Membership problems in braid and Artin groups

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(joint work with Carl-Fredrik Nyberg-Brodda)

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How little we really know!

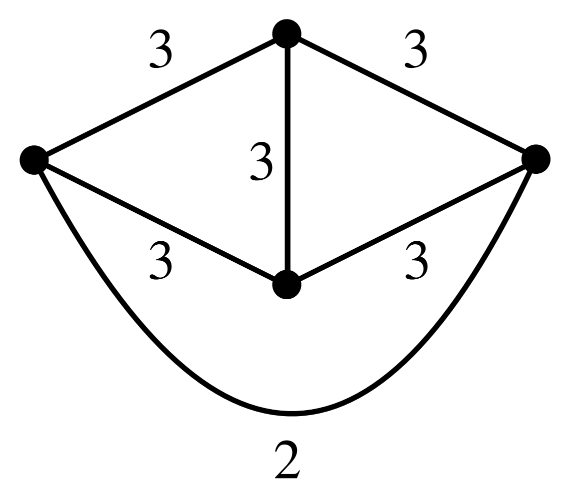
Let G be the group defined by the presentation:

$$G \cong \langle a, b, x, y \quad | \quad ab = ba, \quad axa = xax, \quad aya = yay, \quad bxb = xbx, \quad byb = yby, \quad xyx = yxy \quad \rangle$$

Open problem

Does this group G have decidable word problem? That is, is there an algorithm that takes any two words α, β over $\{a, b, x, y\}^{\pm 1}$ and decides whether or not $\alpha = \beta$ in G?

$\Delta \cong$



- This is a particular case of a more general problem which asks whether the word problem is decidable for all Artin groups.

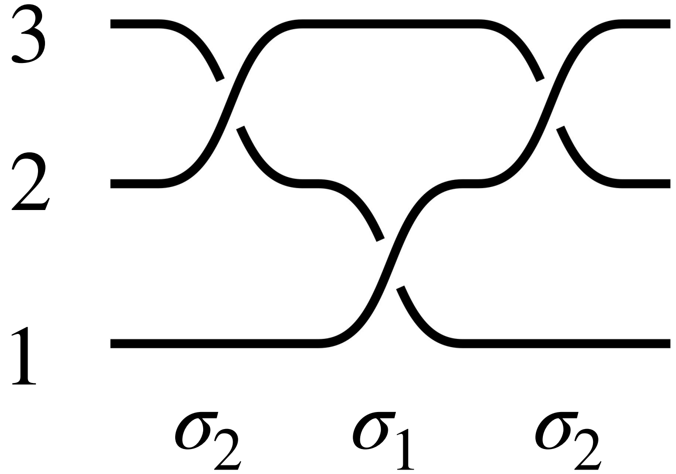
$$G = A(\Delta)$$

The braid group B_n

The braid group B_n has a finite Artin presentation

$$B_n = \langle \sigma_1, \ldots \sigma_{n-1} \mid \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \text{ if } |i-j| \geq 2 \rangle$$

Representing elements graphically



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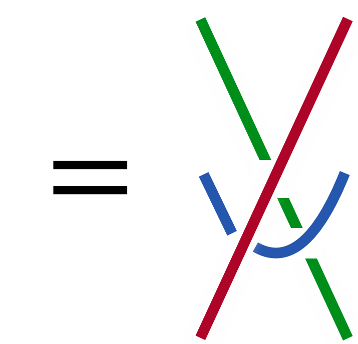

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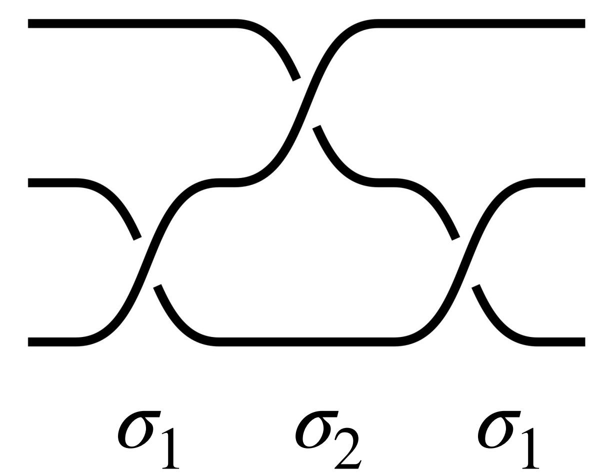
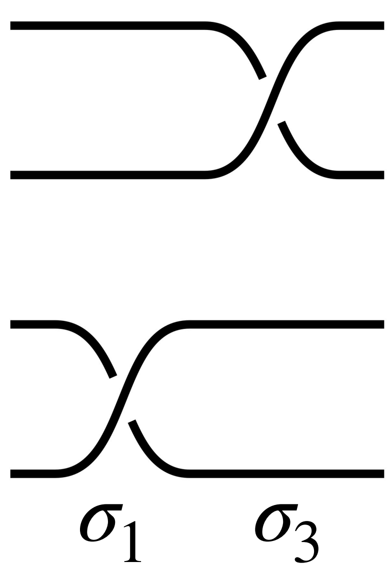


