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Stochastic volatility
in selected models of financial markets

PhD dissertation

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Author's declaration:

I hereby declare that this dissertation is my own work.

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Supervisors' declaration:

The dissertation is ready to be reviewed.

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Abstract

The thesis is focused on various stochastic volatility (SV) models in finance. Such models were introduced in order to overcome some of the drawbacks of the classical Black-Scholes model of the financial market. In the SV models we drop the assumption of constant volatility of the asset price, allowing it to be a stochastic process.

The first part of the thesis is devoted to the Stein and Stein model, i.e. the model where the volatility is an Ornstein-Uhlenbeck process. It was first introduced in 1991, however the original results were derived under the assumption of uncorrelated noises driving the asset price and its volatility. In the thesis we relax this assumption. We establish closed-form formulas for the moments and the Mellin transform of the asset price. These quantities are then applied to numerical option pricing.

In the second part we study inhomogeneous time change equations (TCEs) induced by Markov chains and their applications to stochastic volatility models, namely to regime-switching diffusions. We show the existence and uniqueness of solutions of the TCE and then we investigate the influence of the TCE on the Markov consistency property and Markov structures of processes. First we focus on the time-changed Markov chains, and then we apply the change of time to a diffusion process, obtaining a regime-switching process. We conclude this part by showing the application of the time change to Monte Carlo option pricing.

Keywords: stochastic volatility, Stein and Stein model, change of time, Markov consistency, regime-switching diffusion, option pricing

AMS Subject Classification: 60J27, 60J60, 91G20, 91G30, 91G60, 60H30

Streszczenie

Niniejsza praca dotyczy modeli rynków finansowych ze stochastyczą zmiennością (stochastic volatility models), czyli modeli, w których zmienność ceny akcji opisywana jest przez pewien proces stochastyczny. Powstały one jako odpowiedź na pewne sprzeczności modelu Blacka-Scholsa wynikające z założenia o stałej zmienności (jak np. obserwowany na rynku uśmiech zmienności).

Pierwsza część pracy poświęcona jest modelowi Steina i Steina, w którym zmienność ceny akcji jest opisywana procesem Ornsteina-Uhlenbecka. Został on wprowadzony w 1991 roku, jednak wyniki Steina i Steina uzyskane zostały przy założeniu o nieskorelowaniu szumów rządzących ceną akcji i jej zmiennością. W pracy wyprowadzamy wzory na momenty oraz transformatę Mellina ceny akcji bez założenia o nieskorelowaniu szumów. Wzory te są następnie zastosowane do numerycznej wyceny różnych typów opcji.

W drugiej części pracy rozważane są niejednorodne równania zmiany czasu dla łańcuchów Markowa, oraz ich zastosowanie w modelach zmienności stochastycznej, a ściślej mówiąc – w modelach przełącznikowych (regime-switching diffusions). Pokazujemy istnienie i jednoznaczność rozwiązania równania zmiany czasu oraz badamy wpływ zmian czasu na markowską zgodność i struktury markowskie procesów. Najpierw opisujemy ów wpływ dla samych łańcuchów Markowa, a następnie badamy proces dyfuzji Itô ze zmienionym czasem (taki proces okazuje się być procesem przełącznikowym) i jego struktury markowskie. Rozdział ten jest zakończony prezentacją zastosowań zmiany czasu do wyceny opcji metodą Monte Carlo.

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Contents

Chapter 1

Introduction

1.1   Volatility in financial markets

The main object of study in this dissertation are models of financial markets with *stochastic volatility*, that is models, where the volatility of an asset price is itself a stochastic process. Before introducing particular models, let us start with a more general question – why should we assume that the volatility is not deterministic?

In 1973 Black and Scholes derived their famous formula for option pricing. The model (based on the hypotheses formulated by Samuelson in 1965 [55]) assumes that the stock price follows a geometric Brownian motion. Let us start with introducing some basic assumptions of the model.

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Throughout this chapter we will assume that $\mathbb{P}$ is a risk-neutral (or a *martingale*) measure, i.e. a probability measure equivalent to the real (observed) probability, such that the discounted prices of assets are martingales. Such a measure, when exists, is used for pricing financial instruments in a way that does not lead to arbitrage. In the Black-Scholes model we assume that the market consists of two assets: a risk-free

asset with price B – bank account with constant interest rate $r \geq 0$, and a risky asset with price S following a geometric Brownian motion. More precisely, the dynamics of B are described by

$$B_0 = 1, \qquad \mathrm{d}B_t = rB_t\mathrm{d}t, \tag{1.1.1}$$

whereas the dynamics of S are

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \tag{1.1.2}$$

where W is a standard Brownian Motion and $\sigma > 0$ is constant. The solutions of (1.1.1) and (1.1.2) take the form

$$B_t = \exp(rt), \qquad S_t = S_0 \exp\left((r - \tfrac{1}{2}\sigma^2)t + \sigma W_t\right).$$

The term σ reflects how much the log returns of the price (i.e. $\log(S_t/S_0)$) vary from their mean, and thus is called the *volatility* of the asset price.

Thanks to the fact that in the Black-Scholes model the price of an asset has a log-normal distribution, one can easily derive closed-form formulas for the price of a European call option, that is an option with payoff $(S_T - K)^+$ at maturity time T.

The paper of Black and Scholes was a breakthrough in financial mathematics and since then the model has been widely used by practitioners. Among its main advantages we must point out its simplicity and tractability – thanks to this, many financial derivatives can be priced either analytically or numerically. However, the model has certain drawbacks that push people to come up with more sophisticated ideas.

The problem appears when we try to calibrate the model to the real data. One of the most popular methods of estimating the parameter σ is to take the volatility *implied* by the market, i.e. to choose its value in such a way that the observed prices of European call options coincide with those from the theoretical model. It follows from the pricing formula that there exists exactly one such positive value of σ for each option in the market. Since in the model the volatility is

a parameter describing the behaviour of a particular asset, it should not depend on the characteristics of an option itself (strike price or time to maturity).

And here the main shortcoming of the model comes to light – it appears that the *implied volatility* $\sigma_{imp}(K, \tau)$ varies for options with different values of strike price K and time to maturity $\tau = T - t$. This phenomenon, due to the shape of curves $\sigma_{imp}(K)$ and $\sigma_{imp}(\tau)$ is called the "volatility smile".
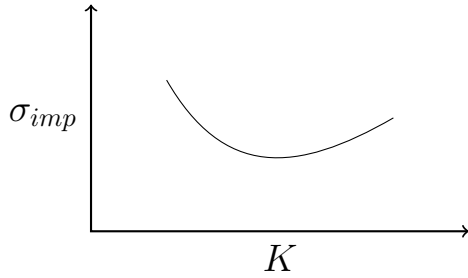
Throughout the years people have been trying to overcome this drawback by introducing more complex models. One of the branches developed over years is a wide class of *stochastic volatility models*, i.e. models where volatility itself is a stochastic process. In greatest generality, the models studied in the dissertation assume the following dynamics of the asset price

$$\mathrm{d}S_t = r_t f_1(S_t)\mathrm{d}t + \sigma_t f_2(S_t)\mathrm{d}W_t,$$

where f_1, f_2 are appropriate deterministic functions, σ is a stochastic process (possibly correlated with the asset price) and r is either a stochastic process or a constant.

There are more stochastic volatility models than one could hope to discuss in a short introduction. In the rest of the section, we will only introduce a few of them, based on their historical importance and later use in the dissertation.

Hull and White model (1987)

One of the first analytically tractable models was proposed by Hull and White [36] in 1987. The variance $v_t := \sigma_t^2$ of the asset price is modelled by a geometric Brownian motion. In other words, the pair (S, v) satisfies

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{v_t}S_t\mathrm{d}W_t^1,$$
$$\mathrm{d}v_t = \mu v_t\mathrm{d}t + \kappa v_t\mathrm{d}W_t^2,$$

where $\mu \in \mathbb{R}$, $\kappa > 0$ are constants and W^1, W^2 are Brownian motions. Under the assumption that W^1 and W^2 are independent, Hull and White determined the price of a European call option in a form of a series. For correlated Brownian motions, i.e., if $\mathrm{d}\langle W^1, W^2\rangle_t = \rho\mathrm{d}t$, for $\rho \in [-1, 1]$, they produced numerical solutions for option prices.

Stein and Stein model (1991)

In [62] Stein and Stein introduced a model where the volatility follows the Ornstein-Uhlenbeck process, i.e. (S, σ) satisfies

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma_t S_t\mathrm{d}W_t^1,$$
$$\mathrm{d}\sigma_t = \lambda(\theta - \sigma_t)\mathrm{d}t + \kappa\mathrm{d}W_t^2,$$

where λ, θ, κ are fixed positive constants. The authors assume that the Brownian motions W^1, W^2 are independent, however, as it is shown in Chapter 3, this restrictive assumption may be relaxed.

In this model σ_t, for each t, has a normal distribution with mean

$$\mathbb{E}\sigma_t = \sigma_0 e^{-\lambda t} + \theta(1 - e^{-\lambda t})$$

and variance

$$\mathrm{Var}(\sigma_t) = \frac{\kappa^2}{2\lambda}\left(1 - e^{-2\lambda t}\right).$$

This property of the model might seem controversial, since it allows for negative values of σ. However, with properly chosen parameters the probability that $\sigma < 0$ is very low. For a more detailed discussion of the properties of the model, see Chapter 3.

Heston model (1993)

In [35] Heston proposed to model the variance $v = \sigma^2$ as a CIR (Cox-Ingersoll-Ross) process, that is

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sqrt{v_t}S_t\mathrm{d}W_t^1,$$
$$\mathrm{d}v_t = \kappa(\theta - v_t)\mathrm{d}t + \xi\sqrt{v_t}\mathrm{d}W_t^2,$$

where W^1, W^2 are (possibly correlated) Brownian motions, and κ, θ and ξ are constants that satisfy

$$\theta > \frac{\xi^2}{2\kappa},$$

so that the process v is positive with probability 1.

The models introduced by Heston [35] and Stein-Stein [62] are amongst the most popular stochastic volatility models. Both are still in use, and both have become the basis of further models, such as models using fractional Brownian motions or Lévy processes instead of the Brownian motion in the dynamics of the volatility (see, e.g, Barndorff-Nielsen and Shephard [6] for stochastic volatility modelled by a Lévy-driven Ornstein-Uhlenbeck process, or [26] and [25] for models with fractional Brownian motion).

Regime-switching models

A slightly different approach to stochastic volatility is demonstrated in regime-switching models. Here, instead of considering σ as a process driven by another

Brownian motion, we assume that it is driven by an independent Markov chain, representing current state of economy, so that

$$\mathrm{d}S_t = \mu(\alpha_t)S_t\mathrm{d}t + \sigma(\alpha_t)S_t\mathrm{d}W_t,$$

where α is a continuous-time Markov chain on a finite state space E, independent of the Brownian motion W. The space E may be seen as the set of possible states of economy (for example bull market vs. bear market), with which the level of volatility in the market is associated. The function $\mu\colon E \to \mathbb{R}$ reflects the sensitivity of an interest rate to the changes of α. Note that, unless the function μ is constant, the model admits a stochastic interest rate. Such models were studied in many monographs and papers, for example [47, 66, 45]. For more details about the model, see Chapter 4.

1.2 Change of time

There are many approaches to tackle the problem of modelling stochastic volatility. One of them is based on the technique of random change of time. The idea is to start with a simpler model where the volatility is deterministic and then use a change of time to introduce randomness to the volatility. By a change of time we mean an increasing, right-continuous family of $[0, \infty]$-valued random variables $(\tau_t)_{t \geq 0}$, such that for all $t \geq 0$, τ_t is a stopping time with respect to a given filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$.

Change of time is a useful tool both from the point of view of the theory of stochastic processes, and from the point of view of applications. It is well known that some processes with complicated structures can be represented as time-changed "simpler" processes (e.g. the famous Dambis-Dubins-Schwarz theorem states that every continuous local martingale M with $M_0 = 0$, $\langle M \rangle_\infty = \infty$ is a time changed Brownian motion) – the same philosophy stands behind applying change of time to financial models. Let us first consider the following example.

Example 1.2.1. Let S be a geometric Brownian motion

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t,$$

where W is an $\mathbb{F}$ - Brownian motion. Consider a change of time τ given by

$$\tau_t = \int_0^t v_s\mathrm{d}s,$$

where the process v is positive, right-continuous and independent of W. Define the process $B_t = \int_0^t \frac{1}{\sqrt{v_s}}\mathrm{d}W(\tau_s)$. Then B is an $(\mathcal{F}(\tau_t))_{t\geq 0}$ – Brownian motion independent of τ, and the process $\hat{S} = S \circ \tau$ satisfies

$$\mathrm{d}\hat{S}_t = rv_t\hat{S}_t\mathrm{d}t + \sigma\sqrt{v_t}\hat{S}_t\mathrm{d}B_t.$$

Thus the term $\sigma\sqrt{v_t}$ is a stochastic volatility, and rv_t is a stochastic interest rate. Moreover, if we assume that $r = 0$ and v is a CIR process, then $(\hat{S}, v)$ follows the Heston model. In order to justify the facts that B is a Brownian motion and that $\hat{S}$ indeed satisfies the equation above, one needs more than these formal calculations. In fact, it requires a deeper theory which is not suitable for an introduction – however, a similar idea is used in the proof of Theorem 4.4.4, and a detailed discussion can be found there.

As shown in the example above, it is possible to obtain a (particular) stochastic volatility model by applying a proper change of time to the geometric Brownian motion. In [18] Carr and Wu generalise this idea – they model the asset price by time-changed exponential Lévy processes. More precisely, in their approach

$$S_t = S_0 e^{rt}\exp(X_{\tau(t)}),$$

where X is a Lévy process and τ is a subordinator (i.e. a nondecreasing Lévy process). Their model was further developed by Linetsky and Mendoza-Arriaga in [48], who assume that X is a d-dimensional process and apply a d-dimensional subordinator τ to X. In this case the change of time is different for each coordinate of the process X. The latter is a particularly interesting technique, since – apart from introducing stochastic volatility to the model – such a change of time may influence the dependence structure of the process X. We will discuss the issues with modelling dependence of multivariate processes in Section 1.3.

The idea of introducing stochastic volatility via change of time may seem unnatural, but it has certain advantages that allow us to calculate (at least numerically) the prices of financial derivatives. The key to obtaining it is the calculation of the characteristic function of a log-price, which may be simplified by introducing change of time. To give the reader a flavour of this idea, let us consider the continuation of Example 1.2.1.

Example 1.2.2. Let S and τ be as in Example 1.2.1. Then the process $\hat{S}_t$ may be written as $\hat{S}_t = S_0 \exp(\hat{X}_t)$, where

$$X_t = \left(r - \frac{1}{2}\sigma^2 \right) t + \sigma W_t \quad \text{and} \quad \hat{X} = X \circ \tau.$$

Then, since τ is independent of W, the characteristic function of $\hat{X}$ can be written as

$$\mathbb{E} \exp\left(iu\hat{X}_t\right) = \mathbb{E} \left(\mathbb{E} \exp\left(iu X_{\tau_t}\right) | \tau_t \right) = \mathbb{E} e^{\tau_t \psi(u)}, \qquad (1.2.1)$$

where ψ denotes the characteristic exponent of X. More precisely, the characteristic function of $\hat{X}$ takes the form

$$\mathbb{E} \exp\left(iu\hat{X}_t\right) = \mathbb{E} \exp\left(\tau_t \cdot \left((r - \tfrac{1}{2}\sigma^2)u - \frac{\sigma^2}{2}u^2 \right) \right).$$

Thus, in order to calculate it one only needs to know the Laplace transform of τ. Note that the equality (1.2.1) holds for any time change τ independent of X. The use of characteristic functions in pricing financial derivatives will be presented in Chapter 2.

1.3 Markov consistency and dependence between stochastic processes

When modelling various phenomena in finance or physics it is convenient (and usually reasonable) to assume that the stochastic processes driving them have the Markov property. A *Markov process* is a process whose future evolution depends only on its current state, not on the path it followed before. In fact, most of the processes mentioned in Section 1.1 are Markov.

In most examples in Section 1.1 we were modelling one-dimensional price processes. Usually it is not difficult to generalize it to the multidimensional case. However, in general it is not always clear how to model the dependence structure of multivariate processes.

When $X = (X_1, \ldots, X_d)$ is a random vector in $\mathbb{R}^d$, the problem of studying dependence between its components reduces to finding an appropriate function, called a *copula function*. More precisely, the Sklar theorem [61] states that there exists such a function C that the cumulative distribution function of the vector $X = (X_1, \ldots, X_d)$ is equal to $C(F_1(\cdot), ..., F_d(\cdot))$, where $F_1, \ldots, F_d$ are the cumulative distribution functions of the coordinates. For example, if $X_1, \ldots, X_d$ are independent, then the resulting C is a product copula, meaning that $C(u_1, \ldots, u_d) = u_1 \cdot \ldots \cdot u_d$. On the other hand, having d one-dimensional random variables $Y_1, \ldots, Y_d$, by applying appropriate copula functions C we can construct all d-dimensional vectors X whose components X_i have the same law as Y_i.

Using this method we can separate the dependence structure of the vector from its marginal distributions. One may ask whether we can use the same reasoning when instead of a random vector, we consider a d-dimensional stochastic process. It appears that in this case we cannot describe dependence structure through a function of marginal distributions – see discussions in Scarsini [56] or in Bielecki et al. [15]. In order to describe the dependence structure of a multivariate process we need to find some relations between the characteristics of its components. The problem of describing structured dependence of stochastic processes was studied in the monograph [14].

If Z is a d-dimensional Markov process, an interesting question is whether its components are Markov themselves. Obviously it is true when the coordinates are independent; however, it may not hold in general.

A Markov process whose components also have Markov property is called *Markov consistent*. More precisely, for a Markov process $Z = (Z^1, \ldots, Z^d)$ on the state space $E_1 \times \cdots \times E_d$, where $E_1, \ldots, E_d$ are Polish spaces, we have the following

definitions (see [13]).

Definition 1.3.1. We say that Z satisfies the *weak* Markovian consistency property with respect to the i-th coordinate if Z^i is a Markov process with respect to its own filtration, i.e. if for any Borel set $B \in \mathcal{B}(E_i)$ and all $t, s > 0$

$$\mathbb{P}(Z^i_{t+s} \in B | \mathcal{F}^{Z^i}_t) = \mathbb{P}(Z^i_{t+s} \in B | Z^i_t),$$

where $\mathbb{F}^{Z^i}$ is the filtration generated by Z^i.

Definition 1.3.2. We say that Z satisfies the *strong* Markovian consistency property with respect to the i-th coordinate if Z^i is a Markov process with respect to the filtration of the process Z, i.e. if for any Borel set $B \in \mathcal{B}(E_i)$ and all $t, s > 0$

$$\mathbb{P}(Z^i_{t+s} \in B | \mathcal{F}^Z_t) = \mathbb{P}(Z^i_{t+s} \in B | Z^i_t),$$

where $\mathbb{F}^Z$ is the filtration generated by the whole process Z.

We say that Z satisfies the weak (strong) Markovian consistency property if it satisfies the weak (strong) Markovian consistency property with respect to all its coordinates.

It is clear that the strong Markovian consistency implies weak Markovian consistency. However, the converse is not always true. One may find examples of weak-only Markov consistent processes in Bielecki et al. [13, Example 3.2].

Consider now d univariate Markov processes $Y^1, \ldots, Y^d$. We would like to construct all d-dimensional Markov processes Z satisfying (weak or strong) Markov consistency property, whose components Z^i have the same finite-dimensional distributions as Y^i. Every such construction will be called a *Markov structure*.

The main advantage of Markov consistency property is that it allows modelling the dependence between Markov processes using rich analytical machinery. The construction of a Markov structure reduces to constructing an infinitesimal generator (or another characteristics of a process) satisfying certain conditions.

Markov structures and Markov consistency find many applications in financial mathematics, whenever we deal with multidimensional Markov processes. For example, in [8] Bielecki, Cousin et al. use it in pricing and hedging Credit Default Swaps (CDS). In [9] Bielecki, A. Vidozzi and L. Vidozzi employ this concept to valuate ratings-triggered corporate bonds, whose cash flows depend on ratings assigned to the issuer by at least two rating agencies. Another applications may be found in Bielecki, Crépey, Jeanblanc [10] or Crépey, Jeanblanc, Zargari [21].

In the world of stochastic volatility models, one may consider a multivariate price process S and a multivariate volatility process σ, and ask whether (S^i, σ^i) is a Markov process itself. Such problems will be considered in Chapter 4.

1.4 Implied volatility and stochastic volatility – practitioners' point of view

"Implied volatility is the wrong number to put in the wrong formula to get the right price." – Riccardo Rebonato [52]

In the real financial markets, the Black Scholes model is often used as a benchmark for pricing financial instruments – the pricing formula is still applied, however, instead of using constant volatility for the whole model, different levels of volatility are plugged in for options with different strikes. Those levels – at least for some of the derivatives – are implied by daily price quotes of specific contracts. However, in order to be able to price various types of derivatives, we would need to construct a volatility model consistent with the market.

