Three-way Clustering: An Advanced Soft Clustering Approach

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Great Moments in Poland

- Friendship
- Meeting Polish friends in Poland after 20+ years of friendship and rough set research especially after the pandemic breakdown.
- Thank you Profs Dominik Slezak, Hung Son Nguyen, Marcin Szczuka, Piotr Wasilewski, ...

Birthday

- Celebrating 80 birthday of Professor Andrzej Skowron, the great scientist, teacher, and friend!
- Wish Prof. Skowron 福如东海, 寿比南山!

Birth Place of Rough Sets

- Rough sets turned 40 years last year.
- Prof Duoqian Miao and I organized a special issues Rough sets: achievements and future directions in International Journal of Approximate Reasoning
  - **Yiyu Yao**,
    The Dao of three-way decision and three-world thinking, IJAR 162:109032, 2023.
  - Guoqiang Wang, Pengfei Zhang, Dexian Wang, Hongmei Chen, **Tianrui Li**,
    Fast attribute reduction via inconsistent equivalence classes for large-scale data, IJAR 163:109039, 2023.
  - **Ryszard Janicki**,
    On Some Generalization of Rough Sets,
    IJAR 163/164:109032:109046, 2023.
  - More to come.

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Granular Computing, Granules, Clusters

- Granule: "a clump of objects (or points) which are drawn together by indistinguishability, similarity, proximity or functionality". (Zadeh 1997)
- Granules: Classes, Clusters

L.A. Zadeh (1997) Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic. Fuzzy Sets and Systems 90:111-127.
J.T. Yao, A.V. Vasilakos, W. Pedrycz (2013) Granular Computing: Perspectives and Challenges. IEEE Transactions on Cybernetics 43(6):1977-1989.

Machine Learning

- Computer algorithms that imitate human learning techniques.
- Computers learn from data vs. Human learn from experiences.
- Learn patterns from old data to generate a model to be used for new data.
- Supervise learning vs. Unsupervised learning vs. Reinforcement learning.
- Classification vs. Clustering

Classification vs. Clustering

- Classification
  - Group objects into predefined classes so that objects with same characteristics are in same classes.
- Clustering
  - Group objects in clusters so that objects in the same cluster are (more) similar (than objects in other groups/clusters).
    - No predefined class labels.
- Classification: "complete" knowledge vs. Clustering: unknown knowledge.

Clustering

- Unsupervised learning as cluster labels are unknown (vs. Supervised learning)
- Two types of clustering: hard clustering, soft clustering.
  - Hard clustering: one data point belongs and only belongs to one cluster.
  - Soft clustering: one data point can belong to different clusters with a probability likelihood.
- Soft in nature: fuzzy, no clear cut, degree, continuous.

  $\Rightarrow$ Three-way Clustering
- Research problems
  - Similarity, distance
  - Inter-cluster vs. intra-cluster.
  - Software engineering: coupling vs. cohesion, functional dependency, density theory.

Three-way Decision Theory

- Thinking in three, analyzing in three, computing in three.
- Initially for decisions: an extension of binary decision-making model with an added 3rd option.
- Thinking in three is natural and popular.
  - Political: Right, Left, Centre
  - Law: Plea guilty, Plea not guilty, Plea nolo contendere (Latin: I do not wish to contend or no contest)
  - Medical: no action, further test, treat (based on testing, test-treatment thresholds)

Yao Y.Y. (2009) Three-way decision: an interpretation of rules in rough set theory. RSKT 2009. LNCS 5589: 642-649.
Yao Y.Y. (2021) The geometry of three-way decision. Applied Intelligence 51:6298–6325.

Three-way Philosophy

- Sartre (萨特): Not to choose is impossible, as not to choose is choosing not to choose. (McCall, 1987)
- Decisions, classifications, clustering.
  - Three levels, three aspects, three approaches, three regions.
  - Three levels of government: federal, provincial and municipal.
  - Sternberg's triangular theory of love: passions, intimacy, and decision/commitment. (Sternberg, 1997)
- In granular computing
  - Multiple levels, Multiple views/angles, Multiple hierarchies/approaches
  - When multiple = three.
- Three means multiple
  - 三思而行，举一反三，事不过三 。。。
  - Multiple authors
    - John Doe et al. OR John Doe, Richard Roe, Mary Moe, et al.

McCall, S. (1987). Decision. Canadian Journal of Philosophy 17(2):261-287.
Sternberg, R.J. (1997). Construct validation of a triangular love scale. European Journal of Social Psychology 27:313–335.

Three Types of Data Analytics to Improve Decision Making

- Descriptive Analytics: What happened?
  - Diagnostic Analytics: Why did this happen?
- Predictive Analytics: What might happen in the future?
- Prescriptive Analytics: What should we do next?

Three-way Classification

- Multi-classification => Binary classification => Ternary classification or tripartition
- Partition approaches: how to partition.
- Partition criteria: what do we mean a good partition.
  - Accuracy?
  - Coverage?
  - Speed?
  - Rule size?
  - Class/region size?
  - Region shifting?

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Three-way Classification (cont.)

- What if the criteria are contradict each other?
  - Conflict analysis?
  - Using game theory?
- In the case of rough sets, especially probabilistic rough sets, determining α, β pair for positive, boundary, and negative regions.
- Game-theoretic rough sets.
- Shadowed sets, determining elevation and reduction operations
- Game-theoretic shadowed sets.
- Game-theoretic three-way decisions.

Zhang, Y and Yao, JT (2020) Game theoretic approach to shadowed sets: A three-way tradeoff perspective. Information Science 507:540-552.
Azam, N; Zhang, Y and Yao, JT (2017) Evaluation functions and decision conditions of three-way decisions with game-theoretic rough sets. European Journal of Operational Research 261: 704-714.
Khan, MT; Azam, N; Khalid, S; Yao, JT (2017)A three-way approach for learning rules in automatic knowledge-based topic models. International Journal of Approximate Reasoning 82:210-226
Zhang, Y and Yao, JT (2017) Gini objective functions for three-way classifications. International Journal of Approximate Reasoning 81:103-114.
Yao, JT and Azam, N (2015) Web-Based Medical Decision Support Systems for Three-Way Medical Decision Making With Game-Theoretic Rough Sets. IEEE Transactions on Fuzzy Systems 23:3-15
Azam, N and Yao, JT (2014) Game-theoretic rough sets for recommender systems. Knowledge-based Systems 72:96-107.

Three-way Clustering

- 3WD applications: three-way classifications, three-way concept analysis, three-way clustering
- Modelling probability likelihood into three categories (three-way):
  - Inside, outside, partial
- Fuzzy membership function: Core: =1, support: > 0.
  - Within a distance = > yes, farther than a distance = > no
  - Distance => fuzzy
- Key issues: distance/similarity function, threshold of three-way cut off
- yes, no, partial, which distance function, similarity function, to what degree consider yes
- Evaluation-based approaches:
  - Based on evaluation functions to generate thresholds.
- Operation-based approach:
  - Generating three-way clustering without thresholds calculation.

