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Structure and representations of Hecke–Kiselman algebras associated to oriented graphs

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Author's declaration:
I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

This dissertation is devoted to the study of combinatorial properties of the class of monoids called Hecke–Kiselman monoids, as well as the structure and irreducible representations of the associated monoid algebras over a field, called Hecke–Kiselman algebras. Every such a monoid HK_Θ is defined by a presentation depending on a finite graph Θ. Here we consider only finite graphs Θ with oriented edges.

The case of the Hecke–Kiselman monoid C_n associated to an oriented cycle of length $n \geqslant 3$ is crucial in the investigation of any infinite-dimensional Hecke–Kiselman algebra. We investigate the surprising ideal structure inside C_n to prove that the associated semigroup algebra $K[C_n]$ is a semiprime Noetherian algebra. The classical ring of quotients is also described. These results are then applied to characterize the radical of any Hecke–Kiselman algebra that satisfies a polynomial identity. Note that the latter condition can be expressed in terms of properties of the corresponding graph. Moreover we characterize all oriented graphs Θ for which the algebra $K[\mathrm{HK}_\Theta]$ is right (left) Noetherian.

Irreducible representations of the Hecke–Kiselman algebra $K[C_n]$ associated to an oriented cycle of length $n \geqslant 3$ are described. They come from either the representations of the semigroups of matrix type occurring in the quotients of the ideal structure of C_n or are one-dimensional and arise from idempotents in a way similar to the representations of finite $\mathcal{J}$-trivial monoids. This result is then applied to the general case of Hecke–Kiselman algebras that satisfy a polynomial identity.

We also find a numerical invariant of the graph Θ that describes the Gelfand–Kirillov dimension of the corresponding algebra $K[\mathrm{HK}_\Theta]$.

Moreover, it is proved that the monoid HK_Θ satisfies a semigroup identity if and only if Θ does not contain two different cycles connected by an oriented path. The explicit construction of a semigroup identity for the monoid associated to a cycle of any length is described.

The obtained results are illustrated with the Hecke–Kiselman algebras of the monoids C_3 and C_4 associated to cycles of length 3 and 4.

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Keywords and phrases: Hecke–Kiselman algebra, monoid, algebra of matrix type, reduced words, PI-algebra, Jacobson radical, irreducible representations, Noetherian algebra, Gelfand–Kirillov dimension, automaton algebra, semigroup identity.

Streszczenie

Niniejsza rozprawa poświęcona jest badaniu własności kombinatorycznych monoidów Hecke–Kislemana oraz struktury i reprezentacji nieprzywiedlnych stowarzyszonych z nimi algebr półgrupowych, które będziemy nazywać algebrami Hecke–Kiselmana. Każdy taki monoid HK_Θ zadany jest przez prezentację związaną z pewnym skończonym grafem Θ. Wszystkie rozpatrywane w pracy grafy są zorientowane.

Okazuje się, że przypadek monoidu C_n związanego ze zorientowanym cyklem długości $n \geqslant 3$ jest kluczowy w badaniu własności dowolnych nieskończonych monoidów oraz algebr Hecke–Kiselmana. Zbadanie własności pewnego ważnego łańcucha ideałów w C_n pozwala nam udowodnić, że algebra półgrupowa $K[C_n]$ nad ciałem K jest półpierwsza oraz Noetherowska. Charakteryzujemy w pracy klasyczny pierścień ułamków tej algebry. Następnie otrzymane rezultaty zostają zastosowane do wyznaczenia radykału dowolnej algebry Hecke–Kiselmana $K[\mathrm{HK}_\Theta]$, która spełnia tożsamość wielomianową. Warunek ten można łatwo wyrazić w języku własności stowarzyszonych grafów. Ponadto podajemy charakteryzację grafów, dla których algebra $K[\mathrm{HK}_\Theta]$ jest prawostronnie oraz lewostronnie Noetherowska.

Kolejnym rozpatrywanym zagadnieniem są reprezentacje nieprzywiedlne. W pracy zostaje udowodnione, że w przypadku algebry Hecke–Kiselmana $K[C_n]$ nad ciałem algebraicznie domkniętym K reprezentacje nieprzywiedlne pochodzą od reprezentacji struktur typu macierzowego występujących w łańcuchu ideałów w C_n lub są jednowymiarowe i stowarzyszone z idempotentami w C_n, analogicznie do dobrze znanego przypadku reprezentacji skończonych $\mathcal{J}$-trywialnych monoidów. Wynik ten umożliwia opisanie reprezentacji w przypadku dowolnych algebr Hecke–Kiselmana spełniających tożsamość wielomianową.

Następnie charakteryzujemy wymiar Gelfanda–Kirillova dowolnych algebr Hecke–Kiselmana za pomocą pewnego numerycznego niezmiennika stowarzyszonych grafów.

Ponadto udowadniamy w pracy, że monoid HK_Θ spełnia tożsamość półgrupową wtedy i tylko wtedy, gdy graf Θ nie zawiera dwóch różnych cykli połączonych zorientowaną ścieżką dowolnej długości. Jest to równoważne temu, że algebra $K[\mathrm{HK}_\Theta]$ spełnia tożsamość wielomianową.

Ostatnia część pracy stanowi ilustrację otrzymanych wyników dla przypadku monoidów C_3 i C_4 stowarzyszonych z cyklami długości 3 oraz 4.

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Słowa kluczowe: algebra Hecke–Kiselmana, monoid, algebra typu macierzowego, postać zredukowana, algebra spełniająca tożsamość wielomianową, radykał Jacobsona, reprezentacja nieprzywiedlna, algebra Noetherowska, wymiar Gelfanda–Kirillova, algebra automatowa, tożsamość półgrupowa.

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Contents

Introduction

Many important algebraic structures occurring in mathematics are defined by imposing generators and relations between them. Celebrated examples include the braid group and the braid monoid, as well as Iwahori–Hecke algebras of Coxeter groups, that play a significant role in particular in knot theory, algebraic combinatorics and quantum groups. In these structures the braid relation, that is the relation $xyx = yxy$ for generators x and y, and its various generalizations occur.

The dissertation is concerned with a class of monoids and their monoid algebras given by finite presentations related to braid relations, depending on associated oriented graphs.

Let Θ be an oriented finite graph with n vertices, denoted by $x_1, \ldots, x_n$ with at most one edge between two vertices and without edges that connect a vertex to itself. Then the corresponding Hecke–Kiselman monoid HK_Θ is defined as follows.

1) HK_Θ is generated by $x_1, \ldots, x_n$ with $x_i^2 = x_i$ for every $i = 1, \ldots, n$.

2) If there is an edge $x_i \to x_j$ in Θ, then $x_i x_j = x_i x_j x_i = x_j x_i x_j$ in the monoid HK_Θ.

3) If vertices x_i and x_j are not connected by an edge, then the relation $x_i x_j = x_j x_i$ is imposed.

This is a special case of a more general definition of Hecke–Kiselman monoids associated to the graphs Θ with both oriented and unoriented edges. If there is an unoriented edge between x_i and x_j, then the braid relation $x_i x_j x_i = x_j x_i x_j$ holds in HK_Θ.

Hecke–Kiselman monoids were introduced by Ganyushkin and Mazorchuk in the paper [18] as a generalization of two classes of finite monoids, one coming from representation theory and the second occurring in convexity theory, that share certain structural properties. Various aspects of such monoids, mainly of a combinatorial nature, have been studied in a series of papers [3–5, 10, 11, 16, 18–20, 29, 33, 39, 40].

Our first motivation for the study of Hecke–Kiselman monoids comes from the fact that these monoids are homomorphic images of 0-Hecke monoids, [42]. The latter come from a specialization of Iwahori–Hecke algebras, occurring naturally in the representation theory of Coxeter groups. For an overview we refer the reader to Chapter 5 of [25].

For any set S and a function $m : S \times S \to \{1, \ldots, +\infty\}$ such that $m(s, s') = m(s', s)$ for all $s, s' \in S$ and $m(s, s') = 1$ if and only if $s = s'$, the corresponding Coxeter group W is given by the (group) presentation

$$\langle s \in S \quad | \quad (ss')^{m(s,s')} = 1 \text{ for all } s, s' \in S \text{ with } m(s,s') \neq +\infty \rangle.$$

The pair (W, S) is called a Coxeter system. Equivalently, a function m in the above definition is often presented as an unoriented graph with the set of vertices identified with S and exactly $m(s, s') - 2$ edges between vertices $s, s' \in S$. Coxeter groups arise in several distant branches of mathematics. As Weyl groups of root systems, they play an important role in Lie theory, in the classification of complex semisimple Lie algebras. In certain cases Coxeter groups can be also interpreted as real reflection groups, [22], interesting from the geometric point of view. Moreover, several combinatorial aspects have been studied, [7].

Iwahori–Hecke algebras of Coxeter systems are deformations of group algebras of these groups, depending on the parameter q. More precisely, for a Coxeter system (W, S), the Iwahori–Hecke algebra $H_q(W, S)$ is a unital algebra over a fixed field generated by elements of the set of generators S, denoted by T_s ($T_1 = 1$) and relations

$$(T_s - q)(T_s + 1) = 0, \text{ for all } s \in S$$

$$(T_s T_{s'})_{m(s,s')} = (T_{s'} T_s)_{m(s,s')} \text{ for all } s \neq s',$$

where the lower index $m(s, s')$ indicates the number of factors in the expressions of the form $T_s T_{s'} T_s \ldots$. Consequently, for $q = 1$ we get the group algebra of (W, S). In the second special case of the specialization at $q = 0$, the Iwahori–Hecke algebra is the monoid algebra of the so-called 0-Hecke monoid. For the function m from the definition of a Coxeter group, such a monoid is generated by the set S and relations of the form $s^2 = s$ and $(ss')_{m(s,s')} = (s's)_{m(s',s)}$. It has been proved that there exists a bijection between the elements of a Coxeter group (W, S) and of the 0-Hecke monoid coming from this group, [55]. Therefore such a monoid can be treated as a semigroup analogue of the associated Coxeter group. The representation theory of 0-Hecke monoids has been characterized by Norton in [42]. These results have been then used to build a rich combinatorial approach to these representations in [21].

Moreover, slightly different families of algebras related to Coxeter systems are also studied, such as for example nil-Coxeter algebras, and their generalizations, [26]. Nil-Coxeter algebras are given by the same relations as in the group algebra of a Coxeter group except for $s^2 = 1$, which is replaced by $s^2 = 0$. Introduction of such classes of algebras is motivated in particular by connections to geometry [15], combinatorics and categorification [26].

Let us also note that the ideal structures inside the Hecke–Kiselman monoids associated to an oriented cycle of any length discovered in [45] have the flavour of the cell ideals of the so called affine cellular algebras [27], that play an important role in representation theory of some classes of algebras, including various Iwahori–Hecke algebras. In this context the study of Hecke–Kiselman monoids seems to be well-founded because of the possible connections and applications to those theories.

Both classes of monoids that led to the definition of the Hecke–Kiselman monoids, namely

0-Hecke monoids and Kiselman's monoids, are examples of finite monoids that are $\mathcal{J}$-trivial, [29]. In particular, their irreducible representations admit a nice description, that can be found for example in [12] and in [53]. In general a characterization of oriented graphs such that the associated Hecke–Kiselman monoids are $\mathcal{J}$-trivial is not known. On the other hand, all finite Hecke–Kiselman monoids associated to oriented graphs and the infinite monoid associated to an oriented cycle of any length have this property, [18] and [11]. In this setting the description of irreducible representations of the latter monoid, obtained in the dissertation, fits into the study of possible generalizations of the representation theory of (not necessarily finite) $\mathcal{J}$-trivial monoids.

What is more, the class investigated in this dissertation plays certain role in representation theory of finite dimensional algebras. Namely, it has been proved in [19] that monoids generated by the so-called projection functors of simple modules over special classes of path algebras of finite quivers are isomorphic to certain finite Hecke–Kiselman monoids. Moreover, a categorical approach to the representation theory is investigated in the paper [20] in the case of Hecke–Kiselman monoids associated to special acyclic graphs.

Hecke–Kiselman monoids are also useful in the study of the mathematical language of computer simulations. Namely, finite Hecke–Kiselman monoids HK_Θ find a natural realization in a combinatorial approach to the so-called sequential dynamical systems defined with the use of the graph Θ, [10].

The next motivation to study Hecke–Kiselman monoids comes from connections with the Yang–Baxter equation. The quantum Yang–Baxter equation is one of the equations arising in mathematical physics that initiated a rapid development of a wide range of studies in several branches of mathematics, such as some aspects of Hopf algebras, knot theory and quantum groups, [24]. As finding all solutions seems to be extremely hard, it was proposed by Drinfel'd in [13] to study a special class of solutions, called set-theoretic solutions. Since then the problem has been extensively explored using various methods built (among others) on: noncommutative rings [51], group theory [14], and semigroup theory [23,32]. In particular, the so-called structure group, monoid and the algebra that can be associated to any set-theoretic solution have been proved extremely useful, [14, 23]. It is well-known that every solution of the Yang–Baxter equation induces a representation of the braid group. Moreover, the special class of idempotent solutions, introduced in [32], induces representations of 0-Hecke monoids and can be used in the study of homological aspects of other important classes of monoids. Note that in the paper [33] of Lebed certain representations of Hecke–Kiselman monoids inspired by the Yang–Baxter equation are constructed. Moreover, representations of the symmetric groups have been already applied in the context of solutions of the Yang–Baxter equation in [34]. The obtained results justify extending research to wider classes of associative algebras that could be used to construct new solutions of Yang–Baxter equation.

These were the main motivations for the research project that has lead to the results of the dissertation. The main aims were to describe structural and combinatorial properties of Hecke–Kiselman algebras and to built representation theory of these algebras. Most of the results of the thesis have been published in [45, 46, 57, 58]. However, some other original results have not been published yet; these include in particular the results of Chapter 7 and of Section 8.5.

Structure and main results of the dissertation

In Chapter 1 we introduce the mathematical notions and the necessary background on noncommutative ring theory and semigroup theory, that will be used in the further parts of the thesis. Next, after giving the definition of the Hecke–Kiselman monoids and algebras, selected known results about the combinatorics on words in these monoids and certain properties of their algebras are described. These include a description of a Gröbner basis and a characterization of Hecke–Kiselman algebras that satisfy a polynomial identity, stated in Theorem 1.63 and Theorem 1.72.

Chapter 2 is devoted to the structure of the Hecke–Kiselman monoids C_n associated to oriented cycles of length $n \geqslant 3$. The importance of this case comes from the fact that the monoid HK_Θ is finite if and only if the graph Θ is acyclic. The main results are Theorem 2.1, Theorem 2.44 and Theorem 2.52. First, a characterization of almost all reduced words in C_n is provided. It is then applied to construct a surprising ideal chain inside this monoid with factors that are, up to finitely many elements, semigroups of matrix type. Lastly, we prove that the semigroup algebras associated to such semigroups are prime algebras. The significant part of this chapter is based on the results obtained during author's master's studies. Therefore, instead of providing full proofs, we often outline only the main ideas. On one hand, this is to explain the nature of the technical auxiliary lemmas, on the other hand, some of these results are also used in the subsequent parts of the thesis. Section 2.4 contains theorem proved during author's PhD studies. The content of this chapter is mostly based on the paper [45], written jointly with Jan Okniński.

In Chapter 3 the radical of Hecke–Kiselman algebras that satisfy a polynomial identity is described. Note that this condition can be expressed in terms of the properties of the corresponding graph. The case of the algebra $K[C_n]$ associated to an oriented cycle turns out to be a crucial step. Using the properties of structures of matrix type from the previous chapter, it is proved that the algebra is semiprime in this case. As a by-product we describe the maximal chains of prime ideals and the classical ring of quotients of $K[C_n]$. We also prove that this algebra is Noetherian. Finally, we proceed inductively to show that every Hecke–Kiselman algebra which is PI has a radical which can be described by a certain congruence in the monoid and the algebra modulo the radical is another Hecke–Kiselman algebra associated to a subgraph of the original graph, admitting a clear description. The main results are collected in Theorem 3.3, Theorem 3.5 and Theorem 3.9. The chapter is based on the joint work with Jan Okniński, [46].

The main result of Chapter 4 is the complete characterization of oriented graphs Θ, for which the Hecke–Kiselman algebra is right (left) Noetherian, Theorem 4.2. As the most difficult case of the algebra associated to an oriented cycle of any length is resolved in the previous chapter, Lemma 4.1 is sufficient to complete the proof. The results of this chapter were obtained in the author's master's thesis and were published in [45].

Chapter 5 is devoted to the irreducible representations of Hecke–Kiselman algebras over an algebraically closed field that satisfy a polynomial identity. As in the case of other properties, the case of the algebra $K[C_n]$ associated to an oriented cycle is crucial. This chapter starts with a characterization of idempotents in C_n. This result had been known before, but we characterize reduced forms of idempotents and provide an alternative proof for the sake of completeness. Then the semigroups of matrix type, discovered inside the monoid C_n in Chapter 2, are exploited to show that irreducible representations of $K[C_n]$ either are induced by those of semigroups of matrix type or are one-dimensional representations arising from idempotents in the way similar to that known for finite $\mathcal{J}$-trivial monoids (see [53]). As a consequence, we are able to characterize all maximal ideals of $K[\mathrm{HK}_\Theta]$ which satisfies a polynomial identity. Irreducible representations of such algebras are then described. The main results of this chapter are collected in Theorem 5.8, Proposition 5.10 and Theorem 5.12. This work was published in the paper [57].

In Chapter 6 we focus on the growth of algebras. It had been known before that the Gelfand–Kirillov dimension of any Hecke–Kiselman algebra is either a finite integer or infinite and it is equal to 1 for the algebra associated to an oriented cycle of any length. In Theorem 6.12 we give a characterization of the Gelfand–Kirillov dimension of $K[\mathrm{HK}_\Theta]$ in terms of a numerical invariant of the graph Θ. Namely, the dimension is equal to the sum of the lengths of paths of certain specific type in the graph Θ and the number of cyclic subgraphs of the graph. The obtained theorem relies on the result asserting that Hecke–Kiselman algebras are automaton, [40]. This part of the dissertation is published in the paper [58].

Chapter 7 is concerned with the semigroup identities of Hecke–Kiselman monoids. The existence of such an identity in the case of finite monoids can be deduced from known results. Therefore we focus on the case when the monoid HK_Θ is infinite. Semigroups of matrix type are used to construct an explicit identity in the case of the monoid C_n associated to a cycle of any length n, Theorem 7.1. Then an inductive construction is used to prove that the Hecke–Kiselman monoid satisfies a semigroup identity if and only if the graph does not contain two different cycles connected by an oriented path, Theorem 7.2. The latter condition is known to be equivalent to the property that the monoid does not contain free submonoids of rank 2 and also to the fact that the corresponding algebra satisfies a polynomial identity.

Chapter 8 illustrates our main results with the Hecke–Kiselman monoid C_3 and the algebra $K[C_3]$ associated to the cycle of length 3 and partially with the monoid C_4 associated to the cycle with 4 vertices. Then a subalgebra Z of the centre of $K[C_3]$ such that $K[C_3]$ is a finite module over Z is described, using two methods. Calculations involve explicit forms of reduced words representing elements of the semigroups of matrix type inside C_3. While we proved that $K[C_n]$ is a finite module over its center for every $n \geq 3$, a description of the center is not known. Thus, we end with several remarks about limitations of the methods used in this example.

Chapter 1

Preliminaries

In this chapter we recall basic notions and results of ring theory and semigroup theory used in the thesis. Moreover we present the definition and preliminary results on Hecke–Kiselman monoids and their algebras.

1.1 Ring theory background

1.1.1 Introduction

Let us introduce the necessary terminology on noncommutative ring theory and theory of radicals. Note that we will touch only a few selected aspects of the theory that will be applied in our context. We refer the reader to the book [38] for a comprehensive overview of the subject.

All considered rings are associative rings, with an identity 1, if not specified otherwise.

If a left or right R-module M has the property that for any ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots$ of submodules of M there exists an integer n such that $M_n = M_{n+1} = \ldots$, then we say that M satisfies the ascending chain condition on submodules.

Definition 1.1. A left (right) R-module M is Noetherian if it satisfies the ascending chain condition on submodules. A ring R is left (right) Noetherian, if it is Noetherian when viewed as a left (right) R-module. Finally, a ring is Noetherian if it is right and left Noetherian.

Example 1.2. 1) Every field is Noetherian.

2) The well-known Hilbert basis theorem states that the polynomial ring $R[x]$ over a Noetherian ring R is also Noetherian.

3) $M_n(R)$ is right Noetherian if and only if R is right Noetherian.

The descending chain condition is defined dually. A left (right) R-module M is Artinian, if it satisfies the descending chain condition on submodules. In particular, a ring R is right (left) Artinian if it is Artinian as right (left) R-module. A ring is Artinian if it is right and left

Artinian. It is easy to check that $\mathbb{Z}$ is Noetherian but not Artinian ring. On the other hand, although these two notions seem to be symmetrical, the famous Hopkins–Levitzki theorem states that every right Artinian ring is right Noetherian. Let us also note that the following proposition holds.

Proposition 1.3. *If M is a finitely generated left module over a left Noetherian (Artinian) ring, then M is a Noetherian (Artinian, respectively) module.*

1.1.2 Prime rings and prime radical

Definition 1.4. An ideal P of a ring R is said to be prime if for any ideals I, J of R from $IJ \subseteq P$ it follows that $I \subseteq P$ or $J \subseteq P$. A ring R is called prime if $\{0\}$ is a prime ideal in R.

Let us mention an alternative equivalent definition. An ideal P of R is prime if for any elements a, b the condition $aRb \subseteq P$ implies that $a \in P$ or $b \in P$. We can also replace in the definition two–sided ideals with one-sided ideals; that is P is prime if for any two left (right) ideals I, J from $IJ \subseteq P$ it follows that $I \subseteq P$ or $J \subseteq P$.

Example 1.5.
1) If R is a commutative ring, then it is prime if and only if it is an integral domain.
2) The matrix ring $M_n(R)$ over a ring R is prime if and only if R is a prime ring.

A minimal prime ideal of R is any prime ideal in R which does not contain properly any other prime ideal of R. It can be proved that every prime ideal contains a minimal prime ideal.

Definition 1.6. The intersection of all prime ideals in a ring R is called the prime radical of R and will be denoted by $\mathcal{P}(R)$.

Example 1.7.
1) If $R = K[x]/(x^2)$, then (x) is the only prime ideal and thus $\mathcal{P}(R) = (x)$.
2) It can be proved that $\mathcal{P}(M_n(R)) = M_n(\mathcal{P}(R))$ for any ring R.

It can be proved that $\mathcal{P}(R)$ is always a nil ideal of R, that is for every $x \in \mathcal{P}(R)$ there exists $n \geqslant 1$ such that $x^n = 0$. Moreover from the definition it follows that every nilpotent ideal (that is an ideal I such that $I^n = \{0\}$ for some $n \geqslant 1$) is contained in $\mathcal{P}(R)$.

Definition 1.8. An ideal Q of a ring R is said to be semiprime if for any ideal I in R the condition $I^2 \subseteq Q$ implies that $I \subseteq Q$. A ring R is semiprime, if $\{0\}$ is a semiprime ideal in R.

An alternative condition involving elements of the ring instead of its ideals is that an ideal I is semiprime if for any element $a \in R$ from $aRa \subseteq I$ it follows that $a \in I$. There is a strong relation between the notions of the prime radical and the semiprime ring. Namely, the following holds.

Proposition 1.9. *A ring R is semiprime if and only if $\mathcal{P}(R) = \{0\}$.*

From the definitions it is easy to see that any direct product of semiprime rings is semiprime. Therefore from the previous example we can construct the following example.

Example 1.10. If R_i are semiprime rings for $i = 1, \dots, k$ then the ring

$$M_{n_1}(R_1) \times M_{n_2}(R_2) \times \cdots \times M_{n_k}(R_k)$$

is semiprime.

Note that from the definition of a semiprime ring it follows that for any $a \in R$ such that $aRa = 0$ we know that $a \in \mathcal{P}(R)$.

1.1.3 Primitive rings and Jacobson radical

We say that a left (right) R-module $M \neq \{0\}$ is simple if it has no submodules different than $\{0\}$ and M. The annihilator of a left (right) R-module M is defined as $\mathrm{ann}_R(M) = \{r \in R : rM = 0\}$ ($\mathrm{ann}_R(M) = \{r \in R : Mr = 0\}$, respectively). A left (right) R-module M is said to be faithful if $\mathrm{ann}_R(M) = \{0\}$.

Definition 1.11. A ring R is called left (right) primitive if it has a simple faithful left (right, respectively) module. An ideal P of R is left primitive if R/P is a left primitive ring.

The following useful proposition holds.

Proposition 1.12. *Left (right) primitive ideals of R are exactly annihilators of simple left (right) R-modules.*

Example 1.13.

1) If R is a commutative ring, then every simple R-module M is of the form R/P for a maximal ideal P such that $P = \mathrm{ann}_R(M)$. It follows that left and right primitive ideals in this case are exactly maximal ideals of R.

2) Every maximal ideal is right and left primitive.

Note that every left (right) primitive ring is prime, but the converse is not true, as for example the polynomial ring $\mathbb{C}[x]$ over complex numbers is prime but not left (right) primitive.

Definition 1.14. The Jacobson radical of a ring R, denoted by $\mathcal{J}(R)$, is the intersection of all left primitive ideals in R.

Alternatively, the Jacobson radical of R can be characterized as the intersection of annihilators of all simple left (or right) R-modules or as the intersection of all maximal left ideals in R.

Despite the fact that left primitive ideals occur in the above definition, the notion of the Jacobson radical is left-right symmetric.

Example 1.15. 1) If $R = K[x]/(x^2)$ for a field K, then $\mathcal{J}(R) = (x)$, as (x) is the only maximal ideal in R.

2) For any ring R it is known that $\mathcal{J}(M_n(R)) = M_n(\mathcal{J}(R))$.

A ring R is called semiprimitive if $\mathcal{J}(R) = \{0\}$. This notion is analogous to the definition of a semiprime ring discussed before.

Let us observe that directly from the definition we get the inclusion $\mathcal{P}(R) \subseteq \mathcal{J}(R)$ for any ring R. In many important cases these two notions of radicals coincide. For example we have the following.

Proposition 1.16. *If R is a left Artinian ring, then $\mathcal{J}(R) = \mathcal{P}(R)$, and it is a nilpotent ideal.*

As in the dissertation we are mainly interested in the case of another class of rings R for which $\mathcal{P}(R) = \mathcal{J}(R)$, we do not dwell into the details of the general theory.

1.1.4 Gelfand–Kirillov dimension

Let us introduce the definitions related to the Gelfand–Kirillov dimension, called also GK dimension, for short. This dimension describes an asymptotic behaviour of the growth of algebras and is a basic tool in the study of noncommutative algebras.

We restrict our attention to finitely generated algebras over a field K. For the definition in full generality and for more information we refer the reader to [28]. Let A be a finitely generated algebra. A generating subspace V of A is any finite dimensional K-subspace such that A is generated by V as a K-algebra. Such a subspace always exists, as we can consider the space spanned by the finite set of generators of A.

Definition 1.17. Let A be a finitely generated algebra over a field K and V be an arbitrary generating subspace of A. The growth function of A with respect to V is given by

$$d_V(n) = \dim_K(V^0 + \ldots + V^n),$$

where $V^k = \mathrm{lin}_K\{v_1 \cdots v_k : v_i \in V \text{ for } i = 1, \ldots, k\}$ for any $k \geqslant 0$, with $V^0 = K$.

The growth function depends on the choice of a generating subspace of an algebra. To obtain an invariant of an algebra we consider the asymptotic behaviour of this function. Arbitrary non-decreasing functions $f, g : \mathbb{N} \to \mathbb{R}_+$ are said to be equivalent, if there are integer constants c_1, c_2 and m_1, m_2 such that $f(n) \leqslant c_1 g(m_1 n)$ and $g(n) \leqslant c_2 g(m_2 n)$ for all but finitely many natural numbers n. Then it can be shown, [28, Lemma 2.1], that if f and g are equivalent, then $\limsup_{n\to\infty} \log_n f(n) = \limsup_{n\to\infty} \log_n g(n)$.

It can be easily verified that if V and W are two generating spaces of an algebra A, then d_V and d_W are equivalent. Consequently, $\limsup_{n\to\infty} \log_n d_V(n)$ does not depend on the choice of V.

Definition 1.18. The Gelfand–Kirillov dimension of a finitely generated algebra A, which we will denote by $\operatorname{GKdim}(A)$, is defined as follows

$$\operatorname{GKdim}(A) = \limsup_{n\to\infty} \log_n d_V(n),$$

where $d_V(n)$ is the growth function of the algebra A with respect to an arbitrary generating subspace V of A. As explained above, the definition does not depend on the choice of V.

Possible values of the Gelfand–Kirillov dimension are in the set $\{0\} \cup \{1\} \cup [2, \infty]$. In particular, there exist algebras with a non-integer dimension. Moreover, an algebra is of zero GK dimension if and only if it is finite dimensional and in the class of finitely generated commutative algebras GK dimension coincides with the classical Krull dimension (defined as the supremum of the lengths of chains of prime ideals).

Example 1.19. In the case of the monoid algebra $K[M]$ associated with a finitely generated monoid M generated by a set Z containing the identity element of M, a generating subspace can be chosen as $V = \operatorname{lin}_K Z$. Then the growth function d_V is the rate of growth of the monoid, that is $d_V(n) = |\{m \in M : m = m_1 \dots m_k, k \leqslant n, m_i \in Z\}|$. Consequently, the GK dimension of the algebra $K[M]$ measures an asymptotic behaviour of the number of elements of M of at most given length. For instance, if M is an infinite cyclic monoid generated by x, then we set $Z = \{1, x\}$ and $d_V(n) = n + 1$. Consequently $\operatorname{GKdim}(K[x]) = 1$.

The following basic lemma describes the relation between the Gelfand–Kirillov dimension of the algebra and its subalgebras and homomorphic images.

Lemma 1.20. *If S is a subalgebra or a homomorphic image of an algebra R then* $\operatorname{GKdim}(S) \leqslant \operatorname{GKdim}(R)$.

1.1.5 Polynomial identity algebras

In the present section we introduce several classical results on the structure of rings that satisfy a polynomial identity. We focus only on the theorems that will be useful in the dissertation.

By an algebra we mean an associative algebra with identity 1. An algebra R over a field K satisfies a polynomial identity if there exists a non-zero polynomial $f(x_1, \dots, x_n)$ in n non commuting variables $x_1, \dots, x_n$ with coefficients in K such that $f(r_1, \dots, r_n) = 0$ for all $r_1, \dots r_n \in R$. For brevity we then say that R is a PI-algebra.

Many natural classes of algebras satisfy a polynomial identity, including commutative algebras, finite dimensional algebras and the algebra $M_n(A)$ of matrices of size $n \times n$ over a commutative algebra A. Indeed, in every commutative algebra the identity $xy - yx = 0$ holds. It is the special case of a family of identities, called standard identities, that are satisfied for any algebra which is finitely generated as a module over its center.

It is easy to show that any subalgebra and any homomorphic image of a PI-algebra satisfies a polynomial identity. The following celebrated theorem, due to Regev, shows that the class of PI-algebras is also closed under taking tensor products, see [49, Theorem 6.1.1].

Theorem 1.21 (Regev). *The tensor product of two PI-algebras also satisfies a polynomial identity.*

The case of PI-algebras that are Noetherian and of finite Gelfand–Kirillov dimension is of special interest in our setting.

First recall the well-known result of Small, Stafford, Warfield, [52], showing in particular that the class of PI-algebras contains all finitely generated algebras with Gelfand–Kirillov dimension 1.

Theorem 1.22 (Small, Stafford, Warfield). *Every finitely generated algebra R of Gelfand–Kirillov dimension one satisfies a polynomial identity. Moreover, if $\mathcal{P}(R)$ is the prime radical of R, then $\mathcal{P}(R)$ is nilpotent and $R/\mathcal{P}(R)$ is a finite module over its Noetherian center.*

By [28, Theorem 10.10], the following result about the relationship between the Gelfand–Kirillov and the classical Krull dimensions of PI-algebras holds. Recall that the classical Krull dimension of R, denoted by $\operatorname{clKdim}(R)$, is the supremum of the lengths of finite chains of prime ideals in the algebra R.

Theorem 1.23. *If R is a finitely generated prime PI-algebra, then* $\operatorname{GKdim}(R) = \operatorname{clKdim}(R)$.

It is clear from the facts mentioned in this section that the matrix algebra $M_n(F)$, where F is a field extension of the field K satisfies a polynomial identity. Anan'in Theorem, [1], asserts that every algebra from an important class of PI-algebras can be embedded into such a matrix algebra.

Theorem 1.24 (Anan'in). *Let R be a finitely generated right Noetherian PI-algebra over a field K. Then R embeds into the matrix ring $M_n(F)$ over a field extension F of K, for some positive integer n.*

We say that a ring D is a division ring if every non-zero element has a right and a left inverse in D. The following Kaplansky theorem, see for instance [50, Theorem 6.1.25], will be useful in the characterization of representations of Hecke–Kiselman algebras.

Theorem 1.25 (Kaplansky). *If R is a left primitive ring satisfying a polynomial identity, then R is isomorphic to the ring of matrices $M_r(D)$, $r \geq 1$, over a division ring D that is finite dimensional over its center.*

Finally, in the class of finitely generated PI-algebras the prime radical and the Jacobson radical, introduced in Sections 1.1.2 and 1.1.3, coincide.

Theorem 1.26. *If R is a finitely generated algebra over a field that satisfies a polynomial identity, then $\mathcal{J}(R) = \mathcal{P}(R)$, where $\mathcal{J}(R)$ is the Jacobson radical of R and $\mathcal{P}(R)$ is the prime radical.*

1.1.6 Rings of quotients

Let us introduce the necessary definitions and properties of rings of quotients. The general idea is to embed the ring into another one with better properties, by making certain elements invertible in the new ring. Localization is one of the crucial techniques used in algebra. One important example is localization of a commutative domain with respect to all non-zero elements, which is called the field of fractions. Another useful case is localization of a commutative ring at any prime ideal. Although the analogous constructions in the non-commutative setting are not always possible, also in this case certain localizations provide a useful tool in the study of the structure of rings. One such important result is the celebrated Goldie's theorem, presented in Theorem 1.32 only in the Noetherian case.

We refer to Chapters 2 and 3 of [38] and Chapter 4 of [30] for further information and proofs of the presented results. The case of algebras satisfying a polynomial identity is considered in Chapter 1.7 of [49].

We restrict the discussion to two special cases of rings of quotients, where the multiplicatively closed set of "denominators" consists of either so-called regular elements or elements from the center of the ring.

A subset S of ring R is multiplicatively closed, if $S \cdot S \subseteq S$ and $1 \in S$.

We say that an element $r \in R$ is right regular in R if $rs = 0$ implies that $s = 0$ for any $s \in R$ and symmetrically left regular if from equality $sr = 0$ it follows that $s = 0$. An element is regular if it is left and right regular.

Definition 1.27. 1) Let S be a multiplicatively closed subset of a ring R consisting of all regular elements in R. Then a ring R' containing R is called a right classical ring of quotients of R, if elements of S are units in R' and every element of R' is of the form rs^{-1} for some $r \in R$ and $s \in S$. We denote this ring of quotients by $Q^r_{cl}(R)$.

2) Let S be the multiplicatively closed subset of a ring R consisting of all regular elements contained in the center of R. Then a ring R' containing R, such that elements of S are units in R' and every element of R' is of the form rs^{-1} for some $r \in R$ and $s \in S$, is called a central ring of quotients, denoted by $Q^r_Z(R)$.

One can define similarly the left analogues of the above definitions.

Note that the right classical ring of quotients does not always exist. For example, let $R = K\langle x, y\rangle$ be the free algebra in two non commuting generators x and y. Then $R \setminus \{0\}$ is multiplicatively closed set consisting of all regular elements. Suppose that $Q^r_{cl}(R)$ exists. Then in particular y is unit in $Q^r_{cl}(R)$, and therefore $y^{-1}x$ can be written in the form rs^{-1} for some $r, s \in Q^r_{cl}(R)$ such that $s \neq 0$. Thus $xs = yr$ in R for some non-zero s, which is impossible.

If right classical ring of quotients of R exists, then the set S, consisting of all regular elements, has the property, called the right Ore condition, that for each $r \in R$ and $s \in S$, there exist $r' \in R$ and $s' \in S$ such that $rs' = sr'$. Similarly, one can also define the left

Ore condition. It turns out that such conditions are also sufficient for the existence of the classical right or left ring of quotients.

Theorem 1.28. *A right (left) classical ring of quotients $Q^r_{cl}(R)$ exists if and only if the multiplicatively closed set of regular elements satisfies the right (left) Ore condition.*

On the other hand, a multiplicatively closed set S of regular central elements of a ring R satisfies evidently the right and left Ore conditions. The central ring of quotients of R always exists.

A right classical ring of quotients Q of a ring R is universal for homomorphisms $\alpha : R \to R'$ such that $\alpha(S)$ consists of units of R', where S is the set of all regular elements of R. Therefore we get the following remark.

Remark 1.29. If there exists a right classical ring of quotients Q of a ring R then it is unique up to an isomorphism.

Similarly, a central ring of quotients of a ring R is universal for homomorphisms $\alpha : R \to R'$ such that $\alpha(S)$ consists of units of R', where S is the set of all regular elements contained in the center of R. Thus it is also unique up to an isomorphism.

Example 1.30. If R is a Noetherian integral domain then the right classical ring of quotients $Q^r_{cl}(R)$ exists. It is enough to show that for any non-zero $a, b \in R$ we have $aR \cap bR \neq \{0\}$. But if we had $aR \cap bR = \{0\}$, then the sum $\Sigma_{m \geqslant 1} b^m aR$ would be direct, contradicting the Noetherian property. Indeed, if $b^k ar_0 + \cdots + b^{k+m} ar_m = 0$ for some k and $m > 0$ such that $r_0, r_m \neq 0$, $r_i \in R$, then because R is an integral domain, $-ar_0 = b(ar_1 + \cdots + ab^{m-1} r_m) \in aR \cap bR = \{0\}$, which leads to a contradiction.

In the thesis we will work with the rings that have both right and left rings of quotients. Then from the universal property satisfied by these rings, we easily get the following lemma.

Lemma 1.31. *If right and left classical (central) rings of quotients exist, they are isomorphic. In this case we speak about the classical ring of quotients (central ring of quotients) and denote this ring by $Q_{cl}(R)$ $(Q_Z(R))$.*

In the forthcoming sections we are interested in the ring of quotients of Noetherian rings. Let us recall the famous Goldie's theorem in the case of Noetherian rings. The theorem gives the characterization of rings with classical rings of quotients that are semisimple. Recall that a ring R is semisimple if and only if every right R-submodule of R is a direct summand of R. It can be proved that a ring R is semisimple if and only if it is left Artinian and semiprimitive.

Theorem 1.32 (Goldie)**.** *If R is a Noetherian ring then the following conditions are equivalent.*

1) R is a semiprime ring.

2) R has a quotient ring $Q_{cl}(R)$ which is semisimple.

One of the aims of Chapter 3 is to find the classical rings of quotients of certain Hecke–Kiselman algebras R. The minimal prime ideals P of these algebras will be constructed and the rings of quotients $Q_{cl}(R/P)$ of the quotient rings R/P will be found. Thanks to the following observation (Corollary 11.44 in [30]), it is then possible to describe the ring of quotients of the algebra R.

Proposition 1.33. *Let R be a semiprime ring with finitely many minimal prime ideals $P_1, \ldots, P_k$. Then R has a classical quotient ring $Q_{cl}(R)$ if and only if all R/P_i have classical quotient rings $Q_{cl}(R/P_i)$. Moreover, if this is the case, then $Q_{cl}(R) \cong \prod_{i=1}^{k} Q_{cl}(R/P_i)$.*

Another natural observation, which can be derived from 3.1.6 in [38], concerning different rings with the same classical ring of quotients, will be useful.

Lemma 1.34. *If R is a prime Noetherian ring, $0 \neq A$ is a two-sided ideal of R and S is a subring of R such that $A \subseteq S \subseteq R$ then S has the same (right) classical quotient ring as R.*

In the present thesis we are interested in the properties of algebras that are PI. Therefore let us recall the following theorem about the classical and central rings of quotients in the case of PI-algebras, see Theorem 1.7.34 in [49].

Theorem 1.35. *If R is a Noetherian semiprime PI-ring then $Q_Z(R) = Q_{cl}(R)$.*

1.2 Basics of semigroup theory

The thesis is devoted to the certain class of monoids and their monoid algebras. In particular, some aspects of semigroup theory are extensively used in the forthcoming chapters. Therefore we now set up the notation and basic definitions of semigroup theory, with emphasis on the representation theory of finite semigroups and semigroup identities. For a complete overview of the topic we refer to the books [9, 43, 53].

Definition 1.36. A semigroup is a set S with an associative binary operation, denoted by $\cdot$, called multiplication. If a semigroup contains an identity element 1, it is called a monoid.

For a semigroup S, we define S^1 as a semigroup obtained from S by adding an identity element 1 to S, if S does not contain an identity and $S^1 = S$, otherwise. An element θ of a semigroup S is called the zero element, if for every $s \in S$ we have $\theta \cdot s = s \cdot \theta = \theta$. By S^0 we denote a semigroup S with zero element adjoined.

Notions of an ideal and Green's relations play a fundamental role in semigroup theory. A subset I of a semigroup S is called a left (right) ideal if $SI \subseteq I$ ($IS \subseteq I$, respectively). We say that I is a two-sided ideal, if it is a right and a left ideal. For example, for any element $s \in S$, we can consider a two-sided principal ideal $S^1 s S^1$ in S generated by this element. It consists of all elements of the form ust for any $u, t \in S^1$.

For every two-sided ideal I of the semigroup S, the Rees quotient semigroup S/I as a set consists of $S \setminus I \cup \{\theta\}$. It has a structure of a semigroup, where multiplication is defined for every $s, t \in S \setminus I$ as follows

$$s \cdot t = \begin{cases} st \text{ if } st \notin I \\ \theta \text{ otherwise} \end{cases} .$$

We assume that for any semigroup S the Rees factor $S/\emptyset$ is the semigroup S with zero adjoined.

Recall that an equivalence relation σ on S is a congruence if $s\sigma t$ for $s, t \in S$ implies that $su\sigma tu$ and $vs\sigma vt$ for any $u, v \in S$. For every congruence σ on a semigroup S the set of σ-classes is a semigroup. We denote this semigroup by S/σ.

Elements m, n of a semigroup S are in Green's relation $\mathcal{R}$ ($\mathcal{L}$) if and only if $mS^1 = nS^1$ ($S^1m = S^1n$, respectively). We say that two elements are in relation $\mathcal{H}$, if they are $\mathcal{R}$ and $\mathcal{L}$–related. Lastly, elements m, n of a semigroup S are in the Green's relation $\mathcal{J}$ (are $\mathcal{J}$–related) if and only if they generate the same two-sided ideal in S^1, that is $S^1mS^1 = S^1nS^1$.

Example 1.37. Consider the multiplicative semigroup $(M_n(K), \cdot)$ of the $n \times n$ matrices over a field K. Then the following can be proved.

1) Two matrices are $\mathcal{J}$–related of and only if they have the same rank.

2) Two matrices A, B are $\mathcal{L}$–related if and only if they have the same row space.

3) Two matrices A, B are $\mathcal{R}$–related if and only if they have the same column space.

The equivalence class of a given element with respect to Green's relation $\mathcal{J}$ ($\mathcal{L}$, $\mathcal{R}$) will be called the $\mathcal{J}$-class ($\mathcal{L}$-class, $\mathcal{R}$-class) of an element. In the set of these classes in a semigroup S let us define a partial order, denoted by $\leqslant_{\mathcal{J}}$ ($\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$, respectively), such that for any two $\mathcal{J}$-classes ($\leqslant_{\mathcal{L}}$, $\leqslant_{\mathcal{R}}$-classes, respectively) L and L' we have $L \leqslant_{\mathcal{J}} L'$ if and only if $S^1LS^1 \subseteq S^1L'S^1$ (similarly, $L \leqslant_{\mathcal{R}} L'$ if and only if $LS^1 \subseteq L'S^1$ and $L \leqslant_{\mathcal{L}} L'$ if and only if $S^1L \subseteq S^1L'$).

We say that a semigroup S is $\mathcal{J}$-trivial if Green's relation $\mathcal{J}$ is equality, that is $S^1mS^1 = S^1nS^1$ implies that $m = n$ in the semigroup S. Notions of $\mathcal{R}$-trivial and $\mathcal{L}$-trivial semigroups are defined analogously.

Example 1.38. Let S be a monoid such that $S = \{1, e\}$ with identity 1 and such that $e^2 = e$. It is clear that S is $\mathcal{J}$-trivial.

Let us note the following useful property of Green's relation $\mathcal{J}$.

Remark 1.39. If M is a $\mathcal{J}$-trivial monoid and I is a two-sided ideal in S, then the Rees quotient semigroup M/I is also $\mathcal{J}$-trivial.

Proof. Let M be a $\mathcal{J}$-trivial monoid and let I be a two-sided ideal in M. Consider u, v from M/I that generate the same principal ideal, that is for some $s, t, p, q \in M \setminus I \cup \{\theta\}$ we have $u = svt$ and $v = puq$ in M/I. If at least one of s, t, p, q is the zero element θ, then it follows easily that $u = v = \theta$. Otherwise we get that $u = svt$ and $v = puq$ also in M, that is $MuM \subseteq MvM$ and $MvM \subseteq MuM$, respectively. Therefore, as M is $\mathcal{J}$-trivial, $u = v$ in M, and thus also in M/I. $\square$

One can also define a direct product of semigroups or monoids as follows.

Definition 1.40. For semigroups (monoids) S and T with operations $\cdot_S$, $\cdot_T$ the direct product of S and T, denoted by $S \oplus T$, is the Cartesian product $S \times T$, together with the binary operation $\cdot : (S \times T) \times (S \times T) \longrightarrow S \times T$ such that $(s, t) \cdot (s', t') = (s \cdot_S s', t \cdot_T t')$. Then $(S \oplus T, \cdot)$ is also a semigroup (monoid).

Now we give a brief exposition of the structure of finite semigroups. In particular we are interested in the semigroup analogue of the Jordan-Hölder theorem. This will be useful in the study of representation theory of semigroups.

Let us define a class of semigroups that do not have non-trivial two-sided ideals.

Definition 1.41. We say that a semigroup without zero is simple, if it does not contain ideals other than S. If a semigroup S contains a zero element θ, it is a 0-simple semigroup, if $\{\theta\}$ and S are its only ideals in S and $S^2 \neq \theta$.

Note that the condition $S^2 \neq \theta$ implies that the semigroup $S = \{\theta, a\}$, where $a^2 = \theta$, is not 0-simple.

Example 1.42. Let $S = \{e_{ij} \in M_n(K) : i, j = 1, \ldots, n\} \cup \{\theta\}$ be the semigroup consisting of all standard basis matrices e_{ij} with 1 in the i row and j column and 0 everywhere else and the zero matrix. Then it can be easily verified that S is a 0-simple semigroup.

An element e of a semigroup is idempotent if $e = e^2$. In the set of idempotents of a semigroup define a partial order such that $e \leqslant f$ if and only if $ef = fe = e$. Idempotent f is called primitive if $f \neq 0$ and $e \leqslant f$ implies that $e = f$ or $e = \theta$ for any idempotent e. For instance, in the semigroup S from Example 1.42 every idempotent of the form e_{ii} is primitive. A semigroup S is completely 0-simple if it is 0-simple and contains a primitive idempotent. Note that for example any finite 0-simple semigroup is completely 0-simple, see [9, Section 2.7].

1.2.1 Finite semigroups, semigroup algebras and their representations

We are now in a position to present the basic theorem describing the ideal structure of finite semigroups.

Theorem 1.43. *If S is a finite semigroup, then it admits a principal series, that is a series*

$$\emptyset = S_{i+1} \subsetneq S_i \subsetneq \cdots \subsetneq S_2 \subsetneq S_1 = S,$$

of ideals S_k, such that every factor S_k/S_{k+1} is either a completely 0-simple semigroup or a null semigroup (namely, a semigroup with zero multiplication) for $k = 1, \ldots, i$.

It turns out that factors from principal series of a finite semigroup are related to equivalence classes with respect to Green's relation $\mathcal{J}$. Let us start with the following notations. If s is an element of a semigroup S, we denote by J_s the $\mathcal{J}$-class in S containing s. For any $\mathcal{J}$-class L of S we denote by $J(L)$ the ideal S^1LS^1 and $I(L) = J(L)\setminus L$. Note that if $s \in I(L)$ and $t \in S$, then $S^1sS^1 \subsetneq J(L)$. As $S^1stS^1 \subseteq S^1sS^1$ ($S^1tsS^1 \subseteq S^1sS^1$), it follows that st, $ts \notin L$. Thus $I(L)$ is an ideal in S, if non-empty. Moreover, it is clear that $I(L) \subseteq J(L)$ is maximal in $J(L)$ in the sense that there are no ideals strictly between $I(L)$ and $J(L)$. Consequently, it can be proved that $J(L)/I(L)$ is either null or a 0-simple semigroup. A factor $J(L)/I(L)$ which is 0-simple (null) is called a 0-simple (null, respectively) factor of S. It turns out that factors of every principal series of a finite semigroup S are of the form $J(L)/I(L)$ for $\mathcal{J}$-classes L in S.

Corollary 1.44. *If S is a finite semigroup, then it admits a principal series*

$$\emptyset = S_{i+1} \subsetneq S_i \subsetneq \cdots \subsetneq S_2 \subsetneq S_1 = S,$$

of ideals S_k, such that every factor S_k/S_{k+1} is of the form $J(L)/I(L)$ for some $\mathcal{J}$-class L in S for $k = 1, \ldots, i$.

Before we move on to the representations, let us set the necessary definitions related to semigroup algebras.

Definition 1.45. For a semigroup S consider the vector space over the field K with the basis consisting of elements of S. A structure of a K–algebra in this vector space is given by the multiplication which is the linear extension of the operation in S. Such an algebra is denoted by $K[S]$ and is called the semigroup algebra of the semigroup S over the field K.

Note that a typical element α of a semigroup algebra $K[S]$ is of the form $\alpha = \sigma_1 s_1 + \cdots + \sigma_k s_k$ with $0 \neq \sigma_i \in K$ and pairwise different $s_i \in S$. Then the support of α is equal to the set $\{s_1, \ldots, s_k\}$.

If a semigroup S contains a zero element θ, then it will be often useful to identify θ with the zero element of K. Such an algebra, defined as $K[S]/K[\theta]$ will be denoted by $K_0[S]$ and called the contracted semigroup algebra. As $K[S] \cong K_0[S] \oplus K[\theta]$, algebras $K[S]$ and $K_0[S]$ have similar structural properties. If we consider an ideal I of S, then the quotient $K[S]/K[I]$ is isomorphic to the contracted algebra $K_0[S/I]$.

For brevity, in the context of representations we always assume that the base field is algebraically closed, without further comments.

By $End_K(V)$ we denote the vector space of all endomorphisms of the finite-dimensional linear space V over the field K. Then $End_K(V)$ has the monoid structure under multiplication.

Definition 1.46. A representation of a semigroup M on a (finite-dimensional) vector space V over a field K is a homomorphism $\varphi : M \longrightarrow End_K(V)$.