Let us take a closer look into how the contracts are priced. Recall that in the Black Scholes model described in 1.1 with constant volatility, the price of a call option $\omega(S_T) = (S_T - K)^+$ at time t is given as

$$C(K, S_t, r, \sigma, T - t) = S_t \mathcal{N}(d_1) - K e^{-r(T-t)} \mathcal{N}(d_2),$$

where $\mathcal{N}$ denotes the cumulative distribution function for the normal distribution and

$$d_1 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}, \quad d_2 = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}.$$

In practice, implied volatility is usually calculated not as a function of K, but as a function of *delta* Δ. Recall that the delta of an option – which is the sensitivity of an option price with respect to the change of the underlying price – in the B-S model is equal to

$$\Delta(K, S_t, T - t, \sigma) := \frac{\partial C}{\partial S_t} = \mathcal{N}(d_1).$$

When traders communicate with brokers, instead of quoting the price of an option in cash, they use the level of its implied volatility, which is then plugged into the Black-Scholes formula. In order to do so, they construct a matrix, which may be seen as a discrete version of an *implied volatility surface*, i.e. the surface $\sigma(T - t, \Delta)$. The columns of the matrix represent the Δ of an option, and the rows – time to maturity $T - t$.

For some deltas (usually $\Delta \in \{0.25, 0.5, 0.75\}$) the prices of options are quoted daily – this is strictly connected to the positions mostly used for hedging, like At The Money, Butterfly and Risk Reversal (see, e.g. Wystup [65] for options in FX markets). This allows us to fill some of the entries of the volatility matrix. The problem arises when there is a need of pricing an unusual contract, for which we can only extrapolate from the data in the matrix. The problem of such extrapolation was studied in many papers (e.g. [19]). However, note that the delta itself is dependent on σ. Hence, in fact, the problem of modelling implied volatility matrix does not reduce to the good choice of interpolation method – it requires a consistent model which does not allow for arbitrage opportunities. In

particular, one needs to be careful about the dependence structure of (S, σ) (see Section 1.3). Stochastic volatility models can be a way to address this problem.

1.5 Discussion of the main results

Before we discuss the main results, let us briefly describe the structure of the thesis.

Chapters 1 and 2 have an introductory character. Chapter 3 is based on a joint work with Jacek Jakubowski and Maciej Wiśniewolski in [37] but it includes additional numerical examples illustrating the techniques developed in the paper. Chapter 4 consists of the author's results from [49] extended by applications of the results to numerical pricing of derivatives.

Let us now discuss the contents of the dissertation in detail.

In Chapter 2 we introduce some numerical techniques of derivative pricing in stochastic volatility models. Those methods are then used further in the dissertation. The need for developing sophisticated numerical methods for pricing is motivated by the fact that for most of stochastic volatility models there are no closed-form formulas neither for the asset price distribution nor for the option price. Section 2.1 is devoted to pricing via Fast Fourier Transform. One of the first papers on applying FFT to pricing options was written by Carr and Madan, see [17]. Their results were then extended by Raible in his PhD dissertation [51]. The main idea, presented in Theorem 2.1.3, is to represent the price of the derivative as a convolution of the so called *modified payoff function* and the density of the logarithmic asset price. Then, knowing the Laplace transforms of the logarithmic price and the modified payoff function, we can easily calculate the Laplace transform of the option price. As a last step, we invert the transform and compute it using the FFT algorithm, which was proposed by Cooley and Tukey in [20] and is one of the most computationally efficient algorithms

to obtain the discrete Fourier transform. Section 2.2 is devoted to numerical option pricing based on calculating a finite sequence of moments of the asset price and then estimating the density given the obtained moments. Knowing the density, we can price any European-style option. This section is based on the results from [22] and [44].

Chapter 3 is devoted to the Stein and Stein model (see Section 1.1) and its main part is based on the joint work with Jacek Jakubowski and Maciej Wiśniewolski [37]. The main results of this part are formulas for moments and the Mellin transform of the asset price; they are then applied in numerical option pricing. Recall that in their original paper [62] Stein and Stein assume that the noises driving the asset's price and its volatility are uncorrelated – in our work we relax this restrictive assumption.

In the first part of the chapter we present three results giving closed formulas of moments of order α of the asset price, under different assumptions on α and on the parameters of the model. These results require different techniques of proof. In Theorem 3.2.1 we consider the process S in the neighbourhood of zero and we find a probability measure under which σ is a Brownian Motion. This allows us to calculate the moments using known formulas for the Laplace transform of some Brownian functionals (see e.g. [46]). The next two results (Theorem 3.2.3 and 3.2.6) are based on the fact that the square of the volatility is a squared radial Ornstein-Uhlenbeck process. The difference between those results comes from different assumptions on the parameters of the model. Hence, different proof strategies are needed for each case. In the first one we can simply apply the results from [39] on the Laplace transform of some functionals of a squared radial Ornstein-Uhlenbeck process. In the second one we cannot directly apply those results, but we use a similar methodology of calculating the moments.

In Theorem 3.2.8 we derive a formula for the Mellin transform of the asset price, which is a generalisation of the result in [59, Appendix A]. Recall that for a positive random variable ξ we define the Mellin transform of ξ as

$$f(z) = \mathbb{E}\xi^z, \quad z \in D,$$

where D is a vertical strip in $\mathbb{C}$ such that ξ^z is integrable for $z \in D$. In fact, the Mellin transform of an asset price is equal to the Laplace transform of its logarithmic price, which is an important quantity, very fruitful in numerical option pricing (see Chapter 2.1).

The rest of the chapter is devoted to the application of the obtained results to option pricing. Apart from applying the numerical methods discussed in Chapter 2, in Proposition 3.3.1 we introduce a method based on the Gil-Pelaez inversion formula [31], which is valid for asymmetric power options (i.e. options with a payoff $\omega(S_t) = (S_T^\alpha - K)^+$). This is yet another application of the Mellin transform in derivatives pricing. In Section 3.3 we compare prices of various European-type options computed using the methods discussed in the chapter. This last part contains examples which were not included in the paper [37].

Another specific example of a stochastic volatility model, namely regime-switching diffusion, is discussed in Chapter 4. The main results come from the author's work [49], however the chapter contains also parts that were not included in the paper. The regime-switching diffusions that appear in the chapter are obtained by applying a change of time to diffusion processes. In fact, the study of such models leads to interesting questions about the time changes themselves, hence the chapter focuses on time changes and some of the more fundamental problems that are not directly related to the regime-switching model.

Throughout the chapter we consider a particular type of change of time, namely a solution of an inhomogeneous time change equation induced by a Markov chain. Such a change of time is defined as a family of random variables satisfying the equation

$$\tau_t = \int_0^t g(s, X_{\tau_s})ds, \tag{1.5.1}$$

for a Borel measurable function $g \colon [0, \infty) \times E \to [0, \infty)$ and a finite, continuous-time Markov chain X. The inhomogeneity refers to the fact that the integrated function g depends on time, which makes the time-changed process $(X_{\tau_t})_{t \geq 0}$ time inhomogeneous.

In the first part of the chapter (Theorem 4.2.3) we prove the existence and uniqueness of solutions of equation (1.5.1) under some mild and very natural assumptions on g. Moreover, we give an explicit construction for the solution, which allows us to simulate the paths of τ. In Theorem 4.2.6 we prove that the time-changed Markov chain $(X_{\tau_t})_{t \geq 0}$ is a (time-inhomogeneous) Markov chain.

The next part of the chapter is devoted to Markov consistency of the time-changed process (see Definition 1.3.2). Consider a two-dimensional Markov chain $X = (X^1, X^2)$ and the change of time induced by it (together with g). In Theorem 4.3.1 and Corollary 4.3.4 we give the conditions that should be satisfied by the function g so that the change of time preserved the Markov consistency property. Moreover, we study whether it is possible to impose Markov consistency by an appropriate change of time. These considerations lead us to define a new class of Markov processes (see Definition 4.3.6), which we call quasi Markov consistent. A process X is called quasi Markov consistent if there exists a non-zero change of time τ such that X_τ is (strongly) Markov consistent. Naturally, such a class contains the class of Markov consistent processes (we can always take the identical change of time), however, in Example 4.3.10 we show that there exists a quasi consistent Markov chain which is not Markov consistent.

In the next part of the chapter we apply the change of time induced by a Markov chain to an n-dimensional diffusion process, which leads to a regime-switching model (see Theorem 4.4.4). In this part we also attempt to answer a reverse question: given an n-dimensional regime-switching diffusion S, when is it possible to represent its coordinates by time-changed diffusions? Such a representation is desirable in many financial applications – not only does it allow us to describe a complicated process as a composition of two simpler processes, but also to consider the coordinates of S without any reference to the dependence structure of the whole process.

The last part of the chapter presents certain examples of applications of the time change to Monte Carlo option pricing. We consider the asset whose price is modelled by the time-changed geometric Brownian Motion. Thanks to the

explicit solution of 1.5.1, we can easily simulate the paths of the time change, and we can use it to Monte Carlo simulations of the asset price process. We analyse time changes for various functions g and then we compare the Monte Carlo prices for those time changes and for several types of options. The Monte Carlo results are new and were not included in the paper [49].

Chapter 2

Option pricing in stochastic volatility models

Unlike in the Black-Scholes model, in most of the stochastic volatility models there are no closed-form formulas for option pricing. Therefore some numerical methods must be applied. In this chapter we give a brief overview of the most popular methods of numerical option pricing, with a special attention to those used in the dissertation. They are all based on computing either the characteristic function of the log price or on the moments of the asset price. The results presented in this chapter are mainly based on the works by Raible [51] (for Section 2.1) and Lin [44] (in Section 2.2).

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a probability space endowed with a filtration satisfying usual conditions. Assume that $\mathbb{P}$ is a martingale measure. Consider a market with a constant interest rate $r \geq 0$ and a risky asset with price $(S_t)_{t \geq 0}$. Throughout this chapter we will assume that S satisfies

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma_t S_t \mathrm{d}W_t,$$

where W is an $\mathbb{F}$ - Brownian motion and σ is a stochastic volatility. We denote by X the log-price of the asset, that is

$$X_t = \log\left(\frac{S_t}{S_0}\right).$$

Recall that by a European (or European-style) option we mean any payoff $\omega(S_T)$ depending only on the value of S at the maturity time T.

2.1 Pricing options with Fast Fourier Transform

The first pricing method that we would like to describe relies on calculating the characteristic function of the logarithmic price of an asset. The price of an option itself is then computed numerically using the Fast Fourier Transform algorithm. We shall start with recalling the definition of the bilateral Laplace transform.

Definition 2.1.1. Let $f \colon \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. The bilateral Laplace transform of f is given by the formula

$$\mathcal{B}\{f\}(z) = \int_{-\infty}^{\infty} e^{-zt} f(t) \mathrm{d}t, \quad z \in D,$$

where D is a vertical strip in $\mathbb{C}$ such that the above integral converges for $z \in D$, i.e. $D = \left\{ z \in \mathbb{C} \colon \int_{-\infty}^{\infty} |e^{-\Re(zt)} f(t)| \mathrm{d}t < \infty \right\}$.

Note that if $f \colon \mathbb{R} \to \mathbb{R}_+$ is a probability density function, then its bilateral Laplace transform is connected with its (generalized) characteristic function (denoted by ϕ_f) by the formula

$$\mathcal{B}\{f\}(z) = \phi_f(iz).$$

The method relies on the following theorem on inverting the Laplace transform. The proof may be found in the book of Widder [64].

Theorem 2.1.2 ([64, Theorem VI.5b]). *Let $f \colon \mathbb{R} \to \mathbb{R}$ be a locally integrable function. Assume there exists $R \in \mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} |e^{-Rt} f(t)| \mathrm{d}t < \infty.$$

Assume additionally that f is of bounded variation in the neighbourhood of $t_0 \in \mathbb{R}$. Then

$$\frac{1}{2}\left(\lim_{t \to t_0^+} f(t) + \lim_{t \to t_0^-} f(t)\right) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{-T}^{T} \mathcal{B}\{f\}(R + yi)e^{(R+yi)t_0}\mathrm{d}y.$$

Consider a European option with payoff $\omega(S_T)$ at fixed time T and let $v(x) := \omega(e^{-x})$ be the modified payoff function. Let $\zeta = -\ln(S_0)$. The following theorem obtained by Raible gives a formula for the price of the option as a function of ζ.

Theorem 2.1.3 ([51, Theorem 3.2]). *Assume that a mapping $x \mapsto e^{-Rx}|v(x)|$ is bounded and integrable for some $R \in \mathbb{R}$, and R is such that $\mathbb{E}e^{-RX_T} < \infty$. Then the price of the option ω at time 0, that is $\Pi_\omega(\zeta)$, is equal to*

$$\Pi_\omega(\zeta) = \frac{e^{\zeta R - rT}}{2\pi} \lim_{s \to \infty} \int_{-s}^{s} e^{iu\zeta}\mathcal{B}\{v\}(R + ui)\phi_{X_T}(-u + Ri)\mathrm{d}u, \qquad (2.1.1)$$

where $\phi_{X_T}(z) := \mathbb{E}e^{izX_T}$, $z \in \mathbb{C}$.

Proof. We will present the sketch of proof for the conveniece of the reader. In [51] the theorem was proved for exponential Lévy models, that is models where $(X_t)_{t \geq 0}$ is a Lévy process. However, this property is not necessary for the proof. Let us denote by f_{X_T} the density function of the log-price X_T. Then the price of the option ω at time 0 as a function of ζ can be formulated as the convolution of f_{X_T} and v. Indeed

$$e^{rT}\Pi_\omega(\zeta) = \mathbb{E}\omega(S_T) = \mathbb{E}\omega(S_0 e^{X_T}) = \mathbb{E}v(\zeta - X_T) = \int_{\mathbb{R}} v(\zeta - x)f_{X_T}(x)dx$$

$$= v * f_{X_T}(\zeta).$$

By the assumptions there exist the bilateral Laplace transforms of v and f_{X_T} at points $R + yi$, $y \in \mathbb{R}$, and since the function $x \mapsto e^{-Rx}|v(x)|$ is bounded, the Laplace transform of Π_ω is equal to

$$\mathcal{B}\{\Pi_\omega\}(R + yi) = e^{-rT}\mathcal{B}\{v\}(R + yi)\mathcal{B}\{f_{X_T}\}(R + yi)$$

$$= e^{-rT}\mathcal{B}\{v\}(R + yi)\phi_{X_T}(-y + Ri)$$

The assertion follows from theorem 2.1.2. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Using this approach one can price any European option as long as it is possible to derive a closed formula for the Laplace transform of its modified payoff function. In Table 2.1 we present the examples of European-style call options together with the Laplace transforms of their modified payoff functions. By $\Pi_\omega(\zeta; K)$ we denote the price of an option with strike K. In the last column we present the relation between the price for general K and the price for $K = 1$. The derivation of the formulas for standard European, symmetric power and self-quanto options can be found in [51, Section 3.4]. The Laplace transform for the asymmetric power options may be obtained following the same method.

Table 2.1: Laplace transforms of the modified payoff functions for European options.

Option	Payoff $\omega(S_T; K)$	Laplace transform $\mathcal{B}\{v(x)\}$, $K = 1$	Price $\Pi_\omega(\zeta; K)$ for general K
Standard European	$(S_T - K)^+$	$\frac{1}{z(1+z)}$, $\Re(z) < -1$	$K\Pi_\omega(\zeta + \ln K; 1)$
Symmetric power	$((S_T - K)^+)^\alpha$	$\frac{\Gamma(-z-\alpha)\Gamma(\alpha+1)}{\Gamma(-z+1)}$, $\Re(z) < -\alpha$	$K^\alpha\Pi_\omega(\zeta + \ln K; 1)$
Asymmetric power	$(S_T^\alpha - K)^+$	$\frac{1}{z(1+z/\alpha)}$, $\Re(z) < -\alpha$	$K\Pi_\omega(\zeta + \frac{\ln K}{\alpha}; 1)$
Self-quanto	$S_T(S_T - K)^+$	$\frac{1}{(z+1)(z+2)}$, $\Re(z) < -2$	$K^2\Pi_\omega(\zeta + \ln K; 1)$

The inverse Laplace transform in the formula (2.1.1) may be approximated by a sum and then calculated using the Fast Fourier Transform algorithm. Given a vector $(g_n)_{n=0,\dots,N-1}$ of complex numbers, the FFT computes the vector $(G_k)_{k=0,\dots,N-1}$, where

$$G_k := \sum_{n=0}^{N-1} e^{2\pi i \frac{nk}{N}} g_n, \qquad (2.1.2)$$

where typically, for the purpose of efficiency, N is an integer power of 2. The algorithm reduces the number of multiplications in the required N summations

from an order of N^2 to that of $N \log_2 N$ and was first introduced by Cooley and Turkey in [20].

In order to apply this method we need to approximate the integral in (2.1.1) by a sum. The integral is of the form

$$\int_{-\infty}^{\infty} e^{iut} g(t) dt,$$

where

$$g(t) = \mathcal{B}\{v(x)\}(R + ti)\phi_{X_T}(-t + Ri).$$

Note that g is a hermitian function in the sense that $g(-t) = \overline{g(t)}$. We approximate the integral using Simpson quadrature. First, we truncate it at points $-A$ and A, where $A \in \mathbb{R}$, and divide the interval $[-A, A]$ into $2N$ small intervals of length $h = \frac{A}{N}$ each. Using Simpson quadrature and the property $g(-t) = \overline{g(t)}$ we obtain

$$\int_{-\infty}^{\infty} e^{iux} g(t) dt \approx \frac{2h}{3} \Re \Big[\sum_{n=0}^{N-1} e^{iunh} g(nh)(3 + (-1)^{n+1} - \delta_n) \Big],$$

where

$$\delta_n = \begin{cases} 1, & n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now taking $u_k = \frac{2\pi k}{A}$ we see that the sum is of the form (2.1.2). By (2.1.1) and Table 2.1 each u_k corresponds to the strike price $K_k = \exp\left(\alpha(\frac{2\pi k}{A} - \zeta)\right)$ in case of asymmetric power options and to $K_k = \exp(\frac{2\pi k}{A} - \zeta)$ in other cases (see the last column in Table 2.1). Thus one computation of FFT returns option prices for the whole sequence of strike prices. Note that $k \geq 0$ implies that in this case we only price out-of-the-money options. However, one can as well compute vector (G_k) for $k = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$, which allows to price also in-the-money options. In fact, the most interesting for the investors are at-the-money options, which correspond to k close to zero. For more details about the application of FFT methods to option pricing see [17] and [51].

2.2 Pricing options based on moments

The pricing method described in the previous section is valid only for the options for which we can derive analytically the Laplace transforms of the modified payoff function. In this section we will present the method which relies on computing moments of the asset price and then estimating the density of the asset price based on the moments. Given the density, we can price numerically any derivative whose payoff is $\omega(S_T)$.

The problem of recovering the unknown density from the sequence of moments was discussed for example in D'Amico et. al [22] and Lin [44]. In particular, Lin proved the following theorem.

Theorem 2.2.1 ([44, Theorem 1]). *Let X be a positive random variable such that $\mathbb{E}X^s < \infty$ for some $s > 0$. Let $(\alpha_j)_{j=0,1,2,\ldots}$ be a sequence of positive and distinct numbers converging to $\alpha \in (0, s)$. Then the sequence of moments $(\mathbb{E}X^{\alpha_j})_{j=0,1,2,\ldots}$ characterizes uniquely the distribution of X.*

In practice we usually deal with finite sequence of moments, and we need to choose a good approximation of the true density. We will use the Maximum Entropy density as the estimator.

Suppose we are given first M elements $\alpha_1, \ldots, \alpha_M$ of the infinite sequence $(\alpha_j)_{j=1,2,\ldots}$ satisfying the conditions from Theorem 2.2.1. Let $\alpha_0 = 0$ and $\mu_j := \mathbb{E}X^{\alpha_j}$ for $j = 0, \ldots, M$. Then we define f_X^M as the solution to the problem of maximization of the entropy

$$H(f) = - \int_0^\infty f(x) \log f(x) \mathrm{d}x \tag{2.2.1}$$

under the constraints

$$\int_0^\infty x^{\alpha_j} f_X^M(x) dx = \mu_j. \tag{2.2.2}$$

In fact (see [22]), the ME density is of the form

$$f_X^M(x) := \exp\Big(-\sum_{j=0}^{M} \lambda_j x^{\alpha_j}\Big), \qquad x \in (0, \infty), \alpha_0 = 0, \qquad (2.2.3)$$

where the coefficients $\lambda_1, \ldots, \lambda_M$ are derived by maximizing 2.2.1 under constraints 2.2.2. For further information on Maximum Entropy density see [22].

Chapter 3

Moments and Mellin Transform in the Stein and Stein model

3.1   Introduction

In this chapter we derive closed-form formulas for the moments and the Mellin transform of the asset price in the Stein and Stein stochastic volatility model. We then apply the results to price power and self-quanto options using numerical methods.

The Stein and Stein model is an example of stochastic volatility model, where the volatility of a financial asset's price is another stochastic process, possibly correlated with the asset price. The subject of stochastic volatility models, very important for financial markets, has been studied in many articles and monographes (see for instance [36, 32, 40, 53]). For most of the stochastic volatility models the closed formulas describing their distribution are not known. Therefore, some numerical techniques have to be applied to obtain derivatives prices, moments and other quantities of the distribution of the asset price (see [60] and [33]). Under some constraints, we deliver closed formulas for moments and the Mellin transform in the Stein and Stein model, one of the important models for applications (see [62] or [29]). In the original paper of Stein and Stein [62],

the closed formulas for asset price distribution were obtained assuming that the noises driving the asset's price and its volatility are uncorrelated. In this chapter we omit this restrictive assumption. The formula for the price of the European call option in the case of correlated noises was obtained by Schöbel and Zhu in [59]. However, our result is more general and may be used for pricing a wide class of financial derivatives.