Isotopic braids are equal



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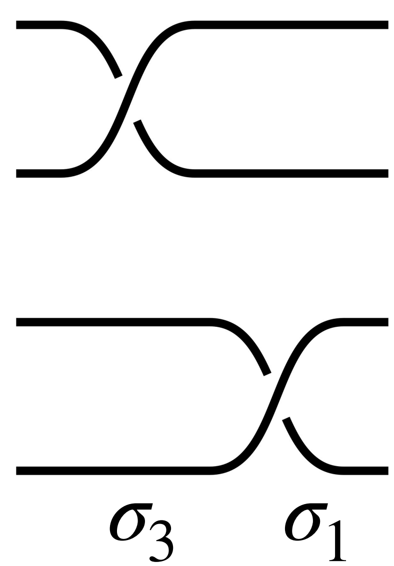
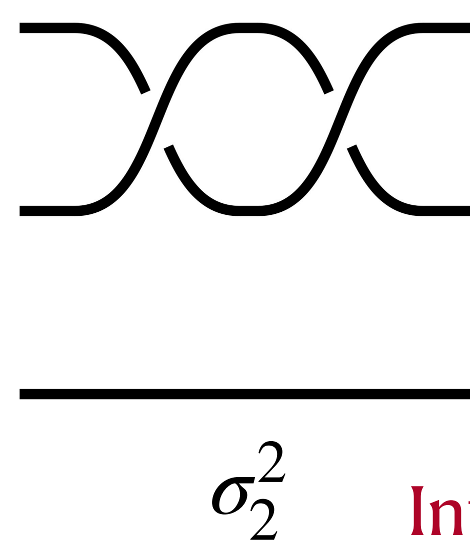
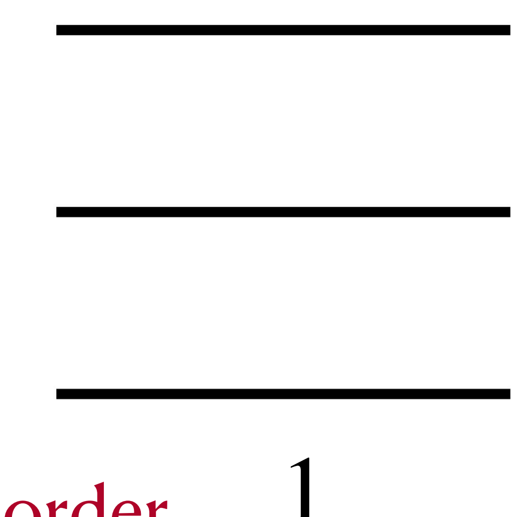


Non-overlapping commute



$\neq$



Infinite order generators

Braid group history and motivation

- Formally introduced and studied by Artin (1925) & implicit in work of Hurwitz (1891) as fundamental groups of configurations spaces.
- Connections with representation theory, algebraic geometry and topology, mathematical physics, Yang–Baxter equation, etc…

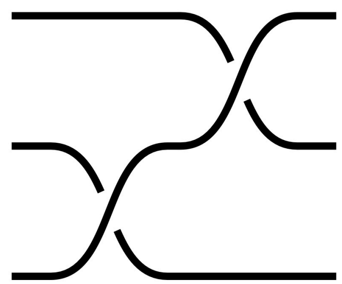
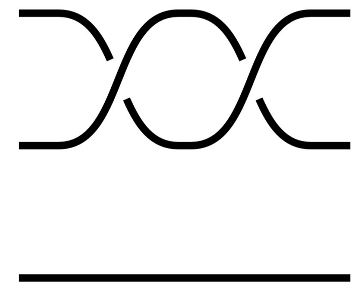
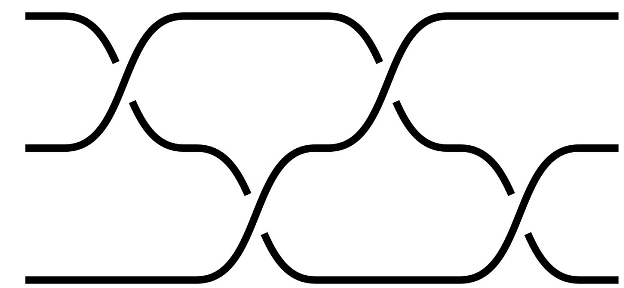
Algorithmic properties of Braid groups

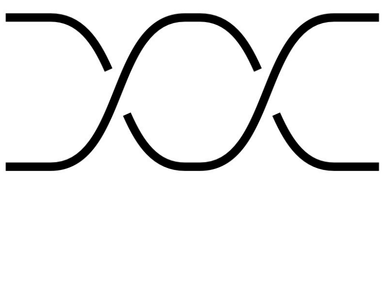
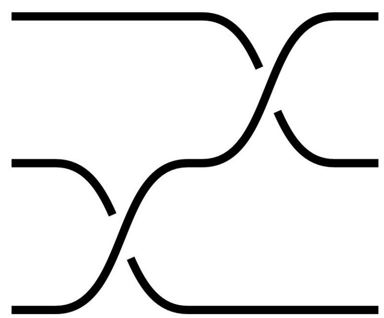
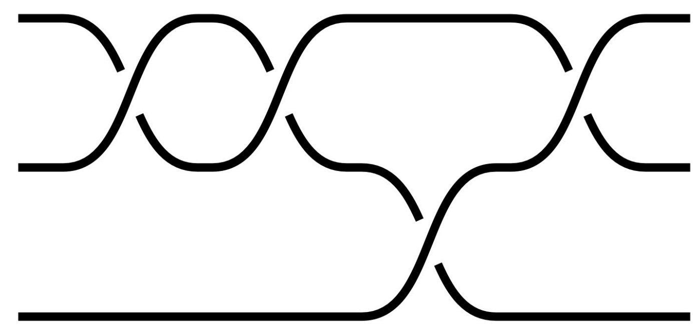
- Word problem is decidable (Artin (1925)).
- Conjugacy problem is decidable (Garside (1969)).
- They are bi-automatic (Charney (1992)) implying that their word problems are solvable in quadratic time and their conjugacy problems are solvable in exponential time.
- The more general “equation solving” problem for braid groups remains open.

The membership problem in B_n

The membership problem in B_n asks if a given braid τ (the target) can be written as a product of some other given set of braids $\delta_1, \ldots, \delta_k$ (the generators).

