ACN

1. $\delta(A) = \limsup_{n \to \infty} \frac{|A \cap [1, n]|}{n}$
   asymptotic density

2. $\delta(A) = \limsup_{n \to \infty} \frac{\sum_{i \in A \cap \mathbb{N}} \frac{1}{i}}{\sum_{i \in \mathbb{N}} \frac{1}{i}}$
   logarithmic density

3. uniform density
   (Borel density)

$\omega(A) = \limsup_{n \to \infty} \left( \max_{k \in \mathbb{Z}} \frac{|A \cap [k+n, k+n]|}{n} \right)$
Def. Abstract upper density

\[ \delta : P(\mathbb{N}) \rightarrow [0,1] \]

with

1. \( \delta(\emptyset) = 1 \)

2. \( F \) finite \( \Rightarrow \delta(F) = 0 \)

3. \( A \subseteq \mathbb{N} \Rightarrow \delta(A) \leq \delta(B) \)

4. \( \delta(A \cup B) \leq \delta(A) + \delta(B) \)
Demonstr

\[ \mathbb{Z}_5 = \{ x : S(A) = 0 \} \]

it is an ideal

on \( \mathbb{Z}_5 \)

(1) \( \overline{0} \in \mathbb{Z}_5 \)

(2) \( \mathbb{N} \cap \mathbb{Z}_5 \)

(3) \( A, B \in \mathbb{Z}_5 \Rightarrow A \cup B \in \mathbb{Z}_5 \)

(4) \( A \subseteq B \in \mathbb{Z}_5 \Rightarrow A \in \mathbb{Z}_5 \).
\[ \text{Eqn} \]

Ideally

on \( N \)

then

\[ S = \frac{1}{\sqrt{P/\omega}} I \]

\[ \uparrow \text{abstract upper density} \]

\[ \mathcal{S} = I. \]
Question (2013)
(G. Grevos)
Is it true that for every ideal \( I \)
there is a "nice" density \( \sigma \) of \( \mathbb{Z} - I \)?

\[ N_{f|e} = \text{translational invariant} \]
If I is translation invariant, the

\[ \delta(\mathbf{a}) = \sum_{\mathbf{A} \in I} \delta(\mathbf{A}) \]

is translation invariant density s.t. \( \mathcal{L}_{\mathbf{I}} = I \).
Def. A is rich if
\[ \forall r \in \mathbb{R}, \exists s \in S : s(A) = r \]

\[ S = \text{rich} = \sigma \text{onto} \]

Thus (Di Nasso – Jin, 2012) (Acta Arith.)

If I is a summable ideal then there is a u.d. I which
Thus

If I has the Poiseuille property, then there is rich and if

\[ \theta \geq I. \]

Proof of Princesso-Jin

- \( I = \mathbb{1} \)
- They show that there is an infinite
I - almost disjoint family in $\mathbb{I}^+$

- Using the function $f$ and this family they construct a rich and
  $\exists \gamma = I$.

- Proof is 2 pages long.
Sketch of our proof

- I w/ B.P.

- There is I-almost disjoint family in $\mathcal{F}$ of cardinality $\mathfrak{c}$.

- Using this I-AD family we construct a rich $\omega$-and $\omega_1$-$\mathcal{G}_0 = \mathcal{I}$.

- Proof is $\frac{1}{2}$ page long.
There is no one with ball 5.

\[ 25 = 5 \]
Exm 1

1. \( J - \max \text{ - ideal} \)

2. \( I = \phi^2 \otimes J \)

\[ \omega x w \]

\[ A \subset I \Rightarrow \forall A(n) \in J. \]

\[ \delta(A) = \sum_{A(n) \in J} \frac{1}{2^n} \]

\[ \text{ren}(S) = [0,1] \]
- J - a. n.d

- \( \overline{Z} = 1 \)

- There is no uncountable \( I - AD \) family in \( I^+ \)
B(0) \cap C(r) \neq \emptyset

B and C are not

I - \alpha \text{-d.}

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Ex. 2

- J_{max} \text{ ideal}
- I = \text{Fin}(X) J

\( A \in \mathcal{C} \Rightarrow \forall \omega \in A(\omega) \in \mathcal{F} \)
There is I-AD

Take Fin-A-Ds family of coordinally I on W.
\[ \phi \otimes \text{Max} \subseteq \text{Fin} \otimes \text{Max} \]

\[ \text{non-mem} \downarrow \]

\[ \text{not B. P.} \]
There is no wid e a.m. of \( \mathcal{J} \)
\[ Z \mathcal{J} = I. \]
Exm 2

I - maximal

Let $\delta \neq I$. $8 \delta = I$

$A \in I \implies \delta(A) = 0$

$A \in I \iff \omega \setminus A \in I$

$1 - \delta(\omega) \leq \delta(A) + \delta(\omega \setminus A)$

$= \delta(A) + 0 = \delta(A) \leq 1$

$A \in I \implies \delta(A) = 1$

$\text{conc}(I) = \{0, 1\}$. not. rich.
Q: How about Lebesgue measureability?