The motivation for our work comes from the variety of applications of both Mellin transform and moments of the asset price. There are many valuation and calibration methods for stochastic volatility models based on computation of moments, for example the generalized method of moments (see [34, 4]), efficient method of moments (see [3, 30]) or pricing methods of financial derivatives (see [50, 43, 22]). The application described in [22] is also discussed in Chapter 2.2. On the other hand, Mellin transform, which is a Laplace transform of the logarithm of the asset price, finds many applications in option pricing, see Chapter 2.1 for some of them and [17, 51, 28, 57] for many more. Other applications of moments in financial models can be found in [2] or [38]. In this chapter we illustrate how the formulas for moments and Mellin transform can be used in derivatives pricing. Some of the numerical methods employed here were already described in Chapter 2.

The chapter is organized as follows. In Section 2 we derive the formulas for moments and Mellin transform of the asset price. In Section 3 we illustrate our results with some numerical examples of pricing financial derivatives. We present different approaches based on computing either Mellin transform or moments of the underlying's price. The first is a generalization of the result obtained by Schöbel and Zhu to the case of asymmetric power options. The second method consists of computing Laplace transform of the modified payoff function and the Mellin transform of the asset price. The pricing formula demands inverting Fourier transform, which can be done using the Fast Fourier Transform algorithm. This technique is widely used for exponential Lévy models (see [17, 51]), and allows to valuate a wide class of European options. In the third approach we use a finite sequence of fractional moments to find the approximate density of the asset price. Given a density we are able to price nu-

merically any derivative whose payoff depends only on the asset price at time T. We end this section with a comparison between the described methods.

3.2 Moments and Mellin transform of the asset price in the Stein and Stein model

We consider a market defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$, $T < \infty$, satisfying the usual conditions and with $\mathcal{F} = \mathcal{F}_T$. Without loss of generality we assume that $r = 0$, so the savings account B is constant and equal to one. Moreover, we assume that the price S_t of the underlying asset at time t has a stochastic volatility σ_t being an Ornstein-Uhlenbeck process, so the dynamics of the proces S is given by

$$\mathrm{d}S_t = \sigma_t S_t \mathrm{d}W_t, \qquad S_0 = 1 \tag{3.2.1}$$

and the vector (S, σ) is given by system of SDE consisting with (3.2.1) and

$$\mathrm{d}\sigma_t = -\lambda \sigma_t \mathrm{d}t + \mathrm{d}Z_t, \qquad \sigma_0 = 1, \tag{3.2.2}$$

where $\lambda > 0$ is a fixed parameter. The processes W, Z are correlated Brownian motions, $\mathrm{d}\langle W, Z \rangle_t = \rho \mathrm{d}t$ with $\rho \in (-1, 1)$. Observe that the process S has the form

$$S_t = e^{\int_0^t \sigma_u \mathrm{d}W_u - \frac{1}{2} \int_0^t \sigma_u^2 \mathrm{d}u}, \tag{3.2.3}$$

and this is the unique strong solution of SDE (3.2.1) on $[0, T]$. The existence and uniqueness follow directly from the assumptions on σ and the well known properties of stochastic exponent (see, e.g., Revuz and Yor [54]). There is no arbitrage on the market so defined, since the process S is a local martingale. For further information on the Stein and Stein model see Chapter 1.1 and [62].

Notice that we can represent W as

$$W_t = \rho Z_t + \sqrt{1 - \rho^2} V_t, \tag{3.2.4}$$

where (V, Z) is a standard two-dimensional Wiener process. Using (3.2.3) and (3.2.4) we can express the moment of order α of S as

$$
\begin{aligned}
\mathbb{E}S_t^\alpha &= \mathbb{E}\exp\left(\alpha\rho\int_0^t \sigma_u dZ_u + \alpha\sqrt{1-\rho^2}\int_0^t \sigma_u dV_u - \frac{\alpha}{2}\int_0^t \sigma_u^2 du\right) \\
&= \mathbb{E}\exp\left(\alpha\rho\int_0^t \sigma_u dZ_u + \frac{\alpha^2(1-\rho^2)-\alpha}{2}\int_0^t \sigma_u^2 du\right).
\end{aligned}
\tag{3.2.5}
$$

3.2.1 Moments of the asset price

In this subsection we present three results giving closed formulas of moments of order α of the asset price, under different assumptions on α and on the parameters of the model. These results require different techniques of proof. In the first one (Theorem 3.2.1) we consider process S in the neighbourhood of zero and we find a probability measure under which σ is a Brownian Motion. This allows us to apply the known formula for the Laplace transform of a vector $(\sigma_t^2, \int_0^t \sigma_s^2 ds)$. The next two results are based on the fact that $R = \sigma^2$ is a squared radial Ornstein-Uhlenbeck process and are valid for all $t \in [0, T]$. Theorem 3.2.3 covers the case when $\alpha\rho \leq 0$, where we can directly apply the formula for the Laplace transform of some functionals of R, which were calculated in [39]. In Theorem 3.2.6 we consider the case when $\alpha\rho \leq 0$, where we use solve a Cauchy problem in order to obtain $\mathbb{E}e^{-\gamma_1 R_t - \gamma_2 \int_0^t R_u du}$.

The first method relies on a direct calculation of Brownian functionals which appear in (3.2.5). For t in a neighborhood of zero we find an exact value of $\mathbb{E}S_t^\alpha$ for some α.

Theorem 3.2.1. *Let, for fixed $\rho \in (-1, 1), \lambda > 0$,*

$$
a_1 = \frac{1 - 2\rho\lambda - \sqrt{1 + 4\lambda(\lambda - \rho)}}{2(1 - \rho^2)}, \quad a_2 = \frac{1 - 2\rho\lambda + \sqrt{1 + 4\lambda(\lambda - \rho)}}{2(1 - \rho^2)}
$$

and b be the unique positive solution of equation $x(1 - e^{-2x}) = 2$. If

   i) $\rho = 0, \alpha \in (a_1, a_2)$

     or

ii) $\rho \neq 0$, $\alpha \in (a_1, \min(a_2, \lambda/\rho))$,

then for $t \in [0, b/\lambda)$,

$$\mathbb{E}S_t^\alpha = \left(\cosh(\gamma t) + \frac{\lambda - \rho\alpha}{\gamma} \sinh(\gamma t) \right)^{-\frac{1}{2}} \exp\left(\frac{(1 + t)(\lambda - \rho\alpha)}{2} \right)$$
$$\times \exp\left(\frac{\gamma^2 + (\lambda - \alpha\rho)\gamma \coth(\gamma t)}{2\gamma \coth(\gamma t) + 2(\lambda - \alpha\rho)} \right),$$

where

$$\gamma := \left(\frac{\lambda^2}{2} - \alpha\rho\lambda + \frac{\alpha}{2} - (1 - \rho^2)\frac{\alpha^2}{2} \right)^{1/2}. \tag{3.2.6}$$

Proof. Define the measure $\mathbb{Q}$ by

$$\left. \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} \right|_{\mathcal{F}_t} = e^{\lambda \int_0^t \sigma_u \mathrm{d}Z_u - \frac{\lambda^2}{2} \int_0^t \sigma_u^2 \mathrm{d}u}.$$

Clearly, for each t, σ_t is a Gaussian random variable with mean $\mathbb{E}\sigma_t = e^{-\lambda t}$ and variance

$$\mathrm{Var}(\sigma_t) = \frac{1}{2\lambda}(1 - e^{-2\lambda t}).$$

By assumption on t we have $\frac{\lambda t}{2}(1 - e^{-2\lambda t}) < 1$, so using the Jensen inequality and the exact value of $\mathbb{E}e^{\frac{t\lambda^2}{2}\sigma_u^2}$ for $u < t$ we obtain

$$\mathbb{E}e^{\frac{\lambda^2}{2}\int_0^t \sigma_u^2 \mathrm{d}u} \leq \mathbb{E}\left(\frac{1}{t} \int_0^t e^{\frac{t\lambda^2}{2}\sigma_u^2} \mathrm{d}u \right) = \frac{1}{t} \int_0^t \mathbb{E}e^{\frac{t\lambda^2}{2}\sigma_u^2} \mathrm{d}u$$

$$= \frac{1}{t} \int_0^t \frac{1}{\sqrt{1 - \frac{t\lambda}{2}(1 - e^{-2\lambda u})}} \exp\left(\frac{t\lambda^2 e^{-2\lambda u}}{2 - t\lambda(1 - e^{-2\lambda u})} \right) \mathrm{d}u$$

$$< \frac{1}{t} \int_0^t \frac{e^{t\lambda^2/2}}{\sqrt{1 - \frac{t\lambda}{2}(1 - e^{-2\lambda u})}} \mathrm{d}u < \infty.$$

Thus, by the Novikov criterion, $\mathbb{Q}$ is a probability measure. Observe, by the Girsanov theorem, that the process σ, under $\mathbb{Q}$, is a Brownian motion starting

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from 1. Formula (3.2.2) implies

$$\int_0^t \sigma_u \mathrm{d}Z_u = \frac{1}{2}\left(\sigma_t^2 - (t+1)\right) + \lambda \int_0^t \sigma_u^2 \mathrm{d}u, \qquad (3.2.7)$$

so, from (3.2.5), we obtain

$$\mathbb{E}S_t^\alpha = e^{-(t+1)\frac{\alpha\rho-\lambda}{2}} \mathbb{E}_{\mathbb{Q}} e^{\frac{\alpha\rho-\lambda}{2}\sigma_t^2 + \frac{2\lambda\alpha\rho-\lambda^2+\alpha^2(1-\rho^2)-\alpha}{2}\int_0^t \sigma_u^2 \mathrm{d}u}.$$

The assumption on α implies that $-\frac{2\lambda\alpha\rho-\lambda^2+\alpha^2(1-\rho^2)-\alpha}{2} \geq 0$ and $\beta := \frac{1}{2}(\lambda - \rho\alpha) \geq 0$, so γ given by (3.2.6) is well defined and

$$\mathbb{E}S_t^\alpha = e^{(1+t)\beta} \mathbb{E} e^{-\beta B_t^2 - \frac{\gamma^2}{2}\int_0^t B_u^2 \mathrm{d}u},$$

Since $\gamma \geq 0$ and $\beta \geq 0$, we can use the form of the Laplace transform of $(B_t^2, \int_0^t B_u^2 \mathrm{d}u)$, where B is a Brownian motion starting from 1 (see Mansuy and Yor [46] p.18). In result we obtain

$$\mathbb{E}S_t^\alpha = e^{(1+t)\beta}\left(\cosh(\gamma t) + \frac{2\beta}{\gamma}\sinh(\gamma t)\right)^{-\frac{1}{2}} \exp\left(\frac{\gamma^2 + 2\beta\gamma \coth(\gamma t)}{2\gamma \coth(\gamma t) + 4\beta}\right),$$

which finishes the proof. □

Remark 3.2.2. The condition $\alpha_2 > 0$ is satisfied provided that

i) $\rho \in (-1,0)$ and $-\rho \leq \lambda(1+\rho)$

or

ii) $\rho \in [0,1)$.

The second method of computing moments of S relies on the closed form of the Laplace transform of vector $(R_t, \int_0^t R_u du)$, where R is a squared radial Ornstein-Uhlenbeck process. This Laplace transform was computed in Proposition 3.17 in [39].

Recall that a radial Ornstein-Uhlenbeck process with parameters $\delta/2-1, \delta \in \mathbb{R}$ and $\kappa > 0$ is given by

$$\zeta_t = x + 2\int_0^t \sqrt{\zeta_s}\mathrm{d}Z_s + \int_0^t (\delta - 2\kappa\zeta_s)\mathrm{d}s.$$

Then by Proposition 3.17 in [39] we have

$$\mathbb{E}e^{-\eta\zeta_t - \gamma\int_0^t \zeta_u \mathrm{d}u} = e^{-x\phi(t,\eta) - \psi(t,\eta)}$$

for $\eta \geq 0, \gamma > 0$, where

$$\phi(t,\eta) = a + \frac{(\eta - a)(2a + \kappa)}{e^{2t(\kappa+2a)}(a + \kappa + \eta) + a - \eta}, \quad a = \frac{\sqrt{\kappa^2 + 2\gamma} - \kappa}{2},$$

$$\psi(t,\eta) = \delta a t - \frac{\delta}{2}\ln\left|1 + \frac{(\eta - a)(1 - e^{2t(\kappa+2a)})}{e^{2t(\kappa+2a)}(a + \eta + \kappa) + a - \eta}\right|.$$

Observe that if σ satisfies (3.2.2), then $R = \sigma^2$ satisfies SDE

$$\mathrm{d}R_t = 2\sqrt{R_t}\mathrm{d}Z_t + (1 - 2\lambda R_t)\mathrm{d}t, \ R_0 = 1, \ t \geq 0, \tag{3.2.8}$$

which is squared radial Ornstein-Uhlenbeck process with parameters $-\frac{1}{2}$ and λ (see [16] p.141). Since σ given by (3.2.2) satisfies (3.2.7), equality (3.2.5) can be expressed in terms of R as follows

$$\mathbb{E}S_t^\alpha = e^{-\frac{\alpha\rho}{2}(1+t)}\mathbb{E}e^{-\gamma_1 R_t - \gamma_2\int_0^t R_u \mathrm{d}u}, \tag{3.2.9}$$

where

$$\gamma_1 = -\frac{\alpha\rho}{2}, \qquad \gamma_2 = -\alpha\rho\lambda + \frac{\alpha}{2} - (1 - \rho^2)\frac{\alpha^2}{2}. \tag{3.2.10}$$

In the following theorem we use the formula from [39], hence in the assumptions we have to guarantee that $\gamma_1 \geq 0$ and $\gamma_2 > 0$. In particular, the theorem covers only the case when $\alpha\rho \leq 0$.

Theorem 3.2.3. *Let, for fixed $\rho \in (-1, 1), \lambda > 0$,*

$$a_1 = \frac{\rho\lambda}{1 - \rho^2}, \quad a_2 = \frac{1 - 3\rho\lambda}{1 - \rho^2}, \quad b_1 = \min(a_1, a_2), \quad b_2 = \max(a_1, a_2),$$

and $\gamma_1 = -\frac{\alpha\rho}{2}$. Assume that $\alpha \in (b_1, b_2)$ and $\alpha\rho \leq 0$. Then, for any $t \geq 0$,

$$\mathbb{E}S_t^\alpha = e^{-\frac{\alpha\rho}{2}(1+t) - \phi(t,\gamma_1) - \psi(t,\gamma_1)},$$

where

$$\phi(t, z) = a + \frac{(z - a)(2a + \lambda)}{e^{2t(\lambda + 2a)}(a + \lambda + z) + a - z},$$

$$\psi(t, z) = at - \frac{1}{2}\ln\left|1 + \frac{(z - a)(1 - e^{2t(\lambda + 2a)})}{e^{2t(\lambda + 2a)}(a + z + \lambda) + a - z}\right|,$$

$$a = \frac{\sqrt{\lambda^2 + 2\gamma_2} - \lambda}{2}.$$

Proof. We use formula (3.2.9). First, by assumptions on α, we conclude that γ_1 and γ_2 given by (3.2.10) satisfy $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$. Therefore, we can use the closed form of Laplace transform of vector $(R_t, \int_0^t R_u \mathrm{d}u)$ given in Proposition 3.17 in [39]. The result follows. □

The last case we consider is $\alpha\rho \geq 0$. We cannot use directly Proposition 3.17 in [39], but we can follow the methodology of obtaining it. We start from the lemma giving uniqueness of solution of some Cauchy problem.

Lemma 3.2.4. *Fix ρ, λ, and $\gamma \geq -\frac{\lambda^2}{2}$. Let $A \leq \frac{\lambda + \sqrt{\lambda^2 + 2\gamma}}{2}$ and $C = \frac{-\lambda + \sqrt{\lambda^2 + 2\gamma}}{2}$. There exists at most one solution on $[0, \infty) \times [-A, C]$ of the following problem*

$$\frac{\partial p(t, z)}{\partial t} = \left(\gamma - 2\lambda z - 2z^2\right)\frac{\partial p(t, z)}{\partial z} - zp(t, z), \quad p(0, z) = e^{-z}.$$

Proof. We follow the idea from Lemma 2.6 in [39]. Assume that p, q are solutions of (3.2.4) and define $g = (p - q)^2$. We have $g(0, z) = 0$ for every $z \in [-A, C]$, and

$$\frac{\partial g(t, z)}{\partial t} = \left(\gamma - 2\lambda z - 2z^2\right)\frac{\partial g(t, z)}{\partial z} - 2zg(t, z).$$

Let $u(t, z) = \int_z^C g(t, v)\mathrm{d}v$. Observe that for $f(z) = \gamma - 2\lambda z - 2z^2$ we have $f(C) = 0$ and $f(z) \geq 0$ for $z \in [-A, C]$. In result, integrating by parts yields

$$\frac{\partial u(t, z)}{\partial t} = f(z)g(t, z)\Big|_z^C + 2\int_z^C (\lambda + v)g(t, v)\mathrm{d}v \leq 2(\lambda + C)u(t, z)$$

for $z \in [-A, C]$, and the assertion follows from Gronwall's lemma. □

Given the lemma, we can compute the moments of S_t in the case when $\alpha\rho \geq 0$, by solving a Cauchy problem connected to the functionals of the squared Ornstein-Uhlenbeck process. We will first formulate the technical assumptions of the theorem.

Let, for fixed $\rho \in (-1,1), \lambda > 0$,

$$a_1 = \frac{1 - 2\rho\lambda - \sqrt{1 + 4\lambda(\lambda - \rho)}}{2(1 - \rho^2)}, \quad a_2 = \frac{1 - 2\rho\lambda + \sqrt{1 + 4\lambda(\lambda - \rho)}}{2(1 - \rho^2)}.$$

(3.2.11)

Assumption 3.2.5. *Let γ_1 and γ_2 be given by (3.2.10) and let a_1, a_2 be given by (3.2.11). Denote by b the unique positive solution of the equation $x(1 - e^{-x}) = 1$. Assume that*

i) $\rho \in [0, \frac{1}{2\lambda})$, $\alpha \in \left[0, \frac{1 - 2\lambda\rho}{1 - \rho^2}\right)$ and α, t are such that

$$\alpha\rho < \min\left(\frac{\lambda}{1 - e^{-2\lambda t}}, \lambda + \sqrt{\lambda^2 + 2\gamma_2}\right)$$

or

ii) $\alpha \in (a_1, a_2)$, $t \in \left(0, \frac{b}{2\lambda}\right)$ and

$$0 \leq \alpha\rho < \min\left(\frac{\lambda}{2(1 - e^{-2\lambda t})}, \lambda + \sqrt{\lambda^2 + 2\gamma_2}\right).$$

Theorem 3.2.6. *Fix $\rho \in (-1,1), \lambda > 0$. Let γ_1 and γ_2 be given by (3.2.10) and assume that the Assumpion 3.2.5 is satisfied. Then*

$$\mathbb{E}S_t^\alpha = e^{-\frac{\alpha\rho}{2}(1+t) - x_0(t,\gamma_1) - at}\sqrt{1 + \frac{(\gamma_1 - a)(1 - e^{2t(2a+\lambda)})}{e^{2t(2a+\lambda)}(\lambda + a + \gamma_1) - (\gamma_1 - a)}},$$

where

$$a = \frac{\sqrt{\lambda^2 + 2\gamma_2} - \lambda}{2},$$

$$x_0(t, \gamma_1) = a + \frac{(\gamma_1 - a)\sqrt{\lambda^2 + 2\gamma_2}}{e^{2t\sqrt{\lambda^2 + 2\gamma_2}}(1 + \sqrt{\lambda^2 + 2\gamma_2}) - (\gamma_1 - a)}.$$

Proof. Recall from (3.2.9) that

$$\mathbb{E}S_t^\alpha = e^{-\frac{\alpha\rho}{2}(1+t)}\mathbb{E}e^{-\gamma_1 R_t - \gamma_2 \int_0^t R_u du},$$

where R is a radial squared Ornstein-Uhlenbeck process and γ_1 and γ_2 are defined by (3.2.10). Note that the assumptions on α, ρ and t imply that

$$\mathbb{E}\int_0^t R_s e^{-2zR_s - 2\gamma_2 \int_0^s R_u du}ds < \infty \qquad (3.2.12)$$

for all $z \in [\gamma_1, C]$, where $C = \frac{-\lambda + \sqrt{\lambda^2 + 2\gamma_2}}{2}$. Define the function

$$p(s, z) = \mathbb{E}e^{-zR_s - \gamma_2 \int_0^s R_u du}, \quad z \in [\gamma_1, C], \ s \in [0, t].$$

Using (3.2.8), Ito's lemma for $e^{-zR_s - \gamma_2 \int_0^s R_u du}$ and (3.2.12), we easily conclude that p satisfies the Cauchy problem

$$\frac{\partial p}{\partial s} = \left(\gamma_2 - 2\lambda z - 2z^2\right)\frac{\partial p}{\partial z} - zp, \quad p(0, z) = e^{-z}, \qquad (3.2.13)$$

for $z \in [\gamma_1, C]$ and $s \in [0, t]$. Lemma 3.2.4 implies that this Cauchy problem has at most one solution on $[\gamma_1, C]$ since we have

$$-2\gamma_1 = \alpha\rho \le \lambda + \sqrt{\lambda^2 + 2\gamma_2} \quad \text{and} \quad \gamma_2 \ge -\frac{\lambda^2}{2}.$$

We find the unique solution of (3.2.12) using the characteristic method. We start in the first step with the special Riccati equation

$$y' = 2y^2 + 2\lambda y - \gamma_2, \ y(0) = x_0.$$

The solution has the form

$$y(s, x_0) = a + \frac{e^{2s\sqrt{\lambda^2 + 2\gamma_2}}(x_0 - a)\sqrt{\lambda^2 + 2\gamma_2}}{\sqrt{\lambda^2 + 2\gamma_2} + (x_0 - a)(1 - e^{2s\sqrt{\lambda^2 + 2\gamma_2}})}, \qquad (3.2.14)$$

where $a = \frac{\sqrt{\lambda^2 + 2\gamma_2} - \lambda}{2}$ (see Eqworld [1, Section 1 point 8] - the special Riccati equation Case 2). It follows from the last equality that x_0 expressed in terms of y is equal to

$$x_0(s, y) = a + \frac{e^{-2s\sqrt{\lambda^2 + 2\gamma_2}}(y - a)\sqrt{\lambda^2 + 2\gamma_2}}{\sqrt{\lambda^2 + 2\gamma_2} + (y - a)(1 - e^{-2s\sqrt{\lambda^2 + 2\gamma_2}})}.$$

Let $v(s, x_0) = p(s, y(t, x_0))$. Using (3.2.13), we obtain for fixed x_0

$$v'(s, x_0) = \frac{\partial}{\partial s} p\big(s, y(s, x_0)\big) + \frac{\partial}{\partial z} p\big(s, y(s, x_0)\big) y'(s, x_0)$$
$$= y(s, x_0) p\big(s, y(s, x_0)\big) = y(s, x_0) v(s, x_0),$$

which has the solution

$$v(s, x_0) = v(0, x_0) e^{-\int_0^s y(u, x_0) \mathrm{d}u} = e^{-x_0 - \int_0^s y(u, x_0) \mathrm{d}u}.$$

It is not difficult to observe that

$$\int_0^s y(u, x_0) \mathrm{d}u = as - \frac{1}{2} \ln \left(\frac{\sqrt{\lambda^2 + 2\gamma_2} + (x_0 - a)(1 - e^{2s\sqrt{\lambda^2 + 2\gamma_2}})}{\sqrt{\lambda^2 + 2\gamma_2}} \right).$$

The final step of characteristic method yields that

$$p(s, z) = v(s, x_0(s, z))$$

$$= e^{-x_0(s, z) - as} \sqrt{ \frac{\sqrt{\lambda^2 + 2\gamma_2} + (x_0(s, z) - a)(1 - e^{2s\sqrt{\lambda^2 + 2\gamma_2}})}{\sqrt{\lambda^2 + 2\gamma_2}} }$$

$$= e^{-x_0(s, z) - as} \sqrt{ 1 + \frac{(z - a)(1 - e^{2s(2a + \lambda)})}{e^{2s(2a + \lambda)}(\lambda + a + z) - (z - a)} }.$$

The assertion follows after inserting $p(t, \gamma_1)$ in the formula (3.2.9). $\qquad \square$

3.2.2   Mellin transform

In this subsection we obtain the Mellin transform of the asset price under more restrictive assumptions. We start with recalling the definition.