Examples

$\delta_1 =$  $\delta_2 =$  $\tau =$ 

  $=$  $= \tau$

$\delta_2 = \sigma_2\sigma_2$ $\quad \delta_1 = \sigma_1\sigma_2$ $\quad \delta_2\delta_1 = \sigma_2\sigma_2\sigma_1\sigma_2$ $\quad = \sigma_2\sigma_1\sigma_2\sigma_1 = \tau$

On the other hand $\quad \tau' :=$  $= \delta_2^{-1}$ cannot be written as a product of δ_1, δ_2 (e.g. since defining relations are length preserving)

The membership problem in B_n

Definition the braid group B_n has decidable (subsemigroup) membership problem if there is an algorithm solving:

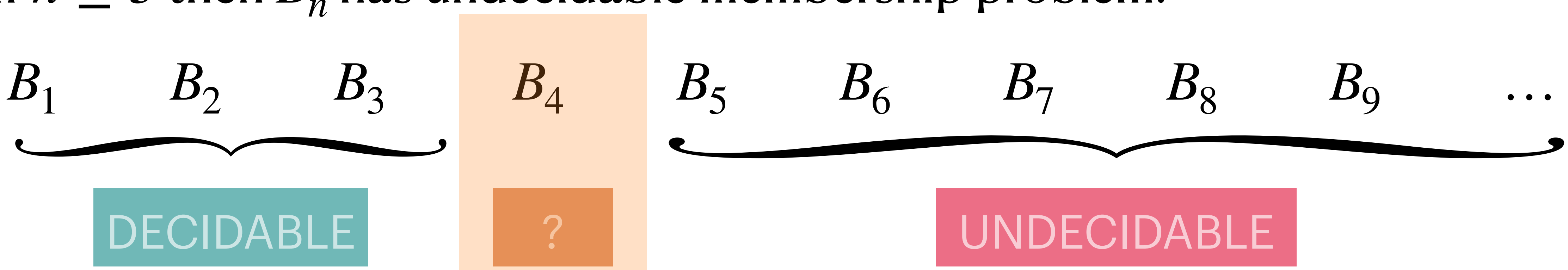
INPUT: A element $\tau \in B_n$ and elements $\delta_1, \ldots, \delta_k \in B_n$.

QUESTION: $\tau \in \langle \delta_1, \ldots, \delta_k \rangle \leq B_n$?

where $\langle \delta_1, \ldots, \delta_k \rangle \leq B_n$ is the subsemigroup generated by $\{\delta_1, \ldots, \delta_k\}$.

Theorem (Potapov 2013)

1. If $n \leq 3$ then B_n has decidable membership problem.
2. If $n \geq 5$ then B_n has undecidable membership problem.



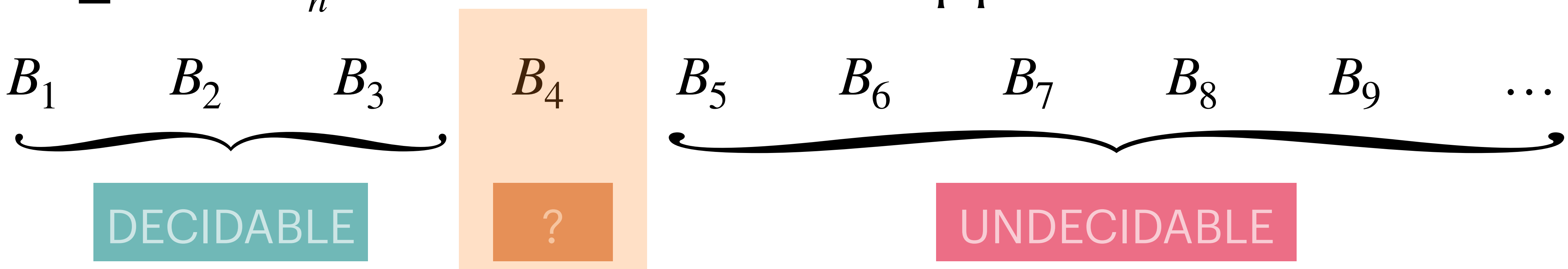
Question (Potapov 2013 + Ko & Potapov 2017)

Does B_4 has decidable membership problem?

The membership problem for small n

Theorem (Potapov 2013)

1. If $n \leq 3$ then B_n has decidable membership problem.
2. If $n \geq 5$ then B_n has undecidable membership problem.



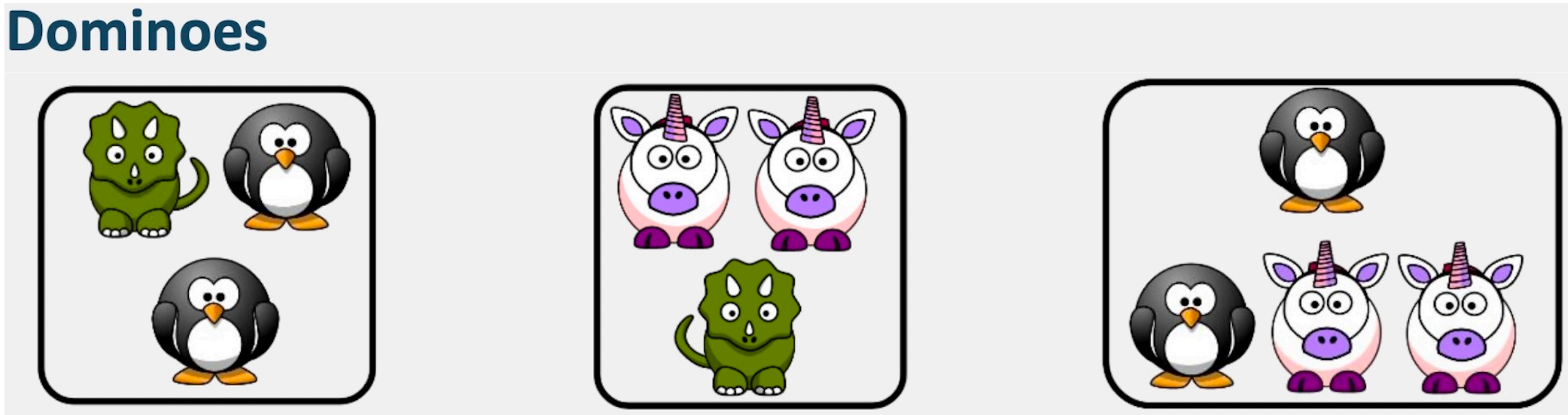
Small n $|B_1| = 1$ and $B_2 \cong \mathbb{Z}$ ✔

