Denoised Monte Carlo for option pricing

9 May 2024

Disclaimer:
The views expressed in this presentation are those of the author alone and do not necessarily represent those of the author's employer.
All errors are the sole responsibility of the author.

- When we price options, with a European payoff $\varphi(S_1,\ldots,S_n)$ with expiry T, where φ is some payoff function and $S_1,\ldots,S_n$ $(n \geq 1)$ are asset prices from the time interval $[0,T]$, we end up with a problem of evaluating the expectation in the pricing formula

$$\textbf{Option Price} = C \times \mathbb{E}\left[\varphi\left(S_1,\ldots,S_n\right)\right] \tag{1}$$

  for some constant C.
- Standard non-simulation numerical techniques to find $\mathbb{E}\left[\varphi\left(S_1,\ldots,S_n\right)\right]$ are
  - a) numerical solution of PDE via finite difference scheme (Cranck-Nickolson, ADI etc.),
  - b) numerical inversion of Fourier or Laplace transforms,
  - c) closed form analytical solution or their approximations,
  - d) other minor, including recent use of neural networks for decoding model parameters from market data.

  but those might fail for various reasons.
- Method of last resort, but often best $\rightarrow$ **Monte Carlo simulation**

- Variance reduction techniques are often used techniques in financial applications of Monte Carlo methods.
- The goal is to maximize the variance ratio (VR), which is

$$\text{VR} = \frac{\text{Crude MC Sample Variance}}{\text{Estimator's Sample Variance}} \tag{2}$$

- In practice, a reduction of Monte Carlo error brings the opportunity to reduce the actual number of simulations, and still preserve good simulation accuracy in variance terms, so that we have a computational efficiency gain.
- Many variance reduction techniques are case specific, but some can be implemented for a broad class of cases, including pricing derivatives with quite general payoffs.
- The joint work (R. Muchorski, A. Daniluk and E. Lakshtanov) provides a new type of variance reduced Monte Carlo estimator for option pricing, applicable to any LSV modelling framework.

Main idea and properties

- The estimator is easy to compute and allows to reduce the Monte Carlo error even by an order of magnitude, which can be shown in several numerical examples.
- It is applicable to a broad class of option payoffs with a European exercise style (possibly multi-asset and path-dependent) and underlying processes (arbitrary pure diffusion, in general).
- The general idea:

1) Given the dynamics of the underlying process, we introduce another, auxiliary process with simplified dynamics (typically an arithmetic or geometric Brownian motion), for which the payoff and simplified dynamics admit a fast and easy calculation of the option price as a function of the underlying.
2) Then we consider the option pricing function as if the dynamics of the underlying followed this auxiliary process.
3) By setting the original underlying process as an argument to this pricing function, we construct a process whose terminal value is the option payoff. At the same time, we explicitly decompose this process into two parts: a martingale and an integral of a drift, which can be calculated explicitly.

The pricing problem (a simple example for better intution)

- Let's consider a simple case with an asset with price S_t at time t, that is governed by the SDE

$$dS_t = rS_t + \theta_t S_t dW_t \tag{3}$$

  where, for simplicity, we assume some constant short rate, Brownian motion W_t and some continuous stochastic process θ_t.
- Suppose we want to price some European payoff $\phi(S_T)$ with maturity T and that the price of the payoff $\phi(S_T)$ is directly related to the quantity $E\phi(S_T) = E^Q\phi(S_T)$, where Q is the pricing measure.
- Unfortunately, in our model the quantity $E\phi(S_T)$ doesn't have a closed form, analytical expression, so we need to use some suitable numerical method to calculate an accurate approximation of it.

- Now the new idea comes into play. Let's consider a model with an asset satisfying a simpler form of SDE (e.g. with Black-Scholes dynamics), so that our postulated dynamics is

$$dS_t = rS_t + \sigma S_t dW_t \tag{4}$$

with, this time, a constant volatility σ.

- By denoting F_t as the time t filtration of the probability space, let's assume that for the simpler type of dynamics the function $\pi(\cdot)$ such that

$$\pi(t, S_t) = E(\phi(S_T)|F_t) \tag{5}$$

is a C^2 function of (t, s).

- **Remark:** We select the danamics of the simple model such that the function $\pi(t, S_t)$ has a known closed form expression.