Definition 3.2.7. For a positive random variable ξ we define the Mellin transform of ξ as

$$f(z) = \mathbb{E}\xi^z, \quad z \in D,$$

where D is a vertical strip in $\mathbb{C}$ such that ξ^z is integrable for $z \in D$.

For fixed λ, ρ we put

$$\alpha^* = \frac{1 - 2\lambda\rho}{1 - \rho^2} \tag{3.2.15}$$

and $A = \{z \in \mathbb{C} : \Re z \in [0 \wedge \alpha^*, 0 \vee \alpha^*]\}$. Next, for $z \in \mathbb{C}$ we define

$$s_1 = s_1(z) = \frac{\rho}{2}z, \quad s_2 = s_2(z) = -\frac{z}{2}\big(z(1 - \rho^2) + 2\lambda\rho - 1\big). \tag{3.2.16}$$

Denote $X_t = \ln S_t$. The next theorem generalizes the result obtained by Schöbel and Zhu in [59, Appendix A].

Theorem 3.2.8. *Fix $\lambda > 0$. Assume $\rho \in (-1, 0] \cup [\frac{1}{2\lambda}, 1)$. Fix $z \in A$ and $t \in [0, T]$. Let*

$$\beta_1 = \sqrt{\lambda^2 + 2s_2}, \qquad \beta_2 = \frac{\lambda - 2s_1}{\beta_1}.$$

Then

$$\mathbb{E}(e^{zX_T}|\mathcal{F}_t) = e^{zX_t - \frac{1}{2}z\rho(\sigma_t^2 + (T-t)) + \frac{1}{2}G(T-t)\sigma_t^2 + H(T-t)}, \tag{3.2.17}$$

where

$$G(t) = \lambda - \beta_1 \frac{\sinh(\beta_1 t) + \beta_2 \cosh(\beta_1 t)}{\cosh(\beta_1 t) + \beta_2 \sinh(\beta_1 t)}, \tag{3.2.18}$$

$$H(t) = \frac{1}{2}\Big(\lambda t - \ell\big(\cosh(\beta_1 t) + \beta_2 \sinh(\beta_1 t)\big)\Big), \tag{3.2.19}$$

and ℓ denotes the branch of a complex logarithm of the function $\cosh(\beta_1 t) + \beta_2 \sinh(\beta_1 t)$.

Proof. The idea of the proof is similar to the one in [59], where the formula was derived for $z = 1 + \phi i$ and $z = \phi i$. Here we give a proof for the whole strip A, taking special care of the assumptions on z and ρ, which are chosen in such a way that the assumptions of the Feynman-Kac theorem are satisfied.

First, note that the assumptions on z and ρ imply $\Re s_1 \le 0$ and $\Re s_2 \ge 0$. Indeed, denoting $z = x + yi$ we get $\Re(s_1) = \frac{x\rho}{2} \le 0$ and

$$\begin{aligned}
\Re s_2 &= -\frac{1}{2}x^2(1 - \rho^2) + \frac{1}{2}x(1 - 2\lambda\rho) + \frac{1}{2}y^2(1 - \rho^2) \\
&\ge -\frac{1}{2}x\big((1 - \rho^2)x - (1 - 2\lambda\rho)\big) \ge 0.
\end{aligned}$$

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Hence

$$\mathbb{E}|e^{s_1\sigma_T^2 - s_2\int_0^T \sigma_t^2 dt}| \le 1.$$

From (3.2.1), (3.2.2) and (3.2.4) we obtain

$$
\mathbb{E}\big(e^{zX_T}\big|\mathcal{F}_t\big) = e^{zX_t}\mathbb{E}\Big[\exp\Big(z\int_t^T \sigma_s dW_s - \frac{1}{2}z\int_t^T \sigma_s^2 ds\Big)\Big|\mathcal{F}_t\Big] = e^{zX_t - \frac{1}{2}z\rho(\sigma_t + (T-t))}
$$

$$
\times \mathbb{E}\Big[\exp\Big(\frac{1}{2}\rho z\sigma_T^2 + z\big((\lambda\rho - \frac{1}{2})\int_0^T \sigma_t^2 dt + \sqrt{1-\rho^2}\int_0^T \sigma_t dV_t\big)\Big)\Big|\mathcal{F}_t\Big]
$$

$$
= e^{zX_t - \frac{1}{2}z\rho(\sigma_t^2 + (T-t))}\mathbb{E}\Big(e^{s_1\sigma_T^2 - s_2\int_0^T \sigma_t^2 dt}\Big|\mathcal{F}_t\Big).
$$

Consider a function $p : \mathbb{R} \times [0, T] \to \mathbb{C}$ satisfying the following PDE

$$
\frac{1}{2}\frac{\partial^2 p}{\partial y^2} - \lambda y\frac{\partial p}{\partial y} + \frac{\partial p}{\partial t} - s_2 y^2 p = 0, \quad y \in \mathbb{R}, \ t \in [0, T) \tag{3.2.20}
$$

$$
p(y, T) = e^{s_1 y^2}.
$$

We will seek the solution of the form

$$
p(y, t) = \exp\Big(\frac{1}{2}A(t)y^2 + B(t)\Big).
$$

By (3.2.20), A and B must satisfy system of equations

$$
\begin{array}{llll}
A' & = & -A^2 + 2\lambda A + 2s_2, & A(T) & = & 2s_1 \\
B' & = & -\frac{1}{2}A, & B(T) & = & 0.
\end{array}
$$

The solution to the first equation is of the form

$$
A(t) = \lambda - \beta_1\frac{(\beta_2 + 1)e^{\beta_1(T-t)} + (\beta_2 - 1)e^{-\beta_1(T-t)}}{(\beta_2 + 1)e^{\beta_1(T-t)} - (\beta_2 - 1)e^{-\beta_1(T-t)}}
$$

(see Eqworld [1, Section 1 point 8] – the special Riccati equation Case 2). Next, integrating the above function we derive the formula for B. Denoting

$$
\gamma_1(u) = \frac{\beta_2 + 1}{\beta_2 - 1} - e^{-2\beta_1(T-t)}
$$

$$
\gamma_2(u) = -\frac{\beta_2 - 1}{\beta_2 + 1} - e^{2\beta_1(T-t)}, \quad t \in [0, T]
$$

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we get

$$B(t) = \frac{1}{2}\lambda(T-t) - \frac{1}{4}\left[\int_{\gamma_1} \frac{1}{u}du + \int_{\gamma_2} \frac{1}{u}du\right]$$
$$= \frac{1}{2}\lambda(T-t) - \frac{1}{2}\ell\big(\cosh(\beta_1(T-t)) + \beta_2\sinh(\beta_1(T-t))\big).$$

Note that from the assumptions on z we have $\Re A(t) \le 0$, so for all $y \in \mathbb{R}$ and all $t \in [0, T]$

$$|p(y,t)| = |e^{\frac{1}{2}A(t)y^2 + B(t)}| \le e^{\Re B(t)}.$$

Hence by the Feynman-Kac formula (see [41, Theorem 5.7.6]) p admits a stochastic representation

$$p(y,t) = \mathbb{E}\big(e^{s_1\sigma_T^2 - s_2\int_t^T \sigma_s^2 ds}\big|\sigma_t = y\big), \quad y > 0, \ t \in [0, T].$$

Thus, since

$$\mathbb{E}\big(e^{zX_T}\big|\mathcal{F}_t\big) = e^{zX_t - \frac{1}{2}z\rho(\sigma_t^2 + (T-t))}p(\sigma_t, t),$$

we obtain (3.2.17), where $G(t) = A(T-t)$, $H(t) = B(T-t)$. $\qquad\square$

Remark 3.2.9. From Theorem 3.2.8, taking $t = 0$ and $z = \alpha$, we obtain the formula for $\mathbb{E}S_T^\alpha$. Moreover, if ρ is as in Theorem 3.2.8 and $\alpha \in [0 \wedge \alpha^*, 0 \vee \alpha^*]$, $\alpha^* = \frac{1-2\lambda\rho}{1-\rho^2}$, then it is easy to check that the assumptions of Theorem 3.2.3 are satisfied and we obtain the formula for $\mathbb{E}S_T^\alpha$ which coincides with the result of Theorem 3.2.8.

Remark 3.2.10. To obtain a continuous branch of the complex logarithms defined in the above theorem it is enough to determine the branches of arguments of the integration paths γ_1 and γ_2 which appeared in the proof. Towards this end one has to calculate how many times the paths circuit 0. Note that this depends on the values of β_1 and β_2, so while integrating the Mellin transform $\mathbb{E}S_T^z$ with respect to $\Im z$ (which will be of our interest for the purpose of pricing financial derivatives) one has to estimate this number on each step of integration.

3.3 Option pricing

In this section we present the applications of moments and the Mellin transform computed before in pricing options in Stein and Stein model. We consider the market defined in the previous section, but we allow S_0 to be any number $c > 0$. Note that in this case the process $\tilde{S}_t := \frac{S_t}{c}$ satisfies the equation

$$\mathrm{d}\tilde{S}_t = \sigma \tilde{S}_t \mathrm{d}W_t, \qquad \tilde{S}_0 = 1.$$

Hence we can apply the theorems from Section 2 to $\tilde{S}$. In particular,

$$S_t = ce^{X_t},$$

where $X_t = \ln(\tilde{S}_t)$. Before we present the numerical results (Section 3.3.2), let us discuss the methods used.

3.3.1 Methods of pricing

In the section we compare three different methods of pricing European options, in which use either moments or the Mellin transform of the asset price.

We will start with presenting the application of Mellin transform in pricing power options via Gil-Pelaez inversion formula. The following proposition generalizes the result obtained in [59] (for $\alpha = 1$) to the case of asymmetric power options, that is options with payoff $(S_T^\alpha - K)^+$. Such generalization is possible due to the Mellin transform derived for the whole strip A in $\mathbb{C}$.

Proposition 3.3.1. *Fix $\lambda > 0$. Let ρ be as in Theorem 3.2.8, i.e. $\rho \in (-1, 0] \cup [\frac{1}{2\lambda}, 1)$. Let α^* be given by (3.2.15) and let $\alpha \in [0 \wedge \alpha^*, 0 \vee \alpha^*]$. Then the price of the asymmetric power option in Stein and Stein model is given by*

$$\mathbb{E}(S_T^\alpha - K)^+ = F_1\left(\frac{1}{c}K^{\frac{1}{\alpha}}\right)\mathbb{E}S_T^\alpha - c^\alpha K F_2\left(\frac{1}{c}K^{\frac{1}{\alpha}}\right), \qquad (3.3.1)$$

where

$$F_k(a) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\left(f_k(u) \frac{\exp(-iu \ln a)}{iu} \right) du, \quad k = 1, 2,$$

and

$$f_1(u) = \frac{f(\alpha + iu)}{f(\alpha)}, \qquad f_2(u) = f(iu),$$

for $f(z) = \mathbb{E}e^{zX_T}$.

Proof. We introduce an equivalent change of measure. Let $\tilde{\mathbb{P}}$ be given by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = \frac{\tilde{S}_T^\alpha}{\mathbb{E}\tilde{S}_T^\alpha}.$$

For $a > 0$ denote

$$F_1(a) = \tilde{\mathbb{P}}(X_T > \ln a), \quad F_2(a) = \mathbb{P}(X_T > \ln a).$$

Then

$$\mathbb{E}(S_T^\alpha - K)^+ = \mathbb{E}S_T^\alpha \, \tilde{\mathbb{P}}\left(X_T > \frac{\ln K}{\alpha} - \ln c \right) - c^\alpha K \mathbb{P}\left(X_T > \frac{\ln K}{\alpha} - \ln c \right)$$
$$= \mathbb{E}S_T^\alpha F_1(K^{\frac{1}{\alpha}}) - c^\alpha K F_2(K^{\frac{1}{\alpha}}).$$

Let f_1 and f_2 be the characteristic functions of x_T respectively in measures $\tilde{\mathbb{P}}$ and $\mathbb{P}$. Then

$$f_1(u) = \mathbb{E}_{\tilde{\mathbb{P}}} e^{iuX_T} = \frac{1}{\mathbb{E}\tilde{S}_T^\alpha} \mathbb{E}\tilde{S}_T^\alpha e^{iuX_T} = \frac{f(\alpha + iu)}{f(\alpha)}.$$

The assertion follows from the Gil-Pelaez inversion formula [31]. $\qquad\square$

Note that numerical computation of the integrals appearing in the proposition is not trivial due to the oscillatory behaviour of the integrand and even for simple quadratures computing one integral is time-consuming. If we are interested in pricing options for a whole sequence of strike prices K_n, this method may appear to be too slow. In the following section we will compare it with two other methods.

The second method relies also on calculating the Mellin transform of the asset price and is a direct application of the Theorem 2.1.3. Recall that for an option with payoff $\omega(S_T)$ we define its modified payoff function as $v(x) = \omega(e^{-x})$. Then – under some technical assumptions (see Chapter 2.1) – the price of an option, as a function of $\zeta := -\ln(S_0)$, can be calculated as

$$
\begin{aligned}
\Pi_\omega(\zeta) &= \frac{e^{\zeta R}}{2\pi} \lim_{s\to\infty} \int_{-s}^{s} e^{iu\zeta} \mathcal{B}\{v(x)\}(R+ui)\phi_{X_t}(-u+Ri)\mathrm{d}u \\
&= \frac{e^{\zeta R}}{2\pi} \lim_{s\to\infty} \int_{-s}^{s} e^{iu\zeta} \mathcal{B}\{v(x)\}(R+ui)f(-R-ui)\mathrm{d}u,
\end{aligned}
\tag{3.3.2}
$$

where $f(z) = \mathbb{E}e^{zX_t} = \mathbb{E}\tilde{S}_T^z$. The result does not depend on R as long as it lies in the right interval (see Theorem 2.1.3).

In order to compute the inverse Laplace transform from 3.3.2 we will use the Fast Fourier Transform algorithm described in Section 2.1. Note that this allows us to compute option's prices for various strikes at once.

The third method is based on computing a finite sequence moments of the asset price $(\mathbb{E}S_T^{\alpha_i})_{i=1,\dots,N} = (\mu_i)_{i=1,\dots,N}$ and then estimating the density of S_T as the Maximum Entropy density f^{ME} subject to the constraints $\int_0^\infty x^{\alpha_j} f^{ME}(x) = \mu_j$ for $j = 1, \dots, N$. Given the density, we can calculate numerically any derivative whose payoff depends only on the price S_T at time T. For more details about the method see Section 2.2 and the references therein.

3.3.2 Numerical examples

In this section we compare prices of various European options computed using methods presented above. The results obtained by all the 3 methods are very close and confirm the accuracy of implementation (see also the discussion below).

We first consider an asymmetric power option in Stein and Stein model with

parameters $S_0 = 100$, $\rho = -0.5$, $\lambda = 1$, where the powers are $\alpha = 1$ (standard European call option), $\alpha = \frac{1}{2}$ and $\alpha = \frac{1}{3}$. We price the options following Maximum Entropy method (ME), Fast Fourier Transform (FFT) and the Gil-Pelaez inversion formula from Proposition 3.3.1 (GP). The results for different values of α are displayed on Figure 3.1.

The prices are computed for a sequence of strike prices $(K_k)_{k=-50,-49,\ldots,50}$, where $K_k = \exp\left(\alpha\left(\frac{2\pi k}{hN} - \zeta\right)\right)$ for $\zeta = \log(S_0)$, $N = 2^{13}$ and the discretization step $h = 0.05$.
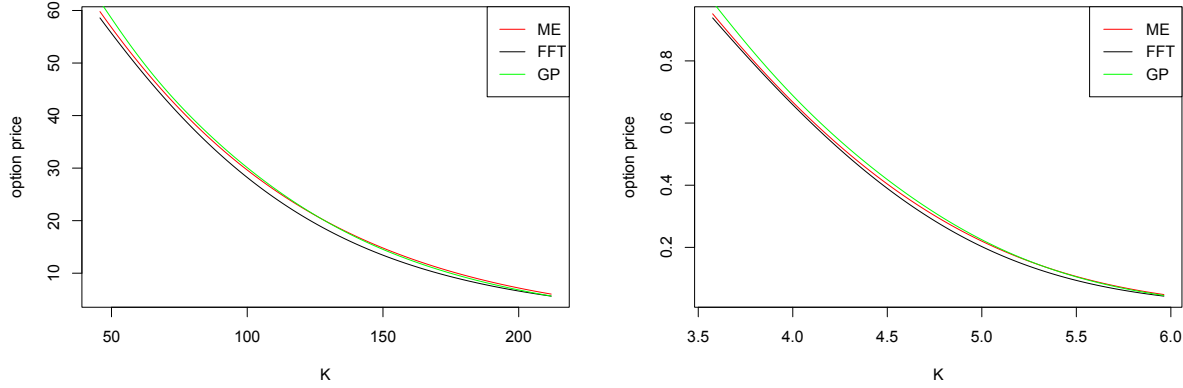

Figure 3.1: Prices of standard European options (left) and asymmetric power options for $\alpha = \frac{1}{3}$ (right) for varying values of strike price K.

All three methods give similar results, however there are some differences, mostly for at-the-money options. For small values of the strike price K the ME method and FFT method prove to be consistent, while the price in GP method increases faster as the strike price tends to zero. In case of the options with high strike prices all three methods give the same results.

In the next example we price self-quanto options for $S_0 = 50$, $\rho = -0.5$ and $\lambda = 1$. Recall that a self-quanto option is an option with payoff $\omega(S_T) = S_T(S_T - K)^+$ (see also Table 2.1). The results are shown in Figure 3.3. In this case the the FFT and ME methods give similar results for K in the neighborhood of 70, however when K decreases, the ME price increases faster than the
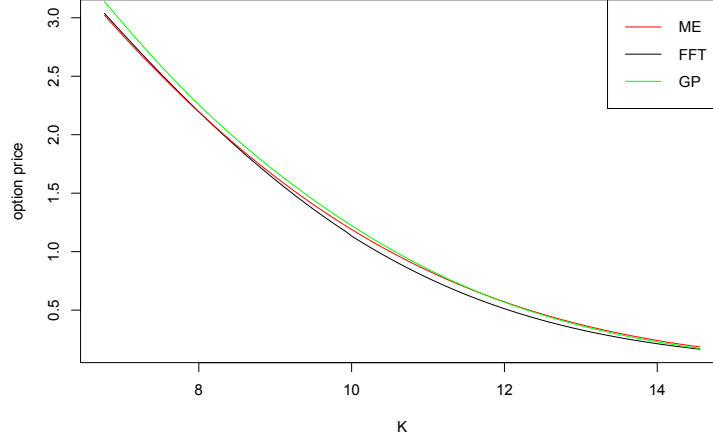
Figure 3.2: Prices of asymmetric power options for $\alpha = \frac{1}{2}$ and varying values of strike price K.

FFT price. Detailed results are presented in Table 3.1, where we separate the cases where the prices are the most consistent.

