University of Warsaw

Faculty of Mathematics, Informatics and Mechanics

Wojciech Górny

Anisotropic least gradient problems

PhD dissertation

Supervisor:
prof. dr hab. Piotr Rybka
Institute of Applied Mathematics and Mechanics
University of Warsaw

Auxiliary Supervisor:
dr Ahmad Sabra
Department of Mathematics
American University of Beirut

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Author's declaration:

I hereby declare that this dissertation is my own work.

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Wojciech Górny

Supervisors' declaration:

The dissertation is ready to be reviewed.

December 17, 2019 ....
prof. dr hab. Piotr Rybka

December 17, 2019 ....
dr Ahmad Sabra

Abstract

The goal of this dissertation is to examine, in the anisotropic least gradient problem, what is the influence of the geometry of the domain and properties of the anisotropy on existence, uniqueness and regularity of the solutions.

In this dissertation we consider three different situations. The first one is the case when ϕ, the function which defines the anisotropy, is a norm. Then, we show existence of solutions to the anisotropic least gradient problem for strictly convex domains and boundary data which are continuous almost everywhere. We are particularly interested in the case when ϕ is not strictly convex; then, the lack of dependence on location of ϕ is particularly important.

In the second case, ϕ may depend also on location, but we assume high regularity and uniform convexity of ϕ. Then, apart from a result analogous to the previous one, we may also use e.g. the maximum principle for minimal surfaces to show that while for fixed boundary data are not necessarily unique, all the solutions have the same frame of superlevel sets.

Finally, in the third case we focus on domains Ω which are not bounded convex sets. We consider two situations: when Ω is a strictly convex set, but it is unbounded, and when the boundary of Ω is not connected. For clarity of the presentation in the second case we restrict most of the discussion to the case, when Ω is an annulus on the plane, which enables us to use methods typical for the optimal transport problem.

Keywords: Least Gradient Problem, Anisotropy, Minimal Surfaces, L^p Regularity, Non-strict Convexity, Nonconvex Domains

AMS Subject Classification: 35J20, 35J25, 35J75, 35J92

Streszczenie

Celem niniejszej pracy jest zbadanie, jaki wpływ na istnienie, jednoznaczność i regularność rozwiązań w anizotropowym zagadnieniu najmniejszego gradientu mają geometria obszaru oraz właściwości zadanej anizotropii.

W pracy poruszane są trzy różne sytuacje. W pierwszej z nich funkcja ϕ zadająca anizotropię jest normą. Wówczas pokazujemy istnienie rozwiązań anizotropowego zagadnienia najmniejszego gradientu dla ściśle wypukłych obszarów oraz danych brzegowych ciągłych prawie wszędzie. Szczególnie interesuje nas przypadek, gdy ϕ nie jest ściśle wypukła; wtedy brak zależności ϕ od położenia gra szczególną rolę.

Drugi przypadek dotyczy sytuacji, gdy ϕ może zależeć także od położenia, jednak zakładamy wysoką regularność oraz jednostajną wypukłość ϕ. Wówczas oprócz wyniku analogicznego do poprzedniego możemy także użyć m.in. zasady maksimum dla powierzchni minimalnych do pokazania, że dla ustalonych danych brzegowych rozwiązania niekoniecznie są jednoznaczne, ale wszystkie rozwiązania mają identyczną strukturę poziomic.

Wreszcie trzeci przypadek dotyczy sytuacji, gdy dziedzina Ω nie jest obszarem ograniczonym oraz wypukłym. Rozważamy dwie sytuacje: kiedy Ω jest zbiorem ściśle wypukłym, ale nieograniczonym, oraz kiedy Ω ma niespójny brzeg. Dla czytelności wywodu w drugim przypadku ograniczamy większość dyskusji do przypadku, kiedy Ω jest pierścieniem na płaszczyźnie, co umożliwia stosowanie metod pochodzących z zagadnienia optymalnego transportu.

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Contents

Chapter 1

Introduction

The main topic of this dissertation is the analysis of several variants of the least gradient problem, which is the variational formulation of the 1-Laplace equation with Dirichlet boundary conditions:

$$\min \left\{ \int_\Omega |Du|, \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \right\}. \tag{LGP}$$

We will explore various aspects of this problem. The first and foremost issue will be the anisotropic case, when instead of the Euclidean total variation $|Du|$ we calculate the total variation with respect to a function $\phi(x,p)$, which among other properties is convex and 1-homogenous with respect to the second variable.

$$\min \left\{ \int_\Omega \phi(x, Du), \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \right\}. \tag{aLGP}$$

Furthermore, much effort will be devoted to understanding the structure of minimisers and their regularity. Finally, we will examine the influence that the shape of the domain Ω has on existence and structure of solutions. In the next Section, we will briefly describe the motivations behind the study of the least gradient problem and its relations to another problems considered in the literature. Later, in Section 1.2, we list the most important results presented in this dissertation and how they are organised into Chapters.

1.1 Motivations

In the calculus of variations minimisation problems for functionals of linear growth have a special place. A programme of analysis of such type of problems was suggested by Giaquinta, Modica and Souček in [23]. In the most general setting, it involves minimisation of the functional of the form

$$F(u) = \int_\Omega f(x, u(x), Du(x))\, dx$$

for a function $f(x, u, p)$ which is convex in p and has linear growth in p, i.e.

$$m|p| \leq f(x, u, p) \leq M(1 + |p|).$$

The best known problem of this kind is minimisation of the area functional. However, the fact that f has linear growth implies that we cannot use neither the machinery of Sobolev spaces for $p > 1$ nor Orlicz spaces. In particular, our most pressing problem will be compactness of families of minimisers. The most natural space for solutions of such problems is the BV space; however, in this case the standard BV theory gives us only compactness of minimisers with respect to the L^1 topology. When we additionally impose Dirichlet boundary condition, we lose compactness, as the trace operator on the BV space is not continuous with respect to L^1 convergence. The least gradient problem is a model problem for understanding the role played by the Dirichlet boundary condition in minimisation problems with linear growth.

In 1969 Bombieri, de Giorgi and Giusti in the paper [9] introduced the concept of least gradient functions - functions which locally minimise the total variation, without taking into account any boundary conditions. The authors proved that boundaries of superlevel sets are area-minimising, which enables the study of minimal surfaces via functions of least gradient. In turn, this led the authors to prove that in dimensions eight and above there exist minimal cones which are not half-spaces and solve the Bernstein problem.

The least gradient problem in the form (LGP) was introduced by Sternberg, Williams and Ziemer in 1992 in the paper [65]. Their main motivation came from numerical methods: in [55], Parks developed a method for approximating an oriented surface with least area having a prescribed boundary (see also [56]). This leads to a least gradient problem with additional constraints. Among the conditions considered in the literature are a bounded slope condition on boundary data or a constraint on the length of ∇u, see [41] and [66]; this approach also has links to optimal design. The main contribution from the authors of [65] is the use of methods from geometrical measure theory to approach the least gradient problem without these additional constraints. The authors established that for continuous boundary data, under a set of conditions on an open bounded set $\Omega \subset \mathbb{R}^N$ slightly weaker than strict convexity, a unique solution to Problem (LGP) exists and it is continuous up to the boundary. Since many of the methods used in this dissertation and in the modern literature descend directly from the methods introduced in [65], we will discuss them at length in Chapter 2.

Another view on problem (LGP) is provided by the p-Laplace equation

$$-\mathrm{div}(|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f,$$

where $1 < p < \infty$. Its variational formulation is the following minimisation problem

$$\min \left\{ \frac{1}{p} \int_\Omega |\nabla u|^p \, dx, \quad u \in W^{1,p}(\Omega), \quad u|_{\partial\Omega} = f \right\}.$$

In particular, the least gradient problem is the formal limit of the variational versions of the p-Laplace problems as $p \to 1$. It was shown by Juutinen in 2004 in the paper [38] that p-harmonic functions indeed converge to least gradient functions as $p \to 1$ (see also [40]). Moreover, in 2014 Mazón, Rossi and Segura de León in the paper [48] gave a precise meaning to the 1-Laplace equation

$$-\mathrm{div}(\frac{Du}{|Du|}) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f$$

and showed that it can be understood as the Euler-Lagrange formulation of the least gradient problem.

The anisotropic least gradient problem in the form (aLGP) was introduced in 2013 by Jerrard, Moradifam and Nachman in the paper [36]. While the problem was stated there in full generality, the main motivation was the weighted least gradient problem, when the anisotropy takes the form $\phi(x, Du) = a(x)|Du|$. It is a dimensional reduction of the conductivity imaging problem, which is the following inverse problem: given a body Ω, we can measure the electrical current and the voltage on its boundary. We want to determine the conductivity inside Ω. In fact, we can assume that we know the modulus of the current density $|J|$ inside Ω, see [37]. If u is the electric potential corresponding to the voltage f on $\partial\Omega$, then the Gauss law takes the form

$$\mathrm{div}(\sigma \nabla u) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f,$$

where σ is the conductivity. By Ohm's law, we have $J = -\sigma \nabla u$. Hence, formally we have

$$\mathrm{div}(-|J| \frac{\nabla u}{|\nabla u|}) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = f.$$

This formal 1-Laplace equation with weight $a = |J|$ has been shown in [54] to be equivalent to the weighted least gradient problem

$$\min \left\{ \int_\Omega a(x)|Du|, \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \right\}. \tag{wLGP}$$

When we solve the weighted least gradient problem, determination of the conductivity σ from the potential u is immediate.

In two dimensions, the least gradient problem has a direct link to optimal transport. As was shown in 2016 by Rybka, Sabra and the author of this dissertation in [31], in two dimensions the least gradient problem is equivalent to the Beckmann problem (also called the free material design problem)

$$\min \left\{ \int_\Omega |p|, \quad p \in \mathcal{M}(\Omega; \mathbb{R}^2), \quad \mathrm{div}\, p = 0, \quad p \cdot \nu|_{\partial\Omega} = g := \partial_\tau f \right\}. \tag{BP}$$

This equivalence has been extended by Dweik and Santambrogio in 2018 in [19] to include anisotropic cases. Problem (BP) has a natural interpretation: given a load applied on the boundary, the goal is to find an optimal distribution of a body to support this load. Moreover, it is in turn equivalent to the optimal transport problem

$$\min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\gamma \,:\, \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\ (\Pi_x)_{\#}\gamma = g^+ \text{ and } (\Pi_y)_{\#}\gamma = g^- \right\} \tag{OTP}$$

where g^+ and g^- are the positive and negative parts of g and the source and target measures are located on $\partial\Omega$, see for instance [60].

Finally, it is worth mentioning that various authors have also considered the parabolic version of this problem, namely the total variation flow, which formally is the equation

$$u_t = \operatorname{div}\left(\frac{Du}{|Du|}\right)$$

equipped with initial and boundary conditions. The motivation for considering the total variation flow comes from image recovery. Its discrete version has been suggested as a way of removing noise from the image while preserving sharp boundaries of the photographed object by Rudin, Osher and Fatemi in 1992 in the paper [57]. Among others, the total variation flow and its discretisation, the Rudin-Osher-Fatemi model, have been studied in [4], [7], [8], [12], [24], [34], [51], and [72]. Various other problems related to the 1-Laplace equations, such as eigenvalue problems, obstacle problems, dynamical boundary conditions, nonlocal versions, and super- and subsolutions were studied for instance in [11], [22], [43], [47] and [61] respectively. Some of the methods used in the study of the problems involving the 1-Laplace equation listed above are similar to the ones in the least gradient problem, usually in the language of Anzelotti pairings, but in this dissertation we focus on the trace constraint which forces us to use a different approach.

1.2 Main results

In this Section, we list shortly the main results in this dissertation and discuss its structure. This PhD thesis is a result of three years of work and all the results have already been published in journals or submitted for publication and uploaded to preprint archives. Here, I list all of them, in order of date of appearance on arXiV: [27]; [26]; [29]; [30]; [17]. I am the sole author of all of them except for [17], which is a result of my collaboration with Samer Dweik from the University of British Columbia.

