Geometry of groups and index theory

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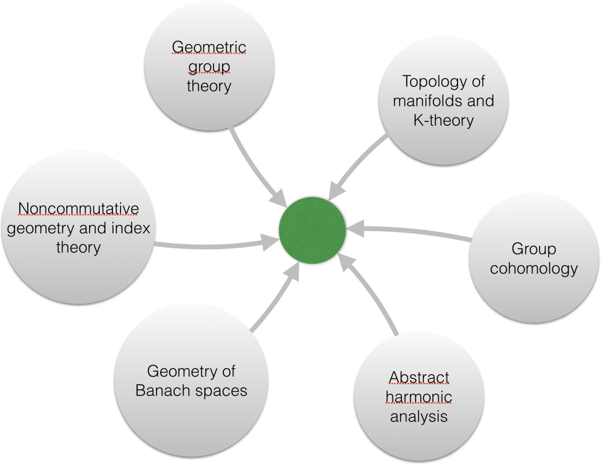


Goal today: overview and show these interactions

What is large scale geometry?

Consider two discrete metric spaces:

1. $\mathbb{Z}$ as a subspace of $\mathbb{R}$,
2. $\mathbb{Z}^2$ as a subspace of $\mathbb{R}^2$,

Both discrete $\Longrightarrow$ 0-dimensional topologically

However: intuitively clear that they share some form of dimensionality of the ambient Euclidean space:

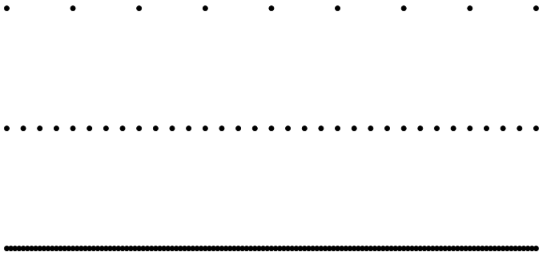
$\mathbb{Z}$ is "1-dimensional" in comparison to $\mathbb{Z}^2$, which is "2-dimensional"

There are geometric phenomena that have similarly global nature - they not depend on any local information (i.e. topology)

How to make this precise?

The big picture - Gromov

Imagine looking at these spaces from an increasingly larger distance:



The points look closer together because of perspective

Looking from ∞: $\mathbb{Z} \simeq \mathbb{R}$ and $\mathbb{Z}^2 \simeq \mathbb{R}^2$

Large scale geometry: the geometry of spaces viewed from ∞

What does $\simeq$ mean in large scale geometry?

Two possibilities:

1. study the actual geometry of the limit at ∞:

$$X_\infty = \text{"}\lim\text{"}(X, \epsilon_n d), \qquad \text{with } \epsilon_n \to 0$$

   Difficulty: defining X_∞ is non-trivial and X_∞ is usually huge.

2. study X itself but adjust the definition of $\simeq$:
   (X, d_X) and (Y, d_Y) are quasi-isometric if

$$\frac{1}{L} d_X(x, x') - C \;\leq\; d_Y(f(x), f(x')) \;\leq\; L d_X(x, x') + C.$$

   additive constant allows for discontinuities and gluing on scale C

   Easier to work with, much more prevalent

Groups as metric spaces

G - discrete group

$S = S^{-1} \subset G$ - finite set generating G

Examples to have in mind:

1. G finite, $S = G$
2. $G = \mathbb{Z}^2, S = \{(1,0), (0,1), (0,0)\}$
3. $G = \mathbb{F}_n$ - free group on n generators, $S = a_1, \ldots, a_n$

Once S is fixed we can view G as a metric space:

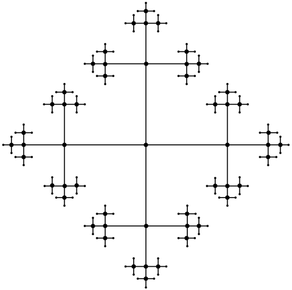
$$d_S(g, h) = \text{smallest number of elements of } S \text{ to write } g^{-1}h$$

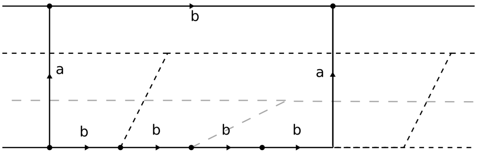
Cayley graphs - another point of view

Define an infinite graph $\mathrm{Cay}(G, S)$, the Cayley graph of (G, S):

- the set of vertices = G,
- vertices $g, h \in G$ are connected by an edge iff $g^{-1}h \in S$.

