The Condensed Homotopy Type of a Scheme

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April 15, 2026

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Section 1

π_1 and Π_∞ - topological spaces and étale site

The topological fundamental group

Let X be a "nice", connected topological space, $x \in X$.
("nice"= locally path-connected and semi-locally simply connected)

Definitions of π_1

1. **Via loops:** $\pi_1^{\mathrm{top}}(X, x) := \{\gamma : [0,1] \to X \mid \gamma(0) = \gamma(1) = x\}/\text{homotopy}.$
2. **Alternatively (Universal Cover):** Let $\tilde{X} \to X$ be the universal covering. Then $\pi_1^{\mathrm{top}}(X, x) \cong \mathrm{Aut}(\tilde{X}/X)^{\mathrm{op}}$.
3. **Alternatively (Galois Correspondence):** Consider $F_x : Cov(X) \to \mathbf{Set}, \quad Y \mapsto Y_x$; Then $\pi_1^{\mathrm{top}}(X, x) \cong \mathrm{Aut}(F_x)$.

Relation to Local Systems

There is an equivalence of categories:

$$\mathbf{Loc}_X(\mathbf{Set}) \simeq \pi_1^{\mathrm{top}}(X, x) - \mathbf{Set}$$

$$\mathbf{Loc}_X(\mathbb{C}) \simeq \mathrm{Rep}_{\mathbb{C}}^{\mathrm{cts}}(\pi_1^{\mathrm{top}}(X, \bar{x}))$$

(Locally constant sheaves $\leftrightarrow$ Representations of fundamental group).

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For "nice" spaces, we want to capture the full homotopy type.

Singular Complex

Let $X \in \mathbf{Top}$. The *singular simplicial set* is:

$$\mathrm{Sing}(X)_\bullet := \mathrm{Hom}_{\mathbf{Top}}(|\Delta^\bullet|, X) \in \mathrm{sSet}$$

where $|\Delta^n| = \{(x_0, \ldots, x_n) \in [0,1]^{n+1} \mid x_0 + \cdots + x_n = 1\}$ is the standard n-simplex.

Geometric Realization

There is a canonical map (counit of adjunction):

$$|\mathrm{Sing}(X)| \longrightarrow X$$

where $|\cdot|$ denotes geometric realization (gluing simplices).

Theorem (Milnor)

If X is a CW-complex (or locally contractible), then $|\mathrm{Sing}(X)| \to X$ is a weak homotopy equivalence.

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∞-categories: quick motivation

Let $[n] = \{0 < 1 < 2 < \cdots < n-1 < n\}$ (lin. ord. set as a category).

The Nerve of a 1-Category

Let $\mathcal{C}$ be an ordinary category. The *nerve* $N(\mathcal{C})$ is the simplicial set where:

$$N(\mathcal{C})_n = \mathrm{Hom}_{\mathrm{Cat}}([n], \mathcal{C})$$

(n-simplices are strings of n composable morphisms).

Both $N(\mathcal{C}), \mathrm{Sing}(X) \in \mathrm{sSet}$ satisfy **weak Kan condition** (inner horn fill.)

Definition

$(\infty, 1)$-**category =** a simplicial set satisfying the weak Kan condition. (Intuitively: objects = vertices; morphisms = edges; composition: defined up to contractible choice of fillers.)
∞-**groupoid = "anima" =** Kan complex = sset satisfying full Kan condition (all horns can be filled)

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Shape of a topos

For loc. contractible X, $\mathrm{Sing}(X) \in \mathbf{Ani}$ is its **"homotopy type"** / **"shape"**.
In general, Borsuk, Artin-Mazur-Friedlander, Töen-Vezzosi, Lurie,.. defines

Shape of an ∞-topos

$$\infty - \mathrm{Topoi} \ni \mathbf{X} \mapsto \Pi_\infty(X) \in \mathrm{Pro}(\mathbf{Ani})$$

(it is defined as $f_\# \mathbf{1}_\mathbf{X}$, where $f_\#$ is a left pro-adjoint to $f^*\colon \mathbf{Ani} \to \mathbf{X}$).