Except for Chapters 1 and 2, which serve as an introduction to the whole thesis, one article corresponds to one chapter in this thesis. The order of appearance of these articles on arXiv is preserved. The content of these articles is essentially unchanged. The differences are as follows: the introductions have been re-written to show in more detail the relationships between Chapters; most of the preliminaries sections have been moved to a unifying Chapter 2; the notation has been made more uniform; also, a few technical results have been moved to the Appendix. On a few occasions, small parts have been re-written to provide more clarity or to give additional commentary to the relationship between results in this thesis and the existing literature. Finally, I will often rely on two earlier articles (which are not part of this thesis) that I was an author or co-author of, namely [28] and [31].

Before we state the main results, let us stress that the least gradient problem is at its core a geometrical problem, as due to the results of [9] we may think of it as a question of existence of a foliation of the set Ω by minimal surfaces in a way that is enforced by the boundary data, except for a set on which the solution is locally constant. A recurring idea in this thesis is that the boundary data impose a structure of the solution, if not the values of the solution itself, and this fact has consequences for existence, uniqueness and regularity of minimisers. Hence, the statement of the problem and validity of most results vary with the geometry of Ω. In this dissertation, general assumptions regarding the set Ω are the following: throughout the entire thesis (except for a part in Chapter 5, which will be addressed separately) we will assume that Ω **is an open bounded set with Lipschitz boundary**. We will often additionally assume some form of convexity of Ω; we note that bounded convex sets automatically have Lipschitz boundary.

The first issue explored in this thesis is the structure of minimisers in the isotropic least gradient problem, which is the main subject of Chapter 3. Since the publication of [65], we have known that for a strictly convex domain and continuous boundary data the solutions to Problem (LGP) are unique. On the other hand, examples provided in [45] and [68] show that for discontinuous boundary data, the solutions need not be unique. However, the structure of the level sets in these examples remains very similar. This led Mazón, Rossi and Segura de León ([48]) to conjecture that the frame of superlevel sets is the same for all minimisers and is determined by the boundary data. Concurrently with this thesis, Moradifam proved in [53] that this frame of superlevel sets is determined by a single vector field. The main result in Chapter 3 is Theorem 3.1.1, which confirms the conjecture of the authors of [48]. Moreover, this conjecture is refined to be a result about pointwise values of solutions of the least gradient problem.

Theorem 3.1.1 Let $\Omega \subset \mathbb{R}^N$, where $2 \leq N \leq 7$, be an open bounded convex set. Let u, v be precise representatives of functions of least gradient in Ω such that $Tu = Tv = h$. Then $u = v$ on $\Omega \backslash (C \cup \mathcal{N})$, where both u and v are locally constant on C and $\mathcal{N}$ has Hausdorff dimension at most $N - 1$.

Most of Chapter 3 is devoted to the proof of Theorem 3.1.1 and to the analysis of its consequences. The proof is very geometrical and relies on the extensive use of a maximum principle for minimal surfaces. In the final Section of Chapter 3, we discuss an approximation of the least gradient problem by the strain-gradient plasticity model, see [2], suggested to the author of this dissertation by Lorenzo Giacomelli from Sapienza Università di Roma.

Chapters 4-6 form the core part of this thesis and concern the anisotropic case. We start with the case when the anisotropy ϕ is a norm. In [36], Jerrard, Moradifam and Nachman discuss the case when the anisotropy may depend on location and it turns out that existence of minimisers is easy - the authors identified a sufficient and almost necessary condition (called the *barrier condition*) for existence of minimisers for continuous boundary data. However, the situation concerning uniqueness and regularity of minimisers is different; the positive results proved in [36] have very strong regularity and uniform convexity requirements on ϕ because of their dependence on classical estimates on the size of singularities of $\phi-$minimal surfaces due to Schoen, Simon and Almgren, see [62].

In Chapter 4, the main effort concerns the two-dimensional situation. Instead of assuming any regularity of ϕ, we instead assume that it is a norm and use the special geometrical situation in two dimensions to discuss uniqueness and regularity of minimisers to the least gradient problem. This variant of the problem is as follows:

$$\min \bigg\{ \int_\Omega \phi(Du), \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \bigg\}, \tag{aLGP2}$$

With a slight abuse of notation, we will say that a norm ϕ is strictly convex if the unit ball $B_\phi(0,1)$ is strictly convex. Restricting our attention to norms, way we are able to consider the situation when ϕ is not a strictly convex norm, even though there are no smooth sets which satisfy the barrier condition. The first of the main results in this Chapter is:

Theorem 4.3.14 Suppose that $\Omega \subset \mathbb{R}^2$ is uniformly convex (in the sense of Definition 4.3.8) and ϕ is a fixed strictly convex norm on $\mathbb{R}^2$. Let $f \in C(\partial\Omega)$ and take ω to be a modulus of continuity of f. Let u be the solution of Problem (aLGP2) with boundary data f. Then $u \in C(\overline{\Omega})$ and it is continuous with modulus of continuity

$$\overline{\omega}(|x-y|) = \omega(c(\Omega)|x-y|^{1/2}).$$

In particular, $C^{0,\alpha}$ regularity of boundary data implies $C^{0,\frac{\alpha}{2}}$ regularity of the solution, which is the optimal Hölder regularity exponent in the isotropic case, see [65]. The second main result of this Chapter is an existence result in the non-strictly convex case:

Theorem 4.4.2 Let $\Omega \subset \mathbb{R}^2$ be an open bounded strictly convex set. Let ϕ be a fixed norm on $\mathbb{R}^2$. Let $f \in C(\partial\Omega)$. Then there exists at least one solution to Problem (aLGP2).

This result is complemented by a series of results showing that given smooth boundary data, the solution need not be unique, but we can find a single minimiser which is continuous. Moreover, we give a detailed discussion on the barrier condition and why it is not suitable in the non-strictly convex setting.

The next issue is existence of minimisers for discontinuous boundary data. The motivation for Chapter 5 is existence of minimisers in the isotropic least gradient problem for BV boundary data in the planar case proved in [28]. In this Chapter, we simultanously extend this result to higher dimensions and anisotropic cases. Recall that we say that the norm ϕ is strictly convex if the unit ball $B_\phi(0,1)$ is strictly convex. The main result is:

Theorem 5.3.1 Let $\Omega \subset \mathbb{R}^N$ be an open bounded strictly convex set. Let ϕ be a strictly convex norm on $\mathbb{R}^N$. Suppose that $f \in L^1(\partial\Omega)$ is a function such that $\mathcal{H}^{N-1}$-almost all points of $\partial\Omega$ are continuity points of f. Then there exists a minimiser to the Problem (aLGP) with boundary data f.

This Theorem is optimal in the sense that even in the planar case there exist counterexamples for existence if we allow the boundary data to be discontinuous on a set of positive $\mathcal{H}^1$-measure, see [64] and [16]. A similar result (Theorem 5.3.6) concerns the case when ϕ may depend on location, but in turn we assume high regularity of ϕ and the barrier condition. Concurrently with this thesis, a very similar result has been proved by Moradifam in [53]. The reasoning behind these results is the following: the fact that the boundaries of superlevel sets of anisotropic least gradient functions are ϕ-minimal surfaces enforces a structure of the solution, hence we can find a natural sequence of approximations which reproduces a similar structure. The second part of Chapter 5 introduces the least gradient problem on unbounded domains; in particular, we focus on the issue of uniqueness of appropriately defined solutions and show that an analogue of Theorem 3.1.1 is not possible in the unbounded case.

The results proved in Chapter 5 enable us to consider the anisotropic least gradient problem for unbounded boundary data, as long as their discontinuity set has $\mathcal{H}^{N-1}$-measure zero. In 2015, Hakkarainen, Korte, Lahti and Shanmugalingam proved that L^∞ bounds on boundary data lead to L^∞ bounds for the solution, see [33]; in Chapter 6, we extend this result and prove L^q estimates on the solution in terms of L^p norm of the boundary data. The main result in this Chapter is about the higher integrability of the minima in the anisotropic least gradient problem.

Theorem 6.2.2 Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary. Suppose that $1 \leq p < \infty$. Let $u \in BV(\Omega)$ be a ϕ-least gradient function such that $Tu = f \in L^p(\partial\Omega)$. Then $u \in L^{\frac{Np}{N-1}}(\Omega)$.

The exponent $\frac{Np}{N-1}$ is shown to be optimal. Again, the rationale behind this result is that the structure of minimisers in the anisotropic least gradient problem is quite rigid, as the boundaries of superlevel sets are ϕ-minimal surfaces. Hence, we can use the isoperimetric inequality to estimate the measure of a superlevel set in terms of its boundary data; its minimality gives us an estimate in terms of boundary data. The exponent comes from the use of the isoperimetric inequality. Moreover, we prove that away from the boundary the solution is in L^∞ in two settings: in the anisotropic least gradient problem on the plane and in the isotropic least gradient problem in any dimension.

Finally, in Chapter 7, we take a much different approach to the least gradient problem. It was shown in [31] by Rybka, Sabra and myself that on convex domains in two dimensions there is a relationship between the least gradient problem and the Beckmann problem, which is in turn related to the optimal transport problem. In [19], Dweik and Santambrogio used this connection to use optimal transport methods to prove regularity estimates for solutions of the least gradient problem. Here, we again use this connection to be able to consider the least gradient problem in an unusual setting: on a nonconvex domain, for which we cannot use the classical results from [65] which require strict convexity of the domain. For clarity, we consider the model case of an annulus. This Chapter is based on [17], which is a result of a collaboration with Samer Dweik from the University of British Columbia.

We start with extending the equivalence between the least gradient problem and the Beckmann problem to the case of the annulus; it turns out that if the boundary of the domain is not connected, as in this case, then solving the Beckmann problem is equivalent to solving simultanously multiple least gradient problems. Since in Chapter 7 we will extensively use methods of optimal transport, we assume that $g \in \mathcal{M}(\partial\Omega)$. The main result is:

Theorem 7.3.4 Let $\Omega \subset \mathbb{R}^2$ be an annulus in the sense of Definition 7.2.1. Consider the following minimisation problems:

$$\inf \left\{ \int_{\overline{\Omega}} |v| : \, v \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^2), \;\; \nabla \cdot v = g \right\} \tag{1.2.1}$$

$$\inf \left\{ \int_{\Omega} |Du| : \, u \in BV(\Omega), \; \partial_\tau (Tu) = g \right\}. \tag{1.2.2}$$

We have $\inf (1.2.1) = \inf (1.2.2)$. Moreover, from each solution $u \in BV(\Omega)$ of (1.2.2), one can construct a solution to (1.2.1). On the other hand, from each solution $v \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^2)$ of (1.2.1), one can construct a solution to (1.2.2), provided that $|v|(\partial\Omega) = 0$.

Using this new equivalence, we prove existence of minimisers under certain admissibility conditions. As the domain is not strictly convex, we use the relationship between the Beckmann problem and the optimal transport problem to change the question of existence of minimisers to either (1.2.1) or (1.2.2) to a question if the transport rays of a corresponding optimal transport problem lie inside the domain or not. The admissibility conditions introduced in Chapter 7 are such that they guarantee that the transport rays lie inside Ω. Moreover, we prove L^p estimates on the transport density, which correspond to $W^{1,p}$ estimates in the corresponding least gradient problem.

Finally, in the Appendix there are a few results concerning general BV theory. The results included concern traces of BV functions and are quite simple, but their proofs are difficult to locate in the literature. They are included in this dissertation for completeness.

Chapter 2

Preliminaries

2.1 Least gradient functions

In this subsection we recall the definition and some properties of least gradient functions; a standard reference is [9] and [25]. Then we prove some results concerning pointwise properties of precise representatives of least gradient functions. The natural setting for considering least gradient functions are BV spaces; for a definition of BV spaces and a reference to standard BV theory, see [3], [20] or [70].