It is clear that this is equivalent to defining a semigroup homomorphism $M \longrightarrow M_k(K)$, where $k = \dim_K(V)$. Every representation of a semigroup M on V can be uniquely extended to a representation of semigroup algebra $K[M]$ on V.

Representations $\varphi : K[M] \longrightarrow End_K(V)$ and $\psi : K[M] \longrightarrow End_K(W)$ are said to be equivalent, if there is a linear isomorphism $\rho : V \longrightarrow W$ such that $\varphi(f) = \rho^{-1}\psi(f)\rho$ for all $f \in K[M]$. In the thesis we are interested in representations up to equivalence classes.

Finally, we will sometimes identify representations of $K[M]$ with (left) $K[M]$-modules without further comments. This is possible thanks to the correspondence described as follows. If $\varphi : K[M] \longrightarrow End_K(V)$ is a representation, then V is left $K[M]$-module defined as $f \cdot v = \varphi(f)(v)$ for every $f \in K[M]$ and $v \in V$. Conversely, if V is a $K[M]$-module, then the induced representation is defined as $\varphi_V : K[M] \longrightarrow End_K(V)$ such that $\varphi_V(f)(v) = f \cdot v$, for every $f \in K[M]$ and $v \in V$. Moreover representations of $K[M]$ are equivalent if and only if the corresponding modules are isomorphic.

We say that a representation $\varphi : K[M] \longrightarrow M_n(K)$ is irreducible if it corresponds to a simple $K[M]$-module.

We now look more closely at the classical theory of irreducible representations of finite monoids, known as Clifford–Munn–Ponizovskii theory. Namely, there exists a bijection between the set of equivalence classes of irreducible representations of a finite monoid and the set of equivalence classes of irreducible representations of its 0-simple principal factors. We refer to Chapter 5 of [9] for proofs and Chapter 4 of [53] for another approach.

Recall that $\leqslant_{\mathcal{J}}$ is an order in the set of Green's $\mathcal{J}$-classes of M defined at the beginning of the present section. We will further write that $L \nleqslant_{\mathcal{J}} L'$ if $J(L) \nsubseteq J(L')$.

Let us define a family of representations of a monoid depending on the $\mathcal{J}$-classes in this monoid.

Definition 1.47. We say that a representation $f : M \to M_n(K)$ of the monoid M has an apex L for a $\mathcal{J}$-class L, if $f(n) \neq 0$ for every (or, equivalently, any) $n \in L$ and $f(m) = 0$ for all $m \in M$ such that $L \nleqslant_{\mathcal{J}} J_m$, where J_m is the $\mathcal{J}$-class containing m.

It follows that a representation f with an apex L has the property that $f(m) \neq 0$ for all $m \in M$ such that $L \leqslant_{\mathcal{J}} J_m$. The first part of the main theorem of Clifford–Munn–Ponizovskii theory states that every irreducible representation of a finite monoid M has an apex. If $f : M \to M_n(K)$ is such a representation with an apex L, then in particular $f(m) = 0$ for all $m \in I(L)$. Therefore f induces a representation $\overline{f}$ of the factor $J(L)/I(L)$. It turns out that then the principal factor $J(L)/I(L)$ has to be 0-simple. Moreover the induced representation $\overline{f} : J(L)/I(L) \to M_n(K)$ is also irreducible. In the opposite direction, for every $\mathcal{J}$-class L such that $J(L)/I(L)$ is 0-simple and for every irreducible representation of this factor semigroup there exists a unique irreducible representation of M with an apex L. More precisely, the following theorem holds.

Theorem 1.48 (Clifford, Munn, Ponizovskii)**.** *Let M be a finite monoid.*

1) *Every irreducible representation of M has an apex.*

2) *If $f : K[M] \to M_n(K)$ is an irreducible representation with an apex L, then the associated principal factor is 0-simple and the induced representation $\overline{f} : J(L)/I(J) \to M_n(K)$ is irreducible.*

3) *For every 0-simple $\mathcal{J}$-class L of M and irreducible representation $\overline{f} : J(L)/I(L) \to M_n(K)$ there exists $e \in K[L]$ such that $\overline{f}(e) = I_n$ and the formula $f(m) = \overline{f}(\overline{me})$, for every $m \in M$, defines an irreducible representation $f : K[M] \to M_n(K)$ with an apex L.*

4) *Two irreducible representations of M are equivalent if and only if they have the same apex L and induce equivalent representations of $MLM/I(L)$.*

As an illustration, we apply the theory for finite $\mathcal{J}$-trivial monoids. We know that then every $\mathcal{J}$-class L contains exactly one element and the quotient $J(L)/I(L) = \{\theta, e\}$ for some e such that $e^2 = \theta$ or $e^2 = e$ in $J(L)/I(L)$. Thus every 0-simple principal factor comes from an idempotent $e^2 = e$. Every irreducible representation $\overline{\phi_e}$ of $\{\theta, e\}$ such that $e^2 = e$ is one-dimensional and such that $\overline{\phi_e}(e) = 1$. Then from Theorem 1.48 we know that the corresponding irreducible representation of M is such that $\phi_e(m) = \overline{\phi_e}(me)$ for every $m \in M$. The following easy proposition, that can be found in [53, Corollary 2.7 (iii)], is useful in describing the corresponding irreducible representations.

Proposition 1.49. *Let M be a $\mathcal{J}$-trivial monoid with an idempotent $e \in M$. Then for every $m \in M$ we have $e \in MmM$ if and only if $em = m$.*

For completeness, we give a proof.

Proof. If $em = m$, then it is clear that $e \in MmM$, as M is a monoid. Assume now that $e \in MmM$, that is $e = xmy$ for some $x, y \in M$. Then $e = exmy$ and thus $MeM = MexM$. From the assumption it follows that $e = ex$. Consequently, $e = e^2 = exmy = emy$, and thus $MemM = MeM$. As M is $\mathcal{J}$-trivial, we get that $em = m$. □

Let us emphasize that the proposition holds for any $\mathcal{J}$-trivial monoid, not necessarily finite. Note also that if $e \notin MmM$, we get that $MeM \nsubseteq MmM$ and then $\phi_e(m) = 0$ in the finite case.

As a consequence, we get the following characterization.

Example 1.50. There is a bijection between (equivalence classes of) irreducible representations of a finite $\mathcal{J}$-trivial monoid M and idempotents in M. For every idempotent e, the corresponding representation $\phi_e : M \to K$ is one dimensional and for every $m \in M$ given by

$$\phi_e(m) = \begin{cases} 1 \text{ if } em = e \\ 0 \text{ if } e \notin MmM \end{cases} .$$

1.2.2 Semigroup identities

Let us recall that for a finite set X we denote by X^* the free semigroup generated by X. Assume that S is a semigroup. We say that S satisfies a semigroup identity of the form $u = v$, where u, v are two different words in X^*, if for any semigroup homomorphism $\varphi : X^* \to S$ the equality $\varphi(u) = \varphi(v)$ holds. Note that if a semigroup satisfies such an identity, then it also satisfies an identity in two variables, that is an identity such that $u = v$ with $u, v \in X^*$ for a two-element set X.

Example 1.51. The free semigroup in two variables does not satisfy any semigroup identity.

It is natural to investigate the relationship between the existence of semigroup identities of the semigroup S and polynomial identities in the corresponding semigroup algebra $K[S]$.

In general, if a finitely generated semigroup S satisfies an identity then its algebra $K[S]$ does not have to be a PI-algebra. For example, every nilpotent group G satisfies a semigroup identity, see [35], [41], but the group algebra $K[G]$ is a PI-algebra only if G is abelian-by-finite [47], Corollary 5.3.8 and Corollary 5.3.10.

On the other hand, it seems that the following is an open problem.

Problem 1.52. Let S be a finitely generated semigroup. Assume that the semigroup algebra $K[S]$ over a field K satisfies a polynomial identity (as usual, in this case we say that $K[S]$ is a PI-algebra). Is it true that the semigroup S satisfies a nontrivial semigroup identity?

The following result is very useful for establishing existence of nontrivial identities in an important class of semigroups.

Theorem 1.53 (Theorem 6.11 in [44])**.** *Let S be a finitely generated subsemigroup of the multiplicative semigroup of the matrix algebra $M_n(K)$ over a field K. Then the following conditions are equivalent.*

1. *S has no free noncommutative subsemigroups,*

2. *S satisfies a semigroup identity.*

We end the section with the following property of semigroup identities, that can be extracted from the proof of Lemma 5.3 in [44].

Proposition 1.54. *If S is a semigroup and I is an ideal of S such that I and S/I satisfy semigroup identities, then the semigroup S also satisfies a semigroup identity.*

We will later use a concrete construction of a semigroup identity in S from identities in the ideal and in the quotient. Therefore, let us prove the proposition.

Proof. Let I be an ideal in the semigroup S such that I and S/I satisfy semigroup identities. As explained earlier we can assume that identities are in two variables. Assume that $u_1(x, y) = v_1(x, y)$ is a semigroup identity in S/I and $u_2(x, y) = v_2(x, y)$ is satisfied in I.

Note that we can assume that u_2 and v_2 are of the same length, changing them into u_2v_2 and v_2u_2, if necessary. Then we claim that

$$u_2(u_1(x,y), v_1(x,y)) = v_2(u_1(x,y), v_1(x,y))$$

is a non-trivial identity satisfied in the semigroup S. Indeed, take any $s,t \in S$. Notice that $u_1(s,t) \in S \setminus I$ if and only if $v_1(s,t) \in S \setminus I$, as $u_1(s,t) = v_1(s,t) \in S/I$ and I is an ideal in S. Thus if $u_1(s,t), v_1(s,t) \in S/I$, then $u_1(s,t) = v_1(s,t) \in S/I$. The equality $u_2(u_1(s,t), v_1(s,t)) = v_2(u_1(s,t), v_1(s,t))$ then follows from the assumption that u_2 and v_2 are of the same length. Moreover, if $u_1(s,t) \in I$, then $v_1(s,t) \in I$. In this case we have $u_2(u_1(s,t), v_1(s,t)) = v_2(u_1(s,t), v_1(s,t))$, because $u_2(x,y) = v_2(x,y)$ is an identity in the ideal I. The assertion follows. $\square$

1.3 Semigroups and algebras of matrix type

As we have seen in Section 1.2.1, completely 0-simple semigroups can be treated as basic building blocks of finite semigroups. It turns out that in the finite case completely 0-simple semigroups can be characterized as semigroups of matrix type that satisfy certain additional conditions. In this section we provide the definition of this class of semigroups, the associated semigroup algebras and describe some of their properties. For the details we refer to [9, Section 3.2] and [43, Chapter 5]. We also introduce the notion of the semigroups of quotients related to such semigroups, see [17].

As we will show in Chapter 2 structures of matrix type occur also in the Hecke–Kiselman monoid associated to an oriented cycle. This will also play a crucial role in further investigation of the structure and properties of Hecke–Kiselman monoids and their algebras in more general cases.

Definition 1.55. Consider a semigroup S, non-empty sets A and B and a matrix $P = (p_{ba})$ of size $B \times A$ over S^1 with zero adjoined. Then the semigroup of matrix type, denoted by $\mathcal{M}^0(S, A, B; P)$, consists of all triples of the form $(s; a, b)$, where $s \in S \cup \{0\}$, $a \in A$, $b \in B$, with zero element θ, identified with all triples of the form $(0, a, b)$. Multiplication is defined by $(s; a, b) \cdot (s'; a', b') = (sp_{ba'}s'; a, b')$ if $p_{ba'} \in S^1$ and θ otherwise.

Semigroup S defined in Example 1.42 is a semigroup of matrix type. Indeed, it is clear that $S = \mathcal{M}^0(\{e\}, [n], [n]; \mathrm{Id}_n)$, where $\{e\}$ is the trivial group, $[n] = \{1, \ldots, n\}$ and Id_n is the identity $n \times n$ matrix.

The famous Rees theorem describes all completely 0-simple semigroups as semigroups of matrix type, that satisfy certain additional conditions.

Theorem 1.56 (Rees theorem)**.** *A semigroup is completely 0-simple if and only if it is isomorphic with a semigroup of matrix type $\mathcal{M}^0(G, X, Y; P)$ over a group G with zero and a sandwich matrix P that contains a non-zero entry in every row and in every column.*

In the dissertation we will also investigate properties of the (contracted) semigroup algebras arising from such semigroups. If A and B are finite sets, the contracted semigroup algebra of a semigroup of matrix type $K_0[\mathcal{M}^0(S, A, B; P)]$ can be interpreted as the algebra of matrices of size $A\times B$ over $K[S]$ with standard addition and the multiplication $\alpha\cdot\beta = \alpha\circ P\circ\beta$, where $\circ$ is the standard matrix product. More generally, for an algebra R, finite sets A and B and matrix P over R the algebra of matrix type, denoted by $\mathcal{M}(R, A, B; P)$, is the algebra of matrices of size $A\times B$ with standard addition and multiplication defined as $X\cdot Y = X\circ P\circ Y$, where $\circ$ is the standard matrix multiplication. If additionally R has an identity and every row and column of the matrix P contains a unit of R, the algebra $\mathcal{M}(R, A, B; P)$ is called a Munn algebra. The ideal structure and the prime radical of $\mathcal{M}(R, A, B; P)$ admit a clear description in terms of ideals and the radical of R. We refer the reader to [43, Chapter 5] for the details.

The next useful tools are the notions of a semigroup of quotients and a completely 0-simple closure of a semigroup. The idea is to embed a given semigroup into a completely 0-simple semigroup, whose structure is in general more approachable.

Let us consider any element a of a completely 0-simple semigroup S. It is known that either $a^2 \neq 0$ and then the $\mathcal{H}$-class H_a containing a is a (maximal) subgroup of S or $H_a^2 = 0$. Moreover, if we represent completely 0-simple semigroup as a semigroup of matrix type $\mathcal{M}^0(G, A, B; P)$, then all $\mathcal{H}$-classes are of the form $H_{ab} = \{(g; a, b) : g \in G\}$ for $a \in A$ and $b \in B$. If $p_{ba} \neq 0$, the the corresponding $\mathcal{H}$-class H_{ab} is a group with the identity element $e = (p_{ba}^{-1}; a, b)$, isomorphic to G.

We say that a subsemigroup of a completely 0-simple semigroup S is uniform if it intersects every non-zero $\mathcal{H}$-class of S.

The following definition, motivated by the connection between certain rings and completely 0-simple semigroups, was introduced in [17].

Definition 1.57. Let S be a subsemigroup of a completely 0-simple semigroup Q. For an element a in a group $\mathcal{H}$-class of Q, denote by a^{-1} its inverse in this group. We say that Q is a completely 0-simple semigroup of quotients of S, if every element $q \in Q$ can be written as $q = ab^{-1}$ and $q = d^{-1}c$ for some $a, b, c, d \in S$ with $b^2 \neq 0$ and $d^2 \neq 0$.

Let S be a completely 0-simple semigroup $\mathcal{M}^0(gr(g), X, Y; P)$, where $gr(g)$ is the infinite cyclic semigroup generated by g, X, Y are finite sets and P is a $Y\times X$ matrix with coefficients in $gr(g)^0$ that contains non-zero entry in every row and every column. Then consider the subsemigroup $U = \mathcal{M}^0(\{g\}^*, X, Y; P)$, where $\{g\}^*$ is the cyclic semigroup generated by g. Note that such a subsemigroup is a uniform subsemigroup of S. It can be easily verified that S is a completely 0-simple semigroup of quotients of U. First note that if $p_{yx'} \neq 0$, then for any $\gamma \in \mathbb{Z}$ we have $(g^\gamma; x', y)^{-1} = (p_{yx'}^{-2}g^{-\gamma}; x', y)$ in S. Moreover we know that for any $y \in Y$, there exists $x' \in X$ such that $p_{yx'} \neq 0$ and similarly we can choose $y' \in Y$ such that $p_{y'x'} \neq 0$. Then for $(g^k; x, y) \in S$, we have $(g^k; x, y) = (g^\alpha; x, y') \cdot ((g^\gamma; x', y))^{-1}$ if and only if $g^\alpha p_{y'x'}p_{yx'}^{-2}g^\gamma = g^k$. As $p_{yx'} \neq 0$ and $p_{y'x'} \neq 0$ it is clear that for any $k \in \mathbb{Z}$ there exist $\alpha, \gamma > 0$

such that the equality $g^{\alpha}p_{y'x'}p_{yx'}^{-2}g^{\gamma} = g^k$ holds. Thus we have shown that S is a semigroup of right quotients of U. The proof for left quotients is similar.

For every uniform subsemigroup U of a semigroup S there exists the smallest completely 0-simple subsemigroup of S containing U, see Proposition 3.1 in [44]. Such a subsemigroup is called a completely 0-simple closure of U. For instance, in our above example the semigroup $\mathcal{M}^0(gr(g); X, Y; P)$ is a completely 0-simple closure of $\mathcal{M}^0(\{g\}^*, X, Y; P)$.

The following lemma will be useful in the calculations of the center of the Hecke–Kiselman algebra associated to cycle of length 3 in Chapter 8.

Lemma 1.58 (Lemma 2.5.1 in [23]). *Let I be an ideal of a semigroup S and let J be a completely 0-simple semigroup of quotients of I such that J is also a completely 0-simple closure of I. Then there is a unique semigroup structure on the disjoint union $\hat{S} = (S \setminus I) \cup J$ that extends the operation on S.*

1.3.1 Representations of Munn algebras

As we have already seen in Section 1.2, in general irreducible representations of a finite semigroup come from irreducible representations of 0-simple principal factors. In this section we characterize irreducible representations of this class of not necessarily finite semigroups, following [44, Section 4.2]. Recall that we assume that K is an algebraically closed field.

Let $\varphi : G \to M_r(K)$ be a representation of a group G. It induces a representation $\overline{\varphi} : \mathcal{M}^0(G, X, Y; P) \to \mathcal{M}(M_r(K), X, Y; \varphi(P))$, where $\varphi(P) = (\varphi(p_{yx}))$ for a sandwich matrix $P = (p_{yx})$. Such a representation is given by $\overline{\varphi}((g; x, y)) = (\varphi(g); x, y)$ for every $(g; x, y) \in \mathcal{M}^0(G, X, Y; P)$. For any natural number r by $X \cdot r$ and $Y \cdot r$ we denote r disjoint copies of sets X and Y, respectively. A matrix $\varphi(P) = (\varphi(p_{yx}))$ with coefficients in $M_r(K)$ can be further identified with a matrix $\overline{P}$ of size $(Y \cdot r) \times (X \cdot r)$ with coefficients in K obtained from $\varphi(P)$ by erasing matrix brackets of all entries $\varphi(p_{yx})$ of $\varphi(P)$. Then the rank of a matrix $\overline{P}$ is defined as $\mathrm{rk}(\overline{P})) = \sup\{t : \overline{P} \in M_{(Y \cdot r) \times (X \cdot r)}(K)$ has an invertible $t \times t$ submatrix$\}$.

By Corollary 4.25 in [44], it follows that in the case of an algebra of matrix type $\widetilde{R} = \mathcal{M}(M_r(K), X, Y; P)$ the quotient $\widetilde{R}/\mathcal{P}(\widetilde{R})$, where $\mathcal{P}(\widetilde{R})$ is the prime radical of $\widetilde{R}$, is isomorphic to $M_t(K)$, provided that $t = \mathrm{rk}(P) < \infty$.

Thus, under certain assumptions, a representation of a group G induces a representation $\mathcal{M}^0(G, X, Y; P) \longrightarrow M_r(K)$. The following theorem holds.

Theorem 1.59 ([44], Theorem 4.26). *Let $S = \mathcal{M}^0(G, X, Y; P)$ be a semigroup of matrix type associated to a group G with a sandwich matrix P that has no zero rows or columns. Assume that $\varphi : G \to M_r(K)$ is an irreducible representation of G such that $rk(\overline{P}) = t < \infty$ for a matrix $\overline{P}$ as described above. Then the induced map*

$$\overline{\varphi} : \mathcal{M}^0(G, X, Y; P) \to \widetilde{R} = \mathcal{M}(M_r(K), X, Y; \varphi(P)) \to \widetilde{R}/\mathcal{P}(\widetilde{R}) \simeq M_t(K)$$

is an irreducible representation. Moreover, every irreducible representation of S arises in this *way and two such representations are equivalent if and only if they are induced by equivalent representations of G.*

Note that the semigroups of matrix type $\mathcal{M}^0(G, X, Y; P)$ fulfilling the assumptions of Theorem 1.59 are exactly completely 0-simple semigroups, see Theorem 1.56.

1.4 Preliminaries on Hecke–Kiselman monoids and algebras

1.4.1 Introduction

In this section we present some preliminaries of the main objects of interest in the thesis, that is Hecke–Kiselman monoids and their algebras. After introducing the definitions, we look more closely at combinatorial aspects of this class of monoids and the structural results on their algebras.

Ganyushkin and Mazorchuk in the paper [18] proposed a study of a new class of monoids, that is a generalization of two another families of semigroups, called Kiselman's semigroups and 0-Hecke monoids. These families consist of $\mathcal{J}$-trivial semigroups generated by idempotents and defined by presentations related to relations similar to braid relations.

Every Hecke–Kiselman monoid is given by a presentation associated with a simple finite graph. A graph is said to be simple finite, if it has finitely many vertices and at most one edge between two different vertices. In particular we do not allow loops, that is edges connecting a vertex with itself. Although in general both oriented and unoriented edges are allowed, we restrict our attention to monoids and algebras associated with graphs with only oriented edges. For any graph Θ, we denote the sets of vertices and edges of Θ by $V(\Theta)$ and $E(\Theta)$, respectively.

Definition 1.60. Let $\Theta = (V(\Theta), E(\Theta))$ be an oriented finite simple graph with n vertices, denoted by $x_1, \ldots, x_n$. The Hecke–Kiselman monoid HK_Θ associated with Θ is given by the following presentation.

(i) HK_Θ is generated by elements $x_i^2 = x_i$, where $1 \leq i \leq n$,

(ii) if the vertices x_i, x_j are not connected in Θ, then $x_i x_j = x_j x_i$,

(iii) if x_i, x_j are connected by an arrow $x_i \to x_j$ in Θ, then $x_i x_j x_i = x_j x_i x_j = x_i x_j$.

By $K[\mathrm{HK}_\Theta]$ we mean the monoid algebra of HK_Θ over a field K.

In certain cases the ground field K does not play any role in our considerations, and then we will denote the algebra $K[\mathrm{HK}_\Theta]$ by A_Θ.

As all considered graphs are finite simple and oriented, from now on by a graph we always mean a simple finite graph with oriented edges, if not stated otherwise.

Example 1.61. 1) If the graph Θ consists of one vertex, then $\mathrm{HK}_\Theta = \langle x \mid x^2 = x \rangle$ has two elements $1, x$.

2) For the graph Θ shown in Figure 1.1, the corresponding Hecke–Kiselman monoid HK_Θ, given by the presentation

$$\begin{aligned}\langle x_1, \ldots, x_n \mid x_i^2 = x_i,\ x_i x_{i+1} = x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ for } i = 1, \ldots, n-1, \\ x_i x_j = x_j x_i \text{ for } n-1 \geqslant |i-j| > 1 \rangle\end{aligned}$$

is the so-called Catalan monoid, [18]. It consists of all functions $f : \{1, 2, \ldots, n+1\} \longrightarrow \{1, 2, \ldots, n+1\}$ which are order–preserving in the sense that $f(i) \leqslant f(j)$ for all $i \leqslant j$ and order–decreasing, that is $f(i) \leqslant i$ for $i \in \{1, \ldots, n+1\}$. Multiplication is given by the composition of maps. The cardinality of this finite monoid is the so-called Catalan number.

$$x_1 \longrightarrow x_2 \dashrightarrow \quad \dashrightarrow x_{n-1} \longrightarrow x_n$$

Figure 1.1: A graph Θ such that HK_Θ is the Catalan monoid

3) We denote by C_n the Hecke–Kiselman monoid associated to an oriented cycle of length n with $n \geqslant 3$, presented in Figure 1.2.

$$\begin{tikzcd}
 & {} \arrow[r, dashed] & x_n \arrow[dr] & \\
x_4 \arrow[ur, dashed] & & & x_1 \arrow[dl] \\
 & x_3 \arrow[ul] & x_2 \arrow[l] &
\end{tikzcd}$$

Figure 1.2: An oriented cycle Θ_n of length $n \geqslant 3$

Then C_n is given by the following presentation:

$$\begin{aligned}\langle x_1, \ldots, x_n \mid x_i^2 = x_i,\ x_i x_{i+1} = x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1} \text{ for } i = 1, \ldots, n-1, \\ x_n x_1 = x_n x_1 x_n = x_1 x_n x_1, \\ x_i x_j = x_j x_i \text{ for } n-1 > |i-j| > 1 \rangle.\end{aligned}$$

It can be proved that if the graph Θ is a disjoint union of Θ_1 and Θ_2, then $\mathrm{HK}_\Theta = \mathrm{HK}_{\Theta_1} \oplus \mathrm{HK}_{\Theta_2}$, that is HK_Θ is a direct product of HK_{Θ_1} and HK_{Θ_2}, see Definition 1.40. In particular, the Hecke–Kiselman monoid HK_Θ is the direct product of monoids associated to connected components of the graph Θ.

We say that two graphs Σ, Θ are isomorphic if there exists a bijection $f : V(\Sigma) \to V(\Theta)$ such that vertices x and y are connected by an oriented (unoriented) edge in Σ if and only if there is an oriented (unoriented, respectively) edge between $f(x)$ and $f(y)$ in Θ.

The following theorem was proved in [18].

Theorem 1.62. *Hecke–Kiselman monoids* HK_Θ *and* HK_Σ*, associated with graphs* Θ *and* Σ*, are isomorphic if and only if the graphs* Θ *and* Σ *are isomorphic.*

1.4.2 Combinatorics on words

Now we give a brief exposition of combinatorial aspects of Hecke–Kiselman monoids and algebras.

For any set X let $F = \langle X \rangle$ be the free monoid generated by this set. Elements of F will be sometimes called words. The number of occurrences of a generator $x \in X$ in a word w is denoted by $|w|_x$. By the support of w, denoted by $\mathrm{supp}(w)$, we mean the set consisting of all $x \in X$ such that $|w|_x > 0$. Moreover, $|w|$ stands for the length of the word w. A word $u \in F$ is said to be a factor of $w \in F$, if w can be written as $w = puq$ for some $p, q \in F$, that is u is a connected subword of w.

So-called Gröbner bases, see for example [56], provide a basic tool in the study of combinatorial aspects of commutative and noncommutative algebras given by (finite) presentations. We briefly introduce this notion in the case of semigroup algebras, as the result characterizing such bases for Hecke–Kiselman algebras will be extensively exploited in our arguments.

Let $\leqslant$ denote the degree-lexicographical order on the free monoid F induced by any well order on the set of generators X. Then for every f in the free algebra $K\langle X \rangle$, the leading monomial of f, denoted by $\overline{f}$ is the largest, with respect to $\leqslant$, monomial occurring in f.

We consider a finitely generated semigroup algebra A over a field K, namely $A = K\langle X \rangle / I$ for some ideal I, which is spanned by elements $w - v$ for some $w, v \in F$. A subset G of the ideal I is called a Gröbner basis of the algebra A, with respect to $\leqslant$ and the given presentation of A, if $0 \notin G$, I is generated by G as an ideal and for every $f \in I$, there is $g \in G$ such that the leading monomial $\overline{g} \in F$ of g is a factor of the leading monomial $\overline{f}$ of f. The set of normal forms of the algebra A (depending on the chosen presentation and order), denoted by $N(A)$ consists of all words that are not leading monomials of elements of the ideal I. It can be verified, that the word $w \in \langle X \rangle$ is normal if and only if it does not have factors that are leading monomials in elements of the Gröbner basis G. Note that the set $N(A)$ of normal words forms a linear basis of A.

So-called diamond lemma provides an useful technique for characterizing normal words in the algebra A, see [6]. Replacing a word w by another word w' such that $w' \leqslant w$, is called a reduction determined by a pair $(w, w') \in F \times F$. Such a pair $(w, w') \in F \times F$ is also called a reduction. We say that $f \in K\langle X \rangle$ is T-reduced, where T is any fixed set of reductions, if the leading monomial of f does not have factors that are the first term, called the leading term, of the pair $(w, w') \in T$. Bergman's diamond lemma (see [6]) states that under certain assumptions the linear span of all monomials that are T-reduced is the set of normal words $N(A)$ of the algebra $A = K\langle X\rangle/I$, where $I = \operatorname{span}_K\{w - w' \mid (w, w') \in T\}$.

We proceed with the notations useful in the context of combinatorics of Hecke–Kiselman monoids and algebras. For any oriented graph Θ, $t \in V(\Theta)$ and $w \in F = \langle V(\Theta)\rangle$ we write $w \nrightarrow t$ if $|w|_t = 0$ and there are no $x \in \operatorname{supp}(w)$ such that $x \to t$ in Θ. Similarly, we define $t \nrightarrow w$: again we assume that $|w|_t = 0$ and there is no arrow $t \to y$, where $y \in \operatorname{supp}(w)$. In the case when $t \nrightarrow w$ and $w \nrightarrow t$, we write $t \nleftrightarrow w$. A vertex $v \in V(\Theta)$ is called a sink vertex if no arrow begins in v. Analogously one defines a source vertex. Sink and source vertices are called terminal vertices.

We are now in a position to present the characterization of a Gröbner basis of Hecke–Kiselman algebras, obtained in [40], Theorem 3.1.

Theorem 1.63. *Let Θ be a graph with vertices $V(\Theta) = \{x_1, x_2, \ldots, x_n\}$. Extend the natural ordering $x_1 < x_2 < \cdots < x_n$ on the set $V(\Theta)$ to the deg-lex order on the free monoid $F = \langle V(\Theta)\rangle$. Consider the following set T of reductions on the algebra $K\langle V(\Theta)\rangle$:*

(i) (twt, tw), for any $t \in V(\Theta)$ and $w \in F$ such that $w \nrightarrow t$,

(ii) (twt, wt), for any $t \in V(\Theta)$ and $w \in F$ such that $t \nrightarrow w$,

(iii) (t_1wt_2, t_2t_1w), for any $t_1, t_2 \in V(\Theta)$ and $w \in F$ such that $t_1 > t_2$ and $t_2 \nleftrightarrow t_1w$.

Then the set $\{w - v,\ \text{where } (w, v) \in T\}$ forms a Gröbner basis of the Hecke–Kiselman algebra A_Θ.

To emphasize the use of the theorem above, whenever we consider the set $N(A_\Theta)$ of normal words of the Hecke–Kiselman algebra $A_\Theta = K[\mathrm{HK}_\Theta]$ that is obtained via reductions from the set T, we will say that the elements of $N(A_\Theta)$ are the reduced words of A_Θ.

The next remark, mentioned in many papers (and also clear from Theorem 1.63), is also relevant.

Remark 1.64. (1) Assume that $x \in V(\Theta)$ is a vertex such that there are no arrows of the form $z \to x$. Then for every word $w \in \mathrm{HK}_\Theta$ the equality $xwx = xw$ holds in HK_Θ.

(2) Assume that $x \in V(\Theta)$ is a vertex such that there are no arrows of the form $x \to z$. Then for every word $w \in \mathrm{HK}_\Theta$ the equality $xwx = wx$ holds in HK_Θ.

Let us look more closely at the Gröbner basis of the Hecke–Kiselman monoid C_n associated to the cycle of length $n \geqslant 3$. Denote its generators by $x_1, \ldots, x_n$. If $i, j \in \{1, \ldots, n\}$ then $x_i \cdots x_j$ denotes the product of all consecutive generators from x_i up to x_j if $i < j$, or down to x_j, if $i > j$.

Consider the degree-lexicographic order on $\langle x_1, \ldots, x_n\rangle$ induced by $x_1 < \cdots < x_n$. Then the Gröbner basis can be described by the following theorem from [40].

Theorem 1.65. *Let $\Theta = C_n$. Let S be the system of reductions in F consisting of all pairs of the form*

(1) $(x_i x_i, x_i)$ *for all* $i \in \{1, \ldots, n\}$,

(2) $(x_j x_i, x_i x_j)$ *for all* $i, j \in \{1, \ldots, n\}$ *such that* $1 < j - i < n - 1$,

(3) $(x_n(x_1 \cdots x_i)x_j, x_j x_n(x_1 \cdots x_i))$ *for all* $i, j \in \{1, \ldots, n\}$ *such that* $i + 1 < j < n - 1$,

(4) $(x_i u x_i, x_i u)$ *for all* $i \in \{1, \ldots, n\}$ *and* $1 \neq u \in F$ *such that* $|u|_i = |u|_{i-1} = 0$. *Here, we write* $i - 1 = n$ *if* $i = 1$, *(we say, for the sake of simplicity, that the word* $x_i u x_i$ *is of type* $(4i)$*)*,

(5) $(x_i v x_i, v x_i)$ *for all* $i \in \{1, \ldots, n\}$ *and* $1 \neq v \in F$ *such that* $|v|_i = |v|_{i+1} = 0$. *Here we write* $i + 1 = 1$ *if* $i = n$, *(and similarly, we say that the word* $x_i v x_i$ *is of type* $(5i)$*)*.

Then the set $\{w - v \mid \text{ for } (w, v) \in S\}$ *is a Gröbner basis of the algebra* $K[C_n]$.

Corollary 1.66. *C_n can be identified with the monoid $\mathcal{R}(S)$ of words in F that are reduced with respect to the system S, with the operation defined for $u, w \in C_n$ by $u \cdot w = \mathcal{R}_S(uw)$, where $\mathcal{R}_S(uw)$ is the S–reduced form of the word uw. More precisely, $\mathcal{R}(S)$ is the set of words in F that do not have factors of the form w_σ, where $\sigma = (w_\sigma, v_\sigma) \in S$.*

For $w, v \in F$, we write $w \xrightarrow{(\eta)} v$ in case $w = u w_\sigma z, v = u v_\sigma z$ for some $u, z \in F$ and an element (w_σ, v_σ) of the set S of reductions of type (η). Here (η) may be one of: (1) – (5), or even more explicitly $(4i)$ or $(5i)$, for some i. More generally, $w \xrightarrow{(\eta)} v$ may also denote a sequence of consecutive reductions of type (η). If clear from the context, $w \to v$ will denote an unspecified sequence of reductions.

Note that for example all words $(x_n \cdots x_1)^k$, $k = 1, 2, \ldots$ are reduced in the monoid C_n. Thus in particular they are pairwise different. This gives a simple argument to the following fact, first proved in [4].

Proposition 1.67. *Hecke–Kiselman monoid C_n associated to cycle of length $n \geqslant 3$ is infinite.*

The following general easy proposition will be relevant in our work.

Proposition 1.68. *If a graph Σ is a subgraph of Θ, then the Hecke–Kiselman monoid* HK_Σ *is a homomorphic image of* HK_Θ.

We proceed with a well-known characterization of finite Hecke–Kiselman monoids associated with oriented graphs, [4].

If the graph Θ_n is an oriented graph with vertices $x_1, \ldots, x_n$ and an oriented edge from x_i to x_j if and only if $i < j$, then the corresponding Hecke–Kiselman monoid HK_{Θ_n} is a finite monoid (called Kiselman's monoid), see [18]. It can be verified that every acyclic oriented graph, that is a graph that does not contain an oriented cycle as a subgraph, is a subgraph of Θ_n for some n. Thus from Proposition 1.68 it follows that for any acyclic graph the Hecke–Kiselman monoid is finite. Moreover, we also get that the Hecke–Kiselman monoid associated to a graph containing an oriented cycle of length $n \geqslant 3$ has an infinite homomorphic image, isomorphic to C_n for some $n \geqslant 3$. Therefore we get the following conclusion.

Theorem 1.69. *If a graph Θ is oriented, then the corresponding Hecke–Kiselman monoid is finite if and only if the graph is acyclic, that is it does not contain any oriented cycle.*

We emphasise that although the above result is easy to prove, the characterization of finite Hecke–Kiselman monoids associated to graphs with both oriented and unoriented edges seems to be extremely difficult. Partial results in this direction, obtained in [4] are combinatorially involved.

Lastly we summarize the relevant material on Green's relations in Hecke–Kiselman monoids.

First, it has been proved in [29, Theorem 22] that Kiselman's monoids are $\mathcal{J}$-trivial. Thus, as observed in [18], from Remark 1.39 we get the following proposition.

Proposition 1.70. *Hecke–Kiselman monoids associated to oriented acyclic graphs are $\mathcal{J}$-trivial.*

Certain combinatorial interpretations of Hecke–Kiselman monoids associated to oriented cycles are used to obtain that these monoids are also $\mathcal{J}$-trivial in [11, Theorem 4.5.3].

Theorem 1.71. *The Hecke–Kiselman monoid associated to an oriented cycle of length $n \geqslant 3$ is $\mathcal{J}$-trivial.*

Let us also mention that the problem of characterization of Green's relations in arbitrary Hecke–Kiselman monoids is still open.

1.4.3 Background on Hecke–Kiselman algebras

While several papers on Hecke–Kiselman monoids focus on combinatorial and semigroup-theoretic aspects, we are also interested in the structure of algebras over a field associated with these monoids. Thus we expand the study, started in [39], of the ring-theoretic structure of Hecke–Kiselman algebras and their representations.

We now outline the main results in this direction. We will focus only on theorems that will play an important role in the thesis.

Oriented graphs Θ such that the corresponding monoid HK_Θ does not contain a free submonoid of rank 2 have been described in [39]. It turns out that this is strictly related to the properties of the corresponding algebra A_Θ; namely to satisfying a polynomial identity and being of finite Gelfand–Kirillov dimension. More precisely, the following theorem has been proved.

Theorem 1.72. *Let Θ be a graph. The following conditions are equivalent.*

(1) Θ *does not contain two different cycles connected by an oriented path of length* ≥ 0,

(2) A_Θ *is an algebra satisfying a polynomial identity,*

(3) $\mathrm{GKdim}(A_\Theta) < \infty$,

(4) the monoid HK_Θ *does not contain a free submonoid of rank* 2.

This will be extremely useful for us, as we often focus on Hecke–Kiselman algebras that are PI. If this is the case, strong structural theorems from Section 1.1.5 can be used.

Consider a finitely generated monoid algebra $A = K\langle X\rangle/I$ and an order $\leqslant$ on the free monoid $F = \langle X\rangle$ which is compatible with the multiplication in the monoid, that is $1 \leqslant w$ for all $w \in F$ and from $v \leqslant w$ it follows that $uv \leqslant uw$ and $vu \leqslant wu$ for all $u, v, w \in F$. Then A is an automaton algebra if the set of normal words $N(A)$ (consisting of all words that are not leading monomials of elements of the ideal I, as defined for deg-lex order in Section 1.4.2) is a regular language for some presentation and an order compatible with multiplication. That means that this set is obtained from a finite subset of F by applying a finite sequence of operations of union, multiplication and operation $*$ defined by $T^* = \bigcup_{i\geq 0} T^i$, for $T \subseteq F$. An expression built recursively from the set of letters from F using operations of union, multiplication and $*$ is called a regular expression. We refer to [56, Chapter 5] for more information on the automaton algebras.

From [56, Theorem 3, p. 97] we get the following useful property.

Theorem 1.73. *The Gelfand–Kirillov dimension of an automaton algebra is either infinite or an integer.*

As it will be explained in Chapter 6, in the finite dimensional case, the dimension is related to certain forms of regular-expressions representations of the regular languages of normal words, [54].

The characterization of Gröbner basis of Hecke–Kiselman algebras from Theorem 1.63 leads to the following corollary, obtained in [40], that will be useful in the calculation of the Gelfand–Kirillov dimension of Hecke–Kiselman algebras in Chapter 6.

Theorem 1.74. *For any oriented graph* Θ *the algebra* A_Θ *is an automaton algebra, with respect to any deg-lex order on the underlying free monoid of rank* n*. Consequently, the Gelfand–Kirillov dimension* $\mathrm{GKdim}(A_\Theta)$ *of* A_Θ *is an integer if it is finite.*

Chapter 2

Structure of the Hecke–Kiselman monoid C_n associated to the cycle of length $n \geqslant 3$

In the present chapter we focus on the structure of the Hecke–Kiselman monoid C_n associated to an oriented cycle of length $n \geqslant 3$. In the first section we give a characterization of reduced forms of all but finitely many elements of the monoid C_n. This is the key technical tool used in our approach. Then we apply this characterization to construct an unexpected chain of ideals inside the monoid with factors that are, up to finitely many elements, semigroups of matrix type. The chain is introduced in Section 2.2. Next we investigate semigroups of matrix type inside the monoid and describe certain involutions of C_n that induce involutions of the quotients of the chain. The main result is summarized in Theorem 2.44. In the last part of the chapter we focus on structural properties of these semigroups of matrix type, that will be extensively used in the investigation of structural properties of the monoid C_n and its monoid algebra $K[C_n]$ over a field K. In particular we show in Theorem 2.52 that the semigroup algebras associated to these semigroups of matrix type are prime.

Results from Sections 2.1, 2.2 and 2.3 were mainly (all except for Proposition 2.13 and Corollary 2.31) obtained during the author's master's studies. In most cases we outline the applied approach and formulate several technical results exploited in the proofs, instead of providing full proofs. Note that some of the technical results will be also applied in the next chapters. The detailed proofs can be found in the paper [45]. The content of Section 2.4 (although also published in [45]) is new.

2.1 The form of (almost all) reduced words in C_n

The problem of characterizing the reduced forms of elements of Hecke–Kiselman monoids has been investigated by several authors, for instance in the papers [5, 33, 40]. In the case of Kiselman's semigroup certain canonical forms of all elements were introduced in [29] and then applied to study the properties and representations of this semigroup. In the paper [40] Gröbner bases have been characterized for any Hecke–Kiselman algebra associated to an oriented graph, see Section 1.4.2 for the details. Another approach in the case of the Hecke–Kiselman monoid associated to a chain or to an oriented cycle of any length is presented in [33], where a diagrammatic interpretation is used. Note that words obtained in [33] are not necessarily reduced in the sense that we use in the present thesis.

The aim of this section is to prove that all elements of the monoid C_n, defined in Example 1.61 3), except for finitely many words, have a very special reduced form with respect to the deg-lex order and the reduction system S introduced in Theorem 1.65. This will be the key to describe the structure and properties of C_n in the next sections, as in view of Corollary 1.66, we may identify the elements of C_n with the reduced words in F.

Note that it is possible to decide whether a given word from free monoid is in the reduced form when a Gröbner basis is given, but in general it is hard to characterize all reduced elements.

We follow the approach of [40], using the language of Gröbner bases and the notation from Section 1.4.2. Recall that F is the free monoid generated by $x_1, \ldots, x_n$. By a prefix (suffix) of the word w we mean any factor $u \neq 1$ such that $w = uv$ ($w = vu$) for some v. For every subset $Z \subseteq F$ by $\mathrm{suff}(Z)$ ($\mathrm{pref}(Z)$, respectively) we denote the set of all suffixes (prefixes, respectively) of elements of Z. For any word $w \in F$, by w^∞ we mean the infinite word $ww\ldots$. The notation introduced after Corollary 1.66 is also used.

Let $q_{n,i} = x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1} \in F$, for $i = 0, \ldots, n-2$ and $n \geqslant 3$. Here we agree that $q_{n,0} = x_n x_{n-1} \cdots x_1$. From Corollary 1.66 it follows that the word $(q_{n,i})^k$ is reduced for every $k \geqslant 0$.

For every $i = 0, \ldots, n-2$ we define two subsets A_i and B_i of F, as follows. First,

$$A_i = \mathrm{suff}(\{(x_{k_s} \cdots x_s)(x_{k_{s+1}} \cdots x_{s+1}) \cdots (x_{k_{i+1}} \cdots x_{i+1})\}),$$

where $s \in \{0, \ldots, i+1\}$, $k_{s+1} < k_{s+2} < \cdots < k_{i+1} \leqslant n-1$, $k_s \leqslant s$ and $k_q > q$ for $q = s+1, \ldots, i+1$.
The convention is that the subset of A_i corresponding to $s = i+1$ has the form $\mathrm{suff}(\{x_{k_{i+1}} \cdots x_{i+1}\})$, where $k_{i+1} \leqslant i+1$. Also, if $s = 0$ then the corresponding subset of A_i has the form $\mathrm{suff}(\{(x_{k_{s+1}} \cdots x_{s+1}) \cdots (x_{k_{i+1}} \cdots x_{i+1})\})$, where $k_{s+1} < k_{s+2} < \cdots < k_{i+1} \leqslant n-1$ and $k_q > q$ for $q = s+1, \ldots, i+1$.
The set B_i is defined by

$$B_i = \mathrm{pref}(\{x_n(x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}) \cdots x_n(x_1 \cdots x_{i_r} x_{n-1} \cdots x_{j_r}) x_n x_{n-1} \cdots x_{j_{r+1}}\}),$$

where $r \geqslant 0$, $i_r < i_{r-1} < \cdots < i_1 < i+1$ and $i+1 < j_1 < j_2 < \cdots < j_{r+1} \leqslant n$.
Here, the subset of B_i corresponding to $r = 0$ has the form $\mathrm{pref}(\{x_n x_{n-1} \cdots x_{j_1}\})$.

The following result characterizes all reduced words that have a factor of the form $q_{n,i}$.

Theorem 2.1. *Assume that w is a reduced word that contains a factor of the form*

$x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1}$ for some $i = 0, \dots, n-2$. Then

$$w = a(x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1})^k b$$

for some $a \in A_i$, $b \in B_i$ and some $k \geq 1$. Moreover, all words of this type are reduced.

We will use the following convention. By *a block* we mean a factor of the form $x_{k_j} \cdots x_j$, for some $j \in \{s, \dots, i+1\}$, appearing in the elements of the set A_i or a factor of the form $x_n(x_1 \cdots x_{i_k} x_{n-1} \cdots x_{j_k})$, for $k \in \{1, \dots, r\}$, or $x_n x_{n-1} \cdots x_{j_{r+1}}$, appearing in the elements of B_i.

The main idea of the proof is to analyse the possible forms of reduced words that satisfy certain additional restrictions, using the Gröbner basis from Theorem 1.65.

Let us consider the word $w = aq_{n,i}^k b$, where $a \in A_i$, $b \in B_i$ and $k \geqslant 1$. Then a factor $a \in A_i$ is exactly the maximal prefix of w that does not contain x_n. Similarly, $b \in B_i$ is a suffix of w which appears after the last occurrence of the factor $q_{n,i}$ in the word. This suggests that it is useful to start with investigating possible occurrences of x_n in the reduced words.

Let us recall several technical lemmas from [45] that lead to the description of possible factors v in the reduced words of the form $x_n v x_n$.

Lemma 2.2. *If $w = x_{n-1}u$ is a reduced word, where $u \in F$ is such that $|u|_n = 0$, then $w = x_{n-1} \cdots x_k$ for some $k \geqslant 1$.*

Lemma 2.3. *If $w = x_n x_1 u$ is a reduced word, where $|u|_n = 0$, then w is of one of the forms*

1. *$w = x_n x_1 x_2 \cdots x_i x_{n-1} \cdots x_j$ for some $1 \leqslant i < j \leqslant n-1$;*
2. *$w = x_n x_1 x_2 \cdots x_i$ for some $1 \leqslant i < n-1$.*

Lemma 2.4. *If $w = x_n u x_n$ is a reduced word, where $|u|_n = 0$, then u is of one of the forms*

1. *$u = x_{n-1} \cdots x_1$;*
2. *$u = x_1 \cdots x_i x_{n-1} \cdots x_j$ for $1 \leqslant i < j \leqslant n-1$.*

The following lemma shows that in the case where $i = 0$ or $i = n-2$, the reduced words with a factor $q_{n,i}$ have an extremely simple form.

Lemma 2.5. *If a reduced word has a factor of the form $x_n x_1 \cdots x_{n-1}$ or $x_n x_{n-1} \cdots x_1$, then it must be a factor of the infinite word $(x_n x_1 \cdots x_{n-1})^\infty$ or $(x_n x_{n-1} \cdots x_1)^\infty$.*

We refer to [45] for the detailed proofs. The reasoning is based on the repeated use of the following observation.

Observation 2.6. 1. If $x_i \cdots x_j x_k$ is reduced, where $i \geqslant j$ and $k > j$, then either $i \neq n$ and $k \geqslant i+1$ or $i = n$, $j = 1$ and $k = n$.

2. If $x_i \cdots x_j x_k$ is reduced, where $i \leqslant j$ and $k < j$, then either $1 \neq i = j$ and $k = j - 1$ or $j = n$ and $k \in \{1, n-1\}$.

3. If $x_k x_i \cdots x_j$ is reduced, where $i \leqslant j$ and $k > i$, then either $i = j$ and $k = i + 1$ or $(i, k) = (1, n)$.

4. If $x_k x_i \cdots x_j$ is reduced, where $i \geqslant j$ and $k < i$, then $k < j$.

5. If $x_n x_1 \cdots x_i x_j$ is reduced, where $1 \leqslant i \leqslant n - 1$, then $j \in \{i + 1, n - 1\}$.

We give a proof as an illustration of the computations used in the lemmas dealing with reduced forms of words in the monoid C_n. Working with the reduction system S from Theorem 1.65, we follow the notation introduced after Corollary 1.66.

Proof. 1. Let $x_i \cdots x_j x_k$ be reduced, where $i \geqslant j$, $k > j$ and $i \neq n$. Suppose that $k < i+1$. Then $j < k < i + 1$ and consequently $x_i \cdots x_j x_k$ has a factor $x_k \cdots x_j x_k$ which can be reduced, namely $x_k \cdots x_j x_k \xrightarrow{(5)} x_{k-1} \cdots x_j x_k$. Now assume that $i = n$. Then, if $j > 1$ we get as in the previous case that $x_i \cdots x_j x_k$ can be reduced using a reduction of type (5). Similarly for $j = 1$ the word is of the form $x_n \cdots x_1 x_k$. Suppose that $k \neq n$, then $x_n \cdots x_1 x_k$ has a factor $x_k x_{k-1} \cdots x_1 x_k$ with $|x_{k-1} \cdots x_1|_{k+1} = 0$, which leads to a contradiction.

2. If $i = j$ and $k < j$, then $x_j x_k$ is reduced for $j \neq 1$ and $k = j - 1$ or for $j = n$ and $k = 1$, as otherwise $x_j x_k \xrightarrow{(2)} x_k x_j$. Assume now that $i < j$ and $k < j$. For $i < k < j$, the word $x_i \cdots x_j x_k$ has a factor $x_k \cdots x_j x_k$, which is not reduced. If $k < i < j$, then for $(k, j) \neq (1, n)$ the word has a factor $x_j x_k \xrightarrow{(2)} x_k x_j$. The assertion follows.

3. Assume first that $i = j$. Then for $(i, k) \neq (1, n)$ and $k \neq i + 1$ we have $x_k x_i \xrightarrow{(2)} x_i x_k$ and thus the word is not reduced. If $i < j$ and $k > i$, then for $(i, k) \neq (1, n)$ and $k \neq i+1$ we have a factor $x_k x_i$ as before. Moreover, if $k = i+1$, the word has a factor $x_{i+1} x_i x_{i+1}$, which is not reduced.

4. If $k \geqslant j$, then $x_k x_i x_{i-1} \cdots x_j$ is not in the reduced form, as it contains a factor $x_k x_i \cdots x_{k+1} x_k$, such that $|x_i \cdots x_{k+1}|_{k-1} = 0$.