(To recover previous notn take $\mathbf{X} := \mathrm{Sh}(X, \mathbf{Ani})$ = sheaves of anima on X)
Fixing a point "$\mathbf{x} \in \mathbf{X}$", $\Pi_\infty(\mathbf{X})$ allows to define

$$\pi_1(\mathbf{X}, \mathbf{x}) \in \mathrm{Pro}(\mathbf{Grp}), \quad \pi_n(\mathbf{X}, \mathbf{x}) \in \mathrm{Pro}(\mathbf{Ab})$$

Monodromy correspondence

If $\mathbf{X}$ is locally contractible, then

$$\mathrm{Fun}(\Pi_\infty(\mathbf{X}), \mathbf{Ani}) \simeq \mathbf{Loc}_\mathbf{X}(\mathbf{Ani})$$

For any $\mathbf{X}$, denoting $\mathbf{Ani}_\pi$ = "π-finite anima" ($\supset \mathbf{FSet}$)

$$\mathrm{Fun}(\widehat{\Pi}_\infty(\mathbf{X}), \mathbf{Ani}_\pi) \simeq \mathbf{lcc}_\mathbf{X}(\mathbf{Ani})$$

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Side note: Exit Paths

π_1 and Π_∞ allow to recover locally constant sheves.
What about constructible sheaves?
Let X "nice" top. sp. and P its finite stratification by loc. closed subsets.

The Exit Path ∞-Category [Lurie, Treumann]

$\mathrm{Exit}^P(X)$ is an ∞-category (quasicategory) where:

- **Objects:** points of X;
- **Morphisms:** "exit paths", i.e. paths $\gamma : [0,1] \to X$ that can only exit a stratum to a larger one, but not return;
- **2-Morphisms:** homotopies respecting the stratification. And so on...

Constructible Sheaves and Homotopy Type

There is an equivalence

$$\mathrm{Sh}^P_{\mathrm{constr}}(X) \simeq \mathrm{Funct}(\mathrm{Exit}^P(X), \mathbf{Ani})$$

The classifying space satisfies:
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Recollection: The étale fundamental group of a scheme

Let X be a connected scheme.

Finite Étale Covers

Étale morphism := flat+unramified $\Leftrightarrow$ smooth of relative dim. 0.
Let Ét_X = category of étale maps $Y \to X$. **This is a site**.
Let FÉt_X = category of finite (=proper+finite fibers) étale maps $Y \to X$.

Étale fundamental group

Fix a geometric point $\bar{x}\colon \operatorname{Spec}(\Omega) \to X$, where $\Omega = \bar{\Omega}$ a field.
Let $F_{\bar{x}}\colon \text{FÉt} \to \textbf{FSet}, \quad Y \mapsto Y(\bar{x})$ be the fiber functor.
Define $\pi_1^{\text{ét}}(X, \bar{x}) = \operatorname{Aut}(F_{\bar{x}})$.
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$\pi_1^{\text{ét}}$ – examples

Example: recovering the Galois group of a field

Let $X = \text{Spec}(k)$ for a field k. Fix $\bar{k}$ separable/algebraic closure of k. This gives a geometric point $\bar{x}$ on X. Then

$$\pi(X, \bar{x}) \simeq \text{Gal}_k$$

This shows that the étale topology is not "locally contractible". Another way to see this: take $X = \text{Spec}(k)$. To "contract it", one needs to pass to k^{sep}. The map $\text{Spec}(k^{\text{sep}}) \to \text{Spec}(k)$ is not étale, it is **pro**-étale. The fiber isn't "discrete", it is pro-discrete/pro-finite:

$$\text{Spec}(k^{\text{sep}}) \times_{\text{Spec}(k)} \text{Spec}(k^{\text{sep}}) \simeq \text{" Gal}_k\text{"} \times \text{Spec}(k^{\text{sep}})$$

Comparison over complex numbers (Riemann Existence Thm)

Let X be a variety (= scheme of finite type) over $\mathbb{C}$.
Let $X^{\text{an}} = X(\mathbb{C})$ with the analytic topology.
There is an isomorphism on profinite completions:

$$\widehat{\pi_1^{\text{top}}(X^{\text{an}})} \xrightarrow{\sim} \pi_1^{\text{ét}}(X)$$

The Artin–Mazur Homotopy Type

Motivation

$\pi_1^{\text{ét}}$ only captures 1-truncated information. How to define $\pi_n^{\text{ét}}$?