Exam 3

1. $J -$ max. ideal

**Def:**

$S(A) = \lim_{n \to \infty} \frac{140n}{n}$

- $\delta$ - rich finitely additive measure
There is no uncountable
\( I \) - A \& family in \( I^+ \)
by \( S \) - is a finite
measurer so c.c. c.

Law of large

Numbers

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} 1_{A_k} = \frac{1}{2} \]

\( A \subseteq \mathbb{N} \)

of measure 1
Let \( A \subseteq \mathbb{N} \) be \( \lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n} = \frac{1}{2} \).

Then

\[ I = \sum_{\gamma \in \mathcal{P}(\omega)} L \]

\( I \) is of measure zero.
To A + Rich

[Di Nasso - Jin, 2018]

(Acta Arith.)

If I is a summable ideal then there is a and \( \delta \) which is rich and translation invariant.

Thus the same as above for translation invariant ideals by the Baire property.
Theorem

There is a translation of the plane that maps the triangle $T$ to the triangle $A \to B, A \to C$ and $B \to C$.

Proof

To every triangle $ABC$, there is a translation that maps $A$ to $B$ and $B$ to $C$.

Given a triangle $ABC$, we can find a translation $T$ such that $T(A) = B$ and $T(B) = C$. Then, $T(C) = A$ because $T$ is a translation.

Thus, $T$ maps $A$ to $B$, $B$ to $C$, and $C$ to $A$. This proves the theorem.
\[ \text{WLOG} \]
\[ \lim_{n \to \infty} (k_{n+1} - k_n) = \infty \]

\[ k_1, k_2, k_3, \ldots \]

\[ I_n = \sum (k_n, x_{n+1}) \]

- Take \( \text{Fin} = \text{AD} \)

- Family \( A \in \mathcal{AC}^w \)

- For \( A \in \mathcal{AC} \)

\[ C_A = \bigcup_{n \in A} I_{2n} \]
\[
0 \leq A = (a_1, a_2, \ldots, a_n) \\
|E| = 2 \\
C = \{ C_A : A \in A \} \\
C_A \in I^T \quad (by \quad Togneyed) \\
C \in I_T^{A-D} (C_{Fin-T-A-D_0}) \\
A \neq B, A, B \in FA \\
k \in \mathbb{Z}, \quad \sqrt{CA \wedge (B+k)_{C_{Fin}}} 
\]
Since $|I_n| \to \infty$

$\exists N \forall n \geq N \quad |I_n| > k$

Now

$\forall n \geq N \quad (I_{2n} + k \subset I_{2n-1} \cup I_{2n} \cup I_{2n+1})$

$\forall n \geq N \quad I_{2n} \cap (I_{2n} + k) = \emptyset$
\( C_\mathcal{A} \cap (C_\mathcal{B} + k) = \)

\[ = \bigcup_{m \in \mathcal{A}} I_{2m} \cap \left( \bigcup_{n \in \mathcal{B}} I_{2n+k} \right) = \]

\[ = \bigcup_{m \in \mathcal{A}} \bigcup_{n \in \mathcal{B}} \left( I_{2m} \cap (I_{2n+k}) \right) \]

\[ \left. \right|_{m = n} \neq \phi \]

\[ \leq \bigcup_{n \in \mathbb{N}} \bigcup_{m \in \mathcal{A} \cap \mathcal{B}} \bigcup_{n \in \mathbb{N}} \left( I_{2m} \cap (I_{2n+k}) \right) \]

\[ \text{Finite} \]

\[ \text{Finite} \]

\[ \text{Finite} \]
Let \( A = \{ A_\alpha : \alpha \in \Sigma \} \)

be \( I \to \mathcal{A} \cdot D \)

family.

Extend \( A \) to maximal \( I \to \mathcal{A} \cdot D \).

Define:

\[
\delta(A) = \sup \{ \sum_\alpha \alpha : \exists k \in \mathbb{Z} \}
\]

\[
CA \wedge (A + k) \in I^+
\]