- We notice that we can call function $\pi(t, s)$ as the 'pricing function' for the simpler model with the simplified dynamics, since at each time t the value $\pi(t, s)$ is related to the time t price of the payoff in the simpler model.
- We also assume that the set of values of the process with dynamics in (3) is a subset of the values of the process with dynamics in (4), where by 'set of values' of a process X we understand

$$\{(t, x) \colon t \in [0, T], x \in X_t(\Omega)\} \tag{6}$$

  This is needed since we further consider the process $\pi(t, S_t)$, but where S_t has dynamics from (3), not (4).
- The idea is to insert the process with dynamics from the original model to the 'pricing function' form the simplified model. The process $\pi(t, S_t)$ is a well defined stochastic process.
- We assume that this process is square integrable so that we don't have to consider local martingales any more.

Next, we write down the Ito formula for process $\pi(t, S_t)$ with the integration over the interval $[0, T]$.

- For the process S_t with dynamics from (4)

$$\begin{aligned} d\pi(t,S_t) &= \frac{\partial \pi(t,S_t)}{\partial t} dt + \frac{\partial \pi(t,S_t)}{\partial s} dS_t + \frac{1}{2}\frac{\partial^2 \pi(t,S_t)}{\partial s^2} d\langle S \rangle_t \\ &= \left(\frac{\partial \pi(t,S_t)}{\partial t} + rS_t \frac{\partial \pi(t,S_t)}{\partial s} + \frac{\sigma^2 S_t^2}{2} \frac{\partial^2 \pi(t,S_t)}{\partial s^2} \right) dt + \frac{\partial \pi(t,S_t)}{\partial s} \sigma S_t dW_t \end{aligned} \tag{7}$$

- For the process S_t with dynamics from (3)

$$\begin{aligned} d\pi(t,S_t) &= \frac{\partial \pi(t,S_t)}{\partial t} dt + \frac{\partial \pi(t,S_t)}{\partial s} dS_t + \frac{1}{2}\frac{\partial^2 \pi(t,S_t)}{\partial s^2} d\langle S \rangle_t \\ &= \left(\frac{\partial \pi(t,S_t)}{\partial t} + rS_t \frac{\partial \pi(t,S_t)}{\partial s} + \frac{\theta_t^2 S_t^2}{2} \frac{\partial^2 \pi(t,S_t)}{\partial s^2} \right) dt + \frac{\partial \pi(t,S_t)}{\partial s} \theta_t S_t dW_t \end{aligned} \tag{8}$$

- By definition of function $\pi(t,s)$ from (5) (with S_t having the simple dynamics from (4)), the process $\pi(t,S_t)$ is a martingale and hence the SDE for this process, expressed in (7), consist a zero drift, namely

$$\frac{\partial\pi(t,S_t)}{\partial t}+rS_t\frac{\partial\pi(t,S_t)}{\partial s}+\frac{\sigma^2S_t^2}{2}\frac{\partial^2\pi(t,S_t)}{\partial s^2}=0 \tag{9}$$

- Hence, the drift vanishes everywhere, and we also obtain a PDE in the time-state variable space

$$\frac{\partial\pi(t,s)}{\partial t}+rs\frac{\partial\pi(t,s)}{\partial s}+\frac{\sigma^2s^2}{2}\frac{\partial^2\pi(t,s)}{\partial s^2}=0 \qquad (t,s):\ t\in[0,T],s\in S_t(\Omega) \tag{10}$$

- Consequently, the function $\pi(t,s)$ satisfies such PDE and we use this fact by rewriting (8) in a simpler form

$$d\pi(t,S_t)=\frac{1}{2}\frac{\partial^2\pi(t,S_t)}{\partial s^2}\left(\theta_t^2-\sigma^2\right)S_t^2dt+\sigma S_t\frac{\partial\pi(t,S_t)}{\partial s}dW_t \tag{11}$$

- Even though we use the pricing function from the simpler model, with S_t having dynamics from the original model from (3), it is true that

$$\pi(T, S_T) = \phi(S_T) \tag{12}$$

- Consequenly, from (12) and (11), we conclude

$$\phi(S_T) = \pi(0, S_0) + \frac{1}{2}\int_0^T \frac{\partial^2 \pi(t, S_t)}{\partial s^2}\left(\theta_t^2 - \sigma^2\right) S_t^2 dt + \int_0^T \sigma S_t \frac{\partial \pi(t, S_t)}{\partial s} dW_t \tag{13}$$

Finally, we apply the expected value to both sides of (13), which leads to

Main Result

$$\begin{aligned} E\phi\left(S_T\right) &= \pi\left(0,S_0\right)+\frac{1}{2}E\left(\int_0^T \frac{\partial^2\pi\left(t,S_t\right)}{\partial s^2}\left(\theta_t^2-\sigma^2\right)S_t^2 dt\right)) \\ &= \pi\left(0,S_0\right)+\frac{1}{2}\int_0^T E\left(\frac{\partial^2\pi\left(t,S_t\right)}{\partial s^2}\left(\theta_t^2-\sigma^2\right)S_t^2\right)dt \end{aligned} \tag{14}$$