Equip $\mathrm{Cay}(G, S)$ with the shortest path metric





The free group and the Baumslag-Solitar group $\langle a, b | b^4 = aba^{-1} \rangle$

with respect to standard presentations

Examples of quasi-isometric spaces

- any bounded/compact metric space $\simeq$ point
- $\mathbb{Z}^n \simeq \mathbb{R}^n$
- $\mathbb{F}_2 \simeq$ 4-regular tree = the universal cover of the figure 8 space

More generally

Theorem (Milnor-Svarc lemma)

M - compact Riemannian manifold, $\widetilde{M}$ universal cover then

$$\pi_1(M) \simeq \widetilde{M}.$$

Index theory

The Atiyah-Singer index theorem

D - differential operator of order m on a closed smooth manifold X

$$D : C^\infty(E) \to C^\infty(F)$$

where E, F - complex vector bundles on X. Locally,

$$D = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

$a_\alpha(x) : E_x \to F_x$ linear transformation

Symbol of D: replace D^i with variables ξ_i and drop lower order terms

$$\sigma(x, \xi) = \sum_{|\alpha| = m} a_\alpha(x) \xi_\alpha.$$

D is elliptic if the symbol $\sigma(x, \xi)$ is an invertible matrix for every $x \in X$ and $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$.

The Atiyah-Singer index theorem

Example (On $X = \mathbb{R}^3$)

Gradient $\nabla : C^\infty(1) \to C^\infty(TX)$, $\nabla = \frac{\partial f}{\partial x_1} e_1 + \frac{\partial f}{\partial x_2} e_2 + \frac{\partial f}{\partial x_3} e_3 \quad \Longrightarrow$

$\sigma(x, \xi) = [\xi_1, \xi_2, \xi_3]^T \Longrightarrow$ not elliptic

Laplacian $\Delta = \nabla^2 : C(1) \to C(1)$, $\Delta = \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_3^2} \quad \Longrightarrow$

$\sigma(x, \xi) = \xi_1^2 + \xi_2^2 + \xi_3^2 \in \mathbb{R} \Longrightarrow$ elliptic

Ellipticity $\Longrightarrow$ $D : C^\infty(E) \to C^\infty(F)$ has finite dimensional kernel and cokernel

(cokernel $D = C^\infty(F) / \operatorname{im} D$)

$$\operatorname{index} D = \dim \ker D - \dim \operatorname{coker} D$$

Topological index

The topological index is a map in K-theory:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
K_0(TX) \arrow[r, "\text{topological index}"] & K_0(pt) \simeq \mathbb{Z} \\
\text{elliptic operators on } X \arrow[u, "\textit{symbol}"] \arrow[ur, "\text{analytical index}"']
\end{tikzcd}$$

Theorem (The Atiyah-Singer Index Theorem)

$$\text{topological index } D = \text{analytical index } D$$

The Baum-Connes conjecture

The Baum-Connes conjecture is a broad generalization of the A-S index theorem.

For G-discrete group the Baum-Connes assembly map ($i = 0, 1$):

$$\mu_i : K_i^G(\underline{E}G) \to K_i(C_r^*(G))$$

$$\left\{ \begin{array}{l} \text{Analytic K-homology:} \\ \text{homotopy classes of} \\ \text{abstract elliptic operators} \\ \text{associated to } G \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} K - \text{theory of the reduced} \\ \text{group } C^*\text{-algebra :} \\ \textit{indices} \end{array} \right\}$$

The Baum-Connes Conjecture

μ_i *is an isomorphism for every finitely generated G.*

The conjecture is a bridge connecting topology and analysis

Baum-Connes: applications

injectivity of $\mu_i \Longrightarrow$ applications in topology

surjectivity of $\mu_i \Longrightarrow$ applications in analysis

Baum-Connes: applications

$$\text{classical } D \quad \mapsto \quad K-\text{homology class } [D] \quad \mapsto \quad \mu_i([D]) \neq 0$$

If μ_i injective after $\otimes\mathbb{Q}$ then the following conjectures are true:

Conjecture (The Novikov conjecture)