Note that for large number M of strike prices the FFT method seems to be the fastest, since it returns all the prices in one computation. The ME method appears to be of similar complexity as FFT, as the time used by the process was (in average) only $30 - 50\%$ longer than in case of FFT. However, the most time-consuming in this approach is computing the density, which can be done once for a given set of parameters. As it was noted before, the GP approach requires calculating M integrals of high computational complexity, making it the most time-consuming among the three methods. In fact, computation via GP formula performed to be in average five times slower than FFT for all types of options.
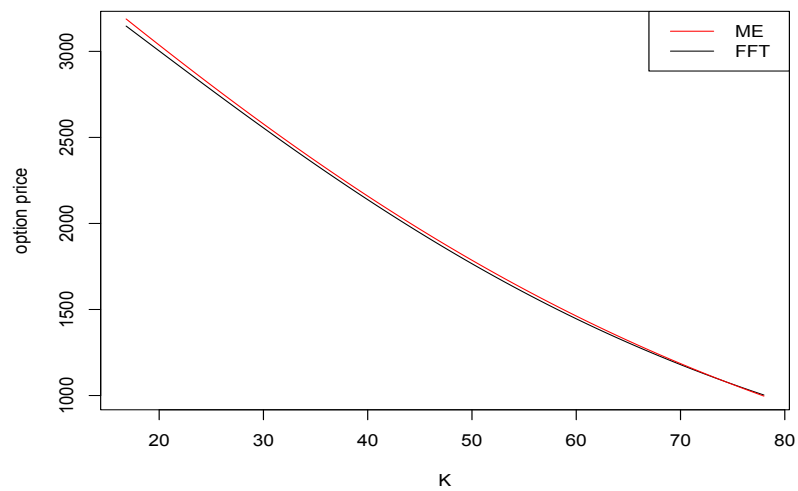
Figure 3.3: Prices of self-quanto options, $S_0 = 50$, $\rho = -0.5$, $\lambda = 1$.
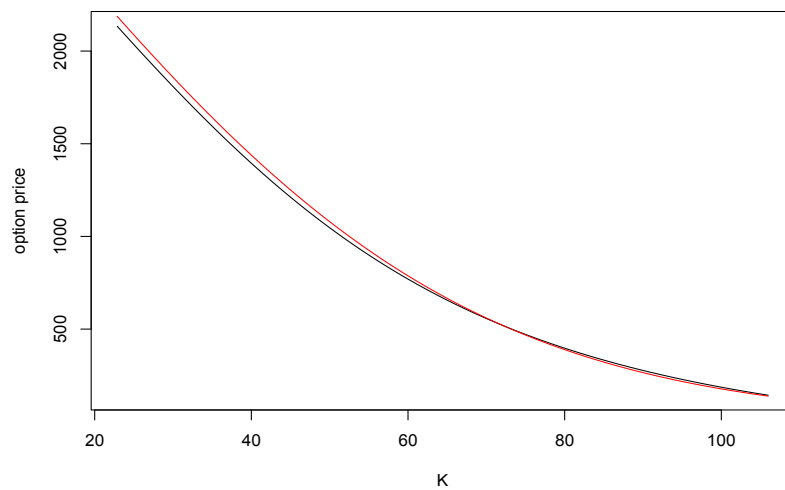

Figure 3.4: Prices of self-quanto options, $S_0 = 50$, $\rho = -0.2$, $\lambda = 2.5$

Table 3.1: Prices of self-quanto options for selected strike prices K and various model parameters

Strike K	$\rho = -0.5,\ \lambda = 1$		$\rho = -0.2,\ \lambda = 2.5$	
	FFT	ME	FFT	ME
50	1765.96	1786.21	1046.70	1079.50
60.10	1443.03	1459.08	767.25	784.84
68.99	1204.09	1211.39	575.94	580.10
71.14	1152.52	1156.97	536.44	537.74
72.24	1127.02	1129.92	517.15	517.07
73.35	1101.74	1102.98	498.17	496.76
74.49	1076.67	1076.18	479.51	476.82
80.42	955.15	944.62	391.16	383.04
85.51	863.13	843.03	326.59	315.71

Chapter 4

Time change equations for Markov chains – Markov consistency and regime-switching diffusions

4.1 Introduction

In this chapter we consider inhomogeneous time change equation induced by Markov chains, that is a family of random variables satisfying the equation

$$\tau_t = \int_0^t g(s, X_{\tau_s})ds,$$

for a Borel measurable function $g\colon [0,\infty) \times E \to [0,\infty)$ and a finite Markov chain X. The inhomogeneity refers to the fact that the integrated function g depends on time, which makes the time-changed process $(X_{\tau_t})_{t\geq 0}$ time inhomogeneous. In the first part of the paper we prove existence and uniqueness of solution of the equation under mild assumptions on g.

The time change equations in the homogeneous case, i.e. when $g(s,x) \equiv g(x)$, were studied for general Markov processes in the book of Ethier and Kurz [27, Chapter 6], and more recently by Krühner and Schnurr in [42]. For the inhomogeneous case, there is a recent paper of Döring et al. [23], who prove the exis-

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tence and uniqueness of solution for general Markov processes. However, their assumptions on g are rather strict and technical, whereas for Markov chains the assumptions on g are quite natural and one can give an explicit construction of solution. Such a construction is very useful for applications, since it allows to simulate the paths of τ, and hence also the paths of $(X_{\tau_t})_{t \geq 0}$ (see Section 4.5).

Change of time is a useful tool both from the point of view of the theory of stochastic processes, and from the point of view of applications. It is well known that some processes with complicated structures can be represented as time-changed "simpler" processes (e.g. the famous Dambis, Dubins, Schwarz theorem states that every continuous local martingale M with $M_0 = 0$, $\langle M \rangle_\infty = \infty$ is a time changed Brownian motion). The inhomogeneous time change equations were recently applied to solve the general Skorohod embedding problem in the paper of Döring et al. [24]. Some of the other techniques in stochastic analysis that are based on change of time may be found in the book of Barndorff-Nielsen and Shiryaev [7]. On the other hand, there are various applications of the change of time in mathematical finance – for example to introduce stochastic volatility to the model (see e.g. Carr, Wu [18] in the context of Lévy processes or Mendoza-Arriaga, Linetsky [48] for multivariate subordination). In particular, representing asset price process with stochastic volatility as time-changed exponential Lévy process (as in [18]) simplifies the calculation of characteristic function of the log-price, which may be used for numerical pricing of financial derivatives. Other applications in mathematical finance are summarized in the article by Swischchuk [63].

In Section 4.2 we prove existence and uniqueness of solution of equation (4.2.1) and study some properties of the time-changed Markov chain. Section 4.3 is devoted to the influence of the change of time on a Markov consistency property of X. This part of our study is a continuation of works by Bielecki et al. in e.g. [11, 12, 13, 15]. Recall that a d-dimensional Markov process Z is Markov consistent if all of its components have Markov property. More precisely, for $i \in \{1, \dots, d\}$ we have the following definition (see [13]).

Definition 4.1.1. We say that Markov process $Z = (Z^1, \dots, Z^d)$ on the state space $E_1 \times \cdots \times E_d$ satisfies strong Markovian consistency property with respect

to the i-th coordinate if Z^i is a Markov process with respect to the filtration of the process Z, i.e. if for any $B \in \mathcal{B}(E_i)$ and all $t, s > 0$

$$\mathbb{P}(Z^i_{t+s} \in B | \mathcal{F}^Z_t) = \mathbb{P}(Z^i_{t+s} \in B | Z^i_t),$$

where $\mathbb{F}^Z$ is the filtration generated by the whole process Z. We say that Z satisfies the strong Markovian consistency property if it satisfies the strong Markovian consistency property with respect to the i-th coordinate for all $i \in \{1, \ldots, d\}$.

Since in the paper we focus mainly on the strong Markov consistency property, we will refer to it as a Markov consistency property.

In this part we formulate the conditions on g for a change of time to preserve Markov consistency property. Moreover, we study whether it is possible to impose Markov consistency property via an appropriate change of time. These considerations lead us to define a new class of Markov processes, which we call quasi Markov consistent. The class consists of Markov processes for which there exists a nontrivial change of time, such that the time-changed process satisfies strong Markov consistency property. Such a class is a generalisation of Markov consistent processes. We formulate sufficient condition for X to be quasi Markov consistent and provide an example of a quasi consistent process which is not Markov consistent.

In Section 4.4 we give a financial application to the studied change of time. We apply the change of time to an n-dimensional diffusion process S, representing the prices of n assets. In this approach τ_t may be regarded as "business time" at calendar time t. The random activity of business day comes from randomness of economy states and causes the prices to have a stochastic volatility. More precisely, the time-changed asset price process is a regime-switching diffusion, where the regimes represent the states of economy. In the second part of this section we consider a reverse problem – given an n-dimensional regime-switching diffusion, we want to represent its coordinates by time-changed diffusions. More precisely, we consider a two-dimensional regime-switching diffusion $S = (S^1, S^2)$ and answer a question whether there exist one-dimensional

diffusions R^1 and R^2 and changes of time τ^1 and τ^2 such that the law of S^i is equal to the law of $R(\tau^i)$ for $i = 1, 2$. Such a representation is desirable in many financial applications – not only does it allow to describe a complicated process as a composition of two simpler processes, but also to consider both coordinates S^1 and S^2 without any reference to the dependence structure of S.

In Section 4.5 the reader will find certain examples of applications of the time change to Monte Carlo option pricing. We consider the asset whose price is modelled by the time-changed geometric Brownian Motion. Thanks to the explicit construction of the time change obtained in Section 4.2, we can easily simulate its paths. The paths then can be used in Monte Carlo simulation of the asset price. The states of the Markov chain that induces the time change can be seen as a reflection of the economic conditions, where some states correspond to higher business activity, and some –lower business activity. The function g in the definition of the TCE is responsible for the sensitivity of the time change to the changes in the market. Therefore we analyse the time changes and the asset price processes for various functions g. We also compare Monte Carlo prices depending on a change of time, for several option types.

4.2 Time change equation and its properties

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space. Throughout this paper we will assume that $\mathbb{F}$ satisfies usual conditions. Let X be a càdlàg $\mathbb{F}$-Markov chain with values in a finite state space E. Therefore it admits an intensity matrix (a generator) being a Q-matrix, which we denote by $\Lambda = [\lambda_y^x]_{x,y \in E}$.

Recall that by a Q-matrix we mean a measurable function $Q \colon E \times E \times \mathbb{R}^+ \to \mathbb{R}$ satisfying conditions:
   (i) $q_{ij}(t) \geq 0$ for $i \neq j$, $i, j \in E$ for all $t > 0$,
  (ii) $\sum_{j \in E} q_{ij} = 0$ for all $i \in E$ and $t > 0$,
 (iii) for all $i, j \in E$ function $t \mapsto q_{ij}(t)$ is locally integrable.

Let $(\tau_t)_{t\geq 0}$ be a family of random variables satisfying the equation

$$\tau_t = \int_0^t g(s, X_{\tau_s})\mathrm{d}s \quad \mathbb{P} - a.s., \tag{4.2.1}$$

where $g \colon [0, \infty) \times E \to [0, \infty)$ is a Borel measurable function such that for all $x \in E$ functions $s \mapsto g(s, x)$ are right-continuous and locally integrable. It is clear that the family $(\tau_t)_{\tau \geq 0}$ satisfying (4.2.1) is nondecreasing and continuous in t with values in $[0, \infty)$.

We shall start with recalling the definition of a change of time.

Definition 4.2.1. A family of random variables $(\theta_t)_{t\geq 0}$ is said to be a random change of time if

   (i) it is a nondecreasing, right-continuous family of $[0, \infty]$-valued random variables

   (ii) θ_t is an $\mathbb{F}$-stopping time for all $t \geq 0$.

Remark 4.2.2. If $\mathbb{F}$ satisfies usual conditions and θ is a change of time, then the time-changed filtration $(\mathcal{F}_{\theta_t})_{t\geq 0}$ also satisfies usual conditions. Indeed, the right continuity is a consequence of right-continuity of $\mathbb{F}$ and the right-continuity and monotonicity of θ. The fact that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets implies that $\mathcal{F}_{\theta_0} \supset \mathcal{F}_0$ contains them as well.

In this section we will show that equation (4.2.1) admits a unique solution which satisfies the definition of a random change of time.

Theorem 4.2.3. *Let X be as above. Suppose $g : [0, \infty) \times E \to [0, \infty)$ is such that the function $s \mapsto g(s, x)$ is right-continuous and locally integrable for all $x \in E$. Then equation (4.2.1) admits a unique solution.*

Proof. We shall start with proving uniqueness of solution. Suppose that τ is a solution to (4.2.1). Let $T_0 = 0$ and $T_1, T_2, \ldots$ be the moments of jumps of X,

i. e. $T_n = \inf\{t > T_{n-1} : X_t \neq X_{T_{n-1}}\}$ for $n \geq 1$. Let us define

$$\rho_0 = 0, \qquad \rho_n = \inf\{t > 0 : \tau_t \geq T_n\}, \quad n = 1, 2, \ldots$$

$$\sigma_0 = 0, \qquad \sigma_n = \inf\left\{t > 0 : \int_{\sigma_{n-1}}^t g(s, X_{T_{n-1}})\mathrm{d}s \geq T_n - T_{n-1}\right\}, \quad n = 1, 2, \ldots$$

We will show that any solution of (4.2.1) is of the form

$$\tau_t = \sum_{n=1}^{\infty} \int_{\sigma_{n-1}\wedge t}^{\sigma_n \wedge t} g(s, X_{T_{n-1}})\mathrm{d}s. \tag{4.2.2}$$

Note that from the definition of ρ_n, $T_n \leq \tau_s < T_{n+1}$ for $s \in [\rho_n, \rho_{n+1})$, so $X_{\tau_s} = X_{T_n}$. It suffices to show that $\rho_n = \sigma_n$ a.s. for all $n \in \mathbb{N}$, which we will prove by induction. For $n = 0$ it is satisfied trivially.

Suppose now that $\rho_n = \sigma_n$ for some $n \in \mathbb{N}$. If $\rho_n = \sigma_n = \infty$, then $\rho_{n+1} = \sigma_{n+1} = \infty$. Assume that $\rho_n = \sigma_n < \infty$. Then for any $s \in [\rho_n, \rho_{n+1})$

$$\int_{\sigma_n}^s g(u, X_{T_n})\mathrm{d}u = \int_{\rho_n}^s g(u, X_{\tau_u})\mathrm{d}u = \tau_s - T_n < T_{n+1} - T_n,$$

so by the definition of σ_{n+1} we have $s < \sigma_{n+1}$. Hence $\rho_{n+1} \leq \sigma_{n+1}$.
We will prove now the converse inequality. Assume first that $\sigma_{n+1} < \infty$. Then by the previous inequality $\rho_{n+1} < \infty$, so

$$\int_{\sigma_n}^{\rho_{n+1}} g(u, X_{T_n})\mathrm{d}u = \int_{\rho_n}^{\rho_{n+1}} g(u, X_{\tau_u})\mathrm{d}u = T_{n+1} - T_n,$$

hence $\rho_{n+1} \geq \sigma_{n+t}$.
Consider the case when $\sigma_{n+1} = \infty$ and suppose $\rho_{n+1} < \infty$. Then

$$T_{n+1} - T_n > \int_{\sigma_n}^{\rho_{n+1}} g(u, X_{T_n})\mathrm{d}u = \int_{\rho_n}^{\rho_{n+1}} g(u, X_{\tau_u})\mathrm{d}u = T_{n+1} - T_n,$$

which is a contradiction. Hence in both cases $\rho_{n+1} = \sigma_{n+1}$. By induction we conclude that $\rho_n = \sigma_n$ a.s. for all $n \in \mathbb{N}$. It then follows that the solution of (4.2.1) is of the form (4.2.2).

To prove the existence we need to show that

$$\tilde{\tau}_t := \sum_{n=1}^{\infty} \int_{\sigma_{n-1} \wedge t}^{\sigma_n \wedge t} g(s, X_{T_{n-1}}) \mathrm{d}s$$

satisfies (4.2.1). Towards this end it suffices to show that for $s \in [\sigma_n, \sigma_{n+1})$, $X(\tilde{\tau}_s) = X(T_n)$. It is easy to see that for all $n \in \mathbb{N}$ on $\{\sigma_n < \infty\}$ we have $\tilde{\tau}(\sigma_n) = T_n$. Hence for $s \in [\sigma_n, \sigma_{n+1})$ we have

$$T_n \leq \tilde{\tau}_s = T_n + \int_{\sigma_n}^{s} g(u, X_{T_n}) du < T_n + (T_{n+1} - T_n) = T_{n+1}$$

and therefore $X(\tilde{\tau}_s) = X(T_n)$.

$\square$

Theorem 4.2.4. *For all $t \geq 0$, τ_t given by (4.2.2), being a unique solution of (4.2.1), is an $\mathbb{F}$ - stopping time.*

Proof. Take any $t \geq 0$ and $u \geq 0$. We will show that $\{\tau_t \leq u\} \in \mathcal{F}_u$. Note that

$$\{\tau_t \leq u\} = \bigcup_{n=1}^{\infty} \{T_{n-1} \leq u < T_n, \ \tau_t \leq u\}$$

$$= \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n-1} \{T_{n-1} \leq u < T_n, \ T_{k-1} \leq \tau_t < T_k\}$$

$$\cup \bigcup_{n=1}^{\infty} \{T_{n-1} \leq u < T_n, \ T_{n-1} \leq \tau_t \leq u\}.$$

Therefore it suffices to investigate sets of the form

$$\{T_{n-1} \leq u < T_n, \ T_{k-1} \leq \tau_t < T_k\} \quad \text{for } k \leq n-1$$

and

$$\{T_{n-1} \leq u < T_n, \ T_{k-1} \leq \tau_t \leq u\}.$$

It is easy to observe that for any $k \in \mathbb{N}$ a random time σ_k is $\mathcal{F}_{T_k}$ - measurable, so

$$\{T_{k-1} \leq \tau_t < T_k\} = \left\{ T_{k-1} \leq T_{k-1} + \int_{\sigma_{k-1}}^{t} g(s, X_{T_{k-1}}) \mathrm{d}s < T_k \right\} \in \mathcal{F}_{T_k} \subset \mathcal{F}_{T_{n-1}}.$$

Hence $\{T_{n-1} \leq u < T_n, \ T_{k-1} \leq \tau_t < T_k\} \in \mathcal{F}_u$. Similarly we show that $\{T_{n-1} \leq u < T_n, \ T_{n-1} \leq \tau_t \leq u\} \in \mathcal{F}_u$. Since t and u were arbitrarily chosen it follows that τ_t is a stopping time for all $t \geq 0$. $\qquad\square$

Having proved the above theorem and bearing in mind that the family $(\tau_t)_{t \geq 0}$ is a.s. nondecreasing and continuous in t, we can conclude with the following corollary.

Corollary 4.2.5. *The family $(\tau_t)_{t \geq 0}$ satisfying (4.2.1) is a random change of time.*

Thanks to the explicit form of solution given in (4.2.2), one can easily simulate τ. Application to simulating time-changed asset price models will be presented in Section 4.5.

Notation. By $\hat{X} = (X_{\tau_t})_{t \geq 0}$ we denote a time-changed process and by $\mathbb{G} = (\mathbb{F}_{\tau_t})_{t \geq 0}$ – the time-changed filtration.

