Martingale inequalities and their applications

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Colloquium WMIM

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Plan of the talk

1. Introduction
2. Unconditional constant of the Haar system
3. Hardy and Sobolev inequalities
4. Estimates for analytic projections
5. Some extensions

1. Introduction

A little bit of history

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- The concept appears in the work of Paul Lévy (1934).
- The name coined by Jean Ville (1939). One of the pioneers.
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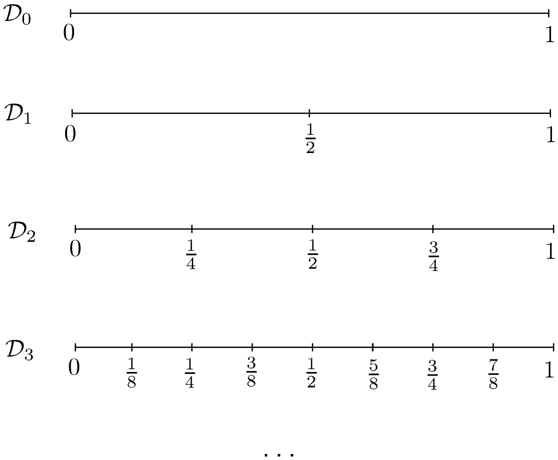
In this talk, we will be interested in martingale inequalities and their applications outside probability theory. Two big names:

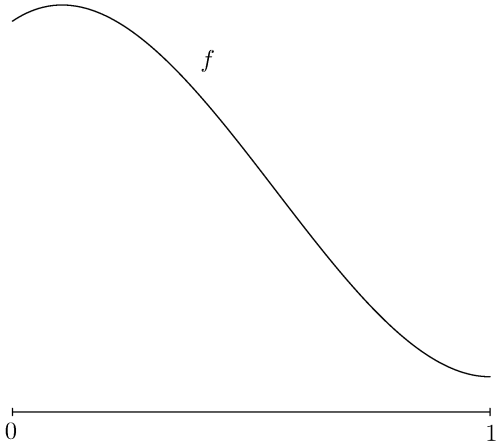
· Joseph Leo Doob,
· Donald Lyman Burkholder.

A dyadic lattice in $[0,1]$



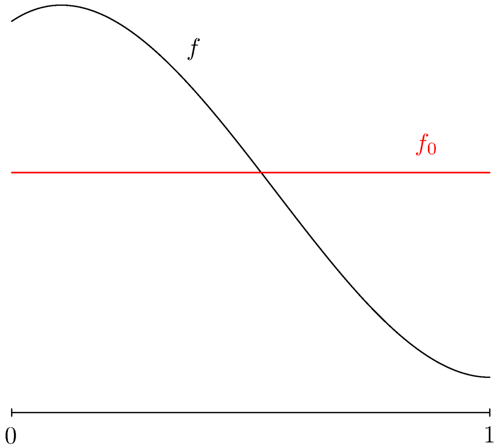
A dyadic martingale

Take an arbitrary integrable function f on $[0, 1]$ ...



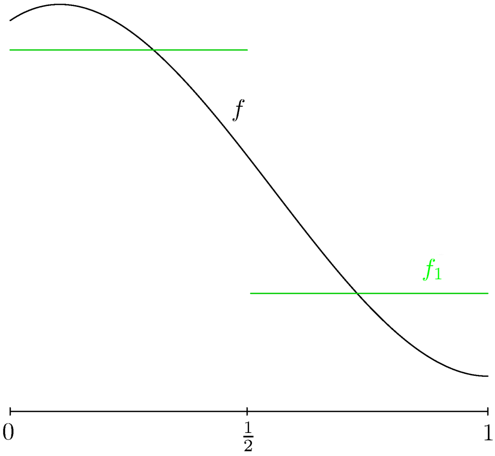
A dyadic martingale

. . . and consider its average with respect to $\mathcal{D}_0$ . . .



