Countable spaces, realcompactness, and cardinal characteristics

Lyubomyr Zdomskyy

TU Wien

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A space $X$ is \textit{realcompact}, if $X$ can be closely embedded into $\mathbb{R}^\kappa$ for some cardinal $\kappa$.

The minimal $\kappa$ as above is denoted by $Exp(X)$.

All Lindelöf spaces are realcompact. Thus all metrizable separable and $\sigma$-compact spaces are realcompact.

$k_c(X)$ is the minimal cardinality of a cover of $X$ by compact subspaces. $k_c^*(X) := k_c(\beta X \setminus X)$. 
Motivation

Theorem (van Douwen 1984)
For every $\kappa \leq \aleph_0$ there exists a metrizable separable space $X$ with $\text{Exp}(X) = \kappa$.

Theorem (Hechler)
$\text{Exp}(\mathbb{Q}) = \aleph_0$.

Question
Which cardinals can be realized as $\text{Exp}(X)$ for a countable crowded space $X$?
Main result

Theorem (AMZ 2023)

Let $\kappa$ be an infinite cardinal. Then there exists a countable crowded space $X$ with $\text{Exp}(X) = \kappa$ iff $p \leq \kappa \leq c$.  

The proof consists of four parts:

- No $\kappa < p$ can serve as $\text{Exp}(X)$ for a countable crowded space;
- Producing an example for $\kappa = p$; the core of the proof;
- Modifying an example for $p$ in such a way that it gets $\text{Exp}$ equal to any given $\kappa \in [p, c]$;
- No $\kappa > c$ can serve as $\text{Exp}(X)$ for a countable crowded space.
Proposition
Let $X$ be a Lindelöf space. Then $\text{Exp}(X) = \max\{w(X), kc^*(X)\}$.

Proof. Set $\kappa = \text{Exp}(X)$ and $\kappa' = \max\{w(X), kc^*(X)\}$.

$\kappa \leq \kappa'$: Fix a compactification $\gamma X$ such that $w(\gamma X) = w(X)$. Fix a compact cover $\{K_\xi : \xi \in \kappa'\}$ of $\gamma X \setminus X$.
Using the lindelöfness, find (exercise) a continuous $f_\xi : \gamma X \to [0, 1]$ for $\xi \in \kappa'$ such that $f_\xi(z) = 0$ for every $z \in K_\xi$ and $f_\xi(z) > 0$ for every $z \in X$. Set $\mathcal{F} = \{f_\xi : \xi \in \kappa'\}$.

Fix a collection $\mathcal{G}$ of size at most $w(\gamma X) = w(X)$ consisting of continuous functions $g : \gamma X \to \mathbb{R}$ that separates points of $\gamma X$.
Define $\phi : \gamma X \to \mathbb{R}^{\mathcal{F} \cup \mathcal{G}}$ by $\phi(z)(f) = f(z)$, where $z \in \gamma X$ and $f \in \mathcal{F} \cup \mathcal{G}$.

$\phi$ is an embedding “thanks” to $\mathcal{G}$. Thus

$$\phi[X] = \phi[\gamma X] \cap ((0, \infty)^{\mathcal{F}} \times \mathbb{R}^{\mathcal{G}}),$$

and hence $\kappa = \text{Exp}(X) = \text{Exp}(\phi[X]) \leq \max\{|\mathcal{F}|, |\mathcal{G}|\} = \kappa'$. 
\( \kappa' \leq \kappa \): Assume that \( X \) is a closed subspace of \((0, 1)^\kappa\), and let \( Z = \text{cl}(X) \), where the closure is taken in \([0, 1]^\kappa\). \( Z \) is a compactification of \( X \).

