REPORT ON THE THESIS

Julia set in radom holomorphic dynamics

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1. Introduction

The present thesis is about randon iteration of holomorphic maps. It splits in two main chapters (and one introduction and one appendix). The first one is dedicated to the study of random iteration of quadratic maps while the second is devoted to random iteration for complex exponential maps.

The different between random iteration and *classical* iteration theory is that, for a given point $z_0 \in \mathbb{C}$ its orbit, $\{z_{n+1}\}_{n\geq 0}$, is not the one given by the n-th iterate of a given function, say f, i.e., $z_n = f^n(z_0)$, but the composition of different maps

$$z_n = (f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1)(z_0).$$

Moreover each f_j is chosen by means of a product distribution probability. We might say that random iteration is a special case of non-autonomous iteration.

As we already said in this thesis the author consider first the case (Chapter 2) where the f_n's are given by (randomly) choosing one function of the family $f_c(z) = z^2 + c$ with $c \in V \subset \mathbb{C}$ while Chapter 3 is devoted to the same sort of study by considering the complex exponential family $f_\lambda(z) = \lambda \exp(z)$ with $c \in \Omega \subset \mathbb{C}$. Although both cases correspond to random iteration it is important to notice that, even in the non-autonomous case where iterates correspond to the same function at each step, the typical dynamics generated for the polynomial case and for the transcendental entire case is very different. The main (or, at least, one) reason is that for (quadratic) polynomials infinity is always a super-attracting fixed point (i.e., no mater the value of $c \in \mathbb{C}$ the iterates of initial conditions in some neighbourhood of infinity U_c, tend to infinity). In contrast, in the transcendental entire enviroment, a direct application of Picard's Theorem implies that points in arbitrary small neighbourhoods of infinity might be sent by iteration to any (with at most one exception) point of $\mathbb{C}$, and so $z = \infty$ is a *source* of chaotic dynamics. The natural questions that arise when we consider random iteration instead of classical iteration is to determine under which assumptions on the sets where the parameters are chosen one might determine the type of typical dynamics (topology of the Julia and Fatou sets, behaviour of almost-all orbits, etc).

2. Main results

Let us first consider the (quadratic) polynomial case. That is, at each step (iterate) we randomly choose a value of c for some bounded Borel set $V \in \mathbb{C}$. The parameter space is defined as $\Omega := V^{\mathbb{N}}$ and we define by $\mathbb{P}$ the product distribution on Ω generated by μ. As usual σ will be the shift map defined on Ω and an element $\omega \in \Omega$ is given by $\omega = (c_0, c_1, \ldots)$. Finally we define $f_\omega := f_{c_0}$ and

$$f_\omega^n = f_{c_{n-1}} \circ f_{c_{n-2}} \circ \cdots \circ f_{c_0}.$$

As in the classical case we denote by $\mathcal{A}_\omega$ the escaping set or basin of attraction of inifnity, by $\mathcal{J}_\omega$ the set of no-normality (Julia set) and by $\mathcal{F}_\omega$ the complement of $\mathcal{J}_\omega$ in $\mathbb{C}$ (the set of normality). The following result was known before this thesis

Theorem 2.1. *Let $\omega \in \mathbb{D}(0,R)^N$, with $R > 0$. Then the Julia set $\mathcal{J}_\omega$ is disconnected if and only if there exists $k \in \mathbb{N}$ such that*

$$f^n_{\sigma^k(\omega)}(0) \longrightarrow \infty,$$

as $n \to \infty$.

It is then natural to ask whether this condition (or a stronger one) implies that $\mathcal{J}_\omega$ is totally disconnected, since, in the classical iteration, the Julia set is disconnected if and only if it is totally disconnected. Of course the value of R should play a role here. On the one hand it was also known the existence of an example such that

$$f^n_{\sigma^k(\omega)}(0) \longrightarrow \infty$$

(for all $k \in \mathbb{N}$) but the Julia set is not totally disconnected. On the other hand if $R \leq 1/4$ (inside the main cardioid) then for every sequence $\omega \in \Omega$ the Julia set is connected while if $R > 1/4$ there exist sequences (for instance $\omega = (1, 1 \ldots)$) for which the Julia set is totally disconnected.