Definition 2.1.1. Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with Lipschitz boundary. We say that $u \in BV(\Omega)$ is a function of least gradient, if for every $v \in BV(\Omega)$ which is compactly supported in Ω we have

$$\int_\Omega |Du| \leq \int_\Omega |D(u+v)|.$$

The assumption on compact support of v is equivalent to the condition that $v \in BV_0(\Omega)$, see [67, Theorem 2.2]. This equivalence is proved using an approximation with functions of the form $v_n = v\chi_{\Omega_n}$ for suitably chosen Ω_n.

Additionally, if a set $E \subset \Omega$ is such that χ_E is a function of least gradient, we say that E is a minimal set.

Definition 2.1.2. We say that $u \in BV(\Omega)$ is a solution to Problem (LGP), if u is a function of least gradient and the trace of u equals f, i.e. for $\mathcal{H}^{N-1}-$almost every $x \in \partial\Omega$ we have

$$\lim_{r \to 0^+} \fint_{B(x,r)\cap\Omega} |f(x) - u(y)| \, dy = 0.$$

Here and in the whole manuscript $\mathcal{H}^{N-1}$ denotes the Hausdorff measure of codimension one. Since least gradient functions are only assumed to be functions of bounded variation, they are defined up to a set of Lebesgue measure zero. Hence, if we want to state any pointwise regularity results, we have to choose a proper representative. The main tool we are going to use is the regularity of boundaries of area-minimising sets; hence, following [65], we employ the convention that a set of bounded perimeter consists of all its points of positive density.

Now, we recall a few classical results concerning least gradient functions, which we will use extensively in this thesis. We start with a theorem by Bombieri, de Giorgi and Giusti, which gives us a link between the function u of least gradient and the regularity of its superlevel sets. Let us stress that due to this result and its anisotropic counterparts the study of superlevel sets will be one of the key tools in this thesis.

Theorem 2.1.3. ([9, Theorem 1]) *Suppose that $\Omega \subset \mathbb{R}^N$ is open and let $u \in BV(\Omega)$ be a function of least gradient in Ω. Then for every $t \in \mathbb{R}$ the set $\{u > t\}$ is minimal in Ω, i.e. the function $\chi_{\{u>t\}}$ is of least gradient.*

The proof is based on the co-area formula, so the Theorem also holds for sets of the form $\{u \geq t\}$. Similarly, all the results below concerning pointwise regularity of least gradient functions could be stated for either $\{u > t\}$ or $\{u \geq t\}$; later, we will use whichever version is more convenient.

This result was later improved in [25, Chapter 10] that in low dimensions ($k \leq 7$), under the convention that a set of bounded perimeter consists of all its points of positive density, the boundary ∂E of a minimal set E is an analytical hypersurface. In higher dimensions, ∂E is regular except for a set of Hausdorff dimension $N - 8$. In particular, if we take the precise representative of a least gradient function u, then in dimensions up to seven E_t is an analytical minimal surface for every t. For this reason, in this thesis we will always assume that if $u \in BV(\Omega)$, we always consider u to be the precise representative of a BV function in order to be able to state any pointwise results.

The next result is a theorem by Miranda and it concerns the stability of families of least gradient functions:

Theorem 2.1.4. ([50, Theorem 3]) *Let $\Omega \subset \mathbb{R}^N$ be open. Suppose that $u_n \in BV(\Omega)$ is a sequence of least gradient functions in Ω convergent in $L^1_{loc}(\Omega)$ to $u \in BV(\Omega)$. Then, u is a function of least gradient in Ω.*

However, this theorem has a very important limitation: as the trace operator is not continuous with respect to L^1 convergence, it gives us no control on the trace of the limit function u. In fact, in this manuscript and other works concerning least gradient functions ([28, 31, 65]) much effort is devoted to prove that the trace of the limit function is correct.

The third result, a theorem by Sternberg, Williams and Ziemer, concerns the existence of minimisers to Problem (LGP) in case when $\Omega \subset \mathbb{R}^N$ is an open bounded strictly convex set. To be precise, assumptions on Ω in [65] are a little weaker: the authors require that $\partial\Omega$ has non-negative mean curvature in the weak sense and that $\partial\Omega$ is not locally minimising with respect to internal variations. In the case when $\partial\Omega$ is smooth, this condition is equivalent to positive mean curvature on a dense subset of $\partial\Omega$; moreover, in two dimensions it is equivalent to strict convexity. For simplicity, we are interested only in strictly convex sets.

Theorem 2.1.5. ([65, Theorems 3.7, 4.1]) *Let $\Omega \subset \mathbb{R}^N$ be an open, bounded, strictly convex set and suppose that $f \in C(\partial\Omega)$. Then there exists a unique minimiser $u \in BV(\Omega)$ to Problem* (LGP) *and additionally $u \in C(\overline{\Omega})$.*

Since the Sternberg-Williams-Ziemer construction of minimisers is a classical type of argument in problems relating to least gradient functions, we will recall its outline. Suppose that we have an open bounded strictly convex set Ω. Let $f \in C(\partial\Omega)$. We take its extension $F \in C(\mathbb{R}^N \backslash \Omega) \cap BV(\mathbb{R}^N \backslash \Omega)$ with compact support and denote $L_t = \{F \geq t\}$. For almost all t, the define the sets E_t as solutions of the problem

$$\min\{P(E, \mathbb{R}^N) : E \backslash \overline{\Omega} = L_t \backslash \overline{\Omega}\},$$

$$\max\{|E| : \ E \text{ is a solution of the above}\}.$$

The result does not depend on the choice of the extension F, so the sets E_t are uniquely defined. Then, the authors use the assumptions on Ω to show that they together with continuity of boundary data imply that $E_t \subset\subset E_s$ for $t > s$. Hence, the function

$$u(x) = \sup \{t \in \mathbb{R} : x \in E_t\}$$

is well-defined, continuous, its trace equals f and up to changing the sets E_t on a set of measure zero we have $E_t = \{u \geq t\}$. Apart from its value as an existence result, this construction is important in itself - it shows that boundaries of superlevel sets provide a foliation of Ω except for a set on which the solution is locally constant. Variants of this construction appear for instance in [71] and in Chapter 5. We will follow the notation introduced in the construction in [65] and in the whole manuscript denote $E_t = \{u \geq t\}$.

Recently, there appeared a few papers on the issue of existence of minimisers to Problem (LGP) for discontinuous boundary data. The first and most straightforward answer comes from a paper [64] by Spradlin and Tamasan and says that in general we cannot expect existence of minimisers for boundary data in $L^\infty(\partial\Omega)$ even in the case when Ω is a disk. The Theorem below has later been extended to the case when Ω is an arbitrary domain with C^2 boundary in [16].

Theorem 2.1.6. ([64, Theorem 1.1]) *Let $\Omega = B(0,1) \subset \mathbb{R}^2$. There exist boundary data $f \in L^\infty(\partial\Omega)$ such that Problem* (LGP) *admits no solution. Moreover, f is a characteristic function of a certain fat Cantor set on $\partial\Omega$.*

The second answer to this problem has been suggested by Mazón, Rossi and Segura de León in [48]. In short, the authors suggest that the boundary condition in the sense of traces is too strong; due to Theorem 2.1.3 the would-be minimisers have a structure which is too rigid and cannot fulfil the required boundary condition. Hence, the authors decided to consider instead a relaxation of the total variation functional, namely a functional defined on $L^{\frac{N}{N-1}}(\Omega)$ by the formula

$$F(u) = \begin{cases} \int_\Omega |Du| + \int_{\partial\Omega} |Tu - f| \, d\mathcal{H}^{N-1} & u \in BV(\Omega) \\ +\infty & u \in L^{\frac{N}{N-1}}(\Omega) \backslash BV(\Omega) \end{cases} \tag{2.1.1}$$

and look for minimisers of the functional F. Then, they provide the following characterisation of the Euler-Lagrange type of minimisers of this functional:

Theorem 2.1.7. ([48, Theorem 2.5]) *Take any $u \in BV(\Omega)$. Then, u is a minimiser of the functional F if and only if there exists a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ with $\|\mathbf{z}\|_\infty \leq 1$ such that the following conditions are satisfied:*

$$-\,div(\mathbf{z}) = 0 \qquad \textit{as distributions}; \tag{2.1.2}$$

$$(\mathbf{z}, Du) = |Du| \qquad \textit{as measures}; \tag{2.1.3}$$

$$[\mathbf{z}, \nu] \in sign(f - u) \qquad \mathcal{H}^{N-1} - \textit{a.e. on } \partial\Omega. \tag{2.1.4}$$

Here, $(\mathbf{z}, Du)$ and $[\mathbf{z}, \nu]$ are understood in the sense of Anzelotti pairings. For an introduction to Anzelotti pairings, see [5].

Note that the boundary condition is considered in a weaker sense than the trace condition; instead, it is interpreted as a weak normal trace of an appropriate vector field. In particular, we see that solutions of the least gradient problem are minimisers of the functional F, but the converse is not necessarily true - this is illustrated by Theorem 2.1.6 and the following example (see also [16, Proposition 2.1]).

Example 2.1.8. Let $\Omega = [0,1]^2 \subset \mathbb{R}^2$. Take $h \in C_c^\infty([0,1])$ with image equal to $[0,1]$. We define the boundary data in the following way: for $y < 1$ set $f(x,y) = 0$ and set $f(x,1) = h(x)$. By Theorem 2.1.3, since Ω is two-dimensional, the boundaries of superlevel sets of a solution u are unions of line segments. However, for $t \in (0,1)$ all of these line segments would have to be contained in the segment $[0,1] \times \{1\}$, so $u \equiv 0$; hence, the solution does not satisfy $Tu = f$, contradiction. Here, the boundary datum f may even be smooth; this phenomenon is dictated by the lack of strict convexity of Ω.

A third, partial answer was provided for the planar case in [28].

Theorem 2.1.9. ([28, Theorem 1.1]) *Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, strictly convex set with C^1 boundary. Then for every $f \in BV(\partial\Omega)$ there exists a solution to Problem* (LGP).

While the proof provided in [28] has important limitations, such as relying on the special properties of $\mathbb{R}^2$ and one-dimensional BV space, the result itself is very natural in the general context of the anisotropic least gradient problem. In fact, it will act as a model for a few of the theorems appearing in this thesis. Furthermore, let us note that Theorem 2.1.6 shows that this result is close to optimal.

2.2 Anisotropic BV spaces

In this Section we recall the notion of a metric integrand and the basic facts and definitions concerning BV spaces, focusing on their anisotropic versions. This entire section is based on the construction in [1].

Definition 2.2.1. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. A continuous function $\phi : \overline{\Omega} \times \mathbb{R}^N \to [0, \infty)$ is called a metric integrand, if it satisfies the following conditions:

(1) ϕ is convex with respect to the second variable for a.e. $x \in \overline{\Omega}$;
(2) ϕ is 1-homogeneous with respect to the second variable, i.e.

$$\forall\, x \in \overline{\Omega}, \quad \forall\, \xi \in \mathbb{R}^N, \quad \forall\, t \in \mathbb{R} \quad \phi(x, t\xi) = |t|\phi(x, \xi);$$

(3) ϕ is comparable to the Euclidean norm on $\overline{\Omega}$, i.e.

$$\exists\, \lambda, \Lambda > 0 \quad \forall\, x \in \overline{\Omega}, \quad \forall\, \xi \in \mathbb{R}^N \quad \lambda|\xi| \leq \phi(x, \xi) \leq \Lambda|\xi|.$$

In particular, ϕ is uniformly elliptic in $\overline{\Omega}$. These conditions apply to most cases considered in the literature, such as the classical least gradient problem, i.e. $\phi(x, \xi) = |\xi|$ (see [65]), the weighted least gradient problem, i.e. $\phi(x, \xi) = g(x)|\xi|$ (see [36]), where $g \geq c > 0$, and l_p norms for $p \in [1, \infty]$, i.e. $\phi(x, \xi) = \|\xi\|_p$ (see [28]).