Construction (Artin–Mazur–Friedlander)

Consider the category **HR(X)** of hypercovers $U_\bullet \to X$ in the étale site.
The **étale homotopy type** is an object in Pro(**Ani**):

$$\Pi_\infty^{\text{ét}}(X) \approx \text{"}\varprojlim_{U_\bullet \to X \in \mathbf{HR}(X)}\text{"} \operatorname{Sing}(|\pi_0(U_\bullet)|) \in \operatorname{Pro}(\mathbf{Ani})$$

Alternatively, use Lurie's shape: $\Pi_\infty^{\text{ét}}(X) = \Pi_\infty(X^{\text{ét}})$ (up to protrunc.).

Properties

- Defines higher invariants: $\pi_n^{\text{ét}}(X) := \pi_n(\Pi_\infty^{\text{ét}}(X))$ (pro-groups).
- **Comparison:** If X is geometrically unibranch (e.g. normal),

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where $\widehat{(-)}$ denotes profinite completion.

Consequence: Algebraic geometry recovers the profinite homotopy type of the underlying complex manifold.

Remark: Exit categories

There is also a profinite “exit path” category (or, a “stratified shape”) $\mathrm{Gal}(X)$ of a qcqs scheme X, defined by Barwick-Glasman-Haine in “Exodromy”, that detects constructible sheaves on $X_{\text{ét}}$.
For X f.t./$\mathbb{C}$, it is related to profinite completions of $\mathrm{Exit}^P(X(\mathbb{C}))$.

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Section 2

The Pro-étale Topology

The proétale site (Bhatt–Scholze)

We now move from étale to proétale. Motivation: "better" for non-finite coefficients. "Better" for non-normal schemes.

Definition: The pro-étale site $X_{\text{proét}}$

Definition: A map $U \to X$ is **weakly étale** if it is flat and $\Delta_{U/X} : U \to U \times_X U$ is flat.

The site $X_{\text{proét}}$ is defined by:

- **Objects:** Weakly étale maps $U \to X$; **Covers:** fpqc covers.

Examples:

- Étale maps, **inv limits of (affine) étale maps**: $\text{Spec}(k^s) \to \text{Spec}(k)$;
- $X \smallsetminus \{x\} \sqcup \text{Spec}(\mathcal{O}^{\text{sh}}_{X,\bar{x}}) \to X$ (for $x \in X$ closed pt)
- If $k = \bar{k}$, then $\text{Spec}(k)_{\text{proét}} \simeq \text{Pro}(\textbf{FSet})$ ("condensed sets").

Key Feature: w-contractible objects

The site $X_{\text{proét}}$ has "enough" local objects that are "weakly contractible".

The proétale site (Bhatt–Scholze)

We now move from étale to proétale. Motivation: "better" for non-finite coefficients. "Better" for non-normal schemes.

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Let's see the usefulness of the proétale topology already for $X = \mathrm{Spec}(\bar{k})$.

TopAb is not abelian

$$\mathbb{R}_{\mathrm{disc}} \to \mathbb{R}_{\mathrm{eucl}}$$

is a continuous bijection but not an isomorphism.
In **TopAb**, the kernel and cokernel are trivial, but the map isn't an iso.

Definition (Clausen–Scholze, Barwick-Haine), modulo set-theory...

Let $\mathcal{C}$ be a category (e.g., $\mathbf{Set}, \mathbf{Ab}, \mathbf{Grp}, \mathbf{Ring}, \mathbf{Ani}$). A **condensed object** in $\mathcal{C}$ is a sheaf

$$F : \mathrm{Pro}(\mathbf{FSet})^{\mathrm{op}} \longrightarrow \mathcal{C}$$

(or, equivalently, on the site of *Extremally Disconnected* sets $\mathbf{Extr} \subset \mathrm{Pro}(\mathbf{FSet})$ – see the next slide), i.e. $\mathrm{Cond}(\mathcal{C}) = \mathrm{Sh}(\mathbf{Extr}, \mathcal{C})$.

Topological spaces give condensed sets

For a (T1) top. space M, we have $\underline{M} = \mathrm{Hom}_{\mathbf{Top}}(-, M) \in \mathrm{Cond}(\mathbf{Set})$.

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Why Condensed?

Simplification via Extremally Disconnected Sets

A compact Hausdorff space S is **extremally disconnected** if the closure of every open set is open.

- (Theorem of Gleason) S is projective in **CompHaus**.
- Any $K \in$ **CompHaus** admits a surjection $S \twoheadrightarrow K$ with $S \in$ **Extr**.

Lemma: A functor $F : \mathbf{Extr}^{op} \to \mathbf{Set}$ is a condensed set iff $F(\emptyset) = *$ and $F(S_1 \sqcup S_2) \cong F(S_1) \times F(S_2)$. (No complicated gluing!)