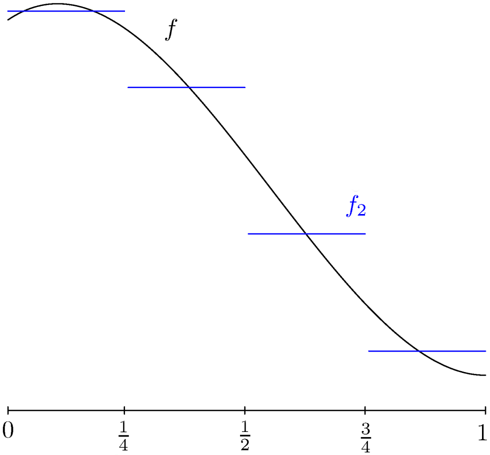
A dyadic martingale

. . . and consider its partial average with respect to $\mathcal{D}_1$ . . .



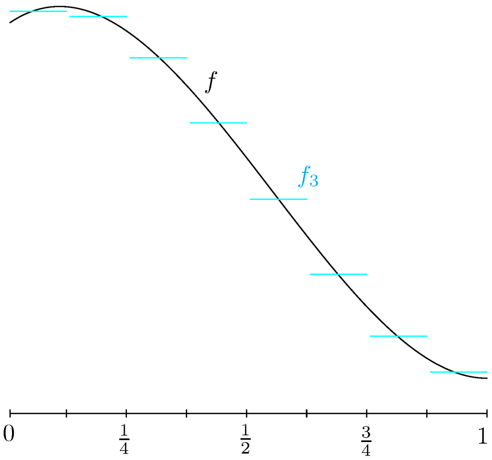
A dyadic martingale

…and consider its partial average with respect to $\mathcal{D}_2$ …

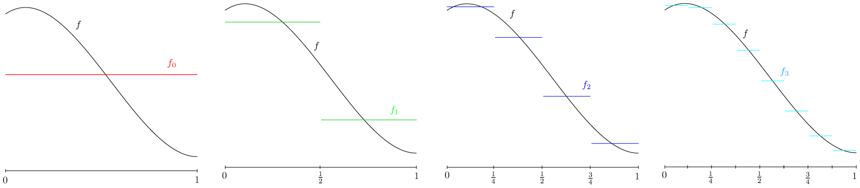


A dyadic martingale

...and consider its partial average with respect to $\mathcal{D}_3$ ...



The obtained sequence $(f_n)_{n\geqslant 0}$ is a dyadic martingale (induced by f).



Towards the general case

One can consider other partitions or other measure spaces.

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Partition $(\mathcal{D}_n)_{n\geqslant 0} \leftrightarrow (\sigma(\mathcal{D}_n))_{n\geqslant 0}$ (called filtration).

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Definition

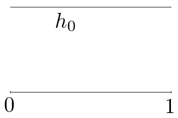
Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $(\mathcal{F}_n)_{n\geqslant 0}$. A sequence $(f_n)_{n\geqslant 0}$ of random variables is a closed martingale, if

$$f_n = \mathbb{E}(f \,|\mathcal{F}_n), \qquad n = 0, 1, 2, \ldots$$

for some integrable random variable f.

2. Unconditional constant of the Haar system

The Haar system on $[0, 1]$ (Haar, 1909)



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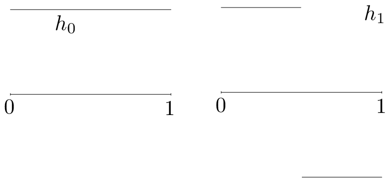

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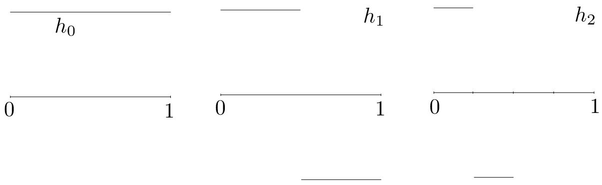

