Variational methods in PDEs

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Unconstrained problems
Consider the following second order elliptic problem

\[-\Delta u + V(x)u = f(u), \quad u : \mathbb{R}^N \rightarrow \mathbb{R}, \ u(x) \rightarrow 0 \text{ as } |x| \rightarrow \infty.\]

We say that $u$ is a weak solution, if

$$\int_{\mathbb{R}^N} \nabla u \cdot \nabla v + V(x)uv \, dx = \int_{\mathbb{R}^N} f(u)v \, dx$$

for all $v$.

All weak solutions are critical points of the energy (Euler-Lagrange) functional

$$\mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2 \, dx - \int_{\mathbb{R}^N} F(u) \, dx$$

with $F(u) := \int_0^u f(s) \, ds$. 
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Variational functionals

Hence, we look for critical points of some nonlinear functional

$$\mathcal{J} : X \to \mathbb{R}$$

defined on some function space $X$ (in applications: some Sobolev-type space).

First idea: look for minimizers of $\mathcal{J}$! Minimizers are critical points, so we will find solutions....

But, if e.g. $F(u) = \frac{1}{p} |u|^p$ with some $p > 2$, we have

$$\mathcal{J}(tu) = t^2 \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x) u^2 \, dx - t^p \frac{1}{p} \int_{\mathbb{R}^N} |u|^p \, dx \to -\infty$$
as $t \to \infty$. The functional is not bounded from below!
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Possible approaches

- Look for other type of solutions: Mountain Pass Theorem, Palais-Smale sequences, ...
- Constrained minimization: look for minimizers on appropriate subsets of $X$ on which the functional is bounded from below. Are such minimizers critical points, and therefore - solutions?
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Then one can expect the existence of a Palais-Smale sequence:

\[ \mathcal{J}(u_n) \to c, \quad \mathcal{J}'(u_n) \to 0, \]

where \( c > 0 \) is some number. Is such a sequence convergent...? Usually not.
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We look for critical points on the following constraint

\[ \mathcal{N} := \{ u \in X \setminus \{0\} : \mathcal{J}'(u)(u) = 0 \} . \]

**Nehari, 1960**

\( \mathcal{N} \) contains all nontrivial critical points of \( \mathcal{J} \).

Properties (under reasonable assumptions, if \( f \) is sufficiently regular):

- \( \mathcal{J} \) is bounded from below on \( \mathcal{N} \).
- \( \mathcal{N} \) is a \( C^{1,1} \) manifold.
- \( \mathcal{N} \) is a natural constraint to \( \mathcal{J} \). Namely - if \( (\mathcal{J}|_{\mathcal{N}})'(u) = 0 \), then \( \mathcal{J}'(u) = 0 \).

**Corollary:** it is enough to look for minimizers of \( \mathcal{J} \) on \( \mathcal{N} \).
Nehari manifold approach

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Does it work when $f$ is not "sufficiently" regular?

- $\mathcal{N}$ may not be a differentiable manifold,
- it makes no sense to write $(\mathcal{I}|_{\mathcal{N}})'(u) = 0$.

**Szulkin, Weth, 2009** There is a homeomorphism $m : S \rightarrow \mathcal{N}$, where $S$ is the unit sphere in $X$.

- Although $m$ is only continuous, it preserves the class of the functional: $\mathcal{J} \circ m$ is of $C^1$ class;
- $S$ is a manifold of $C^{1,1}$ class;
- Minimize $\mathcal{J} \circ m : S \rightarrow \mathbb{R}$! One have the critical point of $\mathcal{J} \circ m$.
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Strongly indefinite problems

For strongly indefinite problems the mentioned methods have their counterparts:

- Mountain Pass Theorem ↔ Linking Theorem (Kryszewski, Szulkin, 1998)
- Nehari manifold ↔ Nehari-Pankov manifold (Pankov, 2005)
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Constrained problems
We consider the following nonlinear Schrödinger wave equation

\[ i \frac{\partial \Psi}{\partial t} = -\Delta_x \Psi - f(|\Psi|)\Psi, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^N, \]

where \( \Psi = \Psi(t, x) \) is the state (wave) function. Looking for solutions of the form (so-called \textit{standing waves})

\[ \Psi(t, x) = e^{-i\lambda t} u(x), \]

where the so-called \textit{soliton} \( u \) vanishes at infinity, leads to the equation

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The normalized problem

We are looking for solutions to the following problem

\[\begin{cases} \ -\Delta u + \lambda u = g(u) \quad \text{in } \mathbb{R}^N, \ N \geq 3, \\
\int_{\mathbb{R}^N} |u|^2 \, dx = \rho > 0, \end{cases}\]

where \(\rho\) is prescribed and \((u, \lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}\) has to be determined.