Nice Categorical Properties

Cond(**Ab**) **is an abelian category.** (unlike **TopAb**!)
As an extra feature, it has compact projective generators.

Example

$\underline{\mathbb{R}_{\mathrm{disc}}} \to \underline{\mathbb{R}_{\mathrm{eucl}}}$ has **non-trivial** cokernel.

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The pro-étale fundamental group

Back to schemes. Let X be connected, locally topologically noetherian.

Geometric Coverings

A map $f : Y \to X$ is a **geometric covering** if: 1. f is étale. 2. f satisfies the valuative criterion of properness.

Let $\mathbf{GeoCov}_X$ be the category of such coverings. $F_{\bar{x}} : \mathbf{GeoCov} \to \mathbf{Set}$.

Theorem (Bhatt–Scholze)

There is a topological group $\pi_1^{\text{proét}}(X, \bar{x})$ ($= \mathrm{Aut}(F_{\bar{x}})$), the **pro-étale fundamental group**, such that:

$$\mathbf{Loc}_{X_{\text{proét}}}(\mathbf{Set}) \simeq \mathbf{GeoCov}_X \simeq \pi_1^{\text{proét}}(X, \bar{x}) - \mathbf{Set}$$

(Equivalence with discrete sets with continuous action).

Comparison with Representations

$$\mathbf{Loc}_{X_{\text{proét}}}(\mathbf{Q}_\ell) \simeq \mathrm{Rep}^{\text{cts}}_{\mathbf{Q}_\ell}(\pi_1^{\text{proét}}(X, \bar{x}))$$

Note: For non-normal schemes, $\pi_1^{\text{ét}}$ is too small to classify these.

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Structure of $\pi_1^{\text{proét}}$: Noohi Groups

What kind of group is it?

$\pi_1^{\text{proét}}(X, \bar{x})$ is a **Noohi group**.

A topological group G is Noohi if the natural map

$$G \longrightarrow \mathrm{Aut}\left(F : (G\text{-}\mathbf{Set}) \to \mathbf{Set}\right)$$

is an isomorphism. ("G is determined by its discrete representations").

Examples: profinite groups, discrete groups, locally profinite groups.

Characterization: Hausd., nbhd basis at 1_G by open subgrps, Raĭkov cplt.

Comparison

- If X is normal: $\pi_1^{\text{proét}}(X) \cong \pi_1^{\text{ét}}(X)$ (profinite)
- If X = nodal curve ($y^2 = x^3 + x^2$) over $\mathbf{Q}$, $\pi_1^{\text{proét}}(X, \bar{x}) = \mathbf{Z} \times \mathrm{Gal}_{\mathbf{Q}}$.
- If X = two smooth curves C_1, C_2 / $\mathrm{Spec}(\bar{k})$ glued at a closed point, then $\pi_1^{\text{proét}}(X) = \pi_1^{\text{ét}}(C_1) *^{\text{Noohi}} \pi_1^{\text{ét}}(C_2)$.

In general, $\pi_1^{\text{proét}}(X)$ is **not** pro-discrete. **Can't get it from usual shape!**

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Section 3

Condensed Homotopy Type

Definition and the basics

Let X be a qcqs scheme.

Definition/formula: condensed homotopy type

$$\Pi_\infty^{\mathrm{cond}}(X) \simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}} \pi_0(X_\bullet) \in \mathrm{Cond}(\mathbf{Ani})$$

for any $X_\bullet \to X$ proétale hypercover by w-contractible schemes. Independent of $X_\bullet$ chosen ("unique proétale cosheaf...") . See also Hemo-Richarz-Scholbach.

Fixing a geometric point $\bar{x}$ on X gives a point $\bar{x}$ of $\Pi_\infty^{\mathrm{cond}}(X)$.

Definition

$$\pi_0^{\mathrm{cond}}(X) \in \mathrm{Cond}(\mathbf{Set}), \quad \mathbf{Extr} \ni S \mapsto \pi_0(\Pi_\infty^{\mathrm{cond}}(X)(S))$$

$$\pi_1^{\mathrm{cond}}(X, \bar{x}) \in \mathrm{Cond}(\mathbf{Grp}), \quad \mathbf{Extr} \ni S \mapsto \pi_1(\Pi_\infty^{\mathrm{cond}}(X)(S), \bar{x})$$

Theorem: comparison with Gal(X)

$$\Pi_\infty^{\mathrm{cond}}(X) \simeq \mathrm{B}^{\mathrm{cond}}\,\mathrm{Gal}(X)$$

where $\mathrm{Gal}(X) =$ the "Galois category" / 'étale exit-path category".