Denote by \( \pi_\xi : [0, 1]^\kappa \rightarrow [0, 1] \) the natural projection on the \( \xi \)-th coordinate. For every \( z \in Z \setminus X \) there exists \( \xi \in \kappa \) such that \( z(\xi) \in \{0, 1\} \). Therefore

\[
Z \setminus X = \bigcup_{\xi \in \kappa} (\pi_\xi^{-1}[\{0, 1\}] \cap Z).
\]

Each \( \pi_\xi^{-1}[\{0, 1\}] \cap Z \) is compact, hence \( \text{kc}^*(X) \leq \kappa \). Since also \( w(X) \leq w((0, 1)^\kappa) = \kappa \), it follows that \( \kappa' \leq \kappa \). \( \square \)
Van Douwen’s and Hechler’s results

Let $X \subset [0, 1]$ be a Bernstein set, $\kappa \leq c$ an infinite cardinal, and $X_\kappa \supset X$ such that $|[0, 1] \setminus X_\kappa| = \kappa$. Then $\text{Exp}(X_\kappa) = \kappa$.

Since $kc^*(\mathbb{Q}) = kc([0, 1] \setminus \mathbb{Q}) = \mathfrak{d}$, $\text{Exp}(\mathbb{Q}) = \mathfrak{d}$. 
Proposition

Let $X$ be a Lindelöf space. Assume that $n \in \omega$ and $X_0, \ldots, X_n$ are Lindelöf subspaces of $X$ such that $X = X_0 \cup \cdots \cup X_n$. Then

$$\text{Exp}(X) \leq \max\{\text{Exp}(X_0), \ldots, \text{Exp}(X_n), w(X)\}.$$ 

Proof.

A straightforward verification that $kc^*(X)$ is also bounded by the maximum above, using a compactification $\gamma X$ of $X$ of weight $w(\gamma X) = w(X)$.
Theorem

Let \( \kappa < \mathfrak{p} \) be an infinite cardinal, and let \( X \) be a countable crowded subspace of \( \omega^\kappa \). Then \( X \) is not closed in \( \omega^\kappa \).

Proof. Define

\[
\mathbb{P} = \{ x \upharpoonright a : x \in X \text{ and } a \in [\kappa]^{< \omega} \}.
\]

Given \( s, t \in \mathbb{P} \), declare \( s \leq t \) if \( s \supseteq t \).

\( \mathbb{P} \) is \( \sigma \)-centered: \( \mathbb{P} = \bigcup_{x \in X} \{ x \upharpoonright a : a \in [\kappa]^{< \omega} \} \). Given \( x \in X \) and \( a \in [\kappa]^{< \omega} \), define

- \( D_x = \{ s \in \mathbb{P} : s(\xi) \neq x(\xi) \text{ for some } \xi \in \text{dom}(s) \} \),
- \( D_a = \{ s \in \mathbb{P} : s = x \upharpoonright b \text{ for some } x \in X \text{ and } b \in [\kappa]^{< \omega} \text{ such that } b \supseteq a \} \).

\( D_x \)'s and \( D_a \)'s are dense in \( \mathbb{P} \). Bell's Theorem yields a filter \( G \) on \( \mathbb{P} \) that meets all of these dense sets. Then \( \bigcup G \in cl(X) \setminus X \), where \( cl \) denotes closure in \( \omega^\kappa \). \( \square \)
Suppose that $X$ is a countable crowded closed subspace of $\mathbb{R}^\kappa$, where $\kappa < p$.

For every $\xi$ pick a countable dense $Q_\xi \subset \mathbb{R} \setminus pr_\xi[X]$.

Then $\mathbb{R} \setminus Q_\xi \equiv \omega^\omega$, and thus $X$ is a closed subspace of $\prod_{\xi \in \kappa}(\mathbb{R} \setminus Q_\xi) \equiv (\omega^\omega)^\kappa$, which is impossible.
There exists a crowded countable $X$ with $Exp(X) = p$, P. 1

Let $\mathbb{P}$ and $\mathbb{P}'$ be posets. $i : \mathbb{P} \longrightarrow \mathbb{P}'$ is a **pleasant embedding**, if

1. $i(1_\mathbb{P}) = 1_{\mathbb{P}'}$,
2. $\forall p, q \in \mathbb{P} \ (p \leq q \rightarrow i(p) \leq i(q))$,
3. $\forall p, q \in \mathbb{P} \ (p \bot q \leftrightarrow i(p) \bot i(q))$.

