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Report on the PhD thesis of Vincent Michielini

Uniformisation and Choice Questions for Regular Languages

Vincent Michielini's thesis work is part of fundamental computer science and more precisely lies at the border between automata theory and mathematical logic. This work is part of a long tradition of tight links between automata and logic. It is focused on the uniformization of relations realized by automata. The thesis contains several interesting results about relations on finite words as well as words whose length is a fixed linear ordering. The first chapter recalls all the necessary basic material. Chapter 2 presents results on relations defined by fragments of the first order theory. Chapter 3 shows some links between uniformization and the algebraic framework of varieties of semigroups. Chapter 4 is devoted to the uniformization of relations whose length is a fixed linear ordering.

Uniformization of relations comes from descriptive set theory and was first considered in the framework of Polish spaces. Uniformizing a relation R contained in some product $X \times Y$ consists in defining a function f from X to Y with the same domain as R and such that its graph is contained in R. Without any further constraints, it boils down to the choice of a unique element $y = f(x)$ in $R(x) = \{y : (x, y) \in R\}$ for each x such that $R(x)$ is non-empty. The question is especially relevant when it is also required that the function f belongs to a given class of complexity, often the same as the class of R. The classes considered in descriptive set theory are topological classes like Borelian and projective classes. It is, for instance, well known that the class $\mathbf{\Pi}^1_1$ of co-analytic sets has the uniformization property. This means that each co-analytic relation can be uniformized by a co-analytic function. The thesis revisits these issues when the relations are realized by synchronous automata, or equivalently defined by $\mathbf{MSO}[<]$. This gives a more combinatorial and logical flavour to the questions.

Chapter 2 of the thesis considers the problem of uniformizing a relation from a class $\mathcal{C}$ by a function in a class $\mathcal{D}$ where classes $\mathcal{C}$ and $\mathcal{D}$ are defined by fragments of the First-Order logic $\mathbf{FO}[<]$. The chapter starts with three negative results showing that $\mathbf{FO}[<]$ does not uniformize $\mathbf{FO}^2[s]$, $\mathbf{FO}[s]$ does not uniformize $\mathbf{FO}^2[\,]$ and $\mathbf{FO}^2[<, s]$ does not uniformize $\mathbf{FO}^2[<] \cap \mathbf{FO}^2[s]$. The three results are obtained through well-chosen counterexamples: the proofs that those relations cannot be uniformized are delicate and particu-

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larly clever. On the positive side, it is shown that **FO**[<] uniformizes **FO**[ ]. The proof is based on a combinatorial characterization of the sets definable in **FO**[ ].

Since the seminal work of Schützenberger and Eilenberg, varieties of languages have been used to classify regular languages. This framework is based on the equivalence between finite automata and finite semigroups. A variety of languages is a class of regular sets closed under some operations which make possible an algebraic characterization of its members by their syntactic semigroup. The main result of Chapter 3 is that the only variety of languages which has the property of uniformization is the variety of all regular languages. This is indeed a very nice result. It could have been expected that robust classes like the one defined by **FO**[<] also had this property but it is not the case. The proof is carried out by showing step by step that more and more languages must be contained in that class. This proof is an illuminating and beautiful piece of work.

Chapter 4 deals with words whose length is a fixed infinite linear ordering. The main result is the equivalence between the uniformization property and the lack of non-trivial automorphism of the fixed ordering. In that case, the uniformization property is also shown to be equivalent to the existence of regular choice functions as well as the ability to logically define each position. This remarkable result is achieved through the expert use of the notion of evaluation tree which yields a very elegant proof. The lack of non-trivial automorphisms is, of course, necessary but it is a striking result that this condition is sufficient.

To summarize, the PhD manuscript contains significant contributions with very nice constructions. His achievements provide evidence that Vincent Michielini has acquired a great mastery of his field of research. I have been impressed by the number of techniques used in the thesis: logical, combinatorial and algebraic. For that reason Vincent Michielini fully deserves to obtain the grade of Doctor. Therefore I give approval for the defense of the PhD.

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