$B_3 \cong \langle x, y \mid x^2 = y^3 \rangle$ is a torus knot group

$\Rightarrow B_3$ is virtually $F_n \times \mathbb{Z}$ with F_n a free group (by e.g. Niblo and Wise (2001))

$\Rightarrow B_3$ has decidable membership problem (by Benois (1969) & Kambites et al. (2007))

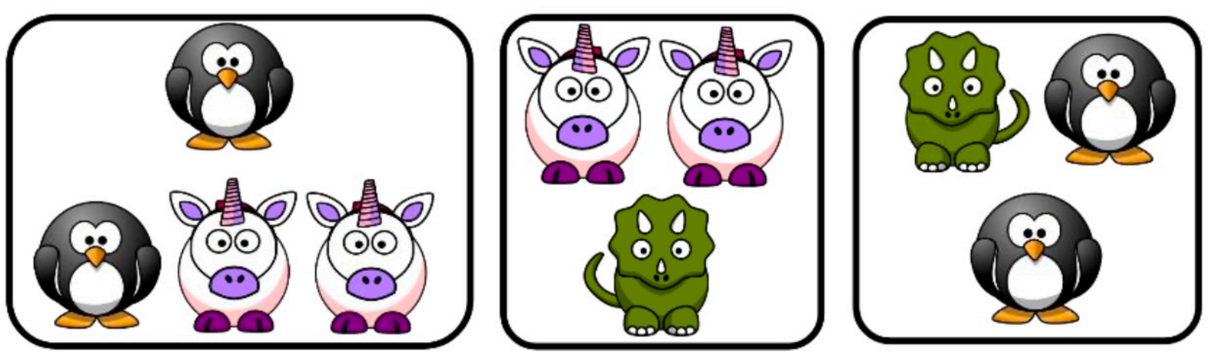
Animal dominoes



The game: Line up dominoes so the top sequence matches the bottom sequence.

Incorrect!

The top does not match the bottom



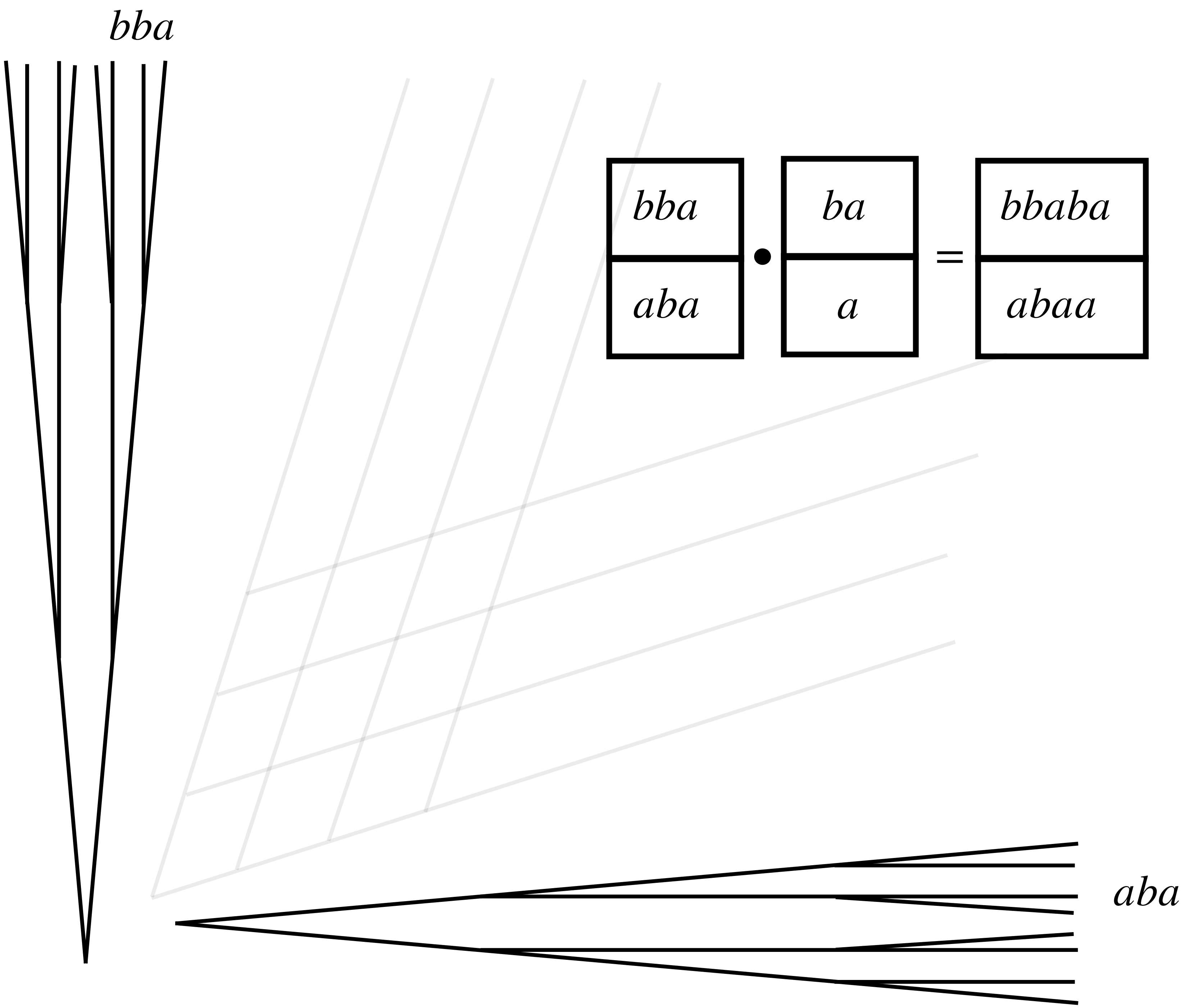
Correct!