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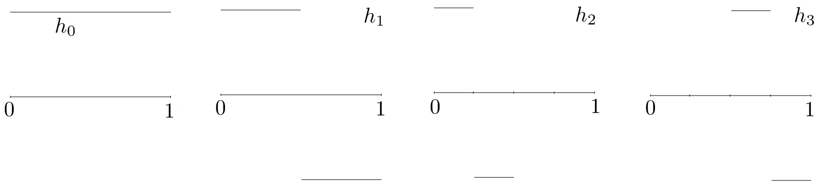

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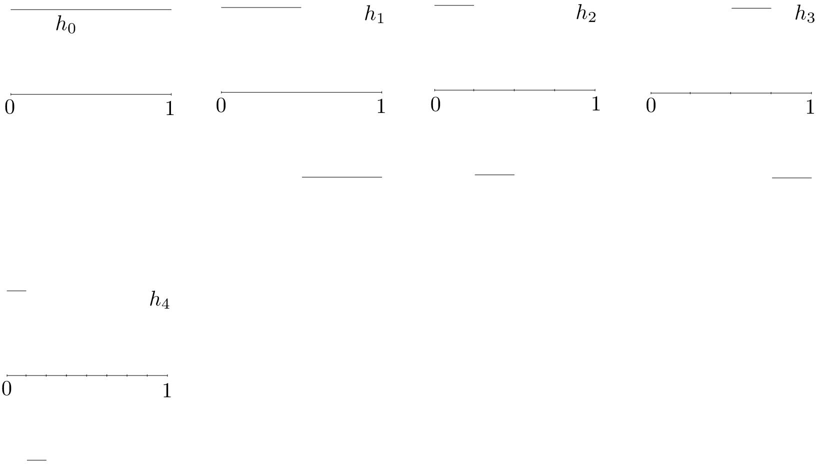

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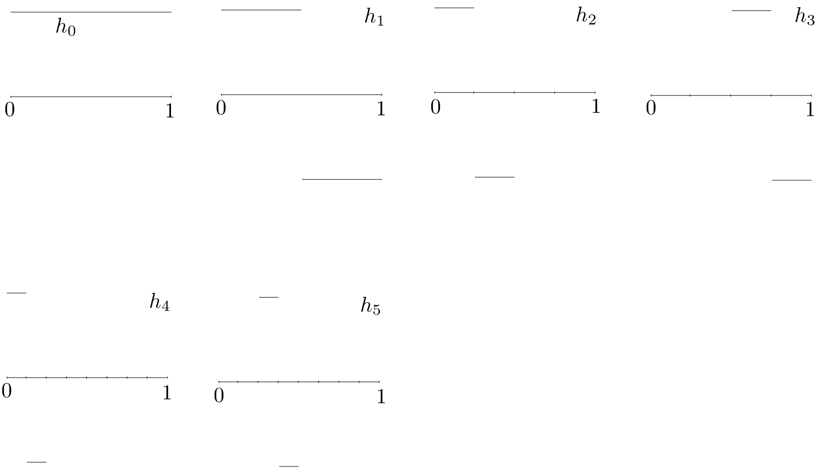

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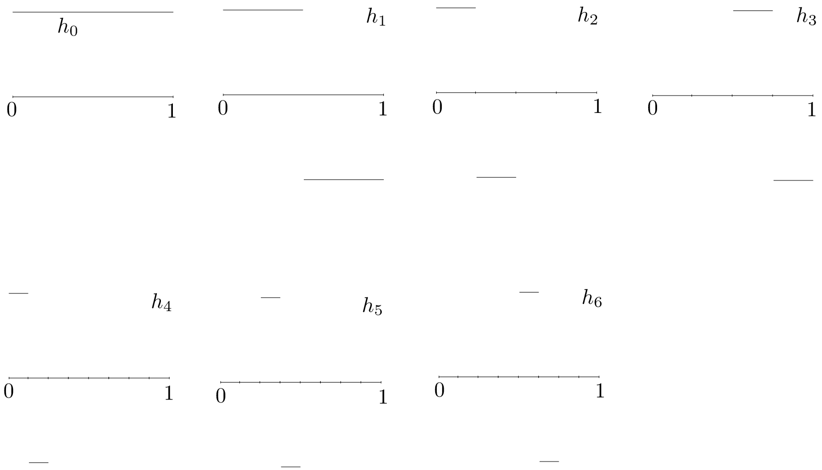