The higher signatures

$$\mathrm{sign}_x(M,u) = \langle \mathcal{L}(M) \cup u^* x, [M] \rangle \in \mathbb{Q}$$

are homotopy invariants for all M with $\pi_1(M) = G$

Conjecture (Gromov's zero-in-the-spectrum conjecture)

0 always in the spectrum of the Laplace-Beltrami operator Δ_n *for some n, acting on the* L_2*-n-forms on the universal cover* $\widetilde{M}$ *of an aspherical Riemannian manifold M*

Conjecture (Gromov-Lawson conjecture)

M^n *closed spin manifold,* $n \geq 5$ *with* $\pi_1(M^n) = G$ *then M cannot carry a metric with positive scalar curvature.*

Baum-Connes: applications

G - infinite group and $g \in G$ has finite order $g^n = e$
$\Longrightarrow$ there is a non-trivial idempotent in the group ring $\mathbb{C}G$:

$$p = \frac{1}{n}\sum_{i=0}^{n-1} r^i g^i \quad \in \mathbb{C}G$$

for $r = n$-th root of unity

Conjecture (Idempotent conjectures)

If G is torsion-free:

1. *Kaplansky: The complex group ring $\mathbb{C}G$ does not have any idempotents except 0 and 1.*
2. *Kadison-Kaplansky: The reduced group C^*-algebra $C^*_r(G) = \overline{\mathbb{C}G}^{\|\cdot\|_r}$ does not have any idempotents except 0 and 1.*

If μ_i surjective then both are true.

Proving the conjecture - "Dirac-dual Dirac"

Need to find a proper G-C^*-algebra A and elements

$$\alpha \in KK^G(A, \mathbb{C}), \qquad \beta \in KK^G(\mathbb{C}, A)$$

such that $\gamma = \beta \otimes_A \alpha = 1 \in KK^G(\mathbb{C}, \mathbb{C})$

$$\begin{tikzcd}
K_*^G(\underline{E}G) \arrow[r, "\otimes_{\mathbb{C}}\beta"] \arrow[d, "\mu_i"] & KK_*^G(\underline{E}G, A) \arrow[r, "\otimes_A \alpha"] \arrow[d, "\mu_i^A"', "\simeq"] & K_*^G(\underline{E}G) \arrow[d, "\mu_i"'] \\
K_*(\mathbb{C} \rtimes_r G) \arrow[r, "\otimes_{\mathbb{C}\rtimes_r G} j_G(\beta)"'] & K_*(A \rtimes_r G) \arrow[r, "\otimes_{\mathbb{C}\rtimes_r G} j_G(\beta)"'] & K_*(\mathbb{C} \rtimes_r G)
\end{tikzcd}$$

Proving the conjecture - “Dirac-dual Dirac”

All this boils down (modulo technical details) to the following question.

Consider two unitary representations of G:

- the trivial representation τ
- the (left) regular representation λ of G on $\ell_2(G)$:

$$\lambda_g f(h) = f(g^{-1}h)$$

  where $f : G \to \mathbb{C}, f \in \ell_2(G), g, h \in G$.

Question

Is there a “path” of “nice” representations connecting τ and λ?

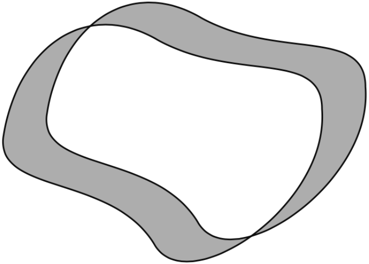
It is here where the geometric input from G becomes important: often the geometry of G is what allows to deduce the existence of an appropriate a path of representations

Amenable groups

G is amenable there is a sequence of finite sets F_n such that

$$\frac{\#(F_n \mathbin{\dot{-}} gF_n)}{\#F_n} \to 0 \qquad \text{for every generator } g \in S.$$

Namely: G has large sets with small boundary



Amenability is a large-scale geometric property of G.