Conclusion: A new MC estimator with good properties

Based on the 'main result', the proposed Monte Carlo, discrete sample estimator can be described as

Input with discretely sampled diffusions

$$\begin{aligned} &S_{t_1}(\omega_k), S_{t_2}(\omega_k), \ldots, S_{t_M}(\omega_k) \quad k = 1, 2, \ldots, N \\ &\theta_{t_1}(\omega_k), \theta_{t_2}(\omega_k), \ldots, \theta_{t_M}(\omega_k) \quad k = 1, 2, \ldots, N \end{aligned} \tag{15}$$

where

- Δ is the step size,
- $t_k = \Delta k$ $(0 = t_0, T = M\Delta)$ compose the discrete set of simualtion dates,
- ω_k corresponds to the simulated path from the k-th simulation (N simulations in total).

Proposed MC estimator

$$\text{MC Estimator}(N, \Delta) = \pi(0, S_0) + \frac{\Delta}{2N} \sum_{k=1}^{N} \sum_{l=0}^{M-1} \frac{\partial \pi(t_l, S_{t_l}(\omega_k))}{\partial s^2} \left(\theta_{t_l}(\omega_k)^2 - \sigma^2 \right) S_{t_l}(\omega_k)^2 \tag{16}$$

Question to the audience: Where does most of the variance reduction come from?

Extensions to the multidimension case

Following analogical reasoning, by assuming asset-specific SDEs

$$dS_{k,t} = rS_{k,t}dt + \theta_{k,t}S_{k,t}dW_{k,t} \quad 1 \leq k \leq n \tag{17}$$

we obtain

$$\begin{aligned} E\left(\phi\left(0, S_{1,T}, \dots, S_{n,T}\right)\right) =& \phi\left(0, S_{1,0}, \dots, S_{n,0}\right) \\ &+ \frac{1}{2}\sum_{i,j=1}^{n}\int_0^T E\left(\frac{\partial^2 \pi(t, S_{1,t}, \dots, S_{n,t})}{\partial s_i \partial s_j}\left(\theta_{i,t}\theta_{j,t} - \sigma_i\sigma_j\right) S_{i,t}S_{j,t}\right) dt \end{aligned} \tag{18}$$

with the multidimensional analogue of our estimator

$$\begin{aligned} &\text{MC Estimator}\left(N, \Delta\right) = \phi\left(0, S_{1,0}, \dots, S_{n,0}\right) \\ &+ \frac{\Delta}{2N}\sum_{k=1}^{N}\sum_{i,j=1}^{n}\sum_{l=0}^{M-1}\frac{\partial^2 \pi\left(t_l, S_{1,t_l}\left(\omega_k\right), \dots, S_{n,t_l}\left(\omega_k\right)\right)}{\partial s_i \partial s_j}\left(\theta_{i,t_l}\left(\omega_k\right)\theta_{j,t_l}\left(\omega_k\right) - \sigma_i\sigma_j\right) S_{i,t_l}\left(\omega_k\right) S_{j,t_l}\left(\omega_k\right) \end{aligned} \tag{19}$$

Reducing the number of simulation steps via stochastic quadratures

- For brevity, considering the single dimension case, in all practical modelling cases the function

$$f\colon (0,T)\to\mathbb{R}\colon f(t):=E\left(\frac{\partial^2\pi(t,S_t)}{\partial s^2}\left(\theta_t^2-\sigma^2\right)S_t^2\right) \tag{20}$$

  is a bounded $C^{+\infty}(0,T)$ function.

- We can approximate the integral over that function with a Gauss-Legendre quadrature (including an affine change of variables), including a change of variables

$$\int_0^T f(t)dt \approx \frac{T}{2}\sum_{l=1}^{L} w_l f(z_l)dt \tag{21}$$

  with quadrature weights w_l and rescaled nodes z_l from $[0,T]$ $(1\le l\le L)$.