Definition 2.2.2. The polar function of ϕ is $\phi^0 : \overline{\Omega} \times \mathbb{R}^N \to [0, \infty)$ defined as

$$\phi^0(x, \xi^*) = \sup \{\langle \xi^*, \xi \rangle \,:\, \xi \in \mathbb{R}^N,\, \phi(x, \xi) \leq 1\}.$$

Definition 2.2.3. Let ϕ be a continuous metric integrand in $\overline{\Omega}$. For a given function $u \in L^1(\Omega)$ we define its $\phi-$total variation in Ω by the formula:

$$\int_\Omega |Du|_\phi = \sup \left\{ \int_\Omega u \operatorname{div} \mathbf{z} \, dx \,:\, \phi^0(x, \mathbf{z}(x)) \leq 1 \text{ a.e.}, \quad \mathbf{z} \in C_c^1(\Omega) \right\}.$$

Another popular notation for the $\phi-$total variation is $\int_\Omega \phi(x, Du)$. We will say that $u \in BV_\phi(\Omega)$ if its $\phi-$total variation is finite in $\overline{\Omega}$; furthermore, let us define the $\phi-$perimeter of a set E as

$$P_\phi(E, \Omega) = \int_\Omega |D\chi_E|_\phi.$$

If $P_\phi(E, \Omega) < \infty$, we say that E is a set of bounded $\phi-$perimeter in Ω.

Remark 2.2.4. By property (3) of a metric integrand, for $u \in L^1(\Omega)$ we have

$$\lambda \int_\Omega |Du| \leq \int_\Omega |Du|_\phi \leq \Lambda \int_\Omega |Du|.$$

In particular, $BV_\phi(\Omega) = BV(\Omega)$ as sets; however, they are equipped with different (but equivalent) norms and corresponding strict topologies. Moreover, the $\phi-$total variation admits the following integral representation:

$$\int_\Omega |Du|_\phi = \int_\Omega \phi(x, \nu^u(x))\, |Du|,$$

where ν^u is the Radon-Nikodym derivative $\nu^u = \frac{dDu}{d|Du|}$. If we take u to be a characteristic function of a set E with a C^1 boundary, we have

$$P_\phi(E, \Omega) = \int_{\partial E \cap \Omega} \phi(x, \nu_E)\, d\mathcal{H}^{N-1},$$

where $\nu_E(x)$ is the (Euclidean) unit vector normal to ∂E at $x \in \partial E$. For the isotropic version of these facts, see [3] or [20]; for the exposition of BV theory in the anisotropic setting and the integral representation formula, see [1].

Moreover, the space $BV_\phi(\Omega)$ satisfies the same basic properties as the space $BV(\Omega)$ defined in the Euclidean case, such as lower semicontinuity of the $\phi-$total variation with respect to convergence in L^1, the co-area formula, and the approximation by smooth functions in the strict topology (also while preserving the trace of the limit).

2.3 Anisotropic least gradient functions

Now, we turn our attention to the precise formulation of Problem (aLGP). Then we recall several known properties of minimisers and discuss differences with respect to the isotropic case, i.e. $\phi(x, \xi) = |\xi|$. This subsection is based on the introduction of the anisotropic least gradient problem in [36] and [46].

From now on, ϕ will denote a continuous metric integrand.

Definition 2.3.1. Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. We say that $u \in BV_\phi(\Omega)$ is a function of $\phi-$least gradient, if for every compactly supported $v \in BV_\phi(\Omega)$ we have

$$\int_\Omega |Du|_\phi \leq \int_\Omega |D(u+v)|_\phi.$$

If ϕ admits a continuous extension to $\mathbb{R}^N$, we may instead assume that v is a BV_ϕ function with zero trace on $\partial\Omega$; see [46, Proposition 3.16].

Additionally, if a set $E \subset \Omega$ is such that χ_E is a function of $\phi-$least gradient, we say that E is a $\phi-$minimal set.

Definition 2.3.2. We say that $u \in BV_\phi(\Omega)$ is a solution to Problem (aLGP), if u is a function of $\phi-$least gradient and the trace of u equals f, i.e. for $\mathcal{H}^{N-1}-$almost every $x \in \partial\Omega$ we have

$$\lim_{r\to 0^+} \fint_{B(x,r)\cap\Omega} |f(x) - u(y)|\, dy = 0.$$

Firstly, since our strategy in the anisotropic least gradient problem will also include considering superlevel sets of $\phi-$least gradient functions, we state an anisotropic version of Theorem 2.1.3. The proof of both implications in this Theorem is based on the the co-area formula.

Theorem 2.3.3. ([46, Theorem 3.19]) *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Assume that ϕ admits a continuous extension to $\mathbb{R}^N$. Take $u \in BV_\phi(\Omega)$. Then, u is a function of $\phi-$least gradient in Ω if and only if χ_{E_t} is a function of $\phi-$least gradient in Ω for almost all $t \in \mathbb{R}$.*

As in the isotropic case, existence and uniqueness of minimisers depend on the geometry of Ω. Again, there are two principal approaches to Problem (aLGP): the first one is to consider that the boundary condition is assumed in the sense of traces and the second one is to consider the relaxation of the anisotropic total variation functional.

Let us interpret the boundary condition in the sense of traces and suppose that the boundary data are continuous. In the isotropic case, we recall that the necessary and sufficient condition, slightly weaker than strict convexity, was introduced in [65]; in two dimensions it is equivalent to strict convexity of Ω and for a set with smooth boundary it reduces to positive mean curvature on a dense subset of $\partial\Omega$. In the anisotropic setting, a sufficient condition inspired by the condition above is the *barrier condition* introduced in [36]:

Definition 2.3.4. ([36, Definition 3]) Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. We say that Ω satisfies the barrier condition (with respect to ϕ) if for every $x_0 \in \partial\Omega$ and sufficiently small $\varepsilon > 0$, if V minimises $P_\phi(\cdot\,; \mathbb{R}^N)$ in

$$\{W \subset \Omega : W \backslash B(x_0, \varepsilon) = \Omega \backslash B(x_0, \varepsilon)\},$$

then

$$\partial V \cap \partial\Omega \cap B(x_0, \varepsilon) = \emptyset.$$

In the isotropic case $\phi(x, \xi) = \|\xi\|_2$ this is equivalent, at least for sets with C^2 boundary, to the condition from [65] mentioned in the previous paragraph. A generalisation of the barrier condition to the setting of metric measure spaces has been introduced in [42].

Theorem 2.3.5. ([36, Theorem 1.1]) *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary which satisfies the barrier condition with respect to ϕ. Let $f \in C(\partial\Omega)$. Then there exists a minimiser $u \in BV_\phi(\Omega)$ to Problem* (aLGP).

Contrary to the isotropic case, when uniqueness of solutions to the least gradient problem for continuous boundary data follows from regularity of minimal sets and purely topological considerations, in the anisotropic case in general we do not know if the solution is unique or continuous. The situation is much more complicated, because regularity of $\phi-$minimal sets is not guaranteed; it is related to the maximum and comparison principles for $\phi-$minimal sets. A typical regularity assumption that will appear on a few occasions in this dissertation is

$$\text{If } E \text{ is a } \phi - \text{minimal set, then } \begin{cases} \mathcal{H}^{N-3}(\text{sing}\, \partial E) < \infty & N \geq 4, \\ \text{sing}\, \partial E = \emptyset & N \leq 3, \end{cases} \tag{H}$$

i.e. ∂E is of class C^2 apart from a set of finite $\mathcal{H}^{N-3}$ measure. In dimensions up to three, the singular set is empty. In fact, it is a regularity assumption on ϕ; the conditions on ϕ which imply (H) involve uniform convexity and regularity somewhat weaker than C^1 in the first (spatial) variable and C^3 in the second (directional) variable. The sufficiency of these conditions is proved in [62]; a detailed discussion can be found in [36], where the authors additionally show that one cannot relax the regularity in the spatial variable. Continuity of solutions is proved in [36] only in low dimensions under the assumption (H) and uniqueness of solutions follows from the following comparison principle:

Theorem 2.3.6. ([36, Theorem 1.2]) *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded set with connected Lipschitz boundary. Let ϕ be a metric integrand which additionally satisfies hypothesis (H). Let $f_1, f_2 \in C(\partial\Omega)$ such that $f_1 \geq f_2$ on $\partial\Omega$. Let u_i be minimisers to Problem (aLGP) corresponding to f_i for $i = 1, 2$. Then $u_1 \geq u_2$ in Ω. In particular, minimisers to Problem (aLGP) with continuous boundary data are unique.*

We note that in the special case when ϕ is a norm on $\mathbb{R}^N$, in Chapters 4 and 5 we will study the meaning of the barrier condition and the uniqueness of minimisers in more detail.

The second approach, introduced in [46], is to consider the relaxation of the anisotropic total variation functional, namely a functional defined on $L^{\frac{N}{N-1}}(\Omega)$ by the formula

$$F_\phi(u) = \begin{cases} \int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega(x))\, |Tu - f|\, d\mathcal{H}^{N-1} & u \in BV(\Omega), \\ +\infty & u \in L^{\frac{N}{N-1}}(\Omega) \backslash BV(\Omega), \end{cases} \tag{2.3.1}$$

where $\nu^\Omega(x)$ is the $\mathcal{H}^{N-1}$-a.e. well defined outer normal to $\partial\Omega$. Then, the author characterises the minimisers of F_ϕ in the following way:

Theorem 2.3.7. ([46, Theorem 3.7]) *Take any $u \in BV_\phi(\Omega)$. Then, u is a minimiser of the functional F_ϕ if and only if there exists a vector field $\mathbf{z} \in L^\infty(\Omega; \mathbb{R}^N)$ with $\phi(x, \mathbf{z}(x)) \leq 1$ a.e. such that the following conditions are satisfied:*

$$\mathbf{z}(x) \in \partial_\xi \phi(x, \nabla u(x)) \qquad \textit{a.e. in } \Omega; \tag{2.3.2}$$

$$- \mathit{div}(\mathbf{z}) = 0 \qquad \textit{as distributions}; \tag{2.3.3}$$

$$(\mathbf{z}, Du) = |Du|_\phi \qquad \textit{as measures}; \tag{2.3.4}$$

$$[\mathbf{z}, \nu] \in \mathit{sign}(f - u)\phi(\cdot, \nu) \qquad \mathcal{H}^{N-1} - \textit{a.e. on } \partial\Omega. \tag{2.3.5}$$

Again, $(\mathbf{z}, Du)$ and $[\mathbf{z}, \nu]$ are understood in the sense of Anzelotti pairings and the boundary condition is considered only in a weak sense instead of the sense of traces.

We stress that in this thesis we will in general interpret the boundary condition in the sense of traces. Much of the effort here is devoted to understanding the implications that the trace requirement has for the structure of minimisers. On a few occasions, we will additionally comment on the formulation in the language of Anzelotti pairings, but this fact will be explicitly mentioned.

2.4 Miscellaneous

In this Section, we list a few unrelated definitions and results, on which we will rely frequently in this thesis. The first one is the concept of $\Gamma-$convergence (see for instance [10]). We will use it in Chapters 3, 4 and 5 when discussing various approximations of least gradient functions.

Definition 2.4.1. Let $F, F_n : X \to [0, \infty]$ be a sequence of functionals on a topological space X. We say that the sequence F_n $\Gamma-$converges to F, what we denote by $\Gamma - \lim_{n\to\infty} F_n = F$, if the following two conditions are satisfied:
(1) For every sequence $x_n \in X$ such that $x_n \to x$ in X we have

$$F(x) \leq \liminf_{n\to\infty} F_n(x_n);$$

(2) For every $x \in X$ there exists a sequence $x_n \to x$ in X such that

$$F(x) \geq \limsup_{n\to\infty} F_n(x_n).$$

An important property of $\Gamma-$convergence is that cluster points of minimisers of F_n are minimisers of F. Furthermore, we extend this notion for continuous families of parameters in the obvious way: F_ε $\Gamma-$converges to F as $\varepsilon \to 0$, if it $\Gamma-$converges for every subsequence.