Examples:

- finite groups are amenable $F_n = G$
- $\mathbb{Z}$ is amenable: $F_n = [-n, n]$

Amenable groups

Theorem (Hulanicki)

G is amenable $\iff$ τ is weakly contained in λ

Weak containment means λ and τ cannot be separated by an open set in the unitary dual of G with the Fell topology:

$v = \dfrac{\chi_{F_n}}{\sqrt{F_n}}$ is a sequence of almost invariant vectors for λ

Unitary dual = equivalence class of irreducible unitary representations of G

The Fell topology is not Hausdorff

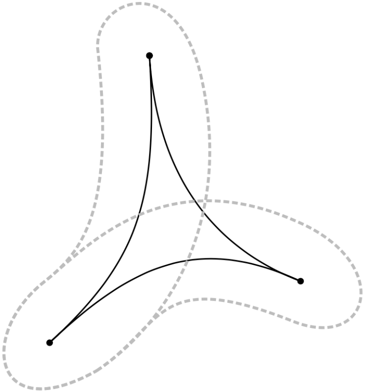
Theorem (Higson-Kasparov 2002)

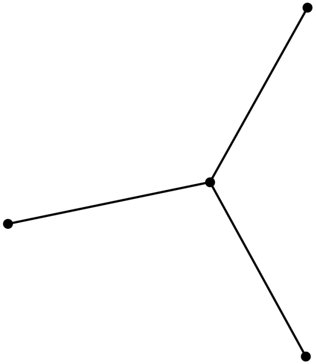
The Baum-Connes conjecture holds for amenable[a] groups.

[a] The statement is actually more general

Hyperbolic groups

G is δ-hyperbolic ($0 \leq \delta < \infty$) if geodesic triangles in the Cayley graph are δ-thin:
one of the sides is always contained in the union of δ-neighborhoods of the other two sides





Hyperbolic groups

Examples:

- free groups $\mathbb{F}_n$
- fundamental groups of hyperbolic manifolds

For hyperbolic groups there are ways to connect τ and λ through a path of representations but in general not unitary ones:

V. Lafforgue gave a technical construction of non-unitary representations induced by certain contraction-like maps on the Cayley graph of a hyperbolic group

Theorem (V. Lafforgue, 2002 and 2012)

G hyperbolic $\Longrightarrow$ the Baum-Connes conjecture holds for G.

[A path of representations on a sufficiently convex Banach space can also be useful]

How to find counterexamples?

For which groups τ and λ cannot be connected by a sufficiently good path of representations?

Classical question:

$$\text{what is the structure of the unitary dual } \widehat{G}?$$

In particular, what are the isolated points? Already extremely hard.

Definition

G has Kazhdan's property (T) if the trivial representation is an isolated point in $\widehat{G}$ with the Fell topology.

(Surjectivity only - no strategies exist for injectivity counterexamples)

Higher rank groups

A classical example of a group with property (T):

$\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$ (Kazhdan 1963)

The Baum-Connes conjecture for $\mathrm{SL}_3(\mathbb{Z})$ is a major open problem.

In light of Lafforgue's work we need versions of property (T) for much more general classes of representations on Hilbert spaces and Banach spaces with good convexity properties:

uniformly convex, non-trivial type, cotype etc.

Spectral gaps

$\rightsquigarrow$ candidates for new counterexamples to a large scale version of the Baum-Connes conjecture

G acts on (M, m) - probability space - ergodically m-preserving

$$L_2(M) = \text{const} \oplus L_2^0(M),$$

where $L_2^0(M) = \{f \in L_2(M) : \int_X f = 0\}$.

Definition

$G \curvearrowright M$ has a spectral gap if $\exists \kappa > 0$ such that $\forall$ generator $s \in S, f \in L_2^0(M)$, $\|f\| = 1$ we have

$$\|f - \pi_s f\| \geq \kappa$$

Here $\pi_g f(x) = f(g^{-1}x)$.

In Fell topology: π and the τ are separated.

Warped cones

(M, d, m) - compact metric probability space

G - acts on M by Lipschitz homeos, m-preserving

Examples to keep in mind:

- $\mathrm{SL}_2(\mathbb{Z}) \curvearrowright \mathbb{R}^2/\mathbb{Z}^2$,
- $\Gamma \curvearrowright G$, where G - compact Lie group, Γ - discrete subgroup

$\mathrm{Cone}(M) \simeq M \times (0, \infty)$ is the Euclidean cone over M, where