A Three-way Cluster Representation

- The universal $U = \{o_1, o_2, o_3, \ldots, o_n\}$ and
  its K distinct clusters $C = \{c_1, c_2, \ldots, c_K\}$.
- Each cluster c_k is depicted by core ($Core(c_k)$) and support ($Support(c_k)$) sets
- $Core(c_k) \subseteq Support(c_k)$ and $Core(c_k)$, $Support(c_k) \subset U$.
- Now we have three parts or regions,

$$Inside(c_k) = Core(c_k), \tag{1}$$

$$Partial(c_k) = Support(c_k) - Core(c_k), \tag{2}$$

$$Outside(c_k) = U - Support(c_k). \tag{3}$$

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Generating Three Regions Based on Thresholds

- With thresholds $0 \leq \beta < \alpha \leq 1$, the three regions are defined as,

$$Inside(c_k) = \{o_i \in U | e(c_k, o_i) \geq \alpha\}, \tag{4}$$

$$Partial(c_k) = \{o_i \in U | \beta < e(c_k, o_i) < \alpha\}, \tag{5}$$

$$Outside(c_k) = \{o_i \in U | e(c_k, o_i) \leq \beta\}. \tag{6}$$

Statistics of Three-way Clustering Publications

- Web of Science, Three-way Clustering
- "three-way decision*" AND clustering OR "three-way cluster*"
- Number of papers: 111(121), total citations 1,655, h-index 20,
- Average per item 14.91 Citing Articles 896
- A few early non CS ones, multi-labeled clustering.

A Bit History

- The term three-way clustering was used in some domain for classification or clustering for three-valued/multi-label data.
- There are eesearch not under the term three-way clustering
- The concept of assigning an object to a cluster in three degrees has been studied in the way of clustering combined with
  - interval sets,
    - H. Yu, Q.F. Zhou, A cluster ensemble framework based on three-way decisions RSKT 2013, LNCS 8171:302-312, 2013.
  - rough sets,
    - M. Joshi, P. Lingras, Y.Y. Yao, C. B. Virendrakumar, Rough, fuzzy, interval clustering for web usage mining, 10th ISDA, pp.397-402, 2010.
    - P. Lingras, Rough set clustering for web mining, IEEE ICFS, pp.1039-1044, 2002.
  - three-way decision, etc.
    - H. Yu, T. Su, X.H. Zeng, A Three-way decisions clustering algorithm for incomplete data, RSKT 2014, LNCS 8818:765-776, 2014.

  in early publications.

With Three-way Decisions

A few examples of applying three-way decision concept in clustering including

- Using vote-based approach to coordinate hard or soft clusters to form three-way clusters,
  - H. Yu, Q.F. Zhou, A cluster ensemble framework based on three-way decisions RSKT 2013, LNCS 8171:302-312, 2013.
- Using a relation-graph based clustering algorithm to obtain different overlapping region types,
  - H. Yu, C. Zhang, F. Hu, An Incremental Clustering Approach Based on Three-Way Decisions, RSCTC 2014 LNCS 8536:152-159, 2014.
- Using 3WD to re-cluster new data to deal with data change,
  - H. Yu, P. Jiao, G.Y. Wang, Y.Y. Yao, Categorizing Overlapping Regions in Clustering Analysis Using Three-way Decisions. WI 2014 Vol. 2: 350-357, 2014.
- Using attribute significance and miss rate to form three region intervals.
  - H. Yu, T. Su, X.H. Zeng, A Three-way decisions clustering algorithm for incomplete data, RSKT 2014, LNCS 8818:765-776, 2014.

WoS Highly Cited Papers

- Yu, H; Zhang, C; Wang, GY (2016) A tree-based incremental overlapping clustering method using the three-way decision theory. KBS 91:189-203. highly cited*
    - Proposing a tree-based incremental overlapping clustering method using the three-way decision theory.
    - Considering both new data and multiple cluster objects.
- Yu, H; Wang, XC; Wang, GY; Zeng XH (2020) An active three-way clustering method via low-rank matrices for multi-view data. IS 507:823-839. highly cited
    - Using low-rank matrices dealing with multi-view data.
    - Based on consensus of low-rank matrix to form three-way.
    - Proposing an active learning strategy.
- Wang, PX and Yao, YY (2018) CE3: A three-way clustering method based on mathematical morphology. KBS 155:54-65.
    - Operation based 3WC.
    - Using contraction and expansion operators inspired from math erosion and dilation.
    - significance and miss rate to form three region intervals.
    - CE3 k-means and CE3 spectral clustering algorithms.
- Afridi, MK; Azam, N; Yao, JT; Alanazi, E (2018) A three-way clustering approach for handling missing data using GTRS. IJAR 98:11-24.
    - Threshold determination is a key issue for evaluation-based 3WC.
    - Using game-theoretic rough sets to determine thresholds.
    - Experimented with missing data sets.

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WoS Highly Cited Papers (Cont.)

- Zhou, J; Lai, ZH; Miao, DQ; Gao, C; Yue, XD (2020) Multigranulation rough-fuzzy clustering based on shadowed sets. IS 507:553-573.
  - Dealing with the uncertainties associated with rough set and fuzzy set-based clustering approaches
  - Using shadowed sets for boundary determination.
  - Optimizing multigranulation approximation regions.
- Yu, H; Jiao, P; Yao, YY; Wang, GY (2016) Detecting and refining overlapping regions in complex networks with three-way decisions.IS 373:21-41 (The term three-way clustering appeared here)
  - Starting with overlapping regions and refine them.
  - Using four macro types and eight micro types of nodes to form three-way regions.
  - With an initial cluster core, expanding it according to a new fitness function, them merge trivial regions.
- Wang, PX; Shi, H; Yang, XB ; Mi, JS (2019) Three-way k-means: integrating k-means and three-way decision. IJMLC 10:2767-2777.
  - Resolving traditional k-means unambiguously assigns an object to a single cluster.
  - An overlap clustering is used to obtain the supports of the clusters.
  - Perturbation analysis is applied to separate the core regions from the supports.
- Zhang, QH; Xia, DY; Liu, KX; Wang, GY (2020) A general model of decision-theoretic three-way approximations of fuzzy sets based on a heuristic algorithm. IS 507:522-539
  - Modification of a decision-theoretic three-way model by replacing 0.5 fuzzy membership function to a variable value c where $0 < c < 1$.
  - Using the elevation and reduction operations in shadowed sets as the loss function.
  - Using particle swarm optimization or PSO as a heuristic algorithm to optimize the loss function.

WoS Highly Cited Papers (3)

- Yu, H; Chen, Y; Lingras, P; Wang, GY (2019) A three-way cluster ensemble approach for large-scale data. IJAR 115, pp.32-49
  - Proposing an efficient three-way cluster ensemble approach based on Spark which has the ability to deal with both hard clustering and soft clustering.
  - Developing a distributed three-way k-means clustering algorithm.
  - Proposing a consensus clustering algorithm based on cluster units and devising various three-way decision strategies to assign small cluster units and no-unit objects.
- Yu, H; Chen, LY and Yao, JT (2021) A three-way density peak clustering method based on evidence theory. KBS 211:106532
  - Improving density peak clustering algorithm.
  - K-nearest with evidence theory to avoid error propagation.
  - 3W-DPET (Density Peak Evidence Theory) algorithm.