Moreover, on multiple occasions we will rely on the monotonicity formula for stationary varifolds. Instead of recalling the full statement, which can be found for instance in [63, Section 17], we state here only the version which fits the setting of the least gradient problem: in codimension one for area-minimising boundaries. In particular, we know that if E is a minimal set in Ω (i.e. χ_E is a function of least gradient), then ∂E is regular except for a set of Hausdorff dimension $N - 8$, hence $P(E, \Omega) = \mathcal{H}^{N-1}(\partial E)$.

Proposition 2.4.2. *Let $E \subset \Omega$ be a set of finite perimeter such that ∂E is locally area-minimising. Then, for each $x \in \Omega$ and $r <$ dist$(x, \partial\Omega)$, the function*

$$f(x,r) = \frac{\mathcal{H}^{N-1}(\partial E \cap B(x,r))}{\omega_{N-1} r^{N-1}}$$

is increasing in r*. In particular, the limit density*

$$\Theta(D\chi_E, x) = \lim_{r \to 0^+} f(x, r)$$

exists and equals at least one at each point of the support of $D\chi_E$*.*

We will only once use the monotonicity formula directly, in Chapter 6. More often, in the proofs of various results concerning pointwise properties of least gradient functions, we will use the following well-known consequence of the monotonicity formula (see for instance [63, Section 39]).

Corollary 2.4.3. *Let* $E \subset \Omega$ *be a set of finite perimeter and suppose that* ∂E *is locally area-minimising in* Ω*. Then, in any ball* $B \subset\subset \Omega$ *there are only finitely many connected components of* ∂E*. In particular, around any point* $x \in \partial E$ *we may find a ball which intersects only one connected component of* ∂E*.*

Finally, the monotonicity formula was used by Sternberg, Williams and Ziemer to prove the following variant of the maximum principle for minimal graphs:

Proposition 2.4.4. ([65, Theorem 2.2]) *Suppose that* $E_1 \subset E_2$ *and let* $\partial E_1, \partial E_2$ *be area-minimising in an open set* U*. Further, suppose* $x \in \partial E_1 \cap \partial E_2 \cap U$*. Then* ∂E_1 *and* ∂E_2 *coincide in a neighbourhood of* x*.*

Let us note that ∂E_1 and ∂E_2 agree on their respective connected components. This variant of the maximum principle will play a crucial part in the proofs of uniqueness and regularity properties of least gradient functions in Chapters 3 and 5.

Chapter 3

(Non)uniqueness of minimisers in the least gradient problem

3.1 Introduction

This Chapter deals with the issue of uniqueness of solutions to the isotropic least gradient problem. It was established in [65] that for continuous boundary data, under a condition on Ω slightly weaker than strict convexity, the solution exists and is continuous up to the boundary. Moreover, a maximum principle argument implies uniqueness of the minimiser. However, if we relax either continuity of boundary data or the conditions on the shape of Ω, two new phenomena arise:

(1) The solution itself might not exist: as we see in Example 2.1.8, without strict convexity of Ω existence may fail even for continuous boundary data. This issue is discussed in [31] and [58], including some positive results on existence. Moreover, Theorem 2.1.6 shows that if the boundary data belong only to $L^\infty(\partial\Omega)$, then the minimiser might not exist even if Ω is a two-dimensional disk; on the other hand, Theorem 2.1.9 shows existence of solutions in the two-dimensional case for BV boundary data.

(2) As pointed out in [48], uniqueness of solutions for discontinuous boundary data may fail even for strictly convex domains. However, all the solutions in the example provided by the authors (attributed to John Brothers and presented here in Example 3.4.2(3)) have very similar structure of superlevel sets; they differ only on a set, on which each of the solutions is constant. This example is the primary motivation for the main result in this Chapter.

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^N$, where $2 \leq N \leq 7$, be an open bounded convex set. Let u, v be precise representatives of functions of least gradient in Ω such that $Tu = Tv = f$. Then $u = v$*

on $\Omega \backslash (C \cup \mathcal{N})$, where both u and v are locally constant on C and $\mathcal{N}$ has Hausdorff dimension at most $N-1$.

Here, we allow that $f \in L^1(\partial\Omega)$ without any direct assumptions on its regularity. Moreover, unlike the known existence results such as Theorems 2.1.5 and 2.1.9, we require only convexity of Ω in place of some form of strict convexity. However, we have an indirect assumption that the set Ω and the function f support at least one solution to the least gradient problem. In this Chapter, we do not address the question of necessary or sufficient conditions for existence of solutions; in addition to the two Theorems mentioned above, we will consider this type of problems at length in Chapters 5 and 7.

In dimension two Theorem 3.1.1 has direct implications for the free material design problem, see [31, Theorem 2.1] for the relationship between the two problems. It implies that all solutions to the free material design problem generated by least gradient functions have voids in the same region. Also, a finer analysis such as in Section 3.4 shows on what lower-dimensional structures the material may concentrate.

The idea that the solutions share the same frame of superlevel sets is not new - apart from the Brothers example, we encourage the reader to compare Theorem 3.1.1 to [53, Theorem 1.1], which states that all solutions to the (anisotropic) least gradient problem in the sense provided by Theorems 2.1.7 and 2.3.7 share the same vector field $\mathbf{z}$ describing the structure of the solution. Earlier, an isotropic version of the this result appeared as part of the proof of [48, Theorem 2.5]. The idea behind Theorem 3.1.1 is similar, but we want to examine in more detail the pointwise properties of functions of least gradient. For this reason, we will use a different technique - instead of the language of Anzelotti pairings, we focus on the structure requirements imposed by the trace condition and use the maximum principle for minimal surfaces. The restrictions for the dimension follow from this: in dimension one, the boundaries of superlevel sets are points and all monotone functions are functions of least gradient; in dimension eight and above, minimal sets may have singularities and we no longer have that boundaries of superlevel sets are analytical minimal surfaces.

This Chapter is organised as follows: in Section 3.2 we prove a few results concerning pointwise properties of precise representatives of least gradient functions. Section 3.3 is devoted to proving the main result of this Chapter, i.e. Theorem 3.1.1. The proof will be performed in two stages; firstly, the claim will be shown in the two-dimensional setting, where the proof faces less geometric difficulties. Then the claim will be shown for any dimension N such that the boundary of the superlevel set is an analytical minimal surface. This proof runs along similar lines, but with more serious geometrical difficulties and its two-dimensional version will act as a toy problem.

In Section 3.4 we use Theorem 3.1.1 to provide a characterisation of the set of solutions in terms of a single solution u_0. The results from this section are most useful in $\mathbb{R}^2$, as we consider certain partitions of sets by minimal surfaces; in dimensions higher than two finding all such partitions is a very hard question, while on the plane it can be turned into an algorithm.

The last Section is Section 3.5, which deals with an approximation of the least gradient problem which takes into account the total mass of the solution. Using the concept of Γ-convergence, we prove that minimisers of the approximate problems converge to a minimisers of least gradient problem with the smallest L^p norm and this convergence is stronger than standard L^p convergence.

Finally, we would like to note that the main technical ingredient in this Chapter, namely Proposition 2.4.4, has been extended by Zuniga in 2019, see [71], to the setting of minimal surfaces with respect to a Riemannian metric generated by a C^2 weight. This suggests that a similar proof could be performed and an analogue of Theorem 3.1.1 could also be valid in the setting of the weighted least gradient problem, namely

$$\min \bigg\{ \int_\Omega a(x)|Du|, \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \bigg\}, \tag{wLGP}$$

where $a \in C^2(\overline{\Omega})$. However, in this thesis we restrict ourselves to the isotropic case.

This Chapter is based on the article [27], of which I am the sole author and which has been published in the Journal of Mathematical Analysis and Applications.

3.2 Preliminaries

This section brings together a few technical results, which will be needed later, but are proved here not to interrupt the reasoning in Section 3.3. As mentioned in the introduction, we work in the framework such that Ω is an open bounded set with Lipschitz boundary. When necessary, we will impose the assumption of convexity or strict convexity of Ω. Furthermore, in many results in Sections 3.2-3.4 we assume that $2 \leq N \leq 7$; this is necessary due to result by Giusti, see the commentary to Theorem 2.1.3 in Chapter 2. A few of the results concerning general BV theory have been moved to the Appendix.

3.2.1 Pointwise properties

In this subsection, we explore pointwise properties of precise representatives of least gradient functions. The following two Lemmata give us some insight about local form of superlevel sets of a least gradient function. Namely, locally there is only one connected component of $\partial\{u \geq t\}$ around any point inside Ω (this statement may obviously fail at the boundary). Secondly, absence of connected components of $\partial\{u \geq t\}$ passing through a given point imply that there are none in some neighbourhood of this point.

We recall Corollary 2.4.3, which was the consequence of the monotonicity formula for minimal surfaces: for every $x \in \Omega$ there exists a ball such that there is at most one connected component of a given minimal surface intersecting this ball. We state this observation in the setting of least gradient functions:

Lemma 3.2.1. *Let $2 \leq N \leq 7$. Let $u \in BV(\Omega)$ be a least gradient function. Let $E_t = \{u \geq t\}$ and take $x \in \partial E_t \subset \Omega$. Then there exists a ball $B(x,r)$ such that there is only one connected component of ∂E_t intersecting this ball.* □

The second Lemma is similar and relates the absence of connected components of ∂E_t passing through $x \in \Omega$ to being in the interior of the superlevel set E_t; due to monotonicity formula, we do not need to consider connected components of ∂E_t in any neighourhood of x.

Lemma 3.2.2. *Let $2 \leq N \leq 7$. Suppose that $u \in BV(\Omega)$ is a function of least gradient. Let $E_t = \{u \geq t\}$. Suppose that $x \in \Omega$ is a point of continuity of u, $u(x) = t$ and $x \notin \partial E_t$. Then there exists a ball $B(x,r) \subset E_t$.*

Proof. We have two possibilities: either $|D\chi_{E_t}|(B(x,r)) > 0$ for all $r > 0$ or for sufficiently small r we have $|D\chi_{E_t}|(B(x,r)) = 0$.

In the first case we set $d = \text{dist}(x, \partial\Omega)$ and take any $r < d$. As for all $r > 0$ we have $|D\chi_{E_t}|(B(x,r)) > 0$, we have at least one (and thus infinitely many) connected component of ∂E_t intersecting $\overline{B(x,r)}$. This contradicts the monotonicity formula for minimal surfaces.

In the second case we take such r. By relative isoperimetric inequality we have either $B(x,r) \subset E_t$ or $B(x,r) \cap E_t = \emptyset$ (remember that we consider the precise representative of u). But the second condition cannot hold, as $u(x) = t$ and x is a point of continuity. □

Remark 3.2.3. By [28, Proposition 3.9] it suffices to assume that $x \notin \partial E_t$ for any $t \in \mathbb{R}$; then x is a point of continuity of u. Moreover, suppose that $u \in BV(\Omega)$ is a function of least gradient, $u(x) = t$, $x \notin \partial\{u \geq t\}$ for any t and $x \notin \partial\{u \leq t\}$ for any t. Then there exists a ball $B(x,r) \subset \{u = t\}$.

Example 3.2.4. However, if $x \notin \partial E_t$ for any $t \in \mathbb{R}$, this does not mean that u is continuous in any open neighbourhood of x; take a nonincreasing function on $[-1, 1]$ defined by the formula

$$f(x) = \begin{cases} 2^{\lfloor \frac{1}{x} \rfloor} & x < 0, \\ 0 & x \geq 0. \end{cases}$$

Now take $\Omega = B(0,1) \subset \mathbb{R}^2$. The function $u(x,y) = f(x)$ is a function of least gradient in Ω. It is continuous at $(0,0)$, yet it is not continuous on any open neighbourhood of (0,0). We see that $(0,0) \notin \partial\{u \geq t\}$ for any t; for $t \leq 0$ it is impossible, because on the whole domain the function is nonnegative. For $t > 0$ we will find $x_0 \in (-1,0)$ such that for $x > x_0$ we have $u(x,y) < t$. Thus, since the function is not constant anywhere near $(0,0)$, by the previous Remark $(0,0) \in \partial\{u \leq 0\}$.