- In terms of the Monte Carlo paths, this translates to a 'stochastic quadrature' approximation, instead of the original estimator

$$\begin{aligned}\text{SQ MC Estimator}(N,\Delta) =&\phi\left(0,S_{1,0},\ldots,S_{n,0}\right)\\ &+\frac{T}{4N}\sum_{k=1}^{N}\sum_{i,j=1}^{n}\sum_{l=1}^{L}w_l\frac{\partial^2\pi(z_l,S_{1,z_l}(\omega_k),\ldots,S_{n,z_l}(\omega_k))}{\partial s_i\partial s_j}\Gamma(i,j,k,l)\end{aligned} \tag{22}$$

$$\Gamma(i,j,k,l)=\left(\theta_{i,z_l}(\omega_k)\,\theta_{j,z_l}(\omega_k)-\sigma_i\sigma_j\right)S_{i,z_l}(\omega_k)\,S_{j,z_l}(\omega_k) \tag{23}$$

  which, as we observe in numerical tests, preserves advantageous variance reduction properties, with only an acceptably small loss of accuracy.

- Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a filtered probability space with filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$, $T > 0$ and let W, $\tilde{W}$ be standard R-dimensional ($R \geq 1$), $\mathbb{F}$ -adapted Wiener processes under $\mathbb{P}$. For a given random variable Z on Ω we denote $\mathbb{E}_t Z := \mathbb{E}^{\mathbb{P}}(Z|\mathbb{F}_t)$. Henceforth, we will also assume that all expected values which appear in formulas exist.
- Consider a pair of continuous, square-integrable, d-dimensional ($R \leq n$) diffusion processes
$$X, \tilde{X} : [0,T] \times \Omega \to C \subset \mathbb{R}^n \tag{24}$$
where $\mathcal{C} = (\underline{B}_1, \overline{B}_1) \times \cdots \times (\underline{B}_n, \overline{B}_n)$ for some $-\infty \leqslant \underline{B}_k < \overline{B}_k \leqslant +\infty, k = 1, \ldots, n$.
- We assume that $\tilde{X}$ attains each its point of C , i.e. for any $t \in [0,T], x \in \mathcal{C}$ the density of $\tilde{X}_t$ at x is positive. Suppose that $X, \tilde{X}$ are solutions of the SDEs
$$dX_t = \mu_t dt + \sigma_t dW_t \tag{25}$$
$$d\tilde{X}_t = \tilde{\mu}\left(t, \tilde{X}_t\right) dt + \tilde{\sigma}\left(t, \tilde{X}_t\right) d\tilde{W}_t \tag{26}$$
where
$$\sigma : [0,T] \times \Omega \to \mathcal{M}(n,R), \quad \tilde{\sigma} : [0,T] \times C^n \to \mathcal{M}(n,R) \tag{27}$$
are respectively some stochastic volatility process and a local volatility function, where $\mathcal{M}(n,R)$ is the space of $n \times R$ matrices. Similarly, μ and $\tilde{\mu}$ are stochastic drift vectors.

- Consider a path-dependent payoff of the form $\pi^*\left((X_s)_{s\in[0,T]}\right)$, which is determined for any continuous path of observed values of process X_t. In practice however, as we observe the values of the process at discrete dates $0 = T_0 < T_1 < \ldots < T_n = T$, we instead further scrutinize the problem of pricing the discrete-payoff analogue

$$Z = \pi\left(X_{T_0}, \ldots, X_{T_n}\right) \tag{28}$$

- As in the European case, we introduce similar variables, however this time defined on sub-intervals, namely

$$\tilde{Z}_{t,k} := \pi_k\left(\tilde{Y}_k, (\tilde{X}_s)_{s\in[t,T]}\right) \quad t \in [T_{k-1}, T_k] \tag{29}$$

where π_k is the payoff observed at time $t \in [T_{k-1}, T_k]$ and $\tilde{Y}_k = \left[\tilde{X}_{T_j}\right]_{1\leqslant j<k}$ is a vector of discrete values of $\tilde{X}_t$, observed until the time t.

- We notice that $\tilde{Z}_{T_k,k} = \tilde{Z}_{T_k,k+1}$, since at time T_k the payoffs π_k and π_{k+1} are defined on the same set of historical and future observations.
- Let's consider the conditional expectation $E_t\tilde{Z}_{t,k}$

$$E_t\tilde{Z}_{t,k} = E\left(\tilde{Z}_{t,k}|\tilde{X}_t, \tilde{Y}_k\right) \quad t \in [T_{k-1}, T_k], \quad 1 \leq k \leq n$$