Three-way Clustering Research

- Discretization of degree of the probability likelihood in soft clustering.
- Exploring various types of discretization functions other than core and support.
- Exploring various soft computing approaches.
- Determining threshold pair with evaluation-based or operation-based approaches
- Examining implications of a given threshold pair, game theory?

My Recent Three-way Clustering Papers

- Yu, H; Chen, LY and Yao, JT (2021) A three-way density peak clustering method based on evidence theory. Knowledge-Based Systems 211:106532.
- Jiang, CM; Li, ZC; Yao, JT (2022) A shadowed set-based three-way clustering ensemble approach. International Journal of Machine Learning and Cybernetics. 13(9):2545–2558.
  - Using possibilistic C-means clustering.
  - Generating three regions with shadowed sets.
  - Algorithm: S-M3WCE (Shadowed set-based Multi-granular Three-way Clustering Ensemble).
- Shah, JA; Azam, N; Alanazi, E; Yao, JT (2022) Image Blurring and Sharpening Inspired Three-way Clustering Approach. Applied Intelligence. 52:18131–18155.
  - Converting hard clusters into grey level images.
  - Applying cluster blur and cluster sharp operations.
  - Extracting three-way clusters based on results of blurring and sharpening operations.
  - Algorithm: BS3WC (blurring and sharpening based three-way clustering)

My Recent Three-way Clustering Papers (cont.)

- Ali, B; Azam, N; Shah, A; Yao, JT (2021) A spatial filtering inspired three-way clustering approach with application to outlier detection. International Journal of Approximate Reasoning 130:1-21.
    - Adopting minimum and maximum filters.
    - RE3WC: reduction and elevation based three-way clustering.
    - Using RE3WC to detect outliers.
- Afridi, MK; Azam, N and Yao, JT (2020) Variance based three-way clustering approaches for handling overlapping clustering. International Journal of Approximate Reasoning 118:47-63.
    - Using different variance based criteria to determine thresholds.
    - Proposing 3WC-OR algorithm (overlapping region).
    - GTRS and GA.
- Ali, B; Azam, N; Yao, JT (2022) A three-way clustering approach using image enhancement operations. International Journal of Approximate Reasoning. 149:1-38.
    - Using an evaluation function to depict the relationship between an object and a cluster.
    - Using blur and sharp options to form three regions.
    - Algorithm: BS3 (blurring and sharpening inspired three-way clustering)
- Afridi, MK; Azam, N; Yao, JT; Alanazi, E (2018) A three-way clustering approach for handling missing data using GTRS. International Journal of Approximate Reasoning 98:11-24.

A Three-way Density Peak Clustering Method based on Evidence Theory

- Improving density peak clustering algorithm.
- K-nearest with evidence theory to avoid error propagation.
- 3W-DPET (3-Way Density Peak Evidence Theory)

Yu, H; Chen, LY and Yao, JT (2021) A three-way density peak clustering method based on evidence theory. Knowledge-Based Systems 211:106532.

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Density Peak Clustering

- Density peak
  - Using density level as a measure.
  - A cluster centre has a higher local density than its neighbours.
  - and a relatively large distance from the other centres.
- Local density

$$\rho_i = \sum_j \chi\left(d_{ij} - d_c\right)$$

$$\rho_i = \sum_j \exp\left(-\frac{d_{ij}^2}{d_c^2}\right)$$

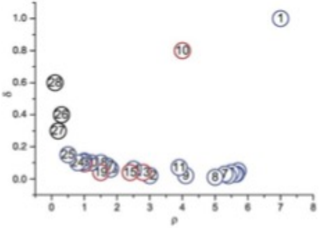
- Distance

$$\delta_i = \begin{cases} \min_{j:\rho_j > \rho_i}\left(d_{ij}\right) & if \exists \rho_j, \rho_j > \rho_i \\ \max d_{ij} & Otherwise \end{cases}$$

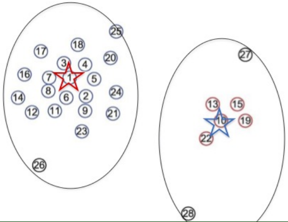
min_j: to find short distance & high density points.

Steps of Density Peak Clustering Algorithm

- Using decision graph (density, distance) to find cluster centres.

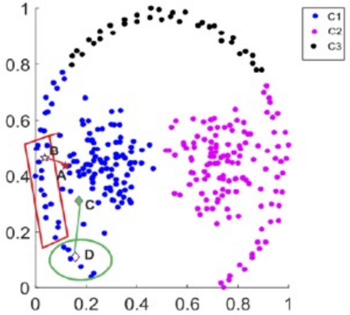


- Assign objects to clusters based on density levels.



Propagation Errors in Density Peak Clustering Algorithm

- If an object is wrongly labeled, it will be propagated in the subsequent assignment.
- no correction mechanism can be implemented to modify it.



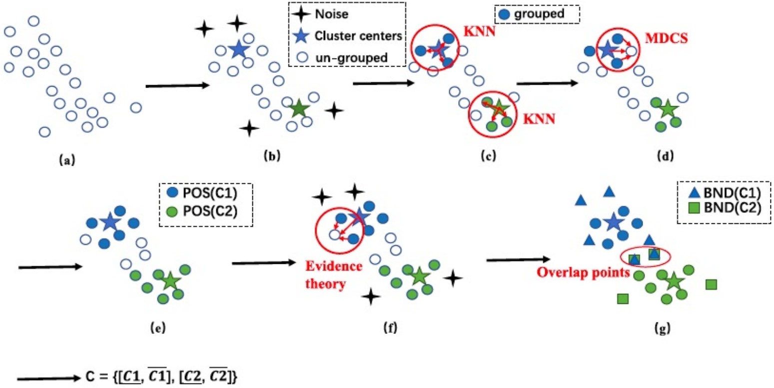
- Local and neighbour information should be considered.
- Not just the label of the nearest object with higher density.

3W-DPET (3-Way Density Peak Evidence Theory)

Following DPC while improving the assignment strategy of non-centre points.

- Determine cluster centres and noises. (b).
- Remove noises and assign points to centres based on kNN. (c). Merge other neighbour points into the positive regions using a midrange distance comparison scheme. (d), (e).
- Allocate remaining points to the boundary regions or negative regions with evidence theory. (f), (g).
  - Apply evidence theory to deal with these objects.
  - For an non-grouped object p, its K-nearest neighbors are actually K pieces of information (or evidence) to assign p into the most possible cluster.

A 3W-DPET Demo Example



Evidence Theory

- Also called Dempster–Shafer theory, reasoning with uncertainty.
- The proposition space Θ includes all possible answers to a problem.
- Each element in 2^Θ represents a proposition.
- Mass function: $m = 2^\Theta \rightarrow [0,1]$ while

$$\sum_{A \in 2^\Theta} m(A) = 1, m(\varnothing) = 0.$$

$m(A)$ is the confidence level to the proposition A. A is a focal element if $m(A) > 0$.