Theorem 4.2.6. *The process $\hat{X}$ is a time inhomogeneous $\mathbb{G}$-Markov chain with the state space E and intensity matrix $\hat{\Lambda}(t) = [g(t, x)\lambda_y^x]_{x,y \in E}$.*

Proof. We will use martingale characterization of finite Markov chains. Obviously $\hat{X}$ is right-continuous and $\mathbb{G}$-adapted. Hence it suffices to show that for all $y \in E$ the process

$$\bar{M}_t^y := \delta_y(\hat{X}_t) - \int_0^t g(u, \hat{X}_u)\lambda_y^{\hat{X}_u} \mathrm{d}u$$

is a $\mathbb{G}$-martingale, where

$$\delta_y(x) = \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases}$$

Fix $y \in E$. Note that by assumptions on g and the fact that E is finite, τ_t is integrable for each t. Indeed,

$$\mathbb{E}\tau_t = \mathbb{E}\int_0^t g(s, \hat{X}_s)\mathrm{d}s \leq \sup_{x \in E}\int_0^t g(s, x)\mathrm{d}s < \infty. \qquad (4.2.3)$$

Thus

$$\mathbb{E}|\bar{M}_t^y| \leq 1 + \sup_{x \in E} |\lambda_y^x| \mathbb{E}\tau_t < \infty, \tag{4.2.4}$$

so $\bar{M}^y$ is integrable. It suffices then to show that $\mathbb{E}\left(\bar{M}_t^y|\mathcal{G}_s\right) = \bar{M}_s^y$ for any $s < t$. Note that by the continuity and monotonicity of τ, for each locally bounded Borel function F we have (see e.g. Appendix 4 in [58])

$$\int_0^{\tau_t} F(s)\mathrm{d}s = \int_0^t F(\tau_u)\mathrm{d}\tau_u = \int_0^t F(\tau_u)g(u, X_{\tau_u})\mathrm{d}u. \tag{4.2.5}$$

In particular

$$\int_0^t g(u, \hat{X}_u)\lambda_y^{\hat{X}_u}\mathrm{d}u = \int_0^{\tau_t} \lambda_y^{X_s}\mathrm{d}s,$$

which implies $\bar{M}_t^y = M_{\tau_t}^y$ for all $t \geq 0$, where $M_t^y := \delta_y(X_t) - \int_0^t g(u, X_u)\mathrm{d}u$. By Doob's optional sampling theorem we have

$$\mathbb{E}\left(M_{\tau_t \wedge n}^y|\mathcal{F}_{\tau_s \wedge n}\right) = M_{\tau_s \wedge n}^y \to_{n \to \infty} M_{\tau_s}^y \text{ a.s.}$$

Thus to finish the proof it suffices to show that $\mathbb{E}\left(M_{\tau_t \wedge n}^y|\mathcal{F}_{\tau_s \wedge n}\right) \to \mathbb{E}\left(M_{\tau_t}^y|\mathcal{F}_{\tau_s}\right)$ as $n \to \infty$. Note that

$$\left|M_{\tau_t \wedge n}^y - M_{\tau_t}^y\right| \leq 2 + \tau_t \sup_{x \in E} |\lambda_y^x|,$$

so by (4.2.3) and the Lebesgue theorem

$$\lim_{n \to \infty} \mathbb{E}\left|M_{\tau_t \wedge n}^y - M_{\tau_t}^y\right| = 0. \tag{4.2.6}$$

Hence

$$\left|\mathbb{E}\left(M_{\tau_t \wedge n}^y|\mathcal{F}_{\tau_s \wedge n}\right) - \mathbb{E}\left(M_{\tau_t}^y|\mathcal{F}_{\tau_s}\right)\right|$$
$$\leq \mathbb{E}\left(\left|M_{\tau_t \wedge n}^y - M_{\tau_t}^y\right||\mathcal{F}_{\tau_s \wedge n}\right) + \left|\mathbb{E}\left(M_{\tau_t}^y|\mathcal{F}_{\tau_s \wedge n}\right) - \mathbb{E}\left(M_{\tau_t}^y|\mathcal{F}_{\tau_s}\right)\right| \longrightarrow_{n \to \infty} 0$$

in probability, where the first component tends to 0 in L^1 by (4.2.6) and the second tends to 0 a.s and in L^1 since the process $(Y_n)_{n \in \mathbb{N}} := \left(\mathbb{E}\left(M_{\tau_t}^y|\mathcal{F}_{\tau_s \wedge n}\right)\right)_{n \in \mathbb{N}}$ is a right-closable martingale. By the uniqueness of the limit in probability we obtain the thesis. $\square$

Example 4.2.7. Let X be a Markov chain with a generator Λ and a state space $E = \{0, 1, \ldots, N\}$. Consider a function $g \colon [0, \infty) \times E \to [0, \infty)$.

$$g(s, x) = sx.$$

Since the function g is obviously nonnegative and continuous in s for each $x \in E$, we can define a change of time τ_t as a solution to the TCE

$$\tau_t = \int_0^t g(s, X_{\tau_s})\mathrm{d}s. \tag{4.2.7}$$

Consider the time-changed Markov chain $\widehat{X}$. Then, by Theorem 4.2.6 the generator matrix of $\widehat{X}$ is given as

$$\widehat{\Lambda}(t) = \left[xt\lambda_y^x \right]_{x,y \in E}.$$

Moreover, the jump times of $\widehat{X}$, i.e. $\sigma_0, \sigma_1, \ldots$ defined in the proof of Theorem 4.2.2 are of the form:

$$\sigma_k = \inf \left\{ t \geq 0 \colon \int_{\sigma_{k-1}}^t s X_{T_{k-1}} \mathrm{d}s \geq T_k - T_{k-1} \right\}$$

$$= \sqrt{\frac{2(T_k - T_{k-1})}{X_{T_{k-1}}} + \sigma_{k-1}^2} \cdot \mathbb{1}_{\{X_{T_k} \neq 0\}} + \infty \cdot \mathbb{1}_{\{X_{T_k} = 0\}}, \quad k = 1, 2, \ldots$$

By induction we can show that

$$\sigma_k = \sqrt{\sum_{i=0}^{k-1} \frac{2(T_{i+1} - T_i)}{X_{T_i}}} \cdot \mathbb{1}_{A_k} + \infty \cdot \mathbb{1}_{A_k^c},$$

where $A_k = \bigcap_{i=0}^{k-1} \{X_{T_i} \neq 0\}$. The solution of 4.2.7 is given by

$$\tau_t = \sum_{n=1}^{\infty} \frac{1}{2} X_{T_{n-1}} \left((\sigma_n \wedge t)^2 - (\sigma_{n-1} \wedge t)^2 \right).$$

Note that 0 is an absorbing state for $\widehat{X}$, which can be seen both from the formula for $\widehat{\Lambda}(t)$ and from the jump times σ_k.

4.3 Markov consistency of a time-changed Markov chain

Throughout this section $X = (X^1, X^2)$ will be a two-dimensional Markov chain on a finite state space $E = E^1 \times E^2$. Let $m = |E^1|$, $n = |E^2|$. The first theorem addresses the question of the conditions which should be satisfied by g associated to the change of time τ so that the time change preserves Markov consistency property with respect to the first coordinate.

Theorem 4.3.1. *Assume that X^1 is an $\mathbb{F}$-Markov chain. Then the following conditions are equivalent:*

(i) $\hat{X}^1$ *is a $\mathbb{G}$-Markov chain,*

(ii) there exists a function $g_1 \colon [0, \infty) \times E^1 \to [0, \infty)$ such that for all non-absorbing states $x^1 \in E^1$

$$\mathbb{1}_{\{\hat{X}^1_t = x^1\}} g(t, x^1, \hat{X}^2_t) = \mathbb{1}_{\{\hat{X}^1_t = x^1\}} g_1(t, x^1) \quad \mathbb{P} \otimes dt\text{- a.e.}$$

Moreover, if (ii) is satisfied, the intensity matrix of $\hat{X}^1$ is of the form

$$\hat{\Lambda}^{(1)}(t) = [g_1(t, x^1) \lambda^{(1)}_{x^1 y^1}]_{x^1, y^1 \in E^1},$$

where $[\lambda^{(1)}_{x^1 y^1}]_{x^1, y^1 \in E^1}$ is an intensity matrix of X^1.

Before we proceed to the proof of the theorem, we need to briefly recall a few notions. Recall that for a càdlàg process V taking values in a finite space $\mathcal{V}$ we can define a counting measure

$$N^V_{vw}((0, t]) := \sum_{0 < s \leq t} \mathbb{1}_{\{V s- = v, V_s = w\}}, \quad v, w \in \mathcal{V}, \ v \neq w.$$

By $\nu^V_{vw}(dt)$ we will denote the compensator of the counting measure $N^V_{vw}(dt)$ associated with a process V. In Bielecki et al. [13], the authors use compensators of counting measures N^V_{vw} in order to give the necessary and sufficient condition for a process V to be a Markov chain. For the convenience of the reader we cite their proposition below.

Proposition 4.3.2 ([13], Proposition 1.3). *An $\mathbb{H}$-adapted, càdlàg process V on a finite state space $\mathcal{V}$ is a Markov chain with respect to $\mathbb{H}$ with infinitesimal generator $Q(t) = [q_{kl}(t)]_{k,l \in \mathcal{V}}$ if and only if the $\mathbb{H}$-compensators of the counting measures N_{kl}^V, $k, l \in \mathcal{V}$, are of the form*

$$\nu_{kl}^V((0,t]) = \int_0^t \mathbb{1}_{\{V_s = k\}} q_{kl}(s) ds.$$

In order to use the above result we need to calculate the compensators of the counting measures for the time-changed process. Let ν_{xy}^X be an $\mathbb{F}$-compensator of N_{xy}^X for $x, y \in E$, $x \neq y$. Then the following lemma holds.

Lemma 4.3.3. *Let $x, y \in E$, $x \neq y$. The $\mathbb{G}$-compensator of the counting measure $N_{xy}^{\hat{X}}$ is of the form*

$$\nu_{xy}^{\hat{X}}((0,t]) = \int_{(0,\tau_t]} \nu_{xy}^X(ds), \quad t > 0.$$

Proof. Since $\hat{X}$ is a $\mathbb{G}$-Markov chain, by Proposition 4.3.2 the $\mathbb{G}$-compensator of $N_{xy}^{\hat{X}}$ is of the form $\nu_{xy}^{\hat{X}}(dt) = \mathbb{1}_{\{\hat{X}_t = x\}} \hat{\lambda}_y^x(t) dt$. Hence, using (4.2.5), we obtain

$$\nu_{xy}^{\hat{X}}((0,t]) = \int_0^t \mathbb{1}_{\{X(\tau_s) = x\}} g(s, X_{\tau_s}) \lambda_y^x ds = \int_0^{\tau_t} \mathbb{1}_{\{X_s = x\}} \lambda_y^x ds = \int_0^{\tau_t} \nu_{xy}^X(ds).$$

$\square$

Now we can proceed to the proof of Theorem 4.3.1.

Proof of Theorem 4.3.1. We will denote $x = (x^1, x^2)$, $y = (y^1, y^2)$ for any $x, y \in E$. By Lemma 4.3.3

$$\nu_{x^1 y^1}^{\hat{X}^1}((0,t]) = \sum_{x^2, y^2 \in E^2} \nu_{xy}^{\hat{X}}((0,t]) = \sum_{x^2, y^2 \in E^2} \int_0^{\tau_t} \nu_{xy}^X(ds) = \int_0^{\tau_t} \nu_{x^1 y^1}^{X^1}(ds)$$

$$= \int_0^{\tau_t} \mathbb{1}_{\{X_s^1 = x^1\}} \lambda_{x^1 y^1}^{(1)} ds = \int_0^t \mathbb{1}_{\{\hat{X}_s^1 = x^1\}} \lambda_{x^1 y^1}^{(1)} g(s, \hat{X}_s) ds.$$

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On the other hand, by Proposition 4.3.2, $\hat{X}^1$ is a $\mathbb{G}$-Markov chain if and only if for all $x^1 \neq y^1$ there exists a locally integrable function $\hat{\lambda}^{(1)}_{x^1y^1} : [0, \infty) \to [0, \infty)$ such that $\nu^{\hat{X}^1}_{x^1y^1}(\mathrm{d}t) = \mathbb{1}_{\{\hat{X}^1_t=x^1\}}\hat{\lambda}^{(1)}_{x^1y^1}(t)\mathrm{d}t$. Hence $\hat{X}^1$ is a $\mathbb{G}$-Markov chain if and only if

$$\forall_{x^1 \neq y^1} \, \exists_{\hat{\lambda}^{(1)}_{x^1y^1}} \text{ such that } \mathbb{1}_{\{\hat{X}^1_t=x^1\}}\lambda^{(1)}_{x^1y^1}g(t, \hat{X}_t) = \mathbb{1}_{\{\hat{X}^1_t=x^1\}}\hat{\lambda}^{(1)}_{x^1y^1}(t) \quad \mathbb{P} \otimes dt\text{- a.e.}$$

(4.3.1)

We will show that (4.3.1) is equivalent to the condition (ii) from the statement of Theorem 4.3.1.

Assuming (ii), we get (4.3.1) by taking $\hat{\lambda}^{(1)}_{x^1y^1}(t) = g_1(t, x^1)\lambda^{(1)}_{x^1y^1}$. Indeed, for an absorbing state x^1 this means $\hat{\lambda}^{(1)}_{x^1y^1}(t) = 0$, so the equality is satisfied trivially, whereas for a non-absorbing state it follows directly from (ii).

Conversely, assume (4.3.1) and take a non-absorbing state x^1. Then there exists $\tilde{y}^1 \in E^1$, $\tilde{y}^1 \neq x^1$ such that $\lambda^{(1)}_{x^1\tilde{y}^1} > 0$. By (4.3.1) there exists a function $\hat{\lambda}^{(1)}_{x^1\tilde{y}^1} : [0, \infty) \to [0, \infty)$ such that

$$\mathbb{1}_{\{\hat{X}^1_t=x^1\}}\lambda^{(1)}_{x^1\tilde{y}^1}g(t, \hat{X}_t) = \mathbb{1}_{\{\hat{X}^1_t=x^1\}}\hat{\lambda}^{(1)}_{x^1\tilde{y}^1}(t).$$

Dividing both sides by $\lambda^{(1)}_{x^1\tilde{y}^1}$ we obtain

$$\mathbb{1}_{\{\hat{X}^1_t=x^1\}}g(t, \hat{X}_t) = \mathbb{1}_{\{\hat{X}^1_t=x^1\}}\frac{\hat{\lambda}^{(1)}_{x^1\tilde{y}^1}(t)}{\lambda^{(1)}_{x^1\tilde{y}^1}}.$$

Since the LHS of the above equality does not depend on $\tilde{y}$, neither does the ratio $\hat{\lambda}^{(1)}_{x^1\tilde{y}^1}(t)/\lambda^{(1)}_{x^1\tilde{y}^1}$, hence we may take

$$g_1(t, x^1) = \begin{cases} \dfrac{\hat{\lambda}^{(1)}_{x^1\tilde{y}^1}(t)}{\lambda^{(1)}_{x^1\tilde{y}^1}} & \text{if } x^1 \text{ is non-absorbing,} \\ 0 & \text{otherwise.} \end{cases}$$

$\square$

Naturally we can formulate the analogue of Theorem 4.3.1 for the second coordinate.

Corollary 4.3.4. *Assume that X^2 is an $\mathbb{F}$-Markov chain. Then the following conditions are equivalent:*

(i) $\hat{X}^2$ is a $\mathbb{G}$-Markov chain,

(ii) there exists a function $g_2 \colon [0, \infty) \times E^2 \to [0, \infty)$ such that for all non-absorbing states $x^2 \in E^2$

$$\mathbb{1}_{\{\hat{X}_t^2 = x^2\}} g(t, \hat{X}_t^1, x^2) = \mathbb{1}_{\{\hat{X}_t^2 = x^2\}} g_2(t, x^2) \quad \mathbb{P} \otimes dt\text{- a.e.}$$

Moreover, if (ii) is satisfied, the intensity matrix of $\hat{X}^2$ is of the form

$$\hat{\Lambda}^{(2)}(t) = [g_2(t, x^2) \lambda^{(2)}_{x^2 y^2}]_{x^2, y^2 \in E^2},$$

where $[\lambda^{(2)}_{x^2 y^2}]_{x^2, y^2 \in E^2}$ is an intensity matrix of X^2.

Corollary 4.3.5. *Assume that X is Markov consistent. Then the process $\hat{X}$ is Markov consistent if and only if condition (ii) from Theorem 4.3.1 and condition (ii) from Corollary 4.3.4 are satisfied for g.*

Our next problem is whether we can impose Markov consistency property by introducing an appropriate change of time. Obviously taking $g \equiv 0$ induces the trivial change of time $\tau \equiv 0$, so the time-changed process $\hat{X} \equiv X_0$ is constant, and thus Markov consistent, regardless of the properties of the original Markov chain X. Henceforth we are interested in changes of time which are not identically zero (or, in general – for other types of changes of time – not constant). We introduce the following class of Markov chains.

Definition 4.3.6. A Markov chain X is called quasi Markov consistent if there exists a change of time $\tau \not\equiv \text{const}$ such that $X_{\tau(\cdot)}$ is Markov consistent.

We will study examples of quasi Markov consistent processes in the framework of inhomogeneous time change equations.

Towards this end we will use condition (M) from Bielecki et al. [15], which is a sufficient (but not necessary) condition for a Markov chain to be strongly Markov consistent. We recall the condition for the convenience of the reader.

Theorem 4.3.7 ([15], Proposition 5.1). *Let $Z = (Z^1, Z^2)$ be a Markov chain on $E^1 \times E^2$ with intensity matrix $Q(t) = [q_{y_1 y_2}^{x^1 x^2}(t)]$. If for all $t \geq 0$*

$$\forall_{x^1 \neq y^1} \forall_{x^2, \tilde{x}^2 \in E^2} \quad \sum_{y^2 \in E^2} q_{y^1 y^2}^{x^1 x^2}(t) = \sum_{y^2 \in E^2} q_{y^1 y^2}^{x^1 \tilde{x}^2}(t),$$

$$\forall_{x^2 \neq y^2} \forall_{x^1, \tilde{x}^1 \in E^1} \quad \sum_{y^1 \in E^1} q_{y^1 y^2}^{x^1 x^2}(t) = \sum_{y^1 \in E^1} q_{y^1 y^2}^{\tilde{x}^1 x^2}(t), \tag{M}$$

then Z is strongly Markov consistent.

Recall that the intensity matrix of the process $\hat{X}$ is of the form

$$\hat{\Lambda}(t) = \left[g(t, x^1, x^2) \lambda_{y^1 y^2}^{x^1 x^2} \right]_{(x^1, x^2), (y^1, y^2) \in E},$$

so condition (M) for the process $\hat{X}$ reads

$$\forall_{t>0} \forall_{x^1 \neq y^1} \forall_{x^2, \tilde{x}^2 \in E^2} \quad g(t, x^1, x^2) \sum_{y^2 \in E^2} \lambda_{y^1 y^2}^{x^1 x^2} - g(t, x^1, \tilde{x}^2) \sum_{y^2 \in E^2} \lambda_{y^1 y^2}^{x^1 \tilde{x}^2} = 0,$$

$$\forall_{t>0} \forall_{x^2 \neq y^2} \forall_{x^1, \tilde{x}^1 \in E^1} \quad g(t, x^1, x^2) \sum_{y^1 \in E^1} \lambda_{y^1 y^2}^{x^1 x^2} - g(t, \tilde{x}^1, x^2) \sum_{y^1 \in E^1} \lambda_{y^1 y^2}^{\tilde{x}^1 x^2} = 0. \tag{$\hat{\text{M}}$}$$

In order to verify whether X is quasi consistent we need to solve the above system of linear equations with respect to g. Note that the system consists of $(m + n)(m - 1)(n - 1)$ equations and only mn unknowns. Let A denote the matrix of the system ($\hat{\text{M}}$) (note that A does not depend on t). Then we have the following easy observation.

Proposition 4.3.8. *If $\operatorname{rank}(A) < mn$, then the Markov chain X is quasi consistent and any nontrivial solution g^* of system ($\hat{\text{M}}$) gives a desired change of time.*

Remark 4.3.9. Since condition (M) is not necessary for Markov consistency, the condition $\operatorname{rank}(A) < mn$ is not necessary for X to be quasi Markov consistent, even if we reduce ourselves to the class of time change equations of the form (4.2.1). However, this condition is easily verifiable by simple computation.

The example below is a version of Proposition [4.3.8](#) for two-element state spaces E^1 and E^2.

Example 4.3.10. Let $E^1 = E^2 = \{0,1\}$ and denote $g_{kl}(t) := g(t,k,l)$ for $k,l \in \{0,1\}$. Then the number of equations and unknowns of $(\hat{\mathrm{M}})$ is equal and the system takes the form

$$
\begin{cases}
g_{00}(t)\left(\lambda_{10}^{00} + \lambda_{11}^{00}\right) - g_{01}(t)\left(\lambda_{10}^{01} + \lambda_{11}^{01}\right) &= 0 \\
g_{10}(t)\left(\lambda_{01}^{10} + \lambda_{00}^{10}\right) - g_{11}(t)\left(\lambda_{01}^{11} + \lambda_{00}^{11}\right) &= 0 \\
g_{00}(t)\left(\lambda_{01}^{00} + \lambda_{11}^{00}\right) - g_{10}(t)\left(\lambda_{01}^{10} + \lambda_{11}^{10}\right) &= 0 \\
g_{01}(t)\left(\lambda_{10}^{01} + \lambda_{00}^{01}\right) - g_{11}(t)\left(\lambda_{10}^{11} + \lambda_{00}^{11}\right) &= 0
\end{cases}
$$

Then $\mathrm{rank}(A) < 4$ if and only if

$$
\begin{aligned}
&\left(\lambda_{10}^{00} + \lambda_{11}^{00}\right)\left(\lambda_{01}^{11} + \lambda_{00}^{11}\right)\left(\lambda_{01}^{10} + \lambda_{11}^{10}\right)\left(\lambda_{10}^{01} + \lambda_{00}^{01}\right) = \\
&\left(\lambda_{10}^{01} + \lambda_{11}^{01}\right)\left(\lambda_{01}^{10} + \lambda_{00}^{10}\right)\left(\lambda_{01}^{00} + \lambda_{11}^{00}\right)\left(\lambda_{10}^{11} + \lambda_{00}^{11}\right).
\end{aligned}
\tag{4.3.2}
$$

Using the result above we can provide an example of a Markov chain which is quasi Markov consistent, but not Markov consistent.