The top matches the bottom

penguin.unicorn.unicorn.dinosaur.penguin

This is the Post Correspondence Problem and is known to be undecidable e.g. there is no algorithm that will take a finite of dominoes and tell you whether or not there is a correct line using these dominoes (each domino can be used as many times as you want).

Large n: Embedding pairs of words $F_2 \times F_2 \hookrightarrow B_n$



- $F_2 = F(a, b)$ = free group
- Makanina (1981) proved the direct product $F_2 \times F_2 \hookrightarrow B_n$ when $n \geq 5$.
- Can be used to simulate animal dominoes game i.e. post correspondence problem.
- Mikhailova (1958) showed $F_2 \times F_2$ contains a fixed subgroup in which membership is undecidable.

Corollary (Makanina (1981) & Mikhailova (1958))

If $n \geq 5$ then B_n has undecidable membership problem i.e. there is no algorithm that takes input $\delta_1, \ldots, \delta_k \in B_n$ and $\tau \in B_n$ and decides whether $\tau \in \langle \delta_1, \ldots, \delta_k \rangle$.

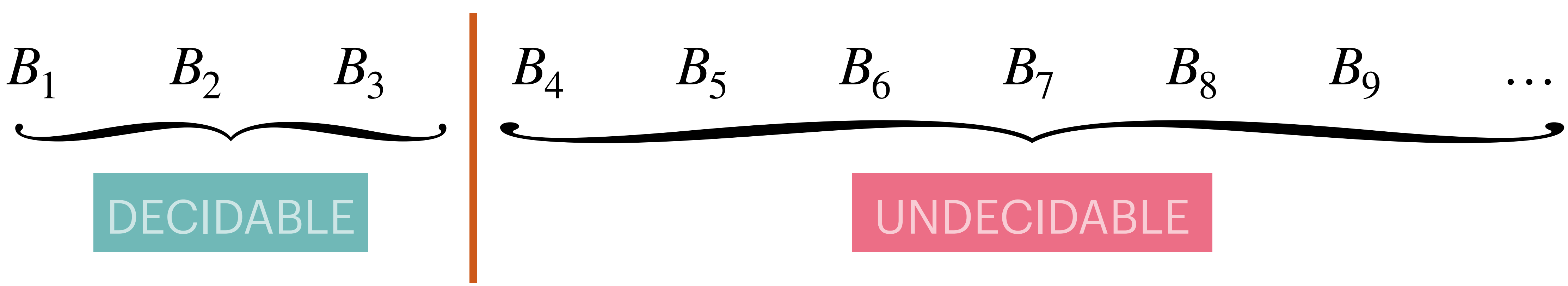
The membership problem in B_4

Quoting from Ko & Potapov 2017

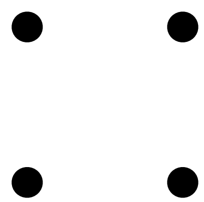
"...there is no embedding from a set of pairs of words into B_4; by Akimenkov (1991). Hence, these problems might be decidable for B_4 since our undecidability proofs for B_5 essentially rely on the embedding from a set of pairs of words into B_5."

Theorem (RDG and Nyberg-Brodda 2025)

The braid group B_4 has an undecidable membership problem. In fact, there is a fixed finitely generated subsemigroup of B_4 in which membership is undecidable. Hence B_n has decidable membership problem if and only of $n \leq 3$.



Membership problem in Right-angled Artin groups (RAAGs)

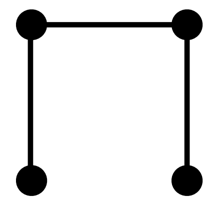


$\langle a_i \mid \rangle$

Benois (1969)

Free groups

DECIDABLE

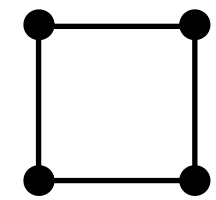


$\langle a_i \mid a_i a_{i+1} = a_{i+1} a_i \rangle$

Lohrey & Steinberg (2008)

$A(P_4)$

UNDECIDABLE

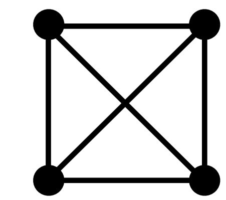


$\langle a_i \mid a_i a_{i+1} = a_{i+1} a_i \pmod 4 \rangle$

Mikhailova (1958)

$A(C_4) = F_2 \times F_2$

UNDECIDABLE



$\langle a_i \mid a_i a_j = a_j a_i \rangle$

Grunschlag (1999)

f.g. abelian groups

DECIDABLE

Definition $A(\Gamma) = \mathrm{Gp}\langle V\Gamma \mid uv = vu$ if and only if $\{u, v\} \in E\Gamma \rangle$.

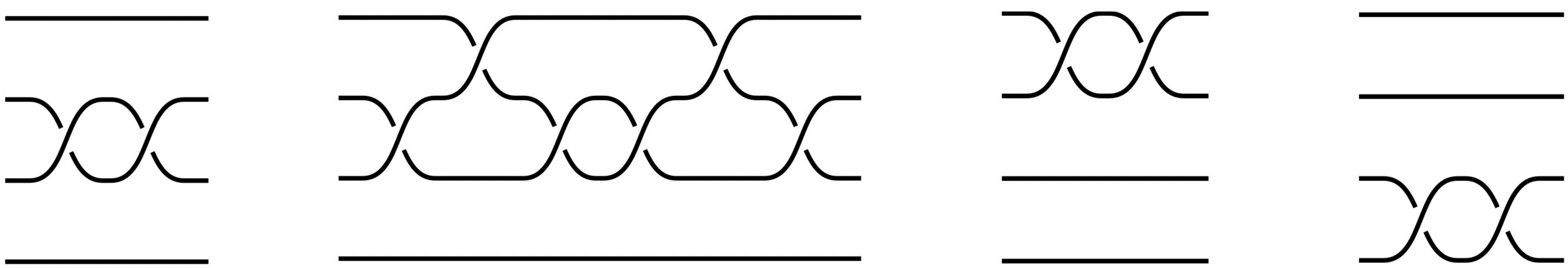
Theorem (Lohrey & Steinberg (2008) A RAAG $A(\Gamma)$ has decidable membership problem $\Leftrightarrow \Gamma$ does not embed P_4 or C_4 as an induced subgraph.