- Combining two pieces of independent evidence with Dempster's combination rule.

$$\begin{aligned} m(A) &= m_1(B_1) \oplus m_2(B_2) \\ &= \begin{cases} \frac{\sum_{B_1 \cap B_2 = A} m_1(B_1) m_2(B_2)}{Y} & \text{if } B_1 \cap B_2 \neq \emptyset \\ 0 & \text{if } B_1 \cap B_2 = \emptyset \end{cases}. \end{aligned}$$

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Algorithm 1: Detect and Assign Core Points using Midrange Distance

Input: cluster centers $\tau = \{\tau_1, \ldots \tau_i \ldots \tau_k\}$, the set $V - \Omega$

Output: positive regions $\{POS(C_1), POS(C_2), \ldots, POS(C_k)\}$

for $i = 1, i \leq k$ **do**

  find K-nearest neighbors of $\tau_i \longrightarrow KNN_{\tau_i}$; $POS(C_i) = [\tau_i; KNN_{\tau_i}]$;

  Initialize $Q = KNN_{\tau_i}$;

  while *Q is not empty* **do**

    Select the head point p;

    find its K-nearest neighbors $KNN_p = \{q_1, \ldots q_j, \ldots, q_K\}$;

    $d_p^{Kmax} = max(d_{pq_1}, d_{pq_2}, \ldots, d_{pq_K})$;

    $d_p^{Kmin} = min(d_{pq_1}, d_{pq_2}, \ldots, d_{pq_K})$;

    $d_p^{mid} = \frac{d_p^{Kmax} + d_p^{Kmin}}{2}$;

    for *each point* q_j *in* KNN_p **do**

      if $d_{pq_j} \leq d_p^{mid}$ **then**

        $POS(C_i) = POS(C_i) \bigcup q_j$;

        Add q_j to the end of Q;

    Remove p from the head of Q;

return to $\{POS(C_1), POS(C_2), \ldots, POS(C_k)\}$;

Algorithm 2: Assign Fringe Points based on Evidence Theory

Input: The set of h fringe points V - $\{POS(C_1), POS(C_2), \ldots, POS(C_k)\}$
Output: $\{BND(C_1), BND(C_2), \ldots, BND(C_k)\}$

for *each fringe point* p_i **do**
- generate K mass functions m^{p_i,q_j} using Eqs. (12)-(13);
- fuse K mass functions: $m^{p_i} = m^{p_i,q_1} \oplus m^{p_i,q_2} \oplus \ldots \oplus m^{p_i,q_K}$ using Eq. (14);
- transform m^{p_i} to the probability distribution μ_{p_i} using Eq. (15);

construct an allocation matrix $AM = [u^j_{p_i}]_{h \times k}$;

while *the maximum* $\mu^j_{p_i}$ *in AM is larger than* $1/k$ **do**
- $x, y \leftarrow arg \max_{i \in [1,h], j \in [1,k]} u^j_{p_i}$;
- $BND(C_y) = BND(C_y) \cap p_x$;
- **for** *each unassigned point* q *satisfying* $p_x \in KNN_q$ **do**
  - construct the new mass function m^{q,p_x}_{new} using Eqs. (12)-(13);
  - $m^q_{new} = m^{q,p_x}_{new} \oplus m^q$ using Eq. (14);
  - transform m^q_{new} into the probability distribution using Eq. (15);
  - update the related values in the matrix AM;

for *each fringe point* p_i **do**
- generate K mass functions m^{p_i,q_j} using Eqs. (12)-(13);
- fuse K mass functions: $m^{p_i} = m^{p_i,q_1} \oplus m^{p_i,q_2} \oplus \ldots \oplus m^{p_i,q_K}$ using Eq. (14);
- transform m^{p_i} to the probability distribution μ_{p_i} using Eq. (15);
- $y \leftarrow arg \max_{j \in [1,k]} \mu^j_{p_i}$;
- $BND(C_y) = BND(C_y) \cap p_i$;
- **if** $\exists t \in [1,k], t \neq y, abs(\mu^y_{p_i} - \mu^t_{p_i}) < \Delta$ **then**
  - $BND(C_t) = BND(C_t) \cap p_i$;

if *the point* p *is still not assigned* **then**
- Find the assigned point q which is nearest to p;
- Assign p to the boundary region of the same cluster as q located;

return to $\{BND(C_1), BND(C_2), \ldots, BND(C_k)\}$;
return to $C = \{[\underline{C_1}, \overline{C_1}], [\underline{C_2}, \overline{C_2}], \ldots, [\underline{C_k}, \overline{C_k}]\}$;

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Experimental Results with Synthetic Data

Flame	Par	ACC	ARI	NMI	Spiral	Par	ACC	ARI	NMI
3W-DPET	7	**1.000**	**1.000**	**1.000**	3W-DPET	8	**1.000**	**1.000**	**1.000**
DPCSA	/	**1.000**	**1.000**	**1.000**	DPCSA	/	**1.000**	**1.000**	**1.000**
DPC-KNN	3	**1.000**	**1.000**	**1.000**	DPC-KNN	5	**1.000**	**1.000**	**1.000**
DPC	0.1	**1.000**	**1.000**	**1.000**	DPC	0.06	**1.000**	**1.000**	**1.000**
K-means	2	0.848	0.483	0.436	K-means	3	0.345	-0.006	0.001
FCM	/	0.850	0.488	0.442	FCM	/	0.340	-0.006	0.000
Path-based	Par	ACC	ARI	NMI	Compound	Par	ACC	ARI	NMI
3W-DPET	8	**0.993**	**0.980**	**0.970**	3W-DPET	5	**0.880**	**0.846**	**0.859**
DPCSA	/	0.823	0.613	0.731	DPCSA	/	0.840	0.828	0.844
DPC-KNN	7	0.760	0.480	0.561	DPC-KNN	2	0.870	0.809	0.852
DPC	0.08	0.753	0.472	0.555	DPC	0.28	0.830	0.833	0.857
K-means	3	0.743	0.462	0.547	K-means	6	0.614	0.544	0.703
FCM	/	0.747	0.465	0.550	FCM	/	0.501	0.406	0.619
R15	Par	ACC	ARI	NMI	Aggregation	Par	ACC	ARI	NMI
3W-DPET	10	**0.998**	**0.996**	**0.997**	3W-DPET	15	**1.000**	**1.000**	**1.000**
DPCSA	/	0.993	0.986	0.989	DPCSA	/	0.973	0.958	0.958
DPC-KNN	5	0.997	0.993	0.994	DPC-KNN	20	0.997	0.996	0.992
DPC	0.04	0.997	0.993	0.994	DPC	0.09	0.998	0.996	0.992
K-means	15	0.892	0.886	0.952	K-means	7	0.788	0.707	0.831
FCM	/	0.997	0.993	0.994	FCM	/	0.778	0.684	0.825
D31	Par	ACC	ARI	NMI	S1	Par	ACC	ARI	NMI
3W-DPET	15	**0.993**	**0.985**	**0.989**	3W-DPET	19	**0.998**	**0.995**	**0.995**
DPCSA	/	0.968	0.935	0.957	DPCSA	/	0.994	0.988	0.988
DPC-KNN	20	0.970	0.940	0.960	DPC-KNN	8	0.995	0.990	0.990
DPC	0.03	0.968	0.936	0.958	DPC	0.03	0.995	0.990	0.990
K-means	31	0.866	0.847	0.934	K-means	15	0.904	0.902	0.957
FCM	/	0.891	0.862	0.936	FCM	/	0.915	0.899	0.955

3W-DPET achieved relatively the best amongst all algorithms.