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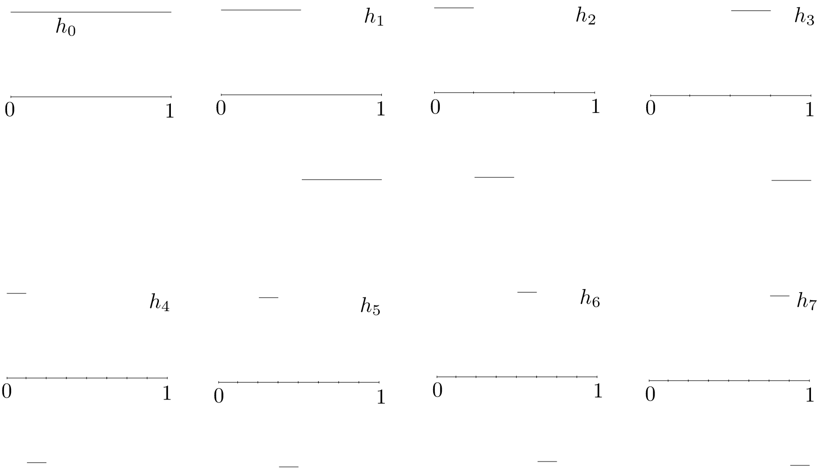

Unconditional basis

The sequence $(h_n)_{n\geqslant 0}$ is a basis of L^p, $1 \leqslant p < \infty$: for any $f \in L^p$,

$$f = \sum_{n=0}^{\infty} a_n h_n \qquad (\text{convergence in } L^p)$$

for some unique coefficients a_0, a_1, a_2, ...

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Theorem (Marcinkiewicz-Paley 1932)

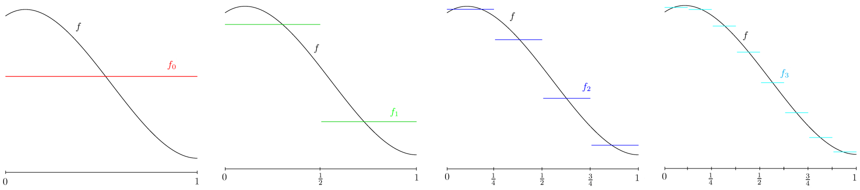
For any $1 < p < \infty$ there is a finite constant c_p such that

$$\left\| \sum_{n=0}^{N} \varepsilon_n a_n h_n \right\|_{L^p} \leqslant c_p \left\| \sum_{n=0}^{N} a_n h_n \right\|_{L^p}$$

for any N, $a_0, a_1, a_2, \ldots a_N \in \mathbb{R}$ and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N \in \{-1, 1\}$.

Dyadic martingale differences

Let $(f_n)_{n\geqslant 0}$ be the dyadic martingale induced by $f \in L^1(0,1)$.



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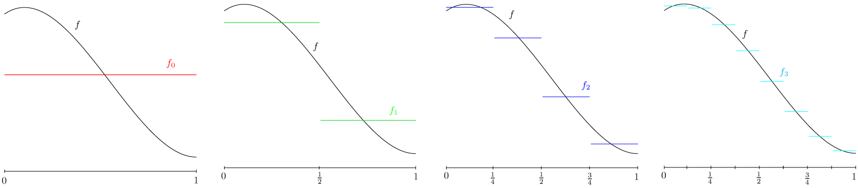