- In fact they show that $A(P_4)$ has a fixed finitely generated subsemigroup in which membership is undecidable.

Tree groups and B_4

Theorem (Droms, Lewin & Servatius (1991))

The subgroup of B_4 generated by $\sigma_2^2, (\sigma_2\sigma_3\sigma_2)^2, \sigma_3^2, \sigma_1^2$ is isomorphic to $A(P_4)$.



Proof that B_4 has undecidable membership problem:

Lohrey & Steinberg (2008) $\Rightarrow \exists$ f.g. $S \leq A(P_4)$ in which membership is undecidable.

Droms, Lewin & Servatius (1991) $\Rightarrow A(P_4) \hookrightarrow B_4$.

Hence $\exists$ f.g. $S \hookrightarrow A(P_4) \hookrightarrow B_4$ in which membership is undecidable.

Therefore B_4 has undecidable membership problem. □

Artin groups

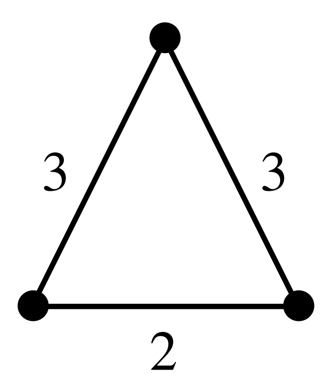
Let Γ be a finite graph with edges labelled by natural numbers greater than or equal to 2. The Artin group $A(\Gamma)$ is the group defined by the presentation with generating set the set $V\Gamma$ of vertices of Γ and a defining relation

$$\underbrace{abababab\ldots}_{m \text{ letters}} = \underbrace{babababa\ldots}_{m \text{ letters}}$$

for each edge that connects vertices a and b and is labelled m.

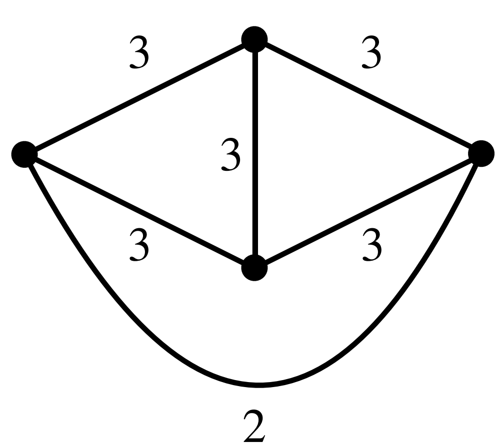
Examples

- RAAGs = Artin groups where all labels equal 2.
- Every braid group B_n is an Artin group e.g. $B_4 = A(\Gamma)$ for $\Gamma =$



- $A(\Delta)$ has presentation

$\Delta \cong$



$$\langle a, b, x, y \mid ab = ba, axa = xax, aya = yay, bxb = xbx, byb = yby, xyx = yxy \rangle$$

Artin groups

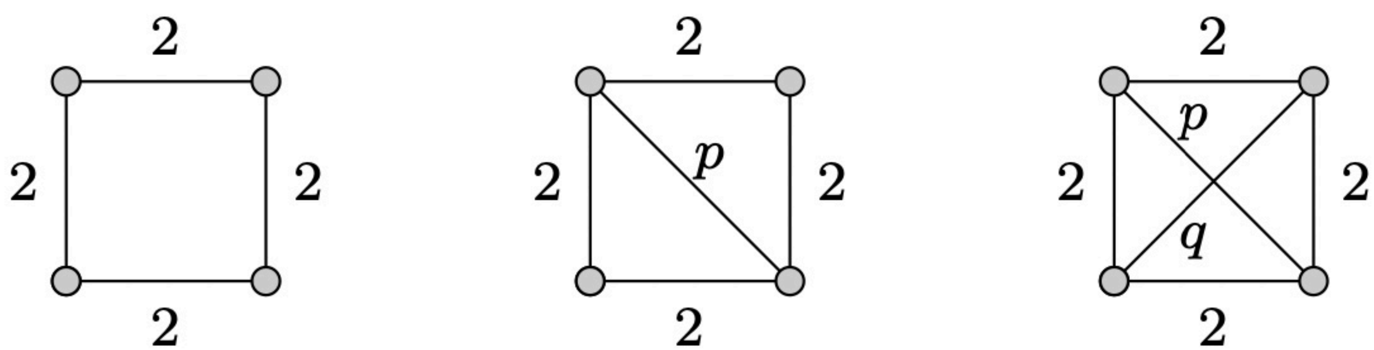
Theorem (RDG and Nyberg-Brodda 2025)

Let $A = A(\Gamma)$ be an Artin group. Then the following are equivalent:

(i) A has decidable membership problem

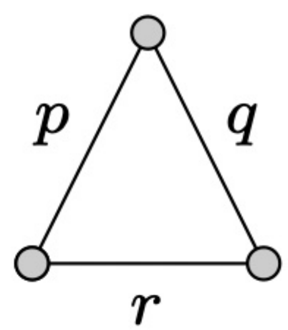
(ii) the graph Γ does not embed any of the following as an induced subgraph:

(a) a (generalized) square of one of the following three forms



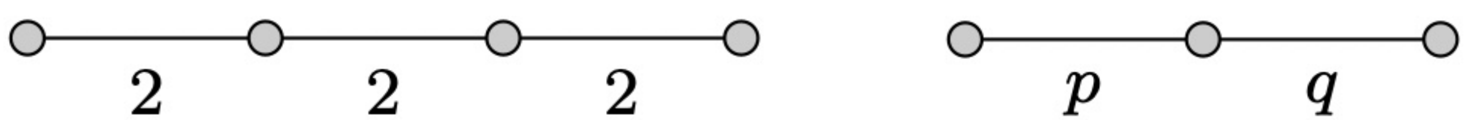
with $p > 2$ and $q > 2$,

(b) a triangle of the form



where at most one of $\{p, q, r\}$ is equal to 2, or

(c) a path of one of the following two forms



where at most one of $\{p, q\}$ is equal to 2.