Example 4.3.11. Let X be a Markov chain on $\{0,1\}^2$ with intensity matrix

$$
\Lambda = \begin{array}{c}
\\
(0,0) \\
(0,1) \\
(1,0) \\
(1,1)
\end{array}
\begin{array}{cccc}
(0,0) & (0,1) & (1,0) & (1,1) \\
\left(\begin{array}{cccc}
\cdot & \beta + c_1 & \beta + c_1 & \alpha \\
\alpha + c_2 & \cdot & \beta & \alpha + c_2 \\
0 & \gamma & -\gamma & 0 \\
\delta & 0 & 0 & -\delta
\end{array} \right)
\end{array}
$$

where $\alpha, \beta, \gamma, \delta > 0$, $c_1 > -\beta$, $c_2 > -\alpha$, $c_1 \neq c_2$ and $\cdot$ means the negative of the sum of other entries in the row. Then the intensity of transition of X^1 from 0 to 1 depends on the state of X^2, namely

$$
\lambda_{10}^{00} + \lambda_{11}^{00} = \alpha + \beta + c_1 \neq \beta + \alpha + c_2 = \lambda_{10}^{01} + \lambda_{11}^{01}.
$$

Moreover, note that for $t > 0$ we have $\mathbb{P}\left(X_t = (0,0)\right) > 0$ and $\mathbb{P}\left(X_t = (0,1)\right) > 0$ regardless of the initial distribution of X. Hence for all $t > 0$

$$
\mathbb{1}_{\{X_t = ((0,0)\}}\left(\lambda_{10}^{00} + \lambda_{11}^{00}\right) \neq \mathbb{1}_{\{X_t = ((0,1)\}}\left(\lambda_{10}^{01} + \lambda_{11}^{01}\right)
$$

with positive probability. Thus by Theorem 1.8 from [13] we conclude that X is not strongly Markov consistent with respect to X^1. By similar argument X is not consistent w.r.t X^2 either. On the other hand, it is easy to see that the condition (4.3.2) is satisfied for X, so X is quasi Markov consistent. We will find such a change of time that $\hat{X}$ is consistent. The solution of $(\hat{M})$ is of the form

$$g_{00}(t) = \frac{\delta}{\beta + \alpha + c_1} f(t)$$

$$g_{01}(t) = \frac{\delta}{\beta + \alpha + c_2} f(t)$$

$$g_{10}(t) = \frac{\delta}{\gamma} f(t)$$

$$g_{11}(t) = f(t)$$

for all $t > 0$, where f is any locally integrable, càdlàg nonnegative function. The intensity matrix of $\hat{X}$ takes the form

$$\Lambda(t) = f(t) \begin{pmatrix} \cdot & \frac{\delta(\beta+c_1)}{\beta+\alpha+c_1} & \frac{\delta(\beta+c_1)}{\beta+\alpha+c_1} & \frac{\delta\alpha}{\beta+\alpha+c_1} \\ \frac{\delta(\alpha+c_2)}{\beta+\alpha+c_2} & \cdot & \frac{\delta\beta}{\beta+\alpha+c_2} & \frac{\delta(\alpha+c_2)}{\beta+\alpha+c_2} \\ 0 & \delta & -\delta & 0 \\ \delta & 0 & 0 & -\delta \end{pmatrix}.$$

One can easily verify that $\hat{X}$ indeed satisfies condition (M) and hence is strongly Markov consistent.

4.4 Change of time in asset price model – regime switching diffusion

In this section we study the time-changed asset price models, where the change of time is a solution of a time change equation (4.2.1) induced by an independent Markov chain. In the first part we will show that changing time in the diffusion process leads to a regime-switching diffusion process.

Consider an n-dimensional asset price process $S = (S^1, \ldots, S^n)$ given as a strong solution of an SDE

$$\mathrm{d}S_t = m(S_t)\mathrm{d}t + \Sigma(S_t)\mathrm{d}W_t, \qquad (4.4.1)$$

where $m \colon \mathbb{R}^n \to \mathbb{R}^n$, $\Sigma \colon \mathbb{R}^n \to \mathbb{R}^{n \times d}$ and W is a d-dimensional standard Brownian motion. Suppose X is a Markov chain on a finite state space E, independent of W, representing states of economy. Let τ be a change of time given by (4.2.1) for an appropriate function g. By $\hat{S}$ we denote the time-changed process of the asset price.

Let us fix our attention for a moment on the financial interpretation of a time change in asset price models. We may see τ_t as business time (or business clock) at calendar time t, which depends on the order flow up to time t. The process $g(t, \hat{X}_t)$ is then interpreted as an instantaneous business activity rate. It may be observed that business activity is influenced by the states of economy at time t (represented by the Markov chain X), but it may also vary in time due to some non-random effects. The fact that g depends on X and is time-inhomogeneous captures both of these phenomena. Intuitively, and what may be seen in the result of Theorem 4.4.4, the more active the business day is, the higher is the volatility of the asset price (see Carr et al. [18] for more detailed interpretation of a general change of time in asset price models).

The next theorem shows that the process $(\hat{S}, \hat{X})$ forms a particular stochastic volatility model, namely a regime-switching diffusion process. Let us start with recalling the definition of such processes (see e.g. [66], Section 2.2).

Definition 4.4.1. Let $(\alpha_t)_{t \geq 0}$ be a continuous-time Markov chain on a finite state space E with intensity matrix Q and let $(W_t)_{t \geq 0}$ be a d-dimensional Brownian motion independent of α. Suppose $a \colon [0, +\infty) \times E \times \mathbb{R}^r \to \mathbb{R}^r$ and $b \colon [0, +\infty) \times E \times \mathbb{R}^r \to \mathbb{R}^{r \times d}$. Then a two-component process (ξ, α) satisfying

$$\mathrm{d}\xi_t = a(t, \alpha_t, \xi_t)\mathrm{d}t + b(t, \alpha_t, \xi_t)\mathrm{d}W_t, \quad t \in (0, T], \qquad (4.4.2)$$

is called a regime-switching diffusion.

Remark 4.4.2. Regime-switching diffusions may also be defined in a more general way, where $(\alpha_t)_{t \geq 0}$ is a right-continuous process on a finite state space E with an x-dependent generator $Q(x)$ (which allows the intensity of jumps of α to depend on ξ). However, for our purposes we will only need the particular case when α is a Markov chain.

In order to ensure the existence and uniqueness of solution of (4.4.2), we will often refer to the assumption on the coefficient functions (weaker than linear growth condition) which we present below. This assuption together with local Lipschitz condition are sufficient to guarantee the existence of a unique solution of (4.4.2) (see Theorem 3.17 in [47]).

Assumption 4.4.3. *There exists a constant $K > 0$ such that for all $(t, k, x) \in [0, T] \times E \times \mathbb{R}^r$*

$$x \cdot a(t, k, x) + \frac{1}{2}|b(t, k, x)|^2 \leq K(1 + |x|^2). \tag{4.4.3}$$

Theorem 4.4.4. *Let S be an asset price process given by (4.4.1), where functions $m \colon \mathbb{R}^n \to \mathbb{R}^n$ and $\Sigma \colon \mathbb{R}^n \to \mathbb{R}^{n \times d}$ are locally Lipschitz and satisfy for all $x \in \mathbb{R}^n$*

$$x \cdot m(x) + \frac{1}{2}|\Sigma(x)|^2 \leq \kappa(1 + |x|^2) \tag{4.4.4}$$

for some constant $\kappa > 0$. Fix a finite time horizon $T < \infty$. Then the time-changed asset price process $\hat{S}$ is a unique strong solution of

$$\mathrm{d}\hat{S}_t = m(\hat{S}_t)g(t, \hat{X}_t)\mathrm{d}t + \Sigma(\hat{S}_t)\sqrt{g(t, \hat{X}_t)}\mathrm{d}B_t, \quad t \in [0, T], \tag{4.4.5}$$

where B is a d-dimensional $\mathbb{G}$-standard Brownian motion independent of X.

Proof. Let Z be a d-dimensional standard Brownian motion with respect to the filtration $\mathbb{G}$, independent of W and of X. Define process B as

$$B_t^i := \int_0^t \frac{1}{\sqrt{g(u, \hat{X}_u)}} \mathbb{1}_{\{g(u, \hat{X}_u) > 0\}} \mathrm{d}W^i(\tau_u) + \int_0^t \mathbb{1}_{\{g(u, \hat{X}_u) = 0\}} \mathrm{d}Z_u^i, \quad i = 1, \dots, d.$$

Then B is a standard $\mathbb{G}$ - Brownian motion. Indeed, (W_{τ_t}) is a continuous $\mathbb{G}$ - local martingale. Moreover, since τ is continuous in t, $\langle W^i_{\tau.} \rangle_t = \langle W^i \rangle_{\tau_t} = \tau_t$ for $i = 1, \ldots, d$ and for $i \neq j$ we have $\langle W^i_{\tau.}, W^j_{\tau.} \rangle_t = \langle W^i, W^j \rangle_{\tau_t} = 0$. Hence

$$\langle B^i \rangle_t = \int_0^t \frac{1}{g(u, \hat{X}_u)} \mathbb{1}_{\{g(u,\hat{X}_u)>0\}} \mathrm{d}\tau_u + \int_0^t \mathbb{1}_{\{g(u,\hat{X}_u)=0\}} \mathrm{d}u =$$

$$= \int_0^t \mathbb{1}_{\{g(u,\hat{X}_u)>0\}} \mathrm{d}u + \int_0^t \mathbb{1}_{\{g(u,\hat{X}_u)=0\}} \mathrm{d}u = t$$

and for $i \neq j$

$$\langle B^i, B^j \rangle_t = 0$$

for all $t \geq 0$. So B is a continuous $\mathbb{G}$ - local martingale and it then follows from Lévy theorem that B is a standard $\mathbb{G}$ - Brownian motion. To see that B is independent of X we need to consider the conditional law of B given X.

First, note that for a nondecreasing, measurable function $h \colon \mathbb{R}_+ \to \mathbb{R}_+$ and a function $f \in L^2([0,T], \mathrm{d}h)$, for any $t \in [0,T]$ we have

$$\int_0^t f(s) \mathrm{d}W_{h(s)} \sim \mathcal{N}\left(0, \int_0^t f^2(s) \mathrm{d}h(s)\right).$$

The proof of it is essentially the same as the proof of the analogous property for Paley-Wiener integral.

Fix $l \in \{1, \ldots d\}$. For $m \in \mathbb{N}$ and $0 < t_1 < \cdots < t_m \leq T$ consider a random vector $(B^l_{t_1}, \ldots, B^l_{t_m})$. Since X is independent of W and of Z, and since τ depends only on X, by the remark above for $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ we have

$$\mathbb{E}\left(e^{iu(B^l_{t_1}, \ldots, B^l_{t_m})} \big| \mathcal{F}^X_\infty\right) = \exp\left(-\frac{1}{2} u^T D(X) u\right),$$

where for any $j, k = 1, \ldots, m$

$$D_{jk}(X) = \int_0^{t_j \wedge t_k} \frac{1}{g(u, \hat{X}_u)} \mathbb{1}_{\{g(u,\hat{X}_u)>0\}} \mathrm{d}\tau_u + \int_0^{t_j \wedge t_k} \mathbb{1}_{\{g(u,\hat{X}_u)=0\}} \mathrm{d}u = t_j \wedge t_k.$$

Thus for all $l \in \{1, \ldots, d\}$ vector $(B_{t_1}^l, \ldots, B_{t_m}^l)$ is independent of $\mathcal{F}_T^X$, and since $m \in \mathbb{N}$ and the sequence $(t_1, \ldots, t_m)$ were arbitrary, the whole process B is independent of X.

To finish the proof we need to show that $\hat{S}$ is a unique strong solution of (4.4.5) with B defined as above. Indeed, by (4.2.5) and Proposition V.1.5 in [54] we have

$$
\begin{aligned}
\hat{S}_t &= S_0 + \int_0^{\tau_t} m(S_u)\mathrm{d}u + \int_0^{\tau_t} \Sigma(S_u)\mathrm{d}W_u \\
&= \hat{S}_0 + \int_0^t m(\hat{S}_u)g(u, \hat{X}_u)\mathrm{d}u + \int_0^t \Sigma(\hat{S}_u)\mathrm{d}W_{\tau_u} \\
&= \hat{S}_0 + \int_0^t m(\hat{S}_u)g(u, \hat{X}_u)\mathrm{d}u + \int_0^t \Sigma(\hat{S}_u)\sqrt{g(u, \hat{X}_u)}\mathrm{d}B_u.
\end{aligned}
$$

Note that (4.4.5) admits a unique solution on a time interval $[0, T]$. Indeed, since m and Σ are locally Lipschitz, the coefficients $a(t, k, x) := m(x)g(t, k)$ and $b(t, k, x) := \Sigma(x)\sqrt{g(t, k)}$ are locally Lipschitz with respect to x uniformly in $k \in E$ and $t \in [0, T]$. Moreover, by (4.4.4) functions a, b satisfy Assumption 4.4.3 with a constant

$$
K = \sup_{(t,k)\in[0,T]\times\mathcal{X}} g(t, k)\kappa.
$$

The uniqueness of solution follows then from Theorem 3.17 in [47]. $\qquad\square$

Remark 4.4.5. One of the motivations for representing stochastic volatility models as time-changed processes is its possible application for calculating characteristic function of the log-price of an asset. This quantity is useful for numerical pricing of financial derivatives (see e.g. Carr, Madan [17]). Consider process $(S_t)_{t\geq0}$ and (any) change of time $(\theta_t)_{t\geq0}$ independent of S. Assume that the process x defined as $x_t := \log\left(\frac{S_t}{S_0}\right)$ is a Lévy process (which – in the diffusion case – is satisfied when S is a Geometric Brownian Motion with drift) and denote by ψ its characteristic exponent. Let the asset price process be modelled by $(\hat{S}_t)_{t\geq0} = (S_{\theta_t})_{t\geq0}$. We would like to calculate the characteristic function of the log-price $\hat{x}_t := \log\left(\frac{\hat{S}_t}{\hat{S}_0}\right) = x_{\theta_t}$. Note that

$$
\mathbb{E}\exp(iu\hat{x}_t) = \mathbb{E}\left(\mathbb{E}\exp(iux_{\theta_t})|\theta_t\right) = \mathbb{E}e^{\theta_t\psi(u)}. \tag{4.4.6}
$$

Thus, in order to calculate the characteristic function of $\hat{x}$ one only needs to know the characteristic function of x and Laplace transform of θ. For more on time-changed exponential Lévy models see Carr, Wu [18].

In the second part of this section we will consider a reverse problem – given an n-dimensional regime-switching diffusion, can we represent its coordinates by time-changed diffusions? For simplicity of notation, without loss of generality take $n = 2$. Consider a regime-switching diffusion (S, Y), where $S = (S^1, S^2)$ and $Y = (Y^1, Y^2)$ is a two-dimensional Markov chain. More precisely, in Theorem 4.4.7 we will answer the following question. Do there exist one-dimensional diffusions R^1 and R^2, i.e. processes satisfying

$$dR_t^i = \mu_i(R_t^i)\mathrm{d}t + \sigma_i(R_t^i)\mathrm{d}B_t^i, \quad i = 1, 2$$

and changes of time τ^1 and τ^2 (induced by one-dimensional Markov chains X^1, X^2), such that for $i = 1, 2$ the law of S^i is equal to the law of $R^i(\tau^i)$?

Such a representation is useful for many reasons. First, note that processes R^i are one-dimensional diffusions independent of the time changes τ^i. Hence instead of considering (S^1, S^2), we may consider separately processes $\hat{R}^1$ and $\hat{R}^2$ without any reference to the dependence structure of the whole process. This phenomenon is fundamental for models based on Markov copulae (Markov structures), which find many applications in financial mathematics – for example in credit risk analysis. The examples of Markov copula models in finance may be found e.g. in the papers by Bielecki, Cousin et al. [8] or by Bielecki, Crépey et al. [10]. Secondly, if process R^i is a Geometric Brownian Motion with drift, then by (4.4.6) one can calculate the characteristic function of the log-price $\log(S_t^i/S_0^i)$ using the characteristic exponent of R^i and Laplace transform of τ^i (see Remark 4.4.5). Moreover, the time changes τ^1 and τ^2 depend only on one dimensional Markov chains, not on the whole chain Y, which simplifies computations by decreasing effectively the number of regimes to consider. The idea of changing time differently for R^1 and R^2 corresponds to the fact that the business time for various assets may differ, also by depending on different factors.

Let $Y = (Y^1, Y^2)$ be a weakly consistent Markov chain (i.e. such that each coordinate of Y is a Markov chain with respect to *its own filtration*) on a state space $E = E^1 \times E^2$, such that the intensities of Y^i are of the form

$$\lambda_{kl}^{(i)}(t) = g_i(t, k)q_{kl}^{(i)}, \quad k, l \in E^i, \ t \in \mathbb{R}, \ i = 1, 2,$$

where $g_i \colon \mathbb{R}_+ \times E^i \to [0, \infty)$ are Borel measurable functions such that for all $x \in E^i$ functions $s \mapsto g_i(s, x)$ are right-continuous and locally integrable. Assume also that $[q_{kl}^{(i)}]_{k,l \in E^i}$ is a Q-matrix for $i = 1, 2$.

Consider a two-dimensional regime-switching diffusion process $(S_t, Y_t)_{t \in [0,T]}$, $T < \infty$, of the form

$$dS_t = m(t, Y_t, S_t)dt + \Sigma(t, Y_t, S_t)dW_t, \tag{4.4.7}$$

where $m \colon \mathbb{R}_+ \times E \times \mathbb{R}^2 \to \mathbb{R}^2$ and $\Sigma \colon \mathbb{R}_+ \times E \times \mathbb{R}^2 \to M_{2 \times 2}(\mathbb{R})$ satisfy

$$m_i(t, k, x) = g_i(t, k_i)\mu_i(x_i), \quad i = 1, 2 \tag{4.4.8}$$

and

$$\sigma_{i1}^2(t, k, x) + \sigma_{i2}^2(t, k, x) = g_i(t, k_i)\sigma_i^2(x_i), \quad i = 1, 2. \tag{4.4.9}$$

where we denote $k = (k_1, k_2) \in E$. Suppose additionally that $\sigma \colon \mathbb{R} \to \mathbb{R}^2$, $\mu \colon \mathbb{R} \to \mathbb{R}^2$ are locally Lipschitz and that σ_i, μ_i satisfy (4.4.4) for $i = 1, 2$. Note that this implies that m, Σ are locally Lipschitz in x and satisfy Assumption 4.4.3, which in turn implies the existence and uniqueness of solution of (4.4.7).

First, we will prove that the components of Y are equal in law to the time-changed Markov chains.

Lemma 4.4.6. *Under the above assumptions there exist Markov chains X^1, X^2, respectively on E^1 and E^2, such that $\mathcal{L}\left(Y^i\right) = \mathcal{L}\left(X_{\tau^i}^i\right)$ for $i = 1, 2$ and τ^i, for $i = 1, 2$, is the solution of the time change equation*

$$\tau_t^i = \int_0^t g_i\left(s, X^i(\tau_s^i)\right) ds. \tag{4.4.10}$$

Proof. We will prove the lemma for $i = 1$. Since $Q^{(1)} := [q_{kl}^{(1)}]_{k,l \in E^1}$ is a Q-matrix, there exists a Markov chain X^1 on the state space E^1 with intensity matrix $Q^{(1)}$ such that $\mathcal{L}(X_0^1) = \mathcal{L}(Y_0^1)$. Let τ^1 be the solution of (4.4.10). From Theorem 4.2.6 we know that the process $\hat{X}^1 := X^1(\tau^1)$ is a Markov chain with intensity matrix $\Lambda^{(1)}(t)$, where $\lambda_{kl}^{(1)}(t) = g_1(t, k) q_{kl}^{(1)}$. Hence $\mathcal{L}(Y^1) = \mathcal{L}(X_{\tau^1}^1)$. $\qquad\square$

It appears that the above conditions are enough to guarantee that the components of S are equal in law to the time-changed diffusion processes.

Theorem 4.4.7. *Let S be a two-dimensional regime-switching diffusion process satisfying (4.4.7), where m and Σ satisfy the assumptions (4.4.8) and (4.4.9). Then*

$$\mathcal{L}\left(S^i\right) = \mathcal{L}\left(R_{\tau^i}^i\right),$$

where τ^i is as in Lemma 4.4.6 and R^i is a unique strong solution of the SDE

$$dR_t^i = \mu_i(R_t^i)\mathrm{d}t + \sigma_i(R_t^i)\mathrm{d}B_t^i, \qquad (4.4.11)$$

where $B = (B^1, B^2)$ is a standard two-dimensional Brownian motion.