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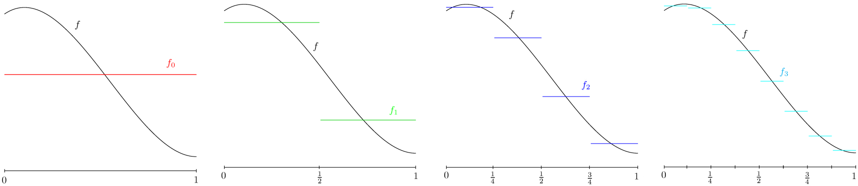
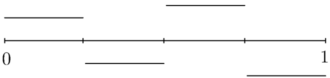


Differences:

$f_2 - f_1$



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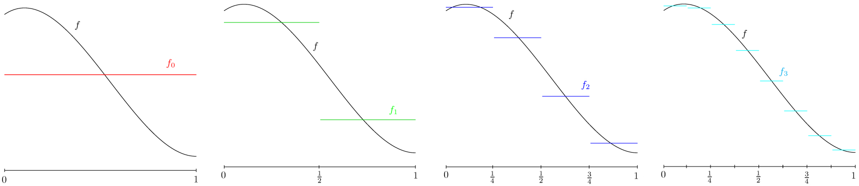
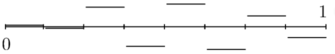


Differences:

$f_3 - f_2$



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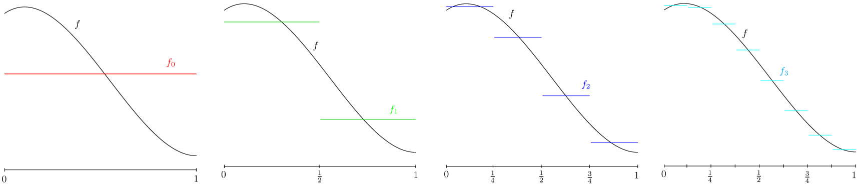

We have

$$f_n = f_0 + (f_1 - f_0) + (f_2 - f_1) + \ldots + (f_n - f_{n-1}) = \sum_{k=0}^{2^n-1} a_k h_k$$

for some coefficients a_0, a_1, a_2, ..., a_{2^n-1}.

A martingale inequality

Theorem (Burkholder 1966, 1984)

Suppose that $(f_n)_{n\geqslant 0}$, $(g_n)_{n\geqslant 0}$ *are martingales such that*

$$|g_0| \leqslant |f_0| \quad \text{and} \quad |g_n - g_{n-1}| \leqslant |f_n - f_{n-1}|, \quad n = 1, 2, \ldots .$$

Then for $1 < p < \infty$ *we have the sharp estimate*

$$\|g_n\|_{L^p} \leqslant B_p \|f_n\|_{L^p}, \qquad n = 0, 1, 2, \ldots,$$

with $B_p = \max\{p-1, (p-1)^{-1}\}$.

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It turns out that the constant is optimal for the Haar system $\rightarrow$ this is the unconditional constant of $(h_n)_{n\geqslant 0}$ in L^p.

In summary

Suppose that $(X, \mathcal{G}, \mu)$ is a measure space, T is some operator acting on measurable functions and we are interested in

$$\|Tf\|_{L^p(X)} \leqslant C_p \|f\|_{L^p(X)}$$

for a given function $f \in L^p(X, \mathcal{G}, \mu)$.

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In 1., one searches for martingales such that the increments $g_n - g_{n-1}$ are dominated by the increments $f_n - f_{n-1}$.

3. Hardy and Sobolev inequalities

Doob's maximal inequality

Theorem (Doob 1940's)

For any $1 < p \leqslant \infty$ *we have the estimate*

$$\Big\| \sup_{n \geqslant 0} |f_n| \Big\|_{L^p} \leqslant \frac{p}{p-1} \|f\|_{L^p}$$

and the constant $p/(p-1)$ *is the best possible.*

Application: Hardy inequality

Theorem (Hardy 1920, Landau 1926)

Suppose that $a_1, a_2, \ldots$ is a sequence of nonnegative numbers. Then

$$\sum_{n=1}^{\infty}\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{\infty} a_n^p.$$

The constant is optimal.