In the time-dependent equation, the mass

\[\int_{\mathbb{R}^N} |\Psi(t, x)|^2 \, dx \quad \text{is independent of } t\]

thus it makes sense to prescribe \(\int_{\mathbb{R}^N} |u|^2 \, dx\) instead of \(\lambda\).
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Variational methods

Let us denote

\[ S = \left\{ u : \int_{\mathbb{R}^N} |u|^2 \, dx = \rho \right\}. \]

Under suitable assumptions, solutions are critical points of the energy functional

\[ \mathcal{J}(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx, \]

where \( G(u) := \int_0^u g(s) \, ds \), on the constraint \( S \) with a Lagrange multiplier \( \lambda \in \mathbb{R} \).
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If $\mathcal{J}$ is bounded from below on $S$, one can just minimize it there. What to do if $\mathcal{J}$ is not bounded from below on $S$?

- Restrict the problem to look for radial solutions (Jeanjean, 1997; Bartsch, de Valeriola, 2013);
- Find another constraint like ”Nehari manifold”? 
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$$-\Delta u + \lambda u = g(u)$$

Nehari manifold:

$$\mathcal{J}'(u)(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \lambda \int_{\mathbb{R}^N} u^2 \, dx - \int_{\mathbb{R}^N} g(u)u \, dx = 0.$$ 

Pohožaev manifold (Pohožaev, 1965):

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, dx = \frac{2N}{N-2} \int_{\mathbb{R}^N} G(u) - \frac{\lambda}{2} u^2 \, dx.$$ 

Idea: take the linear combination of them to rule out \( \lambda! \)
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\[\mathcal{J}'(u)(u) = \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \lambda \int_{\mathbb{R}^N} u^2 \, dx - \int_{\mathbb{R}^N} g(u)u \, dx = 0.\]

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Nehari-Pohožaev manifold

\[ \mathcal{M} = \{ u \neq 0 : M(u) = 0 \}, \]

where

\[ M(u) := \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{N}{2} \int_{\mathbb{R}^N} H(u) \, dx = 0, \]

where \( H(u) := g(u)u - 2G(u) \).

**Idea:** look for solutions in \( \mathcal{M} \cap S \).
\( \mathcal{M} \) is a \( C^1 \)-manifold,

\( \mathcal{I} \) is bounded from below on \( \mathcal{M} \cap S \).

One can use variational techniques to find a kind of Palais-Smale sequence on \( \mathcal{M} \cap S \). Is such a sequence bounded? Convergent? Is the limit still in \( S \)? ...?

It can be done:

- a mini-max approach in \( \mathcal{M} \) based on the \( \sigma \)-homotopy stable family of compact subsets of \( \mathcal{M} \) and some minimax principles (Bartsch, Soave, 2018)

- mountain-pass-type approach connected with the analysis of the ground state energy map (Lu, Jeanjean, 2020)
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The new approach

Assumptions:

- don’t work with radial functions;
- don’t work with Palais-Smale sequences, and avoid the mini-max approach in $\mathcal{M}$ involving strong topological arguments.

The new idea (B., Mederski, 2021):
work in $\mathcal{D} \cap \mathcal{M}$ instead of $\mathcal{S} \cap \mathcal{M}$, where

$$\mathcal{D} := \left\{ u : \int_{\mathbb{R}^N} |u|^2 \, dx \leq \rho \right\}.$$

Obviously $\mathcal{S} \cap \mathcal{M} \subset \mathcal{D} \cap \mathcal{M}$. 
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- \( J \) is bounded from below on \( D \cap M \);

- minimizing sequences \( J(u_n) \to \inf_{D \cap M} J \) are bounded!

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Thank you for your attention!