- Following analogical reasoning as in the European payoff case, there exist functions

$$\pi_1, \ldots, \pi_n : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} : \quad E_t\tilde{Z}_{t,k} = \pi_k\left(t, \tilde{X}_t; \tilde{Y}_k\right) \quad t \in [T_{k-1}, T_k] \tag{30}$$

  where $\tilde{Y}_k$ can be viewed as parameters on which the function π_k depends on.
- We also make an important observation that

$$\pi_k(T_k, \cdot; \tilde{y}_k) \equiv \pi_{k+1}(T_k, \cdot; \tilde{y}_{k+1}) \quad \tilde{y}_k = \tilde{Y}_k(\omega), \quad \tilde{y}_{k+1} = \tilde{Y}_{k+1}(\omega), \quad \omega \in \Omega, \quad 1 \leq k \leq n \tag{31}$$

  which is implied by $\tilde{Z}_{T_k,k} = \tilde{Z}_{T_k,k+1}$.
- Next, we define for $t \in [T_{k-1}, T_k)$, $\quad k(t) = \min\left(k : t \leq T_k\right), \quad 1 \leq k \leq n$

$$V_T = \pi_n\left(T, \tilde{X}_T\right), \quad V_t = \pi_{k(t)}\left(t, X_t; Y_{k(t)}\right) \tag{32}$$

  where, similarly, $Y_k = \left[X_{T_j}\right]_{1 \leqslant j < k}$ is a vector of discrete values of X_t, observed until the time t.

Changing the bump coordinates for gamma sensitivity (1/3)

- Another optimization is possible for payoffs including multiple underlyings ($n > 1$) and assuming a special form of the system of SDEs explaining the asset dynamics in (25)-(26). Namely, if each asset is governed by its own SDE, with Wiener processes that are correlated for a given constant correlation matrix $\rho = [\rho_{ij}]_{i,j=1\ldots d}$ with rank R ($1 \leq R \leq n$), so that

$$d\langle W^i, W^j\rangle_t = d\langle \tilde{W}^i, \tilde{W}^j\rangle_t = \rho_{ij} dt \quad 1 \leq i,j \leq R \tag{33}$$

- Then (25)-(26) can be expressed as

$$dX_t = \mu_t dt + diag(\sigma_t) C dW_t \tag{34}$$

$$d\tilde{X}_t = \tilde{\mu}\left(t, \tilde{X}_t\right) dt + diag(\tilde{\sigma}\left(t, \tilde{X}_t\right)) C d\tilde{W}_t \tag{35}$$

with:

  - $n \times R$ matrix $C = [c_{jk}]_{j=1\ldots n, k=1\ldots R}$, such that $CC^T = \rho$, and
  - diagonal matrices $diag(\sigma_t), diag(\tilde{\sigma}\left(t, \tilde{X}_t\right))$ with, respectively, volatilities $\sigma_t, \tilde{\sigma}\left(t, \tilde{X}_t\right)$ on the diagonal.

- With such simplified dynamics, we can further optimize the calculation of

$$\frac{\partial^2 \pi\left(t_l, S_{1,t_l}(\omega_k), \ldots, S_{n,t_l}(\omega_k)\right)}{\partial s_i \partial s_j}\left(\theta_{i,t_l}(\omega_k)\,\theta_{j,t_l}(\omega_k) - \sigma_i \sigma_j\right) S_{i,t_l}(\omega_k)\, S_{j,t_l}(\omega_k) \quad (36)$$

- At the first glance it may seem that in both cases the associated computational cost is $O(n^2)$, since the formulas involve $n \times n$ Hessian matrices. If each Hessian element was to be calculated using a finite difference method, this would require $n^2 + n + 1$ calls of the pricing function π.
- However, we don't really need to calculate the whole Hessian, but only some related quadratic forms, with their alternative expressions, namely

$$\mathbf{tr}\left(\sigma_t^T \gamma(t,x) \sigma_t\right), \quad \mathbf{tr}\left(\tilde{\sigma}(t,x)^T \gamma(t,x) \tilde{\sigma}(t,x)\right) \quad (37)$$

  in the European payoff case, where

$$\gamma(t,x) := \left[\frac{\partial^2 \pi(t,x)}{\partial x_j \partial x_k}\right]_{j,k=1,\ldots,n} \quad (38)$$

- This can be done in linear time $O(n)$!