We will say that $i$ is a **dense embedding** if it satisfies all of the above conditions plus the following:

4. $i[\mathbb{P}]$ is dense in $\mathbb{P}'$.

Also recall that $\mathbb{P}$ is **separative**, if for all $p, q \in \mathbb{P}$ such that $p \not\leq q$ there exists $r \in \mathbb{P}$ such that $r \leq p$ and $r \bot q$.

$\mathbb{P}$ is **meet-friendly** if whenever $p, q \in \mathbb{P}$ are compatible, $\{p, q\}$ has a greatest lower bound, which we denote by $p \land q$.

Note that $p \land q \in \mathcal{F}$ whenever $\mathcal{F}$ is a filter on $\mathbb{P}$ and $p, q \in \mathcal{F}$.

Notice that $\mathbb{P}$ is meet-friendly iff every centered finite subset $\{p_0, \ldots, p_n\}$ of $\mathbb{P}$ has a greatest lower bound, which we will denote by $p_0 \land \cdots \land p_n$.

Example: $\mathbb{B} \setminus \{0_\mathbb{B}\}$, where $\mathbb{B}$ is a boolean algebra.
There exists a crowded countable \( X \) with \( \text{Exp}(X) = p \), P. 2

If \( \mathbb{P} \) is meet-friendly and \( C \) is a non-empty centered subset of \( \mathbb{P} \),

\[
\mathcal{F} = \{ p \in \mathbb{P} : p_0 \land \cdots \land p_n \leq p \text{ for some } n \in \omega \text{ and } p_0, \ldots, p_n \in C \}
\]

is the (smallest) filter \( \mathcal{F} \) generated by \( C \).

Let \( \mathbb{P} \) and \( \mathbb{P}' \) be meet-friendly. A pleasant embedding \( i : \mathbb{P} \to \mathbb{P}' \) is meet-preserving if

\[
(5) \quad \forall p, q \in \mathbb{P} \left( p \nleq q \to i(p \land q) = i(p) \land i(q) \right).
\]

Lemma

Let \( \mathbb{P} \) be a meet-friendly partial order, and let \( \mathcal{F} \) be a filter on \( \mathbb{P} \). Then the following conditions are equivalent:

(A) \( \mathcal{F} \) is an ultrafilter,

(B) \( \forall p \in \mathbb{P} \setminus \mathcal{F} \exists q \in \mathcal{F} (p \perp q) \).

\( \square \)
There exists a crowded countable $X$ with $\text{Exp}(X) = p$, P. 3

**Lemma**

Let $\mathbb{P}$ and $\mathbb{P}'$ be meet-friendly partial orders, and let $i : \mathbb{P} \to \mathbb{P}'$ be a meet-preserving pleasant embedding. If $\mathcal{G}$ is a filter on $\mathbb{P}'$ then $i^{-1}[\mathcal{G}]$ is a filter on $\mathbb{P}$. □

**Lemma**

Let $\mathbb{P}$ be a meet-friendly partial order, let $\mathbb{B}$ be a boolean algebra, and let $i : \mathbb{P} \to \mathbb{B} \setminus \{0\}$ be a pleasant embedding. Assume that $i[\mathbb{P}]$ generates $\mathbb{B}$ as a boolean algebra. If $\mathcal{U}$ is an ultrafilter on $\mathbb{P}$ then $i[\mathcal{U}]$ generates an ultrafilter on $\mathbb{B} \setminus \{0\}$. □.

Given $a, b \in [\omega]^{<\omega}$, we will write $a \preceq b$ to mean $a \subseteq b$ and $b \setminus a \subseteq \omega \setminus \max(a)$. We will also write $a \prec b$ to mean $a \preceq b$ and $a \neq b$.