5. This is a straightforward consequence of reduction of type (3) from Theorem 1.65.

□

The next lemma will be used to determine the desired shape of the elements of B_i, which are the endings of the considered class of reduced words.

Lemma 2.7. *Let $w \in F$ be of the form $w = x_n x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1} x_n x_1 \cdots x_{i_2} x_{n-1} \cdots x_{j_2}$, where $1 \leqslant i_p < j_p \leqslant n - 1$ for $p = 1, 2$. If w is a reduced word, then*

1. $i_1 \geqslant i_2$,

2. $j_1 \leqslant j_2$.

Moreover, if $i_1 = i_2 = i$*, then* $j_1 = i+1$ *and if* $j = j_1 = j_2$*, then* $i_1 = i_2 = j-1$*. Additionally, if* $i_1 + 1 = j_1$ *and* $i_2 + 1 < j_2$*, then* $j_2 > j_1$*.*

Inequalities in the proof are obtained by excluding non-reduced factors of the form $x_k w x_k$, where $k \in \{1, \ldots, n\}$, $|w|_k = 0$ and $|w|_{k+1} = 0$ or $|w|_{k-1} = 0$ (where for $k = n$ we set $k+1 = 1$ and for $k = 1$ we set $k - 1 = n$).

Applying the above lemmas to the consecutive factors of the form

$$x_n x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1} x_n x_1 \cdots x_{i_2} x_{n-1} \cdots x_{j_2}$$

we get the following corollary.

Corollary 2.8. *If a reduced word* w *is of the form*

$$u x_n (x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}) x_n (x_1 \cdots x_{i_2} x_{n-1} \cdots x_{j_2}) \cdots x_n (x_1 \cdots x_{i_r} x_{n-1} \cdots x_{j_r}) x_n v$$

for some u, v *such that* $|u|_n = |v|_n = 0$*, then it follows that*

$$i_r \leqslant i_{r-1} \leqslant \cdots \leqslant i_1 < j_1 \leqslant j_2 \leqslant \cdots \leqslant j_r.$$

Furthermore, if $i_k + 1 = j_k$ *for some* k*, then* $i_k = i_s$ *and* $j_k = j_s$ *for* $s = 1, \ldots, k-1$*. Moreover, if for some* l *we have* $i_l + 1 < j_l$*, then* $i_r < i_{r-1} < \cdots < i_l < i_l + 1 < j_l < \cdots < j_r$*. If* $l > 1$*, then also* $i_l \leqslant i_{l-1} < j_{l-1} < j_l$*.*

The next few lemmas will be used to deal with the shape of the elements of the set A_i, which are the beginnings of the considered class of reduced words.

Lemma 2.9. *Let* w *be a reduced word such that*

$$w = v(x_{k_s} \cdots x_s) u x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1},$$

where $i = 1, \ldots, n-3$*,* $k_s < s \leqslant i+1$*,* $|u|_j = 0$ *for* $j = 1, \ldots, s$ *and* $|u|_n = |v|_n = 0$*. Then* $v = x_r x_{r+1} \cdots x_{k_s - 1}$ *for some* $r \geqslant 1$*.*

The idea of the proof is to apply Observation 2.6 3, using the assumption that $|v|_n = 1$.

Lemma 2.10. *Assume that a reduced word* w *is of the form*

$$w = u(x_{k_s} \cdots x_s)(x_{k_{s+1}} \cdots x_{s+1}) \cdots (x_{k_{i+1}} \cdots x_{i+1}) x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1}$$

where $s \leqslant i+1$*,* $k_s \geqslant s$*,* $k_q > q$ *for* $q = s+1, \ldots, i+1$*,* $|u|_n = 0$*. Then* $\mathrm{suff}_1(u) \in \{x_{k_s+1}, x_{s-1}\}$*.*

Let $\mathrm{suff}_1(u) = x_j$. If $j < k_s$ then from Observation 2.6 4 it follows that $j \leqslant s-1$. Moreover, for $j > k_s + 1$ the word w has a factor $x_j x_{k_s}$ which is not reduced.

Lemma 2.11. *Assume that a reduced word w is of the form*

$$w = ux_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1}$$

for some $i \in \{1, \ldots, n-3\}$, where $|u|_n = 0$ and u has no factors of the form $x_l \cdots x_j$, where $l < j$. Then u is of the form

$$u = (x_{k_s} \cdots x_s)(x_{k_{s+1}} \cdots x_{s+1}) \cdots (x_{k_{i+1}} \cdots x_{i+1}),$$

where $s \leqslant i+1$, $k_s < k_{s+1} < \cdots < k_{i+1} \leqslant n-1$, $k_s \geqslant s$, $k_q > q$ for $q = s+1, \ldots, i+1$.

The idea of the proof is to exclude factors of the form $x_j u x_j$ such that $|u|_{j-1} = 0$ or $|u|_{j+1} = 0$, where x_j is the suffix of u. Then it is enough to apply Lemma 2.10 several times together with the assumptions.

Lemma 2.12. *If w is a reduced word and $w = ux_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1}$ for some $i = 1, \ldots, n-3$ and some $u \in F$ such that $|u|_n = 0$, then u is of the form*

$$u = (x_{k_s} \cdots x_s)(x_{k_{s+1}} \cdots x_{s+1}) \cdots (x_{k_{i+1}} \cdots x_{i+1}),$$

where $s \leqslant i+1$, $k_{s+1} < k_{s+2} < \cdots < k_{i+1} \leqslant n-1$ and $k_q > q$ for $q = s+1, \ldots, i+1$ (but perhaps $k_s \leqslant s$).

From Lemma 2.11 it follows that $u = z(x_{k_{s+1}} \cdots x_{s+1}) \cdots (x_{k_{i+1}} \cdots x_{i+1})$ for certain z such that $\mathrm{suff}_2(z) = x_{j-1}x_j$ for certain j and $k_{s+1} < k_{s+2} < \cdots < k_{i+1} \leqslant n-1$ and $k_q > q$ for $q = s+2, \ldots, i+1$ and $k_{s+1} > s+1$. Then applying Lemma 2.10 we get two possible suffixes of z, one of which can be easily excluded. Then the assertion follows from Lemma 2.9.

We are ready to show how the proof of Theorem 2.1, published in the paper [45], follows from the above series of technical lemmas.

Proof of Theorem 2.1. From Corollary 1.66 it is clear that all words described in the statement are reduced.

Let $w \in F$ be a reduced word that contains a factor $q_{n,i}$. By Lemma 2.5 the assertion holds for $i = 0, n-2$. Thus we assume further that $i \in \{1, \ldots, n-3\}$. Notice that if the word w has the form $x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1} v$ for some $v \in C_n$, then we must have $\mathrm{pref}(v) = x_n$. Indeed, if $\mathrm{pref}(v) = x_j$ for $j \leqslant i$, then $q_{n,i} x_j \xrightarrow{(4j)} q_{n,i}$. Similarly, if $i+1 \leqslant j \leqslant n-1$, then $q_{n,i} x_j \xrightarrow{(5j)} x_n x_1 \cdots x_i x_{n-1} \cdots x_{j+1} x_{j-1} \cdots x_{i+1}$.

From Lemma 2.4 and Lemma 2.7 we know that if $i = 1, \ldots, n-3$ then w is of the form

$$u(x_n x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}) \cdots (x_n x_1 \cdots x_{i_m} x_{n-1} \cdots x_{j_m}) x_n v,$$

for some m, where $1 \leqslant i_k < j_k \leqslant n-1$ for every k and $|u|_n = |v|_n = 0$.

In view of Corollary 2.8 this implies that w is of the form

$$u(x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1})^k x_n (x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}) \cdots x_n (x_1 \cdots x_{i_r} x_{n-1} \cdots x_{j_r}) x_n v, \quad (2.1.1)$$

where $i_r < i_{r-1} < \cdots < i_1 < i+1$ and $i+1 < j_1 < j_2 < \cdots < j_r$ and $|u|_n = |v|_n = 0$, where the factor of the form $(x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}) \cdots x_n(x_1 \cdots x_{i_r} x_{n-1} \cdots x_{j_r})$ does not have to occur in w (that is, w can be of the form $u(x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1})^k x_n v)$ and then we put $r = 0$.

Notice that $\mathrm{pref}_1(v) \in \{x_1, x_{n-1}\}$, since otherwise w contains a factor $x_n x_s$ for $s < n$, which is not reduced.

If $\mathrm{pref}_1(v) = x_{n-1}$, Lemma 2.2 implies that $v = x_{n-1} \cdots x_{j_{r+1}}$. Moreover, we must have $j_r < j_{r+1}$, as otherwise w has a factor $x_{j_r} x_n \cdots x_{j_r+1} x_{j_r}$ such that $|x_n \cdots x_{j_r+1}|_{j_r-1} = 0$, which is not reduced.

If $\mathrm{pref}_1(v) = x_1$, then by Lemma 2.3 and Corollary 2.8 we get $v = x_1 \cdots x_{i_{r+1}} x_{n-1} \cdots x_{j_{r+1}}$ for $i_{r+1} < i_r$ and $j_{r+1} > j_r$, if $r > 0$. If $r = 0$, then in view of (2.1.1) we have $w = pq$, where $p \in F$ and

$$q = x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1} x_n x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}$$

Corollary 2.8 implies that $i_1 \leqslant i$ and $j_1 \geqslant i+1$. The desired form of the elements of the set B_i follows.

Since $k \geqslant 1$, the desired form of the elements of the set A_i follows by Lemma 2.12. This completes the proof of Theorem 2.1. □

Now we calculate the size of the set A_i occurring in Theorem 2.1, for every $i = 0, \ldots, n-2$ and $n \geqslant 3$, that was calculated in [46].

Proposition 2.13. *For any $i \in \{0, \ldots, n-2\}$ and $n \geqslant 3$ we have $|A_i| = \binom{n}{i+1}$.*

Proof. For $i = n-2$ the assertion follows from Lemma 2.5. Thus next we assume that $i \leqslant n-3$.
By the description of the set A_i from Theorem 2.1 it is clear that every element w of A_i is exactly of one of the forms

1. $w = (x_{k_s} \cdots x_s)(x_{k_{s+1}} \cdots x_{s+1}) \cdots (x_{k_{i+1}} \cdots x_{i+1})$ where $i+1 \geqslant s \geqslant 1$, $s+1 < k_{s+1} < \cdots < k_{i+1} \leqslant n-1$ and $s \geqslant k_s$; for $s = i+1$ we assume that $w = (x_{k_{i+1}} \cdots x_{i+1})$ with $i+1 \geqslant k_{i+1}$;

2. $w = (x_{k_s} \cdots x_s)(x_{k_{s+1}} \cdots x_{s+1}) \cdots (x_{k_{i+1}} \cdots x_{i+1})$ where $i+1 \geqslant s \geqslant 1$, $s < k_s < \cdots < k_{i+1} \leqslant n-1$;

3. $w = 1$.

Choose $1 \leqslant s \leqslant i+1$ and $0 \leqslant i \leqslant n-3$. Then the elements w from Case 1. are in a bijection with strictly increasing sequences $(k_s, \ldots, k_{i+1})$ of natural numbers such that $1 \leqslant k_s \leqslant s < s+2 \leqslant k_{s+1} < \cdots < k_{i+1} \leqslant n-1$. It is easy to see that there exist exactly $s\binom{n-s-2}{i-s+1}$ sequences of the above form. Similarly, elements w of the form as in Case 2. are in a bijection with strictly increasing sequences $(k_s, \ldots, k_{i+1})$ of natural numbers such that $s+1 \leqslant k_s < \cdots < k_{i+1} \leqslant n-1$. There are exactly $\binom{n-s-1}{i-s+2}$ such sequences.

It follows that

$$|A_i| = 1 + \sum_{s=1}^{i+1} \left(\binom{n-s-1}{i-s+2} + s\binom{n-s-2}{i-s+1} \right).$$

Thus, it is enough to prove that $1 + \sum_{s=1}^{i+1}(\binom{n-s-1}{i-s+2} + s\binom{n-s-2}{i-s+1}) = \binom{n}{i+1}$ for $n \geqslant 3$ and $0 \leqslant i \leqslant n-3$.

Moreover, if $i = n-3$, then by a direct calculation we get that

$$1 + \sum_{s=1}^{n-2} \left(\binom{n-s-1}{n-s-1} + s\binom{n-s-2}{n-s-2} \right) = \binom{n}{n-2},$$

as desired.

It is easy to check that

$$1 + \sum_{s=1}^{i+1} \left(\binom{n-s-1}{i-s+2} + s\binom{n-s-2}{i-s+1} \right) = \sum_{k=0}^{i+1} (i+2-k)\binom{n-i-3+k}{k}.$$

Indeed, substituting $k = i+1-s$ in the sum in the left hand side, we get that this sum is equal to

$$\begin{aligned} 1 + \sum_{k=0}^{i} \binom{n-i-2+k}{k+1} + \sum_{k=0}^{i} (i+1-k)\binom{n-i-3+k}{k} &= \\ = 1 + \sum_{k=1}^{i+1} \binom{n-i-3+k}{k} + \sum_{k=0}^{i} (i+1-k)\binom{n-i-3+k}{k} &= \\ = \sum_{k=0}^{i+1} \binom{n-i-3+k}{k} + \sum_{k=0}^{i+1} (i+1-k)\binom{n-i-3+k}{k} &= \\ = \sum_{k=0}^{i+1} (i+2-k)\binom{n-i-3+k}{k}&, \end{aligned}$$

as claimed.

We proceed by induction on n to prove that

$$\sum_{k=0}^{i+1} (i+2-k)\binom{n-i-3+k}{k} = \binom{n}{i+1}.$$

For $i = 0$ and arbitrary $n \geqslant 3$ we have $1 + \binom{n-2}{1} + \binom{n-3}{0} = \binom{n}{1}$ and the assertion follows. If $n = 3$, then we have $0 \leqslant i \leqslant 0$, so the proposition holds.

Assume now that the equality is true for some n and every $i \leqslant n-3$. Consider the sum

$$\sum_{k=0}^{i+1}(i+2-k)\binom{(n+1)-i-3+k}{k}$$

for $n-2 > i > 0$. Using $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$ if $k \geqslant 1$ and $\binom{m+1}{0} = \binom{m}{0}$ we get

$$\begin{aligned}\sum_{k=0}^{i+1}&(i+2-k)\binom{(n+1)-i-3+k}{k} = \\ &= \sum_{k=0}^{i+1}(i+2-k)\binom{n-i-3+k}{k} + \sum_{k=1}^{i+1}(i+2-k)\binom{n-i-3+k}{k-1}.\end{aligned}$$

From the induction hypothesis it follows that the first sum is equal to $\binom{n}{i+1}$. Substituting $m = k-1$ and $j = i-1$ we get

$$\sum_{k=1}^{i+1}(i+2-k)\binom{n-i-3+k}{k-1} = \sum_{m=0}^{j+1}(j+2-m)\binom{n-j-3+m}{m}.$$

From the induction hypothesis it follows that the above sum is equal to $\binom{n}{i}$. Now, using $\binom{n}{i+1} + \binom{n}{i} = \binom{n+1}{i+1}$ we get

$$\sum_{k=0}^{i+1}(i+2-k)\binom{(n+1)-i-3+k}{k} = \binom{n}{i+1} + \binom{n}{i} = \binom{n+1}{i+1}$$

and the assertion follows. $\square$

Our next aim is to show that Theorem 2.1 describes all but finitely many elements of the monoid C_n. Let us introduce the following crucial notation.

Definition 2.14. For every $i = 0, \ldots, n-2$ we denote by $\tilde{M}_i$ the following set

$$\tilde{M}_i = \{aq_{n,i}^k b \in C_n : a \in A_i, b \in B_i, k \geqslant 1\}. \tag{2.1.2}$$

(the set of reduced forms of elements of C_n that have a factor $q_{n,i}$). Define also $\tilde{M} = \bigcup_{i=0}^{n-2} \tilde{M}_i$.

Corollary 1.66 ensures that two elements $w, w' \in \tilde{M}$ are equal in C_n if and only if the equality $w = w'$ holds in the free monoid F generated by $x_1, \ldots, x_n$. In particular, we can write $\tilde{M} \subseteq C_n$. This identification will be often used without further comment.

Proposition 2.15. *$C_n \setminus \tilde{M}$ is a finite set.*

The set $C_n \setminus \tilde{M}$ consists of all words in C_n that do not contain a factor of the form $q_{n,i} = x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1}$ for $i = 0, \ldots, n-2$. The idea of the proof that there are finitely many such words is to estimate the maximal possible length of reduced words without a factor $q_{n,i}$ for any $i \in \{0, \dots n-2\}$. If x_n does not occur in the reduced form of the word w, then, as factor $x_i u x_i$ with $|u|_{i-1}$ is not reduced, it can be proved that $|w|_m \leqslant m$ for every $m \in \{1, \dots, n-1\}$. Otherwise, it is possible to estimate the number of occurrences of the generator x_n using the form of any reduced word $x_n u x_n$ with $|u|_n = 0$ from Lemma 2.4 and Corollary 2.8. We conclude that there is a number $C = C(n)$ such that if w is in $C_n \setminus \tilde{M}$, then $|w| < C$. As there are only finitely many words w with $|w| < C$, the assertion follows. We refer to Section 2 of [45] for the detailed proof.

As a consequence, using the characterization of sets $\tilde{M}_i$ in Definition 2.14 we get an alternative calculation of the Gelfand–Kirillov dimension of the Hecke–Kiselman algebra associated to an oriented cycle obtained in Example 2 from [39].

Remark 2.16. The Hecke–Kiselman algebra $K[C_n]$ over a field K is of the Gelfand–Kirillov dimension one.

We end the section with an easy observation of independent interest. It will be later useful in the proofs of various properties of the Hecke–Kiselman monoids C_n by induction on n.

Lemma 2.17. *Assume that $y_1, y_2, \dots, y_{n-1}$ are the consecutive vertices of a cyclic graph C_{n-1}. Consider an epimorphism ϕ from the free monoid $Y = \langle y_1, \dots, y_{n-1} \rangle$ to the submonoid $\langle x_2, \dots, x_{n-1}, x_n x_1 \rangle$ of F defined by*

$$\phi(y_i) = \begin{cases} x_{i+1}, & \text{for } 1 \leqslant i \leqslant n-2, \\ x_n x_1, & \text{for } i = n-1. \end{cases}$$

Then ϕ induces a homomorphism $\overline{\phi} : C_{n-1} \longrightarrow C_n$ given by the formula $\overline{\phi}(w) = \phi(w)$, for every $w \in \langle y_1, \dots, y_{n-1} \rangle$. Moreover, $\overline{\phi}$ determines an isomorphism

$$C_{n-1} \cong \langle x_2, \dots, x_{n-1}, x_n x_1 \rangle \subseteq C_n.$$

Proof. By a straightforward computation it is verified in [40], Lemma 4, that $\overline{\phi}$ is a homomorphism. We claim that if a word $w = w(y_1, \dots, y_{n-1})$ is reduced in the sense of the reduction system S as in Theorem 1.65, defined with respect to the deg-lex order extending $y_1 < \dots < y_{n-1}$ in the free monoid $Y = \langle y_1, \dots, y_{n-1} \rangle$, then the word $w(x_2, \dots, x_{n-1}, x_n x_1)$ is reduced with respect to the system S in the free monoid $F = \langle x_1, \dots, x_n \rangle$.

If $w \in Y$ then it is clear that if $\phi(w)$ contains a factor that is the leading term of a reduction of type (1) in Theorem 1.65, then also w contains such a factor. Assume that $\phi(w)$ contains a factor $x_j x_i$ of type (2). Then w contains a factor $y_{j-1} y_{i-1}$. Assume that $\phi(w)$ has a factor $x_i u x_i$ that is of type (4) or (5). If $i = 1$ or $i = n$ then $\phi(w)$ has a factor $x_n x_1 v x_n x_1$. If v does not contain x_2 (x_{n-1}, respectively) then $\phi^{-1}(v)$ does not contain y_1 (y_{n-2}, respectively), and we are done. If $i \neq 1, n$, and u does not contain x_{i+1} (x_{i-1}, respectively) then $\phi^{-1}(u)$ does not contain y_i (respectively, y_{i-2} if $i > 2$; and if $i = 2$ then $\phi^{-1}(u)$ does not contain y_{n-1}), as desired. Assume that $\phi(w)$ contains a factor of the form $x_n(x_1 \cdots x_i)x_j$ for $i, j \in \{1, \ldots, n\}$ such that $i + 1 < j < n - 1$. Then w contains a factor $y_{n-1}(y_1 \cdots y_{i-1})y_{j-1}$ or $y_{n-1}y_{j-1}$, and the assertion follows as well. This proves the claim.

Therefore $\overline{\phi}$ is injective. The result follows. □

2.2 An ideal chain

Our next goal is to introduce a special ideal chain in the monoid C_n that is strongly related to certain structures of matrix type. This will be essential when dealing with the structure and properties of the algebra $K[C_n]$, and consequently of every Hecke–Kiselman algebra, in the forthcoming sections. We refer to [45, Section 2] for the detailed proofs.

In view of Corollary 1.66 we identify elements of C_n with the (unique) reduced forms of words in F.

Definition 2.18. For every $i = 0, \ldots, n-2$ let us introduce

$$I_i = \{w \in C_n : C_n w C_n \cap \langle q_{n,i} \rangle = \emptyset\}.$$

We also define $I_{-1} = I_0 \cup C_n q_{n,0} C_n$.

It is clear that every I_i is an ideal in C_n, if it is non-empty. We claim that $I_{n-2} = \emptyset$. This is a consequence of the following observation.

Lemma 2.19. *Let $w \in C_n$. If $k = 1, \ldots, n$ then the reduced form of $(x_{k+1} \cdots x_n x_1 \cdots x_k)w$ is a factor of the infinite word $(x_n x_1 \cdots x_{n-1})^{\infty}$. Moreover $(x_{k+1} \cdots x_n x_1 \cdots x_k)w$ has a prefix of the form $x_{k+1} \cdots x_n x_1 \cdots x_k$.*

To prove the lemma it is enough to analyse the possible reduced forms of the word $(x_{k+1} \cdots x_n x_1 \cdots x_k)x_j$ for any index $j \in \{1, \ldots, n\}$. We omit the details.

It follows that for any word w, the reduced form of the element $(x_n x_1 \cdots x_{n-1})w$ is a factor of $(x_n x_1 \cdots x_{n-1})^{\infty}$. Consequently, we know that $(x_n x_1 \cdots x_{n-1})w \notin I_{n-2}$. Therefore we get the following corollary.

Corollary 2.20. $I_{n-2} = \emptyset$

A dual version of Lemma 2.19 also holds. In order to prove this, we introduce a natural involution of the monoid C_n that will be useful also later.

Definition 2.21. Denote by $\langle x_1, \ldots, x_n \rangle$ the free monoid generated by $x_1, \ldots, x_n$. Let $\tau : \langle x_1, \ldots, x_n \rangle \longrightarrow \langle x_1, \ldots, x_n \rangle$ be the involution such that

$$\tau(x_i) = \begin{cases} x_{n-i} & \text{for } i \neq n \\ x_n & \text{for } i = n \end{cases}.$$

It is easy to see that τ preserves the set of defining relations of C_n. Hence, it determines an involution of C_n, also denoted by τ.

As $\tau(w(x_{k+1}\cdots x_n x_1\cdots x_k)) = x_{n-k}x_{n-k+1}\cdots x_n x_1\cdots x_{n-k-1}\tau(w)$, from Lemma 2.19 we know that the reduced form of this word is a factor of $(x_n x_1\cdots x_{n-1})^\infty$ with a prefix of the form $x_{n-k}\cdots x_n x_1\cdots x_{n-k-1}$. It can be easily verified that the image of any factor of $(x_n x_1\cdots x_{n-1})^\infty$ under τ is also a factor of $(x_n x_1\cdots x_{n-1})^\infty$. Therefore, applying τ to the element $\tau(w(x_{k+1}\cdots x_n x_1\cdots x_k))$, we obtain the following dual version of the lemma.

Lemma 2.22. *Let $w \in C_n$. If $k = 0,\dots,n-1$ then the reduced form of $w(x_{k+1}\cdots x_n x_1\cdots x_k)$ is a factor of the infinite word $(x_n x_1\cdots x_{n-1})^\infty$. Moreover, $w(x_{k+1}\cdots x_n x_1\cdots x_k)$ has a suffix of the form $x_{k+1}\cdots x_n x_1\cdots x_k$.*

Let us return to properties of the ideals I_i for $i = 0,\dots,n-3$.

Lemma 2.23. $I_{i+1} \subseteq I_i$ *for* $i = 0,\dots,n-3$.

It can be verified, using the reductions from Theorem 1.65, that for every $i = 0,\dots,n-3$ and $l \geqslant 1$ there exist $u,v \in C_n$ such that $uq_{n,i}^l v \in \langle q_{n,i+1}\rangle$, that is $q_{n,i}^l \notin I_{i+1}$. Suppose that there exists $w \in I_{i+1}\setminus I_i$. Then in particular for some $u,v \in C_n$ and $l \geqslant 1$ we have $uwv = q_{n,i}^l$. Thus there are also u',v' for which $u'uwvv' \in \langle q_{n,i+1}\rangle$, that is $w \notin I_{i+1}$. The assertion follows.

Therefore we get the following chain of ideals in C_n

$$I_{n-3} \lhd \cdots \lhd I_0 \lhd I_{-1} \lhd C_n.$$

In general, it is difficult to use the definition of the ideals I_i to determine the minimal j such that a given element w of C_n satisfies $w \in I_j$. In order to prove that certain elements of C_n are contained in I_i we introduce an auxiliary chain of ideals. It turns out that the following representation introduced in [4], and generalized in [33], is useful in our setting. Let $\mathrm{Map}(\mathbb{Z}^n,\mathbb{Z}^n)$ denote the monoid of all functions $\mathbb{Z}^n \longrightarrow \mathbb{Z}^n$, under composition. Consider the homomorphism $f : C_n \longrightarrow \mathrm{Map}(\mathbb{Z}^n,\mathbb{Z}^n)$ which is defined on generators x_i of C_n as follows.

$$f(x_i)(m_1,\dots,m_n) = \begin{cases} (m_1,\dots m_{i-1}, m_{i+1}, m_{i+1},\dots,m_n) \text{ for } i \neq n \\ (m_1,\dots,m_{n-1}, m_1+1) \text{ for } i = n. \end{cases}$$

If $w \in C_n$ then the components of $f(w)(m_1,\dots,m_n)$ are polynomials in the variables $m_1,\dots,m_n$. Let $\mathrm{supp}(f(w))$ be the minimal subset N of the set $M = \{1,\dots,n\}$ such that for every $(m_1,\dots,m_n) \in \mathbb{Z}^n$ the components of $f(w)(m_1,\dots,m_n)$ are polynomials depending on the variables with indices from the set N. So $|\mathrm{supp}(f(w))|$ denotes the number of variables on which the value of $f(w)$ depends. For example, if $f(w)(m_1,\dots,m_n) = (m_1,\dots,m_{i-1},m_{i+1},m_{i+1},\dots,m_n)$, then $\mathrm{supp}(f(w)) = \{1,\dots,i-1,i+1,\dots,n\}$ and thus $|\mathrm{supp}(f(w))| = n-1$.

It can be proved using direct computations and induction on $k \geqslant 1$ that the value of $|\mathrm{supp}(f((q_{n,i})^k))|$ does not depend on k.

Lemma 2.24. *For every $k \geqslant 1$ and $i = 0,\dots,n-2$ we have* $|\mathrm{supp}(f(q_{n,i}^k))| = n-i-1$.

For every $i = -1, \dots, n-2$ consider the following set

$$Q_i = \{w \in C_n : |\operatorname{supp}(f(w))| \leqslant n - i - 2\}.$$

Then Q_i is an ideal in C_n for $i < n-2$ because for every $x, y, w \in C_n$ we have $\operatorname{supp}(f(xw)) \subseteq \operatorname{supp}(f(w))$ and $\operatorname{supp}(f(wy)) \subseteq \operatorname{supp}(f(w))$. Thus, we get the following chain of ideals

$$\emptyset = Q_{n-2} \subseteq Q_{n-3} \subseteq \cdots \subseteq Q_{-1} \subseteq C_n.$$

This chain is strongly related to the ideals I_j introduced in this section. Indeed, if we had $w \in Q_i \setminus I_i$, then from the definition of I_i for some $u, v \in C_n$ we have $uwv = q_{n,i}^l$ for some $l \geqslant 1$. On the other hand, from Lemma 2.24 we know that $q_{n,i}^l \notin Q_i$, which implies that also $w \notin Q_i$. Therefore we have proved the following.

Lemma 2.25. *For every $i = 0, \dots, n-2$ we have $Q_i \subseteq I_i$.*

Now we exploit Lemma 2.25 to show that certain families of elements are in I_i.

Lemma 2.26. *For all $n - 1 \geqslant j > i + 1 \geqslant 1$ we have*

1. $x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1} \in I_{i-1}$;
2. $w = x_j \cdots x_{i+2} x_n x_1 \cdots x_{i+1} x_{n-1} \cdots x_{j+1} \in I_i$;

where for $j = n-1$ we put $w = x_{n-1} \cdots x_{i+2} x_n x_1 \cdots x_{i+1}$.

The first part is a direct consequence of Lemmas 2.25 and 2.26. The second is also based on Lemma 2.25 and the computation of $f(w)$ for words w as in the lemma.

The following is a direct consequence of the definition of the ideals I_i and of Lemma 2.26.

Corollary 2.27. *For every $i \in \{0, \dots, n-2\}$ we have $q_{n,i}^k \in I_{i-1} \setminus I_i$. Moreover, for all $i \in \{1, \dots, n-2\}$ and $j \in \{i+1, \dots, n-1\}$ we have $x_j \cdots x_{i+1} x_n x_1 \cdots x_i x_{n-1} \cdots x_{j+1} \in I_{i-1} \setminus I_i$.*

The following will be crucial for the results of the next section, where we investigate the sets $\tilde{M}_i \cup \{\theta\}$ from Definition 2.14 after factoring out the ideal I_i.

Theorem 2.28. *Let $w \in C_n$. Then for every $i \in \{0, \dots, n-2\}$ we have $q_{n,i} w q_{n,i} \in \{q_{n,i}^k : k \geqslant 2\} \cup I_i$.*

Let us make a few comments concerning the proof. For the details we refer to [45]. The idea is to proceed by induction on the length of the (reduced) word w. The assertion is clearly true if w is the trivial word 1. For non-trivial $w = x_j w'$ for some $j \in \{1, \dots, n\}$, we investigate the reduced form of $q_{n,i} w q_{n,i}$. We use techniques as in Observation 2.6 to show that for $j \leqslant i$ the word $q_{n,i} w q_{n,i}$ can be reduced to the word of the form $q_{n,i} w' q_{n,i}$ and in this case the assertion holds by the inductive hypothesis. By similar methods we obtain that for $i < j < n$ the element $q_{n,i} w q_{n,i}$ can be rewritten to the form with a factor

$x_{j-1}\cdots x_{i+2}x_nx_1\cdots x_{i+1}x_{n-1}\cdots x_j$, which is contained in I_i in view of Lemma 2.26. If $j=n$ we analyse a suffix of w. More precisely, we write $w=w''x_nw'$ with $|w'|_n=0$ and use Lemmas 2.2 and 2.3 to obtain the possible form w'. Then, depending on this form, either we can apply the inductive hypothesis to end the proof or Lemma 2.26 to show that $q_{n,i}wq_{n,i}\in I_i$.

2.3 Structures of matrix type

Our next aim is to refine the information on the ideal chain

$$\emptyset = I_{n-2} \lhd I_{n-3} \lhd \cdots \lhd I_0 \lhd I_{-1} \lhd C_n$$

of C_n defined in the previous section. We will show that every factor I_{j-1}/I_j, for $j=0,\ldots,n-2$, is, up to finitely many elements, a semigroup of matrix type over a cyclic semigroup and also that C_n/I_{-1} is finite. Namely, the elements of the family $\tilde{M}_j$, described in Definition 2.14, with a zero element adjoined, treated as elements of the Rees factor I_{j-1}/I_j, form a semigroup of matrix type. Using certain natural involutions on C_n, we will also show that the corresponding sandwich matrices are square matrices and they are symmetric. In particular, this means that, for every j, there is a bijection between the sets A_j and B_j, which is not clear directly from the description obtained in Theorem 2.1. The details of all omitted proofs can be found in [45, Section 3].

Recall the definition of the sets $\tilde{M}_i$ and $\tilde{M}$ from Definition 2.14, describing sets arising from Theorem 2.1. For every $i=0,\ldots,n-2$ we write

$$\tilde{M}_i = \{aq_{n,i}^k b \in C_n : a\in A_i, b\in B_i, k\geqslant 1\}.$$

Recall that we identify elements of C_n with the corresponding reduced words. Hence, $\tilde{M} = \bigcup_{i=0}^{n-2}\tilde{M}_i$ consists of elements of C_n that have (in the reduced form) a factor of the form $q_{n,i}$, for some i. Moreover, from Proposition 2.15 we know that almost all elements of C_n are in this set.

Certain generalizations of the involution from Definition 2.21 that preserve the ideals I_{i-1} and sets $\tilde{M}_i$, for $i\in\{0,\ldots,n-2\}$, will be useful in this context. In particular, they can be used to establish an internal symmetry of every set $\tilde{M}_i$.

Definition 2.29. Let $\tau : C_n \longrightarrow C_n$ be the involution defined in Definition 2.21, that is $\tau(x_i)=x_{n-i}$ for $i\neq n$ and $\tau(x_n)=x_n$. Denote, as before, by $\langle x_1,\ldots,x_n\rangle$ the free monoid generated by $x_1,\ldots,x_n$. Let $\sigma : \langle x_1,\ldots,x_n\rangle \longrightarrow \langle x_1,\ldots,x_n\rangle$ be the automorphism such that $\sigma(x_i)=x_{i+1}$ for every $i=1,\ldots,n$, where we put $x_{n+1}=x_1$. It is easy to check that σ preserves the set of defining relations of C_n. Hence, σ can be viewed as an automorphism of C_n. Therefore, the map $\sigma^i\tau$ also is an involution of C_n, for $i=0,\ldots,n-1$. We will denote the involution $\sigma^{i+1}\tau$ by χ_i, for $i=0,\ldots,n-1$.

It can be easily computed that $\chi_i(q_{n,i})=q_{n,i}$. Moreover, investigating possible reduced forms of $\chi_i(q_{n,i}^m b)$ and $\chi_i(a q_{n,i}^m)$ for $m \geqslant 1$, $a \in A_i$, $b \in B_i$, exploiting the fact that $\chi_i^2 = id$ we get the following important corollary.

Corollary 2.30. *The involution χ_i satisfies: $\chi_i(q_{n,i}^k) = q_{n,i}^k$ for every $k \geqslant 1$, $\chi_i(A_i) = B_i, \chi_i(B_i) = A_i$, and in particular $\chi_i(\tilde{M}_i) = \tilde{M}_i$.*

As noticed in Lemma 2.5, if $i = 0$ or $i = n-2$, then reduced words in C_n that have a factor of the form $q_{n,i}$ must come from the infinite word $(q_{n,i})^\infty$. It is then clear that for such a word s we can find w, $z \in C_n$ such that $wsz \in \langle q_{n,i} \rangle$. The latter property remains valid for all $i \in \{0, \ldots, n-2\}$.

From Corollary 2.30 it follows in particular that χ_i determines a bijection between the sets A_i and B_i. Thus, from Proposition 2.13 we know also the cardinality of B_i, which is not obvious from the definitions of these sets.

Corollary 2.31. *The set B_i has exactly $\binom{n}{i+1}$ elements, for every $i \in \{0, \ldots, n-2\}$.*

Theorem 2.32. *Let $i \in \{0, \ldots, n-2\}$. Then:*

1) for every $a \in A_i$ there exists $w \in C_n$ such that $wa \in \langle q_{n,i} \rangle$;

2) for every $b \in B_i$ there exists $w \in C_n$ such that $bw \in \langle q_{n,i} \rangle$.

Note that in view of Corollary 2.30 it is enough to prove assertion 2). Moreover if the theorem holds for some word then it also holds for every prefix of this word. We proceed by induction on the number of blocks in $b \in B_i$, introduced just after the formulation of Theorem 2.1. We construct a specific word $v \in C_n$ such that the reduced form of bv is $q_{n,i}b'$ with $b' \in B_i$ with smaller number of blocks. Namely, if

$$b = x_n(x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}) \cdots x_n(x_1 \cdots x_{i_r} x_{n-1} \cdots x_{j_r}) x_n x_{n-1} \cdots x_{j_{r+1}},$$

then $v = x_{i_1+1} \cdots x_i x_{j_1-1} \cdots x_{i+1}$, where for $i_1 = i$, we put $w' = x_{j_1-1} \cdots x_{i+1}$, satisfies the required conditions. The necessary calculations can be found in [45]. From the inductive hypothesis we get that for some $w' \in C_n$ we have $b'w' \in \langle q_{n,i} \rangle$. It follows that $bvw' \in \langle q_{n,i} \rangle$, which completes the proof.

As a consequence, we are able to place the set $\tilde{M}_i$ in the ideal chain from Section 2.2.

Proposition 2.33. *For every $i \in \{0, \ldots, n-2\}$ we have $\tilde{M}_i \subseteq I_{i-1} \setminus I_i$.*

From Corollary 2.27 we know that $\tilde{M}_i \subseteq I_{i-1}$. Moreover from Theorem 2.32 and the definition of I_i it follows that $\tilde{M}_i \subseteq C_n \setminus I_i$.

Applying the above corollary and Proposition 2.15 we get that the sets $(I_{i-1} \setminus I_i) \setminus \tilde{M}_i$ are finite.

Corollary 2.34. *For every $i \in \{0, \ldots, n-2\}$ the set $(I_{i-1} \setminus I_i) \setminus \tilde{M}_i$ is finite.*

By Proposition 2.15 we know that $C_n \setminus \tilde{M}$ is a finite set. Moreover, Proposition 2.33 implies that for every $i = 0, \ldots, n-2$ we have $\tilde{M}_i \subseteq I_{i-1} \setminus I_i \subseteq I_{-1}$, so that also $\tilde{M} \subseteq I_{-1}$. Our next observation follows.

Corollary 2.35. *C_n/I_{-1} is a finite semigroup.*

Now let us investigate some useful properties of the automorphism σ and the involutions χ_i introduced in Definition 2.21.

Lemma 2.36. *For every $i \in \{0, \ldots, n-3\}$ we have $\sigma(I_i) = I_i$. Moreover, $\sigma(w) \in \tilde{M}_i$ for almost all $w \in \tilde{M}_i$, if $i \in \{0, \ldots, n-2\}$.*

As $\sigma^n = id$, for the first assertion it is enough to to prove that $\sigma(C_n \setminus I_i) \subseteq C_n \setminus I_i$. If $w \in C_n \setminus I_i$, that is for some $u, v \in C_n$ we have $uwv = q_{n,i}^m$, then a direct computation shows that $\sigma(uwv)$ is of the form $aq_{n,i}^{m-1}b$ for certain $a \in A_i$, $b \in B_i$. Thus we get the assertion from Theorem 2.32. The second part now follows from Proposition 2.33 and Corollaries 2.34 and 2.35.

Lemma 2.37. *For every $i \in \{1, \ldots, n-2\}$ and every non-negative integer m we have $\sigma^m\tau(I_{i-1} \setminus I_i) = I_{i-1} \setminus I_i$ and $\sigma^m\tau(I_{i-1}) = I_{i-1}$.*

First, Corollary 2.30 and the definition of the ideals I_i are applied to show that $\sigma^{i+1}\tau(I_i) = I_i$. Then, as $\sigma^n = id$, it follows that $\tau(I_i) = I_i$ for every $i \in \{1, \ldots, n-2\}$. Consequently, from Lemma 2.36 we get the assertion.

In the two extreme cases, namely for $i = 0$ and $i = n-2$, the description of $\tilde{M}_i$ is quite simple (see Lemma 2.5). In particular, $\tilde{M}_{n-2}$ coincides with the set of all factors of the word $(x_n x_1 \cdots x_{n-1})^\infty$, that contain a factor $x_n x_1 \cdots x_{n-1}$. Moreover, from Lemmas 2.19 and 2.22 it can be proved that $\tilde{M}_{n-2}$ is a two-sided ideal in C_n. More precisely, the following holds.

Corollary 2.38. *$\tilde{M}_{n-2} = C_n(x_n x_1 \cdots x_{n-1})C_n$.*

In the second extreme case, namely when $i = 0$, we have $\tilde{M}_0 \subseteq C_n q_{n,0} C_n \subseteq I_{-1}$. Moreover, equality holds modulo the ideal I_0, as proved in the following lemma.

Lemma 2.39. *$I_{-1} = \tilde{M}_0 \cup I_0$.*

Recall that I_{-1} is defined as $I_0 \cup C_n q_{n,0} C_n$. Since I_{-1} is an ideal in C_n and $\tilde{M}_0 \subseteq C_n(q_{n,0})C_n$, it is clear that $\tilde{M}_0 \cup I_0 \subseteq I_{-1}$. Note also that $q_{n,0} \in \tilde{M}_0$. To prove the opposite inclusion, it is enough to check that $\tilde{M}_0 \cup I_0$ is a two-sided ideal. Moreover, Corollary 2.30 and Lemma 2.37 imply that if $\tilde{M}_0 \cup I_0$ is a one-sided ideal, then it has to be also a two-sided ideal. Explicit computations of reduced forms of wx_j for any $w \in \tilde{M}_0$ and $j \in \{1, \ldots, n\}$ are then used to show that $wx_j \in \tilde{M}_0 \cup I_0$.

We are now in a position to improve slightly the assertion of Theorem 2.32.

Corollary 2.40. *Let $i \in \{0, \ldots, n-2\}$. Then*

1. *for every $a \in A_i$ there exists $w \in \tilde{M}_i$ such that $wa \in \langle q_{n,i} \rangle$,*

2. *for every $b \in B_i$ there exists $v \in \tilde{M}_i$ such that $bv \in \langle q_{n,i} \rangle$.*

Consequently, $x \mapsto wx$ is an injective map $a\langle q_{n,i}\rangle B_i \longrightarrow \langle q_{n,i}\rangle B_i$, and $x \mapsto xv$ is an injective map $A_i\langle q_{n,i}\rangle b \longrightarrow A_i\langle q_{n,i}\rangle$.

It is sufficient to prove the first part, then the second follows from Corollary 2.30. Consider any $u \in C_n$ such that $ua = q_{n,i}^m$ for some $m \geqslant 1$. Such a word exists according to Corollary 2.40. The idea is to show that for some $k \geqslant 1$ the element $q_{n,i}^k u$ is in $\tilde{M}_i$. Words $q_{n,i}^k u$ are pairwise different, for $k \geqslant 1$ (as $q_{n,i}^k ua = q_{n,i}^{k+m}$ from the choice of u). From Corollary 2.15 it follows that for some $k \geqslant 1$ we have $q_{n,i}^k u \in \tilde{M}$. Moreover it can be checked that $q_{n,i}^k u \in I_{i-1} \setminus I_i$ and thus $w = q_{n,i}^k u \in \tilde{M} \cap (I_{i-1} \setminus I_i) = \tilde{M}_i$ is such that $wa \in \langle q_{n,i} \rangle$.

Now let us define semigroups M_i of matrix type, with properties described in the beginning of this section. We know from Theorem 2.28 that if $u = aq_{n,i}^k b, w = a'q_{n,i}^{k'}b' \in \tilde{M}_i$, then either $uw = aq_{n,i}^m b' \in \tilde{M}_i$ for some $m \geqslant 2$ or $uw \in I_i$. In particular the result implies that the following semigroups are well-defined.

Definition 2.41. Let $i \in \{0, \ldots, n-2\}$. Consider the set $M_i = \tilde{M}_i \cup \{\theta\}$ with operation defined, for any $u = aq_{n,i}^k b, w = a'q_{n,i}^{k'}b' \in \tilde{M}_i$, by

$$uw = \begin{cases} aq_{n,i}^k ba'q_{n,i}^{k'}b' & \text{if } q_{n,i}ba'q_{n,i} \in \langle q_{n,i} \rangle \\ \theta & \text{if } q_{n,i}ba'q_{n,i} \in I_i \end{cases}$$

and $w\theta = \theta w = \theta$ for every $w \in M_i$. Then the definition is correct and M_i is a semigroup under this operation.

These semigroups can be interpreted as Rees factor semigroups. Namely, for $i \leqslant n-3$, I_i is an ideal of C_n, and we may consider the factor semigroup C_n/I_i. In other words, C_n/I_i is the semigroup $(C_n \setminus I_i) \cup \{\theta\}$ with zero θ and with operation

$$s \cdot t = \begin{cases} st & \text{if } st \notin I_i \\ \theta & \text{if } st \in I_i. \end{cases}$$

While $I_{n-2} = \emptyset$, for every subsemigroup J of C_n we define $J/I_{n-2} = J^0$; the semigroup J with zero adjoined. Notice that $J_i = \tilde{M}_i \cup I_i$ is a subsemigroup of I_{i-1} by Theorem 2.28 and Proposition 2.33. Thus, our definition yields $M_i = J_{i-1}/I_i \subseteq C_n/I_i$.

In the extreme cases, from Lemma 2.39 we know that $I_{-1}/I_0 = M_0$ and Lemma 2.19 implies that $\tilde{M}_{n-2} = M_{n-2} \setminus \{\theta\}$ is an ideal in C_n.

Corollary 2.42. *M_i is a semigroup of matrix type. Namely, $M_i \cong \mathcal{M}^0(Q_i, A_i, B_i; P_i)$, where P_i is a matrix of size $B_i \times A_i$ with coefficients in $\langle q_{n,i} \rangle \cup \{\theta\}$ and Q_i is an infinite cyclic semigroup generated by $q_{n,i}$.*

The sandwich matrix $P_i = (p_{ba})$ is defined as follows

$$p_{ba} = \begin{cases} (q_{n,i})^{\alpha-2} & \text{if } q_{n,i}baq_{n,i} = q_{n,i}^{\alpha} \in \langle q_{n,i} \rangle \\ \theta & \text{if } q_{n,i}baq_{n,i} \in I_i. \end{cases} \tag{2.3.1}$$

Then it can be verified that the map $\phi : M_i \longrightarrow \mathcal{M}^0(Q_i, A_i, B_i; P_i)$, given by the formula $\phi(aq_{n,i}^k b) = (q_{n,i}^k; a, b)$ and $\phi(\theta) = \theta$, is indeed an isomorphism of the semigroups M_i and $\mathcal{M}^0(Q_i, A_i, B_i; P_i)$.

Remark 2.43. Assume that $q_{n,i}baq_{n,i} = q_{n,i}^{\alpha}$ for some $a \in A_i, b \in B_i$. Then

$$q_{n,i}^{\alpha} = \chi_i(q_{n,i}^{\alpha}) = \chi_i(q_{n,i})\chi_i(ba)\chi_i(q_{n,i}) = q_{n,i}\chi_i(a)\chi_i(b)q_{n,i}.$$

By Corollary 2.30, χ_i determines a bijection between the sets A_i and B_i. Hence, from the definition of P_i in the formula (2.3.1) it follows that the matrix P_i is symmetric, if the ordering of the elements of the set A_i corresponds to the ordering of their images under χ_i. In particular such an ordering is chosen in all lemmas in Chapter 8.

The main results of this section can be now summarized as follows.

Theorem 2.44. *C_n has a chain of ideals*

$$\emptyset = I_{n-2} \lhd I_{n-3} \lhd \cdots \lhd I_0 \lhd I_{-1} \lhd C_n$$

with the following properties

1. *for $i = 0, \ldots, n-2$ there exist semigroups of matrix type $M_i = \mathcal{M}^0(Q_i, A_i, B_i; P_i)$, such that $M_i \subseteq I_{i-1}/I_i$, where Q_i is the cyclic semigroup generated by $q_{n,i}$, P_i is a square symmetric matrix of size $B_i \times A_i$ with $|A_i| = |B_i| = \binom{n}{i+1}$ and with coefficients in $\langle q_{n,i} \rangle \cup \{\theta\}$;*

2. *for $i = 1, \ldots, n-2$ the sets $(I_{i-1}/I_i) \setminus M_i$ are finite;*

3. *$I_{-1}/I_0 = M_0$;*

4. *$\tilde{M}_{n-2} = M_{n-2} \setminus \{\theta\} \lhd C_n$;*

5. *C_n/I_{-1} is a finite semigroup.*

We postpone the illustration of the results of this section to Chapter 8. Note that the main idea of our approach is to use the properties of the sandwich matrices P_i to investigate the structure of semigroups of matrix type M_i and their algebras. This extends the classical approach used in the case of finite semigroups and their algebras, see [9]. The difficulty is that in general it seems to be extremely hard to calculate the coefficients of these sandwich matrices. Despite this, we are still able to derive a number of significant general results without knowing exactly the coefficients of P_i.

2.4 Properties of semigroups of matrix type inside C_n

Our approach to the study of the structure of the Hecke–Kiselman monoid C_n and the monoid algebra $K[C_n]$, for any field K, and $n \geqslant 3$ is to derive the results from certain properties of the quotients arising from the ideal chain in Theorem 2.44. Thus, in the present section we investigate the properties of semigroups of matrix type M_i and their semigroup algebras $K[M_i]$ over any field K.

Lemma 2.45. *M_i is a right ideal in C_n/I_i for every $i = 0, 1, \ldots, n-2$.*

Proof. Let $aq_{n,i}^k b \in \tilde{M}_i$ and take any generator $x_r \in C_n$. Assume that the element $aq_{n,i}^k bx_r$ is not in $\tilde{M}_i$. We claim that then $aq_{n,i}^k bx_r \in I_i$. Let b' be the reduced form of bx_r. If $b' = x_j\bar{b}$ for some word $\bar{b}$, where $j \leqslant i+1$, then using reduction (4) from Theorem 1.65 we get that $aq_{n,i}^k b'$ can be reduced to $aq_{n,i}^k \bar{b}$. Therefore we can assume that a prefix of b' is equal to x_j, for some $j > i+1$. If $j < n$, then it can be calculated that $aq_{n,i}^k bx_r$ can be rewritten as a word with a factor of the form $x_{j-1} \cdots x_{i+2}x_n x_1 \cdots x_{i+1}x_{n-1} \cdots x_j$ and this element is in I_i by Lemma 2.26. Let us now consider the case when x_n is a prefix of b'. As we assume that $aq_{n,i}^k b' \notin \tilde{M}_i$, this word can be rewritten in C_n as an element without the factor $q_{n,i}$. From Theorem 1.65 it is easy to see that to obtain a word without such a factor one has to use a reduction of type (5). Therefore $aq_{n,i}^k b'$ can be written as a word with a prefix of the form $aq_{n,i}^k x_n v x_j$, where $|x_n v|_j = |x_n v|_{j+1} = 0$. Moreover, for $j \leqslant i$ or $j = n-1$ the generator x_{j+1} occurs in $q_{n,i}x_n$ after x_j, thus the reduction of x_j of type (5) is not possible in this case. Therefore $n-1 > j \geqslant i+1$. It follows from Lemma 2.3 that such a prefix is of the form $aq_{n,i}^k x_n x_1 \cdots x_j$. Therefore this element has a factor $x_n x_1 \cdots x_i x_{n-1} \cdots x_{i+1} x_n x_1 \cdots x_j$ for some $n-1 > j \geqslant i+1$. It can be checked (using the reductions from Theorem 1.65) that the latter word can be rewritten as an element with the factor $x_{n-1} \cdots x_{j+1} x_n x_1 \cdots x_j$, which is in $I_{j-1} \subseteq I_i$, by Lemma 2.26. The assertion follows. $\square$

From Lemma 2.37 and Corollary 2.30 we know that, for every $i = 0, \ldots, n-2$, the semigroup C_n/I_i is endowed with a natural involution χ_i which leaves M_i invariant. Thus from the above lemma we get the following corollary.