Changing the bump coordinates for gamma sensitivity (3/3)

Lemma

Let f be a real function that is twice differentiable in $x \in \mathbb{R}^n$, and let $A = diag(\tilde{c})C$, where $diag(\tilde{c})$ is a diagonal matrix with vector $\tilde{c} = [\tilde{c}_1, \ldots, \tilde{c}_n]$ on the diagonal, with positive entries. We define the linear mapping

$$\tau(u) : \mathbb{R}^R \to \mathbb{R}^n \quad \tau(u) := Au^T \tag{39}$$

where u^T is the column vector of the arguments $u \in \mathbb{R}^n$ of τ. Then for

$$u(x) = (u_1(x), \ldots, u_R(x)) : \tau(u(x)) = x \tag{40}$$

holds

$$\nabla^2 (f \circ \tau)(u(x)) = \sum_{i=1}^{R} \frac{\partial^2 (f \circ \tau)}{\partial u_i^2}(u(x)) = \mathbf{tr}\left(A^T \kappa(x) A\right) \tag{41}$$

where

$$\kappa(x) := \left[\frac{\partial^2 f(x)}{\partial x_j \partial x_k}\right]_{j,k=1,\ldots,n} \tag{42}$$

- We performed numerical tests to assess the accuracy of approximations, including the variance reduction effect.
- We focus on only few examples, which concern pricing results only (e.g. without sensitivities).
- For comparison we examined the following types of option payoffs

Description	Payoff
Vanilla: European plain vanilla call	$Z = \max(0, X_T - K)$
Barrier: Down-and-out barrier call (no rebate)	$Z = \max(0, X_T - K) \cdot 1_{m_T > H}, \quad m_T = \min_{0 \leqslant t \leqslant T} X_t$
Asian: Asian call (with quarterly averaging)	$Z = \max(0, X_{avg} - K), \quad X_{avg} = \frac{1}{4T} \sum_{k=1}^{4T} X_{k/4}$
Basket: Call on an basket (equally weighted)	$Z = \max(0, X_{avg} - K), \quad X_{avg} = \frac{1}{10} \sum_{k=1}^{10} X_T^{(k)}$
Rainbow: Call on a maximum of 3 assets	$Z = \max(0, X_{max} - K), \quad X_{max} = \max_{k=1,2,3} X_T^{(k)}$

- For each option type, calculations were done for 2 maturities: $T = 1$ and $T = 5$ and for 2 strikes: at-the-money-forward (approximately) and out-of-the-money.

- For each options type calculations were performed assuming 2 different underlying dynamics, described by Heston and SABR models and parametrized as follows:

Name	Dynamics	Parameters
Heston	$dX_t = X_t\left(rdt + \sqrt{v_t}dW_t\right)$ $dv_t = \kappa(\theta - v_t)dt + \gamma\sqrt{v_t}dZ_t$	$X_0 = 100, r = 0.05$ $\theta = v_0, \kappa = 5, \gamma = 0.3, \rho = -0.1$
SABR	$dX_t = v_t X_t^{\beta} dW_t$ $dv_t = \alpha v_t dZ$	$\alpha = 0.4, \beta = 0.5, \rho = 0$

- In terms of initial values, in all cases we took $X_0 = 100$. In respect of v_0 we differentiate between single-asset and multi-asset options. In case of Vanilla, Barrier and Asian options we took $v_0 = 0.01$ for Heston and $\sigma_0 = 2.5$ for SABR, respectively. In case of basket options, we assumed each asset to have a different v_0 taking the values:
For Heston:

$$v_0 = 0.0036, 0.0049, 0.0064, 0.0081, 0.01, 0.0121, 0.0144, 0.0169, 0.0196, 0.0225 \tag{43}$$

For SABR:

$$v_0 = 1.8, 2.0, 2.2, 2.4, 2.6, 2.8, 3.0, 3.2, 3.4, 3.6 \tag{44}$$

- In case of rainbow options we only tested the SABR model, assuming step size $\Delta t = 0.0002$, maturity $T = 1$ and 3 assets having different v_0 values $v_0 = 2, 2.5, 3$, with equal correlation across all asset price pairs $\rho_{i,j} \equiv 0.4$ $(i \neq j)$ and independent stochastic volatility processes.
- As the simplified, tractable dynamics we used either Black-Scholes or Bachelier model, which in our formalism correspond to $\tilde{\sigma}_t = \tilde{\sigma} X_t^B$, where $B = 1$ for Black-Scholes and $B = 0$ for Bachelier. The values of $\tilde{\sigma}$ were chosen in accordance with the original dynamics, so asset volatilities at the inception coincide $\tilde{\sigma}_0 = \sigma_0$, which is summarized in the following table:

$\tilde{\sigma}_0$	Black-Scholes	Bachelier
Heston	$\sqrt{v_0}$	$100 \cdot \sqrt{v_0}$
SABR	$0.1 \cdot v_0$	$10 \cdot v_0$

Monte Carlo prices (1/2)

Tabela: PV estimate for various option payoffs and simplified dynamics. Comparison of Crude MC average over 1,000,000 simulations with our Riemann sum and Gauss-Legendre quadrature methods, using 5,000 simulations.