The following result states that there can be only countably many t such that $\partial\{u > t\} \neq \partial\{u \geq t\}$. Here and in the whole manuscript, for a Borel set A we denote by $|A|$ its Lebesgue measure $\mathcal{L}^N(A)$.

Lemma 3.2.5. *Let $2 \leq N \leq 7$. Suppose that $u \in BV(\Omega)$ is a function of least gradient. We have $\partial\{u > t\} \neq \partial\{u \geq t\}$ if and only if $|\{u = t\}| > 0$.*

Proof. Suppose that $\partial\{u > t\} \neq \partial\{u \geq t\}$. Obviously $\{u > t\} \subset \{u \geq t\}$ and by Theorem 2.1.3 their boundaries are minimal surfaces. We have two possibilities: either there is a connected component S of $\partial\{u \geq t\}$ such that $S \cap \partial\{u > t\} = \emptyset$ or there is not. In the first case we easily see, for example using Lemma 3.2.1, that $|\{u \geq t\} \backslash \{u > t\}| = |\{u = t\}| > 0$. In the second case, let us see that by Proposition 2.4.4, if a connected component of $\partial\{u \geq t\}$ and a connected component of $\partial\{u > t\}$ intersect, then they are equal. Thus, the second case cannot happen.

Now, we prove the other implication. Suppose that $|\{u = t\}| > 0$. Take a point $x \in \partial\{u = t\} \subset (\partial\{u \geq t\} \cup \partial\{u \leq t\})$. By the previous Remarks, since u is not constant in any neighbourhood of x, we have exactly one of the sets $\partial\{u \geq t\}$ and $\partial\{u \leq t\}$ passing through x. □

3.2.2 The weak maximum principle

The following result proved in [28] is a weak maximum principle for least gradient functions, as it states that each of the level superlevel sets cannot have compact support in Ω, i.e. the maximum value is attained on the boundary. In fact, it is just Theorem 2.1.3 combined with the regularity theory for area-minimising boundaries.

Proposition 3.2.6. ([28, Proposition 3.4]) *Let $\Omega \subset \mathbb{R}^N$, where $N \leq 7$, be an open set and suppose $u \in BV(\Omega)$ is a function of least gradient. Then for every $t \in \mathbb{R}$ we have $\partial\{u \geq t\} = \bigcup_{i \in I} S_{t,i}$, where $S_{t,i}$ are smooth minimal surfaces without boundary in Ω and this union is locally finite.*

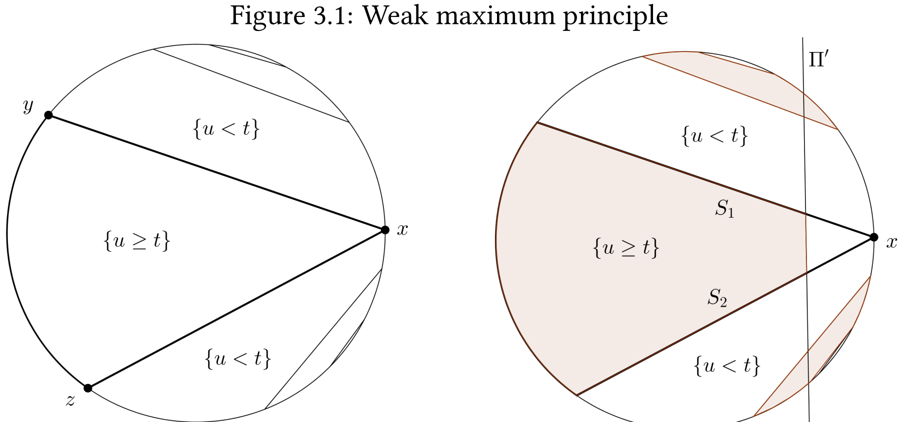
Unfortunately, the weak maximum principle as presented in Proposition 3.2.6 is not enough for our considerations. The second and main result in this subsection, Proposition 3.2.8, is an improvement of the weak maximum principle in which we consider the geometry of the superlevel sets of a least gradient function near the boundary of Ω: it states that two connected components of a superlevel set cannot intersect even on $\partial\Omega$. The first result is two-dimensional and is recalled to serve as a toy model for the more general one. Note that these results require additionally convexity of Ω; however, they do not require strict convexity of Ω.

Lemma 3.2.7. ([28, Proposition 3.5]) *Let $\Omega \subset \mathbb{R}^2$ be an open bounded convex set with Lipschitz boundary and suppose $u \in BV(\Omega)$ is a function of least gradient. Let $E_t = \{u \geq t\}$. Then for every $t \in \mathbb{R}$ for every point $x \in \partial\Omega$ there is at most one line segment belonging to ∂E_t which ends at x.*

The essence of the proof is to find a convenient competitor in the least gradient problem. Using the consequence of the monotonicity formula, Corollary 2.4.3, we may reduce the situation to what is depicted on Figure 3.1 on the left hand side: without loss of generality two connected components of ∂E_t ending at x, i.e. line segments $\overline{xy}$ and $\overline{xz}$, are adjacent and in the sense that a connected component of the set E_t lies between them. Thus, in the region enclosed by the line segments $\overline{xy}, \overline{xz}$ and the arc $\overline{yz} \subset \partial\Omega$ not containing x we have $\chi_{E_t} = 1$ and $\chi_{E_t} = 0$ on the two sides of the triangle. Then, χ_{E_t} is not a function of least gradient: the function $\widetilde{\chi_{E_t}} = \chi_{E_t} - \chi_{\Delta xyz}$ has the same trace and strictly smaller total variation due to the triangle inequality, which contradicts Theorem 2.1.3.

In the more general case, we have to state the result and its proof more carefully. There are two main reasons: firstly, the only connected minimal surface in two dimensions is a line segment, which divides Ω into two simply-connected open sets. This may fail in higher dimensions: for a simple example, consider Ω to be a ball in $\mathbb{R}^3$ and $\partial\{u \geq t\}$ to be a catenoid. Secondly, we may not use the triangle inequality and we have to rely on projections, so the geometrical part becomes more complicated.

Figure 3.1: Weak maximum principle



Proposition 3.2.8. *Let $\Omega \subset \mathbb{R}^N$, where $2 \leq N \leq 7$, be a convex set with Lipschitz boundary and suppose $u \in BV(\Omega)$ is a function of least gradient. Then, for every $t \in \mathbb{R}$ the boundary*

of the set $E_t = \{u \geq t\}$ is a union of minimal surfaces $S_{t,i}$, without self-intersections, with closures pairwise disjoint in $\overline{\Omega}$.

Proof. We only have to prove that intersection does not take place on $\partial\Omega$. Let $x \in \overline{S_1} \cap \overline{S_2} \cap \partial\Omega$, where S_1 and S_2 are two different connected components of ∂E_t. We know that S_1 divides Ω into two disjoint parts Ω_1^+ and Ω_1^-; similarly S_2 divides Ω into Ω_2^+ and Ω_2^-. Among these, due to Proposition 3.2.6, there is only one set of the form $\Omega_1^\pm \cap \Omega_2^\pm$, which lies between S_1 and S_2, i.e. has both of these sets as parts of its boundary. We consider the case when it is $\Omega_1^+ \cap \Omega_2^+$; the other situations are handled similarly.

We may assume that S_1 and S_2 are adjacent minimal surfaces in ∂E_t: by Corollary 2.4.3 there may be only finitely many connected components of a minimal surface in any compact subset of Ω. Here, if we take $\overline{(\Omega_1^+ \cap \Omega_2^+)} \cap \{y : \text{ dist}(y, \partial\Omega) \geq d\}$ for sufficiently small d as this compact set, we see that there may be only finitely many connected components of ∂E_t between S_1 and S_2. Thus, up to renumbering of S_i, the surfaces S_1 and S_2 are adjacent. Without loss of generality $\Omega_1^+ \cap \Omega_2^+ \subset E_t$ and it is a connected component of E_t.

If $\partial\Omega$ is smooth, consider the hyperplane Π tangent to $\partial\Omega$ at x. Since $\partial\Omega$ is only Lipschitz, such a hyperplane might not exist; in that case, take any of the supporting hyperplanes such that the normal ν to Π points inside Ω. Now, take $s > 0$ small enough and consider the hyperplane $\Pi' = \Pi + s\nu$. There are two subsets of Ω bounded by $\partial\Omega$ and Π'. Let G be the one such that $x \in \overline{G}$.

Theorem 2.1.3 implies that χ_{E_t} is a function of least gradient in Ω. Now, consider a competitor χ_F constructed in the following way:

- in $\Omega \backslash (\Omega_1^+ \cap \Omega_2^+)$ we have $F = E_t$;

- in $\Omega_1^+ \cap \Omega_2^+$ we take $F = E_t \backslash G$.

This situation is presented on Figure 3.1 on the right hand side. Here, the set F is the shaded region. The characteristic function χ_F constructed this way obviously satisfies $T\chi_{E_t} = T\chi_F$. Moreover, let us see that

$$|D\chi_F|(\Omega) = |D\chi_F|(\Omega \backslash (\Omega_1^+ \cap \Omega_2^+)) + \mathcal{H}^{N-1}(S_1 \cap (\Omega \backslash G)) +$$
$$+ \mathcal{H}^{N-1}(S_2 \cap (\Omega \backslash G)) + \mathcal{H}^{N-1}((\Pi') \cap (\Omega_1^+ \cap \Omega_2^+)) <$$
$$< |D\chi_{E_t}|(\Omega \backslash (\Omega_1^+ \cap \Omega_2^+)) + \mathcal{H}^{N-1}(S_1) + \mathcal{H}^{N-1}(S_2) = |D\chi_{E_t}|(\Omega),$$

as the first summands are the same and projection onto Π' delivers strict inequality in the remaining summands. We have reached a contradiction with Theorem 2.1.3. $\square$

Finally, let us see that convexity of Ω in Lemma 3.2.7 and Proposition 3.2.8 cannot be relaxed. In the following Example the set Ω is star-shaped, but it is not convex, so the weak maximum principle fails on $\partial\Omega$. Slightly modifying this example, we may ensure smoothness of $\partial\Omega$ and of the boundary data, but this plays no role in the failure of the weak maximum principle.

Example 3.2.9. Denote by φ the angular coordinate in the polar coordinates on the plane. Let $\Omega = B(0,1)\backslash(\{\frac{\pi}{4} \leq \varphi \leq \frac{3\pi}{4}\} \cup \{0\}) \subset \mathbb{R}^2$, i.e. the unit ball with one quarter removed. Take the boundary data $f \in L^1(\partial\Omega)$ to be

$$f(x,y) = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0. \end{cases}$$

Then the solution to the least gradient problem is the function (defined inside Ω)

$$u(x,y) = \begin{cases} 1 & y \geq 0 \\ 0 & y < 0, \end{cases}$$

in particular $\partial\{u \geq 1\}$ consists of two horizontal line segments whose closures intersect at the point $(0,0) \in \partial\Omega$. Note that in this example the set Ω is star-shaped, but it is not convex.