Corollary 2.46. *M_i is a two-sided ideal of C_n/I_i for every $i = 0, 1, \ldots, n-2$.*

The main aim of this section is to prove that all algebras $K[M_i]$ are prime. We start with the extreme cases, namely $K[M_{n-2}]$ and $K[M_0]$.

Remark 2.47. For every $n \geqslant 3$ the algebras of matrix type $K[M_0]$ and $K[M_{n-2}]$ defined for $K[C_n]$ are prime.

Proof. Write $R = K[M_0]$ and suppose that $x \in R$ is a non-zero element such that $xRx = 0$. Then x can be uniquely written in the form

$$x = \sum_{i \in I} \sigma_i u_i q_{n,0}^{n_i} v_i,$$

for some finite set I, where $\sigma_i \neq 0$ are elements of the field K, $n_i \geqslant 0$, $u_i \in A_0$, $v_i \in B_0$. In $F = \langle x_1, \ldots, x_n \rangle$ consider the deg-lex order induced by $x_1 < \cdots < x_n$. Let $u_0 q_{n,0}^m v_0$ be the leading term in the support of x. We may assume that its coefficient is equal to 1. From Theorem 2.1 it follows that u_0 and v_0 must be a suffix, and a prefix respectively, of $q_{n,0}$. Hence, there exist words p, q such that $pu_0 = v_0 q = q_{n,0}$ holds in the free monoid F. Then for all elements $w \neq u_0 q_{n,0}^m v_0$ in the support of x we have $q_{n,0}^{m+2} = pu_0 q_{n,0}^m v_0 q > pwq$ in F. If $xRx = 0$, then also $pxqRpxq = 0$. In particular, $(pxq)^2 = 0$. On the other hand in $K[M_0]$

$$pxq = q_{n,0}^{m+2} + \sum_{y_i < q_{n,0}^{m+2}} \rho_i y_i,$$

where $i \in I \setminus \{0\}$, $\rho_i \in K$, y_i is the reduced form of the word $pu_i q_{n,0}^{n_i} v_i q$. In particular, for every y_i we have $y_i < q_{n,0}^{m+2}$. Since the reduced form of $q_{n,0}^{m+2} q_{n,0}^{m+2}$ is $q_{n,0}^{2m+4}$, for every pair (y_i, y_j) such that $(y_i, y_j) \neq (q_{n,0}^{m+2}, q_{n,0}^{m+2})$ the reduced form y_{ij} of $y_i y_j$ satisfies $y_{ij} < q_{n,0}^{2m+4}$. In particular, the leading term of $(pxq)^2$ is equal to $q_{n,0}^{2m+4}$ and it is non-zero. This contradiction shows that $K[M_0]$ is semiprime. This implies that the sandwich matrix P_0 is not a zero divisor in the corresponding matrix ring $M_n(K[\langle q_{n,0} \rangle])$, see Theorem 2.44. Since $K[\langle q_{n,0} \rangle]$ is a domain, it follows from Section 1.3 that $K_0[M_0]$ is prime. A similar argument can be applied for $K[M_{n-2}]$. □

It turns out that the the proof in general case is more complicated. To show that $K[M_i]$ are prime for all $i = 0, \ldots, n-2$ we will use the following observation.

Proposition 2.48. *Assume that $t \in \{1, \ldots, n-3\}$ and $\alpha \in K_0[M_t]$ is such that $\alpha x_i = 0$ in $K_0[M_t]$ for every $i \in \{1, \ldots, n\}$. Assume also that every $w \in \operatorname{supp}(\alpha)$ is of the form $q_{n,t}^k b$, where $k \geqslant 1$ and $b \in B_t$. Then $\alpha = 0$.*

In order to prove it, we need some preparatory technical lemmas. We assume that $t \in \{1, \ldots, n-3\}$. Moreover, we will suppose that a non-zero $\alpha \in K_0[M_t]$ is given that satisfies the hypotheses of the proposition. The aim is to come to a contradiction.

Roughly speaking, the first lemma describes the reduced form of any word of type wx_r for w in block form (see the convention introduced after Theorem 2.1), namely

$$q_{n,t} x_n (x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}) \cdots x_n (x_1 \cdots x_{i_k} x_{n-1} \cdots x_{j_k}),$$

where $i_k < i_{k-1} < \cdots < i_1 < t+1 < j_1 < \cdots < j_k$ and x_r is such that $n-1 \geqslant r \geqslant j_k - 1$ or $r \leqslant i_k + 1$. This means that x_r cannot be pushed to the left by using only reductions (2) or (3) in such a way that $wx_r = q_{n,t} x_n (x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}) \cdots x_n x_r (x_1 \cdots x_{i_k} x_{n-1} \cdots x_{j_k})$ in C_n.

Lemma 2.49. *Let $t \in \{1, \ldots, n-3\}$. Consider the word w of the form*

$$w = q_{n,t} x_n (x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}) \cdots x_n (x_1 \cdots x_{i_k} x_{n-1} \cdots x_{j_k}),$$

where $i_k < i_{k-1} < \cdots < i_1 < t+1 < j_1 < \cdots < j_k$. The word $q_{n,t}$ is also assumed to be of the above type for $k = 0$. Moreover, in every w we use the convention that $i_0 = t, j_0 = t+1$. Let

$r \geqslant t$ be such that $n-1 \geqslant r \geqslant j_k - 1$ or $r \leqslant i_k + 1$ (so in the latter case $t \leqslant r \leqslant i_k + 1 \leqslant t+1$). Then the following holds:

1. *if $n-1 \geqslant r > j_k$, then $wx_r \in I_t$;*
2. *if $r = j_k$, then $wx_r = w$ in C_n;*
3. *if $j_k = r+1$, then either $wx_r = w$ in C_n or the word wx_r is reduced;*
4. *if $j_k > r+1$, $r = t$, $i_k = t-1$, then either (for $k=1$) the word wx_r has the reduced form $q_{n,t}x_nx_1 \cdots x_t x_{n-1} \cdots x_{j_k}$ or (for $k \geqslant 2$) $wx_r \in I_t$;*
5. *if $j_k > r+1$, $r = t$, $i_k = t$, then $wx_r = w$ in C_n;*
6. *if $j_k > r+1$, $r = t+1$, $i_k = t$, then $wx_r \in I_t$.*

Proof. Parts 2. and 5. are clear.

To prove part 1., we proceed by induction on k (the number of blocks in the word w). Let $n-1 \geqslant r > t+1$. If $k=0$ then $w = q_{n,t}$ and

$$\begin{aligned} wx_r &\xrightarrow{(5r)} x_nx_1 \cdots x_t x_{n-1} \cdots x_{r+1}x_{r-1} \cdots x_{t+1}x_r \xrightarrow{(2)} x_nx_1 \cdots x_t(x_{r-1} \cdots x_{t+1})(x_{n-1} \cdots x_r) \\ &\xrightarrow{(3)} (x_{r-1} \cdots x_{t+2})(x_nx_1 \cdots x_{t+1})(x_{n-1} \cdots x_r). \end{aligned}$$

From Lemma 2.26 we obtain $(x_{r-1} \cdots x_{t+2})(x_nx_1 \cdots x_{t+1})(x_{n-1} \cdots x_r) \in I_t$, as desired.
So, assume that the assertion holds for every $m < k$, where $k \geqslant 1$. Consider

$$wx_r = q_{n,t}x_n(x_1 \cdots x_{i_1}x_{n-1} \cdots x_{j_1}) \cdots x_n(x_1 \cdots x_{i_k}x_{n-1} \cdots x_{j_k})x_r,$$

for $r > j_k$. Then we have

$$\begin{aligned} x_n(x_1 \cdots x_{i_k}x_{n-1} \cdots x_{j_k})x_r &\xrightarrow{(5r)} x_nx_1 \cdots x_{i_k}x_{n-1} \cdots x_{r+1}x_{r-1} \cdots x_{j_k}x_r \\ &\xrightarrow{(2)} x_nx_1 \cdots x_{i_k}x_{r-1} \cdots x_{j_k}x_{n-1} \cdots x_r \end{aligned}$$

From the assumptions we know that $j_k > i_k + 1$ and $r-1 < n-1$, so the following reduction holds:

$$x_nx_1 \cdots x_{i_k}x_{r-1} \cdots x_{j_k}x_{n-1} \cdots x_r \xrightarrow{(3)} x_{r-1} \cdots x_{j_k}x_nx_1 \cdots x_{i_k}x_{n-1} \cdots x_r.$$

By the assumptions $j_k < r \leqslant n-1$ and $j_{k-1} < j_k$, so $j_{k-1} < r-1 \leqslant n-1$. I_t is an ideal in C_n, so from the above calculation and the induction hypothesis for the element

$$v = q_{n,t}x_n(x_1 \cdots x_{i_1}x_{n-1} \cdots x_{j_1}) \cdots x_n(x_1 \cdots x_{i_{k-1}}x_{n-1} \cdots x_{j_{k-1}})$$

(a word with $k-1$ blocks) the following holds in C_n

$$wx_r = vx_{r-1} \cdots x_{j_k}x_nx_1 \cdots x_{i_k}x_{n-1} \cdots x_r \in I_t.$$

Hence part 1. follows.

To prove part 3., assume that $j_k = r+1$. Recall that $i_0 = t$ and $j_0 = t+1$. It follows that for $k=0$ we have $r=t$. In this case $q_{n,t}x_r \xrightarrow{(4t)} w$. Hence, we can assume that $k \geqslant 1$. Then

$$wx_r = q_{n,t}x_n(x_1 \cdots x_{i_1}x_{n-1} \cdots x_{j_1}) \cdots x_n(x_1 \cdots x_{i_k}x_{n-1} \cdots x_{j_k})x_{j_k-1}.$$

If $j_{k-1} < j_k - 1$, then we see that the above word is reduced.
Hence, assume now that $j_{k-1} = j_k - 1$. Then the word wx_r has a factor of the form $x_{j_k-1}x_nx_1 \cdots x_{i_k}x_{n-1} \cdots x_{j_k}x_{j_k-1}$. If $i_k + 1 < j_k - 1$, then

$$x_{j_k-1}x_nx_1 \cdots x_{i_k}x_{n-1} \cdots x_{j_k}x_{j_k-1} \xrightarrow{(4(j_k-1))} x_{j_{k-1}}x_nx_1 \cdots x_{i_k}x_{n-1} \cdots x_{j_k}.$$

It follows that $wx_r = w$, which ends the proof of part 3 in the case $i_k + 1 < j_k - 1$.
Finally, if $i_k + 1 \geqslant j_k - 1$, then $i_k < j_{k-1} = j_k - 1 \leqslant i_k + 1$, so that $j_{k-1} = i_k + 1$. Hence $i_k \leqslant i_{k-1} < t+1 \leqslant j_{k-1}$ implies that $i_{k-1} = i_k = t$, $j_k = t+2$. It follows that wx_r is reduced. This proves part 3.

In the proof of the remaining assertions (parts 4. and 6.) we can assume that $k \geqslant 1$, because for $k=0$ it is impossible to have $t+1 = j_k > r+1$ and $r \in \{t, t+1\}$.

To prove part 4., assume that $j_k > r+1$, $r = t$, $i_k = t-1$. Then from the definition of w we obtain that $k \in \{1,2\}$ and either $w = q_{n,t}x_nx_1 \cdots x_tx_{n-1} \cdots x_{j_1}x_nx_1 \cdots x_{t-1}x_{n-1} \cdots x_{j_2}$, where $j_2 > j_1 > t+1$ or $w = q_{n,t}x_nx_1 \cdots x_{t-1}x_{n-1} \cdots x_{j_1}$. In the first case

$$\begin{aligned} wx_t &\xrightarrow{(2)} q_{n,t}x_nx_1 \cdots x_tx_{n-1} \cdots x_{j_1}x_nx_1 \cdots x_tx_{n-1} \cdots x_{j_2} \\ &\xrightarrow{(5t)} \cdots \xrightarrow{(51)}\xrightarrow{(5n)} q_{n,t}x_{n-1} \cdots x_{j_1}x_nx_1 \cdots x_tx_{n-1} \cdots x_{j_2}. \end{aligned}$$

From part 1. applied to $q_{n,t}$ and $r = n-1$ we get $wx_r \in I_t$.
In the second case $wx_t \xrightarrow{(2)} q_{n,t}x_nx_1 \cdots x_tx_{n-1} \cdots x_{j_1}$ and the last word is reduced.

To prove part 6., assume that $j_k > r+1$, $r = t+1$, $i_k = t$. Then from the definition of w it follows that $k=1$ and

$$\begin{aligned} wx_{t+1} &\xrightarrow{(2)} q_{n,t}x_nx_1 \cdots x_{t+1}x_{n-1} \cdots x_{j_1} \xrightarrow{(5(t+1))} x_nx_1 \cdots x_tx_{n-1} \cdots x_{t+2}x_nx_1 \cdots x_{t+1}x_{n-1} \cdots x_{j_1} \\ &\xrightarrow{(5t)} \cdots \xrightarrow{(51)}\xrightarrow{(5n)} x_{n-1} \cdots x_{t+2}x_nx_1 \cdots x_{t+1}x_{n-1} \cdots x_{j_1} \in I_t \end{aligned}$$

by Lemma 2.26. Hence the assertion follows. □

We continue under the assumptions of Proposition 2.48. By Theorem 2.1, every $w \in \mathrm{supp}(\alpha)$ must satisfy one of the following conditions:

(i) $x_nx_1 \cdots x_{i_{s-1}}x_{n-1} \cdots x_{j_{s-1}}x_nx_1 \cdots x_{i_s}x_{n-1} \cdots x_{j_s} \in \mathrm{suff}(w)$, where $i_s < i_{s-1} < t+1 < j_{s-1} < j_s \leqslant n-1$, or $i_s = i_{s-1} = t$ and $j_{s-1} = t+1 < j_s$,

(ii) $x_nx_1 \cdots x_{i_{s-1}}x_{n-1} \cdots x_{j_{s-1}}x_nx_1 \cdots x_{i_s}x_{n-1} \cdots x_{j_s}x_nx_1 \cdots x_{i_{s+1}} \in \mathrm{suff}(w)$, where

$i_{s+1} < i_s < i_{s-1} < t+1 < j_{s-1} < j_s \leqslant n-1$,
or $i_s = i_{s-1} = t > i_{s+1}$ and $j_{s-1} = t+1 < j_s$;
or $q_{n,t}x_n x_1 \cdots x_{i_{s+1}} \in \mathrm{suff}(w)$ with $i_{s+1} \leqslant t$,

(iii) $x_n x_1 \cdots x_{i_{s-1}} x_{n-1} \cdots x_{j_{s-1}} x_n x_{n-1} \cdots x_{j_s} \in \mathrm{suff}(w)$, where $2 \leqslant i_{s-1} < t+1 < j_{s-1} < j_s \leqslant n$,

(iv) $x_n x_1 x_{n-1} \cdots x_{j_{s-1}} x_n x_{n-1} \cdots x_{j_s} \in \mathrm{suff}(w)$, where $2 \leqslant t+1 < j_{s-1} < j_s \leqslant n$,

(v) $b = x_n x_{n-1} \cdots x_{j_s}$, where $t+1 < j_s \leqslant n$,

(vi) $b = 1$, i.e. $w = q_{n,t}^k$.

Hence, we can write $\alpha = \alpha_{(i)} + \alpha_{(ii)} + \alpha_{(iii)} + \alpha_{(iv)} + \alpha_{(v)} + \alpha_{(vi)}$, where $\mathrm{supp}(\alpha_k)$ consists of all words of the form (k) listed above, which are in the support of the element α. We will prove that for every $k \in \{(i), \ldots, (vi)\}$ the element $\alpha_{(k)}$ is zero, which will contradict the supposition that $\alpha \neq 0$.
First, we prove the following result concerning αx_1.

Lemma 2.50. *Let α be as described above. Then*

1. $\alpha_{(vi)} = 0$;

2. $\alpha_{(i)} = \alpha_{(i),i_s=1}$;

3. $\alpha_{(ii)} = \alpha_{(ii),i_{s+1}=1}$,

where $\alpha_{(i)} = \alpha_{(i),i_s=1} + \alpha_{(i),i_s>1}$ and $\mathrm{supp}(\alpha_{(i),i_s=1})$ consists of all words from the support of $\alpha_{(i)}$ with $i_s = 1$, while $\mathrm{supp}(\alpha_{(i),i_s>1})$ does not contain such words; similarly $\alpha_{(ii),i_{s+1}=1}$ involves all words from the support of $\alpha_{(ii)}$ with $i_{s+1} = 1$ (see the description of $\alpha_{(i)}$, $\alpha_{(ii)}$).

Proof. We know that $\alpha x_1 = 0$ in $K_0[M_t]$. We calculate the reduced forms of wx_1 for all $w \in \mathrm{supp}(\alpha_k)$, for $k \in \{(i), \ldots, (vi)\}$. It will be more convenient to consider certain suffixes of the given word w.

- $x_n x_1 \cdots x_{i_s} x_{n-1} \cdots x_{j_s} x_1 \xrightarrow{(41)} x_n x_1 \cdots x_{i_s} x_{n-1} \cdots x_{j_s}$, so $\alpha_{(i)} x_1 = \alpha_{(i)}$;

- $x_n x_1 \cdots x_{i_{s+1}} x_1 \xrightarrow{(41)} x_n x_1 \cdots x_{i_{s+1}}$, whence $\alpha_{(ii)} x_1 = \alpha_{(ii)}$;

- $x_n x_1 \cdots x_{i_{s-1}} x_{n-1} \cdots x_{j_{s-1}} x_n x_{n-1} \cdots x_{j_s} x_1$
$$\xrightarrow{(*)} \begin{cases} x_n x_1 \cdots x_{i_{s-1}} x_{n-1} \cdots x_{j_{s-1}} x_n x_1 \text{ for } j_s = n \\ x_n x_1 \cdots x_{i_{s-1}} x_{n-1} \cdots x_{j_{s-1}} x_n x_1 x_{n-1} \cdots x_{j_s} \text{ for } j_s < n, \end{cases}$$
where $(*)$ denotes equality in the first case and reduction (2) in the second case. We see that in the first case ($j_s = n$) the obtained word is reduced of type (ii) with $i_{s+1} = 1$. In the second case ($j_s < n$) the word is reduced of type (i) with $i_s = 1$.

- $x_n x_1 x_{n-1} \cdots x_{j_{s-1}} x_n x_{n-1} \cdots x_{j_s} x_1 \xrightarrow{(2)} x_n x_1 x_{n-1} \cdots x_{j_{s-1}} x_n x_1 x_{n-1} \cdots x_{j_s}$
$\xrightarrow{(51),(5n)} x_{n-1} \cdots x_{j_{s-1}} x_n x_1 x_{n-1} \cdots x_{j_s}$.

  In this case the obtained form of the element wx_1 has a factor of the form

$$q_{n,t} x_n (x_1 \cdots x_{i_1} x_{n-1} \cdots x_{j_1}) \cdots x_n (x_1 \cdots x_{i_k} x_{n-1} \cdots x_{j_k}) x_{n-1},$$

  where $k = s - 2$, $j_k < n - 1$ (notice that $s \geqslant 2$). Assertion 1. of Lemma 2.49 implies that $wx_1 = 0$ in $K_0[M_t]$ for every $w \in \operatorname{supp}(\alpha_{(iv)})$.

- $x_n x_{n-1} \cdots x_{j_s} x_1 \xrightarrow{(*)} \begin{cases} x_n x_{n-1} \cdots x_{j_s} x_1 & \text{for } j_s = n \\ x_n x_1 x_{n-1} \cdots x_{j_s} & \text{for } j_s < n, \end{cases}$
  where $(*)$ denotes equality in the first case and reduction (2) in the second case. We see that in the first case the word wx_1 is of the reduced form (ii) with $i_{s+1} = 1$, whereas in the second case we obtain a reduced word of type (i) with $i_s = 1$.

- $q_{n,t} x_1 \xrightarrow{(41)} q_{n,t}$, so $\alpha_{(vi)} x_1 = \alpha_{(vi)}$.

From the above calculations we see that in $K_0[M_t]$

$$0 = (\alpha_{(ii)} + \alpha_{(iii),j_s=n} x_1 + \alpha_{(v),j_s=n} x_1) + (\alpha_{(i)} + \alpha_{(iii),j_s<n} x_1 + \alpha_{(v),j_s<n} x_1) + \alpha_{(vi)}.$$

It is clear that the terms from the last component $\alpha_{(vi)}$ are the only terms of type (vi) in the above sum, so $\alpha_{(vi)} = 0$. Moreover, reduced forms of elements from $\alpha_{(i)} + \alpha_{(iii),j_s=n} x_1 + \alpha_{(v),j_s=n} x_1$ are of type (i), whereas reduced forms of words in the sum $\alpha_{(ii)} + \alpha_{(iii),j_s<n} x_1 + \alpha_{(v),j_s<n} x_1$ are of type (ii). It follows that these sums are 0 in $K_0[M_t]$. It is not difficult to see that every word from $\operatorname{supp}(\alpha_{(iii),j_s=n} x_1)$ and $\operatorname{supp}(\alpha_{(v),j_s=n} x_1)$ has a reduced form ending with $x_n x_1$, so $\alpha_{(ii)} = \alpha_{(ii),i_{s+1}=1}$. Similarly, every (reduced) word from $\operatorname{supp}(\alpha_{(iii),j_s<n} x_1)$ and $\operatorname{supp}(\alpha_{(v),j_s<n} x_1)$ has a suffix of the form $x_n x_1 x_{n-1} \cdots x_j$ for some j, so $\alpha_{(i)} = \alpha_{(i),i_s=1}$. The assertion follows. □

It follows that $\operatorname{supp}(\alpha) = \operatorname{supp}(\alpha_{(i),i_s=1}) \cup \operatorname{supp}(\alpha_{(ii),i_{s+1}=1}) \cup \operatorname{supp}(\alpha_{(iii)}) \cup \operatorname{supp}(\alpha_{(iv)}) \cup \operatorname{supp}(\alpha_{(v)})$. Let $m = \min\{j_s : w \in \operatorname{supp}(\alpha)\}$, with j_s defined for every word w as in cases (i)-(vi) listed before Lemma 2.50. Then $n \geqslant m \geqslant t + 1 \geqslant 2$. By our assumption on α, also $\alpha x_{m-1} = 0$. We calculate the reduced form of words wx_{m-1}, where $w \in \operatorname{supp}(\alpha)$. By $s_{(k)}$ we mean an appropriately chosen suffix of the word from the support of $\alpha_{(k)}$. We consider the following two cases.
Case I. Assume that $m = j_s$.

1. First, suppose that $j_{s-1} = j_s - 1$. Then

   (a) $s_{(i)} x_{m-1} = x_n x_1 \cdots x_{i_{s-1}} x_{n-1} \cdots x_{j_s - 1} x_n x_1 \cdots x_{i_s} x_{n-1} \cdots x_{j_s} x_{j_s - 1}$
   If $i_s + 1 < j_s - 1$, then $s_{(i)} x_{m-1} \xrightarrow{(4j_s)} s_{(i)}$.

Otherwise we have $i_s + 1 = j_s - 1$, which implies that $j_s - 2 \leqslant i_{s-1} < j_s - 1$, so it follows easily that $i_{s-1} = t, j_{s-1} = t+1$. In this case $s_{(i)}x_{m-1} = q_{n,t}x_nx_1\cdots x_tx_{n-1}\cdots x_{t+1} = q_{n,t}^2$. Consequently, $s_{(i)}x_{m-1}$ is of the form (vi) in this case.

(b) $s_{(ii)}x_{m-1} = x_nx_1\cdots x_{i_{s-1}}x_{n-1}\cdots x_{j_{s-1}}x_nx_1\cdots x_{i_s}x_{n-1}\cdots x_{j_s}x_nx_1x_{j_s-1}$.
Suppose that $j_{s-1}+1 = j_s \leqslant 3$. Since $j_{s-1} \geqslant t+1$, it follows that $t=1$ and $s_{(ii)}$ must be the word $x_nx_1x_{n-1}\cdots x_2x_nx_1x_{n-1}\cdots x_3x_nx_1$, which is not reduced. Therefore we can assume that $n-1 \geqslant j_s > 3$ and

$$s_{(ii)}x_{m-1} \xrightarrow{(3)} x_nx_1\cdots x_{i_{s-1}}x_{n-1}\cdots x_{j_{s-1}}x_nx_1\cdots x_{i_s}x_{n-1}\cdots x_{j_s}x_{j_s-1}x_nx_1.$$

It is clear that the reduced word $x_nx_1\cdots x_{i_{s-1}}x_{n-1}\cdots x_{j_{s-1}}x_nx_1\cdots x_{i_s}x_{n-1}\cdots x_{j_s}$ is of the form (ii). From the previous case we obtain

$$s_{(ii)}x_{m-1} = \begin{cases} s_{(ii)} \text{ if } i_s+1 < j_s-1 \\ q_{n,t}^2x_nx_1 \text{ otherwise .}\end{cases}$$

Thus $s_{(ii)}x_{m-1}$ has the reduced form either equal to $s_{(ii)}$ or ending with $x_jx_nx_1$ with $j \leqslant m$.

(c) $s_{(iii),(iv),(v)}x_{m-1} = x_{j_s-1}x_nx_{n-1}\cdots x_{j_s}x_{j_s-1} \xrightarrow{(4(j_s-1))} x_{j_s-1}x_nx_{n-1}\cdots x_{j_s}$. It follows that for every w of the form (iii), (iv) or (v) we have $wx_{m-1} = w$.

2. Secondly, assume that $j_{s-1} < j_s - 1$. Then

   (a) for every w of the form (i) wx_{m-1} is reduced;

   (b) since $j_s - 1 > j_{s-1} \geqslant 2$, then

$$\begin{aligned} s_{(ii),i_{s+1}=1}x_{m-1} &= x_nx_1\cdots x_{i_{s-1}}x_{n-1}\cdots x_{j_{s-1}}x_nx_1\cdots x_{i_s}x_{n-1}\cdots x_{j_s}x_nx_1x_{j_s-1} \\ &\xrightarrow{(3)} x_nx_1\cdots x_{i_{s-1}}x_{n-1}\cdots x_{j_{s-1}}x_nx_1\cdots x_{i_s}x_{n-1}\cdots x_{j_s}x_{j_s-1}x_nx_1. \end{aligned}$$

   It follows that the reduced form of wx_{m-1}, where $w \in \mathrm{supp}(\alpha_{(ii),i_{s+1}=1})$, has a suffix of the form $x_jx_nx_1$ with $j \leqslant m$;

   (c) similarly, it is clear that wx_{m-1} is reduced for every $w \in \mathrm{supp}(\alpha_k)$, where $k \in \{(iii),(iv),(v)\}$.

3. Assume that $s_{(ii)}x_{m-1} = q_{n,t}x_nx_1x_{m-1}$. In this case $m = j_s = t+1$. If $t=1$, then $s_{(ii)}x_t = s_{(ii)}$ in C_n. Moreover, if $t \geqslant 3$ then $q_{n,t}x_nx_1x_t \xrightarrow{(4t)} q_{n,t}x_nx_1$, so also $s_{(ii)}x_t = s_{(ii)}$. Finally, if $t=2$, it is easy to see that wx_t is in the reduced form.

We summarize the foregoing observations as follows.

Corollary 2.51. *Let $m = j_s$ be as described above. Consider an element w from the support of $\alpha_{(i),i_s=1}$, $\alpha_{(iii)}$, $\alpha_{(iv)}$ or $\alpha_{(v)}$.*

1. *If $j_{s-1} = j_s - 1$, then $wx_{m-1} = w$ in C_n or wx_{m-1} is of the form (vi).*

2. *If $j_{s-1} < j_s - 1$, then wx_{m-1} is reduced.*

Assume now that w is in the support of $\alpha_{(ii),i_{s+1}=1}$.

3. *If $q_{n,t}x_nx_1 \in \mathrm{suff}(w)$, then either (for $t \neq 2$) $wx_{m-1} = w$ in C_n or wx_{m-1} has the reduced form $q_{n,2}^Mx_nx_1x_2$, for some $M \geqslant 1$.*

4. *If $j_{s-1} = j_s - 1$, then either $wx_{m-1} = w$ in C_n or wx_{m-1} is of the form $q_{n,t}^Mx_nx_1$, where $M \geqslant 1$.*

5. *If $j_{s-1} < j_s - 1$, then $w = vx_nx_1$ for some reduced word v and wx_{m-1} has the reduced form $vx_{m-1}x_nx_1$.*

In particular, words from the supports of $\alpha_{(i),i_s=1}$, $\alpha_{(ii),i_{s+1}=1}$, $\alpha_{(iii)}$, $\alpha_{(iv)}$ and $\alpha_{(v)}$ multiplied by x_{m-1} have reduced forms ending with $x_{n-1}\ldots x_j$, $x_jx_nx_1$ or $x_jx_nx_1x_2$, where $j \leqslant m$.

Case II. Now assume that $m < j_s$. In particular $m-1 < n-1$.

We claim that if w is a word in the support of $\alpha_{(i),i_s=1}$, $\alpha_{(iii)}$, $\alpha_{(iv)}$ or $\alpha_{(v)}$, then wx_{m-1} is 0 in $K_0[M_t]$ or its reduced form has a suffix of the form $x_{n-1}\cdots x_j$ for some $j > m$. Moreover, if w is in $\mathrm{supp}(\alpha_{(ii),i_{s+1}=1})$, then wx_{m-1} is 0 or $\mathrm{suff}(wx_{m-1}) = x_{n-1}\cdots x_jx_nx_1$ for some $j > m$. The idea is to reduce words by pushing x_{m-1} to the left and then to use Lemma 2.49. By $w_{(k)}$ we denote a suffix of a word of type (k).

(a) $w_{(i)}x_{m-1} = q_{n,t}x_n(x_1\cdots x_{i_1}x_{n-1}\cdots x_{j_1})\cdots x_n(x_1\cdots x_{i_s}x_{n-1}\cdots x_{j_s})x_{m-1}$. As long as $j_k - 1 > m-1 > i_k+1$ $(k = 1,\ldots,s+1)$ we use reductions (2) and (3) to push x_{m-1} to the left. After this procedure we obtain a word with a prefix vx_{m-1}, where v is exactly a word from Lemma 2.49, for some k_0 and $r = m-1$. By the assumption $j_s \neq m$ (hence, it is impossible that $k_0 = s$ and $j_{k_0} = r+1$), so applying Lemma 2.49 we obtain that $w_{(i)}x_{m-1}$ is either in I_t or its reduced form ends with $x_{n-1}\cdots x_{j_s}$, $j_s > m$.

(b) Since $t+1 \leqslant m < j_s$, we must have $w \neq q_{n,t}^Mx_nx_1$. Then

$$w_{(ii)}x_{m-1} = x_nx_1\cdots x_{i_{s-1}}x_{n-1}\cdots x_{j_{s-1}}x_nx_1\cdots x_{i_s}x_{n-1}\cdots x_{j_s}x_nx_1x_{m-1}.$$

If $m-1 = 1$, then $w_{(ii)}x_{m-1} = w_{(ii)}$ in C_n, and hence it has a suffix $x_{n-1}\cdots x_{j_s}x_nx_1$, $j_s > m$.
If $m-1 = 2$, then $j_s > 3$ and $t \leqslant 2$. From the form of $w_{(ii)}$ we see that in this case $i_s > 1$ and of course $i_s \leqslant 2$. It follows that $i_s = 2$. Then

$$w_{(ii)}x_{m-1} \xrightarrow{(42),(41),(4n)} x_nx_1\cdots x_{i_{s-1}}x_{n-1}\cdots x_{j_{s-1}}x_{n-1}\cdots x_{j_s}x_nx_1x_2.$$

Applying assertion 1. of Lemma 2.49, it follows that this word is in I_t.

If $m-1>2$ then $s_{(ii)}x_{m-1} \xrightarrow{(3)} x_nx_1\cdots x_{i_{s-1}}x_{n-1}\cdots x_{j_{s-1}}x_nx_1\cdots x_{i_s}x_{n-1}\cdots x_{j_s}x_{m-1}x_nx_1$. Using the observation made in the previous case and Lemma 2.49, we get that either $w_{(ii)}x_{m-1}\in I_t$ or its reduced form has a suffix $x_{n-1}\cdots x_{j_s}x_nx_1$, for $j_s>m$.

(c) Every word $w\in \mathrm{supp}(\alpha_{(iii)})\cup\mathrm{supp}(\alpha_{(iv)})\cup\mathrm{supp}(\alpha_{(v)})$ can be written as $w=vx_nx_{n-1}\cdots x_{j_s}$, where v has a block form as in Lemma 2.49. Then $wx_{m-1} \xrightarrow{(2)} vx_{m-1}x_nx_{n-1}\ldots x_{j_s}$. Pushing x_{m-1} to the left by using reductions (2) and (3) we can apply Lemma 2.49. It follows that either $wx_{m-1}\in I_t$ or its reduced form has a suffix $x_nx_{n-1}\cdots x_{j_s}$, for $j_s>m$.

This completes the proof of the claim made at the beginning of Case II.

By our assumptions (of Proposition 2.48), we know that $\alpha x_{m-1}=0$ in $K_0[M_t]$. From the above discussion it follows that for every $w\in\mathrm{supp}(\alpha)$ either wx_{m-1} is 0 (and it is possible only if $m<j_s$) or a suffix of the reduced form of wx_{m-1} is equal to $x_jx_nx_1$, $x_jx_nx_1x_2$ (only if $w\in\mathrm{supp}(\alpha_{(ii)})$), or to $x_{n-1}\cdots x_j$. Moreover, $j\leqslant m$ if and only if in the word w we have $j_s=m$ (see the description of possible types of words). From the property that $\alpha x_{m-1}=0$ it follows that after multiplying by x_{m-1} the sum of all elements of the support of α with $j_s=m$ vanishes.

Assume that v, z are reduced words such that $j_{s_v}=j_{s_z}=m$ (here j_{s_v}, j_{s_z} are defined for v and z as in the list of possible types (i)-(vi) listed before Lemma 2.50) and $vx_{m-1}=zx_{m-1}$ holds in C_n. We will now use the proof of Corollary 2.51 to conclude that $v=z$. Let u be the reduced form of $vx_{m-1}=zx_{m-1}$.

- Assume u is of type (i), (iii), (iv) or (v).

  If u has a suffix x_{m-1}, then it follows that $j_{s-1}<j_s-1$ and vx_{m-1}, zx_{m-1} are reduced, so that $v=z$. Otherwise $vx_{m-1}=v$ and $zx_{m-1}=z$, so also $v=z$.

- Assume u is of type (vi).

  Then $j_{s-1}=j_s-1$ and $z=v$ are of the form $q^M_{n,t}x_nx_1\cdots x_tx_{n-1}\ldots x_{t+2}$.

- Assume u is of type (ii).

  If $m\neq t+1$, then it follows that $\mathrm{suff}(u)=x_jx_nx_1$ for $j\in\{m-1,m\}$. If $j=m$, then $v=vx_{m-1}$ and $zx_{m-1}=z$ in C_n, so the assertion holds. If $j=m-1$, it follows that for $v=v_0x_nx_1$, we have $vx_{m-1}=v_0x_{m-1}x_nx_1$ in C_n and the latter word is in the reduced form. It is clear that $v=z$ also in this case.

  Otherwise $m=t+1$. Then, as $j_s=m=t+1$, we are in the case as in part 3. of Corollary 2.51. It is clear that in this case if $vx_{m-1}=zx_{m-1}$, then $v=z$.

We have shown that for every pair of words v,z with $j_{s_v}=j_{s_z}=m$ if $vx_{m-1}=zx_{m-1}$, then $v=z$. This implies that $\mathrm{supp}(\alpha)$ has no words with $j_s=m$, which contradicts the definition of m. Hence, the assertion of Proposition 2.48 has been proved.

For any K-algebra A, recall that by $\mathcal{P}(A)$ we denote the prime radical of A.

Theorem 2.52. *For every $t = 0, 1, \ldots, n-2$, the algebra $K_0[M_t]$ is prime.*

Proof. In view of Remark 2.47, $K_0[M_{n-2}]$ and $K_0[M_0]$ are prime. In particular, the result holds for $n = 3$. We proceed by induction on n. Assume that $n > 3$. Moreover, we may assume that $1 \leqslant t \leqslant n-3$.

First, we show that $K_0[M_t]$ is semiprime. Suppose that $\alpha K_0[M_t]\alpha = 0$ for some non-zero $\alpha \in K_0[M_t]$. Then, by Theorem 2.28, for every $u, w \in M_t$ we have $\operatorname{supp}(u\alpha w) \subseteq a\langle q_{n,t}\rangle b$ for some $a \in A_t, b \in B_t$. By Corollary 2.40, if $u\alpha w \neq 0$, then there exist $u', w' \in M_t$ such that $0 \neq u'u\alpha ww' \in \langle q_{n,t}\rangle$, and $u'u\alpha ww' \in \mathcal{P}(K[\langle q_{n,t}\rangle]) = 0$. It follows that $u\alpha w = 0$ for every $u, w \in M_t$. Thus, either $\alpha M_t = 0$ or $M_t\alpha w = 0$ for some $w \in M_t$ such that $\alpha w \neq 0$. This means that $\alpha \circ P_t = 0$ or $P_t \circ \alpha w = 0$ ($\circ$ stands for the ordinary matrix multiplication, where $K_0[M_t]$ is interpreted as a subset of the matrix algebra $M_{|A_t|}(K[\langle q_{n,t}\rangle])$. Since P_t is a symmetric matrix by Remark 2.43, we may assume that $\alpha \circ P_t = 0$ for some non-zero $\alpha \in K_0[M_t]$. Then α can be chosen so that $\operatorname{supp}(\alpha) \subseteq a\langle q_{n,t}\rangle B_t$ for some $a \in A_t$. Hence, Corollary 2.40 allows us to assume that $\operatorname{supp}(\alpha) \subseteq \langle q_{n,t}\rangle B_t$. Finally, we may assume that $|\operatorname{supp}(\alpha)|$ is minimal possible.

We claim that $\alpha x_1 = 0$ in $K_0[M_t]$. By Corollary 2.46, $\alpha x_1 M_t = 0$ in $K_0[M_t]$. From the proof of Lemma 2.50 we know that $\alpha_{(iv)}x_1 = 0$ in $K_0[M_t]$ and $vx_1 \in \langle q_{n,t}\rangle B_t$ for every $v \in \operatorname{supp}(\alpha) \setminus \operatorname{supp}(\alpha_{(iv)})$. So, αx_1 inherits the hypotheses on α. Therefore, the minimal choice of α allows us to assume that $\alpha_{(iv)} = 0$. Moreover, $\alpha x_1 \in K[\langle x_2, \ldots, x_{n-1}, x_n x_1\rangle]$. But, from Lemma 2.17 we know that the latter is isomorphic to $K[C_{n-1}]$. Moreover, under this identification, $\operatorname{supp}(\alpha x_1)$ is contained in a single row of the matrix structure $M_{t-1}^{(n-1)}$ defined for the monoid C_{n-1} as in Definition 2.41. It is easy to see that $\alpha x_1 M_{t-1}^{(n-1)} = 0$ in $K_0[M_{t-1}^{(n-1)}]$. The inductive hypothesis implies that $\alpha x_1 = 0$. This proves the claim.

From Lemma 2.36 it follows that replacing α by $q_{n,t}^k\alpha$, for some $k \geqslant 1$, if necessary, we may assume that $\sigma(\alpha) \in K_0[M_t]$ and hence we get that $\sigma(\alpha)$ lies in a single row of the matrix structure $K_0[M_t]$. In other words, there exists $a \in A_t$ such that $\operatorname{supp}(\sigma(\alpha)) \subseteq a\langle q_{n,t}\rangle B_t$. Then, by Corollary 2.40, there exists $z \in M_t$ such that $\operatorname{supp}(z\sigma(\alpha)) \subseteq \langle q_{n,t}\rangle B_t$. The proof of Lemma 2.50 implies that for every $w \in \operatorname{supp}(\sigma(\alpha))$ either $wx_1 \in a\langle q_{n,t}\rangle B_t$ or $wx_1 = 0$ in $K_0[M_t]$. Therefore, by the previous paragraph, $z\sigma(\alpha)x_1 = 0$. Hence, Corollary 2.40 implies that also $\sigma(\alpha x_n) = \sigma(\alpha)x_1 = 0$. Consequently, $\alpha x_n = 0$.

Repeating this argument, we get that $\alpha x_i = 0$ in $K_0[M_t]$ for every $i = 1, \ldots, n$. From Proposition 2.48 it now follows that $\alpha = 0$, a contradiction. Thus, we have proved that $K_0[M_t]$ is semiprime. This implies that the sandwich matrix P_t is not a zero divisor (in the corresponding matrix ring $M_{n_t}(K[\langle q_{n,t}\rangle])$, where $n_t = |A_t|$). Since $K[\langle q_{n,t}\rangle]$ is a domain, it is known that $K_0[M_t]$ must be prime, see [43, Chapter 5]. $\square$

It follows that every matrix $P_t, t = 0, \ldots, n-2$, has a non-zero determinant, which seems to be inaccessible by a direct proof.

Corollary 2.53. *Sandwich matrices P_i are invertible in $M_{n_i}(K(q_{n,i}))$ for all $i = 0, \ldots, n-2$ and $n \geqslant 3$.*

Chapter 3

The radical of the PI Hecke–Kiselman algebras

In the present chapter we continue the study of the structure of Hecke–Kiselman algebras over a field K. Namely, it is shown that the algebra associated to oriented cycle of any length is semiprime and its central localization is a finite direct product of matrix algebras over the field of rational functions $K(x)$. More generally, the radical is described in the case of PI-algebras, and it is shown that it comes from an explicitly described congruence on the underlying Hecke–Kiselman monoid. Moreover, the algebra modulo the radical is again a Hecke–Kiselman algebra and it is a finite module over its center. The content of this chapter was published in the paper [46].

3.1 The radical and Noetherian property of $K[C_n]$ for any $n \geqslant 3$

Now we are in a position to study the radical of the Hecke–Kiselman algebra $K[C_n]$ associated to an oriented cycle of length $n \geqslant 3$. Note that from Theorem 1.72 we know that $K[C_n]$ is PI. Therefore its prime radical and Jacobson radical coincide, see Theorem 1.26. We also show that the algebra is right and left Noetherian.

Let us set the notation from the previous chapters. We use the Gröbner basis characterized in Theorem 1.65 and identify elements of the monoid C_n with their reduced forms from Theorem 2.1 without further comments. Inside C_n there are special $n-1$ sets, denoted by $\tilde{M}_i$ for $i = 0, \ldots, n-2$, as in Definition 2.14 and their sum $\bigcup_{i=0}^{n-2} \tilde{M}_i$ is denoted by $\tilde{M}$. Recall that the complement $C_n \setminus \tilde{M}$ is finite. Moreover, in C_n there is a chain of ideals $\emptyset = I_{n-2} \lhd I_{n-3} \lhd \cdots \lhd I_{-1}$ with the surprising properties described in Theorem 2.44. In particular, recall that by M_i we denote certain semigroup of matrix type inside the quotient I_{i-1}/I_i for $i = 0, \ldots, n-2$. It is isomorphic to $\mathcal{M}^0(\langle q_{n,i} \rangle, A_i, B_i; P_i)$ for certain sets $|A_i| = |B_i| = \binom{n}{i+1}$ and the sandwich matrix P_i that is not a zero-divisor in $M_n(K(q_{n,i}))$ (Corollary 2.53). From Theorem 2.52 we know that the contracted semigroup algebras $K_0[M_i]$ are prime.

We start with a further investigation of properties of the algebras $K_0[M_i]$.

Lemma 3.1. *For every $n \geqslant 3$ and $i \in \{0, \dots, n-2\}$ the algebra $K_0[M_i]$ does not have non-zero finite dimensional ideals.*

Proof. Let J be a non-zero ideal in $K_0[M_i]$. Then, as from Theorem 2.52 we know that this algebra is prime, there exist $v, w \in M_i$ such that $vJw \neq 0$. Indeed, if we had $M_iJM_i = 0$, then also $K_0[M_i]JK_0[M_i] = 0$. On the other hand, as $K_0[M_i]$ is a prime algebra, then it follows that either $K_0[M_i] = 0$ or $J = 0$, which leads to a contradiction. Hence, the matrix type structure of $K_0[M_i]$ implies simply (see also Corollary 2.40) that there exist $v', w' \in M_i$ such that $0 \neq v'vJww' \subseteq K[q_{n,i}]$. Then, clearly, $J \cap K[q_{n,i}]$ is infinite dimensional and consequently, also J has infinite dimension. $\square$

In the next lemma we exploit the properties of $\tilde{M}_i$, primeness of the algebras $K_0[M_i]$ and the fact that C_n is $\mathcal{J}$-trivial, see Theorem 1.71, to show that the radical of $K[C_n]$ is zero.

Lemma 3.2. *Assume that J is a finite dimensional ideal of $K[C_n]$. Then $J = 0$. In particular, the left annihilator $A = \{\alpha \in K[C_n] : \alpha K[\tilde{M}] = 0\}$ of $K[\tilde{M}]$ in $K[C_n]$ is zero. Moreover, $K[C_n]$ is a semiprime algebra.*

Proof. Suppose that $J \neq 0$ is a finite dimensional ideal of $K[C_n]$. First, we claim that a non-zero element $\alpha \in J$ can be chosen so that for every $i = 1, \dots, n$ we have $wx_i = w$ for all $w \in \operatorname{supp}(\alpha)$ or $\alpha x_i = 0$.

Since J is finite dimensional, the set Z consisting of all k-tuples $\{z_1, \dots, z_s\}$, such that $\operatorname{supp}(\beta) = \{z_1, \dots, z_s\}$ for some $0 \neq \beta \in J$ and $s \geqslant 1$ is finite. Let $0 \neq \alpha \in J$ be such that $|\operatorname{supp}(\alpha)|$ is minimal possible. Let $\operatorname{supp}(\alpha) = \{v_1, \dots, v_k\}$.

Let $\mathcal{R}$ denote the Green's relation on the monoid C_n, that is two elements x, y of C_n are in this relation if and only if $xC_n = yC_n$. Consider the $\mathcal{R}$-order $\leq_{\mathcal{R}}$ on C_n; in other words, we write $w \leq_{\mathcal{R}} v$ if $wC_n \subseteq vC_n$, see Section 1.2. Then define a relation $\preceq$ on C_n^k by: $(u_1, \dots, u_k) \preceq (w_1, \dots, w_k)$ if $u_i \leq_{\mathcal{R}} w_i$ for every $i = 1, \dots, k$.

Now, by the choice of α, for every $x \in C_n$ we have that either $\alpha x = 0$ or $\operatorname{supp}(\alpha x) = \{v_1x, \dots, v_kx\}$ and in the latter case $(v_1x, \dots, v_kx) \preceq (v_1, \dots, v_k)$. Since the set Z introduced above is finite, we may further choose an element α for which the k-tuple $(v_1, \dots, v_k)$ is minimal possible with respect to $\preceq$. Then $v_i\mathcal{R}v_ix$ for every i. Since the monoid C_n is $\mathcal{J}$-trivial by Theorem 1.71 it follows that for every j we either have $wx_j = w$ for every $w \in \operatorname{supp}(\alpha)$ or $\alpha x_j = 0$, as claimed.

Next, assume that $\beta \in K[C_n]$ is a non-zero element such that $wx_1 = w$ holds in C_n for every $w \in \operatorname{supp}(\beta)$. Then $|w|_1 > 0$ for every such w. Write $w = w_0x_1w_1$, for some reduced words w_0, w_1 such that $|w_1|_1 = 0$. We claim that then $|w_1|_n = 0$. Indeed, if $w_1 = ux_nv$ with $|v|_n = 0$, then $wx_1 = w_0x_1ux_nvx_1$ and then the only possible reduction that allows to decrease the length of this word (needed in order to get $wx_1 = w$ in C_n) is of the form $x_1zx_1 \to zx_1$, where z is a prefix of ux_nv containing ux_n. But then we do not get $wx_1 = w$ in C_n because

x_1 appears after the last occurrence of x_n in the reduced form of wx_1, a contradiction. So $|w_1|_n = 0$, as claimed.

Assume first that $|w|_n > 0$. Write $w = sx_ntx_1w_1$, for some reduced words s, t (so $w_0 = sx_nt$) such that $|t|_n = 0$. Then also $|t|_1 = 0$ because w is reduced. Hence, either $wx_n = sx_ntx_1w_1x_n$ is a reduced word with $|wx_n|_n \geqslant 2$ (if $|tw_1|_{n-1} > 0$) or $wx_n = w$ in C_n and the reduced form of $wx_n = w$ does not end with generator x_n (if $|tw_1|_{n-1} = 0$).

Next, consider the case when $|w|_n = 0$. It is clear that in this case wx_n is a reduced word, and $|wx_n|_n = 1$. Together with the previous paragraph of the proof this implies that $wx_n \neq w'x_n$ in C_n for all $w, w' \in \operatorname{supp}(\beta)$ with $w \neq w'$.

We have proved that the hypotheses on β imply that $\beta x_n \neq 0$.

Now, we apply this observation to the element α. Because of the choice of α, we get that if $\alpha x_1 = \alpha$ then $\alpha x_n = \alpha$. Using linear extension of the automorphism σ from Definition 2.29, and noting that $\sigma(\alpha)$, as an element of the finite dimensional ideal $\sigma(J)$ of $K[C_n]$, inherits the hypotheses on α, we get that $\sigma(\alpha)x_1 = \sigma(\alpha)$, so that $\sigma(\alpha)x_n = \sigma(\alpha)$, by the above argument applied to $\sigma(\alpha)$ in place of α. Thus, $\alpha x_{n-1} = \alpha$, by applying σ^{-1}. Repeating this argument several times, we then get $\alpha x_j = \alpha$ for every j. A similar argument shows that if $\alpha x_k \neq 0$ for some k, then $\alpha x_j \neq 0$ for every j. However, $\alpha = \alpha x_n x_1 x_2 \cdots x_{n-1} \in J \cap K[\tilde{M}_{n-2}]$, (see Lemma 2.19), a finite dimensional ideal of $K[\tilde{M}_{n-2}]$, because $x_n x_1 \cdots x_{n-1} \in \tilde{M}_{n-2} \subseteq \tilde{M}$ and $\tilde{M}_{n-2}$ is an ideal of C_n, as we know from Corollary 2.38. Therefore, Lemma 3.1 implies that $\alpha = 0$. This contradiction shows that we may assume that $\alpha x_j = 0$ for every j.

Let $w \in \operatorname{supp}(\alpha)$ be maximal with respect to the order $\leq_{\mathcal{R}}$. If x_j is the last letter of the (reduced form of the) word w then, as $\alpha x_j = 0$, we have $w = wx_j = w'x_j$ in C_n, for some $w' \in \operatorname{supp}(\alpha)$ such that $w \neq w'$. This implies that $w \leq_{\mathcal{R}} w'$, so by the choice of w we get $w = w'$, a contradiction. Therefore $J = 0$.