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Comparisons

Prodiscrete/profinite completions

There are natural equivalences

$$\mathrm{L}^{\mathsf{pro}}(\Pi_\infty^{\mathrm{cond}}(X)) \simeq \Pi_{<\infty}^{\text{ét}}(X) \qquad \text{and} \qquad (\Pi_\infty^{\mathrm{cond}}(X))_\pi^\wedge \simeq \widehat{\Pi}_\infty^{\text{ét}}(X) \ .$$

Theorem

Let X be a qcqs scheme. Then

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($\mathbf{wLoc}_X = \mathbf{Loc}_X$ for (topologically) noeherian schemes.

Theorem

Let X connected topologically noetherian scheme. Then

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Example

Let W w-contractible. Then $\Pi_\infty^{\mathrm{cond}}(W) = \underline{\pi_0(W)} \in \mathrm{Cond}(\mathbf{Ani})$.

Example

k - field. Then

$$\Pi_\infty^{\mathrm{cond}}(\mathrm{Spec}(k)) \simeq \mathrm{B\,Gal}(\mathrm{Spec}(k)) \simeq^{\text{B-G-H}} \mathrm{B}(\mathrm{BGal}_k) \simeq \mathrm{BGal}_k$$

In particular, $\pi_1^{\mathrm{cond}}(\mathrm{Spec}(k), \mathrm{Spec}(\bar{k})) = \underline{\mathrm{Gal}_k}$

(can be also shown a bit more directly)

Theorem

X - compact topological space, $A = \mathrm{Map}_{\mathrm{cts}}(X, \mathbb{C})$. Then

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Connected components

One may describe $\pi_0(\Pi_\infty^{\mathrm{cond}}(X))$ "somewhat explicitly".

Corollary

Suppose that X has finitely many irreducible components. Then

$$\pi_0^{\mathrm{cond}}(X) \tilde{\to} \pi_0(X)$$

Corollary (related to an upcoming project!!)

Let X scheme s.t. $|X|$ is a valuative spectral space (more common for adic spaces). Then

$$\pi_0^{\mathrm{cond}}(X) \tilde{\to} X^{\mathrm{sep}}$$

Example: Berkovich disk as connected components

Pick a ring R such that $|\mathrm{Spec}(R)| \simeq \mathbb{D}^{1,\mathrm{adic}}_{\mathbf{Q}_p}$ (adic disc). Then

$$\pi_0^{\mathrm{cond}}(\mathrm{Spec}(R)) = \underline{\mathbb{D}^{1,\mathrm{Berk}}}$$

Example

For $X =$ "schematic Warsaw circle", $\pi_0^{\mathrm{cond}}(X)$ is non-quasiseparated.

Connected components

One may describe $\pi_0(\Pi_\infty^{\mathrm{cond}}(X))$ "somewhat explicitly".

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Suppose that X has finitely many irreducible components. Then
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Non-qs problems of π_1^{cond}

For a topological group G and $H < G$, let

H^{nc} = group-theoretic normal closure of H in G
$H^{\mathrm{tnc}} = \overline{H^{\mathrm{nc}}}$ = topological normal closure of H in G
$\widehat{\mathrm{Fr}}_W$ = free profinite group on a (discrete) set W.

Proposition

Let $N = \langle \widehat{\mathbb{Z}}(a) | a \in \mathbb{C} \rangle^{\mathrm{nc}} < \widehat{\mathrm{Fr}}_{\mathbb{C}}$.
Then there is a s.e.s. of (abstract) groups

$$1 \to N \to \widehat{\mathrm{Fr}}_{\mathbb{C}} \to \pi_1^{\mathrm{cond}}(\mathbb{A}^1_{\mathbb{C}}, \eta)(*) \to 1$$

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$$\pi_1^{\mathrm{cond}}(\mathbb{A}^1_{\mathbb{C}}, \eta)(*) \neq 1$$

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Integral descent and $\pi_1^{\mathrm{cond,qs}}$ of normal schemes