Theorem (Hardy 1920, Landau 1926)

Suppose that $a_1, a_2, \ldots$ is a sequence of nonnegative numbers. Then for any N,

$$\sum_{n=1}^{N}\left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^p \leqslant \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{N} a_n^p.$$

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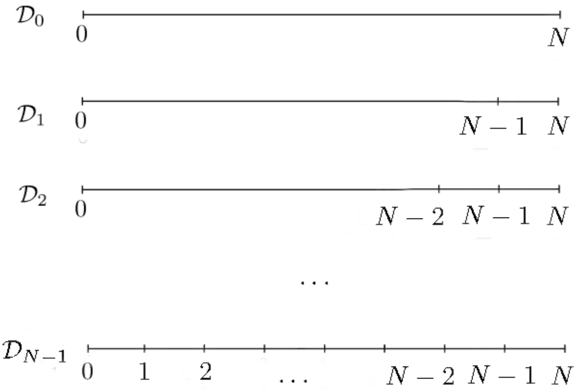
Partitions


$\mathcal{D}_0$
0
N
$\mathcal{D}_1$
0
$N-1$
N
$\mathcal{D}_2$
0
$N-2$
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...
$\mathcal{D}_{N-1}$
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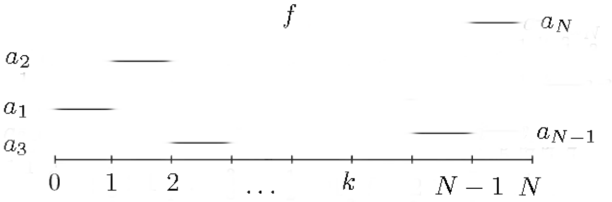
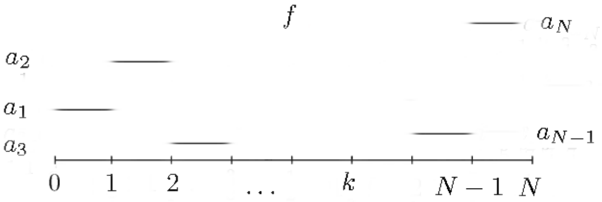
2
...
$N-2$
$N-1$
N

Reduction to Doob's estimate





Fix $k \geqslant 1$. On $(k-1, k)$, we have

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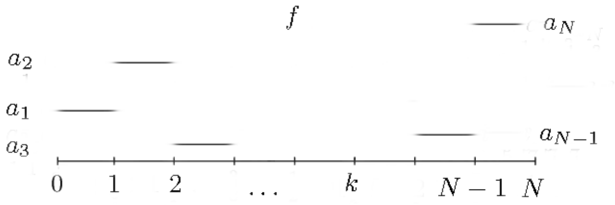

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Hence, summing over k,

$$\sum_{k=1}^{N} \left(\frac{a_1 + a_2 + \ldots + a_k}{k} \right)^p \leqslant \sum_{k=1}^{N} \int_{k-1}^{k} \sup_n |f_n|^p.$$

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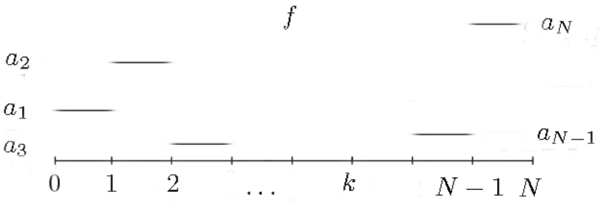

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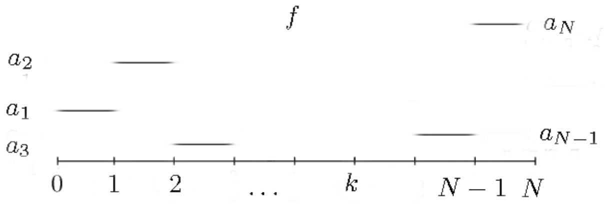

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Continuous-time extension

Theorem (Hardy 1920, Landau 1926)