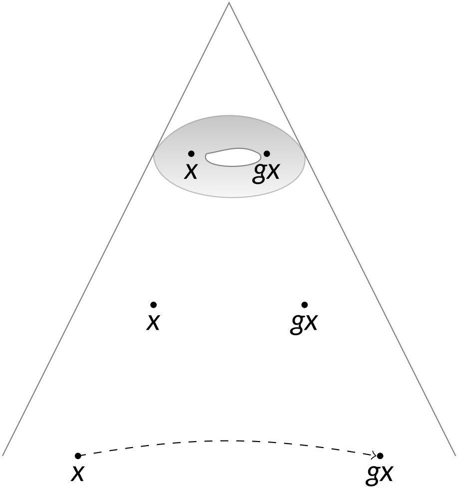
$$d_{\mathrm{Cone}}|_{M \times \{t\}} = t \cdot d$$

Definition

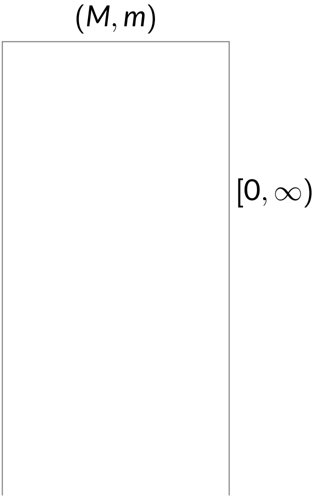
The warped metric $d_{\mathcal{O}}$ is the largest metric on $M \times (0, \infty)$ satisfying:

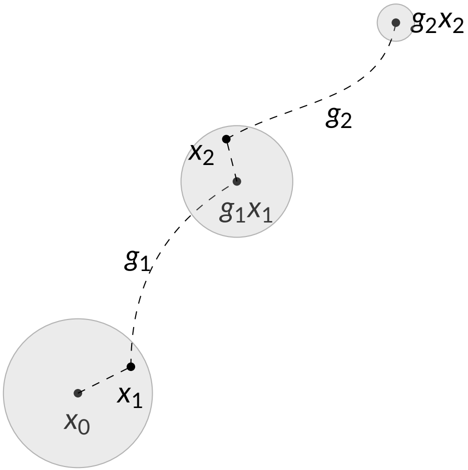
$$d_{\mathcal{O}}(x, y) \leq d_{\mathrm{Cone}(M)}(x, y) \quad \text{and} \quad d_{\mathcal{O}}(x, gx) \leq |g|.$$

metric



equip with product measure





dynamics of the action $\rightsquigarrow$ geometric properties of the warped cone

Ghost projections

We have: $L_2[0,\infty) \subseteq L_2(\mathcal{O}_\Gamma M)$

Theorem (Cornelia Druţu-PN, 2015)

$G \curvearrowright M$ ergodically with a spectral gap. Then the orthogonal projection

$$P : L_2(\mathcal{O}_\Gamma M) \to L_2[0,\infty)$$

is a non-compact ghost projection and a limit of finite propagation operators.

Proof via convergence of a random walk on G

The properties of the operator are important from the index-theoretic perspective - they are characteristic for the Roe C^*-algebra, whose K-theory is the target for the coarse index map

Being a ghost means the operator is “locally invisible at infitnity"
For matrix (kernel) operator this means matrix (kernel) is c_0.

[ghost projection] $\in$ K-theory cannot be an index

Many classical result about spectral gaps for actions:
Margulis, Sullivan, Drinfeld - motivated by the Ruziewicz problem

Theorem (Bourgain-Gamburd 2008)

For many appropriately chosen free subgroups $\mathbb{F}_n \subseteq SU(2)$ the action of $\mathbb{F}_n$ on $SU(2)$ has a spectral gap.

Note that

$$SU(2) \simeq_{diffeo} S^3$$

Example

The warped cones

- $\mathcal{O}_{\mathbb{F}_n} \mathrm{SU}(2)$
- $\mathcal{O}_{\mathrm{SL}_2(\mathbb{Z})} \mathbb{T}^2$

have non-compact ghost projections

Conjecture

The coarse Baum-Connes assembly map is not surjective for a warped cone over an action with a spectral gap.

ghost projection $\rightsquigarrow$ K-theory class not in the image of coarse index map

Using the Bourgain-Gamburd theorem we would have such counterexamples obtained from modifying the metric on the 4-dimensional Euclidean space:

$$\mathcal{O}_{\mathbb{F}_n}\,\mathrm{SU}(2) \simeq (\mathbb{R}^4, d_{\mathcal{O}})$$

Thank you!