Let $A = \{\alpha \in K[C_n] : \alpha K[\tilde{M}] = 0\}$ be the left annihilator of $K[\tilde{M}]$ in $K[C_n]$. Suppose that $0 \neq \beta \in A \cap K[\tilde{M}]$. Let i be the minimal integer such that $\operatorname{supp}(\beta) \cap \tilde{M}_i \neq \emptyset$. Passing to $K[C_n]/I_i$, we get a non-zero element $\overline{\beta} \in K_0[M_i]$ such that $\overline{\beta} K_0[M_i] \overline{\beta} = 0$. From Theorem 2.52 it then follows that $\overline{\beta} = 0$, which leads to a contradiction. Thus $A \cap K[\tilde{M}] = 0$. Similarly, suppose that there exists $0 \neq \gamma \in \mathcal{P}(K[C_n]) \cap K[\tilde{M}]$. Take minimal integer i such that $\operatorname{supp}(\gamma) \cap \tilde{M}_i \neq \emptyset$. Passing to $K[C_n]/I_i$ as before, we get $0 \neq \overline{\gamma} \in \mathcal{P}(K_0[C_n/I_i]) \cap K_0[M_i] \subseteq \mathcal{P}(K_0[M_i])$. This contradicts Theorem 2.52. So $\mathcal{P}(K[C_n]) \cap K[\tilde{M}] = 0$ and it follows that A and $\mathcal{P}(K[C_n])$ are finite dimensional, because $C_n \setminus \tilde{M}$ is finite (Proposition 2.15). Hence, the assertion follows. $\square$

As a corollary we can easily prove that the Hecke–Kiselman algebra associated to an oriented cycle of any length is right and left Noetherian. The main result of the present section states as follows.

Theorem 3.3. *The Hecke–Kiselman algebra $K[C_n]$ associated to an oriented cycle of length $n \geqslant 3$ is a semiprime Noetherian PI-algebra.*

Proof. From Remark 2.16 we get that $K[C_n]$ has Gelfand–Kirillov dimension 1, see also Example 2 of [39]. In view of Lemma 3.2, we thus know that $K[C_n]$ is a semiprime algebra of

Gelfand–Kirillov dimension 1. Applying Theorem 1.22 we conclude that $K[C_n]$ is a PI-algebra and a Noetherian algebra as a finite module over its Noetherian center. $\square$

3.2 Prime ideals and the ring of quotients of $K[C_n]$

Now our aim is to describe the prime spectrum and characterize the classical ring of quotients of the algebra $K[C_n]$, for any $n \geqslant 3$.
We start with the definition of a family of ideals in $K[C_n]$ that, as we will show, consists of all minimal primes of the algebra.

For any $i = 0, \dots, n-2$, let J_i be a maximal among all ideals of $K[C_n]$ such that $I_i \subseteq J_i$ and the intersection $K[q_{n,i}] \cap J_i$ is zero. Such an ideal exists from Zorn's lemma, as $K[q_{n,i}] \cap I_i = 0$.

Theorem 3.4. *An ideal J_i defined above is uniquely defined for $i = 0, \dots, n-2$. Moreover ideals J_i are the only minimal prime ideals in $K[C_n]$ and $J_0 \cap J_1 \cap \dots \cap J_{n-2} = 0$.*

Proof. Suppose that for some $a, b \in K[C_n]$ we have $aK[C_n]b \subseteq J_i$ and $a, b \notin J_i$, contradicting the primeness of J_i. Then, from $J_i \subsetneq J_i + (a)$, $J_i \subsetneq J_i + (b)$ and from the definition of J_i it follows that there exist non-zero v and w in $(J_i + (a)) \cap K[q_{n,i}]$ and $(J_i + (b)) \cap K[q_{n,i}]$, respectively. As $K[q_{n,i}]$ is an integral domain, it follows that $0 \neq vw \in (J_i + (a))(J_i + (b)) \cap K[q_{n,i}] \subseteq J_i \cap K[q_{n,i}]$, which contradicts $J_i \cap K[q_{n,i}] = 0$. Therefore every J_i is prime.

We know that $J_i \cap K[q_{n,i}] = 0$ and thus $K[q_{n,i}]$ embeds into the quotient $K[C_n]/J_i$. Moreover, $\operatorname{GKdim} K[C_n] = 1$. It follows from Example 1.19 and Lemma 1.20 that $\operatorname{GKdim} K[C_n]/J_i \geqslant \operatorname{GKdim} K[x] = 1$. Then, as $\operatorname{GKdim} K[C_n] = 1$, we know that $\operatorname{GKdim} K[C_n]/J_i = 1$. Therefore we get from Theorem 1.23 that $1 = \operatorname{clKdim} K[C_n]/J_i = \operatorname{GKdim} K[C_n]/J_i = \operatorname{GKdim} K[C_n]$, so J_i is a minimal prime ideal of $K[C_n]$.

We claim that that ideals J_i are the only minimal prime ideals in $K[C_n]$ and $\bigcap_{i=0}^{n-2} J_i = 0$. First observe that as $K[q_{n,i}] \cap K[I_i] = 0$, algebra $K[q_{n,i}]$ embeds into $K[C_n]/K[I_i]$. We also know that $K_0[M_i] \lhd K[C_n]/K[I_i]$ is a prime algebra (Theorem 2.52). Moreover, from Theorem 2.28 it follows that $q_{n,i} K_0[M_i] q_{n,i} \subseteq K[q_{n,i}]$. Denote by $\overline{J_i}$ the image of J_i in $K[C_n]/K[I_i]$. Then $\overline{J_i}$ is a prime ideal of $K[C_n]/K[I_i]$ and thus $q_{n,i}(K_0[M_i] \cap \overline{J_i}) q_{n,i} \subseteq K[q_{n,i}] \cap \overline{J_i}$. Therefore $K[q_{n,i}] \cap J_i = 0$ implies that $K[\tilde{M}_i] \cap J_i = 0$.

Denote by J the intersection $J = J_0 \cap J_1 \cap \ldots \cap J_{n-2}$. Suppose that $\alpha \in J \cap K[\tilde{M}]$ is a non-zero element. Then we can write $\alpha = \alpha_0 + \ldots + \alpha_{n-2}$, where $\alpha_k \in K[\tilde{M}_k]$ for every k. Take minimal i such that $\alpha_i \neq 0$. Then $i = n-2$ would imply $\alpha = \alpha_{n-2} \in K[\tilde{M}_{n-2}] \cap J = 0$, which leads to a contradiction. Therefore $i < n-2$. Consider $0 \neq \alpha_i = \alpha - (\alpha_{i+1} + \ldots + \alpha_{n-2})$. On the one hand, we have that $\alpha_i \in K[\tilde{M}_i]$. On the other hand $\alpha_{i+1} + \ldots + \alpha_{n-2} \in K[\tilde{M}_{i+1}] + \ldots + K[\tilde{M}_{n-2}] \subseteq K[I_i] \subseteq J_i$ and thus $\alpha - (\alpha_{i+1} + \ldots + \alpha_{n-2}) \in J_i$. Therefore $0 \neq \alpha_i \in K[\tilde{M}_i] \cap J_i = 0$. The obtained contradiction shows that $J \cap K[\tilde{M}] = 0$, as claimed. As $C_n \setminus \tilde{M}$ is finite, we get that J is a finite dimensional ideal of $K[C_n]$. From Lemma 3.2 it follows that $J = 0$.

Now, for any prime ideal Q of $K[C_n]$, from $J_0 \cap J_1 \cap \cdots \cap J_{n-2} \subseteq Q$ it follows that $J_i \subseteq Q$ for some i. Thus the ideals J_i are the only minimal prime ideals in $K[C_n]$. Consequently, for every i the ideal J_i is uniquely defined. $\square$

Now we are in a position to characterize the classical quotient ring of $K[C_n]$.

Theorem 3.5. *The classical quotient ring of the semiprime Noetherian PI-algebra $K[C_n]$, for any $n \geqslant 3$, is isomorphic to $\prod_{i=0}^{n-2} M_{n_i}(K(x))$, where $n_i = \binom{n}{i+1}$, for $i = 0, \ldots, n-2$.*

Recall that the classical quotient ring of a semiprime Noetherian PI-algebra is its central localization, see Theorem 1.35.

Proof. Let J_i be a minimal prime ideal of $K[C_n]$ as in Theorem 3.4. It follows from the proof of this theorem that $K_0[M_i]$ is a prime algebra such that $K_0[M_i] \triangleleft K[C_n]/J_i$. Thus the classical rings of quotients of $K_0[M_i]$ and $K[C_n]/J_i$ are equal, see Lemma 1.34.

Moreover, from Corollary 2.53 it is known that all sandwich matrices P_i are invertible, when treated as matrices in $M_{n_i}(K(q_{n,i})) \cong M_{n_i}(K(x))$. Thus from [43, Proposition 4.13] we know that the algebra of matrix type $\mathcal{M}^0(K(q_{n,i}), A_i, B_i; P_i)$ is isomorphic to $M_{n_i}(K(q_{n,i}))$, where $n_i = \binom{n}{i+1}$ and this algebra is the classical ring of quotients of the prime algebra $K_0[M_i] = \mathcal{M}^0(K[Q_i], A_i, B_i; P_i)$, where Q_i is the cyclic semigroup generated by $q_{n,i}$. Consequently, $Q_{cl}(K[C_n]/J_i) \cong M_{n_i}(K(x))$, where $n_i = \binom{n}{i+1}$.

From Proposition 1.33 and Theorem 3.4 we obtain that $K[C_n]$ has the classical ring of quotients which is isomorphic to $\prod_{i=0}^{n-2} M_{n_i}(K(x))$, where $n_i = \binom{n}{i+1}$, for $i = 0, \ldots, n-2$. This completes the proof of the theorem. $\square$

We continue with the investigation of the prime spectrum of $K[C_n]$. As a consequence of theorem of Kaplansky (Theorem 1.25) we also obtain that primitive ideals are exactly maximal ideals in our case.

Theorem 3.6. *Every maximal chain of prime ideals in the Hecke–Kiselman algebra $K[C_n]$ is of the form $J_i \subsetneq P$ for $i = 0, \ldots, n-2$, for prime ideals J_i as above and a maximal ideal P in $K[C_n]$, depending on i. Every maximal ideal Q in $K[C_n]$ contains J_i for some i. Left (right) primitive ideals in $K[C_n]$ are precisely the maximal ideals.*

Proof. From Theorem 3.4 we know that J_i are the only minimal prime ideals of $K[C_n]$. Moreover, the algebra $K[C_n]$ is a Noetherian semiprime PI-algebra of Gelfand–Kirillov dimension one, Theorem 3.5. As we explained in the proof of Theorem 3.4, we have that $\operatorname{clKdim} K[C_n]/J_i = \operatorname{GKdim} K[C_n]/J_i = 1$, so J_i is not a maximal ideal of $K[C_n]$. Hence, for every $i \in \{0, \ldots, n-2\}$ there exists a prime ideal P such that $J_i \subsetneq P$. Then from the definition of the classical Krull dimension we have $\operatorname{clKdim} K[C_n]/P = 0$ and, as $K[C_n]/P$ is a finitely generated prime ring which is PI, we get that $\operatorname{clKdim} K[C_n]/P = \operatorname{GKdim} K[C_n]/P$. Now $\operatorname{GKdim} K[C_n]/P = 0$ implies that $K[C_n]/P$ is a finite dimensional prime algebra. Therefore P is a maximal ideal in $K[C_n]$ by Exercise 10.4 in [31]. Clearly for every i there may be many maximal ideals P containing J_i. In particular, from Kaplansky theorem, (left) right primitive ideals of $K[C_n]$ are exactly such ideals P. $\square$

Let us note that in Chapter 5 we will show that the maximal ideals of $K[C_n]$ play an important role in the study of irreducible representations of the algebra. Therefore we will further investigate certain properties of these ideals in the next chapter.

3.3 General case

Our second main result describes the radical of a Hecke–Kiselman algebra $K[\mathrm{HK}_\Theta]$, as well as the algebra modulo the radical, in the case of PI-algebras. So, assume that Θ is a finite oriented graph such that $K[\mathrm{HK}_\Theta]$ is a PI-algebra. This is equivalent to saying that Θ does not contain two cyclic subgraphs (i.e. subgraphs which are cycles) connected by an oriented path, Theorem 1.72.

Recall that the Jacobson radical of a finitely generated PI-algebra R is nilpotent, see Theorem 1.26. However, we note that for $R = K[\mathrm{HK}_\Theta]$ this can also be derived from our proof.

We start with the definition of a congruence on HK_Θ that will be crucial in the description of the radical. For the basic definitions see Section 1.2.

Definition 3.7. Denote by ρ the congruence on HK_Θ generated by all pairs (xy, yx) such that there is an arrow $x \to y$ that is not contained in any cyclic subgraph of Θ. If there is no such a pair then we assume that ρ is the trivial congruence.

Example 3.8. Let Θ be an oriented graph presented in Figure 3.1.

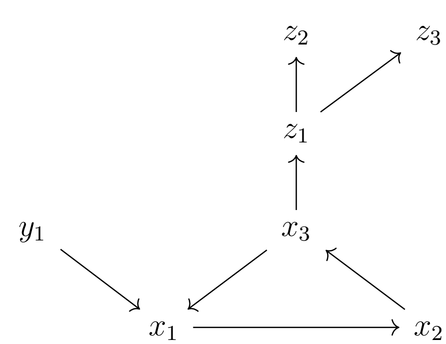

Figure 3.1: A graph Θ such that $K[\mathrm{HK}_\Theta]$ has non-trivial radical

Then the congruence ρ is the congruence generated by pairs (y_1x_1, x_1y_1), (z_1z_2, z_2z_1), (z_1z_3, z_3z_1) and (x_3z_1, z_1x_3). Moreover, the semigroup $\mathrm{HK}_\Theta /\rho \cong \mathrm{HK}_{\Theta'}$, where Θ' is the graph in Figure 3.2.

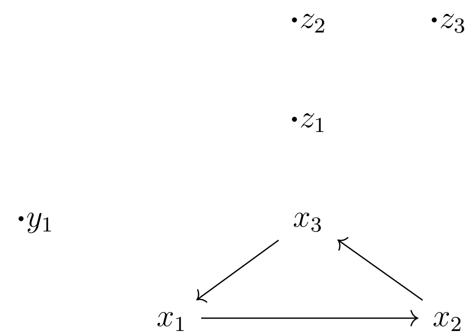

Figure 3.2: A graph Θ' such that $\mathrm{HK}_\Theta/\rho \cong \mathrm{HK}_{\Theta'}$

For any congruence σ on a semigroup S by $I(\sigma)$ we denote the ideal in the semigroup algebra $K[S]$ (where K is a field) which is the K-span of the set of elements of the form $s-t$ for all pairs $(s,t) \in S \times S$ with $s\sigma t$. Such an ideal is the kernel of the natural epimorphism $K[S] \to K[S/\sigma]$.

Let Θ be an oriented graph such that $K[\mathrm{HK}_\Theta]$ is PI. Denote by Θ' the subgraph of Θ obtained by deleting all arrows $x \to y$ that are not contained in any cyclic subgraph of Θ. Then $\mathrm{HK}_{\Theta'} \cong \mathrm{HK}_\Theta/\rho$ and from Theorem 1.72 it follows that the connected components of Θ' are either singletons or cyclic subgraphs. In particular, the algebras of such connected components are semiprime, see Theorem 3.3. We are now in a position to state the main result.

Theorem 3.9. *Assume that Θ is a finite oriented graph such that $K[\mathrm{HK}_\Theta]$ is a PI-algebra. Let Θ' be the subgraph of Θ obtained by deleting all arrows $x \to y$ that are not contained in any cyclic subgraph of Θ and let ρ be the congruence on HK_Θ from Definition 3.7. Then*

1. *the Jacobson radical $\mathcal{J}(K[\mathrm{HK}_\Theta])$ of $K[\mathrm{HK}_\Theta]$ is equal to the ideal $I(\rho)$ determined by ρ,*
2. *$K[\mathrm{HK}_\Theta]/\mathcal{J}(K[\mathrm{HK}_\Theta]) \cong K[\mathrm{HK}_{\Theta'}]$ and it is the tensor product of algebras $K[\mathrm{HK}_{\Theta_i}]$ of the connected components $\Theta_1, \ldots, \Theta_m$ of Θ', each being isomorphic to $K \oplus K$ or to the algebra $K[C_j]$, for some $j \geqslant 3$,*
3. *$K[\mathrm{HK}_{\Theta'}]$ is a finitely generated module over its center.*

Proof. Suppose that a vertex $x \in V(\Theta)$ is a source vertex. In other words, there is an arrow $x \to y$ for some $y \in V(\Theta)$ but there are no arrows of the form $z \to x$. For any $w \in \mathrm{HK}_\Theta$ consider the element $\beta = (xy - yx)w(xy - yx) \in K[\mathrm{HK}_\Theta]$. Since x is a source vertex, we know that $xvx = xv$ in HK_Θ for every $v \in \mathrm{HK}_\Theta$. Hence $xwxy = xwy, xwyx = xwy$ (Remark 1.64). Similarly, $xywxy = xywy$ and $xywyx = xywy$. Therefore $\beta = 0$. It follows that $xy - xy \in \mathcal{P}(K[\mathrm{HK}_\Theta])$.

If x is a sink, that is there is an arrow $z \to x$ for some $z \in V(\Theta)$ but there are no arrows of the form $x \to y$ in the graph Θ, a symmetric argument shows that $xz - zx \in \mathcal{P}(K[\mathrm{HK}_\Theta])$

for all z such that $z \to x$ in Θ. Let ρ_1 be the congruence generated by all pairs (xy, yx) such that x or y is either source or sink and there is an arrow $x \to y$ that is not contained in any cyclic subgraph of Θ. Equivalently, we may consider the graph Γ_1 obtained by erasing in Θ all such arrows $x \to y$ and $z \to x$ as above. Then $K[\mathrm{HK}_{\Gamma_1}] \cong K[\mathrm{HK}_\Theta]/I(\rho_1)$. We have shown that $I(\rho_1) \subseteq \mathcal{P}(K[\mathrm{HK}_\Theta])$. Repeating this argument finitely many times we easily get that $I(\rho) \subseteq \mathcal{P}(K[\mathrm{HK}_\Theta])$ (and our argument shows that $I(\rho)$ is nilpotent, because Θ is finite).

Since we know that $\mathcal{J}(K[\mathrm{HK}_\Theta]) = \mathcal{P}(K[\mathrm{HK}_\Theta])$, to prove the first assertion of the theorem it is now enough to check that $K[\mathrm{HK}_{\Theta'}]$ is semiprime. $\mathrm{HK}_{\Theta'}$ is the direct product of all HK_{Θ_i}, where Θ_i, $i = 1, \ldots, m$, are the connected components of Θ'. From Theorem 1.72 we know that each HK_{Θ_i} is either a band with two elements (if Θ_i has only one vertex) or it is isomorphic to C_k for some $k \geqslant 3$. In the former case $K[\mathrm{HK}_{\Theta_i}] \cong K \oplus K$, in the latter $K[\mathrm{HK}_{\Theta_i}]$ is a finitely generated module over its center, see Theorem 1.22. It follows that $K[\mathrm{HK}_{\Theta'}]$ is a direct product of tensor products of algebras that are all finitely generated modules over their center. Thus $K[\mathrm{HK}_{\Theta'}]$ is a finitely generated module over its center.

Let Q_i be the classical ring of quotients of $K[\mathrm{HK}_{\Theta_i}]$. If $\Theta_i = C_{m_i}$ for some m_i then we know that Q_i is a central localization of the form described in Theorem 3.5. Clearly, $\mathrm{HK}_{\Theta'}$ is the direct product $\prod_{i=1}^{m} \mathrm{HK}_{\Theta_i}$. Then in the localization $Q = Q_1 \otimes \cdots \otimes Q_m$ of $K[\mathrm{HK}_{\Theta'}] \cong \bigotimes_{i=1}^{m} K[\mathrm{HK}_{\Theta_i}]$ each of the factors is isomorphic to $K \oplus K$ or to $\prod_{j=0}^{m_i-2} M_{r_j}(K(x))$, where $r_j = \binom{m_i}{j+1}$. Therefore, Q is isomorphic to a finite direct product of rings isomorphic to matrix rings over certain commutative Noetherian integral domains. Thus it is a semiprime Noetherian ring. From Proposition 10.34 in [30] it follows that $K[\mathrm{HK}_{\Theta'}]$ is semiprime, because Q is its central localization. It is now clear that $K[\mathrm{HK}_{\Theta'}] \cong K[\mathrm{HK}_\Theta]/\mathcal{P}(K[\mathrm{HK}_\Theta])$. The result follows. $\square$

Chapter 4

Noetherian property of Hecke–Kiselman algebras

In this section we characterize Noetherian Hecke–Kiselman algebras $K[\mathrm{HK}_\Theta]$ of arbitrary oriented graphs Θ. Recall that in Theorem 3.3 it has been proved that the algebra $K[C_n]$ is right and left Noetherian for every $n \geqslant 3$. Therefore we start the present chapter with a description of some simple obstacles to the Noetherian property of the Hecke–Kiselman algebras associated to graphs that are cycles with one adjoined arrow. As a consequence, we are able to characterize Noetherian Hecke–Kiselman algebras in Theorem 4.2. The results of this chapter are published in [45].

Lemma 4.1. *Let Θ be the graph obtained by adjoining the arrow $y \to x_1$ to the cyclic graph C_n: $x_1 \to x_2 \to \cdots \to x_n \to x_1$, as shown in Figure 4.1. Then the monoid HK_Θ does not satisfy the ascending chain condition on left ideals, and it does not satisfy the ascending chain condition on right ideals.*

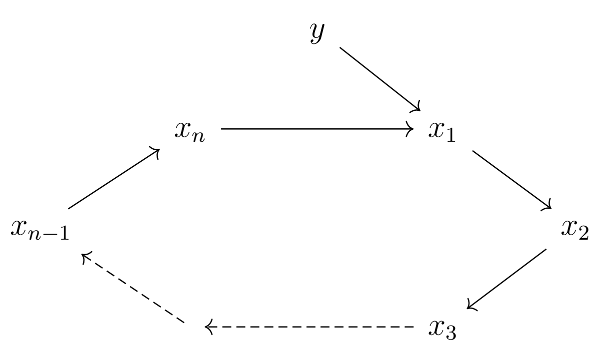

Figure 4.1: A graph Θ such that $K[\mathrm{HK}_\Theta]$ is not Noetherian

Proof. Write $w_k = (x_n x_{n-1} \cdots x_1)^k y$, for $k = 1, 2, \ldots$. It is clear that w_k cannot be rewritten in the monoid HK_Θ except for applying relations of the form $x_i^2 = x_i, y^2 = y$. Therefore $w_k \notin \bigcup_{i=1}^{k-1} w_i \,\mathrm{HK}_\Theta$ for $k \geqslant 2$. Hence, HK_Θ does not satisfy acc on right ideals.

Let $\phi : \langle x_1, \ldots, x_n, y\rangle \longrightarrow \langle x_1, \ldots, x_n\rangle$ be the homomorphism such that $\phi(w)$ is obtained from w by erasing all occurrences of y (compare with Proposition 1.68). Consider the following subsets of the free monoid $F = \langle x_1, \ldots, x_n, y\rangle$: $Z_k = \{\phi^{-1}((x_1^{i_1} x_n^{i_n} \cdots x_2^{i_2})^k) \mid i_j \geqslant 1 \text{ for every } j\}$, for $k = 1, 2, \ldots$, and

$$R_k = \{wyvz \mid w \in \langle x_1, x_2\rangle, v \in \langle x_2, y\rangle\langle x_1, y\rangle, z \in Z_k, |wv|_2 \geqslant 1\}.$$

We claim that R_k is closed under defining relations of the monoid HK_Θ. It is easy to see that the set R_k is closed under relations of the form $x = x^2$ and under $xz = zx$ for generators x, z not connected in the graph Θ (the only such factors of a word $u \in R_k$ can be of the form yx_j, x_jy, where $j \geqslant 2$). Moreover, u does not have factors of the form x_jx_i with $i = 3, \ldots, n$ and $j \neq i+1$ (modulo n). So we do not have to consider relations $x_ix_{i+1}x_i = x_{i+1}x_ix_{i+1} = x_ix_{i+1}$ for $i = 2, \ldots, n$. It is also easy to see that every relation $yx_1y = yx_1$, $x_1yx_1 = yx_1$ and $x_1yx_1 = yx_1y$ leaves R_k invariant. Finally, every relation $x_2x_1x_2 = x_1x_2$, $x_1x_2x_1 = x_1x_2$ and $x_1x_2x_1 = x_2x_1x_2$ leaves R_k invariant. This proves the claim.

Define $v_k = x_1x_2y(x_1x_n \cdots x_2)^k$, for $k = 1, 2, \ldots$. Notice that $v_k \in R_k$ but $v_k \notin FR_i$ for $i < k$. It follows that $v_k \notin \bigcup_{i=1}^{k-1} \mathrm{HK}_\Theta\, v_i$, for every $k \geqslant 2$. Therefore HK_Θ does not satisfy acc on left ideals. □

Now we are in a position to prove the main theorem.

Theorem 4.2. *Let Θ be a finite oriented graph. Then the following conditions are equivalent*

1) *$K[\mathrm{HK}_\Theta]$ is right Noetherian,*

2) *$K[\mathrm{HK}_\Theta]$ is left Noetherian,*

3) *each of the connected components of Θ is either an oriented cycle or an acyclic graph.*

Proof. Assume that condition 3) is satisfied. From Theorem 1.72 we know that HK_Θ is a PI-algebra. In order to prove conditions 1) and 2) we proceed by induction on the number k of connected components of Θ. If $k = 1$ then the assertion follows from Theorem 3.3 and from the fact that HK_Θ is finite if Θ is an acyclic graph (Theorem 1.69). Assume that $k > 1$. Let Θ_1 be a connected component of Θ and let $\Theta_2 = \Theta \setminus \Theta_1$. Clearly, HK_Θ is a direct product of HK_{Θ_1} and HK_{Θ_2}, so that $K[\mathrm{HK}_\Theta] \cong K[\mathrm{HK}_{\Theta_1}] \otimes K[\mathrm{HK}_{\Theta_2}]$. By the induction hypothesis, HK_{Θ_i} is (right and left) Noetherian and it is a PI-algebra, for $i = 1, 2$. Then $K[\mathrm{HK}_\Theta]$ is a Noetherian algebra by [2], Proposition 4.4 (which says that every finitely generated right Noetherian PI-algebra is a universally right Noetherian algebra).

Assume that 3) is not satisfied. Then Θ contains a subgraph Θ' that is of the form described in Lemma 4.1 or the graph Θ'' obtained from Θ' by inverting all arrows. It is easy to see that in this case $K[\mathrm{HK}_{\Theta'}]$, respectively $K[\mathrm{HK}_{\Theta''}]$, is a homomorphic image of $K[\mathrm{HK}_\Theta]$. Moreover, Θ' and Θ'' are antiisomorphic. Therefore, Lemma 4.1 implies that $K[\mathrm{HK}_\Theta]$ is neither right nor left Noetherian. The result follows. □

From the proof it actually follows that the conditions in Theorem 4.2 are satisfied if and only if the monoid HK_Θ has acc on right (left) ideals.

Since $K[C_n]$ is a PI-algebra (Theorem 1.72), we derive the following consequence from Theorem 1.24 of Anan'in. It is of interest in view of the results on faithful representations of various special classes of Hecke–Kiselman monoids, obtained in several papers, for instance [16, 18, 29].

Corollary 4.3. *If an oriented graph Θ satisfies the conditions of Theorem 4.2, then $K[\mathrm{HK}_\Theta]$ embeds into the matrix algebra $M_r(L)$ over a field L, for some $r \geqslant 1$.*

Proof. If the graph Θ satisfies the conditions of Theorem 4.2, then $K[\mathrm{HK}_\Theta]$ is a tensor product of finitely many PI-algebras. From Regev Theorem 1.21 we know that the algebra $K[\mathrm{HK}_\Theta]$ is also a PI-algebra. Thus, the assertion follows from Theorem 1.24 and Theorem 4.2. □

Chapter 5

Irreducible representations of PI Hecke–Kiselman algebras

The chapter is devoted to irreducible representations of Hecke–Kiselman algebras that satisfy polynomial identities. It turns out that the case of the algebra associated to an oriented cycle is a crucial step. To characterize representations in this case we revisit the ideal structure from Theorem 2.44. The structures of matrix type M_i occurring in the quotients have the flavour of the principal factors of a finite semigroup, thus we are able to build a class of irreducible representations of C_n from the representations of these factors. It turns out that the remaining representations arise from idempotents in C_n, in a way similar to that known for the so-called $\mathcal{J}$-trivial finite monoids, see [53] and Example 1.50. In the present chapter we consider Hecke–Kiselman algebras $K[\mathrm{HK}_\Theta]$ over an algebraically closed field K. We assume without additional comments that the graph Θ does not contain two different cycles connected by an oriented path. This condition is equivalent to the fact that the corresponding Hecke–Kiselman algebra satisfies a polynomial identity, see Theorem 1.72. The results have already been published in the paper [57].

5.1 Idempotents in C_n

We are going to characterize all idempotents in the Hecke–Kiselman monoid associated to the cycle of length n. This will be an intermediate step in our approach to a description of all irreducible representations of this monoid. Recall that idempotents in the Hecke–Kiselman monoids associated to any oriented graph have been characterized in [16]. We provide an alternative proof for monoids associated to an oriented cycle of any length.

In this case the result relies on Theorem 1.71 and on the characterization of idempotents in the Hecke–Kiselman monoids which do not contain any oriented cycles, described in [18] with the use of [29]. More precisely, the following theorem describes idempotents in the Hecke–Kiselman monoid associated to any acyclic graph.

Theorem 5.1 ([18], Theorem 1 (iii), [29], Proposition 11)**.** *Assume that Γ is an oriented graph with the set of vertices $\{1,\ldots,n\}$ such that if $i \to j$ in Γ then $i < j$. Then HK_Γ has exactly 2^n idempotents. More precisely, every idempotent is of the form e_X, where for every $X \subseteq \{1,\ldots,n\}$ such that $X = \{i_1,\ldots,i_j\}$ with $i_1 < i_2 < \cdots < i_j$, we denote by e_X the element $x_{i_1}\cdots x_{i_j}$ (for $X = \emptyset$ we set $e_X = 1$).*

Let us recall from Section 1.4.2 that for any word w in the free monoid by its support we mean the set of generators that occur in w. Note that the supports of any two words representing the same element in the Hecke–Kiselman monoid are equal. Therefore, the support of an element of the Hecke–Kiselman monoid can be defined as the support of any word in the free monoid representing this element.

Let C_n be the Hecke–Kiselman monoid associated to the cycle of length n with the corresponding set of generators $\{x_1,\ldots,x_n\}$.

We start with the following crucial observation.

Lemma 5.2. *Idempotents in C_n are not of full support, that is there are no idempotents in which all generators occur.*

Proof. Suppose that e is an idempotent of full support in the monoid C_n. Then we claim that $ex_i = e$ for $i = 1,\ldots,n$. Indeed, as e has full support, e is of the form $e_1x_ie_2$ for some elements e_1 and e_2. Then equalities $e = e^2 = ee_1x_ie_2$ imply that $eC_n = ee_1C_n$ and $eC_n = ee_1x_iC_n$. Moreover, the monoid C_n is $\mathcal{J}$-trivial, see Theorem 1.71. Therefore $e = ee_1$ and $e = ee_1x_i$. Consequently $e = ex_i$, as claimed.

It follows that $e = ex_nx_1x_2\cdots x_{n-1}$. From the description of the family $\tilde{M}_{n-2}$ in Definition 2.14 we get that $x_nx_1\cdots x_{n-1} \in M_{n-2} = \tilde{M}_{n-2}\cup\{\theta\}$ and therefore, from Theorem 2.44 4, $e = ex_nx_1x_2\cdots x_{n-1}$ is also in $\tilde{M}_{n-2}$. On the other hand $M_{n-2} = \mathcal{M}^0(Q_{n-2}, A_{n-2}, B_{n-2}; P_{n-2})$ is the semigroup of matrix type associated to cyclic semigroup generated by $q_{n,n-2}$ and a matrix P_{n-2} with coefficients in the monoid generated by $q_{n,n-2}$ with zero adjoined. Such a semigroup does not contain non-zero idempotents, which contradicts $e \in \tilde{M}_{n-2}$. The assertion follows. $\square$

We are ready to list the reduced forms of all idempotents in C_n, using Lemma 5.2 and Theorem 5.1.

From Lemma 5.2 every idempotent in C_n is not of full support. Therefore the full subgraph Γ' (of the cycle) whose vertices correspond exactly to the support of an idempotent is an acyclic graph. Moreover such an idempotent can be treated as an idempotent in the Hecke–Kiselman monoid $\mathrm{HK}_{\Gamma'}$ with maximal possible support. Theorem 5.1 implies that there exists exactly one such idempotent. To find its form, we need to order the set of vertices $V(\Gamma')$ in such a way that if $i \to j$ in Γ' then $i < j$. Denote $V(\Gamma') = \{i_1,\ldots,i_k\}$ with $i_1 < \cdots < i_k$. Let us consider two cases.

1) If $\{1,n\} \nsubseteq V(\Gamma')$ then the arrow $n \to 1$ is not in Γ', hence $i \to j$ in Γ' implies that $i < j$.

2) If $\{1,n\} \subseteq V(\Gamma')$ then there exists $t \in \{1,\ldots,k-1\}$ such that $1,\ldots,t \in V(\Gamma')$ and $t+1 \notin V(\Gamma')$. In particular, the arrow $t \to t+1$ is not contained in Γ'. Therefore, it is possible to shift vertices of the cycle graph in such a way that the the arrow $n \to 1$ is not contained in the graph Γ' with shifted indices. Formally, Γ' is isomorphic to the acyclic subgraph Σ of the cycle of length n such that $V(\Sigma) = \{i_{t+1}-t, i_{t+2}-t, \ldots, i_k - t, i_1+(n-t), i_2+(n-t), \ldots, i_t+(n-t)\}$, via the isomorphism $i_j \mapsto i_j - t$ if $i_j > t$ and $i_j \mapsto i_j + n - t$ otherwise, for $j = 1,\ldots,k$. As $i_{t+1} > t+1$, it follows that $1 \notin V(\Sigma)$ and then this case reduces to case 1). The form of idempotents in $\mathrm{HK}_{\Gamma'}$ in this case follows, using Theorem 5.1, from the described isomorphism between Σ and Γ'.

Therefore, we have proved the following:

Corollary 5.3. *Idempotents in C_n are exactly the elements that can be written in the form e_X for some $X \subseteq \{1,\ldots,n\}$ such that $X \neq \{1,\ldots,n\}$, where e_X is defined as follows.*

1) If $X = \emptyset$ then $e_X = 1$.

2) If $\{1,n\} \nsubseteq X \neq \emptyset$, then $e_X = x_{i_1}x_{i_2}\cdots x_{i_j}$ where $X = \{i_1,\ldots,i_j\}$ with $i_1 < i_2 < \cdots < i_j$.

3) If $\{1,n\} \subseteq X$, let $k \in \{1,\ldots,n-1\}$ be such that $\{1,2,\ldots,k\} \subseteq X$ and $k+1 \notin X$. Then $e_X = x_{i_1}\cdots x_{i_s}x_1\cdots x_k$ where $i_s = n$ and $X = \{1,\ldots,k,i_1,\ldots,i_s\}$ with $k+1 < i_1 < \cdots < i_s = n$.

We now place idempotents from C_n in the chain of ideals I_i for $i = -1,\ldots,n-2$ from Theorem 2.44.

Proposition 5.4. *For every idempotent e_X in C_n such that $|X| \geqslant 2$ we have $e_X \in I_{|X|-2} \setminus I_{|X|-1}$. Moreover $x_i \notin I_{-1}$ for $i \in \{1,\ldots,n\}$.*

Proof. To prove the first statement, we will use the homomorphism $f : C_n \longrightarrow Map(\mathbb{Z}^n, \mathbb{Z}^n)$ defined after Lemma 2.23 in Section 2.2. We also follow the notation introduced in this section. From Lemma 2.25 we get that to show that $e_X \in I_{|X|-2}$ it is enough to check that $e_X \in Q_{|X|-2}$. This follows from the following technical lemma.

Lemma 5.5. *Assume that $w = x_{i_1}\cdots x_{i_k} \in C_n$ is such that for every $j,l \in \{1,\ldots,k\}$ if $j < l$ then $0 < i_l - i_j < n-1$ or $i_j - i_l \geqslant 2$. Then* $\mathrm{supp}(f(w)) = \{1,\ldots,n\} \setminus \{i_1,\ldots,i_k\}$.

Proof. We proceed by induction on the length of w. If $|w| = 1$ the claim is a straightforward consequence of the definition of $\mathrm{supp}(f(w))$, see Section 2.2. Assume now that the assertion holds for some $k-1$, where $k \geqslant 2$. Consider the word $w = x_{i_1}\cdots x_{i_k} \in C_n$ satisfying the condition from the lemma. Then $f(w)(m_1,\ldots,m_n) = f(x_{i_1})(f(x_{i_2}\cdots x_{i_k})(m_1,\ldots,m_n))$. Applying the inductive hypothesis to the word $w' = x_{i_2}\cdots x_{i_k}$ we obtain that $\mathrm{supp}(f(x_{i_2}\cdots x_{i_k})) = \{1,\ldots,n\} \setminus \{i_2,\ldots,i_k\}$. Moreover, as $i_1 \neq i_l$ for $l \geqslant 2$, we know that $f(x_{i_2}\cdots x_{i_k})(m_1,\ldots,m_n)$ has m_{i_1} on the i_1th coordinate. It is enough to check that m_{i_1} does not occur on any other coordinate of $f(x_{i_2}\cdots x_{i_k})(m_1,\dots,m_n)$. Indeed, every $f(x_j)$ changes only jth coordinate of the sequence on which it acts. Therefore, $f(w')(m_1,\dots,m_n)$ could have m_{i_1} on the coordinate different than i_1th, only if m_{i_1} had been rewritten firstly on the (i_1-1)th coordinate in the case $i_1\neq 1$ or on the nth coordinate if $i_1=1$. Thus we get that either $x_{i_1-1}\in\{x_{i_2},\dots,x_{i_k}\}$ or $i_1=1$ and $x_n\in\{x_{i_2},\dots,x_{i_k}\}$, which is not true as $i_l-i_1\in\{-n+1,\dots,-2\}\cup\{1,\dots,n-2\}$ for every $l>1$. It follows that $\operatorname{supp}(f(w))=\operatorname{supp}(f(w'))\setminus\{i_1\}=\{1,\dots,n\}\setminus\{i_1,\dots,i_k\}$. □

Remark 5.6. Elements satisfying conditions of Lemma 5.5 are exactly idempotents of C_n described in Corollary 5.3.

Proof. Let $w=x_{i_1}\cdots x_{i_k}\in C_n$ be a (reduced) word in C_n such that for every $j,l\in\{1,\dots,k\}$ if $j<l$ then $0<i_l-i_j<n-1$ or $i_j-i_l\geqslant 2$. If $i_1<\cdots<i_k$, then $(i_1,i_k)\neq(1,n)$ and thus w is indeed of the form 2) from Corollary 5.3. Let $1\leqslant s<k$ be such that $w=x_{i_1}\cdots x_{i_s}x_{i_{s+1}}\cdots x_{i_j}$, where $i_1<\cdots<i_s$ but $i_{s+1}>i_s$. Then we know that $i_{s+1}-i_s\geqslant 2$. It follows that $i_s=n$ and $i_{s+1}=1$, as otherwise the word w would contain a non-reduced factor of type (2) in Theorem 1.65. Thus $w=x_{i_1}\cdots x_nx_1x_{i_{s+2}}\cdots x_{i_k}$. From Lemma 2.3 and the fact that $|x_{i_{s+2}}\cdots x_{i_k}|_j=0$ for $j=\{i_1-1,n-1\}$ it then follows that w is of the form 3) in Corollary 5.3. The assertion follows. □

To continue with the proof of Proposition 5.4 note that from the remark and Lemma 5.5 we get that $|\operatorname{supp}(f(e_X))|=n-|X|$, and thus $e_X\in I_{|X|-2}$ for $|X|\geqslant 2$. Let us now check that $e_X\notin I_{|X|-1}$. We have an automorphism σ of C_n such that $\sigma(x_i)=x_{i+1}$ for every $i=1,\dots,n$ (where we agree that $x_{n+1}=x_1$), see Definition 2.29. By Lemma 2.36, it has the property that $\sigma(I_i)=I_i$ for $i\in\{0,\dots,n-3\}$. Applying the automorphism σ a few times if necessary, we may assume that the idempotent e_X is of the form $x_{i_1}\cdots x_{i_j}$ with $i_1<\cdots<i_j$ and $i_j=n-1$. Indeed, if $e_X=x_{i_1}\cdots x_{i_j}$ is such that $i_1<\cdots<i_j$ and $(i_1,i_j)\neq(1,n)$, then $\sigma^{n-1-i_j}(e_X)=x_{n-(i_j+1)+i_1}\cdots x_{n-(i_j+1)+i_j}$ is of the required form as $1\leqslant n-(i_j+1)+i_k<n$ for $k=1,\dots,j$. Moreover, if $e_X=x_{i_1}\cdots x_{i_s}x_nx_1\cdots x_k$ with $k+1<i_1<\cdots<i_s$ then $\sigma^{n-1-k}(e_X)=x_{i_1-k-1}\cdots x_{i_s-k-1}x_{n-k-1}\cdots x_{n-1}$, as $n+1\leqslant i_j+n-k-1<2n$ and thus $\sigma^{n-1-k}(e_X)$ is of the required form in this case, as well.
So, let e_X be of the form $x_{i_1}\cdots x_{i_j}$ with $i_1<\cdots<i_j$ and $i_j=n-1$. Then consider the element

$$w=(x_{i_1-1}x_{i_1-2}\cdots x_1)(x_{i_2-1}x_{i_2-2}\cdots x_2)\cdots(x_{i_j-1}x_{i_j-2}\cdots x_j).$$

Note that if $i_k=k$ for some $1\leqslant k\leqslant j$ then $i_1=1,\dots,i_k=k$. If $i_k=k$ then by $(x_{i_k-1}x_{i_k-2}\cdots x_k)$ we understand the trivial word.
Since $x_{i_{m+1}}\cdots x_{i_j}$ commutes with $x_{i_m-1}\cdots x_m$ for $m=1,\dots,j-1$, the element e_Xw can be written in C_n in the form

$$(x_{i_1}x_{i_1-1}\cdots x_1)(x_{i_2}x_{i_2-1}\cdots x_2)(x_{i_3}x_{i_3-1}\cdots x_3)\cdots(x_{i_j}x_{i_j-1}\cdots x_j).$$

From the description of the sets A_i in Theorem 2.1, it follows directly that $e_Xw\in A_{j-1}$. Theorem 2.32 gives that there exists $u\in C_n$ such that $ue_Xw=(q_{n,j-1})^N$ for some $N\geqslant 1$.

By the definition of the ideal I_{j-1} this implies that $e_X \notin I_{j-1} = I_{|X|-1}$. The assertion follows. The second part of the proposition is clear from the definition of I_{-1}. $\square$

5.2 Irreducible representations of the Hecke–Kiselman algebra $K[C_n]$

Our aim is to investigate the irreducible representations of the Hecke–Kiselman algebra $K[C_n]$. Let us recall that the field K is algebraically closed.

Our approach is based on Theorem 2.44. Although monoids C_n are infinite for all $n \geqslant 3$, the general idea is motivated by the representation theory of finite semigroups. Namely, as explained in Section 1.2.1 if S is a finite semigroup, then the irreducible representations of S can be obtained in terms of irreducible representations of the 0-simple principal factors of the semigroup S, which are semigroups of matrix type, see Theorem 1.48.

Moreover the Hecke–Kiselman monoid C_n is $\mathcal{J}$-trivial, Theorem 1.71. Representation theory of finite $\mathcal{J}$-trivial monoids (more generally finite $\mathcal{R}$-trivial monoids) can be easily described, see Corollary 5.7 in [53] or [12]. In particular, in the $\mathcal{J}$-trivial case representations can be parametrized by idempotents, Example 1.50. We will construct two types of representations of $K[C_n]$: those coming from the representations of $K_0[M_i]$, and those related to the idempotents in C_n. The sandwich matrices P_i are invertible as matrices in $M_{n_i}(K(q_{n,i}))$, see Corollary 2.53. Recall that here $K(q_{n,i})$ stands for the field of rational functions in the indeterminate $q_{n,i}$, and thus $K_0[M_i]$ are almost Munn algebras, Section 1.3. Simple modules over such algebras can be described, see Section 1.3.1. In Section 5.3 we will extend this result to our setting.

If P is a maximal ideal of $K[C_n]$, then from Theorem 1.25 $K[C_n]/P \simeq M_r(D)$ for $r \geqslant 1$ and a division algebra D. On the other hand, as explained in the proof of Theorem 3.6, $\mathrm{GKdim}(K[C_n]/P) = \mathrm{clKdim}(K[C_n]/P) = 0$ and thus $K[C_n]/P$ is finite dimensional. Therefore, assuming that the field K is algebraically closed, we get $D = K$. In particular, the irreducible representation corresponding to the maximal ideal P is the natural homomorphism $K[C_n] \to K[C_n]/P$. Conversely, the kernel of any irreducible representation of $K[C_n]$ is a (left) primitive ideal. Therefore, by Theorem 3.6, it is also a maximal ideal of $K[C_n]$. Consequently, there exists a bijection between maximal ideals and irreducible representations of the algebra. Bearing that in mind we describe the representation theory of $K[C_n]$ using both: maximal ideals and direct constructions of irreducible representations, interchangeably.

First, we are going to prove that every maximal ideal in $K_0[M_i]$ extends to a maximal ideal in the algebra $K[C_n]$.

Proposition 5.7. *Assume that I is a maximal ideal in $K_0[M_i]$. Then there exists a unique maximal ideal $\tilde{I}$ in $K[C_n]$ such that $\pi(\tilde{I}) \cap K_0[M_i] = I$ and $K_0[M_i]/I \simeq K[C_n]/\tilde{I}$, where $\pi : K[C_n] \to K[C_n]/K[I_i]$ is the natural homomorphism.*

Proof. We treat the semigroup M_i as a subsemigroup in I_{i-1}/I_i. Then we know from

Corollary 2.46 that M_i is a two-sided ideal in C_n/I_i. Let I be the ideal in $K_0[M_i]$ such that the algebra $K_0[M_i]/I$ is simple. Then, as $K_0[M_i]$ is an ideal in $K_0[C_n/I_i]$, we have that $I \subseteq K_0[C_n/I_i]IK_0[C_n/I_i] \subseteq K_0[M_i]$. Therefore either $I = K_0[C_n/I_i]IK_0[C_n/I_i]$ or $K_0[C_n/I_i]IK_0[C_n/I_i] = K_0[M_i]$. On the other hand

$$\begin{aligned}&(K_0[C_n/I_i]IK_0[C_n/I_i])^3 \subseteq \\ &\quad (K_0[C_n/I_i]IK_0[C_n/I_i])I(K_0[C_n/I_i]IK_0[C_n/I_i]) \subseteq \\ &\quad K_0[M_i]IK_0[M_i] \subseteq I.\end{aligned}$$

It follows that $I = K_0[C_n/I_i]IK_0[C_n/I_i]$. Therefore $\overline{I} = \pi^{-1}(I)$ is an ideal in $K[C_n]$, where $\pi : K[C_n] \to K[C_n]/K[I_i]$ is the natural homomorphism. Moreover, from $\pi(\overline{I}) \cap K_0[M_i] = I$ (as $M_i \setminus \{0\} = \tilde{M}_i \subseteq I_{i-1} \setminus I_i$) we get that $K_0[M_i]/I \lhd K[C_n]/\overline{I}$. As $K_0[M_i]/I$ is a simple PI-algebra, by Kaplansky Theorem 1.25 it follows that $K_0[M_i]/I$ is an algebra with an identity. What is more, it is an ideal of $K[C_n]/\overline{I}$, so that $K_0[M_i]/I = (K[C_n]/\overline{I}) \cdot f$ for a central idempotent f. Indeed, let f be the identity of $K_0[M_i]/I$. Then for any $x \in K[C_n]/\overline{I}$ we have $fx, xf \in K_0[M_i]/I$ and thus $fx = fxf = xf$, that is f is central idempotent. As $K_0[M_i]/I$ is simple, it follows that $K_0[M_i]/I = (K[C_n]/\overline{I})f$. If we consider the natural epimorphism $\varphi : K[C_n] \twoheadrightarrow (K[C_n]/\overline{I})\cdot f$ then $\tilde{I} = \ker\varphi$ is an ideal in $K[C_n]$ such that $K[C_n]/\tilde{I} \simeq K_0[M_i]/I$. The uniqueness of the ideal $\tilde{I}$ is a direct consequence of the construction because $\bar{I}$ is the ideal of $K[C_n]$ generated by I. □

The next step is to investigate any irreducible representation of the algebra $K[C_n]$, not necessarily arising from the representation of $K_0[M_i]$ in the way described by Proposition 5.7. Let us consider any maximal ideal P of $K[C_n]$. Recall that the minimal prime ideals J_i ($i = 0, \ldots, n-2$) of $K[C_n]$ have been described in Theorem 3.4. From Theorem 3.6 we know that, $J_i \subseteq P$ for some $i \in \{0, \ldots, n-2\}$ and thus also $I_i \subseteq P$ ($I_i \subseteq J_i$ from the definition). Take the minimal $i \geqslant 0$ such that $I_i \subseteq P$. Then either $K[\tilde{M}_i] \nsubseteq P$ or $K[\tilde{M}_i] \subseteq P$.

Assume first that $K[\tilde{M}_i] \nsubseteq P$. Then $P \cap K[\tilde{M}_i] \neq \{0\}$, as otherwise $P \cap K[q_{n,i}] = \{0\}$ and $I_i \subseteq P$ which, together with maximality of the ideal P and the definition of J_i, implies that $P = J_i$. That contradicts Theorem 3.6. Then $\pi(P) \cap K_0[M_i] \neq \{0\}$, where π is the natural homomorphism $K[C_n] \to K[C_n]/K[I_i]$, is an ideal in $K_0[M_i]$ such that $K_0[M_i]/(\pi(P) \cap K_0[M_i]) \lhd K[C_n]/P \simeq M_j(K)$ for some $j \geqslant 1$. Therefore we get that $K_0[M_i]/(\pi(P) \cap K_0[M_i]) = K[C_n]/P$. In particular, in this case the maximal ideal P comes from a maximal ideal in $K_0[M_i]$, in the way described in Proposition 5.7.

Now, let us consider the second case, namely $K[\tilde{M}_i] \subseteq P$.

If $i = 0$ and $I_{-1} \subseteq P$, then every irreducible representation corresponding to the maximal ideal P comes from the irreducible representation of the finite monoid C_n/I_{-1}. As C_n is a $\mathcal{J}$-trivial monoid, also C_n/I_{-1} is $\mathcal{J}$-trivial, as explained in Remark 1.39. From Section 5.1 it follows that there are exactly $n+1$ idempotents in $C_n \setminus I_{-1}$ (generators $x_1, \ldots, x_n$ and 1).