Given a subset $\mathcal{C}$ of $[\omega]^{\omega}$ with the SFIP, define

$$\mathbb{P}(\mathcal{C}) = \{(a, F') : a \in [\omega]^{<\omega} \text{ and } F' \in [\mathcal{C}]^{<\omega}\}.$$
Order \( \mathbb{P}(\mathcal{C}) \) by declaring \((a, F) \leq (b, G)\) if the following conditions hold:

- \( b \preceq a, \ G \subseteq F \),
- \( a \setminus b \subseteq \bigcap G \).

This is the standard partial order that generically produces a pseudointersection of \( \mathcal{C} \). \( \mathbb{P}(\mathcal{C}) \) is meet-friendly:

If \((a, F) \not\leq (b, G)\), then \((a \cup b, F \cup G)\) is the greatest lower bound of \(\{(a, F), (b, G)\}\).

Recall that \( \mathcal{A} \subset [\omega]^\omega \) is independent, if \( \bigcap_{i \in n} A^\delta_i \) is infinite for any injective \( \langle A_i : i \in n \rangle \in \mathcal{A}^n \) and \( \langle \delta_i : i \in n \rangle \in \{0, 1\}^n \), where \( A^0 = \mathcal{A} \) and \( A^1 = \omega \setminus \mathcal{A} \).

**Proposition (Nyikos)**

There exists an independent family of size \( p \) with no pseudointersection.

**Proof.** Fix an independent family \( \mathcal{A} \) of size \( p \), subset \( \mathcal{C} \) of \( [\omega]^\omega \) of size \( p \) with the SFIP and no pseudointersection. Let \( \mathcal{A} = \{A_\xi : \xi < p\} \) and \( \mathcal{C} = \{C_\xi : \xi < p\} \) be injective enumerations. Set \( \Delta^+ = \{(m, n) \in \omega \times \omega : m \leq n\} \). Then \( \{(A_\xi \times C_\xi) \cap \Delta^+ : \xi < p\} \) is as required. \( \square \)
There exists a crowded countable $X$ with $\text{Exp}(X) = p$, P. 5

Now we can pass to the actual construction

Fix an independent family $\mathcal{A}$ of size $p$ with no pseudointersection. Wlog, for every $n \in \omega$ there exists $A \in \mathcal{A}$ such that $n \notin A$. Set $\mathbb{P} = \mathbb{P}(\mathcal{A})$. For $a \in [\omega]^{<\omega}$, denote by $\mathcal{U}_a$ the filter on $\mathbb{P}$ generated by $\{ (a, F) : F \in [\mathcal{A}]^{<\omega} \}$.

**Claim 1.** Each $\mathcal{U}_a$ is an ultrafilter on $\mathbb{P}$.

**Proof.** Enough to check that if $(b, G) \in \mathbb{P}$ is compatible with every element of $\mathcal{U}_a$, then $b \leq a$ and $a \setminus b \subseteq \bigcap G$, hence $(a, G) \leq (b, G')$. $\square$

**Claim 2.** $\mathbb{P}$ is separative.

**Proof.** Routine, using the independence of $\mathcal{A}$. $\square$

Given $p \in \mathbb{P}$, we set $p \downarrow = \{ q \in \mathbb{P} : q \leq p \}$. $U \subseteq \mathbb{P}$ is open if $p \downarrow \subseteq U$ for every $p \in U$.

$RO(\mathbb{P})$ is the regular open algebra of $\mathbb{P}$.

The map $i : \mathbb{P} \rightarrow RO(\mathbb{P}) \setminus \{0\}$ such that $i(p) = p \downarrow$ for $p \in \mathbb{P}$, is known to be well-defined, dense and meet-preserving embedding, and the following stronger form of condition (2) holds:

(2') $\forall p, q \in \mathbb{P} \left( p \leq q \iff i(p) \leq i(q) \right)$. 
There exists a crowded countable $X$ with $\text{Exp}(X) = \mathfrak{p}$, P. 6