For any $1 < p < \infty$ *and* $f \in L^p(0,\infty)$ *we have the sharp bound*

$$\int_0^\infty \left| \frac{1}{x} \int_0^x |f(y)| dy \right|^p dx \leqslant \left(\frac{p}{p-1} \right)^p \int_0^\infty |f(x)|^p dx.$$

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Theorem (Bliss 1930)

For $1 < p < q$, *put* $\alpha = q/p - 1$ *and let* $f \in L^p(0,\infty)$. *Then*

$$\int_0^\infty x^\alpha \left(\frac{1}{x} \int_0^x |f(y)| dy \right)^q dx \leqslant C_{p,q} \left(\int_0^\infty |f(x)|^p dx \right)^{q/p},$$

where the optimal constant is

$$C_{p,q} = \frac{1}{q - \alpha - 1} \left[\frac{\alpha \Gamma(q/\alpha)}{\Gamma(1/\alpha)\Gamma((q-1)/\alpha)} \right]^\alpha.$$

Application: Sobolev imbedding theorem

Theorem (Sobolev 1938, Talenti 1976)

For any $1 \leqslant p < d$ *and any* $u \in C_0^1(\mathbb{R}^d)$*, we have*

$$\|u\|_{L^q(\mathbb{R}^d)} \leqslant C_{p,d} \|\nabla u\|_{L^p(\mathbb{R}^d)},$$

where $q = pd/(d-p)$ *and the best constant* $C_{p,d}$ *is*

$$C_{p,d} = \pi^{-1/2} d^{-1/p} \left(\frac{p-1}{d-p} \right)^{1-1/p} \left(\frac{\Gamma(1+d/2)\Gamma(d)}{\Gamma(d/p)\Gamma(1+d-d/p)} \right)^{1/d}.$$

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Idea of proof: Suffices for $u(x) = f(|x|) \rightarrow$ Bliss' inequality.

4. Estimates for analytic projections

Analytic projection

Suppose that f is a trigonometric polynomial

$$f(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}, \qquad \theta \in (-\pi, \pi].$$

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The analytic projection P_+ and the co-analytic projection P_- are

$$P_+ f(\theta) = \sum_{n \geqslant 0} c_n e^{in\theta}, \qquad P_- f(\theta) = \sum_{n<0} c_n e^{in\theta}.$$

Why analytic/co-analytic ?

We may treat f as a function on the unit circle $\mathbb{T} \subset \mathbb{C}$:

$$f(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta} \qquad \rightarrow \qquad f(\zeta) = \sum_{n=-N}^{N} c_n \zeta^n, \qquad \zeta = e^{i\theta} \in \mathbb{T}.$$

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Then f extends to a harmonic function u_f on the unit disc.

Why analytic/co-analytic ?

We may treat f as a function on the unit circle $\mathbb{T} \subset \mathbb{C}$:

$$f(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta} \qquad \rightarrow \qquad f(\zeta) = \sum_{n=-N}^{N} c_n \zeta^n, \qquad \zeta = e^{i\theta} \in \mathbb{T}.$$

Then f extends to a harmonic function u_f on the unit disc.

We have $u_f(z) = u_{P_-f}(z) + u_{P_+f}(z)$ and u_{P_+f}, $\overline{u_{P_-f}}$ are analytic.

L^p norms

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Theorem (Riesz 1927)

If $1 < p < \infty$, then there is $C_p < \infty$ such that

$$\|P_+ f\|_{L^p(-\pi,\pi)} \leqslant C_p \|f\|_{L^p(-\pi,\pi)}.$$

If $p \leqslant 1$ or $p = \infty$, then the bound does not hold with any $C_p < \infty$.

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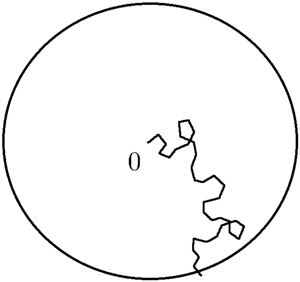
Theorem (Hollenbeck–Verbitsky 2000)

For $1 < p < \infty$, the best C_p is $(\sin(\pi/p))^{-1}$.