By Example 1.50, there exists a bijection between isomorphism classes of irreducible representations of the monoid C_n/I_{-1} and idempotents $e \in C_n \setminus I_{-1}$. More precisely, let e be such an idempotent and consider $\overline{I}(e) = \{m \in C_n/I_{-1} : e \notin (C_n/I_{-1})m(C_n/I_{-1})\}$, that is the ideal of non-generators of the principal ideal $(C_n/I_{-1})e(C_n/I_{-1})$. Then the corresponding one-dimensional irreducible representation $\varphi_e : K[C_n/I_{-1}] \to K$ is given for $m \in C_n/I_{-1}$ by

$$\varphi_e(m) = \begin{cases} 0 & \text{if } m \in \overline{I}(e) \\ 1 & \text{otherwise} \end{cases} .$$

Moreover, different idempotents lead to non-isomorphic representations.

Then the induced irreducible representation of $K[C_n]$ is given by $K[C_n] \twoheadrightarrow K[C_n/I_{-1}] \xrightarrow{\varphi_e} K$ and thus leads to a maximal ideal P in $K[C_n]$ associated to e. All possible maximal ideals P such that $I_{-1} \subseteq P$ are of the above form. In addition, for different idempotents $e \in C_n \setminus I_{-1}$ we get different ideals. Indeed, from Proposition 1.49, it follows that for every idempotent e and $m \in C_n/I_{-1}$ we have $m \notin \overline{I}(e)$ if and only if $em = e$. Therefore if $\overline{I}(e) = \overline{I}(f)$ for idempotents e and f, then $e \notin \overline{I}(f)$ and $f \notin \overline{I}(e)$, that is $fe = f$ and $ef = e$. We get that f and e are $\mathcal{J}$–related in the $\mathcal{J}$-trivial monoid C_n/I_{-1}, which means that $e = f$. It follows that for different idempotents we get different maximal ideals P.

Assume now that $K[\tilde{M}_i] \subseteq P$ and $i > 0$, that is $I_{i-1} \nsubseteq P$. Consider the finite semigroup $I_{i-1}/(I_i \cup \tilde{M}_i)$. In this case every simple $K[C_n]$-module with annihilator P is also a simple $K[I_{i-1}/(I_i \cup \tilde{M}_i)]$-module. From Theorem 1.48 it follows that every such a module W has an apex, that is an idempotent $e \in I_{i-1}/(I_i \cup \tilde{M}_i)$ satisfying conditions $eW \neq 0$ and $K[\overline{I}(e)] = Ann(W)$, where $\overline{I}(e) = \{w \in I_{i-1}/(I_i \cup \tilde{M}_i) : e \notin (I_{i-1}/(I_i \cup \tilde{M}_i))w(I_{i-1}/(I_i \cup \tilde{M}_i))\}$. From the description of idempotents in Section 5.1 it is clear that there exists an idempotent $e \in I_{i-1}/(I_i \cup \tilde{M}_i)$. Using the characterization of simple $K[I_{i-1}/(I_i \cup \tilde{M}_i)]$-modules and knowing that there are finitely many idempotents in $I_{i-1}/(I_i \cup \tilde{M}_i)$, we can choose a minimal one with respect to the $\mathcal{J}$–relation, defined in Section 1.2, in $I_{i-1} \setminus (I_i \cup \tilde{M}_i)$ such that $e \notin P$.

Next, consider the ideal $N_e = I_i \cup \tilde{M}_i \cup I(e)$ in C_n, where $I(e) = \{w \in C_n : e \notin C_n w C_n\}$. The monoid C_n is $\mathcal{J}$-trivial and from Proposition 1.49 we have that for every $m \in C_n$ either $em = e$ or $m \in I(e) \subseteq N_e$ and symmetrically either $me = e$ or $m \in I(e)$. It follows that $K[N_e] \subseteq P$, $\{\theta, e\}$ is a two-sided ideal in C_n/N_e and thus in particular $K[C_n]/K[N_e] \simeq Ke \oplus (K[C_n]/K[N_e])(1-e)$. Therefore in this case the irreducible representation φ of $K[C_n]$ is one-dimensional and given by $K[C_n] \twoheadrightarrow K[C_n/N_e] \xrightarrow{\varphi_e} K$, where for any $m \in C_n \setminus N_e$

$$\varphi_e(m) = \begin{cases} 1 \text{ if } em = e \\ 0 \text{ if } em \in N_e \end{cases} .$$

As in the previous case, for every choice of an idempotent $e \in I_{i-1} \setminus (\tilde{M}_i \cup I_i)$ we get a different maximal ideal P_e and in such a way we get all maximal ideals in this case.

Thus we have proved the following theorem.

Theorem 5.8. *Let $\varphi : K[C_n] \to M_j(K)$ be an irreducible representation of the Hecke–Kiselman algebra $K[C_n]$ over an algebraically closed field K. If $\varphi(K[I_{n-3}]) \neq 0$ set $i = n-2$.*

Otherwise take the minimal $i \in \{-1, \dots, n-3\}$ *such that* $\varphi(K[I_i]) = 0$.

1) *If* $i \geqslant 0$ *then*

   a) *either* $\varphi(K[\tilde{M_i}]) \neq 0$ *and the representation* φ *is induced by a representation of* $K_0[M_i]$ *as described in Proposition 5.7;*

   b) *or* $\varphi(K[\tilde{M_i}]) = 0$ *and the representation* φ *is one-dimensional and induced by an idempotent* $e \in I_{i-1} \setminus I_i$. *For any* $m \in C_n$ *it is given by*

$$\varphi(m) = \begin{cases} 1 \text{ if } em = e \\ 0 \text{ if } em \in N_e \end{cases},$$

   where $N_e = I_i \cup \tilde{M_i} \cup I(e)$ *is the ideal in* C_n *such that* $I(e) = \{w \in C_n : e \notin C_n w C_n\}$.

2) *If* $i = -1$ *then the representation* φ *is one-dimensional and induced by an idempotent* $e \in C_n \setminus I_{-1}$. *It is given for* $m \in C_n$ *by*

$$\varphi(m) = \begin{cases} 0 & \text{if } m \in \overline{I}(e) \cup I_{-1} \\ 1 & \text{otherwise} \end{cases},$$

   where $\overline{I}(e) = \{m \in C_n \setminus I_{-1} : e \notin (C_n/I_{-1})m(C_n/I_{-1})\}$.

5.3 Irreducible representations of $K_0[M_i]$

In view of Theorem 5.8, in order to complete our discussion of irreducible representations of $K[C_n]$, the final step is to characterize irreducible representations of $K_0[M_i]$. Recall that the field K is algebraically closed. We start with a construction of certain family of such representations using the representation theory of Munn algebras, see Section 1.3.1. To characterize irreducible representations of semigroups $M_i = \mathcal{M}^0(Q_i, A_i, B_i; P_i)$ we cannot use Theorem 1.59 directly, as the infinite cyclic semigroup Q_i generated by $q_{n,i}$ is not a group. On the other hand, note that Q_i is contained in the cyclic group generated by $q_{n,i}$, denoted by $\mathrm{gr}(q_{n,i})$. As we prove in the next two propositions, all representations of M_i come from representations of the semigroup of matrix type associated to such a group $\mathrm{gr}(q_{n,i})$, sets A_i, B_i and the sandwich matrix P_i. Moreover, every irreducible representation of the latter semigroup restricts to a representation of M_i.

Proposition 5.9. *Consider the semigroup of matrix type* $M_i = \mathcal{M}^0(Q_i, A_i, B_i; P_i)$ *from Theorem 2.44. Then for every* $\lambda \in K^*$ *there exists a unique irreducible representation* $\psi_\lambda : K_0[M_i] \to M_r(K)$, *for some* $1 \leqslant r \leqslant |A_i|$, *induced by the irreducible representation of* $gr(q_{n,i})$ *such that* $q_{n,i} \mapsto \lambda$. *Such a representation is the restriction of the representation* $\overline{\varphi}_\lambda : \mathcal{M}^0(gr(q_{n,i}), A_i, B_i; P_i) \to M_r(K)$ *described in Theorem 1.59 coming from the mentioned irreducible representation of* $gr(q_{n,i})$.

Proof. We know that $|A_i| = |B_i|$ and the sandwich matrix P_i, is invertible as a matrix in $M_{n_i}(K(q_{n,i}))$, see Corollary 2.53. Therefore, every algebra $K_0[M_i]$ embeds into the algebra of matrix type $K_0[\mathcal{M}^0(gr(q_{n,i}), A_i, B_i; P_i)] \simeq \mathcal{M}(K[q_{n,i}, q_{n,i}^{-1}], A_i, B_i; P_i)$.

For any $\lambda \in K^*$ consider the homomorphism $\varphi_\lambda : K[q_{n,i}] \to K$ such that $q_{n,i} \mapsto \lambda$. Every such a homomorphism extends to an irreducible representation of the group $gr(q_{n,i})$. Therefore, from Theorem 1.59, it induces an irreducible representation $\mathcal{M}(K[q_{n,i}, q_{n,i}^{-1}], A_i, B_i, P_i) \to \mathcal{M}(K, A_i, B_i; \overline{P}_i) \to M_r(K)$, which is an epimorphism, where $\overline{P}_i = (\overline{p}_{b,a})$ with $\overline{p}_{b,a} = \varphi(p_{b,a})$ for every $a \in A_i$, $b \in B_i$. The map $\overline{\varphi}_\lambda : \mathcal{M}(K[q_{n,i}, q_{n,i}^{-1}], A_i, B_i; P_i) \to \mathcal{M}(K, A_i, B_i; \overline{P}_i)$ is given by $\overline{\varphi}_\lambda(q_{n,i}^k; a, b) = (\lambda^k; a, b)$ for all $k \in \mathbb{Z}$, $a \in A_i, b \in B_i$. It is enough to check that the restriction of the above map to $K_0[M_i]$ is also an epimorphism, as then it gives an irreducible representation $\psi_\lambda : K_0[M_i] \to M_r(K)$. To show this, let us notice that for any $\alpha \in K$, $a \in A_i$, $b \in B_i$ we have $(\alpha\lambda^{-1}q_{n,i}; a, b) \in K_0[M_i]$ and $\overline{\varphi}_\lambda(\alpha\lambda^{-1}q_{n,i}; a, b) = \alpha\lambda^{-1}(\lambda; a, b) = (\alpha; a, b)$. The claim now follows from the fact that $\mathcal{M}(K, A_i, B_i; \overline{P}_i) \to M_r(K)$ is also an epimorphism. $\square$

From the definition of the sets A_i, B_i that can be found before Theorem 2.1 it follows that the trivial word 1 is in both sets A_i and B_i. Therefore, using formula (2.3.1) for coefficients of sandwich matrices $P_i = (p_{ba})$ of the semigroups of matrix type M_i from the proof of Corollary 2.42, we get that $p_{11} = 1$. Then $M_{1,1} = \{(q_{n,i}^k; 1, 1) \in M_i : k \geqslant 1\} \subseteq M_i$ is isomorphic to the infinite cyclic semigroup.

The irreducible representations of M_i described in Proposition 5.9 come from representations of the completely 0–simple closure of M_i, namely $cl(M_i) = \mathcal{M}^0(gr(q_{n,i}), A_i, B_i; P_i)$. In particular, the image of $(q_{n,i}; 1, 1)$ is equal to λe for an idempotent matrix e of rank 1 and some $\lambda \in K^*$. Now we show that every irreducible representation of M_i is of such a form and extends to a representation of $\mathcal{M}^0(gr(q_{n,i}), A_i, B_i; P_i)$.

Proposition 5.10. *Every irreducible representation $\varphi : K_0[M_i] \to M_r(K)$ of $K_0[M_i]$ is such that $\varphi(q_{n,i}; 1, 1) = \lambda e$, where $\lambda \in K^*$ and e is an idempotent of rank 1. Such a representation φ can be uniquely extended to an irreducible representation of $K_0[\mathcal{M}^0(gr(q_{n,i}), A_i, B_i, P_i)]$.*

Consider any irreducible representation $\varphi : K_0[M_i] \to M_r(K)$, where $M_i = \mathcal{M}^0(Q_i, A_i, B_i; P_i)$. In particular, as K is algebraically closed, the representation is onto. The first step of the proof is to investigate the image of $M_{1,1}$ under φ.

Let us notice that $\varphi(M_{1,1})$ is non-zero. Indeed, suppose that in particular $\varphi(q_{n,i}; 1, 1) = 0$. For any $(q_{n,i}^k; a, b) \in M_i$ either $p_{ba} = 0$ or $p_{ba} = q_{n,i}^\alpha$ for some $\alpha \geqslant 0$. In the first case $\varphi((q_{n,i}^k; a, b)^{N_0}) = \varphi(0; a, b) = 0$ for $N_0 \geqslant 2$ and in the latter we have $(q_{n,i}^k; a, b)^{N_0} = (q_{n,i}^{N_0k+(N_0-1)\alpha}; a, b)$ with $N_0k + (N_0 - 1)\alpha \geqslant 3$ for $N_0 \geqslant 3$. Moreover, from the definition (equation 2.3.1) of sandwich matrices P_i it follows that $p_{11} = 1$. Therefore, if $\varphi(q_{n,i}; 1, 1) = 0$, we would have $\varphi((q_{n,i}^k; a, b)^{N_0}) = \varphi(q_{n,i}; a, 1)\varphi(q_{n,i}^{N_{ba}}; 1, 1)\varphi(q_{n,i}; 1, b) = 0$, where $N_{ba} = N_0k + (N_0 - 1)\alpha - 2 \geqslant 1$ for all $N_0 \geqslant 3$. Then $\varphi(M_i)$ would be nil and thus nilpotent in the monoid $(M_r(K), \cdot)$, see Proposition 2.14 in [44], which leads to a contradiction. A similar argument shows that $\varphi(q_{n,i}; 1, 1)$ cannot be nilpotent.

Lemma 5.11. *The image $\varphi(K[M_{1,1}])$ is a commutative integral domain. Consequently, $\varphi(K[M_{1,1}]) \simeq K$.*

Proof. As noted above, we have $p_{11} = 1$ in the sandwich matrix P_i, which implies that $K[M_{1,1}]$ is isomorphic to $K[q_{n,i}]q_{n,i}$ and in particular $\varphi(K[M_{1,1}])$ is commutative.
Suppose now that $\varphi(K[M_{1,1}])$ is not an integral domain, that is $uv = 0$ for some non-zero $u, v \in \varphi(K[M_{1,1}])$ with $u = \varphi(\alpha)$, $v = \varphi(\beta)$, where $\alpha = (f; 1, 1)$, $\beta = (g; 1, 1)$. Define ideals in $K_0[M_i]$ as follows:

$$I = \{y = (y_{ab}) \in K_0[M_i] : y_{ab} \in K[q_{n,i}]f \text{ for all } a \in A_i, b \in B_i\},$$

$$J = \{y = (y_{ab}) \in K_0[M_i] : y_{ab} \in K[q_{n,i}]g \text{ for all } a \in A_i, b \in B_i\}.$$

Then $IJ \subseteq \{y = (y_{ab}) \in K_0[M_i] : y_{ab} \in K[q_{n,i}]fg \text{ for all } a \in A_i, b \in B_i\}$ and as $\varphi(I) \neq 0$, $\varphi(J) \neq 0$, it follows that $\varphi(I) = \varphi(J) = M_r(K)$. Therefore,

$$\varphi((q_{n,i}; 1, 1)IJ(q_{n,i}; 1, 1)) = \varphi(q_{n,i}; 1, 1)M_r(K)\varphi(q_{n,i}; 1, 1) \neq 0.$$

On the other hand $\varphi((q_{n,i}; 1, 1)IJ(q_{n,i}; 1, 1)) \subseteq \varphi((K[q_{n,i}]fg; 1, 1)) = 0$, as $\varphi((f; 1, 1)(g; 1, 1)) = uv = 0$, which leads to a contradiction. Consequently, $\varphi(K[M_{1,1}])$ is a finite dimensional domain and, as K is algebraically closed, it follows that $\varphi(K[M_{1,1}]) \simeq K$. $\square$

Now we are ready to prove Proposition 5.10.

Proof. As explained earlier, if we denote $q = (q_{n,i}; 1, 1)$, then $\varphi(q)$ is non-zero and not nilpotent. Therefore from Proposition 1.3 in [44] it follows that for k big enough $\varphi(q)^k$ lies in a maximal subgroup of the monoid $(M_n(K), \cdot)$. Then $\varphi : q^k K_0[M_i]q^k \to \varphi(q)^k M_r(K)\varphi(q)^k$ and $\varphi(q)^k M_r(K)\varphi(q)^k = eM_r(K)e$ for some idempotent e, see Lemma 1.10 in [44]. Moreover, by Lemma 5.11, $q^k K_0[M_i]q^k$ is contained in a commutative ring $K[M_{1,1}]$, thus it follows that $\mathrm{rk}(e) = 1$. Now, we have that $\varphi(q) = g \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix} g^{-1}$ for a matrix $a_0 \in M_p(K)$, such that $p \in \{1, \dots, r\}$, $\mathrm{rk}(a_0^2) = \mathrm{rk}(a_0)$, a nilpotent matrix $a_1 \in M_{r-p}(K)$, where $g \in Gl_r(K)$ and $\varphi(q^k) = g \begin{pmatrix} a_0^k & 0 \\ 0 & 0 \end{pmatrix}^{-1} g^{-1}$. As $\varphi(q^k) \in eM_r(K)e$ for an idempotent e of rank 1, it follows that $p = 1$ and we can assume that $a_0 \in K^*$. Moreover, if $a_1 \in M_{r-1}(K)$ is non-zero, then $g \begin{pmatrix} a_0^k & 0 \\ 0 & 0 \end{pmatrix} g^{-1}$, $g \begin{pmatrix} a_0 & 0 \\ 0 & a_1 \end{pmatrix} g^{-1} \in \varphi(K[M_{1,1}])$, so $\dim_K \varphi(K[M_{1,1}]) \geqslant 2$, which contradicts the assertion of Lemma 5.11. Thus in particular $\varphi(q_{n,i}; 1, 1) = \lambda e$ for $\lambda \in K^*$ and e is an idempotent with $\mathrm{rk}(e) = 1$.
Now consider the irreducible representation $\varphi : K_0[M_i] \to M_r(K)$ such that $\varphi(q_{n,i}; 1, 1) = \lambda e$. As $M_i = \{(q_{n,i}^m; a, b) : m \geqslant 1, a \in A_i, b \in B_i\}$ and the sandwich matrix P_i has $p_{11} = 1$, we have $\varphi(q_{n,i}^m; a, b) = \varphi(q_{n,i}; a, 1)\varphi(q_{n,i}^{m-2}; 1, 1)\varphi(q_{n,i}; 1, b) = \varphi(q_{n,i}; a, 1)\lambda^{m-2}e\varphi(q_{n,i}; 1, b)$ for $m \geqslant 3$. Let us define the extension $\overline{\varphi} : K_0[\mathcal{M}^0(gr(q_{n,i}), A_i, B_i; P_i)] \to M_r(K)$ for any $(q_{n,i}^p; a, b)$,

where $p \in \mathbb{Z}$, $a \in A_i$, $b \in B_i$ by

$$\overline{\varphi}(q_{n,i}^p; a, b) = \varphi(q_{n,i}; a, 1)\lambda^{p-2} e\varphi(q_{n,i}; 1, b).$$

To verify that $\overline{\varphi}$ is a homomorphism take any $(q_{n,i}^l; a, b)$, $(q_{n,i}^m; a', b') \in K_0[M_i]$. Then either $(q_{n,i}^l; a, b)(q_{n,i}^m; a', b') = (q_{n,i}^{l+m+n_{ba'}}; a, b')$ if $p_{ba'} = q_{n,i}^{n_{ba'}}$ or $(q_{n,i}^l; a, b)(q_{n,i}^m; a', b') = \theta$ if $p_{ba'} = 0$. In the first case also $(q_{n,i}; 1, b) \cdot (q_{n,i}; a', 1) = (q_{n,i}^{n_{ba'}+2}; 1, 1)$. We then get

$$\begin{aligned}
\overline{\varphi}(q_{n,i}^l; a, b)\overline{\varphi}(q_{n,i}^m; a', b') &= \varphi(q_{n,i}; a, 1)\lambda^{l-2} e\varphi((q_{n,i}; 1, b) \cdot (q_{n,i}; a', 1))\lambda^{m-2} e\varphi(q_{n,i}; 1, b') = \\
&= \varphi(q_{n,i}; a, 1)\lambda^{l-2} e\varphi(q_{n,i}^{n_{ba'}+2}; 1, 1)\lambda^{m-2} e\varphi(q_{n,i}; 1, b') = \\
&= \varphi(q_{n,i}; a, 1)\lambda^{l+m+n_{ba'}-2} e\varphi(q_{n,i}; 1, b') = \\
&= \overline{\varphi}(q_{n,i}^{l+m+n_{ba'}}; a, b').
\end{aligned}$$

Moreover $(q_{n,i}^l; a, b)(q_{n,i}^m; a', b') = \theta$ if and only if $(q_{n,i}; 1, b)(q_{n,i}; a', 1) = \theta$, and thus in the second case

$$\overline{\varphi}(q_{n,i}^l; a, b)\overline{\varphi}(q_{n,i}^m; a', b') = \overline{\varphi}((q_{n,i}^l; a, b)(q_{n,i}^m; a', b')) = 0.$$

Therefore $\overline{\varphi}$ is a homomorphism. Let us denote $N_i = \{(q_{n,i}^k; a, b) : k \geqslant 3, a \in A_i, b \in B_i\} \subseteq M_i$. Then $K_0[N_i]$ is an ideal in $K_0[M_i]$ and it is clear that $\overline{\varphi}|_{K_0[N_i]} = \varphi|_{K_0[N_i]}$. Moreover $\varphi(K_0[N_i]) = M_r(K)$ and in particular $\varphi(f) = 1$ for some $f \in K_0[N_i]$. Then for any $g \in K_0[M_i]$ we have $\varphi(g) = \varphi(g)\varphi(f) = \varphi(gf)$. It follows that φ is uniquely determined by $\varphi|_{K_0[N_i]}$. Therefore $\overline{\varphi}|_{K_0[M_i]} = \varphi$, in other words the irreducible representation φ can be extended to the representation $\overline{\varphi}$ of $K_0[\mathcal{M}^0(gr(q_{n,i}), A_i, B_i; P_i)]$. From Theorem 1.59 it follows that every irreducible representation of $K_0[\mathcal{M}^0(gr(q_{n,i}), A_i, B_i; P_i)]$ is uniquely determined (up to equivalence) by its value on the element $(q_{n,i}; 1, 1) \in M_i$. Therefore the extension of the constructed representation is unique. $\square$

5.4 Irreducible representations of PI Hecke–Kiselman algebras

In this section we describe irreducible representations of arbitrary Hecke–Kiselman algebras satisfying a polynomial identity. Recall from Theorem 1.72, that this condition can be characterized by the property that the corresponding graph does not contain two cyclic subgraphs (that is subgraphs which are oriented cycles) connected by an oriented path.

The radical of the algebra $K[\mathrm{HK}_\Theta]$, denoted by $\mathcal{J}(K[\mathrm{HK}_\Theta])$, in this case was described in Theorem 3.9. Assume that Θ' is the subgraph of Θ obtained by deleting all arrows $x \to y$ that are not contained in any cyclic subgraph of Θ. Every connected component of Θ' is either a singleton or an oriented cycle. Then $K[\mathrm{HK}_\Theta]/\mathcal{J}(K[\mathrm{HK}_\Theta]) \cong K[\mathrm{HK}_{\Theta'}]$ and it is the tensor product of algebras $K[\mathrm{HK}_{\Theta_i}]$ of the connected components $\Theta_1, \ldots, \Theta_m$ of Θ', each being isomorphic to $K \oplus K$ or to the algebra $K[C_j]$, for some $j \geqslant 3$.

Theorem 5.12. *Assume that Θ is a finite oriented graph such that $K[\mathrm{HK}_\Theta]$ is a PI-algebra and Θ' is the subgraph of Θ as described above, with the connected components $\Theta_1, \ldots, \Theta_m$.*

Then the maximal ideals of $K[\mathrm{HK}_\Theta]$ are in a bijection with maximal ideals of $K[\mathrm{HK}_{\Theta'}]$. The latter maximal ideals are of the form

$$\sum_{i=1}^{m} K[\mathrm{HK}_{\Theta_1}] \otimes \cdots \otimes K[\mathrm{HK}_{\Theta_{i-1}}] \otimes P_i \otimes K[\mathrm{HK}_{\Theta_{i+1}}] \otimes \cdots \otimes K[\mathrm{HK}_{\Theta_m}]$$

for maximal ideals P_i of $K[\mathrm{HK}_{\Theta_i}]$.

In particular, from Theorem 5.12 and the Kaplansky theorem (Theorem 1.25) it follows that all irreducible representations of the Hecke–Kiselman algebra $K[\mathrm{HK}_\Theta]$ are determined by representations of the algebras associated to the connected components $\Theta_1, \ldots, \Theta_m$ of the subgraph Θ' obtained by erasing all arrows not contained in any cyclic subgraph of Θ. Every such component is either an oriented cycle of length $j \geqslant 3$ or a singleton. Note that maximal ideals (irreducible representations) of $K[C_j]$ have been characterized in Theorem 5.8 and Section 5.3 and if Θ_i is a singleton, then $K[\mathrm{HK}_{\Theta_i}] \cong K \oplus K$ has two obvious maximal ideals.

Then every irreducible representation of $K[\mathrm{HK}_\Theta]$ is of the form

$$K[\mathrm{HK}_\Theta] \to K[\mathrm{HK}_{\Theta_1}] \otimes \cdots \otimes K[\mathrm{HK}_{\Theta_m}] \to M_{r_1}(K) \otimes \cdots \otimes M_{r_m}(K) \xrightarrow{\simeq} M_{r_1 \cdots r_m}(K),$$

where the first map is a natural epimorphism $K[\mathrm{HK}_\Theta] \to K[\mathrm{HK}_\Theta]/\mathcal{J}(K[\mathrm{HK}_\Theta]) \simeq K[\mathrm{HK}_{\Theta_1}] \otimes \cdots \otimes K[\mathrm{HK}_{\Theta_m}]$, and the second homomorphism is the natural homomorphism $\psi_1 \otimes \cdots \otimes \psi_m$ for the irreducible representations $\psi_i : K[\mathrm{HK}_\Theta] \to M_{r_i}(K)$ for $i = 1, \ldots, m$.

Proof of Theorem 5.12. As we know that $K[\mathrm{HK}_\Theta]/\mathcal{J}(K[\mathrm{HK}_\Theta]) \cong K[\mathrm{HK}_{\Theta'}]$ it is clear that there exists a one-to-one correspondence between maximal ideals of $K[\mathrm{HK}_\Theta]$ and those of $K[\mathrm{HK}_{\Theta'}]$.

So, it remains to find all maximal ideals in $K[\mathrm{HK}_{\Theta'}]$. Assume that Θ' has only two connected components, that is $K[\mathrm{HK}_{\Theta'}] = R_1 \otimes R_2$, where $R_i = K[\mathrm{HK}_{\Theta_i}]$ are isomorphic to either $K[C_j]$, for some $j \geqslant 3$, or $K \oplus K$. The general case can be proved analogously. Let P be a maximal ideal of $K[\mathrm{HK}_{\Theta'}]$ and $\pi : R_1 \otimes R_2 \to (R_1 \otimes R_2)/P$ the natural projection. Since $R_1 \otimes R_2$ is a PI-algebra over an algebraically closed field, from Kaplansky theorem it follows that $(R_1 \otimes R_2)/P \simeq M_r(K)$ for some $r \geqslant 1$. Denote $\bar{R}_1 = \pi(R_1 \otimes K)$ and $\bar{R}_2 = \pi(K \otimes R_2)$. Recall that the Jacobson radical of a finitely generated PI-algebra is nilpotent, as mentioned in Section 3.3. We claim that the algebras $\bar{R}_i$ are semisimple. Indeed, let N_1/P be a nilpotent ideal in $\bar{R}_1$. Then $N_1(1 \otimes R_2)$ is a nilpotent ideal in $K[\mathrm{HK}_{\Theta'}]$, as $N_1(1 \otimes R_2) = (1 \otimes R_2)N_1$. It follows that $N_1 = 0$ and $\bar{R}_1$ is semisimple, as it is finite dimensional. A symmetric argument shows that $\bar{R}_2$ is also semisimple. If $\bar{R}_i$ was not simple, then it would contain a non-trivial central idempotent. This idempotent would be then also central in $K[\mathrm{HK}_{\Theta'}]/P$, a contradiction. Therefore $\bar{R}_i \simeq M_{r_i}(K)$ for $i = 1, 2$. Projection $\pi : K[\mathrm{HK}_{\Theta'}] \to K[\mathrm{HK}_{\Theta'}]/P$

factors through $K[\mathrm{HK}_{\Theta'}] \xrightarrow{\bar{\pi}_1} \bar{R}_1 \otimes \bar{R}_2 \xrightarrow{\bar{\pi}_2} K[\mathrm{HK}_{\Theta'}]/P$. Moreover, as $\bar{R}_1 \otimes \bar{R}_2 \simeq M_{r_1 r_2}(K)$, $\bar{\pi}_2$ is an isomorphism. Then we can assume that $\pi : K[\mathrm{HK}_{\Theta'}] \twoheadrightarrow M_r(K)$ where $r = r_1 r_2$. If we denote $\pi_1 = \pi|_{R_1 \otimes K}$ and $\pi_2 = \pi|_{K \otimes R_2}$, then $\ker(\pi_i) = P_i$ are maximal ideals in R_i. It can be easily checked that $P = P_1 \otimes R_2 + R_1 \otimes P_2$.

Conversely, if $P = P_1 \otimes R_2 + R_1 \otimes P_2$ for maximal ideals $P_1 \triangleleft R_1$, $P_2 \triangleleft R_2$, then $(R_1 \otimes R_2)/P \simeq R_1/P_1 \otimes R_2/P_2$. As R_i are finitely generated PI-algebras it follows that $R_1/P_1 \otimes R_2/P_2 \simeq M_{r_1}(K) \otimes M_{r_2}(K) \simeq M_{r_1 r_2}(K)$. Therefore P is indeed a maximal ideal in $K[HK_{\Theta'}]$. $\square$

Chapter 6

Gelfand–Kirillov dimension of Hecke–Kiselman algebras

In this chapter we describe the Gelfand–Kirillov dimension of Hecke–Kiselman algebras associated to oriented graphs in terms of numerical invariants of the underlying graph. The results of this chapter come from [58]. Our main theorem can be seen as a natural continuation of in [39, Theorem 1], see Theorem 1.72. The methods rely extensively on the property discovered in [40], namely on the fact that Hecke–Kiselman algebras are automaton.

6.1 Growth of automaton algebras

Now let us restrict our attention to automaton algebras. Recall from Section 1.4.3 that an algebra A is automaton if its set of normal forms $N(A)$ (with respect to certain set of generators and an ordering in this set) is a regular language. What is more, from Theorem 1.73 we know that the Gelfand–Kirillov dimension of every algebra with this property is infinite or it is an integer. The value of GK dimension is related to certain forms of regular-expression representations of the regular languages of normal words in the algebra. We reformulate the results of [54] in the language of normal forms and Gelfand–Kirillov dimension. In particular, we sketch the idea of an alternative proof of Theorem 3 from [54], omitting interpretation of regular languages as those recognised by finite automaton.

Following notation from [54], recall that the density function of a regular language $L \subseteq \langle X\rangle$, where $\langle X\rangle$ is the free monoid over the set X, is defined as $p_L(n) = |L \cap X^n|$, that is the number of elements in L of length n. Given two functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants C and n_0 such that $f(n) \leqslant Cg(n)$ for every $n \geqslant n_0$. Function $f(n)$ is $\Omega(g(n))$ if there is a sequence $n_i \to \infty$ of natural numbers and positive constant C such that $f(n_i) \geqslant Cg(n_i)$ for every $i \geqslant 1$.

The density function $p_{N(A)}$ of the regular language $N(A)$ of normal words of an automaton algebra A satisfies

$$d_V(n) = \sum_{i=1}^{n} p_{N(A)}(i),$$

where d_V is the growth function of A with respect to the generating subspace V spanned by the chosen generators of A. Consequently, the density function is $O(n^{k-1})$ for some $k \geqslant 1$ precisely when the growth function of A with respect to the chosen set of generators is $O(n^k)$ and thus $\mathrm{GKdim}(A) \leqslant k$.

The operation $*$ defined by $T^* = \bigcup_{i\geq 0} T^i$, for $T \subseteq F$, will be sometimes called a Kleene star. Similarly we define $T^+ = \bigcup_{i\geq 1} T^i$ for $T \subseteq F$. If $T = \{w\}$ for some $w \in F$, then we write $T^* = w^*$ and $T^+ = w^+$.

It can be checked that languages L described by regular expressions with minimal number of nested star operator equal at least two (for example of the form $(u^*w^*)^*$ for some non-trivial words u, w such that one is not a power of another) have exponentially many words of length n for infinitely many n. On the other hand, if an automaton algebra has finite GK dimension, from Theorem 1.73 it follows that the number of normal words of length at most n is $O(n^k)$ for some $k \geqslant 0$. As a consequence, in this case $N(A)$ can be described by expressions without nested Kleene stars. We get, using so-called disjunctive normal form, that $N(A)$ can be represented as a finite union of expressions $v_0 w_{i_1}^* v_1 w_{i_2}^* v_2 \dots v_{s-1} w_{i_s}^* v_s$ for some $s \geqslant 0$. Moreover, as it is shown in [54], a regular-expression representation can be chosen in such a way that $s \leq k$, provided that A has GK dimension at most k. It may be also easily checked that the growth of a finite sum of such expressions with $s \leq k$ is at most k. Theorem 3 in [54] can be now rephrased as follows.

Theorem 6.1. *The Gelfand–Kirillov dimension of an automaton algebra A is not bigger than k for some $k \geqslant 0$ if and only if the set of normal words $N(A)$ can be represented as a finite union of regular expressions of the following form*

$$v_0 w_{i_1}^* v_1 w_{i_2}^* v_2 \dots v_{s-1} w_{i_s}^* v_s, \tag{6.1.1}$$

with $v_0, \dots, v_s \in F$, $w_{i_1}, \dots, w_{i_s} \in F$ and $0 \leqslant s \leqslant k$.

Unfortunately representation of the form (6.1.1) is not unique. Moreover, without further assumptions we cannot conclude that GK dimension is equal to the maximal number s occurring in the description as in the theorem, as illustrated by the following example.

Example 6.2. The polynomial algebra $A = K[x]$ is generated by the set $\{1, x\}$. Let us choose the well ordering in $\langle x \rangle$ compatible with multiplication such that $x^k < x^l$ if and only if $k < l$. Then the set of normal words $N(A)$ can be represented by the regular expression x^*. This set can be also described by x^*x^*. The growth function d_V associated with a generating subspace $V = \mathrm{lin}_K\{1, x\}$ is given by $d_V(n) = n + 1$. Thus we get $\mathrm{GKdim}(A) = 1$.

In the next simple observation we show, following Lemma 1 in [54], that under certain natural assumptions the rate of growth of a set described by an expression (6.1.1) is polynomial of degree s.

Observation 6.3. If a regular expression $v_0 w_1^* v_1 w_2^* v_2 \dots v_{s-1} w_s^* v_s$ has the property that for any two different substitutions of non-negative powers of words $w_1, \dots, w_s$ we get distinct words, then number $d(n)$ of words of length at most n in the language described by this expression is $\Omega(n^s)$.

Proof. From the assumptions every element w of this family of words is uniquely determined by s non-negative integers $(n_1, \dots, n_s)$ such that $w = v_0 w_1^{n_1} v_1 w_2^{n_2} v_2 \dots v_{s-1} w_s^{n_s} v_s$. The number of words of length at most n is thus not smaller than the number of elements of the set $\{(n_1, \dots, n_m) : n_i \in \mathbb{Z}_+, n_1 + \dots + n_s \leqslant \frac{n-q}{K}\}$, where q is the length of the word $v_0 \cdots v_s$ and K is the maximum of lengths of w_i, $i = 1, \dots, s$. For all n such that K divides $n - q$ the cardinality of such a set is $\binom{\frac{n-q}{K}+s}{s}$, which is a polynomial of degree s. The assertion follows. $\square$

6.2 The main result

Now we focus on the Gelfand–Kirillov dimension of Hecke–Kiselman algebras associated to oriented graphs. As the result does not depend on a field K we will denote the algebra $K[\mathrm{HK}_\Theta]$ associated to a field K by A_Θ. The reasoning relies on two results known earlier. Namely, in the paper [39] algebras of finite Gelfand–Kirillov dimension have been characterized. Namely, Hecke–Kiselman algebra A_Θ has finite GK dimension if and only if the graph does not contain two different oriented cycles connected by an oriented edge of length $\geqslant 0$, see Theorem 1.72. Moreover, as it has been proved in [40], algebras A_Θ associated to oriented graphs Θ are automaton for any choice of degree-lexicographic order on the underlying free monoid. We will also investigate the combinatorics of words in the Hecke–Kiselman monoids. In this context Gröbner bases of the algebras A_Θ from paper [40] will be extensively used. To emphasize the use of Theorem 1.63, whenever we consider the set $N(A_\Theta)$ of normal words of the Hecke–Kiselman algebra A_Θ that is obtained via reductions from the set T, we will say that the elements of $N(A_\Theta)$ are the reduced words of A_Θ.

Recall from Section 1.4.2 that for any oriented graph Θ with a set of vertices denoted by X, $t \in X$ and $w \in F = \langle X \rangle$ we write $w \nrightarrow t$ if $|w|_t = 0$ and there are no $x \in \mathrm{supp}(w)$ such that $x \to t$ in Θ. Similarly, we define $t \nrightarrow w$: we assume that $|w|_t = 0$ and there is no arrow $t \to y$, where $y \in \mathrm{supp}(w)$. In the case when $t \nrightarrow w$ and $w \nrightarrow t$, we write $t \nleftrightarrow w$. Let us recall that a vertex $v \in X$ is a sink vertex if no arrow begins in v. Analogously one defines a source vertex. Sink and source vertices are called terminal vertices.

In the reminder of this section we assume that an oriented graph Θ does not contain two different cycles connected by an oriented path of length $\geqslant 0$, which means that the corresponding Hecke–Kiselman algebra A_Θ has finite GK dimension. From Theorem 1.74 and results from Section 6.1, to determine this dimension we need to investigate regular expressions of the form (6.1.1) describing normal words in the algebra.

If a graph Θ is acyclic, then the corresponding monoid is finite and consequently, the Gelfand–Kirillov dimension of underlying algebra is zero, see Theorem 1.69.

Every vertex of Θ that belongs to some cycle will be called a cycle vertex, or a cycle generator of HK_Θ. Any vertex that is not a cycle vertex will be called a non-cycle vertex (respectively a non-cycle generator).

We begin with a general observation which says that in any family of normal words of the form (6.1.1) factors $w_{i_j}^*$ correspond to certain words with the support in one of the cycles of the graph Θ.

Observation 6.4. Let Θ be a graph such that A_Θ is of finite Gelfand–Kirillov dimension. Let $C_1, \ldots, C_k$ be the set of disjoint simple cycles in Θ, where C_l is of the form

$$x_{1,l} \to x_{2,l} \to \ldots \to x_{n_l,l} \to x_{1,l},$$

for some $n_l \geqslant 3$ and $1 \leq l \leq k$. Assume any degree-lexicographic order on F such that we have $x < y$ for some $x \in C_r$ and $y \in C_s$ if and only if either $r < s$, or if ($r = s$ and $x = x_{p,r}, y = x_{q,r}$, for $p < q$). Assume that for some $1 \neq w \in F$, the words $w^m \in F$ are reduced with respect to the reduction set T in Theorem 1.63 (constructed with respect to the chosen deg-lex order) for every $m \geqslant 1$. Then w is a factor of the infinite word of the form $(q_{N,i})^\infty$ of full support, where $x_1 \to x_2 \to \ldots \to x_N \to x_1$ is one of the cycles C_k with $N = n_k$, $q_{N,i} = x_N(x_1 \ldots x_i)(x_{N-1} \ldots x_{i+1})$ and $i \in \{0, \ldots, N-2\}$. Here we assume that $q_{N,0} = x_N x_{N-1} \ldots x_1$.

Proof. Let $w \neq 1$ be such that the word w^m is reduced for every $m \geqslant 1$. Suppose that $y \in \mathrm{supp}(w)$ is a non-cycle vertex of Θ. First, we will show that then the support of w would also contain either a source or sink vertex. If y is not a terminal vertex, from conditions (i) and (ii) in Theorem 1.63, it follows that there exist $u_1, z_1 \in V(\Theta)$, $u_1 \neq z_1$, such that $u_1 \to y$, $z_1 \leftarrow y$ in Θ and $u_1, z_1 \in \mathrm{supp}(w)$. Similarly, if u_1 is not a sink vertex, then there exists $u_2 \in \mathrm{supp}(w)$ such that $u_2 \to u_1$. Symmetrically, if z_1 is not a source vertex, then $z_2 \leftarrow z_1$ in Θ for some $z_2 \in \mathrm{supp}(w)$. Moreover $\{u_1, u_2\} \cap \{z_1, z_2\} = \emptyset$, because y is a non-cycle vertex and $z_2 \notin \{y, z_1\}$, $u_2 \notin \{y, u_1\}$. We continue this procedure until at least one of the chosen vertices is either terminal or cycle vertex. As the graph is finite, after finitely many steps we obtain a path $u_s \to \cdots \to u_1 \to y \to z_1 \to \cdots \to z_r$ such that $u_1, \ldots, u_s, z_1, \ldots, z_r \in \mathrm{supp}(w)$ and either u_s is a cycle vertex, or a source vertex and, similarly, either z_r is a cycle vertex, or a sink vertex. From Theorem 1.72 and the assumption that A_Θ is of finite Gelfand–Kirillov dimension, the graph Θ does not contain two cycles connected by a path and thus it follows that u_s and z_r cannot be both cycle vertices. Therefore, either u_s is a source or z_r is a sink, as claimed. However, according to Theorem 1.63 a sink or source vertex may occur in a reduced word at most once. Since w^2 is reduced and contains at least two occurrences of u_s and z_r, they cannot be terminal vertices, which leads to a contradiction.

We have proved that the entire support of w consists of cycle generators. Call these cycles $C_1, \ldots, C_q$. Since the Gelfand–Kirillov dimension of A_Θ is finite, no vertex can belong to two cycles and if two elements in the support of w belong to different cycles, they are not connected in Θ by an oriented path. From Theorem 1.63 and from the assumed deg-lex order on F it follows that $w = w_1 w_2 \dots w_q$, where $\mathrm{supp}(w_p) \subseteq V(C_{i_p})$ for pairwise different cycles C_{i_p} for $p = 1, \dots, q$. Yet, as w^m is reduced, for all $m \geq 1$ it easily follows that $q = 1$, so the support of w belongs entirely to a single cycle. Say that this cycle C is of the form $x_1 \to x_2 \to \dots \to x_N \to x_1$. Suppose that there exists x_i which is not in the support of w. Take an index i such that $x_i \notin \mathrm{supp}(w)$ but $x_{i-1} \in \mathrm{supp}(w)$, where for $i = 1$ we take $i - 1 = N$. Then w^2 contains a factor of the form $x_{i-1} u x_{i-1}$ such that $x_i \notin \mathrm{supp}(u)$. From the description of the Gröbner basis in Theorem 1.63 it follows that then w^2 is not reduced. This means that $\mathrm{supp}(w) = \{x_1, \dots, x_N\}$. From Proposition 2.15 it follows that if w^n is reduced for every $n \geqslant 1$, then for some $m \geqslant 1$ the word w^m is of the form $a q_{N,i}^k b$, where $i \in \{0, \dots, N-2\}$, $k \geqslant 1$ and a and b are members of an explicitly described finite families of words from Theorem 2.1. Then, from the assumption, w^{2m} has the reduced form $a q_{N,i}^k b a q_{N,i}^k b$. In particular, this word has a factor $q_{N,i}$ and therefore, from Theorem 2.1, it follows that ba is either of the form $q_{N,i}$ or the trivial word 1. Consequently, as w is a prefix and suffix of $w^m = a q_{N,i}^k b$, it is also a factor of the infinite word of the form $(q_{N,i})^\infty$ for some $i \in \{0, \dots, N-2\}$. The assertion holds. $\square$

From the observation it follows in particular that if there are no cyclic subgraphs in the graph, then a regular language of normal words of the corresponding algebra is described by expressions without Kleene stars. Thus, applying Theorem 1.73, we get an alternative proof that Hecke–Kiselman algebra of acyclic graph is finite dimensional.

After an introductory observation concerning words in the Hecke–Kiselman monoids with the property that their positive powers are in the reduced form, we find the maximal possible number of occurrences of certain non-cycle generators in reduced words in A_Θ. We are interested in those non-cycle vertices that are connected by an oriented path with at least one cyclic subgraph. As we will show, GK dimension depends only on such vertices.

Definition 6.5. A full subgraph Θ' of a graph Θ whose set of vertices consists of all cycle vertices and all vertices connected with at least one cycle by an oriented path will be called the maximal cycle-reachable subgraph of Θ.

Example 6.6. In the graph Θ presented below, the maximal cycle-reachable subgraph is the full subgraph represented by solid edges.

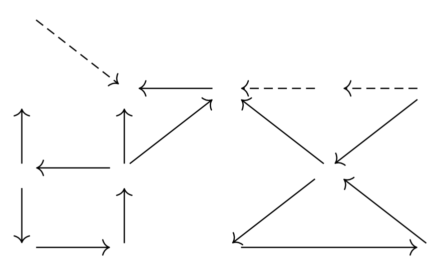

Figure 6.1: A graph Θ with the maximal cycle-reachable subgraph represented by solid edges

Let us recall that we assume that graph Θ does not contain two different cycles connected by an oriented path. In particular, for any vertex $x \in V(\Theta')$ that is not contained in any cycle, if there exists a path from x to a cycle (from a cycle to x, respectively), then all paths between x and all cycles are from x to the cycles (from the cycles to x, respectively).

We agree that for any vertex x there exists exactly one path of length 0 with the end (or beginning) in x.

Lemma 6.7. *Let Θ be an oriented graph with cycles denoted by $C_1, \ldots, C_k$, and let Θ' be its maximal cycle-reachable subgraph. For every vertex $x \in V(\Theta') \setminus (V(C_1) \cup \ldots \cup V(C_k))$ either all oriented paths between x and any cycle lead from x into cycles or all lead from cycles into x. Denote by k_x the number of oriented paths in Θ of non-negative length with the end in the vertex x in the first case, and the number of oriented paths of non-negative length with the beginning in x in the latter case. Then, in every reduced word in HK_Θ, the element x occurs at most k_x times.*

Proof. Let x be any vertex contained in the maximal cycle-reachable subgraph Θ' of the graph Θ but not contained in the cycles $C_1, \ldots, C_k$. Assume first that there are oriented paths from the cycles into x. To prove the statement we proceed by induction on the maximal length $l(x)$ of a path starting at x in the graph Θ.

If $l(x) = 0$ then x is a sink vertex in the graph Θ and thus there are no edges starting at x. Then for any $w \in \mathrm{HK}_\Theta$ we have $xwx = wx$ (see Remark 1.64) and thus x can occur at most once in any reduced word.

Assume now that $l(x) > 0$ for some $x \in V(\Theta') \setminus (V(C_1) \cup \ldots \cup V(C_k))$ and let $z_1, \ldots, z_m$ be the set of all vertices in Θ such that there is an edge $x \to z_i$ for every $i = 1, \ldots, m$. Then from the definition of the maximal cycle-reachable subgraph it follows that all $z_1, \ldots, z_m$ are also in Θ'. Moreover, for $i = 1, \ldots, m$ we have $l(z_i) < l(x)$. By the inductive hypothesis every z_i occurs in any reduced word at most k_{z_i} times, where k_{z_i} is number of paths starting at z_i. We know that if a word of the form xwx with $|w|_x = 0$ is reduced in HK_Θ then in particular $x \to y$ for some $y \in \mathrm{supp}(w)$, as otherwise $x \nrightarrow w$ and $xwx = wx$ in HK_Θ. It follows that at least one of $z_1, \ldots, z_m$ occurs between any two generators x. As already explained, every z_i occurs in any reduced word at most k_{z_i} times. Therefore x can occur at most $k_{z_1} + \ldots + k_{z_m} + 1$ times in any reduced word. On the other hand, in Θ there is exactly one path of length 0 starting at x. Every other path starting from x uniquely determines a path starting from one of $z_1, \ldots, z_m$ and every path p starting at z_i defines a path starting with $x \to z_i$ and followed by p. Thus, in total there are exactly $k_{z_1} + \ldots + k_{z_m} + 1$ paths starting from x in the graph Θ. The assertion follows.

The case where there exist paths from x to a cycle can be treated by a symmetric argument, using induction on the maximal length of a path that ends in x. $\square$

Note that for every non-cyclic vertex x in the maximal cycle-reachable subgraph Θ' such that all paths between x and the cycles lead from the cycles into x (from x into the cycles, respectively) the number k_x of all paths in Θ starting (ending, respectively) at x is the same as the number of such paths in Θ'.

Our next step is to use Lemma 6.7 to show that every regular expression of the form $w_1^* v_1 w_2^* \ldots v_{s-1} w_s^*$ which describes reduced words in the algebra A_Θ can be expressed using at most certain number of stars. To do so we need to introduce certain order in the set of vertices of Θ. For the rest of the present section we will assume that such an order had been chosen.

Definition 6.8 (Order on vertices of the graph)**.** Let Θ be a graph with the cycles $C_1, \dots C_k$ of length $n(j) \geqslant 3$ for $j = 1, \dots, k$ and let Θ' be its maximal cycle-reachable subgraph. For every vertex x of Θ' that is not contained in any cycle, denote by k_x, as before, the number of oriented paths of length $\geqslant 0$ in Θ with either the end or the beginning in x, depending on the direction of paths between x and the cycles. In the set of these vertices define any order such that if $k_x < k_y$ holds, then $y < x$.

Let C_j be of the form $x_{1,j} \to \dots \to x_{n(j),j} \to x_{1,j}$ for some $n(j) \geqslant 3$ and $j = 1, \dots, k$. In the set of all cycle vertices introduce the order such that $x_{i,j} < x_{l,m}$ if ether $j < m$ or $j = m$ and $i < l$. Moreover, assume that all cycle vertices are smaller than any vertex outside the cycles.

Finally, choose any order in the set of vertices of Θ that are not in Θ', for example such that all these vertices are bigger than the vertices of Θ'.

Let us note that it is possible to define the number k_x for non-cycle vertices from Θ', and the order which satisfies all above conditions, provided that the graph Θ does not contain two different cycles connected by an oriented path.

In the next technical lemma we characterize the possible form of a family of reduced words described by w^*vw^*, with $\mathrm{supp}(w) \subseteq V(C_n)$ for some n.