Let $\mathcal{B}$ be the boolean subalgebra of $RO(\mathbb{P})$ generated by $i[\mathbb{P}]$, and let $Z$ be the Stone space of $\mathcal{B}$. Given $b \in \mathcal{B}$, we will denote by $[b] = \{V \in Z : b \in V\}$ the corresponding basic clopen subset of $Z$. It follows that each $i[U_a]$ generates an ultrafilter on $\mathcal{B}$, which we will denote by $\mathcal{V}_a$. Finally, set

$$X = \{\mathcal{V}_a : a \in [\omega]^{<\omega}\}.$$

**Claim 3.** $Z$ is crowded.

*Proof.* This is equivalent to showing that $\mathcal{B}$ has no atoms, which follows from $\mathbb{P}$ having no atoms and (2'). ■

**Claim 4.** $X$ is a countable dense subset of $Z$.

*Proof.* $\bigcup_{a \in [\omega]^{<\omega}} U_a = \mathbb{P}$, and hence $\bigcup_{a \in [\omega]^{<\omega}} \mathcal{V}_a = \mathcal{B} \setminus \{0\}$. ■

It follows from Claims 3 and 4 that $X$ is a countable crowded space, and that $Z$ is a compactification of $X$. Furthermore, $w(X) \leq w(Z) = |\mathcal{B}| = \mathfrak{p}$. Since $\text{Exp}(X) \geq \mathfrak{p}$, it requires to show that $kc(Z \setminus X) = kc^*(X) \leq \mathfrak{p}$. 

17 / 20
There exists a crowded countable $X$ with $\text{Exp}(X) = p$, P. 7

Fix an enumeration $\mathcal{A} = \{A_\xi : \xi \in p\}$. For every $\xi$ set

$$U_\xi = \bigcup_{a \in [\omega]<\omega} [(a, A_\xi) \downarrow],$$

an open subset of $Z$.

**Claim 5.** $X = \bigcap_{\xi \in p} U_\xi$.

**Proof.** $\subseteq$ is straightforward. In order to prove $\supseteq$, pick $V \in \bigcap_{\xi \in p} U_\xi$. Thus, for every $\xi \in p$ we can fix $a_\xi \in [\omega]<\omega$ such that $(a_\xi, A_\xi) \downarrow \in V$. Set $U = i^{-1}[V]$, and observe that $U$ is a filter on $\mathcal{P}$. Also, $(a_\xi, A_\xi) \in U$ for all $\xi$.

Set $a = \bigcup_{\xi \in p} a_\xi$. $a$ is finite, because $a \setminus \max(a_\xi) \subseteq A_\xi$ for all $\xi$.

Fix $\xi \in p$ such that $a = a_\xi$, We check that $U = U_a$, which would give that $V = V_a$, thus concluding the proof.

Since $U_a$ is an ultrafilter, it will be enough to show that $U_a \subseteq U$. So pick $(a, F) \in U_a$, where $F = \{A_{\xi_0}, \ldots, A_{\xi_k}\}$. Note that

$$(a, F \cup \{A_\xi\}) = (a_{\xi_0}, A_{\xi_0}) \land \cdots \land (a_{\xi_k}, A_{\xi_k}) \land (a_\xi, A_\xi) \in U,$$

which clearly implies $(a, F) \in U$, as desired.
Using the CDH property of $\mathbb{R}^\kappa$ for $\kappa < p$, we get that any countable dense $X \subset \mathbb{R}^\kappa$ has a closed copy of any countable space of weight $\leq \kappa$, is a topological group, and $Exp(X) = \emptyset$.

Using our Main Tool one can show that $Exp(X) = \kappa$ for any countable dense $X \subset \mathbb{R}^\kappa$, provided that $\emptyset \leq \kappa \leq c$. In particular, this is true for dense countable subgroups of $\mathbb{R}^\kappa$.

This motivates the following

**Question**

*For which cardinals $\kappa$ such that $p \leq \kappa < \emptyset$ does there exist a countable crowded topological group (homogeneous space) $X$ such that $Exp(X) = \kappa$?*
Thank you for your attention.