A martingale proof

We consider an ε-random walk $(W_n)_{n\geqslant 0}$ in $\mathbb{C}$, started at 0 and stopped upon leaving the unit disc.



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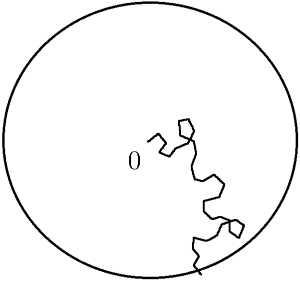

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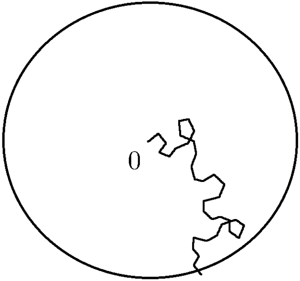

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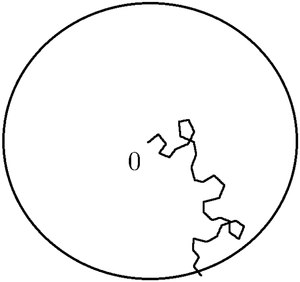

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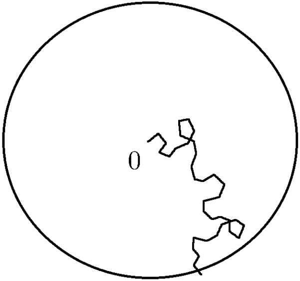

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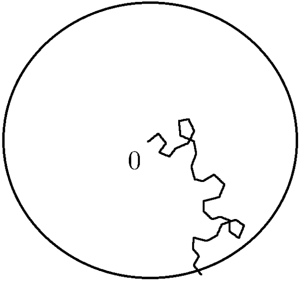

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5. Some extensions

Martingale approach can be used to study wider classes of singular integral operators and Fourier multipliers on $\mathbb{R}^d$.

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Riesz transforms on $\mathbb{R}^d$: for $j = 1, 2, \ldots, d$,

$$R_j f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}} \text{ p.v.} \int_{\mathbb{R}^d} \frac{x_j - y_j}{|x-y|^{d+1}} f(y) \mathrm{d}y.$$

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Theorem (Calderón–Zygmund 1956, Iwaniec–Martin 1996, Pichorides 1972)

We have

$$\|R_j\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} = \begin{cases} \tan\left(\frac{\pi}{2p}\right) & \text{if } 1 < p \leqslant 2, \\ \cot\left(\frac{\pi}{2p}\right) & \text{if } 2 \leqslant p < \infty. \end{cases}$$

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Theorem (Nazarov–Volberg 2001, Geiss–Montgomery-Smith–Saksman 2010)

For $j \neq k$, we have

$$\|R_j R_k\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} = \frac{1}{2}\min\{p-1, (p-1)^{-1}\}.$$

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Theorem (Bañuelos–O. (2013))

For arbitrary complex coefficients $(a_{jk})_{1\leqslant j,k\leqslant d}$, the norm

$$\left\| \sum_{j,k} a_{jk} R_j R_k \right\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)}$$

is equal to ....

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Let T be a general singular integral operator on $\mathbb{R}^d$:

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Theorem (Coifman–Fefferman 1974, Hÿtonen 2012)

For any $1 < p < \infty$ and any weight w satisfying Muckenhoupt's condition A_p, we have

$$\|T\|_{L^p(w) \to L^p(w)} \leqslant C_{p,w,T}.$$

All the above problems can be studied in other function spaces (weak-type estimates, Lorenz-norm estimates, LlogL inequalities, *BMO* estimates, etc.).

Given an $n \times n$ matrix A, its upper triangular projection is

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Towards noncommutative analysis

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$\rightarrow$ matrix martingales $\rightarrow$ noncommutative harmonic analysis ....

The last slide

Thank you for your attention.