Experimental Results with Real-world Data

Iris	Par	ACC	ARI	NMI	Libras	Par	ACC	ARI	NMI
3W-DPET	7	**0.987(1)**	**0.960**	**0.941**	3W-DPET	7	**0.508(1)**	**0.376**	0.628
DPCSA	/	0.967(2)	0.904	0.885	DPCSA	/	0.458(3)	0.309	0.626
DPC-KNN	2	0.960(3.5)	0.886	0.862	DPC-KNN	11	0.497(2)	0.361	**0.634**
DPC	0.17	0.960(3.5)	0.886	0.862	DPC	0.53	0.453(4)	0.317	0.601
K-means	3	0.847(6)	0.678	0.716	K-means	15	0.443(5)	0.300	0.583
FCM	/	0.893(5)	0.729	0.743	FCM	/	0.133(6)	0.069	0.276
Wine	**Par**	**ACC**	**ARI**	**NMI**	**Parksins**	**Par**	**ACC**	**ARI**	**NMI**
3W-DPET	7	**0.978(1)**	**0.931**	**0.909**	3W-DPET	6	**0.862(1)**	**0.432**	**0.396**
DPCSA	/	0.910(4)	0.741	0.753	DPCSA	/	0.821(4)	0.269	0.274
DPC-KNN	16	0.904(5)	0.727	0.743	DPC-KNN	1	0.851(2.5)	0.391	0.365
DPC	0.42	0.882(6)	0.672	0.710	DPC	0.06	0.851(2.5)	0.391	0.365
K-means	3	0.942(3)	0.838	0.826	K-means	2	0.628(6)	0.049	0.234
FCM	/	0.949(2)	0.850	0.834	FCM	/	0.672(5)	0.111	0.235
Seeds	**Par**	**ACC**	**ARI**	**NMI**	**Credit**	**Par**	**ACC**	**ARI**	**NMI**
3W-DPET	4	**0.933(1)**	**0.815**	**0.794**	3W-DPET	8	0.419(3)	0.021	0.010
DPCSA	/	0.881(6)	0.687	0.667	DPCSA	/	**0.428(1.5)**	**0.020**	**0.010**
DPC-KNN	3	0.914(2)	0.766	0.734	DPC-KNN	3	**0.428(1.5)**	**0.020**	**0.010**
DPC	0.08	0.905(3)	0.745	0.719	DPC	0.43	0.404(4)	0.032	0.026
K-means	3	0.888(5)	0.700	0.671	K-means	3	0.398(5)	0.021	0.027
FCM	/	0.900(4)	0.727	0.691	FCM	/	0.387(6)	0.024	0.028
Thyroid	**Par**	**ACC**	**ARI**	**NMI**	**Waveform**	**Par**	**ACC**	**ARI**	**NMI**
3W-DPET	2	0.823(3)	0.421	0.446	3W-DPET	3	**0.663(1)**	**0.309**	**0.370**
DPCSA	/	0.749(6)	0.212	0.291	DPCSA	/	0.524(5)	0.135	0.152
DPC-KNN	1	0.781(4)	0.273	0.317	DPC-KNN	2	0.635(3)	0.223	0.218
DPC	0.6	0.763(5)	0.220	0.285	DPC	0.48	0.645(2)	0.255	0.256
K-means	3	0.884(2)	0.619	0.598	K-means	3	0.512(6)	0.252	0.364
FCM	/	**0.907(1)**	**0.693**	**0.666**	FCM	/	0.628(4)	0.353	0.374

3W-DPET achieved the highest on 6 out of 8 datasets.

A Shadowed Set-based Three-way Clustering Ensemble Approach

- Using possibilistic C-means clustering.
- Generating three regions with shadowed sets.
- Algorithm: S-M3WCE (Shadowed set-based Multi-granular Three-way Clustering Ensemble).

Jiang, CM; Li, ZC; Yao, JT (2022) A shadowed set-based three-way clustering ensemble approach. International Journal of Machine Learning and Cybernetics. 13(9):2545–2558.

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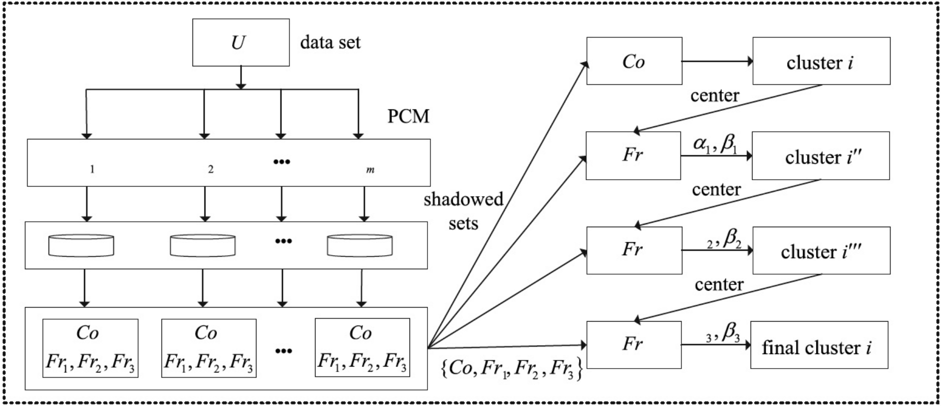
Shadowed set-based Multi-granular Three-way Clustering Ensemble

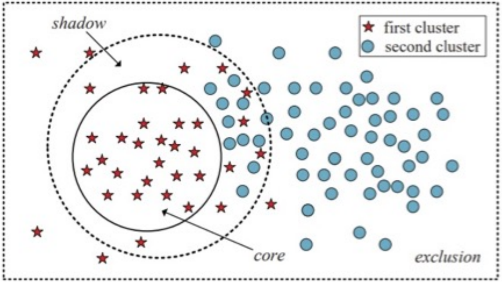
- Generate a set of clustering results with Possibilistic C-means Clustering (PCM).
- Map PCM membership degree into the set {0, [0, 1], 1} (shadowed sets).
- Divided objects into different approximation regions by analyzing the uncertainty relationship between objects and clusters.
- Determine cluster centres by the objects in the core region $C_o(Ci)$.
- Assign objects in fringe regions, $F_{r_1}(C_i)$, $F_{r_2}(C_i)$, and $F_{r_3}(C_i)$ to the $C_o(C_i)$ by shadowed sets.
- Objects that cannot be divided are left in fringe regions.