Theorem

Let X - qc topologically noetherian scheme.
Assume X is geometrically unibranch (e.g. normal). Then

$$\pi_1^{\mathrm{cond}}(X, \bar{x})^{\mathsf{qs}} \xrightarrow{\sim} \pi_1^{\text{ét}}(X, \bar{x})$$

Theorem: integral hyperdescent for $\Pi_\infty^{\mathrm{cond}}$

Let $X_\bullet \twoheadrightarrow Y$ be an integral hypercover. Then the canonical map

$$\operatorname*{colim}_{k \in \Delta^{\mathrm{op}}} \Pi_\infty^{\mathrm{cond}}(X_k) \xrightarrow{\simeq} \Pi_\infty^{\mathrm{cond}}(Y) \text{ in } \mathrm{Cond}(\mathbf{Ani})$$

Proposition: free condensed vs free topological products

Let $G_1, \ldots, G_m \in \mathrm{Pro}(\mathbf{Grp}^{\mathrm{fin}})$ and $r \in \mathbf{N}$. Then

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is an isomorphism in $\mathrm{Cond}(\mathbf{Grp})$.

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van Kampen formula for $\pi_1^{\mathrm{cond,qs}}$

Theorem

X "reasonable" ($\in$ [Nagata, topologically noetherian]), connected
$\nu \colon X^\nu = \sqcup_i X_i^\nu \to X$ normalization.
Dual graph Γ: $E = \pi_0(X^{2\nu})$, $V = \pi_0(X^\nu)$. Then

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where H is given by "global" and rather standard / expected relations.

Example

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Kurosh subgroup theorem holds!

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Kurosh subgroup theorem holds!

Subgroups of Free Products: Algebraic vs. Topological

Abstract Kurosh Subgroup Theorem

Let $G = *_{i\in I} G_i$ be the free product of groups. If $H \le G$, then there exists a free group F and a set of representatives D such that:

$$H \cong F * \left(\mathop{*}_{i\in I,\, d\in D} (H \cap dG_i d^{-1}) \right).$$

Corollary: Finite Subgroups

Let $G = (*_i A_i) * (*_j B_j)$, where $\{A_i\}$ are finite groups and $\{B_j\}$ are torsion-free. If $K \le G$ is a **finite** subgroup, then K resides in a conjugate of a finite factor:

$$\exists i, \exists g \in G \quad \text{such that} \quad K \subseteq gA_i g^{-1}.$$

Theorem: Compact Subgroups (Topological)

Let $G = *^{\mathrm{top}}_{i\in I} G_i$ be the free topological product of Hausdorff topological groups. If $K \subseteq G$ is a **compact** subgroup, then:

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Fundamental exact sequence(s)

Theorem

Let $f\colon X \to S$ be a morphism between qcqs schemes, and $\bar{s} \to S$ a geom. pt. If $\dim(S) = 0$, then the sequence

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is a fiber sequence in Cond(**Ani**).

Corollary

For X qcqs and geometrically connected

$$1 \to \pi_1^{\mathrm{cond}}(X_{\bar{k}}) \to \pi_1^{\mathrm{cond}}(X) \to G_k \to 1$$

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Upcoming work

Let X be a (qcqs) rigid/adic space over a non-archimedean field K (think "a p-adic analytic space over a field like $\mathbb{Q}_p$").
Its étale topos is "spectral" (in the adic formalism) and "coherent" – our machinery should work well.
We've seen that

$$\pi_0^{\mathrm{cond}}(X) \simeq |X^{\mathrm{Berk}}|$$

i.e. a (usually) non-trivial compact, locally path connected Hausdorff space.
We have a *groupoid* $\Pi_1^{\mathrm{cond}}(X)$, but the fundamental *groups* will heavily depend on the base point. How to get a single π_1 (for a connected X)?

Thm (Achinger-L.-Youcis 2022, building on de Jong)

There is a notion of a "geometric arc" on X: a mixture of an actual arc on $|X^{\mathrm{Berk}}|$ and étale data: a compatible system of isomorphisms of fibre functors along the arc. Can define the property of "unique (geometric) arc lifting". Then

$$\{Y \to X \quad | \quad \text{p.p. étale} + \text{satisfies UGAL after pullback to any curve}\}$$

has a good Galois theory $\Rightarrow$ we get a π_1^{arc} (independent of the base point)

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Let X be a locally "wep-connected" topological space. (More general than "locally path connected". One can take e.g. $X =$ "Hawaiian earring"!). Consider *semicoverings*

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