Dynamics	Payoff	Simplified	Maturity	Strike	Crude MC	Riemann	Legendre	Z-score
Heston	Vanilla	Black-Scholes	1Y	105	4.1317	4.1555	4.1579	1.56
Heston	Vanilla	Bachelier	1Y	105	4.1293	4.1499	4.1522	1.36
Heston	Vanilla	Black-Scholes	1Y	112	1.6215	1.6323	1.6356	0.97
Heston	Vanilla	Bachelier	1Y	112	1.6225	1.6336	1.6327	0.68
Heston	Vanilla	Black-Scholes	5Y	128	11.5222	11.5450	11.5396	0.57
Heston	Vanilla	Bachelier	5Y	128	11.5223	11.5513	11.5663	1.43
Heston	Vanilla	Black-Scholes	5Y	149	4.5438	4.5421	4.5373	-0.25
Heston	Vanilla	Bachelier	5Y	149	4.5353	4.5607	4.5635	0.78
Heston	Asian	Bachelier	1Y	103	2.8355	2.8530	2.8523	1.47
Heston	Asian	Bachelier	1Y	106	1.5900	1.6049	1.6043	1.34
Heston	Asian	Bachelier	5Y	114	6.3486	6.3651	6.3683	1.06
Heston	Asian	Bachelier	5Y	129	1.7202	1.7308	1.7331	0.78
Heston	Barrier	Black-Scholes	1Y	105	3.6048	3.6213	3.6250	1.59
Heston	Barrier	Black-Scholes	1Y	112	1.4587	1.4709	1.4747	1.24
Heston	Barrier	Black-Scholes	5Y	128	8.0206	7.9970	7.9949	-1.04
Heston	Barrier	Black-Scholes	5Y	149	3.4004	3.3735	3.3650	-1.64
Heston	Basket	Bachelier	1Y	105	2.7273	2.7336	2.7336	1.15
Heston	Basket	Bachelier	1Y	112	0.5639	0.5678	0.5679	1.23
Heston	Basket	Bachelier	5Y	128	7.3833	7.3836	7.3849	0.11
Heston	Basket	Bachelier	5Y	149	1.4158	1.4050	1.4057	-1.03

Tabela: PV estimate for various option payoffs and simplified dynamics. Comparison of Crude MC average over 1,000,000 simulations with our Riemann sum and Gauss-Legendre quadrature methods, using 5,000 simulations.

Dynamics	Payoff	Simplified	Maturity	Strike	Crude MC	Riemann	Legendre	Z-score
SABR	Vanilla	Black-Scholes	1Y	100	10.0623	10.1307	10.1308	1.60
SABR	Vanilla	Bachelier	1Y	100	10.0524	10.1398	10.1419	2.08
SABR	Vanilla	Black-Scholes	1Y	118	3.9621	3.9872	3.9870	0.66
SABR	Vanilla	Bachelier	1Y	118	3.9673	3.9877	3.9883	0.55
SABR	Vanilla	Black-Scholes	5Y	100	22.9696	23.1963	23.1915	0.99
SABR	Vanilla	Bachelier	5Y	100	22.9937	23.2097	23.2169	0.99
SABR	Vanilla	Black-Scholes	5Y	146	9.9313	9.9381	9.9620	0.14
SABR	Vanilla	Bachelier	5Y	146	9.9614	9.9435	9.9269	-0.15
SABR	Asian	Bachelier	1Y	100	6.8722	6.9142	6.9141	1.88
SABR	Asian	Bachelier	1Y	108	3.7524	3.7936	3.7924	1.93
SABR	Asian	Bachelier	5Y	100	13.7470	13.7576	13.7672	0.23
SABR	Asian	Bachelier	5Y	135	2.8534	2.8121	2.8073	-0.56
SABR	Barrier	Black-Scholes	1Y	100	7.7704	7.8016	7.8016	1.23
SABR	Barrier	Black-Scholes	1Y	118	3.2642	3.2857	3.2857	0.72
SABR	Barrier	Black-Scholes	5Y	100	15.8369	15.8120	15.7985	-0.44
SABR	Barrier	Black-Scholes	5Y	146	6.5874	6.5224	6.5333	-0.41
SABR	Basket	Bachelier	1Y	100	7.3148	7.3398	7.3399	1.68
SABR	Basket	Bachelier	1Y	118	1.8283	1.8357	1.8358	0.71
SABR	Basket	Bachelier	5Y	100	16.1575	16.1202	16.1218	-0.61
SABR	Basket	Bachelier	5Y	146	4.2730	4.2318	4.2412	-0.51
SABR	Rainbow	Black-Scholes	1Y	100	20.3989	20.4451	20.4297	0.54
SABR	Rainbow	Black-Scholes	1Y	118	9.5983	9.6501	9.6216	0.43

Tabela: Standard error of PV estimate and variance reduction ratio. Comparison of Crude MC and our Gauss-Legendre quadrature method for 5,000 simulations.