The Framework of S-M3WCE



Construct Three-way Clustering through Shadowed Sets

- Use the possibilistic C-means clustering or PCM.
- Each cluster contains two subsets/regions: core and shadowed regions.

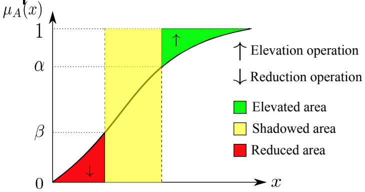


Shadowed Sets

- Shadowed sets are symbolic representation of numeric fuzzy sets
- A shadowed set: a mapping from the universe $U\{0, [0, 1], 1\}$ based on a membership function $\mu_A(x)$ and a pair of thresholds (α, β), i.e., $S : U \to \{0, [0, 1], 1\}$,

$$S_{\mu_A}(x) = \begin{cases} 1, & \mu_A(x) \geq \alpha, \\ 0, & \mu_A(x) \leq \beta, \\ [0,\ 1], & \beta < \mu_A(x) < \alpha. \end{cases}$$

- Elevation and reduction operations



Zhang, Y; Yao, JT (2020) Game theoretic approach to shadowed sets: a three-way tradeoff perspective. Information Sciences. 507:540-552.

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Construct Three-way Clustering through Shadowed Sets

- Given a set of data $U = \{_1,_2, \ldots,_n\}$, objects in U are partitioned into k clusters, $C = \{C_1, C_2, \ldots, C_k\}$ by PCM.
- The $C_j(x)$ denotes the membership degree between x and C_j.
- With a therehod pair (α, β), three operations are defined as

$$EO = \left\{\mu_{C_j}(x) = 1 \mid \mu_{C_j}(x) \geq \alpha\right\},$$
$$RO = \left\{\mu_{C_j}(x) = 0 \mid \mu_{C_j}(x) \leq \beta\right\},$$
$$SO = \left\{\mu_{C_j}(x) = [0,1] \mid \beta < \mu_{C_j}(x) < \alpha\right\}$$

- Three regions are formed.

$$\text{core}(C_j) = \left\{x \in U \mid \mu_{C_j}(x) = 1\right\},$$
$$\text{shadow}(C_j) = \left\{x \in U \mid \mu_{C_j}(x) = [0,1]\right\},$$
$$\text{exclusion}(C_j) = \left\{x \in U \mid \mu_{C_j}(x) = 0\right\}.$$

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Uncertainty Analysis of Attribution Relationship

Construct two kinds of shadowed set-based multi-granulation models.

- The pessimistic multi-granulation lower approximation set is constructed as,

$$\underline{\sum_{i=1}^{m} \pi_i^p}(C_j) = \bigcap_{i=1}^{m} \underline{apr}^{\pi_i^p}(C_j)$$
$$= \left\{x \in U \middle| core_{\pi_1}(C_j) \wedge core_{\pi_2}(C_j) \wedge \cdots \wedge core_{\pi_m}(C_j)\right\}$$
$$= \left\{x \in U \middle| \mu_{C_j}^{\pi_1}(x) = 1 \wedge \mu_{C_j}^{\pi_2}(x) = 1 \wedge \cdots \wedge \mu_{C_j}^{\pi_m}(x) = 1\right\}.$$

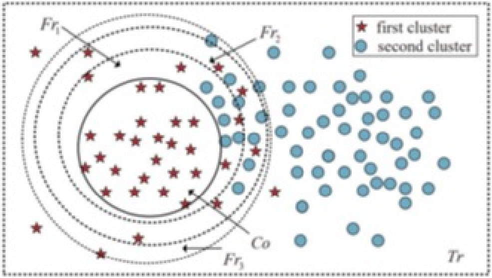
- The optimistic multi-granulation lower approximation set is constructed as,

$$\underline{\sum_{i=1}^{m} \pi_i^o}(C_j) = \bigcup_{i=1}^{m} \underline{apr}^{\pi_i^o}(C_j)$$
$$= \left\{x \in U \middle| core_{\pi_1}(C_j) \vee core_{\pi_2}(C_j) \vee \cdots \vee core_{\pi_m}(C_j)\right\}$$
$$= \left\{x \in U \middle| \mu_{C_j}^{\pi_1}(x) = 1 \vee \mu_{C_j}^{\pi_2}(x) = 1 \vee \cdots \vee \mu_{C_j}^{\pi_m}(x) = 1\right\}.$$

Partial Order of Four Approximation Sets

$$Co(C_j) \succ Fr_1(C_j) \succ Fr_2(C_j) \succ Fr_3(C_j)$$

- Fr_1: the difference between optimistic upper set and pessimistic lower set.
- Fr_2: the difference between optimistic upper set and optimistic lower set.
- Fr_3: the difference between optimistic lower set and pessimistic upper set.



Formation of Four Approximation Sets

$$
\begin{aligned}
Co(C_j) &= \underline{\sum_{i=1}^{m}} \pi_i^p(C_j), \\
Fr_1(C_j) &= \underline{\sum_{i=1}^{m}} \pi_i^o(C_j) - \underline{\sum_{i=1}^{m}} \pi_i^p(C_j) \\
&= \{x \in U \mid x \in \underline{\sum_{i=1}^{m}} \pi_i^o(C_j) \wedge x \notin \underline{\sum_{i=1}^{m}} \pi_i^p(C_j)\}, \\
Fr_2(C_j) &= \overline{\sum_{i=1}^{m}} \pi_i^o(C_j) - \underline{\sum_{i=1}^{m}} \pi_i^o(C_j) \\
&= \{x \in U \mid x \in \overline{\sum_{i=1}^{m}} \pi_i^o(C_j) \wedge x \notin \underline{\sum_{i=1}^{m}} \pi_i^o(C_j)\}, \\
Fr_3(C_j) &= \overline{\sum_{i=1}^{m}} \pi_i^p(C_j) - \overline{\sum_{i=1}^{m}} \pi_i^o(C_j) \\
&= \{x \in U \mid x \in \overline{\sum_{i=1}^{m}} \pi_i^p(C_j) \wedge x \notin \overline{\sum_{i=1}^{m}} \pi_i^o(C_j)\}.
\end{aligned}
$$

Fringe Regions Objects

- Calculate cluster centre with objects in the core region.

$$v_j = \frac{\sum_{k=1}^{N} x_k}{|C_j|}.$$

- Calculate membership degree $\mu_{C_k}(x_i)$ between objects x_i and cluster C_k where d_{ik} denotes distance between object x_i and cluster C_k

$$\mu_{C_k}(x_i) = \frac{1}{\sum_{j=1}^{C}(\frac{d_{ik}}{d_{jk}})^{\frac{2}{m-1}}}.$$

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S-M3WCE Algorithm

Input: A set of objects $U = \{x_1, x_2, ..., x_n\}$ and the number of clusters k.
Output: The results of three-way clustering ensemble.