3.3 Uniqueness

This Section is devoted to proving the main result of this Chapter, namely uniqueness of solutions of the least gradient problem except for a set where the solution is locally constant. The proof is valid in dimensions up to seven, so that boundaries of superlevel sets are analytic minimal surfaces. However, much of the proof is simplified in the planar case. In the beginning, let us underline the fact that **we are always dealing with exact representatives** of least gradient functions, and thus we may discuss pointwise properties of least gradient functions. Our main tools will be Theorem 2.1.3, connecting least gradient functions to minimal surfaces, and a variant of the maximum principle for minimal graphs as stated in Proposition 2.4.4. Now, we recall the statement of Theorem 3.1.1:

Theorem 3.1.1. *Let $\Omega \subset \mathbb{R}^N$, where $2 \leq N \leq 7$, be an open bounded convex set. Let u, v be precise representatives of functions of least gradient in Ω such that $Tu = Tv = f$. Then $u = v$ on $\Omega\backslash(C \cup \mathcal{N})$, where both u and v are locally constant on C and $\mathcal{N}$ has Hausdorff dimension at most $N-1$.*

For the whole Section we introduce the following notation: let $u, v \in BV(\Omega)$ be two functions of least gradient with the same trace. Let $E_t = \{u \geq t\}$ and $F_s = \{v \geq s\}$. The proof will consist of four major steps:

1. We prove that if $\partial E_t \cap \partial F_t \neq \emptyset$, then these sets coincide on their respective connected components; this gives a partition of Ω.

2. We look at the structure of the set $E_t \backslash F_t$.

3. We use this knowledge to prove that for $t \neq s$ we have $\partial E_t \cap \partial F_s \subset J_u \cup J_v$.

4. We introduce a singular set $\mathcal{N}$ with Hausdorff dimension $N-1$. We infer local properties of u and v from the steps above; case-by-case analysis proves uniqueness outside of $C \cup \mathcal{N}$.

The proof is much easier to visualise in the two-dimensional case. This is most striking in Step 3 of the proof, therefore Step 3 will be proved in two stages: firstly in a two-dimensional setting with far fewer technical difficulties, secondly in the general setting with the two-dimensional proof serving as an illustration. Furthermore, after Step 3 we present its variant, denoted by Step 3a, which allows us to eliminate the possibility of u being constant in some ball $B(x,r)$, while v is not constant in this ball.

Proof of Theorem 3.1.1.

Step 1. Let $x \in \partial E_t \cap \partial F_t$. Then the respective connected components of ∂E_t and ∂F_t coincide.

We begin with noting that Step 1 remains the same for $2 \leq N \leq 7$. From Proposition 3.2.6 we know that ∂E_t is an at most countable union of (smooth) minimal surfaces which do not intersect inside Ω. By Lemma 3.2.7 (in the two-dimensional case) or by Proposition 3.2.8 (in the general case) they do not intersect on $\partial\Omega$.

Let $w = \min(u,v)$. We assumed $Tu = Tv = f \in L^1(\partial\Omega)$. By Corollary A.0.3 w is another function of least gradient with boundary data f. Consider $H_t = \{\min(u,v) \geq t\} = E_t \cup F_t$. Let $x \in \partial E_t \cap \partial F_t$ and let S_u, S_v be connected components of ∂E_t and ∂F_t respectively containing x.

Using Lemma 3.2.1 we can find a ball $B(x,r) \subset \Omega$ that intersects only S_u and S_v among all connected components of ∂E_t and ∂F_t. Now we have two possibilities:

(1) For every sequence $\rho_n \to 0$ we have $S_u \cap B(x,\rho_n) \neq S_v \cap B(x,\rho_n)$. In this case every neighbourhood of x intersects $\{u, v < t\}$, thus $x \in \partial H_t$.

(2) There is an open ball $B(x,\rho)$ with $\rho < r$ such that $S_u \cap B(x,\rho) = S_v \cap B(x,\rho)$.

Now, if there is another point $x' \in S_u \cap S_v$ such that condition (1) holds with x' in place of x, then $x' \in \partial H_t$ and we may proceed to the next paragraph. If condition (2) holds for every $y \in S_u \cap S_v$, then as the intersection of two minimal surfaces is a closed set in Ω, we have $S_u = S_v$.

By the reasoning above, we have $x \in \partial H_t$ (or $x' \in \partial H_t$). As $E_t \subset H_t$, by Proposition 2.4.4 we have that $S_u = S_w$, where S_w is the connected component of ∂H_t containing x. Similarly, as $F_t \subset H_t$, we have $S_v = S_w$; thus $S_u = S_v$.

Step 2. The structure of $E_t \backslash F_t$ for all but countably many $t \in \mathbb{R}$.

By the Alexander duality theorem, see [32, Theorem 27.10], each of the surfaces $S_u \subset \partial E_t$ divides Ω into two disjoint open sets Ω_+ and Ω_- (in two dimensions one may use the Jordan curve theorem). If any other connected component of E_t or F_t intersects $\Omega_\pm$, then by Step 1 it entirely lies in $\Omega_\pm$. Now take any connected component of ∂E_t or ∂F_t which lies in $\Omega_\pm$ (if such exists) and it divides $\Omega_\pm$ again into two sets. This way, since by Proposition 3.2.6 the union of the connected components of ∂E_t and ∂F_t is locally finite, we obtain a decomposition of Ω into at most countably many pairwise disjoint connected open sets Ω_i.

By Lemma A.0.4, the set $E_t \backslash F_t$ may not touch the boundary of $\partial\Omega$ on a set of positive $\mathcal{H}^{N-1}$-measure for all but countably many t; it may happen only if $f = t$ on a set of positive measure. From now on, assume that t is such that the level set $\{f = t\} \subset \partial\Omega$ has zero Hausdorff measure of codimension one.

Under this assumption, the boundary of C_t, a connected component of $E_t \backslash F_t$, cannot consist of parts of $\partial\Omega$ of positive area. Thus ∂C_t is an at most countable union of minimal surfaces $S_i \subset \partial E_t$ and $T_j \subset \partial F_t$. Let S_i and T_j be the connected components of ∂C_t which belong to ∂E_t and ∂F_t respectively. We may say that S_i and T_j **interlace**, as $\overline{S_i} \cap \overline{S_j} = \emptyset$ for $i \neq j$ due to Lemma 3.2.7, while $\overline{S_i}$ and $\overline{T_j}$ must overlap for some i and j. One way to imagine this is, in the two-dimensional setting, that C_t is a $2n-$sided polygon such that the even sides belong to ∂E_t and odd sides belong to ∂F_t; a three-dimensional example could be S_1 to be a part of a vertical catenoid and T_1 and T_2 be two horizontal disks. Here, C_t is the set bounded by these three surfaces.

As χ_{E_t} is a function of least gradient in Ω, taking as a competitor the function $\chi_{E_t} - \chi_{C_t}$ (note that $T\chi_{C_t} = 0$) we obtain that

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-1}(S_i) \leq \sum_{j=1}^{\infty} \mathcal{H}^{n-1}(T_j).$$

Similarly, since χ_{F_t} is a function of least gradient, taking the function $\chi_{F_t} + \chi_{C_t}$ as a competitor we have

$$\sum_{j=1}^{\infty} \mathcal{H}^{n-1}(T_j) \leq \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(S_i).$$

This implies that for every t the set C_t satisfies what we will call the **Green's formula**, i.e. we have

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-1}(S_i) = \sum_{j=1}^{\infty} \mathcal{H}^{n-1}(T_j). \tag{3.3.1}$$

The motivation for the name *Green's formula* comes from the formulation in the language of Anzelotti pairings as written in Theorem 2.1.7: formula (3.3.1) corresponds to the fact that the structure vector field $\mathbf{z}$ has zero divergence, so its flow through the closed surface ∂C_t equals zero.

Step 3: a model two-dimensional case. For all but countably many t, s such that $t \neq s$ we have $\partial E_t \cap \partial F_s \subset J_u \cup J_v$.

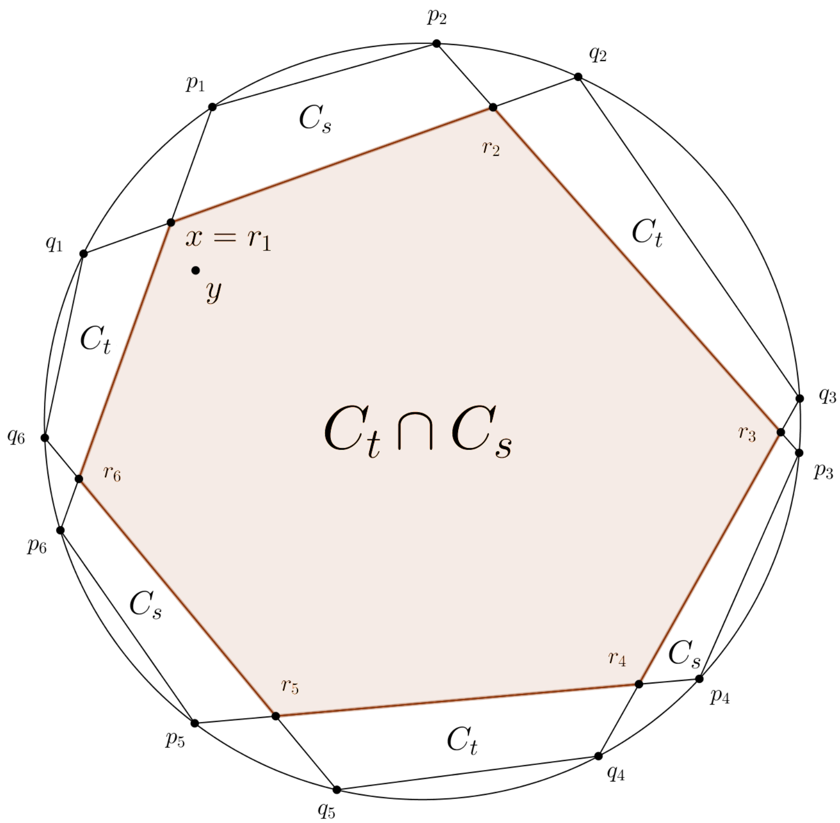
Without loss of generality we have $s < t$. Let t, s be as in Step 2, i.e. such that $E_t \backslash F_t$ does not touch $\partial\Omega$ on a set of positive measure, so the interlacing condition and Green's formula are satisfied. Suppose that $x \in \partial E_t \cap \partial F_s$ and that u, v are continuous at x. Obviously $x \in E_t \backslash F_t$. Consider C_t, the connected component of $E_t \backslash F_t$ containing x. As $x \in \partial E_t \cap \partial F_s$ and u, v are continuous at x, there is a point y in the neighbourhood of x such that $y \in E_s \backslash F_s$. Similarly, let C_s be a connected component of $E_s \backslash F_s$ containing y.

The proof of this Step is much more clear in dimension two and we may rely on triangle inequality in place of Proposition 2.4.4. Step 2 implies that the polygon C_s has at most countably many vertices p_i and (due to interlacing condition) its sides consist alternately of connected components of ∂E_s and ∂F_s. Since the two-dimensional case serves only as a toy problem for the general case, let us assume for simplicity that the number of vertices is finite and equals N_s. Furthermore, in order to make this model problem more clear, we make a further simplification that $N_t = N_s$.

We choose the numbering of the vertices so that $p_i p_{i+1}$ belongs to $S_{s,i}$ for odd i and to $T_{s,i}$ for even i. We employ the notation that $p_1 = p_{N_s+1}$ and so on. Similarly, the polygon C_t has at most countably many vertices q_i and its sides consist alternately of connected components of ∂E_t and ∂F_t. Again, we choose the numbering of the vertices so that $q_i q_{i+1}$ belongs to $S_{t,i}$ for odd i and to $T_{t,i}$ for even i. Again, we employ the notation that $p_1 = p_{N_t+1}$. The situation is as presented on Figure 3.2; for a possible exception, see the next paragraph. The polygons are positioned so that in the region cut off by a side $p_i p_{i+1} \subset \partial E_s \cap \partial C_s$ (i.e. this line segment divides Ω into two open sets and we look at the set not containing y) of the polygon C_s there is one connected component of $\partial E_t \cap \partial C_t$. The same holds for F_t and F_s. In this setting, the points r_i are intersections between sides of the two polygons, such that r_1 and r_2 lie on $q_1 q_2$, r_2 and r_3 lie on $p_2 p_3$ and so on. The enumeration is chosen so that $x = r_1$.

