**Zadanie 1.** We consider approximation of the integral

\[ \mathcal{J}(f) = \int_{-1}^{1} f(x) \, dx \]

by quadrature rule

\[ Q(f) = \sum_{j=1}^{4} a_j f(x_j), \]

where \( x_1 = -1, x_4 = 1, \) and the other parameters are arbitrary. Determine the highest degree of precision \( r \) of quadratures of such type. (By the degree of precision of a quadrature we mean the maximum \( r \) such that \( \mathcal{J}(w) = Q(w) \) for all polynomials \( w \) of degree \( \leq r - 1 \)).

**Zadanie 2.** Find all solutions to the initial value problem

\[ x' - \frac{6x}{t} = \frac{24x^{2/3}}{t}, \quad x(1) = -27. \]

Remark: a change of variable of the type \( x = x^n \) might be useful.

**Zadanie 3.** Let \( f: \mathbb{R} \to \mathbb{R} \) be a continuous function satisfying \( f(t) \geq 0 \) for all \( t \in \mathbb{R} \). Consider the following Cauchy problem

\( (\ast) \quad \frac{dx}{dt} = x \left( 1 - x - f(t) \right), \quad x(0) = x_0. \)

(1) Show that solutions to Cauchy problem \( (\ast) \) exist (locally) and are unique.

(2) Prove that every solution to \( (\ast) \) for \( x_0 \geq 0 \) is well defined for all \( t \in [0, +\infty) \), and

\[ \lim_{t \to +\infty} x(t) \leq 1. \]

(3) Is it true that for an arbitrary function \( f \) satisfying the problem’s assumptions and \( x_0 < 0 \), each solution to \( (\ast) \) is well defined for all \( t \geq 0 \)?

**Zadanie 4.** Let \( f \) be a function, \( f: [a, b] \to \mathbb{R} \). We assume \( f \) is a Darboux function. Moreover, we assume that for every rational number \( c \in f([a, b]) \) the set \( f^{-1}(c) \) is closed. Prove that \( f \) is continuous.

**Zadanie 5.** Let \( f: \mathbb{R}_+ \to \mathbb{R}_+ \) be a continuous function such that for all \( x \in \mathbb{R}_+ \) the condition \( \lim_{n \to +\infty} f(nx) = 0 \) holds. Prove that \( \lim_{t \to +\infty} f(t) = 0 \).

**Zadanie 6.** Let \( P \) be a finite \((n\text{ elements})\) commutative ring with unity. Prove that if the commutative group of \( P \) is cyclic, then the ring is isomorphic to the ring \( \mathbb{Z}_n \).

**Zadanie 7.**

(1) Prove that if \( H \) is a subgroup of a finite group \( G \), then the union of all subgroups conjugate to \( H \) is not equal to \( G \) (that is: \( G \) is not equal to the union of subgroups of the form \( gHg^{-1} \)).
(2) Prove that the group $\text{GL}_n(\mathbb{C})$ of invertible $n \times n$ matrices with complex entries is equal to the union of all subgroups conjugate to the subgroup of upper triangular matrices.

Zadanie 8. Prove that if every chain in a countable partially ordered set has a supremum, then also every directed subset has a supremum. (we say that $X$ is a directed subset, if for all $a, b \in X$ there exists a $c \in X$, such that $a, b \leq c$).

Zadanie 9. Let $\overline{G}$ be the complement of graph $G$, and $\text{diam}(G)$ be its diameter, i.e., the greatest distance between any pair of vertices. We assume that $\text{diam}(G) = \infty$ if $G$ is not connected. Prove that $\min(\text{diam}(G), \text{diam}(\overline{G})) \leq 3$ for every graph $G$.

Zadanie 10. Let $G = (V, E)$ be a directed acyclic graph. Propose an efficient algorithm that computes, for each vertex of the graph $G$, the length of the longest path that avoids this vertex.

An $O(|E| \cdot |V|)$-time solution will be awarded 50% of the points. A full score can be obtained by an $O(|V| + |E| \log |V|)$-time solution.

Zadanie 11. There are two relations:

\begin{itemize}
  \item Person(id, name, surname, age)
  \item Friend(person1, person2)
\end{itemize}

with primary keys:

\begin{itemize}
  \item Person.id
  \item Friend.{person1, person2}
\end{itemize}

and foreign keys:

\begin{itemize}
  \item Friend.person1 -> Person.id
  \item Friend.person2 -> Person.id
\end{itemize}

Write a query in SQL'92 which produces every person who is neither the youngest nor the oldest when compared to the group of its friends. We give more points for solutions without subqueries.

Zadanie 12. A domino brick is a triple $(n, c, n')$ where $n$ and $n'$ are natural numbers and $c$ is an element of a fixed finite set $C$ of colours. A domino game is a finite set of domino bricks. A play of a game $G$ is any finite sequence $(n_1, c_1, n_2, c_2, n_3, \ldots, c_k, n_{k+1})$ where $k \geq 0$ and $(n_i, c_i, n_{i+1})$ is a domino brick in $G$ for $i = 1, 2, \ldots, k$. We allow multiple use of the same brick. The palette of a game $G$ is the set of all possible sequences of colours $(c_1, c_2, \ldots, c_k)$ that appear in some plays of $G$. Is true that for every domino game $G$ the palette of $G$ is a regular language? Explain your answer.

Zadanie 13. Is there a sentence of first-order logic over the signature consisting of one binary relational symbol $\leq$ that distinguishes the structure of ordered real numbers $(\mathbb{R}, \leq)$ from the structure of ordered rational numbers $(\mathbb{Q}, \leq)$? Explain your answer.

Zadanie 14. A set $A \subseteq \mathbb{N}$ is recursively enumerable iff there exists a computable function $f: \mathbb{N} \to \mathbb{N}$ such that $A$ is the set of values of $f$, and recursive
iff there exists an algorithm running in finite time deciding whether $x$ given on input is an element of $A$. Show that every infinite recursively enumerable subset of $\mathbb{N}$ has an infinite subset that is recursive.

**Zadanie 15.** For two given graphs $(V, E_1)$ and $(V, E_2)$, take a random permutation $\pi : V \to V$, and set $E_2' = \{\{\pi(v), \pi(w)\} : \{v, w\} \in E_2\}$. Let $X = |E_1 \cap E_2'|$. Give an algorithm computing the expected value and variance of $X$. For full credit your algorithm should run in linear time.

**Zadanie 16.** For a set of integers $A = \{a_1, a_2, \ldots, a_k\}$ such that $a_1 < a_2 < \cdots < a_k$, we define the maxgap operation as:

$$\text{maxgap}(A) = \max\{a_{i+1} - a_i : i = 1, \ldots, k - 1\}.$$ 

Provide an efficient implementation (in C or Pascal) of the following procedures/functions that work on a set of integers $A$ that changes dynamically:

- **Init**($A$, $B$, $n$) - Initializes the data structure with a set $B \subseteq \{1, \ldots, n\}$ such that $1 \in B$ and $n \in B$. This procedure is called only once, in the beginning.
- **Delete**($A$, $x$) - Updates the set $A$ by removing an element $x \in A$ such that $1 < x < n$.
- **MaxGap**($A$) - Gives as a result $\text{maxgap}(A)$.

You can choose which data structure to use to store the set $A$. Analyse the complexity of your implementation in terms of $n$. You can assume that $n$ fits in a standard integer type.

The grade of your solution will be based primarily on the correctness of the code and the time complexity of operations Delete and MaxGap.