Generate a set of individual clustering $\Pi = \{\pi_1, \pi_2, ..., \pi_m\}$ using PCM algorithm.
for $\pi \in \Pi$ **do**
  for C_i *in* π **do**
    Calculate optimal threshold α_i and β_i according to Formula 7.
    for x *in* U **do**
      if $\mu_{C_i}(x) \geq \alpha_i$ **then**
        Divide x into $core(C_i)$.
      end
      else if $\mu_{C_i}(x) \leq \beta_i$ **then**
        Divide x into $exclusion(C_i)$.
      end
      else if β_i ¡ $\mu_{C_i}(x)$ ¡ α_i **then**
        Divide x into $shadow(C_i)$.
      end
      // Determine the lower approximation region and upper approximation region of C_j.
      $\underline{apr}(C_j) = core(C_j)$.
      $\overline{apr}(C_j) = core(C_j) \cup shadow(C_j)$.
    end
  end
end
// Uncertainty analysis of attribution relationship.
for C_i *in* C **do**
  $\underline{\sum_{i=1}^{m} \pi_i^p(C_j)} = \bigcap_{i=1}^{m} \underline{apr}^{\pi_i^p}(C_j)$, $\overline{\sum_{i=1}^{m} \pi_i^p(C_j)} = \bigcup_{i=1}^{m} \overline{apr}^{\pi_i^p}(C_j)$,
  $\underline{\sum_{i=1}^{m} \pi_i^o(C_j)} = \bigcup_{i=1}^{m} \underline{apr}^{\pi_i^o}(C_j)$, $\overline{\sum_{i=1}^{m} \pi_i^o(C_j)} = \bigcap_{i=1}^{m} \overline{apr}^{\pi_i^o}(C_j)$.
  // Divide all objects into four different approximation regions $Co(C_i)$, $Fr_1(C_i)$, $Fr_2(C_i)$ and $Fr_3(C_i)$ using Formula 12.
  for $Fr_j(C_i)$ *in* $\{Fr_1(C_i), Fr_2(C_i), Fr_3(C_i)\}$ **do**
    Calculate new centers and membership degree using Formula 14 and Formula 15.
    Calculate the optimal thresholds α_i and β_i using Formula 7.
    // Divide object in $Fr_j(C_i)$ using shadowed sets.
    for x *in* $Fr_j(C_i)$ **do**
      if $\mu_{C_i}(x) \geq \alpha_i$ **then**
        Divide x into $Co(C_i)$.
      end
      $Fr'_j(C_i) = Fr_j(C_i) - Co(C_i)$, $Fr_{j+1}(C_i) = Fr'_j(C_i) \vee Fr_{j+1}(C_i)$.
    end
  end
end

Experimental Results

- Experimented 12 datasets, including 4 synthetic sets sets and 8 real-world sets.
- Three evaluation criteria: clustering accuracy (ACC), adjusted rand index (ARI), and normalized mutual information (NMI).
- Algorithms for comparison: voting-based approach, similarity partition (CSPA), weighted shared nearest neighbor graph (WSNNG), evidence accumulation (EA), the voting-based three-way clustering ensemble algorithm (TWCE), and the rough set-based incremental fuzzy clustering ensemble learning algorithm (IFCERS).
- S-M3WCE achieved highest ACC amongst 3 out 4 synthetic sets and 7 out 8 real-world sets. The overall average was the first.
- The average ranking with statistic analysis of ACC, ARI, and NMI, suggested that S-M3WCE outperformed comparison algorithms.

Image Blurring and Sharpening Inspired Three-way Clustering Approach

- An operation-based approach.
- Using image inspired cluster blur and sharp operators.
- Three-way clusters are constructed without threshold determination.
- Three steps
  - Converting hard clusters into grey level images.
  - Applying cluster blur and cluster sharp operations.
  - Extracting three-way clusters based on results of blurring and sharpening operations.

Shah, JA; Azam, N; Alanazi, E; Yao, JT (2022) Image Blurring and Sharpening Inspired Three-way Clustering Approach, Applied Intelligence. 52:18131-18155

Blurring and Sharpening Operations

- Let $f(x, y)$ represent the grey scale intensity at coordinates (x, y).
- The blurring operation of pixel (x, y) is defined as,

$$\underset{Blur}{f}(x, y) = \frac{1}{mn} \sum_{(s,t) \in S_{xy}} f(s, t) \tag{7}$$

where S_{xy} is a neighborhood of size $m \times n$ centered at pixel (x, y).
- The blurring operation will change the pixel (x, y) to an average pixel value of the neighborhood pixels S_{xy}.
- The sharpening operation of pixel (x, y) is defined as,

$$\underset{Sharp}{f}(x, y) = f(x, y) + k \times [f(x, y) - \underset{Blur}{f}(x, y)] \tag{8}$$

k is a scaling factor for controlling the sharpening effect.
- Sharpening operation: subtracting the blurred image from the original image.
- Sharp images are produced when the edges are added to the original image.

Blurring and Sharpening Operations

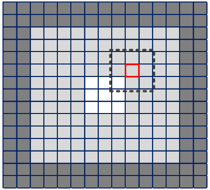

Figure: After blurring

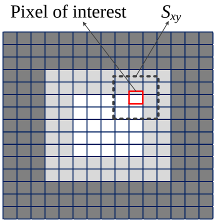

Figure: Input image

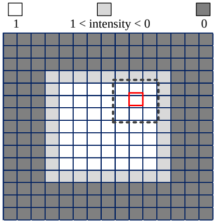

Figure: After sharpening

- The light grey boxes correspond to the blurry or unclear part of the object while the dark grey boxes are assumed to be the background.
- The blurring operation reduces the sharp/clear details and results in reduced size of the clear part of the object.
- The sharpening operation increases the sharp/clear details and results in a reduced blurry region and an increased size of the clear part of the object.

Three-way Clustering using Blurring and Sharpening

- Step 1: Conversion of hard clusters into images.
- Step 2: Applying cluster blur and cluster sharp operations.
- Step 3: Extraction of three-way clustering using cluster blur and cluster sharp operations.

Step 1: Conversion of Hard Clusters into Images

- $U = \{o_1, o_2, o_3, \ldots, o_n\}$: a finite set of data points with N attributes.
- $C = \{c_1, c_2, \ldots, c_K\}$: a partition of the data points.
- There will be s^N samples if we divide each attribute into s number of equidistant parts or samples.
- Each sample is treated as a pixel of an image $(x_1, \ldots, x_N)$.
- The set P: the set of all pixels of the image.
- The intensity of the pixel $(x_1, x_2, x_3, \ldots, x_n)$ to cluster c_k,

$$Image_{c_k}(x_1, \ldots, x_n) = \frac{|O_{(x_1,\ldots,x_n)}|}{\max\limits_{(x_1,\ldots,x_n)\in P}(|O_{(x_1,\ldots,x_n)}|)} \times \frac{|O_{(x_1,\ldots,x_n)} \cap c_k|}{|O_{(x_1,\ldots,x_n)}|} \qquad (9)$$

- The relative density of data points associated with the pixel X The relationship of the same data points with that of cluster c_k.

