Tensor Random Fields

Martin Ostoja-Starzewski

Department of Mechanical Science and Engineering,
Institute for Condensed Matter Theory, and Beckman Institute
University of Illinois at Urbana-Champaign, USA

Deterministic models of continuum mechanics tackled by boundary value problems may fail for various reasons, especially in multiscale problems. Viewed from the standpoint of random microstructures, probabilistic models such as stochastic partial differential equations (SPDE) and stochastic finite elements (SFE) should naturally involve tensor-valued random fields (TRF) with generally anisotropic realizations and non-trivial correlation functions or variograms. We discuss current work and outline open problems.

Overwhelmingly, the commonly employed SPDE and SFE models either employ one scalar-valued RF to represent a tensorial material property (conductivity, elasticity…) or simply postulate a TRF to have white-noise correlations. The setting is wide-sense homogeneous and isotropic, second-order, mean-square continuous fields. We give explicit representations of the most general correlation functions of TRFs of 1st, 2nd, 3rd, and 4th ranks in 2d or 3d [1,2]. Employing polyadics, such TRFs can readily be constructed from rapidly simulated scalar random fields, to provide inputs to stochastic boundary value problems.

Balance laws of continuum theories (such as incompressibility or equilibrium) offer constraints on the correlation functions of dependent fields (e.g., displacement, velocity, strain, stress…) of various ranks in classical continua and, similarly, in conductivity, electricity, or magnetism. Analogously, one can establish the consequences for TRFs of rotation, curvature-torsion… in stochastic micropolar theories. When there is interest in TRF of constitutive properties (e.g., conductivity, stiffness), experiments can be used to determine/calibrate the correlation functions.

Besides "conventional" correlation structures, this strategy can be used to generate TRFs with fractal and Hurst characteristics, i.e., with multi-scale randomness and free of the restriction to self-similarity. Out of a large menu of correlation functions in probability theory, two models can accomplish that: Cauchy and Dagum. In each case, the correlation function is controlled by two independent parameters, one specifying the fractal dimension and another the Hurst exponent. The current research extends our earlier work on scalar-valued RFs (including random processes) in vibration problems, rods and beams with random properties under random loadings, elastodynamics, wavefronts, fracture, buckling, homogenization of random media, statistical turbulence, and contact mechanics.

1. A. Malyarenko and M. Ostoja-Starzewski, *Tensor-Valued Random Fields for Continuum Physics*, Cambridge University Press, 2019.
2. A. Malyarenko, M. Ostoja-Starzewski, and A. Amiri-Hezaveh, *Random Fields of Piezoelectricity and Piezomagnetism*, Springer, 2020.