Lemma 6.9. *If a family of reduced words is described by a regular expression of the form u^*vw^* with $\mathrm{supp}(u), \mathrm{supp}(w) \subseteq V(C_n)$ for a cycle C_n, then either v contains a vertex connected by an edge with C_n or this family of words can be expressed by a sum of finitely many regular expressions of the form pr^*q or p, for some words p, q and r.*

Proof. Let u^*vw^* be the regular expression describing reduced words with $\mathrm{supp}(u), \mathrm{supp}(w) \subseteq V(C_n)$ for a cycle C_n. First we claim that either $\mathrm{supp}(v) \subseteq V(C_n)$ or v contains a non-cycle vertex. Indeed, by Definition 6.8 of the order on the vertices of Θ and the fact that the graph does not contain two different cycles connected by an oriented path, generators corresponding to the vertices from different cycles commute. Consequently, every reduced word w such that $\mathrm{supp}(w) \subseteq V(C_1) \cup \dots \cup V(C_k)$ has elements from different cycles grouped in such a way that if $w = w_1 \cdots w_j$ with $w_i \in V(C_{n(i)})$ and $w_l \in V(C_{n(l)})$, then $n(i) \leqslant n(l)$ for all $i < l$. Thus if the family of words u^*vw^* is reduced and such that $\mathrm{supp}(u), \mathrm{supp}(w) \subseteq V(C_n)$, then either $\mathrm{supp}(v) \subseteq V(C_n)$ or v contains a non-cycle generator.

Let us now consider the first case, that is u^*vw^* consists of reduced words and $\mathrm{supp}(u)$, $\mathrm{supp}(v)$, $\mathrm{supp}(w) \subseteq V(C_n)$. We proceed to show that then u^*vw^* can be expressed as a finite sum of expressions with at most one Kleene star. From Observation 6.4 it follows that u and w are factors of the infinite words $(x_N(x_1 \dots x_{l_1})(x_{N-1} \dots x_{l_1+1}))^\infty$ and $(x_N(x_1 \dots x_{l_2})(x_{N-1} \dots x_{l_2+1}))^\infty$, denoted shortly by q^∞_{N,l_1} and q^∞_{N,l_2}, for some $l_1, l_2 \in \{0, \dots, N-2\}$, where N is a number of vertices of cycle C_n. Moreover, as there exist $j, k, m \geqslant 1$ and

$l \in \{0, \ldots, N-2\}$ such that $u^j v w^k$ is of the form $aq_{N,l}^m b$, it follows that $l = l_1 = l_2$. Furthermore, v is a factor of the same infinite word $(x_N(x_1 \ldots x_l)(x_{N-1} \ldots x_{l+1}))^\infty$. Thus we can write $u = aq_{N,l}^{\alpha_1} b$, $v = aq_{N,l}^{\beta} b'$ and $u = a' q_{N,l}^{\alpha_2} b'$ for non-negative α_i, β and words a, a', b, b' that are suffixes and prefixes of the word $q_{N,l}$, respectively, of length at most $N-1$. Thus both ba and $b'a'$ are either the trivial word 1 or are of the form $q_{N,l}$. Then $u^* v w^*$ is equal to the set $\{aq_{N,l}^{l_1\beta_1 + l_2\beta_2 + \beta_3} b' : l_1, l_2 \geqslant 0\}$, for some positive integers β_i $(i = 1, 2, 3)$, where $\beta_1 = \alpha_1$ if $ba = 1$ and $\beta_1 = \alpha_1 + 1$ otherwise, and $\beta_2 = \alpha_2$ if $b'a' = 1$ and $\beta_2 = \alpha_2 + 1$ otherwise. From Proposition 2.2 in [48] it follows that there exist a positive integer n_0 and a finite set D such that $\{l_1\beta_1 + l_2\beta_2 + \beta_3\} = \{n_0 + kd : k \geqslant 0\} \cup D$, where $d = \gcd(\beta_1, \beta_2)$. We thus get easily that $u^* v w^*$ can be written as a finite sum of regular expressions with at most one star $*$.

Now assume that a family of reduced words described by $u^* v w^*$ is such that v contains a non-cycle vertex and $\mathrm{supp}(u), \mathrm{supp}(w) \subseteq V(C_n)$. We can write $v = v_s z v_c$ for words v_s, v_c and a non-cycle vertex z such that $\mathrm{supp}(v_c) \subseteq \bigcup_{j=1}^{k} V(C_j)$. Suppose that z is not connected by an edge with a cycle C_n. Consider the first occurrence of a vertex x such that $x \in V(C_n)$ in the word $v_c w$. Then the word vw contains a factor of the form $zv'x$ with $\mathrm{supp}(v_c) \subseteq \bigcup_{j \neq n} V(C_j)$. Furthermore, $x < z$ and $zv' \nleftrightarrow x$. Consequently, vw contains a factor which can be reduced using reduction (iii) from Theorem 1.63. The obtained contradiction shows that for every family of reduced words of the form $u^* v w^*$ with $\mathrm{supp}(u), \mathrm{supp}(w) \subseteq V(C_n)$ and $\mathrm{supp}(v) \nsubseteq V(C_n)$, for a cycle C_n, factor v contains at least one vertex connected by an edge with C_n. Thus, the result follows. $\square$

We are now ready to estimate Gelfand–Kirillov dimension, using Theorem 1.73.

Corollary 6.10. *If Θ is an oriented graph with the cycles $C_1, \ldots, C_k$ such that the corresponding Hecke–Kiselman algebra has finite Gelfand–Kirillov dimension, then*

$$\mathrm{GKdim}\, A_\Theta \leqslant \sum_{j=1}^{k} \left(\sum_{x \in \mathcal{A}_j} k_x + 1 \right),$$

where $\mathcal{A}_j$ consists of all vertices of Θ that are connected by an edge with the cycle C_j for $j = 1, \ldots, k$.

Proof. From Theorem 6.1 we know that the set of normal (reduced) words of A_Θ is a finite union of regular expressions of the form $v_0 w_{i_1}^* v_1 w_{i_2}^* v_2 \ldots v_{s-1} w_{i_s}^* v_s$. Moreover, it is enough to prove that every family of such form can be expressed as a regular expression with $s \leqslant \sum_{j=1}^{k} \sum_{x \in \mathcal{A}_j} (k_x + 1)$.

From Observation 6.4 for every n we have $\mathrm{supp}(w_n) \subseteq V(C_{j(n)})$, for some $j(n) \in \{1, \ldots, k\}$ and w_n are factors of the word $(x_N(x_1 \ldots x_i)(x_{N-1} \ldots x_{i+1}))^\infty$ of full support, where $x_1 \to x_2 \to \ldots \to x_N \to x_1$ is one of the cycles C_j with $N = n(j)$ and $i \in \{0, \ldots, N-2\}$.

By Lemma 6.9 we can rewrite the considered family of words in such a way that between any two w_i, w_j $(i, j \in \{1, \ldots, s\})$ such that $\mathrm{supp}(w_i), \mathrm{supp}(w_j) \subseteq V(C_n)$ for some $n \in$

$\{1,\ldots,k\}$ there is a non-cycle vertex z which is connected by an edge with C_n, that is $z \in \mathcal{A}_n$. By Lemma 6.7, all vertices z with this property occur at most $\sum_{x\in\mathcal{A}_n} k_x$ times in total in any reduced word of A_Θ. Consequently, in the regular expression of the above form, for every $j = 1,\ldots,k$, factors of the form w^* with $\operatorname{supp}(w) \subseteq V(C_j)$ occur at most $\sum_{x\in\mathcal{A}_j} k_x + 1$ times. As for every $n \in \{1,\ldots,s\}$ we have that $\operatorname{supp}(w_n) \subseteq V(C_{n(j)})$ for some $n(j)$, it follows that $s \leqslant \sum_{j=1}^{k}\left(\sum_{x\in\mathcal{A}_j} k_x + 1\right)$. Therefore, from Theorem 6.1 we get $\operatorname{GKdim} A_\Theta \leqslant \sum_{j=1}^{k}\left(\sum_{x\in\mathcal{A}_j} k_x + 1\right)$, as claimed. □

Our next step is to construct a family of reduced words of the algebra A_Θ described by a regular expression with exactly $s = \sum_{j=1}^{k}\left(\sum_{x\in\mathcal{A}_j} k_x + 1\right)$ stars and such that for different substitutions of stars with positive integers we get different elements. As for every word w we have $w^*w = w^+$, we will write w^+ instead of w^*w and we refer to the number of stars in the regular expression even if $+$ is used.

Let Θ be a graph with cycles $C_1,\ldots,C_k$ of the length $i_j \geqslant 3$ for $j \in \{1,\ldots,k\}$. Denote by Θ' the maximal cycle-reachable subgraph of Θ. We assume that the set of vertices of Θ is ordered as in Definition 6.8.

We construct a family of reduced words in HK_Θ via an insertion process that is described below.

Step 1. First we insert subsequent vertices contained in the cycle-reachable subgraph Θ' of the graph Θ that are not cycle vertices to certain words, starting from the trivial word 1. At every step a chosen generator y is inserted at the beginning of the word and directly after every vertex of the (previously constructed) word that is connected by an edge with y. Every vertex y occurs exactly k_y times in the constructed word. Note that at this stage the resulting word is not necessarily reduced. The procedure is described precisely as follows.

As Θ does not contain two different cycles connected by an oriented path, either there is at least one terminal vertex y with $k_y = 1$ or the graph is a disjoint union of cycles $C_1,\ldots,C_k$. If the latter case holds we set $w' = 1$, where 1 is a trivial word and go to Step 2.

Now we consider the case when there are some terminal vertices in Θ'. Note that a vertex y from Θ' is terminal exactly if $k_y = 1$. Let $y_1^{(1)} < \ldots < y_{n_1}^{(1)}$ be the set of all vertices in Θ' such that $k_{y_i^{(1)}} = 1$ and define

$$w_1 = y_1^{(1)} y_2^{(1)} \cdots y_{n_1}^{(1)}.$$

Next, take the biggest (with respect to the order defined in Definition 6.8) vertex $y^{(2)} \in V(\Theta')$ that is not contained in any cycle of the graph and that has not been used yet in w_1. We can assume that all paths between the cycles and $y^{(2)}$ lead from the cycles into $y^{(2)}$. Otherwise, all such paths lead from $y^{(2)}$ into the cycles and the reasoning is symmetric. If for some non-cycle vertex $z \in V(\Theta')$ we have $y^{(2)} \to z$, then $k_z < k_{y^{(2)}}$ and thus $y^{(2)} < z$. By the choice of $y^{(2)}$ it follows that $z \in \{y_1^{(1)},\ldots,y_{n_1}^{(1)}\}$. Moreover, there are exactly $k_{y^{(2)}} - 1$ (recall that $k_{y^{(2)}}$ is the number of paths starting at z) generators in w_1 that are connected by an edge with $y^{(2)}$. Let w_2 be the word that is formed from w_1 by inserting the generator

$y^{(2)}$ in such a way that it is the first letter of w_2 and $y^{(2)}$ also directly follows in w_2 every $y_j^{(1)}$ that is connected by an edge $y^{(2)} \to y_j^{(1)}$ with $y^{(2)}$ in Θ'. Generator $y^{(2)}$ occurs in w_2 exactly $k_{y^{(2)}}$ times. Additionally, every generator z used in the word w_2 occurs in this word exactly k_z times.

Similarly, if we have already constructed the word w_i for some $i > 1$, then in the next step we insert to this word several copies of the largest non-cycle generator $y^{(i+1)} \in V(\Theta')$ that is not in the support of w_i yet. In the word w_i every generator z occurs k_z times. We know that every z such that $y^{(i+1)} < z$ is already in the support of w_i. In particular every generator z for which $k_z < k_{y^{(i+1)}}$ is in w_i. As explained above, we can assume that all directed paths connecting the cycles and $y^{(i+1)}$ start from the cycles. Therefore, if we have $y^{(i+1)} \to p$ in the graph Θ', then $p \in \mathrm{supp}(w_i)$. Define the word w_{i+1} by inserting $y^{(i+1)}$ to w_i at the beginning and also directly after every generator $z \in \mathrm{supp}(w_i)$ such that $y^{(i+1)} \to z$ in Θ'. In such a word w_{i+1} the element $y^{(i+1)}$ occurs exactly $\sum\limits_{y^{(i+1)} \to z} k_z + 1$ times. Let us note that all paths starting at $y^{(i+1)}$ in the graph Θ are either the path of length 0 or are uniquely determined by a path starting at z for some z such that $y^{(i+1)} \to z$. Consequently, in the word w_{i+1} the element $y^{(i+1)}$ occurs exactly $\sum\limits_{y^{(i+1)} \to z} k_z + 1 = k_{y^{(i+1)}}$ times.

After finitely many steps as described above we get a word w' whose support contains every non-cycle generator z of Θ' and with the property that every $z \in \mathrm{supp}(w')$ occurs in w' exactly k_z times.

Step 2. Now we insert cycle vertices into the word w' constructed in Step 1. The idea relies on a slight modification of the previous step. Namely, we insert regular expressions of the form $w_0 w^* w_1$ with $\mathrm{supp}(w_0), \mathrm{supp}(w_1), \mathrm{supp}(w) \subseteq V(C_j)$ (w_0 and w_1 vary depending on the insertion place), for a cycle C_j, at the beginning of the constructed regular expression and directly after every vertex connected by an edge with C_j. The procedure is repeated for every cycle, starting from the cycle with the biggest vertices in the sense of ordering from Definition 6.8. It can be precisely described as follows.

For every cycle C_i ($i = 1, \ldots, k$) with vertices $x_{1,i}, \ldots, x_{n,i}$ for some $n \geqslant 3$ denote by c_i the reduced word of the form $x_{1,i} \cdots x_{n,i}$.

We can write $w' = v_1 \cdots v_{m+1}$, where every v_i is the word of minimal possible length that ends with an element z_i connected by an edge with the cycle C_k (possibly with $v_{m+1} = 1$) for $i = 1, \ldots, m$. Note that we have $m = \sum_{x \in \mathcal{A}_k} k_x$ if $\mathcal{A}_k$ is non-empty and $m = 0$ otherwise.

For every vertex z_i connected by an edge with the cycle C_k of length n, we may choose $j(i) \in \{1, \ldots, n\}$ such that either $z_i \to x_{j(i),k}$ or $x_{j(i),k} \to z_i$. Then we define the regular expression (that is certain family of words) r_k as follows:

$$\begin{aligned} c_k^+ (x_{1,k} \ldots x_{j(1)-1,k}) v_1 (x_{j(1),k} \cdots x_{n,k}) c_k^+ (x_{1,k} \ldots x_{j(2)-1,k}) \cdots \\ \cdots c_k^+ (x_{1,k} \ldots x_{j(m)-1,k}) v_m (x_{j(m),k} \cdots x_{n,k}) c_k^+ v_{m+1}. \end{aligned}$$

In this expression Kleene star $*$ occurs exactly $m_k = \sum_{x \in \mathcal{A}_k} k_x + 1$ times, where $\mathcal{A}_k$ consists of all vertices x that are connected by an edge with the cycle C_k in Θ'. If $\mathcal{A}_k$ is empty, that is there are no vertices connected by an edge with the cycle C_k and $w' = v_1$ we define the regular expression r_1 as $c_k^+ v_1$. Then we also assume that $\sum_{x\in\mathcal{A}_k} k_x = 0$ and thus Kleene star $*$ occurs exactly $1 = \sum_{x\in\mathcal{A}_k} k_x + 1$ times.

Next we repeat this procedure for every cycle of the graph Θ. More precisely, at every step we rewrite the constructed regular expression r_j as $v_1 \cdots v_{m+1}$, where $v_1, \ldots, v_m$ are regular expressions of minimal possible length that end with an element z_i connected by an edge with the cycle C_{j-1} (perhaps with $v_{m+1} = 1$). If there are no vertices connected by an edge with C_{j-1}, we set $r_j = v_1$, that is $m = 0$. Note that we have $m = \sum_{x\in\mathcal{A}_{j-1}} k_x$, where for empty $\mathcal{A}_{j-1}$ we put $\sum_{x\in\mathcal{A}_{j-1}} k_x = 0$. For every vertex z_i connected by an edge with the cycle C_{j-1} of length n, we may choose $j(i) \in \{1, \ldots, n\}$ such that either $z_i \to x_{j(i),j-1}$ or $x_{j(i),j-1} \to z_i$. Then define the regular expression r_{j-1} as:

$$\begin{aligned} &c_{j-1}^+(x_{1,j-1} \ldots x_{j(1)-1,j-1}) v_1 (x_{j(1),j-1} \cdots x_{n,j-1}) c_{j-1}^+ (x_{1,j-1} \ldots x_{j(2)-1,j-1}) \cdots \\ &\qquad \cdots c_{j-1}^+ (x_{1,j-1} \ldots x_{j(m)-1,j-1}) v_m (x_{j(m),j-1} \cdots x_{n,j}) c_{j-1}^+ v_{m+1}. \end{aligned} \tag{6.2.1}$$

As before, if $\mathcal{A}_{j-1}$ is empty, we set $r_{j-1} = c_{j-1}^+ r_j$. Then expression r_{j-1} contains exactly $m_{j-1} = m_j + \sum_{x\in\mathcal{A}_{j-1}} k_x + 1$ Kleene stars.

This way we construct a regular expression r_1 that contains exactly $m_1 = m_2 + \sum_{x\in\mathcal{A}_1} k_x + 1 = \sum_{j=1}^k \left(\sum_{x\in\mathcal{A}_j} k_x + 1\right)$ stars. We will show that r_1, treated as a family of words, consists of reduced words of HK_Θ. This will be crucial to get the lower bound for the Gelfand–Kirillov dimension of the algebra A_Θ.

Lemma 6.11. *Words* (6.2.1) *are reduced in A_Θ with respect to the system introduced in Theorem 1.63. Consequently,* $\operatorname{GKdim} A_\Theta \geqslant \sum_{j=1}^k \left(\sum_{x\in\mathcal{A}_j} k_x + 1\right)$.

Proof. We claim that no leading term of reductions of the form (i)–(iii) listed in Theorem 1.63 appears as a factor of a word w from the family described by the regular expression r_1.

We start with reductions of type (i) and (ii). First consider any factor of w of the form tvt for some generator t and any word v such that $t \notin \operatorname{supp}(v)$. We need to show that then there are vertices $x, y \in \operatorname{supp}(v)$ such that $x \to v$ and $v \to y$.

Assume first that t is a cycle vertex, let $t \in V(C_j)$ for a cycle C_j with vertices $x_{1,j}, \ldots, x_{n,j}$ and some $j \in \{1, \ldots, k\}$. Consider the image of elements of the family described by a regular expression (6.2.1) under the natural projection $\varphi_j : \mathrm{HK}_\Theta \to \mathrm{HK}_{C_j}$ onto the Hecke–Kiselman monoid associated to the cycle C_j, such that $\varphi_j(x) = 1$ for all $x \notin V(C_j)$.

By the construction, every such image is a factor of $(x_{1,j} \cdots x_{n,j})^\infty$. Thus if t is a cycle vertex x_i, then $x_{i-1}, x_{i+1} \in \operatorname{supp}(v)$, where for $i = 1$ and $i = n$ we set $i - 1 = n$ and $i + 1 = 1$, respectively. In particular it is then impossible to have $t \nrightarrow v$ or $t \nleftarrow v$. Therefore, we may consider any t that is not in the cycle and we claim that in every factor tvt the set $\operatorname{supp}(v)$ contains elements p and q connected by an edge with t such that $t \to p$ and $q \to t$.

Note that every sink or source vertex x either is not contained in the maximal cycle-reachable subgraph Θ' of the graph or $k_x = 1$. Consequently, it occurs at most once in every word described by the considered regular expression. Thus we know that t is neither a sink nor a source vertex.

Now assume that t is non-cycle and not terminal vertex from Θ'. Assume first that all oriented paths connecting t with the cycles lead from the cycles to t. For any $z \to t$ contained in the graph Θ' we have $z < t$. From the construction of the family of words it follows that such z is inserted into the word between any two occurrences of t, that is $z \in \operatorname{supp}(v)$ and the leading term from the reduction (i) in Theorem 1.63 is impossible. The other way round, the generator t is inserted into the regular expression at the beginning and directly after any vertex y such that $t \to y$ (y are inserted before t). In particular, all such generators y occur between any two t's. It follows directly that no leading term of a reduction of type (ii) appears as a factor of w. The case when all oriented paths lead from t to the cycles can be handled in much the same way.

Now we consider reductions of type (iii). We claim that w does not contain any factor t_1vt_2 such that $t_1 > t_2$ and $t_2 \not\leftrightarrow t_1v$. If t_1 is contained in any of the cycles, then $t_1 > t_2$ implies that also t_2 is a cycle vertex.

Let a word w be described by a regular expression (6.2.1). By the construction, for every factor of w of the form $px_{i,j}$, where $x_{i,j}$ is a cycle vertex and p is a word such that $p \not\leftrightarrow x_{i,j}$, the word p consists of cycle vertices $x_{l,m}$ such that $m < j$. In particular we have $g < x_{i,j}$ for every $g \in \operatorname{supp}(p)$. Thus there is no factor of the above form with t_2 being a cycle element.

In consequence, we can assume that both t_1 and t_2 are non-cycle vertices.

We claim that no word w_i from the first part of the construction of regular expression r_1 has a factor of type (iii) from Theorem 1.63. To do so, we proceed by induction on i. First observe that the assertion holds for $i = 1$, as generators in w_1 are in the increasing order. Hence, assume that the claim holds for some w_i and denote by $y^{(i+1)}$ the vertex inserted in the next step, that is $\operatorname{supp}(w_{i+1}) \setminus \operatorname{supp}(w_i) = \{y^{(i+1)}\}$. Then every factor t_1vt_2 such that $t_1 > t_2$ and $t_2 \not\leftrightarrow t_1v$ in w_{i+1} would have $t_2 = y^{(i+1)}$ because by the inductive hypothesis w_i does not have such factors and all elements of $\operatorname{supp}(w_i)$ are bigger than $y^{(i+1)}$. On the other hand, in w_{i+1} the element directly before $y^{(i+1)}$ is connected by an edge with $y^{(i+1)}$. Thus in w_{i+1} every factor of the form $t_1vy^{(i+1)}$ with $t_1 > y^{(i+1)}$ is such that the last generator of t_1v is connected by an edge with $y^{(i+1)}$. The inductive assertion holds.

Consequently, we know that the word w', built in the first step of the construction, does not contain factors of type (iii). The regular expression r_1 is obtained from w' by inserting only cycle generators. Every factor t_2vt_1 with $t_2 > t_1$ and $t_2 \not\leftrightarrow t_1w$ would therefore start or end with a cycle vertex, that is either t_1 or t_2 is a cycle vertex. This is not possible as we explained earlier. We have proved that any w described by the regular expression r_1 does not contain factors of the form (iii) in the Theorem 1.63, as claimed. The first part of lemma follows.

As every word described by the regular expression r_1 is reduced, two different words are equal in the algebra A_Θ if and only if they are equal as elements of free monoid generated by the vertices of Θ. Moreover, every word w in the set denoted by r_1 is uniquely determined by

m positive integers $(n_1, \dots, n_m)$, where $m = \sum_{j=1}^{k} \left(\sum_{x \in \mathcal{A}_j} k_x + 1 \right)$ and $n_1, \dots, n_m$ are powers of consecutive words c contained in cycles corresponding to $^+$ in (6.2.1). From Observation 6.3 it follows that the number of elements of length at most n in r_1 is $\Omega(n^m)$. Consequently, we get that $\operatorname{GKdim} A_\Theta \geqslant \sum_{j=1}^{k} \left(\sum_{x \in \mathcal{A}_j} k_x + 1 \right)$. $\square$

Corollary 6.10 and Lemma 6.11 are summarized in the following theorem that describes the Gelfand–Kirillov dimension of the Hecke–Kiselman algebra associated to any oriented graph without two different cycles connected by an oriented path.

Theorem 6.12. *Let Θ be an oriented graph with the cycles $C_1, \dots, C_k$ for some $k \geqslant 1$ without two different cycles connected by an oriented path. In particular, for any non-cyclic vertex x connected by an oriented path with a cycle either all paths between x and cycles are directed from x into the cycles or all begin at the cycles. Denote by $\mathcal{A}_j$ the set of vertices of the graph that are connected by an edge with the cycle C_j for $j = 1, \dots, k$. For any $x \in \mathcal{A}_j$ let k_x be the number of oriented paths of length $\geqslant 0$ in Θ that start with x if all paths between C_j and x start with the cycle vertices and oriented paths that end with x otherwise. Then*

$$\operatorname{GKdim} A_\Theta = \sum_{j=1}^{k} \left(\sum_{x \in \mathcal{A}_j} k_x + 1 \right),$$

where $\sum_{x \in \mathcal{A}_j} k_x + 1$ is equal to 1 if $\mathcal{A}_j$ is an empty set. Lastly, if the graph Θ does not contain any cycle, then $\operatorname{GKdim} A_\Theta = 0$.

6.3 An example

Let us illustrate concepts from Theorem 6.12 and its proof for the oriented graph Θ presented in the picture.

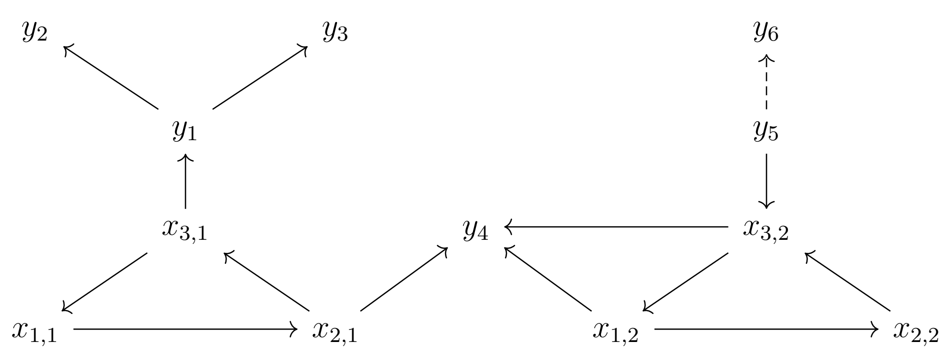

Figure 6.2: A graph Θ with the maximal cycle-reachable subgraph marked by solid edges

The maximal cycle-reachable subgraph Θ' is the full subgraph of Θ with all vertices except y_6. The edges of Θ' are denoted by solid arrows, whereas the complement is denoted by dashed ones.

For the non-cycle vertices in Θ' named as in the picture we have: $k_{y_2} = k_{y_3} = k_{y_4} = k_{y_5} = 1$ and $k_{y_1} = 3$. Denote the cycle with vertices $x_{i,1}$, $i = 1, 2, 3$ by C_1 and let C_2 be the cycle $x_{1,2} \to x_{2,2} \to x_{3,2} \to x_{1,2}$. Then the sets $\mathcal{A}_1$ and $\mathcal{A}_2$ consisting of the vertices connected by an edge with the cycles are $\mathcal{A}_1 = \{y_1, y_4\}$ and $\mathcal{A}_2 = \{y_4, y_5\}$. We get that $\sum_{x\in\mathcal{A}_1} k_x + 1 = 5$ and $\sum_{x\in\mathcal{A}_2} k_x + 1 = 3$.

From Theorem 6.12 we obtain the following corollary.

Corollary 6.13. *The Gelfand–Kirillov dimension of the Hecke–Kiselman algebra A_Θ associated to the graph Θ in Figure 6.2 is* 8.

Following Lemma 6.11 let us construct a family of reduced words in A_Θ described by a regular expression with exactly 8 Kleene stars.

In the set of vertices of Θ we introduce the following order.

- Cycle vertices are such that $x_{1,1} < x_{2,1} < x_{3,1} < x_{1,2} < x_{2,2} < x_{3,2}$.
- For non-cyclic vertices we may choose any order such that y_1 is the smallest one. Assume that $y_1 < y_2 < y_3 < y_4 < y_5 < y_6$.
- All cycle vertices are smaller than non-cyclic ones, that is $x_{3,2} < y_1$.

Then the word w' without cycle vertices built in the first part of the construction is of the form $y_1y_2y_1y_3y_1y_4y_5$. Note that each element y_j of the support of this word occurs in it exactly m_{y_j} times. Next denote by c_i the word $x_{1,i}x_{2,i}x_{3,i}$ for $i = 1, 2$. We have that every vertex of c_1 is smaller than any vertex of c_2. The regular expression r_2 is $c_2^+ y_1y_2y_1y_3y_1y_4c_2^+ x_{1,2}x_{2,2}y_5x_{3,2}c_2^+$. Finally, the regular expression r_1 with exactly 8 stars and consisting of reduced words has the following form:

$$(c_1^+x_{1,1}x_{2,1})(c_2^+)\underline{y_1}(x_{3,1}c_1^+x_{1,1}x_{2,1})\underline{y_2y_1}(x_{3,1}c_1^+x_{1,1}x_{2,1})\underline{y_3y_1}(x_{3,1}c_1^+x_{1,1})\underline{y_4}(x_{2,1}x_{3,1}c_1^+)(c_2^+x_{1,2}x_{2,2})\underline{y_5}(x_{3,2}c_2^+).$$

The consecutive factors of w' constructed in the first step are underlined for clarity.

Chapter 7

Semigroup identities of Hecke–Kiselman monoids

In the present chapter we focus on semigroup identities in Hecke–Kiselman monoids. First we characterize finite graphs Θ such that the monoid HK_Θ satisfies a non-trivial identity. The second aim is to discover certain concrete identities satisfied by such monoids. Bases of identities holding in the monoids associated to certain class of acyclic graphs have been described in [3]. In particular, a concrete identity of the Hecke–Kiselman monoid associated to any finite oriented graph which is acyclic can be derived. Namely, in this case the Hecke–Kiselman monoid is a homomorphic image of the Kiselman's semigroup K_n, for some $n \geqslant 2$. Therefore, for example, if the acyclic graph Θ has n vertices, then the identity $(xy)^n x = (xy)^n$ is satisfied in HK_Θ. Recall from Theorem 1.69 that the Hecke–Kiselman monoid associated to an oriented graph is finite if and only if the graph is acyclic. Thus we focus on the case of monoids associated to oriented graphs containing a cycle, i.e. infinite Hecke–Kiselman monoids.

Let us recall several results useful in the context of semigroup identities of Hecke–Kiselman monoids. Theorem 1.72 in particular characterizes oriented graphs Θ such that the monoid HK_Θ contains a free submonoid of rank 2. It follows from this theorem that if the the graph Θ contains two different cycles connected by an oriented path, then HK_Θ does not satisfy any semigroup identity. We will show that if the graph does not contain such subgraph, then a semigroup identity holds in the corresponding Hecke–Kiselman monoid.

Our result can be put in a broader perspective. Namely, we will show that the answer to Problem 1.52 is positive in the case of Hecke–Kiselman monoids associated to oriented graphs.

One of important classes of semigroups are subsemigroups of the multiplicative semigroup of matrices over a field. Theorem 1.53 provides a useful characterization of finitely generated semigroups of this type that satisfy semigroup identities. To apply this theorem in the case of Hecke–Kiselman monoids, Anan'in Theorem 1.24, which gives a sufficient condition for a PI-algebra to embed into the matrix ring over a field, turns out to be helpful. Finally, we will use a very transparent characterization of Noetherian Hecke–Kiselman algebras, obtained in Theorem 4.2.

7.1 Identities in the monoid associated to an oriented cycle

As we will prove in the main theorem, identities in Hecke–Kiselman monoids HK_Θ in the general case can be constructed from those in the monoids associated to oriented cycles and to graphs with exactly one vertex and no edges, that are subgraphs of the given graph Θ. In the latter case the possible identity is clear from the definition. Therefore we start with the identities in the monoid associated to an oriented cycle. Let us denote by C_n the Hecke–Kiselman monoid associated to a cycle of length $n \geqslant 3$. Exploring the ideal chain in C_n described in Theorem 2.44 and using Lemma 3.5 from [16] we construct explicitly a semigroup identity in this monoid.

First, let us recall from Theorem 2.44 that in C_n there exists a chain of ideals $\emptyset = I_{n-2} \subseteq I_{n-3} \subseteq \cdots \subseteq I_{-1} \subseteq C_n$ and semigroups of matrix type $M_i \subseteq I_{i-1}/I_i$ for $i = 0, \dots, n-2$, such that the sets $(I_{i-1}/I_i) \setminus M_i$ and C_n/I_{-1} are finite. We denote by $\tilde{M}_i$ the set $M_i \setminus \{\theta\}$, treated as a subset of C_n. Then for all $i \in \{0, \dots, n-2\}$ we have $|(I_{i-1}/I_i) \setminus M_i| \leqslant N$, where $N = |C_n \setminus \bigcup_{i=0}^{n-2} \tilde{M}_i| + 1$ is the constant from the proof of Proposition 2.15. From Lemma 3.5 (1) in [16] it follows that for every word s of full support (that is, all generators of C_n occur in s) positive powers $s, s^2, \dots$ are pairwise different. Therefore, for any $s \in I_{i-1}/I_i$ of full support there exists $k \leqslant N$ such that $s^k \in \tilde{M}_i \cup I_i$ and similarly for any $s \in C_n/I_{-1}$ of full support there exists $k \leqslant N$ such that $s^k \in I_{-1}$. It follows from Corollary 2.46 in the first case that s^N is also either in $\tilde{M}_i$ or in I_i and in the latter case $s^N \in I_{-1}$. Note that N depends only on n, but is independent of $s \in C_n$ of full support and $i \in \{0, \dots, n-2\}$. In what follows we assume without loss of generality that $N \geqslant n-1$.

Theorem 7.1. *Let us define the following family of words in the free semigroup $\{s,t\}^*$ of rank 2.*

$$f_1^{(1)}(s,t) = s^N t^N s^{2N},\, f_2^{(1)}(s,t) = s^{2N} t^N s^N$$

$$f_1^{(i)}(s,t) = f_1^{(1)}\left(f_1^{(i-1)}(s,t), f_2^{(i-1)}(s,t)\right), f_2^{(i)}(s,t) = f_2^{(1)}\left(f_1^{(i-1)}(s,t), f_2^{(i-1)}(s,t)\right) \text{ for } i \geqslant 2.$$

Then the Hecke–Kiselman monoid C_n satisfies the identity

$$f_1^{(n-1)}((st)^N, (ts)^N) = f_2^{(n-1)}((st)^N, (ts)^N)$$

for any $n \geqslant 3$.

Proof. The construction of an identity relies on the proof of Proposition 1.54. Namely, it can be obtained from this proof that if an identity $u(s,t) = v(s,t)$ is satisfied in the quotient S/J for an ideal J of S, and $f(s,t) = g(s,t)$ holds in J, where f and g have the same length, then $f(u(s,t), v(s,t)) = g(u(s,t), v(s,t))$ is an identity in the semigroup S.

Let $L \triangleleft C_n$ consist of all words of full support. Then in the monoid C_n there exists a chain of ideals

$$\emptyset = (I_{n-2} \cap L) \subseteq (I_{n-3} \cap L) \subseteq \cdots \subseteq (I_{-1} \cap L) \subseteq C_n.$$

Therefore, it is sufficient to construct identities in the quotients $(I_{i-1} \cap L)/(I_i \cap L)$ for $i = 0, \ldots, n-2$, where in the case $i = n-2$ we set $(I_{n-3} \cap L)/(I_{n-2} \cap L) = (I_{n-3} \cap L)$, and in $C_n/(L \cap I_{-1})$. We will show that in every quotient $(I_{i-1} \cap L)/(I_i \cap L)$ the identity $f_1^{(1)}(s,t) = f_2^{(1)}(s,t)$ is satisfied. For brevity, let us denote $I_i \cap L = L_i$ for $i = -1, \ldots, n-2$. As explained in the beginning of this section, there exists N such that for every $w \in L_{n-3}$ we have $w^N \in \tilde{M}_{n-2} \cup I_{n-2} = \tilde{M}_{n-2}$, where $M_{n-2} = \tilde{M}_{n-2} \cup \{\theta\}$ is the semigroup of matrix type $\mathcal{M}^0(Q_{n-2}, A_{n-2}, B_{n-2}; P_{n-2})$ associated to the infinite cyclic semigroup Q_{n-2}, see Theorem 2.44. In particular for any $s, t \in M_{n-2}$ the word sts is either 0 or both s and sts are contained in a maximal subgroup, isomorphic to $\mathbb{Z}$, of the completely 0-simple closure of M_{n-2}, see Section 1.3. Therefore $ssts = stss$ is satisfied in the semigroup M_{n-2}. Consequently, $s^{2N}t^N s^N = s^N t^N s^{2N}$ is an identity in L_{n-3}.

Similarly, let us construct an identity in the quotient L_i/L_{i+1} for $i \in \{-1, \ldots, n-4\}$. As already explained, for every $w \in L_i$ we have $w^N \in \tilde{M}_{i+1} \cup I_{i+1}$. Therefore for any $s, t \in L_i/L_{i+1}$ either at least one of s^N, t^N is in I_{i+1} and then both $s^{2N}t^N s^N$ and $s^N t^N s^{2N}$ lie in I_{i+1} and thus are zero in L_i/L_{i+1} or $s^N, t^N \in M_{i+1}$, where M_{i+1} is the semigroup of matrix type over an infinite cyclic semigroup. Then, as in the previous case, we have $s^{2N}t^N s^N = s^N t^N s^{2N}$. Therefore $s^{2N}t^N s^N = s^N t^N s^{2N}$ is an identity in L_i/L_{i+1}.

Consequently $f_1^{(n-2-i)}(s,t) = f_2^{(n-2-i)}(s,t)$ is an identity in L_i for $i = -1, \ldots, n-3$.

Lastly, let us construct an identity in the quotient C_n/L_{-1}. For any $s, t \in C_n/L_{-1}$ if st, or equivalently ts, is of full support, then $(st)^N$ and $(ts)^N$ are both in $L \cap I_{-1} = L_{-1}$. Otherwise, from Lemma 3.5 (2), in [16], it follows that $(st)^N = (ts)^N$ is the zero element of the finite Hecke–Kiselman monoid HK_Θ associated to the acyclic full subgraph Θ of the oriented cycle of length n, whose vertices are exactly the generators occurring in the word st (or equivalently ts). Therefore $(st)^N = (ts)^N$ is the identity in C_n/L_{-1}.

The assertion of the theorem now follows from the fact that $f_1^{(n-1)}(s,t) = f_2^{(n-1)}(s,t)$ is the identity in L_{-1} and $(st)^N = (ts)^N$ in C_n/L_{-1}. □

Note that, as we will show in Section 8.4 in the case of the cycle of length 3, the identity from Theorem 7.1 is not necessarily of the smallest possible degree. For example, it is sometimes possible to construct an identity as in the proof of the theorem, with the constant N smaller than this calculated from the estimate in the proof of Proposition 2.15.

7.2 General case

As before, by Θ we understand a finite oriented graph. For any such a graph Θ let us denote by $V(\Theta)$ the set of its vertices and by $E(\Theta)$ the set of its edges. We identify elements of the monoid HK_Θ with the set of reduced words from the free monoid generated by $V(\Theta)$, using the Gröbner basis from Theorem 1.63.

Remark 1.64, describing a convenient reduction of the words of the form xwx in HK_Θ, for $x \in V(\Theta)$ and any word w such that x is terminal (there are no arrows $x \longrightarrow z$ or $z \longrightarrow x$ in the graph Θ) will be used in the proof of Theorem 7.2.

Consider the homomorphism $\varphi : \mathrm{HK}_\Theta \longrightarrow \mathrm{HK}_\Theta$ given by $\varphi(y) = y$ for all $y \in V(\Theta)$ such that $y \neq x$ and $\varphi(x) = 1$, where x is a fixed vertex such that there are no arrows of the form $z \to x$ ($x \to z$, respectively). For any word w in the free monoid generated by the set $V(\Theta)$ denote by $\overline{w}$ the image of w under such a homomorphism. Then for any $w \in \mathrm{HK}_\Theta$ and a word $v \in \mathrm{HK}_\Theta$ that contains the generator x in the support we get from Remark 1.64 that $vw = v\overline{w}$ ($wv = \overline{w}v$, respectively) in HK_Θ.

We are now in a position to prove the main theorem, which provides an affirmative solution of the Problem 1.52 in the case of Hecke–Kiselman algebras.

Theorem 7.2. *For a finite oriented graph Θ the following conditions are equivalent.*

(1) Θ does not contain two different cycles connected by an oriented path of length $\geqslant 0$,

(2) the Hecke–Kiselman monoid HK_Θ satisfies a semigroup identity.

Proof. As mentioned in the introduction, implication (2) $\Longrightarrow$ (1) follows directly from Theorem 1.72 and the fact that the free submonoid of rank 2 does not satisfy a semigroup identity. To prove that if condition (1) holds then the monoid HK_Θ satisfies a semigroup identity we proceed by induction on the number of edges in the graph Θ that are not contained in any cyclic subgraph of Θ (that is, a subgraph which is an oriented cycle). Let us denote this number by n_Θ.

If $n_\Theta = 0$, then from the hypothesis (1) it follows that the graph Θ is a disjoint union of oriented cycles and graphs with exactly one vertex and no edges. It follows then from Theorem 4.2 and Theorem 1.72 that $K[\mathrm{HK}_\Theta]$ is a finitely generated right Noetherian PI-algebra. Consequently, it embeds into a matrix ring over a field, see Theorem 1.24. As HK_Θ is a finitely generated monoid, we can assume that $\mathrm{HK}_\Theta \subseteq M_n(L)$ for some finitely generated field L. From the Theorem 1.72 and Theorem 1.53 it follows that in this case the monoid HK_Θ satisfies a semigroup identity. Moreover, such an identity can be explicitly constructed from the identities satisfied in monoids associated to cyclic subgraphs of the graph Θ and identities of the monoids associated to subgraphs of Θ with exactly vertex and no edges. The identities in the first case were obtained in Theorem 7.1, whereas monoids of the latter type satisfy the identity $st = ts$.

Assume now that $n_\Theta > 0$. From the description of the graph Θ and the assumption $n_\Theta > 0$ it follows that there exists a vertex x that is either a source vertex or a sink vertex. In other words, x is such a vertex that either there is an arrow $x \to y$ for some $y \in V(\Theta)$ but there are no arrows of the form $z \to x$ (x is a source vertex) or there is an arrow $y \to x$ for some $y \in V(\Theta)$ but there are no arrows of the form $x \to z$ (x is a sink vertex).

Let x be a source vertex. Consider the subgraph $\Theta_0 \subseteq \Theta$ such that $V(\Theta_0) = V(\Theta) \setminus \{x\}$ and $E(\Theta_0)$ consists of all edges from $E(\Theta)$ that are of the form $y \to z$ where $y \neq x$, that is Θ_0 is the graph Θ with vertex x and all edges of the form $x \to y$ for some $y \in V(\Theta)$ removed. By the induction hypothesis we know that HK_{Θ_0} admits a semigroup identity $\alpha(s,t) = \beta(s,t)$ for some different words α, β from the free monoid generated by s and t. From Remark 1.64 it follows that in the reduced form of elements of HK_Θ the generator x occurs at most once. In other words, every reduced element is of the form either $w \in \mathrm{HK}_{\Theta_0}$ or wxv, where w, v are elements of HK_{Θ_0}. It is clear that if $s, t \in \mathrm{HK}_{\Theta_0}$, then also $\alpha(s,t), \beta(s,t) \in \mathrm{HK}_{\Theta_0}$ and by the definition the identity $\alpha(s,t) = \beta(s,t)$ holds in HK_Θ in this case.

Assume now that $s, t \in \mathrm{HK}_\Theta$ are such that either s or t contains x. As without loss of generality we can assume that α and β both contain s and t, this is equivalent to the condition that the reduced form of $\alpha(s,t)$ is wxv, where $w, v \in \mathrm{HK}_{\Theta_0}$ and to the condition that the reduced form of $\beta(s,t)$ is pxq, where $p, q \in \mathrm{HK}_{\Theta_0}$. For a word $w \in \mathrm{HK}_\Theta$ let us denote by $\overline{w}$ the image of w under the homomorphism φ described in the comments before the formulation of Theorem 7.2. From those comments we get that $\alpha(s,t)\beta(s,t) = \alpha(s,t)\overline{\beta(s,t)}$ and $\alpha(s,t)\alpha(s,t) = \alpha(s,t)\overline{\alpha(s,t)}$. Moreover, it is clear that $\overline{\alpha(s,t)} = \alpha(\overline{s}, \overline{t}) = \beta(\overline{s}, \overline{t}) = \overline{\beta(s,t)}$, as $\overline{s}, \overline{t}$ can be treated as elements of HK_{Θ_0}. Therefore $\alpha(s,t)\beta(s,t) = \alpha(s,t)\overline{\beta(s,t)} = \alpha(s,t)\overline{\alpha(s,t)} = \alpha(s,t)\alpha(s,t)$, that is the following identity is satisfied

$$\alpha(s,t)\beta(s,t) = \alpha(s,t)\alpha(s,t). \tag{7.2.1}$$

Symmetric arguments applied to the case when x is a sink vertex show that the following identity holds

$$\beta(s,t)\alpha(s,t) = \alpha(s,t)\alpha(s,t). \tag{7.2.2}$$

It follows easily that the identity

$$\alpha(s,t)\beta(s,t)\alpha(s,t) = \alpha(s,t)\alpha(s,t)\alpha(s,t)$$

holds for any $s, t \in \mathrm{HK}_\Theta$. This proves the inductive assertion. □

Note that the proof allows us to construct inductively an identity in HK_Θ, for any graph Θ as in the theorem, from identities satisfied in the Hecke–Kiselman monoid associated to an oriented cycle.

Chapter 8

Working example: Hecke–Kiselman algebras $K[C_3]$ and $K[C_4]$

Now we illustrate the results of the previous chapters for Hecke–Kiselman monoids and algebras associated to cycles with small number of vertices, that is for the cycle of length 3 and of length 4. Moreover, we describe a subalgebra Z of the Hecke–Kiselman algebra $K[C_3]$ of the center of $K[C_3]$ such that $K[C_3]$ is a finitely generated module over Z.

8.1 Structure of the monoid C_3

Let us start with the description of the structure of the monoid C_3. For simplicity, write $x_1 = a$, $x_2 = b$, $x_3 = c$. Recall that

$$C_3 = \langle a, b, c : a^2 = a, b^2 = b, c^2 = c, ab = aba = bab, bc = bcb = cbc, ca = cac = aca \rangle.$$

From Theorem 1.65 in the case of $K[C_3]$ it follows that the set $\{aa - a, bb - b, cc - c, cac - ca, aca - ca, bcb - bc, cbc - bc, aba - ab, bab - ab\}$ forms a Gröbner basis of the algebra $K[C_3]$ with respect to the deg-lex order induced by $a < b < c$. Let us list the set of reductions in the following way

(1) (aa, a), (bb, b), (cc, c);

(2) (cac, ca), (cbc, bc), (bab, ab);

(3) (bcb, bc), (aba, ab), (aca, ca).

For $w, v \in \langle a, b, c \rangle$ write $w \to v$ in case v can by obtained from w by unspecified reductions. Reduction of a word w of type (η), where $(\eta \in \{1, 2, 3\})$ means that w can be rewritten as v, where $w = uw_\sigma z, v = uv_\sigma z$ for some $u, z \in \{a, b, c\}^*$ and an element (w_σ, v_σ) of the set S of reductions of type (η). In this case we also write that $w = v$ in $K[C_3]$, if unambiguous. As a natural consequence we obtain the following observation.

Lemma 8.1. *The reduced form of every element of C_3 is a factor of one of the following infinite words: $(cab)^\infty$, $(cba)^\infty$.*

From Theorem 2.44, applied for $n = 3$, we get that the monoid C_3 has an ideal chain

$$I_0 \subseteq I_{-1},$$

such that $I_0 = \{w \in C_3 : C_3 w C_3 \cap \langle cba \rangle = \emptyset\}$ and $I_{-1} = I_0 \cup \tilde{M}_0$, where $\tilde{M}_0$ consists of all factors of $(cba)^\infty$, that have cba as a subfactor. Moreover, we denote by $\tilde{M}_1$ a set consisting of all factors of $(cab)^\infty$ that have a subfactor cab.

Lemma 8.2. *Let T be the cyclic semigroup generated by $t = cab$. Then $M_1 = (C_3 cab C_3)^0$ is a semigroup of matrix type $\mathcal{M}^0(T, A_1, B_1; P_1)$, where $A_1 = \{1, b, ab\}$, $B_1 = \{1, c, ca\}$, with sandwich matrix (with coefficients in T^1)*

$$P_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & t \\ 1 & t & t \end{pmatrix}.$$

Similarly, for I_{-1} we have

Lemma 8.3. *Let S be the cyclic semigroup generated by $s = cba$. Then the semigroup $M_0 = I_{-1}/I_0$ is a semigroup of matrix type $\mathcal{M}^0(S, A_0, B_0; P_0)$, $A_0 = \{1, a, ba\}$, $B_0 = \{1, c, cb\}$, with sandwich matrix (with coefficients in $S^1 \cup \{\theta\}$)*

$$P_0 = \begin{pmatrix} 1 & 1 & \theta \\ 1 & \theta & s \\ \theta & s & s \end{pmatrix}.$$

Recall that the rows of P_i are indexed by the set B_i, and columns by the set A_i. For simplicity, we identify the elements of these sets with $1, 2, 3$, in the order in which these elements were listed. For example, the $(3, 3)$-entry of the sandwich matrix P_0 corresponds to the pair (cb, ba).

The above two lemmas follow directly from Theorem 2.44. To indicate computations that are used to determine the coefficients of the sandwich matrices, let us focus on P_0. For simplicity, if $\alpha \in A_0$, $\beta \in B_0$, then we write

$$\overline{p_{\beta\alpha}} = \begin{cases} (cba)\beta\alpha(cba) & \text{if } \beta\alpha \in \langle s \rangle \\ \theta & \text{if } \beta\alpha \in I_0, \end{cases}$$

that is if $\overline{p_{\beta\alpha}} = s^k$, then $p_{\beta\alpha} = s^{k-2}$ and if $\overline{p_{\beta\alpha}} = \theta$, then also $p_{\beta\alpha} = \theta$. Then, for example $\overline{p_{(cb)(ba)}} = (cba)cbba(cba) \xrightarrow{(1)} (cba)cba(cba) \xrightarrow{(2)} (cba)^3$. So, $p_{(cb)(ba)} = s$.

Recall from Chapter 2 that $\tilde{M}_i$ are subsets of C_3, whereas $M_i = \tilde{M}_i \cup \{\theta\}$ with θ being the zero element, are subsemigroups in the quotients, $M_i \subseteq I_{i-1}/I_i$.

We derive the following consequence for the algebras $K_0[M_0]$ and $K_0[M_1]$.

Corollary 8.4. *Algebras $K_0[M_0]$ and $K_0[M_1]$ are of matrix type. Namely, we have $K_0[M_1] = \mathcal{M}(K[T], A_1, B_1; P_1)$ and $K_0[M_0] = \mathcal{M}(K[S], A_0, B_0; P_0)$, where T and S are the cyclic semigroups generated by $t = cab$ and $s = cba$, respectively.*

It is easy to see that $\det P_1 = -(t-1)^2 \neq 0$ and $\det P_0 = -s(s+1) \neq 0$, whence P_1 and P_0 are not zero divisors in $M_3(K[T])$ and $M_3(K[S])$. From standard results about algebras of matrix type, see also Section 1.3, we obtain the following.