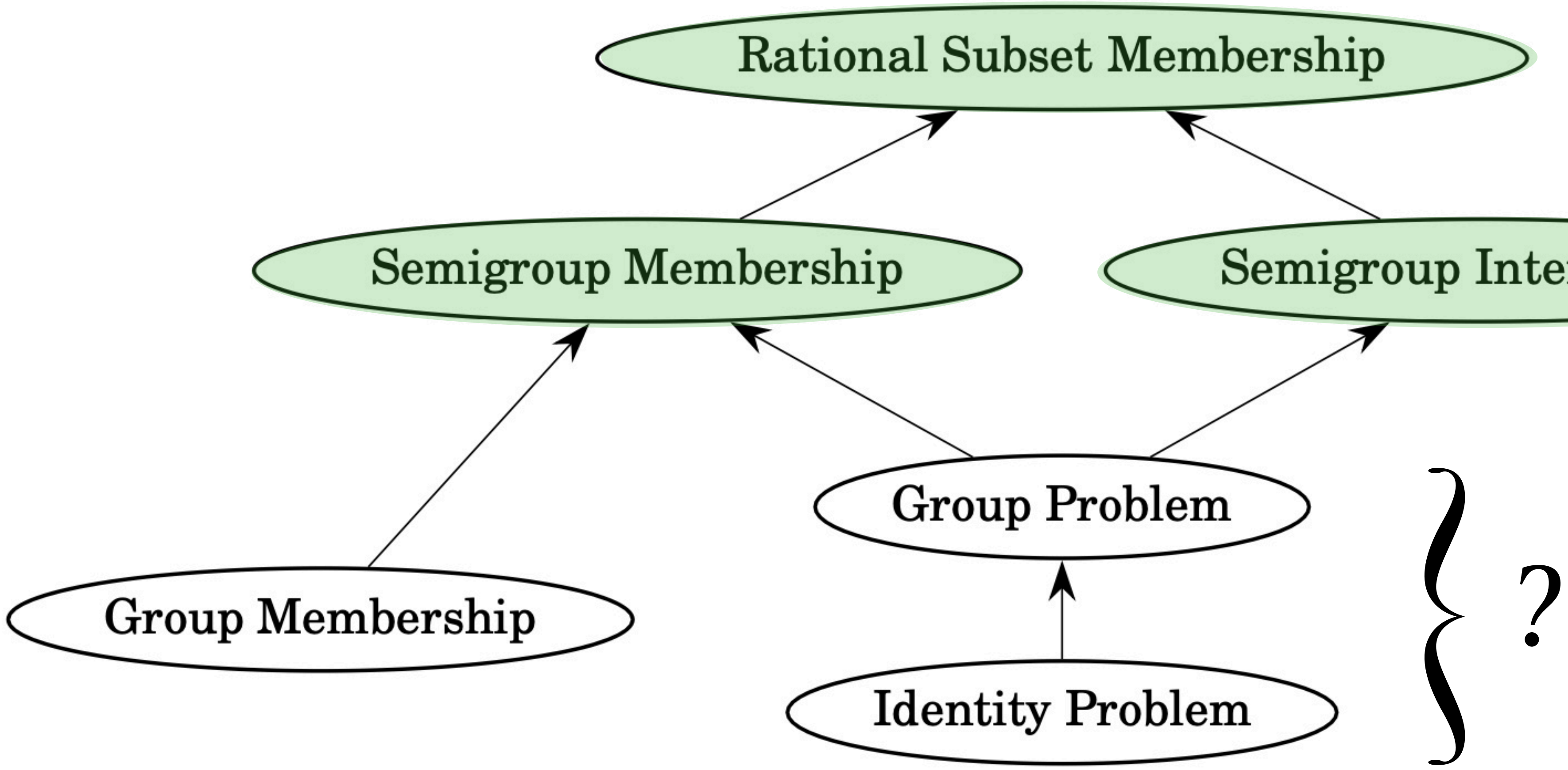
Artin groups - notes on the proof

- A group G is subgroup separable (also called LERF) if for every finitely generated subgroup H of G, and element $g \in G \backslash H$ there is a finite index subgroup K of G, which contains H but $g \notin K$.
- In general for finitely presented groups:

  [subgroup separable $\not\Rightarrow$ decidable membership problem] and

  [decidable membership problem $\not\Rightarrow$ subgroup separable].
- However, a key step in the proof of our result above is showing that for Artin groups subgroup separable is equivalent to having decidable membership problem.
- Then we may apply results of Almeida and Lima (2021 + 2024) who classified the subgroup separable Artin groups.
- See also Foniqi (2024) for further exploration of these connections.

Other properties for Artin groups



Theorem (RDG and Nyberg-Brodda 2025)

Let $A = A(\Gamma)$ be an Artin group. Then the following are equivalent:

(i) A has decidable (subsemigroup) membership problem;

(ii) A has decidable rational subset membership problem;

(iii) A has decidable semigroup intersection problem;

(iv) Γ does not embed any of the forbidden subgraphs in the above theorem.

The identity problem for B_n

Definition the braid group B_n has decidable identity problem if there is an algorithm solving:

INPUT: The identity element $1 \in B_n$ and elements $\delta_1, \ldots, \delta_k \in B_n$.

QUESTION: $1 \in \langle \delta_1, \ldots, \delta_k \rangle \leq B_n$?

where $\langle \delta_1, \ldots, \delta_k \rangle \leq B_n$ is the subsemigroup generated by $\{\delta_1, \ldots, \delta_k\}$.