Proof. We will prove the theorem for $i = 1$. By Lemma 4.4.6 there exists a Markov chain X^1 and a change of time τ^1 such that $\mathcal{L}\left(Y^1\right) = \mathcal{L}\left(X_{\tau^1}^1\right)$ and τ^1 is the solution of the time change equation

$$\tau_t^1 = \int_0^t g_1\left(s, X^1(\tau_s^1)\right) \mathrm{d}s.$$

Note that since σ_1, μ_1 are locally Lipschitz and satisfy (4.4.4), the equation (4.4.11) admits a unique solution. Let B^1 be a standard Brownian motion independent of X^1 and let R^1 be the unique solution of (4.4.11). Denote $\hat{R}^1 = \left(R^1(\tau_t^1)\right)_{t \geq 0}$ and $\hat{X}^1 = \left(X^1(\tau_t^1)\right)_{t \geq 0}$. Then by Theorem 4.4.4 $(\hat{R}^1, \hat{X}^1)$ is a regime-switching diffusion given by the SDE

$$\mathrm{d}\hat{R}^1{}_t = \mu_1(\hat{R}_t^1) g_1(t, \hat{X}_t^1)\mathrm{d}t + \sigma_1(\hat{R}_t^1)\sqrt{g_1(t, \hat{X}_t^1)}\mathrm{d}\tilde{B}_t^1,$$

with $\tilde{B}^1$ independent of $\hat{X}^1$. On the other hand, by (4.4.7) and assumption (4.4.8), S^1 is a strong solution to

$$\mathrm{d}S_t^1 = \mu_1(S_t^1)g_1(t, Y_t^1)\mathrm{d}t + \sigma_{11}(t, Y_t, S_t)\mathrm{d}W_t^1 + \sigma_{12}(t, Y_t, S_t)\mathrm{d}W_t^2.$$

Let Z be a one-dimensional standard Brownian motion independent of W and of Y. Define

$$\tilde{W}_t^1 = \int_0^t \mathbb{1}_{\{\sigma_1(S_u^1)g_1(u,Y_u^1)\neq 0\}} \left(\frac{\sigma_{11}(u, Y_u, S_u)}{\sigma_1(S_u^1)\sqrt{g_1(u, Y_u^1)}}\mathrm{d}W_u^1 + \frac{\sigma_{12}(u, Y_u, S_u)}{\sigma_1(S_u^1)\sqrt{g_1(u, Y_u^1)}}\mathrm{d}W_u^2 \right)$$
$$+ \int_0^t \mathbb{1}_{\{\sigma_1(S_u^1)g_1(u,Y_u^1)=0\}}\mathrm{d}Z_u.$$

Then the process $\tilde{W}^1$ is a continuous local martingale. Moreover, by assumption (4.4.9) we see that

$$\left\langle \tilde{W}^1 \right\rangle_t = t,$$

so $\tilde{W}^1$ is a standard Brownian motion. Note that, since Y is independent of (W^1, W^2, Z), the latter is a Wiener process with respect to the filtration

$$\mathbb{H} = \left(\mathcal{F}_t^{(W^1, W^2, Z)} \vee \mathcal{F}_\infty^Y \right)_{t \geq 0}.$$

Thus $\tilde{W}^1$ is an $\mathbb{H}$ - standard Brownian motion. In particular, $\tilde{W}^1$ is independent of $\mathcal{H}_0 = \mathcal{F}_\infty^Y$. Thus the processes $\tilde{W}^1$ and Y are independent. Moreover

$$\mathrm{d}S_t^1 = \mu_1(S_t^1)g_1(t, Y_t^1)\mathrm{d}t + \sigma_1(S_t^1)\sqrt{g_1(t, Y_t^1)}\mathrm{d}\tilde{W}_t^1.$$

Thus, since $\mathcal{L}\left(Y^1\right) = \mathcal{L}(\hat{X}^1)$, we see that $\mathcal{L}\left(S^1\right) = \mathcal{L}(\hat{R}^1)$. $\qquad\square$

4.5 Monte Carlo simulations of the time-changed process

In this section we will present the application of change of time to Monte Carlo pricing of European options. Let S be a geometric Brownian motion

$$\mathrm{d}S_t = rS_t\mathrm{d}t + \sigma S_t\mathrm{d}W_t, \tag{4.5.1}$$

and let τ be a change of time given as a solution to the TCE (4.4.10) for an appropriate function g and a Markov chain X. Then by Theorem 4.4.4 the time-changed asset price process $\widehat{S}$ is a regime-switching diffusion

$$\mathrm{d}\widehat{S}_t = rg(t, \widehat{X}_t)\widehat{S}_t\mathrm{d}t + \sigma\sqrt{g(t, \widehat{X}_t)}\widehat{S}_t\mathrm{d}B_t.$$

The quantity $rg(t, \widehat{X}_t)$ plays a role of a stochastic interest rate, while $\sigma\sqrt{g(t, \widehat{X}_t)}$ – a stochastic volatility.

Consider a European-style option $\omega(\widehat{S}_T)$ with maturity T. Then its price at time 0 is calculated as

$$\Pi_\omega = \mathbb{E}e^{-\int_0^T rg(t,\widehat{X}_t)\mathrm{d}t}\omega(\widehat{S}_T) = \mathbb{E}e^{-r\tau_T}\omega(S_{\tau_T}).$$

Hence, to be able to price it using Monte Carlo simulations, instead of simulating regime-switching diffusion and the stochastic interest rate, we only need to simulate geometric Brownian motion S and the change of time τ. In order to simulate the time change τ, we first draw simulation of X according to its generator matrix Λ, and then use the explicit solution of the TCE given in (4.2.2). The Markov chain X and the GBM S are simulated using standard techniques, see e.g. [5]. Note that this procedure, i.e. simulating GBM and change of time, would be very useful also for pricing Asian or American options, whose prices depend on the whole path of $\widehat{S}$.

We will start with presenting some results of simulations of time-changed Markov chains and time-changed geometric Brownian motions for particular changes of time. We consider Markov chain X on a state space $E = \{1, 2, 3\}$, starting from 1 and following the intensity matrix

$$\Lambda = \begin{pmatrix} -3 & 2 & 1 \\ 4 & -6 & 2 \\ 0.5 & 1.5 & -2 \end{pmatrix}.$$

In addition, we consider changes of time $\tau_1, \ldots, \tau_4$ given by equations

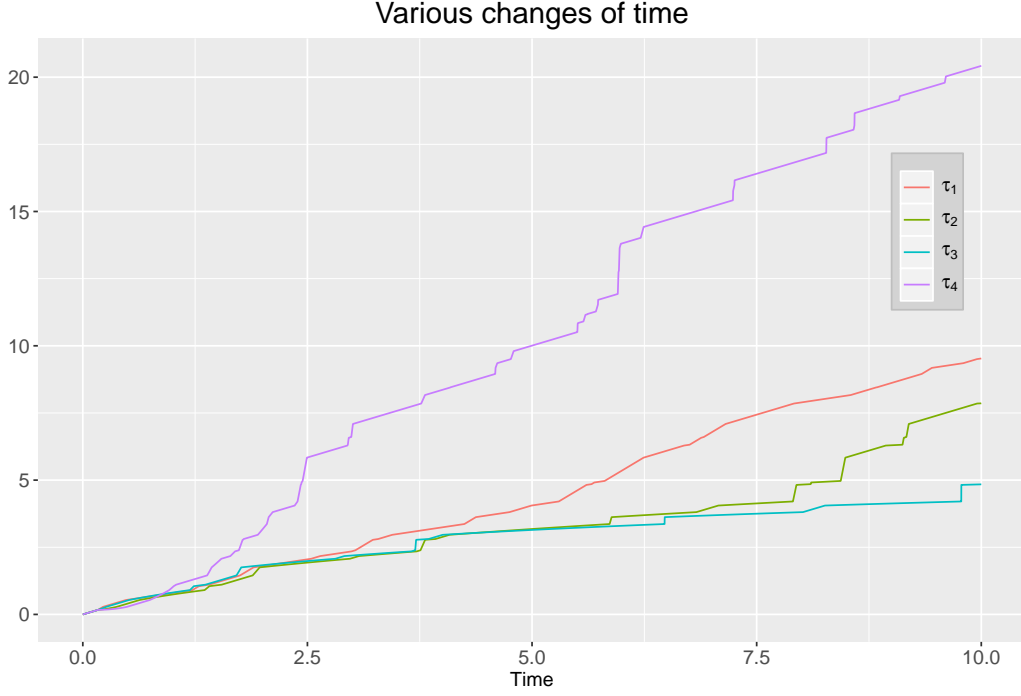$$\tau_i(t) = \int_0^t g_i\left(s, X_{\tau_i(s)}\right)\mathrm{d}s, \quad i = 1, \ldots, 4$$

Figure 4.1: Sample paths of changes of time $\tau_1, \ldots, \tau_4$.

for functions $g_1, \ldots, g_4$, which are presented in Table 4.1. Note that only the first function g_1 is time-independent, which makes τ_1 time-homogeneous, whereas τ_2, τ_3 and τ_4 are inhomogeneous changes of time.

Table 4.1: Functions corresponding to changes of time $\tau_1, \ldots, \tau_4$.

	$k = 1$	$k = 2$	$k = 3$
$g_1(t,k)$	1	$\frac{1}{2}$	2
$g_2(t,k)$	1	$\frac{1}{2}t^{-1/2}$	$2t$
$g_3(t,k)$	1	$\frac{1}{t+1}$	e^t
$g_4(t,k)$	1	$2t$	$3t^2$

Let S be given by 4.5.1, with $S_0 = 100$, $r = 0.01$ and $\sigma = 0.1$. The process S will be time-changed according to $\tau_1, \ldots, \tau_4$. We draw simulations of X and S and then we simulate τ_i, $i = 1, \ldots, 4$ using (4.2.2). The graphs of these changes of
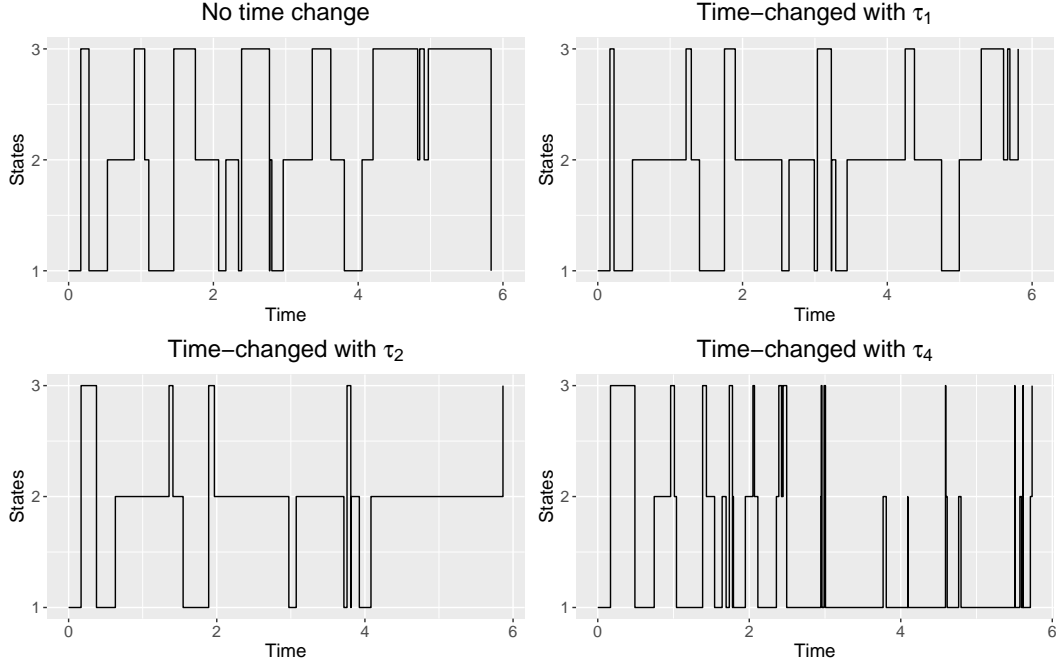
Figure 4.2: Sample paths of Markov chain $(X_t)_{t\geq 0}$ and time-changed Markov chains $X_{\tau_1(t)}$, $X_{\tau_2(t)}$ and $X_{\tau_4(t)}$.

time and sample paths of time-changed Markov chains are presented in Figures 4.1 and 4.2 respectively. The graphs of sample paths of asset price processes $S_{\tau_i(\cdot)}$ are depicted in Figure 4.3.

Let us denote by $T_1, T_2, \ldots$ the jump times of X and by $\rho_1^i, \rho_2^i, \ldots$ – the jump times of $X_{\tau_i(\cdot)}$ for $i = 1, \ldots, 4$. Note that in state $k = 1$ the clock "runs normally", i.e. if $X_{T_j} = 1$ for some $j \in \mathbb{N}$, then

$$T_{j+1} - T_j = \rho_{j+1}^i - \rho_j^i \quad \text{a.s. for all } i = 1, \ldots, 4.$$

From the formula for g_4 we see that the time change τ_4 speeds up in both states $k = 2$ and $k = 3$, however the acceleration is smaller in $k = 2$. For the other time changes state $k = 2$ is decelerating and $k = 3$ – accelerating. Naturally, time-changed Markov chains tend to stay longer in the decelerating states and

jump out quickly of accelerating states (which is also noticeable in Figure 4.2), which results in τ_2 and τ_3 being on average much slower.

This categorization of states may reflect possible economic condition, where $k = 1$ represents a "neutral state", $k = 2$ – lower business activity and $k = 3$ – higher business activity. Functions $g_1, \ldots, g_4$ are responsible for the sensitivity of the transaction flow to the changes of the market.
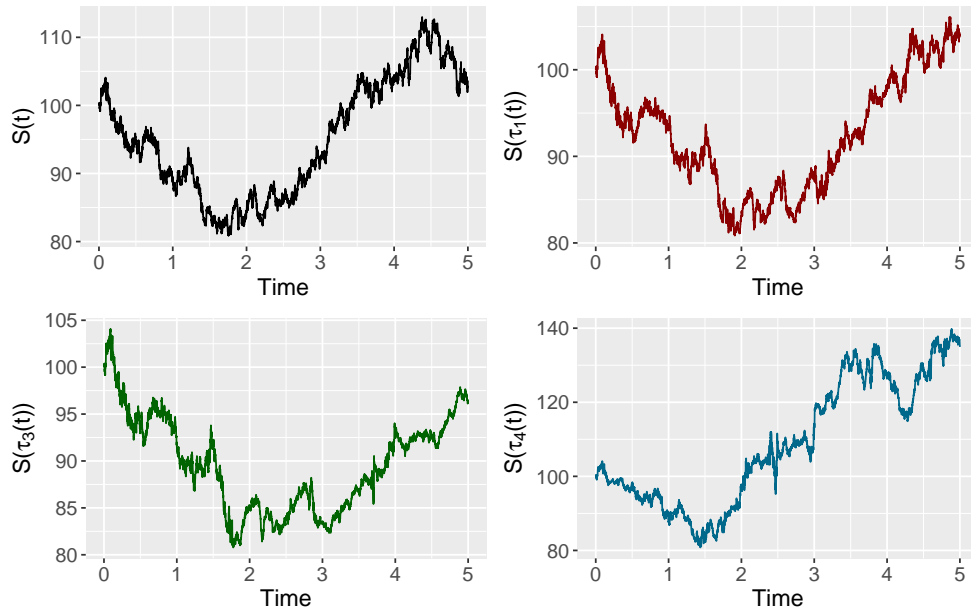

Figure 4.3: Sample paths of asset price process S (top left) and of time-changed asset prices $S_{\tau_1(t)}$ (top right), $S_{\tau_3(t)}$ (bottom left) and $S_{\tau_4(t)}$ (bottom right).

Similarly to Chapter 3.3, we price three kinds of options:

- Standard European call option, $\omega(S_T) = (S_T - K)^+$

- Power options $\omega(S_T) = (S_T^\alpha - K)^+$ for $\alpha = 1.3$

- Self-quanto options $\omega(S_T) = S_T(S_T - K)^+$.

For each kind we set the maturity $T = 3$ years.

We run $N = 50000$ simulations and then price the options for various strike prices K, for each time change $\tau_1, \ldots, \tau_4$ separately. In order to compare the results with the classical Black-Scholes model, we also introduce the time change $\tau_0(t) = t$. The prices of options for those time changes are presented in Figures 4.4, 4.5 and 4.6. One can also find the exact prices for selected strikes in Table 4.2.

Table 4.2: Option prices for various changes of time.

Option	European call	Self-quanto	Power, $\alpha = 1.3$
Strike K	100	100	400
S (Black-Scholes)	8.380	1064.047	44.367
$S(\tau_1)$	7.485	930.552	38.857
$S(\tau_2)$	7.171	874.893	37.581
$S(\tau_3)$	7.193	881.072	37.549
$S(\tau_4)$	11.730	1652.923	62.718

For all kinds of options, the prices for change of time τ_4 are dominating other prices. This result is intuitive, since this is the only accelerating change of time, which makes the volatility of the time-changed process on average the highest. For τ_1, τ_2 and τ_3, the prices are lower than in the Black-Scholes model, which is consistent with the fact that those changes of time are decelerating. Note also that the difference between the prices for τ_2 and τ_3 is negligible for most cases.
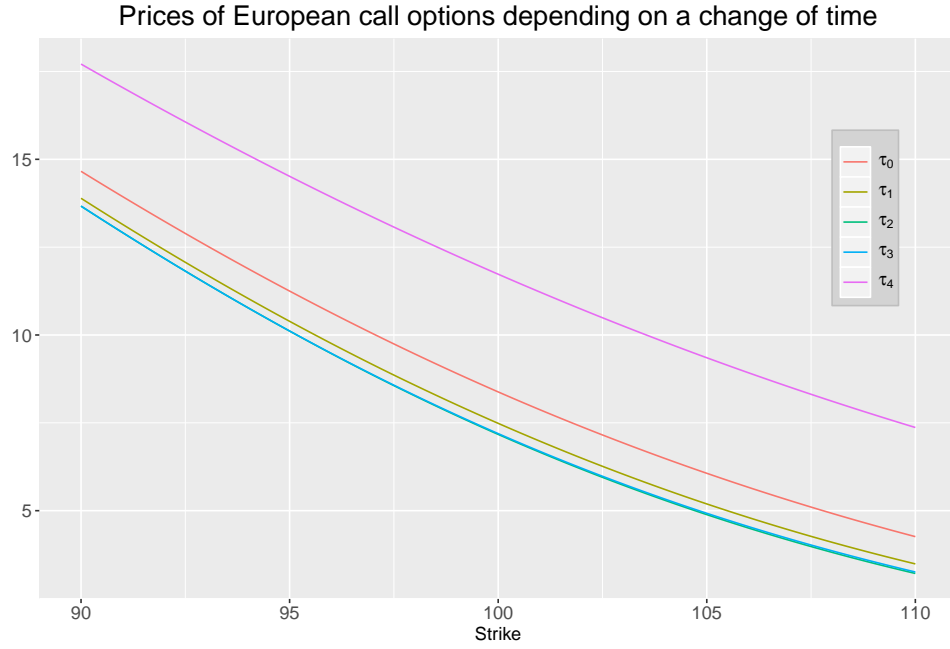
Figure 4.4: Prices of standard European call options on $\widehat{S}$ for various time changes $\tau_0, \ldots, \tau_4$. τ_0 (red line) corresponds to the Black-Scholes model.
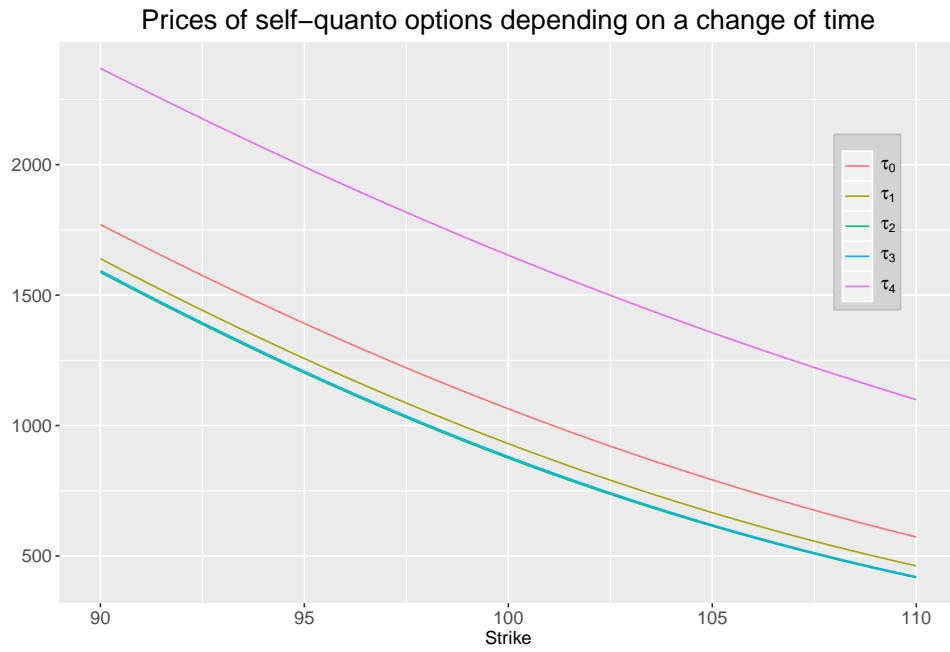
Figure 4.5: Prices of self-quanto options on $\widehat{S}$ for various time changes $\tau_0, \ldots, \tau_4$. τ_0 (red line) corresponds to the Black-Scholes model.
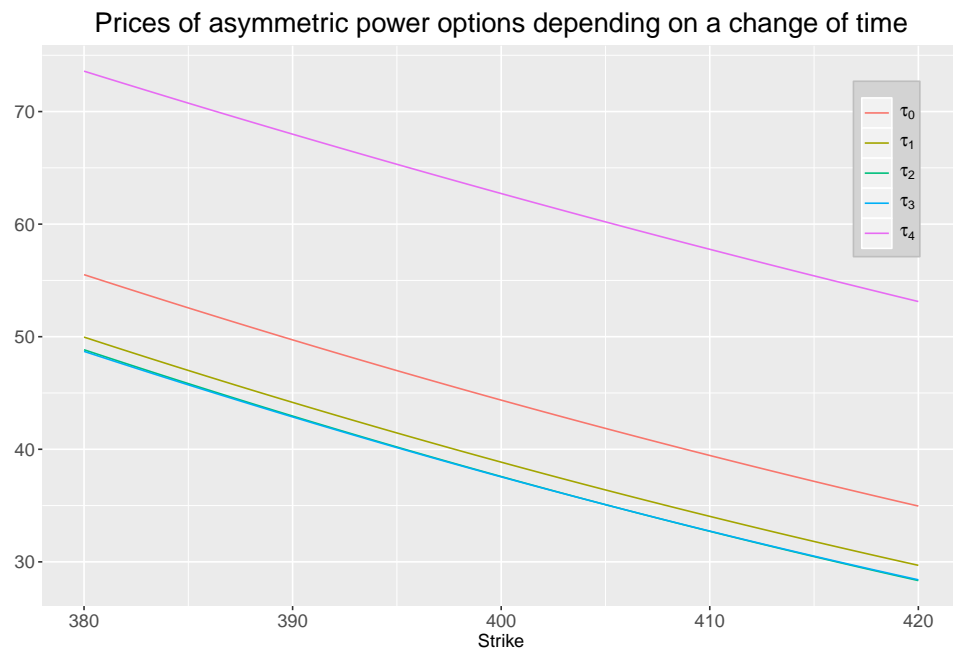
Figure 4.6: Prices of asymmetric power options on $\widehat{S}$, for $\alpha = 1.3$ for various time changes $\tau_0, \ldots, \tau_4$. τ_0 (red line) corresponds to the Black-Scholes model.

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