Dynamics	Payoff	Simplified	Maturity	Strike	Crude MC	Legendre	Variance Ratio
Heston	Vanilla	Black-Scholes	1Y	105	0.0914	0.0154	35.1
Heston	Vanilla	Bachelier	1Y	105	0.0915	0.0155	34.7
Heston	Vanilla	Black-Scholes	1Y	112	0.0602	0.0139	18.6
Heston	Vanilla	Bachelier	1Y	112	0.0602	0.0143	17.7
Heston	Vanilla	Black-Scholes	5Y	128	0.2700	0.0240	126.4
Heston	Vanilla	Bachelier	5Y	128	0.2697	0.0240	126.1
Heston	Vanilla	Black-Scholes	5Y	149	0.1776	0.0229	60.2
Heston	Vanilla	Bachelier	5Y	149	0.1772	0.0337	27.6
Heston	Asian	Bachelier	1Y	103	0.0613	0.0105	33.8
Heston	Asian	Bachelier	1Y	106	0.0475	0.0101	22.0
Heston	Asian	Bachelier	5Y	114	0.1430	0.0155	84.6
Heston	Asian	Bachelier	5Y	129	0.0781	0.0156	25.0
Heston	Barrier	Black-Scholes	1Y	105	0.0890	0.0110	65.8
Heston	Barrier	Black-Scholes	1Y	112	0.0578	0.0123	22.1
Heston	Barrier	Black-Scholes	5Y	128	0.2482	0.0175	200.2
Heston	Barrier	Black-Scholes	5Y	149	0.1596	0.0184	74.9
Heston	Basket	Bachelier	1Y	105	0.0573	0.0036	249.9
Heston	Basket	Bachelier	1Y	112	0.0261	0.0026	97.2
Heston	Basket	Bachelier	5Y	128	0.1626	0.0068	579.7
Heston	Basket	Bachelier	5Y	149	0.0723	0.0084	74.4

Monte Carlo errors (2/2)

Tabela: Standard error of PV estimate and variance reduction ratio. Comparison of Crude MC and our Gauss-Legendre quadrature method for 5,000 simulations.

Dynamics	Payoff	Simplified	Maturity	Strike	Crude MC	Legendre	Variance Ratio
SABR	Vanilla	Black-Scholes	1Y	100	0.2342	0.0395	35.2
SABR	Vanilla	Bachelier	1Y	100	0.2343	0.0398	34.6
SABR	Vanilla	Black-Scholes	1Y	118	0.1557	0.0358	18.9
SABR	Vanilla	Bachelier	1Y	118	0.1557	0.0366	18.1
SABR	Vanilla	Black-Scholes	5Y	100	0.7679	0.2172	12.5
SABR	Vanilla	Bachelier	5Y	100	0.7712	0.2197	12.3
SABR	Vanilla	Black-Scholes	5Y	146	0.6262	0.2129	8.7
SABR	Vanilla	Bachelier	5Y	146	0.6299	0.2316	7.4
SABR	Asian	Bachelier	1Y	100	0.1536	0.0194	62.5
SABR	Asian	Bachelier	1Y	108	0.1166	0.0191	37.4
SABR	Asian	Bachelier	5Y	100	0.3658	0.0832	19.3
SABR	Asian	Bachelier	5Y	135	0.2072	0.0817	6.4
SABR	Barrier	Black-Scholes	1Y	100	0.2212	0.0199	123.9
SABR	Barrier	Black-Scholes	1Y	118	0.1443	0.0282	26.2
SABR	Barrier	Black-Scholes	5Y	100	0.6022	0.0760	62.7
SABR	Barrier	Black-Scholes	5Y	146	0.4593	0.1278	12.9
SABR	Basket	Bachelier	1Y	100	0.1618	0.0096	285.1
SABR	Basket	Bachelier	1Y	118	0.0822	0.0089	84.5
SABR	Basket	Bachelier	5Y	100	0.4215	0.0498	71.5
SABR	Basket	Bachelier	5Y	146	0.2452	0.0602	16.6
SABR	Rainbow	Black-Scholes	1Y	100	0.3154	0.0546	33.4
SABR	Rainbow	Black-Scholes	1Y	118	0.2442	0.0515	22.5