, so is a (non-metrizable) $\mathbb{R}^{<\omega}$-manifold.

It can be shown that the $\mathbb{R}^{<\omega}$-manifolds $\mathbb{C}P^\infty$, $\mathbb{R}P^\infty$, $\mathbb{R}_+P^\infty$ are pairwise non-homeomorphic (because of different homotopical properties of complements $Y_0 \setminus Y_n$ of their canonical skeleta).

The distinguishing topological property of the space $\mathbb{R}_+P^\infty$ is possessing a superskeleton $(Y_n)_{n \in \omega}$ such that for every $n < m$ in $\omega$ the complement $Y_n \setminus Y_m$ is contractible.
**Fact:** The space $\mathbb{R}_+ P^\infty$ has a superskeleton $(Y_n)_{n \in \omega}$ such that for every $n < m$ in $\omega$ the complement $Y_n \setminus Y_m$ is contractible.

This fact and the topological characterization of $\mathbb{Q}P^\infty$ suggests the following topological characterization of the space $\mathbb{R}_+ P^\infty$.

**Conjecture**

A Hausdorff topological space $X$ is homeomorphic to $\mathbb{R}_+ P^\infty$ iff $X$ has a superskeleton $(X_n)_{n \in \omega}$ such that for every $n$ the set $X_{n+1}$ is a $Z$-set in $X_n$ and the space $X_n \setminus X_m$ is homeomorphic to $\mathbb{R}^{<\omega}$.

A closed subset $A$ of a topological space $X$ is called a $Z$-set in $X$ if the set $C([0, 1]^\omega, X \setminus A)$ is dense in the function space $C([0, 1]^\omega, X)$, endowed with the compact-open topology.

**Remark:** It can be shown that the spaces $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$, $\mathbb{R}_+ P^\infty$ contain dense subspaces, homeomorphic to $\mathbb{Q}P^\infty$. 
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**Remark:** It can be shown that the spaces $\mathbb{R}P^\infty$, $\mathbb{C}P^\infty$, $\mathbb{R}_+P^\infty$ contain dense subspaces, homeomorphic to $\mathbb{Q}P^\infty$. 
T. Banakh, Ya. Stelmakh,
*A universal coregular countable second-countable space*,
Thank you!