Under this scenario, one of the main results of Chapter 2 in this thesis is the following. Consider $\mathbb{P} := \bigotimes_{n=0}^{\infty} \lambda_R$ where each λ_R is the normalized Lebesgue measure on $\mathbb{D}(0,R)$. Denote by

$$\mathcal{T} := \{\omega \in \Omega \mid \mathcal{J}_\omega \text{ is totally disconnected}\}.$$

Theorem A. *Let $R > 1/4$. Consider $\Omega = \mathbb{D}(0,R)^{\mathbb{N}}$. Then $\mathbb{P}(\mathcal{T}) = 1$. Or, equivalently, a typical (metrically) Julia set $\mathcal{J}_\omega$ is totally disconnected.*

This result may seem to indicate that to have disconnected Julia set with random iteration one needs to *intersect* the exterior of the Mandelbrot set (as it happens in classical iteration). Next result it is surprising. Denote, as usual, $\mathcal{M}_0 \subset \mathcal{M}$ the main cardioid of the Mandelbrot set.

Theorem B. *Let $\Omega = \mathcal{M}_0^{\mathbb{N}}$. Then for almost every sequence $\omega \in \Omega$ the Julia set $\mathcal{J}_\omega$ is totally disconnected.*

The final part of Chapter 2 is devoted to study the properties of the maximal measure defined on random quadratic Julia sets (Green functions).

Chapter 3 is devoted to random complex exponentials. So now $\omega = (\lambda_0, \lambda_1, \ldots)$ and $f_\lambda(z) = \lambda \exp(z)$. Notice that for all maps of the complex exponential family there is a unique asymptotic value (no singular values) and it is independent of λ: $z = 0$. For classical iteration a new phenomena on the topology of the Julia set (with respect to (quadratic) polynomials) may occur since the Julia set can be the whole plane. This happens for instance for all $\lambda > 1/e$. Dynamically, for all those parameters the orbit of $z = 0$ tends to infinity under (classical) iteration. Finally, observe that for $\lambda = 1/3$ the map with a unique parabolic fixed point, while for $\lambda(0, 1/e)$ the maps have a unique attracting fixed point p_λ.

Previous to this thesis (and coherently with the previous paragraph about classical iteration) the following result was known.

Theorem 2.2. *Let $M > \hat{\lambda} > 1/e$ be positive real numbers. Let $\omega = (\lambda_n)_{n=0}^{\infty}$ be a sequence of real numbers satisfying $M > \lambda_n > \hat{\lambda} > 1/e$, for all $n \geq 0$. Then $\mathcal{J}_\omega = \mathbb{C}$.*

The first result of this chapter (this is not random iteration but non-autonomous iteration) shows that even in the case of ω have parameters arbitrarily close to $\lambda = 1/e$ the Julia set might be the whole complex plane.

Theorem C. *There exists a constant $C > 0$ such that if $\omega = (\lambda_n)_{n=0}^{\infty}$ is a sequence satisfying*

$$\lambda_n = \frac{1}{e} + \frac{C}{n^p}, \quad p < \frac{1}{2},$$

then $\mathcal{J}_\omega = \mathbb{C}$.

Finally the following result is a nice contribution in terms of the dynamics of the asymptotic value. Notice that in classical iteration, if the orbit of $z = 0$ tends to infinity under iteration (for complex exponentials) then $\mathcal{J}_\omega = \mathbb{C}$.

Theorem D. *There exists a sequence $\omega = (\lambda_n)_{n=0}^{\infty}$ satisfying $\lambda_n > 0$ such that*

$$\lim_{n\to\infty} f^n_{\sigma^k(\omega)}(0) = \infty, \quad \textit{for all } k \in \mathbb{N},$$

but $\mathcal{J}_\omega \neq \mathbb{C}$.

3. Final consideration and evaluation

The thesis is well written. The authors takes care in the introduction of explaining the basics of the theory of random iteration versus classical one. In all explanations the results on random iteration are perfectly motivated for the results on classical one and so the questions (and answers) he addressed are well motivated. The proofs included in the thesis are carefully presented.

Chapter 2 is clearly the core of the thesis and the proof of the main results it is far from trivial. There are ingredients coming from probability theory, potential theory, dynamics, complex analysis, etc. So, all together, is a good training for the author. On the other hand the last theorem (Theorem B) in Chapter 3 is also a very interesting result since it runs against the intuition, at least at the first thought.

Finally I emphasise that Chapter 2 and Chapter 3, the core of this thesis are (extended) versions of two publications in prestigious journals like *Ergodic Theory and Dynamical Systems* and *Fundamenta Mathematicae*, respectively. The second one with the author of the thesis as a single author of the paper. This is clearly a signal of the value of the thesis and the mature of the author.

It would be interesting to see if some of the arguments, proofs and results on this thesis can be considered under the presence of two singular values. For instance what can be said for cubic polynomials? Or for functions like $F_\lambda(z) = \lambda z^2 \exp(z)$?

Hence I conclude my report by congratulating the author and the advisor. I think it is a piece of work following the international standards of thesis in mathematics. The thesis is more than sufficient to grant a Ph. D.

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