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Converting Data Points to Grey Scale Image

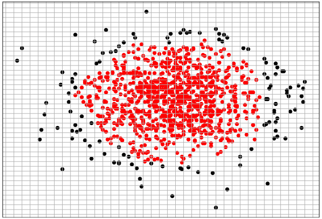

Figure: Data Points on a Grid

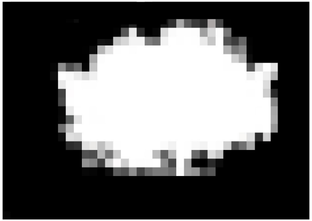

Figure: Gray Scale Image

- The central white pixels: the clear part of a cluster; the outer light grey pixels: the unclear or outer sparse region of the cluster.
- A subspace has a high density and high association with a cluster, then it will be assigned an intensity reflecting a lighter shade of grey (a colour close to white).
- A subspace has a low density or low association with a cluster, then it will be assigned an intensity reflecting a darker shade of grey (a colour close to black).

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Step 2: Applying Cluster Blur and Cluster Sharp Operations

- $Neig_q(x_1, \ldots, x_n)$: the set of q neighbouring pixels of a pixel $(x_1, \ldots, x_n)$.
- The cluster blur operation on an image pixel corresponding to cluster c_k is defined as,

$$CB_{c_k}(x_1, \ldots, x_n) = \frac{1}{q} \sum_{(y_1,\ldots,y_n) \in Neig_q(x_1,\ldots,x_n)} (Image_{c_k}(x_1, \ldots, x_n)) \tag{10}$$

- The blurring operation for a pixel will be1 when all the neighbours of the same pixel have a grey level intensity of 1,

$$CB_{c_k}(x_1, \ldots, x_n) = \begin{cases} 1 & \forall_{(y_1,\ldots,y_n) \in Neig_q(x_1,\ldots,x_n)} \; Image_{c_k}(x_1, \ldots, x_n) = 1 \\ < 1 & otherwise \end{cases} \tag{11}$$

Step 2: Applying Cluster Blur and Cluster Sharp Operations (II)

- The cluster sharp operation on an image pixel to cluster c_k is defined as,

$$CS_{c_k}(x_1, \dots, x_n) = Image_{c_k}(x_1, \dots, x_n) + \Big(Image_{c_k}(x_1, \dots, x_n) - CB_{c_k}(x_1, \dots, x_n)\Big) \tag{12}$$

- The cluster sharp for a pixel produce a value of 1 in any one of the following three cases.
  - Case 1: When the cluster blur for the same pixel produces a value of 1, $CB_{c_k}(x_1, , , , x_n) = 1$.
  - Case 2: The pixel itself has an intensity of 1. The value of $CS_{c_k}(x_1, ..., x_n)$ will always be greater than 1 which is truncated to obtain a value of 1.
  - Case 3: If the pixel intensity is greater than the mean intensity of its neighbouring pixels.

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Step 3: Extraction of 3WC using Cluster Blur and Cluster Sharp Operations

- The core set consists of only those objects whose corresponding pixel after the cluster blur operation results in a value of 1.

$$Core(c_k) = \{o_i \in U | o_i \in O_{(x_1,\ldots,x_n)} \wedge CB_{c_k}(x_1, \ldots, x_n) = 1\} \tag{13}$$

- The support set consists of those objects whose corresponding pixel value after cluster sharp operation results in a value of 1.

$$Support(c_k) = \{o_i \in U | o_i \in O_{(x_1,x_2,\ldots,x_n)} \wedge CS_{c_k}(x_1, x_2, \ldots, x_n) = 1\} \tag{14}$$

Step 3: Extraction of 3WC using Cluster Blur & Cluster Sharp Operations (II)

- The three regions of 3WC based on the sets of $Core(c_k)$ and $Support(c_k)$ are defined as,

$$\begin{aligned} Inside(c_k) &= Core(c_k) \\ &= \{o_i \in U | o_i \in O_{(x_1,\ldots,x_n)} \wedge CB_{c_k}(x_1, \ldots, x_n) = 1\} \quad (15) \\ Outside(c_k) &= U - Support(c_k) \\ &= \{o_i \in U | o_i \in O_{(x_1,\ldots,x_n)} \wedge CS_{c_k}(x_1, \ldots, x_n) \neq 1\} \quad (16) \\ Partial(c_k) &= Support(c_k) - Core(c_k) \\ &= \{o_i \in U | o_i \in O_{(x_1,\ldots,x_n)} \wedge CS_{c_k}(x_1, \ldots, x_n) = 1 \wedge \\ &\quad CB_{c_k}(x_1, \ldots, x_n) \neq 1\} \quad (17) \end{aligned}$$

Blurring and Sharpening based Three-way Clustering Algorithm

Algorithm 1 BS3WC Algorithm

Input A universal set $U = \{o_1, o_2, o_3, \ldots, o_n\}$, sampling parameter $s > 1$ and initial partition $C = \{c_1, c_2, \ldots, c_K\}$

Output A set of family of three-way clusters C'.

for each $c_k \in C$ **do**
    Obtain $Image_{c_k}$ corresponding to cluster c_k
    Apply cluster blur and cluster sharp operations on $Image_{c_k}$
    $Core(c_k) = \{o_i \in U | o_i \in O_{(x_1,\ldots,x_n)} \wedge Blur_{c_k}(x_1, \ldots, x_n) = 1\}$
    $Support(c_k) = \{o_i \in U | o_i \in O_{(x_1,x_2,\ldots,x_n)} \wedge Sharp_{c_k}(x_1, x_2, \ldots, x_n) = 1\}$
    $Inside(c_k) = \{o_i \in U | o_i \in O_{(x_1,\ldots,x_n)} \wedge Blur_{c_k}(x_1, \ldots, x_n) = 1\}$
    $Outside(c_k) = \{o_i \in U | o_i \in O_{(x_1,\ldots,x_n)} \wedge Sharp_{c_k}(x_1, \ldots, x_n) \neq 1\}$
    $Partial(c_k) = \{o_i \in U | o_i \in O_{(x_1,\ldots,x_n)} \wedge Sharp_{c_k}(x_1, \ldots, x_n) = 1 \wedge Blur_{c_k}(x_1, \ldots, x_n) \neq 1\}$
end for
$C' = \Big\{\Big(\text{Inside}(c_1), \text{Partial}(c_1), Outside(c_1)\Big), \ldots, \Big(\text{Inside}(c_K), \text{Partial}(c_K), Outside(c_K)\Big)\Big\}$

Concluding Remarks

- Granular computing => Granules => Clusters = Three-way clustering.
- Three-way clustering:
  - Evaluation-based
  - Operation-based
- Rough sets, fuzzy sets, shadowed sets, interval sets ...
- Image inspired, math inspired, density peak, evidence theory, k-means ....
- Game theoretic approaches.
- Three-way clustering is a hot research topic.

Three-way Clustering: An Advanced Soft Clustering Approach

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