Theorem (Potapov 2013)

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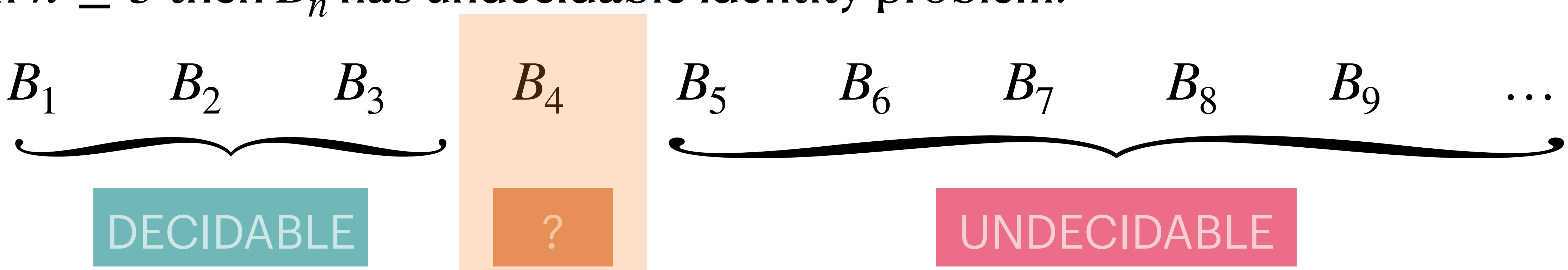


Question (Potapov 2013 + Ko & Potapov 2017)

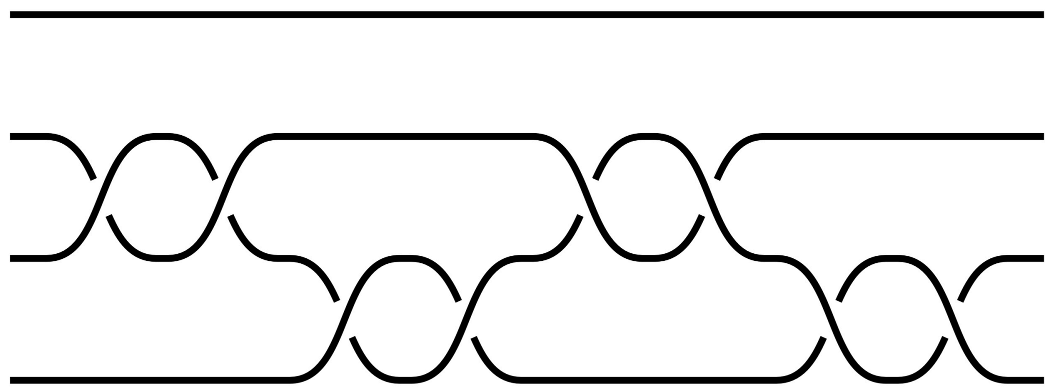
Does B_4 has decidable identity problem?

This problem remains open!

Undecidable fixed target membership problem

Theorem (RDG and Nyberg-Brodda 2025)

Let τ_0 be the following fixed element from the braid group B_4

$\tau_0 =$  $= \sigma_2^2\sigma_1^2\sigma_2^{-2}\sigma_1^{-2}$

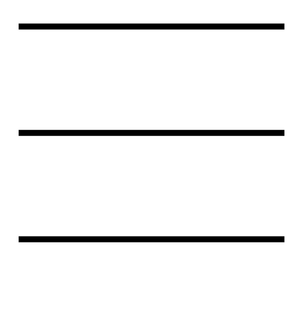
Then there is no algorithm that can solve the following:

INPUT: The fixed element τ_0 above, and elements $\delta_1, \ldots, \delta_k \in B_4$.

QUESTION: $\tau_0 \in \langle \delta_1, \ldots, \delta_k \rangle \leq B_n$?

where $\langle \delta_1, \ldots, \delta_k \rangle \leq B_n$ is the subsemigroup generated by $\{\delta_1, \ldots, \delta_k\}$.

Remark

The identity problem for B_4 asks whether we can replace τ_0 by the identity braid  $\in B_4$ in the above result?

Fixed target membership problem - proof ideas

Lemma (RDG and Nyberg-Brodda 2025)

Let G be a finitely generated group, and let $F_{x,y,z}$ be the free group with basis $\{x, y, z\}$. If the fixed-target membership problem for xy^{-1} in the free product $G * F_{x,y,z}$ is decidable, then the (subsemigroup) membership problem in G is decidable.

Corollary

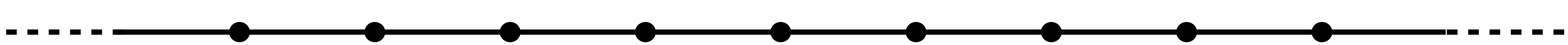
The fixed-target membership problem for xy^{-1} in $A(P_4) * F_{x,y,z}$ is undecidable.

Final step of the B_4 undecidable fixed target membership problem proof

Adapt results of Droms, Lewin & Servatius (1991) to find appropriate embeddings

$$A(P_4) * F_{x,y,z} \hookrightarrow A(P_\infty) \hookrightarrow A(P_4) \hookrightarrow B_4$$

where $\tau_0 = xy^{-1}$ and P_∞ is the bi-infinite line



Open problems / future directions

- Does B_4 has decidable identity problem?
- Classify the Artin groups with decidable identity problem.
- Does B_4 have decidable subgroup membership problem?
  - Note: The RAAG $A(C_5) \hookrightarrow B_4$ and it is unknown whether $A(C_5)$ has decidable subgroup membership problem.
- Classify Coxeter groups with decidable subsemigroup membership problem.
- In particular does the the right-angled Coxeter group of the pentagon with presentation (subscripts taken mod 5)
  $$\langle a_0, a_1, a_2, a_3, a_4 \mid a_i^2 = 1, \quad a_i a_{i+1} = a_{i+1} a_i \quad (i \in \{0,1,2,3,4\}) \rangle$$
  have decidable subsemigroup membership problem?
  - Note: This group contains a hyperbolic surface subgroup of finite index and the subsemigroup membership problem is open for surface groups.