Corollary 8.5. *Semigroup algebras $K_0[M_0]$ and $K_0[M_1]$ are prime.*

8.2 Structure of the monoid C_4

For simplicity, we write $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = d$. Recall that C_4 has the following presentation

$$C_4 = \langle a, b, c, d : a^2 = a, b^2 = b, c^2 = c, d^2 = d, ab = aba = bab, bc = bcb = cbc,$$
$$cd = cdc = dcd, da = dad = ada, ac = ca, bd = db\rangle.$$

The form of the sets $A_0, B_0, A_1, B_1, A_2, B_2$ follows directly from Theorem 2.1.

Lemma 8.6. *If an element of C_4 has a factor of the form $w_0 = dcba$, $w_1 = dacb$ or $w_2 = dabc$ then it is of the form $\alpha_i w_i^k \beta_i$, with $k \geqslant 1$, $\alpha_i \in A_i$, $\beta_i \in B_i$, where*

1. *$A_0 = \{1, a, ba, cba\}$, $B_0 = \{1, d, dc, dcb\}$;*
2. *$A_1 = \{1, b, cb, acb, ab, bacb\}$, $B_1 = \{1, d, da, dac, dc, dacd\}$;*
3. *$A_2 = \{1, c, bc, abc\}$, $B_2 = \{1, d, da, dab\}$.*

From Theorem 2.44 we know that C_4 has a chain of ideals

$$\emptyset = I_2 \subseteq I_1 \subseteq I_0 \subseteq I_{-1},$$

with semigroups of matrix type M_0, M_1 and M_2, such that

1. $M_2 = (C_4 dabc C_4)^0 \subseteq I_1/I_2$ and the set $(I_1/I_2) \setminus M_2$ is finite,
2. $M_1 = \{\alpha(dacb)^k\beta : \alpha \in A_1, \beta \in B_1, k \geqslant 1\} \cup \{\theta\} \subseteq I_0/I_1$ and the set $(I_0/I_1) \setminus M_1$ is finite,
3. $M_0 = \{\alpha(dcba)^k\beta : \alpha \in A_0, \beta \in B_0, k \geqslant 1\} \cup \{\theta\} = I_{-1}/I_0$,
4. $C_4 \setminus I_{-1}$ is finite, where $I_{-1} = I_0 \cup C_4 dcba C_4$.

We present these structures of matrix type below. A simple verification is left to the reader.

Lemma 8.7. *Let Q_2 be the cyclic semigroup generated by $s = dabc$. Then the ideal generated by s in C_4, with a zero adjoined, that is $M_2 = (C_4 dacb C_4)^0$, is a semigroup of matrix type $\mathcal{M}^0(Q_2, A_2, B_2; P_2)$, where $A_2 = \{1, c, bc, abc\}$, $B_2 = \{1, d, da, dab\}$, with sandwich matrix (with coefficients in Q_2^1)*

$$P_2 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & s \\ 1 & 1 & s & s \\ 1 & s & s & s \end{pmatrix}.$$

Lemma 8.8. *Let Q_1 be the cyclic semigroup generated by $s = dacb$. Then M_1 is a semigroup of matrix type $\mathcal{M}^0(Q_1, A_1, B_1; P_1)$ where $A_1 = \{1, b, cb, acb, ab, bacb\}$, $B_1 = \{1, d, dc, dac, da, dacd\}$, with sandwich matrix (with coefficients in $Q_1^1 \cup \{\theta\}$)*

$$P_1 = \begin{pmatrix} 1 & 1 & \theta & \theta & 1 & \theta \\ 1 & 1 & \theta & s & \theta & s \\ \theta & \theta & \theta & s & s & s \\ \theta & s & s & s & s & \theta \\ 1 & \theta & s & s & \theta & \theta \\ \theta & s & s & \theta & \theta & s^2 \end{pmatrix}.$$

Lemma 8.9. *Let S be the cyclic semigroup generated by $s = dcba$. Then M_0 is a semigroup of matrix type $\mathcal{M}^0(Q_0, A_0, B_0; P_0)$, where $s = dcba$, $A_0 = \{1, a, ba, cba\}$, $B_0 = \{1, d, dc, dcb\}$, with sandwich matrix*

$$P_0 = \begin{pmatrix} 1 & 1 & \theta & \theta \\ 1 & \theta & \theta & s \\ \theta & \theta & s & s \\ \theta & s & s & \theta \end{pmatrix}.$$

We get the following consequence.

Corollary 8.10. *Algebras $K_0[M_2]$, $K_0[M_1]$ and $K_0[M_0]$ are algebras of matrix type. Namely, $K_0[M_2] = \mathcal{M}(K[Q_2], A_2, B_2; P_2)$, $K_0[M_1] = \mathcal{M}(K[Q_1], A_1, B_1; P_1)$ and $K_0[M_0] = \mathcal{M}(K[Q_0], A_0, B_0; P_0)$, where Q_2, Q_1, Q_0 are the cyclic semigroups generated by $s_2 = dabc$, $s_1 = dacb$, and by $s_0 = dcba$, respectively.*

A direct computation shows that $\det P_2 = -(s_2 - 1)^3 \neq 0$. Similarly, one can see that $\det P_1 = -s_1^3(s_1 + 1)^3 \neq 0$ and $\det P_0 = -s_0^2(s_0 - 1) \neq 0$, so that the matrices P_i are not zero divisors in the corresponding matrix rings $M_{n_i}(K[Q_i])$, for $i \in \{0, 1, 2\}$. Therefore, as in Corollary 8.5, we get

Corollary 8.11. *Semigroup algebras $K_0[M_2]$, $K_0[M_1]$ and $K_0[M_0]$ are prime.*

Note that we proved in Theorem 2.52 that all algebras $K_0[M_t]$ coming from monoids C_n, $n \geqslant 3$ are prime. However, the proof for arbitrary n is much more involved, since the determinants of the corresponding matrices cannot be easily computed.

8.3 Irreducible representations of $K[C_3]$

Let us illustrate the results of Chapter 5 with the case of the Hecke–Kiselman monoid C_3 associated to the cycle of length 3. We start with the representations induced by the irreducible representations of the semigroups of matrix type inside the Hecke–Kiselman algebra $K[C_3]$ described in Lemmas 8.2 and 8.3. They are the restrictions of the representations of the completely 0-simple closures of M_i, isomorphic to $\mathcal{M}^0(gr(t), A_1, B_1; P_1)$ and $\mathcal{M}^0(gr(s), A_0, B_0; P_0)$ for M_1 and M_0, respectively. Irreducible representations of the latter come from the representations of their maximal subgroups. In our case, these maximal subgroups are infinite cyclic groups, as described in Section 5.3. Here we use the classical approach presented in Chapter 5.4 of [9], in particular Theorem 5.37, with certain computations omitted.

For any semigroup S, we denote by S^0 the semigroup S with zero element adjoined.

For every fixed $\lambda \in K^*$ we consider the irreducible representation ψ_λ of M_1 described in Proposition 5.9. It is induced by the representation of the cyclic group $\mathrm{gr}(t)$ given by $t \mapsto \lambda$. If $\lambda \neq 0, 1$, we have that $\overline{P}_1$ is a matrix of rank 3 and therefore we get a family of representations $\psi_\lambda : K_0[M_1] \to M_3(K)$. In this case the epimorphism $\mathcal{M}(K, A_1, B_1; \overline{P}_1) \to M_3(K)$ is given by $A \mapsto A \circ \overline{P}_1$. Therefore, the representation ψ_λ is given for every $(t^k; x, y) \in M_1$ by

$$(t^k; x, y) \mapsto M_{(\lambda^k; x,y)} \circ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & \lambda \\ 1 & \lambda & \lambda \end{pmatrix},$$

where $\circ$ is the standard matrix multiplication and $M_{(\lambda^k;x,y)} \in M_3(K)$ is the matrix with the only non-zero entry (x, y) equal to λ^k.

For $\lambda = 1$ the matrix $\overline{P}_1$ is of rank 1 and therefore we get the one-dimensional representation $\psi_1 : K_0[M_1] \to K$, such that $\psi_1(t^k; x, y) = 1$ for all $k \geqslant 1$, $x \in A_1$, $y \in B_1$, see Theorem 5.37 in [9].

Similarly, for every fixed $\lambda \in K^*$ consider the irreducible representation ψ_λ of M_0 (see Lemma 8.3) described in Proposition 5.9. It is induced by the representation of $\mathrm{gr}(t)$ given by $t \mapsto \lambda$. If $\lambda \neq 0, -1$ then we have that $\overline{P}_0$ is a matrix of rank 3 and therefore we get a family of representations $\psi_\lambda : K_0[M_0] \to M_3(K)$. The representation ψ_λ is given for every $(s^k; x, y) \in M_0$ by

$$(s^k; x, y) \mapsto M_{(\lambda^k; x,y)} \circ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & \lambda \\ 0 & \lambda & \lambda \end{pmatrix},$$

where $\circ$ is the standard matrix multiplication and $M_{(\lambda^k;x,y)} \in M_3(K)$ is the matrix with the only non-zero entry (x, y) equal to λ^k.

Moreover, for $\lambda = -1$ the matrix $\overline{P}_0 = (\overline{p}_{yx})$ has rank 2 and therefore the corresponding representation $\psi_{-1} : K_0[M_0] \to M_2(K)$ is two-dimensional. To give a formula for this representation we use Theorem 5.37 from [9]. Firstly, we determine $r_x \in K$ and $q_y \in K$ for $x \in A_0$

and $y \in B_0$ satisfying the condition

$$q_y r_x = \widetilde{\psi_{-1}}(p_{yx}) - \widetilde{\psi_{-1}}(p_{y1} \cdot p_{1x})$$

for $x \in \{1, a, ba\}$ and $y \in \{1, c, cb\}$, where $\widetilde{\psi_{-1}} : \mathrm{gr}(s)^0 \to K$ is the map such that $\widetilde{\psi_{-1}}(s^k) = (-1)^k$ for all $k \in \mathbb{Z}$ and $\widetilde{\psi_{-1}}(0) = 0$. Namely, $(r_1, r_a, r_{ba}) = (0, -1, -1)$ and $(q_1, q_c, q_{cb}) = (0, 1, 1)$ satisfy this condition. Consequently, we obtain that the representation ψ_{-1} is given by

$$(s^k; x, y) \mapsto \begin{pmatrix} \widetilde{\psi_{-1}}(p_{1x} s^k p_{y1}) & \widetilde{\psi_{-1}}(p_{1x} s^k) q_y \\ r_x \widetilde{\psi_{-1}}(s^k p_{y1}) & r_x \widetilde{\psi_{-1}}(s^k) q_y \end{pmatrix},$$

for all $k \in \mathbb{Z}$, $x \in \{1, a, ba\}$ and $y \in \{1, c, cb\}$.

Note also that for $\lambda = 0$ and any i the induced homomorphism $K_0[M_i] \to \mathcal{M}(K, A_i, B_i; \overline{P_i})$ is the zero map.

From the results in Section 5.1 it follows that $\{1, a, b, c, ab, bc, ca\}$ is the set of idempotents in C_3 and $1, a, b, c \in C_3 \setminus I_{-1}$, $ab, bc, ca \in I_0 \setminus I_1$ (note that $I_1 = \emptyset$). Thus, we get (Theorem 5.8) that irreducible representations of $K[C_3]$ either come from the representations of $K_0[M_0]$ or $K_0[M_1]$ described above or are one-dimensional representations associated to one of the idempotents in the monoid C_3.

8.4 Semigroup identity

Now we apply the results of Chapter 7 to construct a semigroup identity in C_3.

From the description of reduced words in C_3 in Lemma 8.1 and the definitions of semigroups of matrix type M_0 and M_1, it follows easily that for every word $w \in C_3$ of full support $w^2 \in \tilde{M}_0 \cup \tilde{M}_1$. Thus, in the construction of a semigroup identity in C_3 from Theorem 7.1 we can set $N = 2$. Therefore in C_3 the identity $u(s,t) = v(s,t)$ is satisfied, where

$$u(s,t) = ((st)^4(ts)^4(st)^8)^2((st)^8(ts)^4(st)^4)^2((st)^4(ts)^4(st)^8)^4$$

$$v(s,t) = ((st)^4(ts)^4(st)^8)^4((st)^8(ts)^4(st)^4)^2((st)^4(ts)^4(st)^8)^2.$$

As we noted in Chapter 7, straightforward application of the proof of Theorem 7.1 leads to an identity which is not necessarily of the smallest possible order. Indeed, in the case of cycle of length 3 the constant N from the proof of Theorem 7.1, equal to $|C_3 \setminus (\tilde{M}_0 \cup \tilde{M}_1)|$, is 18. On the other hand, for every word $w \in C_3$ of full support we have that $w^2 \in \tilde{M}_0 \cup \tilde{M}_1$ and thus $N = 2$ can be used in our construction.

8.5 Center of $K[C_3]$

From Theorem 1.22 and Lemma 3.2 it follows that the algebra $K[C_3]$ is a finite module over its center. Therefore it is a natural problem to characterize the center of this algebra. As it seems to be difficult to find the whole center of $K[C_3]$, our aim is to describe a subalgebra Z of the center of $K[C_3]$ such that $K[C_3]$ is a finitely generated module over Z.

Recall from Section 8.1 that $\tilde{M}_0$ consists of all factors of $(cba)^\infty$ that contain cba, whereas $\tilde{M}_1$ is an ideal of C_3 generated by cab.

Let us start with the following simple modification of Theorem 2.28 in the case of the monoid C_3, that can be proved by a straightforward computation.

Lemma 8.12. *For every word $w \in C_3$ we have $(cba)w(cba) \in \{(cba)^k : k \geqslant 2\} \cup \tilde{M}_1$.*

Consider the subsemigroup of C_3 of the form $S_3 = \tilde{M}_0 \cup \tilde{M}_1$. From the above lemma it follows that $S_3/\tilde{M}_1$ is isomorphic to $I_{-1}/I_0 = M_0$.

In particular we can replace the ideal I_0 with M_1 in Lemma 8.3, obtaining the following result.

Lemma 8.13. *Let S be the cyclic semigroup generated by $s = cba$. Then the semigroup $M_0 = S_3/\tilde{M}_1$ is a semigroup of matrix type $\mathcal{M}^0(S, A_0, B_0; P_0)$, $A_0 = \{1, a, ba\}$, $B_0 = \{1, c, cb\}$, with sandwich matrix (with coefficients in $S^1 \cup \{\theta\}$)*

$$P_0 = \begin{pmatrix} 1 & 1 & \theta \\ 1 & \theta & s \\ \theta & s & s \end{pmatrix}.$$

Let us consider the completely 0-simple closure of the subsemigroup $M_1 = (C_3 cab C_3)^0$; it is isomorphic to $\mathcal{M}^0(gr(t), A_1, B_1; P_1)$, see Section 1.3. Then, by Lemma 1.58 we know that there is a unique semigroup structure on the disjoint union $\widehat{S_3} = (S_3 \setminus \tilde{M}_1) \cup \mathcal{M}^0(gr(t), A_1, B_1; P_1)$ that extends the operation from S_3. Indeed, $\tilde{M}_1$ is an ideal of the semigroup S_3 and $\mathcal{M}^0(gr(t), A_1, B_1; P_1)$ is a completely 0-simple semigroup of quotients of M_1. Then $K[\widehat{S_3}] = K[S_3 \setminus \tilde{M}_1] + K[\mathcal{M}^0(gr(t), A_1, B_1; P_1)]$, with $K[\widehat{S_3}]/K[\mathcal{M}^0(gr(t), A_1, B_1; P_1)]$ isomorphic to $K_0[M_0]$, where by $K[S_3 \setminus \tilde{M}_1] + K[\mathcal{M}^0(gr(t), A_1, B_1; P_1)]$ we mean the direct sum of linear subspaces.

Algebras $K_0[M_0]$ and $K[\mathcal{M}^0(gr(t), A_1, B_1; P_1)]$ are of matrix type. Namely, $K_0[M_0] = \mathcal{M}(K[S], A_0, B_0; P_0)$ and $K_0[M_1] = \mathcal{M}(K[t, t^{-1}], A_1, B_1; P_1)$ where S is the cyclic semigroup generated by $s = cba$ and $K[t, t^{-1}]$ is the Laurent polynomials ring, where $t = cab$.

As already noted, matrices P_1 and P_0 are invertible as matrices in $M_3(K(t))$ and $M_3(K(s))$, respectively. We have

$$P_1^{-1} = \frac{1}{t-1} \begin{pmatrix} t & \theta & -1 \\ \theta & -1 & 1 \\ -1 & 1 & \theta \end{pmatrix}$$

and

$$P_0^{-1} = \frac{1}{s(s+1)} \begin{pmatrix} s^2 & s & -s \\ s & -s & s \\ -s & s & 1 \end{pmatrix}.$$

Moreover, we know that $\mathcal{M}(K(t), A_1, B_1; P_1)$ and $\mathcal{M}(K(s), A_0, B_0; P_0)$ are isomorphic to $M_3(K(t))$ and $M_3(K(s))$, respectively. Isomorphisms $M_3(K(t)) \to \mathcal{M}(K(t), A_1, B_1; P_1)$ and $M_3(K(s)) \to \mathcal{M}(K(s), A_0, B_0; P_0)$ are given by $x \longmapsto x \circ P_i^{-1}$, where $\circ$ is the standard matrix multiplication, see [44, Proposition 4.13].

Therefore

$$Z(\mathcal{M}(K(t), A_1, B_1; P_1)) = K(t)P_1^{-1} \text{ and } Z(\mathcal{M}(K(s), A_0, B_0; P_0)) = K(s)P_0^{-1}.$$

It follows that

$$K[t]t(t-1)P_1^{-1} = K[t]t \begin{pmatrix} t & \theta & -1 \\ \theta & -1 & 1 \\ -1 & 1 & \theta \end{pmatrix} \subseteq Z(\mathcal{M}(K[T], A_1, B_1; P_1)$$

and

$$K[s]s^2(s+1)P_0^{-1} = K[s]s \begin{pmatrix} s^2 & s & -s \\ s & -s & s \\ -s & s & 1 \end{pmatrix} \subseteq Z(\mathcal{M}(K[S], A_0, B_0; P_0).$$

Conversely, we know that

$$Z(\mathcal{M}(K[T], A_1, B_1; P_1)) = Z(\mathcal{M}(K(t), A_1, B_1; P_1)) \cap \mathcal{M}(K[T], A_1, B_1; P_1))$$

and

$$Z(\mathcal{M}(K[S], A_0, B_0; P_0) = Z(\mathcal{M}(K(s), A_0, B_0; P_0)) \cap \mathcal{M}(K[S], A_0, B_0; P_0)).$$

It then follows that if

$$f(t)P_1^{-1} = \begin{pmatrix} \frac{f(t)t}{t-1} & \theta & \frac{-f(t)}{t-1} \\ \theta & \frac{-f(t)}{t-1} & \frac{f(t)}{t-1} \\ \frac{-f(t)}{t-1} & \frac{f(t)}{t-1} & \theta \end{pmatrix} \in \mathcal{M}(K[T], A_1, B_1; P_1),$$

then $f(t) = t(t-1)g(t)$ for some $g(t) \in K[t]$, that is

$$K[t]t(t-1)P_1^{-1} = K[t]t \begin{pmatrix} t & \theta & -1 \\ \theta & -1 & 1 \\ -1 & 1 & \theta \end{pmatrix} = Z(\mathcal{M}(K[T], A_1, B_1; P_1).$$

Similarly, we have

$$K[s]s^2(s+1)P_0^{-1} = K[s]s \begin{pmatrix} s^2 & s & -s \\ s & -s & s \\ -s & s & 1 \end{pmatrix} = Z(\mathcal{M}(K[S], A_0, B_0; P_0).$$

Elements of $\langle t\rangle(t-1)P_1^{-1}$ correspond to elements of the algebra $K[C_3]$ of the form

$$t_k = (cab)^{k+1} - (cab)^k ca - b(cab)^k c + b(cab)^k ca - ab(cab)^k + ab(cab)^k c,$$

for $k \geqslant 1$.
Similarly, $\langle s\rangle s(s+1)P_0^{-1}$ correspond to elements of $K[C_3]$ of the form

$$s_k = (cba)^{k+2} + (cba)^{k+1}c - (cba)^{k+1}cb + a(cba)^{k+1} - a(cba)^{k+1}c + a(cba)^{k+1}cb - \\ -ba(cba)^{k+1} + ba(cba)^{k+1}c + ba(cba)^k cb$$

for $k \geqslant 1$.

We want to use the structures of matrix type within $K[S_3]$ to characterize the center of $K[S_3]$. To find the elements of this center it is more convenient to consider the following extension of the algebra $K[S_3]$ and of the algebra $K[\widehat{S_3}]$ introduced after Lemma 8.13.

Lemma 8.14. *$R = K[S_3 \setminus \tilde{M}_1] + \mathcal{M}(K(t), A_1, B_1; P_1)$ (direct sum as a subspaces) has a natural structure of an algebra, which extends the structure of $K[\widehat{S_3}] = K[S_3 \setminus \tilde{M}_1] + K[\mathcal{M}^0(gr(t), A_1, B_1; P_1)]$.*

Proof. We need to define $q \cdot p$ and $p \cdot q$, where $q \in K[S_3 \setminus \tilde{M}_1]$ and $p \in \mathcal{M}(K(t), A_1, B_1; P_1)$. By linearity we can assume that $q \in S_3 \setminus \tilde{M}_1$ and p is of the form $(\frac{t^k}{g(t)}; x, y)$, where $k \geqslant 0$, $\frac{t^k}{g(t)} \in K(t)$, $x \in A_1$, $y \in B_1$. Then it is clear that $\frac{t^k}{g(t)}$ can be written in the form $\frac{t^k}{g(t)} = t\tilde{p}(t)t$ for $\tilde{p}(t) = \frac{t^{k-2}}{g(t)} \in K(t)$ and $p = (t; x, 1) \cdot (\tilde{p}(t); 1, 1) \cdot (t; 1, y)$ in $\mathcal{M}(K(t), A_1, B_1; P_1)$. Therefore we can identify $K(t)$ with the $\mathcal{H}$-class $H_{1,1} = \{(f(t); 1, 1) : f(t) \in K(t)\}$. It follows that $\mathcal{M}(K(t), A_1, B_1; P_1)$ is the disjoint union

$$\bigcup_{i \in A_1, j \in B_1} a_i H_{1,1} b_j \cup \{\theta\},$$

where $a_i \in \{(t; 1, 1), (t; b, 1), (t; ab, 1)\}$, $b_i \in \{(t; 1, 1), (t; 1, c), (t; 1, ca)\}$. Let us denote by $\widehat{t}$ the word cab in the monoid C_3. Because M_1 is an ideal in S_3 and $\widehat{t} \in M_1$, we know that $qx\widehat{t}, \widehat{t}yq \in M_1 \subseteq \mathcal{M}(K(t), A_1, B_1; P_1)$. Let us define $q \cdot p = (qx\widehat{t}) \cdot (\tilde{p}(t)t; 1, y)$ and $p \cdot q = (t\tilde{p}(t); x, 1) \cdot (\widehat{t}yq)$, where $qx\widehat{t}$ and $\widehat{t}yq$ mean products in S_3 and $\cdot$ is the product in the algebra $\mathcal{M}(K(t), A_1, B_1; P_1)$. It follows from the arguments very similar to the proof of Lemma 2.5.1 in [23] (see also Lemma 1.58) that this operation is associative. Moreover, the construction in the latter lemma shows that the definition extends the structure of the algebra on $K[\widehat{S_3}]$. $\square$

Our aim is to use this lemma to understand how elements of the centers of algebras of matrix type correspond to the center of $K[S_3]$. We know that $\mathcal{M}(K(t), A_1, B_1; P_1)$, isomorphic to $M_3(K(t))$, is an ideal of the algebra $R = K[S_3 \setminus \tilde{M_1}] + \mathcal{M}(K(t), A_1, B_1; P_1)$ (see the proof of Lemma 8.14) and $\mathcal{M}(K(t), A_1, B_1; P_1)$ has the unit of the form $e = P_1^{-1}$. Therefore $R = (1-e)R \oplus eR$ with $(1-e)R = K[S_3 \setminus M_1] \subseteq \mathcal{M}(K(s), A_0, B_0; P_0)$, $eR = \mathcal{M}(K(t), A_1, B_1; P_1)$ and $K[S_3] \hookrightarrow K[\widehat{S_3}] \subseteq R$. From the proof of Theorem 3.5 we know that $K[S_3]$ and $K[C_3]$ have the same classical (central) ring of quotients, which is isomorphic to $M_3(K(s)) \times M_3(K(t))$. Therefore $Z(K[S_3]) \subseteq Z[K[C_3]) \subseteq Z(M_3(K(s)) \times M_3(K(t)))$. Similarly, $K[S_3]$ and R have the same classical (central) ring of quotients and thus also $Z(K[S_3]) \subseteq Z(R) = Z((1-e)R) \oplus Z(eR)$. We know that $Z((1-e)R) \oplus Z(eR) \cong Z(\mathcal{M}(K[S], A_0, B_0; P_0)) \oplus Z(\mathcal{M}(K(t), A_1, B_1; P_1))$. It follows that the inclusion $K[S_3] \hookrightarrow \mathcal{M}(K(s), A_0, B_0; P_0) + \mathcal{M}(K(t), A_1, B_1; P_1)$ is given by $x \longmapsto (1-e)x + ex$. As we have already seen $lin_K\{t_k : k \geqslant 1\} = Z(\mathcal{M}(K[T], A_1, B_1; P_1) = Z(eR) \cap K[S_3]$. Secondly, $Z((1-e)R) \cap K[S_3]$ is given by elements of the form $x - ex$, where $x \in Z(\mathcal{M}(K(s), A_0, B_0; P_0)) \cap K[S_3]$ and $ex \in K[S_3]$, that is $Z((1-e)R) \cap K[S_3] = lin_K\{s_k - P_1^{-1}s_k : k \geqslant 1\} \cap K[S_3]$. Therefore we get that

$$Z = Z(K[S_3]) = lin_K\{s_k - P_1^{-1}s_k : k \geqslant 1\} \cap K[S_3] + lin_K\{t_k : k \geqslant 1\}.$$

It turns out that for every $k \geqslant 1$ we have $s_k - P_1^{-1}s_k \in K[S_3]$. We start with an easy computational lemma.

Lemma 8.15. *(1) If $k \geqslant 1$ is odd, then*

- *(a)* $b(cba)^k = b(cab)^{\frac{k-1}{2}}ca$;
- *(b)* $ca(cba)^k = (cab)^{\frac{k+1}{2}}$;
- *(c)* $(cba)^k b = (cab)^{\frac{k+1}{2}}$;
- *(d)* $(cba)^k ca = b(cab)^{\frac{k-1}{2}}ca$.

(2) If $k \geqslant 1$ is even, then

- *(a)* $b(cba)^k = b(cab)^{\frac{k}{2}}$;
- *(b)* $ca(cba)^k = (cab)^{\frac{k}{2}}ca$;
- *(c)* $(cba)^k b = b(cab)^{\frac{k}{2}}$;
- *(d)* $(cba)^k ca = (cab)^{\frac{k}{2}}ca$.

Proof. In each case we proceed by induction on k using the relations in $K[C_3]$. For instance, consider case (a). For $k = 1$ we have $b(cba) \to bca = b(cab)^0 ca$. Assume now that the assertion holds for some $k \geqslant 1$. If $k+1$ is even using the inductive step we get $b(cba)^{k+1} = b(cba)^k cba \to b(cab)^{\frac{k-1}{2}} cacba \to b(cab)^{\frac{k-1}{2}} cab = b(cab)^{\frac{k+1}{2}}$. Similarly, for odd $k+1$ we calculate $b(cba)^{k+1} = b(cba)^k cba = b(cab)^{\frac{k}{2}} cba \to b(cab)^{\frac{k}{2}} ca$. This proves the inductive assertion. $\square$

Let us denote $u_k = (cab)^{k+1} + b(cab)^k ca + ab(cab)^k c$ and $v_k = b(cab)^k c + (cab)^k ca + ab(cab)^k$. Then in particular $t_k = u_k - v_k$.

Proposition 8.16. *For every $k \geqslant 1$ we have $P_1^{-1} s_k \in K[S_3]$. More precisely, if $k = 2n$ for some $n \geqslant 1$ we have $P_1^{-1} s_{2n} = u_n$, and for $k = 2n-1$, where $n \geqslant 1$ we get that $P_1^{-1} s_{2n-1} = v_n$.*

Proof. We start with the calculation of $(t-1)P_1^{-1} s_k \subseteq K[C_3]$. The element $(t-1)P_1^{-1}$ can be interpreted as $t_0 = cab - ca - bc + bca - ab + abc \in K[C_3]$. We aim to find the reduced form of $t_0 s_k$.

1. $cab \cdot s_k$
   Using reductions we get that $cabba \to caba \to cab$. It follows that

   $$cab(((cba)^{k+1}c - a(cba)^{k+1}c) + (a(cba)^{k+1}cb - (cba)^{k+1}cb) + (-ba(cba)^{k+1} + a(cba)^{k+1})) = 0.$$

   Therefore

   $$cab \cdot s_k = cab((cba)^{k+2} + ba(cba)^{k+1}c + ba(cba)^k cb).$$

2. $-ca \cdot s_k$
   Using the fact that $cacba \to caba$ and $caa \to ca$ we get that

   $$-ca \cdot s_k = -ca(a(cba)^{k+1} + ba(cba)^{k+1}c + ba(cba)^k cb).$$

3. $-bc \cdot s_k$
   Similarly, we have $bca(cba)^{k+1} = bcba(cba)^{k+1}$, $bca(cba)^{k+1}c = bcba(cba)^{k+1}c$, $bc(cba)^{k+1}cb = bcba(cba)^k cb$ and therefore

   $$-bc \cdot s_k = -bc((cba)^{k+2} + (cba)^{k+1}c + a(cba)^{k+1}cb).$$

4. $bca \cdot s_k$
   Using the reductions it is easy to check that $bca(cba)^{k+2} = bcaba(cba)^{k+1}$, $bca(cba)^{k+1}c = bcaa(cba)^{k+1}c$, $bca(cba)^{k+1}cb = bcaa(cba)^{k+1}cb$ and we get that

   $$bca \cdot s_k = bca(a(cba)^{k+1} + ba(cba)^{k+1}c + ba(cba)^k cb).$$

5. $-ab \cdot s_k$
   We have that $ab(cba)^{k+1}c = aba(cba)^{k+1}c$, $ab(cba)^{k+1}cb = aba(cba)^{k+1}cb$, $a(cba)^{k+1} = aba(cba)^{k+1}$. Thus

   $$-ab \cdot s_k = -ab((cba)^{k+2} + ba(cba)^{k+1}c + ba(cba)^k cb).$$

6. $abc \cdot s_k$
   One can check that $abca(cba)^{k+1} = abcba(cba)^{k+1}$, $abca(cba)^{k+1}c = abcba(cba)^{k+1}c$,

$abc(cba)^{k+1}cb = abcba(cba)^k cb$, so

$$abc \cdot s_k = abc((cba)^{k+2} + (cba)^{k+1}c + a(cba)^{k+1}cb)$$

Moreover the following holds: $cabba(cba)^{k+1}c = caba(cba)^{k+1}c$, $cabba(cba)^k cb = caba(cba)^k cb$, $b(cba)^{k+2} = bca(cba)^{k+1}$, $ab(cba)^{k+2} = abc(cba)^{k+2}$, $abba(cba)^{k+1}c = abc(cba)^{k+1}c$, $bcab(cba)^k cb = bca(cba)^{k+1}cb$.

It follows that

$$\begin{aligned} t_0 s_k =& cab(cba)^{k+2} - ca(cba)^{k+1} - bc(cba)^{k+2} - bc(cba)^{k+1}c + bcab(cba)^{k+1}c + \\ &+ bcab(cba)^k cb - ab(cba)^k cb + abca(cba)^{k+1}cb. \end{aligned}$$

If k is even then using Lemma 8.15 we calculate that

$$t_0 s_k = (cab)^{\frac{k+4}{2}} - (cab)^{\frac{k+2}{2}} - b(cab)^{\frac{k}{2}}ca + b(cab)^{\frac{k+2}{2}}ca - ab(cab)^{\frac{k}{2}}c + ab(cab)^{\frac{k+2}{2}}c.$$

It follows that $t_0 s_k$ corresponds to the matrix

$$\begin{pmatrix} t^{\frac{k+2}{2}}(t-1) & \theta & \theta \\ \theta & \theta & t^{\frac{k}{2}}(t-1) \\ \theta & t^{\frac{k}{2}}(t-1) & \theta \end{pmatrix}.$$

Therefore for even k the element $P_1^{-1}s_k = u_{\frac{k}{2}}$ and in particular $P_1^{-1}s_k \in K[S_3]$ in this case.

Similarly, for odd k we have

$$t_0 s_k = (cab)^{\frac{k+3}{2}}ca - (cab)^{\frac{k+1}{2}}ca - b(cab)^{\frac{k+1}{2}}c + b(cab)^{\frac{k+3}{2}}c - ab(cab)^{\frac{k+1}{2}} + ab(cab)^{\frac{k+3}{2}}.$$

It follows that $t_0 s_k$ corresponds to the matrix

$$\begin{pmatrix} \theta & \theta & t^{\frac{k+1}{2}}(t-1) \\ \theta & t^{\frac{k+1}{2}}(t-1) & \theta \\ t^{\frac{k+1}{2}}(t-1) & \theta & \theta \end{pmatrix}.$$

Thus for odd k the element $P_1^{-1}s_k = v_{\frac{k+1}{2}}$ and in particular $P_1^{-1}s_k \in K[S_3]$ in this case too. $\square$

We obtain the following characterization of the center of $K[S_3]$.

Corollary 8.17. *The center Z of $K[S_3]$ is equal to $lin_K\{t_k : k \geqslant 1\} + lin_K\{s_{2k-1} - v_k : k \geqslant 1\} + lin_K\{s_{2k} - u_k : k \geqslant 1\}$.*

It is also possible to calculate the center of $K[S_3]$ in a slightly more direct way. We will show that the center Z is equal to $lin_K\{t_k : k \geqslant 1\} + lin_K\{s_{2k-1} - v_k : k \geqslant 1\} + lin_K\{s_{2k} - u_k : k \geqslant 1\}$, as in Corollary 8.17.

The following few remarks will be crucial. As we have already explained every element of the center $Z(K[S_3])$ is contained in the center of the Hecke–Kiselman algebra $K[C_3]$. It follows that $w \in Z(K[S_3])$ if and only if it commutes with every generator of C_3. Moreover, the image of w under the natural projection $K[S_3] \to K[S_3]/K[M_1] = K_0[M_0]$ is in the center of $K_0[M_0]$. That means that every element of the center of $K[S_3]$ is of the form $w = w_1 + \sum_k \beta_k s_k$, where $w_1 \in K[M_1]$, $s_k \in K[M_0]$ and $\beta_k \in K$ for every k. The idea is to investigate the equations of the form $aw = wa$, $bw = wb$ and $cw = wc$.

Recall that $u_k = (cab)^{k+1} + b(cab)^k ca + ab(cab)^k c$ and $v_k = b(cab)^k c + (cab)^k ca + ab(cab)^k$. Then in particular $t_k = u_k - v_k$.

We know that w is of the form $w_1 + \sum_k \beta_k s_k$, where $w_1 \in K[M_1]$, $s_k \in K[M_0]$ and $\beta_k \in K$ for every k. Every w_1 can be written in the following form

$$w_1 = \sum_{i \in A_1, j \in B_1} \Big(\sum_{k_{i,j} \geqslant 1} \alpha_{i,j} i (cab)^{k_{i,j}} j \Big)$$

(1) Firstly we will see what follows from the equality $aw = wa$. It is easy to check that for any $v \in \{(cab)^*, ab(cab)^*, (cab)^* ca, ab(cab)^* ca\}$ we have $av = va$, so we can skip terms of these forms in our equation. Moreover for every $k \geqslant 1$ we know that $as_k - s_k a \in K[M_1]$. Therefore the only elements of the support of as_k that are left are $\{aba(cba)^{k+1}, aba(cba)^{k+1}c, aba(cba)^k cb\}$ and the only elements of the support of $s_k a$ that will not be cancelled are $\{(cba)^{k+1}ca, a(cba)^{k+1}ca, ba(cba)^k ca\}$.

Using Lemma 8.15 we calculate that the remaining part of $\beta_k as_k$ is of the form

$$\begin{cases} \beta_k ab(cab)^{\frac{k}{2}} c \text{ for even } k; \\ \beta_k ab(cab)^{\frac{k+1}{2}} c \text{ for odd } k. \end{cases}$$

Similarly, the remaining part of $\beta_k s_k a$ is of the following form

$$\begin{cases} \beta_k b(cab)^{\frac{k+1}{2}} ca \text{ for odd } k; \\ \beta_k b(cab)^{\frac{k}{2}} ca \text{ for even } k. \end{cases}$$

Hence, equality $aw = wa$ is equivalent to

$$\sum \alpha_{b,1} ab(cab)^{k_{b,1}} + \sum \alpha_{1,c}(cab)^{k_{1,c}}c + \sum \alpha_{b,c} ab(cab)^{k_{b,c}}c + \sum \alpha_{b,ca} ab(cab)^{k_{b,ca}}ca$$
$$+ \sum \alpha_{ab,c} ab(cab)^{k_{ab,c}}c + \sum_{2|k} \beta_k ab(cab)^{\frac{k}{2}}c + \sum_{2\nmid k} \beta_k ab(cab)^{\frac{k+1}{2}}c =$$
$$\sum \alpha_{b,1} b(cab)^{k_{b,1}} + \sum \alpha_{1,c}(cab)^{k_{1,c}}ca + \sum \alpha_{b,c} b(cab)^{k_{b,c}}ca + \sum \alpha_{b,ca} b(cab)^{k_{b,ca}}ca$$
$$+ \sum \alpha_{ab,c} ab(cab)^{k_{ab,c}}ca + \sum_{2|k} \beta_k b(cab)^{\frac{k}{2}}ca + \sum_{2\nmid k} \beta_k b(cab)^{\frac{k+1}{2}}ca.$$

Therefore

(i) $\alpha_{b,1} = 0$;

(ii) $\alpha_{1,c} = 0$;

(iii) $\sum \alpha_{b,ca} ab(cab)^{k_{b,ca}}ca = \sum \alpha_{ab,c} ab(cab)^{k_{ab,c}}ca$;

(iv) $\sum \alpha_{b,c} ab(cab)^{k_{b,c}}c + \sum \alpha_{ab,c} ab(cab)^{k_{ab,c}}c + \sum_{2|k} \beta_k ab(cab)^{\frac{k}{2}}c + \sum_{2\nmid k} \beta_k ab(cab)^{\frac{k+1}{2}}c = 0$;

(v) $\sum \alpha_{b,c} b(cab)^{k_{b,c}}ca + \sum \alpha_{b,ca} b(cab)^{k_{b,ca}}ca + \sum_{2|k} \beta_k b(cab)^{\frac{k}{2}}ca + \sum_{2\nmid k} \beta_k b(cab)^{\frac{k+1}{2}}ca = 0$.

From (iii) it follows that we can assume that $\alpha_{b,ca} = \alpha_{ab,c} := \alpha_i$. Therefore it is easy to see that (iv) and (v) are equivalent.

(2) Now we investigate equality $bw = wb$, assuming that $\alpha_{b,1} = 0$, $\alpha_{1,c} = 0$ and $\alpha_{b,ca} = \alpha_{ab,c} := \alpha_i$.

It is easy to check that for any $v \in \{b(cab)^*c, ab(cab)^*, ab(cab)^*c\}$ we have $bv = vb$, so we can skip terms of these forms in our equation. Moreover for every $k \geqslant 1$ we know that $as_k - s_k a \in K[M_1]$. Therefore the only elements of the support of bs_k that are left are $\{b(cba)^{k+2}, b(cba)^{k+1}c, b(cba)^{k+1}cb\}$ and the only elements of the support of $s_k b$ that will not be cancelled are $\{(cba)^{k+2}b, a(cba)^{k+1}b, ba(cba)^{k+1}b\}$.

Using Lemma 8.15 we calculate that the remaining part of $\beta_k bs_k$ is of the form

$$\begin{cases} \beta_k b(cab)^{\frac{k}{2}}ca \text{ for even } k; \\ \beta_k b(cab)^{\frac{k+1}{2}}ca \text{ for odd } k. \end{cases}$$

Similarly, the remaining part of $\beta_k s_k b$ is of the following form

$$\begin{cases} \beta_k (cab)^{\frac{k+3}{2}} \text{ for odd } k; \\ \beta_k (cab)^{\frac{k+2}{2}} \text{ for even } k. \end{cases}$$

Hence, we get the following equation

$$\begin{aligned}\sum \alpha_{1,1} b(cab)^{k_{1,1}} + \sum \alpha_{1,ca} b(cab)^{k_{1,ca}} ca + \sum \alpha_{b,ca} b(cab)^{k_{b,ca}} ca \\ + \sum \alpha_{ab,ca} ab(cab)^{k_{ab,ca}} ca + \sum_{2|k} \beta_k b(cab)^{\frac{k}{2}} ca + \sum_{2\nmid k} \beta_k b(cab)^{\frac{k+1}{2}} ca = \\ \sum \alpha_{1,1} (cab)^{k_{1,1}} + \sum \alpha_{1,ca} (cab)^{k_{1,ca}+1} + \sum \alpha_{b,ca} b(cab)^{k_{1,ca}+1} \\ + \sum \alpha_{ab,ca} ab(cab)^{k_{ab,ca}+1} + \sum_{2|k} \beta_k (cab)^{\frac{k+2}{2}} + \sum_{2\nmid k} \beta_k (cab)^{\frac{k+3}{2}}.\end{aligned}$$

Therefore

(i) $\sum \alpha_{1,1} b(cab)^{k_{1,1}} = \sum \alpha_{b,ca} b(cab)^{k_{1,ca}+1}$;

(ii) $\sum \alpha_{1,ca} b(cab)^{k_{1,ca}} ca + \sum \alpha_{b,ca} b(cab)^{k_{b,ca}} ca + \sum_{2|k} \beta_k b(cab)^{\frac{k}{2}} ca + \sum_{2\nmid k} \beta_k b(cab)^{\frac{k+1}{2}} ca = 0$;

(iii) $\alpha_{ab,ca} = 0$;

(iv) $\sum \alpha_{1,1} (cab)^{k_{1,1}} + \sum \alpha_{1,ca} (cab)^{k_{1,ca}+1} + \sum_{2|k} \beta_k (cab)^{\frac{k+2}{2}} + \sum_{2\nmid k} \beta_k (cab)^{\frac{k+3}{2}} = 0$.

From (i), (iii) and part (1) we get that

$$\begin{aligned}w_1 = \sum_i \alpha_i ((cab)^{i+1} + b(cab)^i ca + ab(cab)^{i+1} c) + \sum \alpha_{ab,1} ab(cab)^{k_{ab,1}} + \\ + \sum \alpha_{1,ca} (cab)^{k_{1,ca}} ca + \sum \alpha_{b,c} b(cab)^{k_{b,c}} c.\end{aligned}$$

It follows also from (i) that (ii) and (iv) are equivalent.

(3) Now we investigate equality $cw = wc$, assuming parts (1) and (2).

It is easy to check that for any $v \in \{b(cab)^* c, (cab)^* ca, b(cab)^* ca\}$ we have $cv = vc$, so we can skip terms of these forms in our equation. Moreover for every $k \geqslant 1$ we know that $cs_k - s_k c \in K[M_1]$. Therefore the only elements of the support of cs_k that are left are $\{ca(cba)^{k+1}, ca(cba)^{k+1}c, ca(cba)^{k+1}cb\}$ and the only elements of the support of $s_k c$ that will not be cancelled are $\{(cba)^{k+1}cbc, a(cba)^{k+1}cbc, ba(cba)^k cbc\}$.

Using Lemma 8.15 we calculate that the remaining part of $\beta_k cs_k$ is of the form

$$\begin{cases} \beta_k (cab)^{\frac{k+2}{2}} \text{ for even } k; \\ \beta_k (cab)^{\frac{k+3}{2}} \text{ for odd } k. \end{cases}$$

Similarly, the remaining part of $\beta_k s_k c$ is of the following form

$$\begin{cases} \beta_k ab(cab)^{\frac{k+1}{2}} c \text{ for odd } k; \\ \beta_k ab(cab)^{\frac{k}{2}} c \text{ for even } k. \end{cases}$$

Hence, we get the following

$$\sum \alpha_i((cab)^{i+1} + b(cab)^i ca + (cab)^{i+1}c) + \sum \alpha_{ab,1}(cab)^{k_{ab,1}+1} + \\ + \sum_{2|k} \beta_k (cab)^{\frac{k+2}{2}} + \sum_{2\nmid k} \beta_k (cab)^{\frac{k+3}{2}} = \\ = \sum \alpha_i((cab)^{i+1}c + b(cab)^i ca + ab(cab)^i c) + \sum \alpha_{ab,1} ab(cab)^{k_{ab,1}} c + \\ + \sum_{2|k} \beta_k ab(cab)^{\frac{k}{2}} c + \sum_{2\nmid k} \beta_k ab(cab)^{\frac{k+1}{2}} c$$

Therefore we have

(i) $\sum \alpha_i (cab)^{i+1} + \sum \alpha_{ab,1}(cab)^{k_{ab,1}+1} + \sum_{2|k} \beta_k (cab)^{\frac{k+2}{2}} + \sum_{2\nmid k} \beta_k (cab)^{\frac{k+3}{2}} = 0;$

(ii) $\sum \alpha_i ab(cab)^i c + \sum \alpha_{ab,1} ab(cab)^{k_{ab,1}} c + \sum_{2|k} \beta_k ab(cab)^{\frac{k}{2}} c + \sum_{2\nmid k} \beta_k ab(cab)^{\frac{k+1}{2}} c = 0.$

It is clear that equations (i) and (ii) are equivalent.

Equalities $(3(i))$ and $(2(iv))$ give us that $\{\alpha_{1,ca} : \alpha_{1,ca} \neq 0\} = \{\alpha_{ab,1} : \alpha_{ab,1} \neq 0\}$. Moreover, from $(1(iv))$ and $(2(ii))$ it follows that $\{\alpha_{1,ca} : \alpha_{1,ca} \neq 0\} = \{\alpha_{b,c} : \alpha_{b,c} \neq 0\}$. Lastly, we have that if $\{\alpha_{1,ca} : \alpha_{1,ca} \neq 0\} = \{\alpha_{ab,1} : \alpha_{ab,1} \neq 0\} = \{\alpha_{b,c} : \alpha_{b,c} \neq 0\}$, then $(1(iv))$, $(1(v))$, $(2(ii))$, $(2,(iv))$, $(3(i))$ and $(3(ii))$ are equivalent.

More precisely, we get that $w = w_1 + \sum_k \beta_k s_k$, where

$$w_1 = \sum_i \alpha_i((cab)^{i+1} + b(cab)^i ca + ab(cab)^{i+1}c) + \sum_j \gamma_j(ab(cab)^j + (cab)^j ca + b(cab)^j c)$$

and

$$\sum_i \alpha_i (cab)^i + \sum_j \gamma_j (cab)^j + \sum_{2|k} \beta_k (cab)^{\frac{k}{2}} + \sum_{2\nmid k} \beta_k (cab)^{\frac{k+1}{2}} = 0.$$

Comparing coefficients of $(cab)^n$ it follows that $\alpha_n + \beta_{2n-1} + \beta_{2n} + \gamma_n = 0$ for any $n \geqslant 1$.

Therefore $w \in Z(K[S_3])$ if and only if

$$w = \sum_i \alpha_i t_i + \sum_k \beta_{2k-1}(s_{2k-1} - u_k) + \sum_k \beta_{2k}(s_{2k} - u_k)$$

for some $\alpha_i, \beta_j \in K$, where

$$u_k = (cab)^{k+1} + b(cab)^k ca + ab(cab)^k c$$

and

$$v_k = b(cab)^k c + (cab)^k ca + ab(cab)^k.$$

It follows that $Z(K[S_3]) = lin_K\{t_k : k \geqslant 1\} + lin_K\{s_{2k-1} - u_k : k \geqslant 1\} + lin_K\{s_{2k} - u_k : k \geqslant 1\}$.

We found the center Z of the algebra $K[S_3]$ that is contained in the center of the algebra $K[C_3]$. As mentioned at the beginning of the present section, the algebra $K[C_3]$ is a finite module over its center. Moreover, as we shall prove, $K[C_3]$ is a finite module over the characterized part of the center.

Corollary 8.18. *Hecke–Kiselman algebra $K[C_3]$ is a finite module over Z.*

Proof. From Theorem 3.3 it follows that $K[S_3]$ is a (finitely generated) semiprime algebra of Gelfand–Kirillov dimension 1 and therefore it is finitely generated module over its center Z, see Theorem 1.22. We know that $C_3 \setminus S_3$ is finite, so $K[S_3] \subseteq K[C_3]$ is finite module extension. Thus $K[C_3]$ is finite module over Z. $\square$

8.5.1 Limitations of the method

Let us consider the general case of the Hecke–Kiselman monoid C_n associated to the cycle of length n, for any $n \geqslant 3$. In contrast to the relatively simple form of the elements of C_3, the characterization of almost all elements of C_n, for any $n \geqslant 3$, obtained in Theorem 2.1 is quite complicated. Therefore it would be difficult to find the center of $K[C_n]$ by direct computations, as in the second method used in the case of $K[C_3]$.
On the other hand, the structures of matrix type hidden in the monoid C_n, see Theorem 2.44, could be used to characterize the center in the general case. Namely, one can consider the subsemigroup $S_n = \bigcup_{i=0}^{n-2} \tilde{M}_i$ in the monoid C_n. The set $C_n \setminus S_n$ is finite for every $n \geqslant 3$, see Proposition 2.15. Thus it is clear that an analogue of Corollary 8.18 holds in the general case, that is the Hecke–Kiselman algebra $K[C_n]$ is a finite module over the center Z of $K[S_n]$. We know that $\tilde{M}_{n-2}$ is an ideal in S_n and an analogue of Lemma 8.14 can also be proved. More precisely, the sum of linear subspaces $R = K[S_n \setminus \tilde{M}_{n-2}] + \mathcal{M}(K(t_{n-2}), A_{n-2}, B_{n-2}; P_{n-2})$ has a natural structure of algebra, which extends the structure of $K[S_n]$. To find the center of $\mathcal{M}(K[T_{n-2}], A_{n-2}, B_{n-2}; P_{n-2})$ in particular we have to calculate the inverse of the sandwich matrix $P_{n-2} \in M_n(K(t_{n-2}))$ which however seems to be extremely hard in the general case. To find the center of $K[S_n \setminus \tilde{M}_{n-2}]$ it would be natural to consider first $\tilde{M}_{n-3} \subseteq S_n \setminus \tilde{M}_{n-2}$, then $\tilde{M}_{n-4} \subset S_n \setminus (\tilde{M}_{n-2} \cup \tilde{M}_{n-3})$ and so on. Unfortunately several problems occur. Firstly, for $n > 3$ the subset $S_n \setminus \tilde{M}_{n-2}$ is no longer a subsemigroup in S_n and moreover if $w \in M_i$ and $u \in S_n$ then wu, uw are not necessarily in M_i for $i \neq n-2$ (for example it can be easily checked that $(x_n q_{n-3})(x_{n-1} \cdots x_2 x_n q_1) \in M_{n-2}$). Secondly, the exact form of the sandwich matrices $P_i \in M_{\binom{n}{i+1}}(K(t_i))$ is really difficult to calculate and thus also we do not know the inverses of these matrices. Therefore the arguments described in this chapter cannot be generalized directly.

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