The structure of these sets can differ from Figure 3.2 in the following way: some of the line segments may coincide when we have a jump, for instance $q_3 q_4 = p_3 p_4$. For now, let us assume this is not the case and this will be discussed later.

Figure 3.2: The sets C_t and C_s



Let us look at the little trapezoids at the sides of $C_t \cap C_s$. By triangle inequality for every i we have

$$|p_{2i-1}p_{2i}| < |p_{2i-1}r_{2i-1}| + |r_{2i-1}r_{2i}| + |r_{2i}p_{2i}|;$$

$$|q_{2i}q_{2i+1}| < |q_{2i}r_{2i}| + |r_{2i}r_{2i+1}| + |r_{2i+1}q_{2i+1}|.$$

We sum up these inequalities and use the collinearity of $q_{2i-1}, r_{2i-1}, r_{2i}, q_{2i}$ and the collinearity of $p_{2i}, r_{2i}, r_{2i+1}, p_{2i+1}$ to obtain

$$\sum_i |p_{2i-1}p_{2i}| + \sum_i |q_{2i}q_{2i+1}| < \sum_i |p_{2i}p_{2i+1}| + \sum_i |q_{2i-1}q_{2i}|. \tag{3.3.2}$$

This contradicts Green's formula: in the notation of Step 2, we have $S_{s,i} = p_{2i-1}p_{2i}$, $T_{s,j} = p_{2j}p_{2j+1}$, $S_{t,i} = q_{2i-1}q_{2i}$ and $T_{t,j} = q_{2j}q_{2j+1}$. Thus application of equation (3.3.1) for C_t and C_s implies that in equation (3.3.2) there should be an equality, contradiction.

Finally, let us go back to the case where some of the line segments coincide. Then the corresponding inequality ceases to be strict. However, at least one inequality is strict: the inequality for $i = 1$, since we assumed that there is no jump at x. Thus, the proof still holds.

Step 3: the general case. For all but countably many t, s such that $t \neq s$ we have $\partial E_t \cap \partial F_s \subset J_u \cup J_v$.

We proceed similarly to the two-dimensional case. In place of the triangle inequality, we will use the maximum principle for minimal surfaces. We are going to prove the statement by contradiction: without loss of generality we have $s < t$. Again, let t, s be as in Step 2, i.e. such that $E_t \backslash F_t$ does not touch $\partial\Omega$ on a set of positive measure, so the interlacing condition and Green's formula are satisfied. Suppose that $x \in \partial E_t \cap \partial F_s$ and that u, v are continuous at x. Obviously $x \in E_t \backslash F_t$. Consider C_t, the connected component of $E_t \backslash F_t$ containing x. By Step 2 the boundary of C_t consists of at most countably many minimal surfaces, $S_{t,i}$ and $T_{t,j}$, the connected components of ∂E_t and ∂F_t respectively. As $x \in \partial E_t \cap \partial F_s$ and u, v are continuous at x, there is a point y in the neighbourhood of x such that $y \in E_s \backslash F_s$. Similarly, let C_s be a connected component of $E_s \backslash F_s$ containing x.

Without loss of generality assume that $x \in S_{t,1}$. The surface $S_{t,1}$ divides Ω into two parts: $\Omega^+_{t,1}$ and $\Omega^-_{t,1}$. $\Omega^-_{t,1}$ is the part of Ω which locally close to $S_{t,1}$ contains $\{u < t\}$. If $\Omega^-_{t,1} \cap \partial E_s = \emptyset$, then $\Omega^-_{t,1} \subset E_s$; but this contradicts Step 2 for C_s, a connected component of $E_s \backslash F_s$. Thus there is a connected component $S_{s,1}$ of ∂E_s in $\Omega^-_{t,1}$ (additionally we may pick the one closest to $S_{t,1}$). Since u is continuous at x, by Proposition 2.4.4 $S_{s,1} \cap S_{t,1} = \emptyset$, i.e. $S_{s,1} \subset \Omega^-_{t,1}$. This reasoning mirrors the last paragraph of the two-dimensional proof.

The boundary of C_s contains $S_{s,1}$. Similarly to the reasoning above, using Proposition 2.4.4 we prove that $S_{s,i} \subset \overline{\Omega^-_{s,i}}$ and $T_{t,j} \subset \overline{\Omega^-_{t,j}}$. Similarly as in the two-dimensional case, here we cannot exclude the case that $T_{t,j} = T_{s,j}$.

Now, both C_t and C_s satisfy Green's formula. Explicitly, from equation (3.3.1) we have

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-1}(S_{t,i}) = \sum_{j=1}^{\infty} \mathcal{H}^{n-1}(T_{t,j}),$$

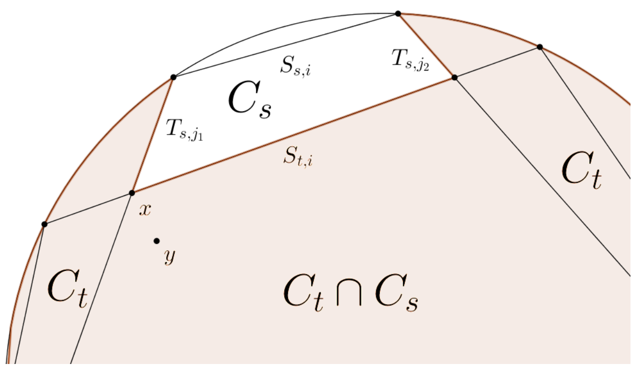
$$\sum_{i=1}^{\infty} \mathcal{H}^{n-1}(S_{s,i}) = \sum_{j=1}^{\infty} \mathcal{H}^{n-1}(T_{s,j}).$$

We argue as in the penultimate paragraph of the two-dimensional case that we may consider the situation which is presented on Figure 3.2. Let us look at $S_{s,i}$ and $T_{t,j}$, i.e. these connected components of ∂C_t and ∂C_s which lay outward with respect to y, i.e. if we draw any Jordan curve from y to any point in $S_{s,i}$, then it intersects a point from $S_{t,i}$; similarly, if we draw any Jordan curve from y to any point in $T_{t,j}$, then it intersects a point from $T_{s,j}$. Finally, we will notice that

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-1}(S_{s,i}) + \sum_{j=1}^{\infty} \mathcal{H}^{n-1}(T_{t,j}) < \sum_{i=1}^{\infty} \mathcal{H}^{n-1}(S_{t,i}) + \sum_{j=1}^{\infty} \mathcal{H}^{n-1}(T_{s,j}).$$

Take the surface $S_{s,i}$. It divides Ω into $\Omega^+_{s,i}$ and $\Omega^-_{s,i}$. Since χ_{E_s} is a function of least gradient, then its localised version $\chi_{E_s \cap \Omega^+_{s,i}}$ is as well. Let G be the set bounded by $S_{s,i}$, $S_{t,i}$ and these $T_{s,j}$ which intersect S_{t_i}. Consider a competitor χ_F, where F is the set $(E_s \cap \Omega^+_{s,i})\backslash G$. The situation is presented on Figure 3.3, which is a zoomed-in version of Figure 3.2; the set F is the shaded region, the set G is the trapezoid on the top and $\Omega^+_{s,i}$ is everything below the line segment $S_{s,i}$.

Figure 3.3: $\partial E_t \cap \partial F_s \subset J_u \cup J_v$



Since $\chi_{E_s \cap \Omega^+_{s,i}}$ is a function of least gradient, then by comparing it to χ_F we obtain

$$\mathcal{H}^{N-1}(S_{s,i}) \leq \sum_j \mathcal{H}^{N-1}(T_{s,j} \cap \Omega^-_{t,i}) + \mathcal{H}^{N-1}(S_{t,i} \cap \overline{G}).$$

Moreover, this inequality is strict. If it was not strict, then the surface consisting of parts of $T_{s,j}$ and $S_{t,i}$, i.e. the boundary of G minus $S_{s,i}$, would be a minimal surface. But then by Proposition 2.4.4 it equals $S_{t,i}$, as it intersects $S_{t,i}$ and $G \subset \Omega^-_{t,i}$.

This contradicts the Green's formula for C_t and C_s, so our claim is proved.

Step 3a. Let $s \neq t$. Suppose that u is constant in a ball $B(x,r)$ with value s and that $x \notin J_v$. Then $x \notin \partial F_t$.

Assume the contrary: let $x \in \partial F_t$. Without loss of generality $s > t$. As $x \notin J_v$, by [28, Proposition 3.9] u is continuous at x. Hence, as $x \in \partial F_t$, in every ball around x there is a connected component S_τ of ∂F_τ for τ in some interval (t, t') (without loss of generality $t < t' < s$). We choose τ_1 and τ_2 in the interval (t, t') such that $\tau_1 < \tau_2$ and such that the set $\{f = \tau_1\} \cup \{f = \tau_2\} \subset \partial\Omega$ has $\mathcal{H}^{N-1}-$measure zero. We set C_{τ_1} and C_{τ_2} to be the connected components of $E_{\tau_1} \backslash F_{\tau_1}$ and $E_{\tau_2} \backslash F_{\tau_2}$ containing x. By Step 2 the sets C_{τ_1} and C_{τ_2} satisfy Green's formula and the interlacing condition, so (as S_{τ_1} intersects the ball $B(x,r)$ and $B(x,r) \subset E_{\tau_2}$) S_{τ_1} and ∂E_{τ_2} intersect and the geometrical situation for C_{τ_1} and C_{τ_2} is identical to the one in Step 3. We repeat the reasoning from Step 3 to obtain an inequality analogous to (3.3.2); this inequality is an equality if and only if $S_{\tau_1} = S_{\tau_2}$. While in Step 3 we concluded that it is not possible from the assumption that $x \notin J_u$, here it is not possible (up to choosing different τ_1, τ_2 closer to t) because x was a point of continuity of u; if $S_{\tau_1} = S_{\tau_2}$ for all $\tau_1, \tau_2 \in (t, t')$, then also $S_{\tau_1} = S_t$ and u has a jump at x, contradiction. This contradicts the Green's formula for C_{τ_1} and C_{τ_2}, so our claim is proved.

Step 4. Finally, we may define the set $\mathcal{N}$. At first, recall that J_u denotes the jump set of u and that for any function $u \in BV(\Omega)$ we have $\dim_H J_u = N - 1$.

By Lemma 3.2.5 we have $\partial\{u \geq t\} \neq \partial\{u > t\} = \partial\{u \leq t\}$ for at most countably many $t \in \mathbb{R}$. Similarly, the set $\{f = t\} \subset \partial\Omega$ has positive Hausdorff measure of codimension one for at most countably many t. Let us denote the (at most countable) set of $t \in \mathbb{R}$ satisfying either of these conditions by T_u. Let

$$B_u = \bigcup_{t \in T_u} (\partial\{u \geq t\} \cup \partial\{u \leq t\}).$$

We observe that this set has Hausdorff dimension at most $N - 1$: each of the sets $\partial\{u \geq t\}$ is a minimal surface with finite $\mathcal{H}^{N-1}$-measure, and the set B_u is an at most countable union of such sets (as the function from Example 3.2.4 shows, it does not have to have finite $\mathcal{H}^{N-1}$-measure). Now, we define a set (with Hausdorff dimension at most $N - 1$)

$$\mathcal{N} = J_u \cup J_v \cup B_u \cup B_v.$$

Take $x \in \Omega \backslash \mathcal{N}$. We have four possibilities:
1. $x \in \partial E_t \cap \partial F_t$. Then, as u and v are continuous at x, we have $u(x) = v(x) = t$.
2. $x \in \partial E_t \cap \partial F_s$ for $s \neq t$. This case is excluded by Step 3 of the proof.
3. $x \in \partial F_t$, $x \notin \partial E_s$ for any $s \in \mathbb{R}$. By Lemma 3.2.2 u is constant on some ball $B(x, r)$ with value s_0. If $t = s_0$, then, as u is continuous at x, we have $u(x) = t = s_0 = v(x)$. The case when $t \neq s_0$ is excluded by Step 3a of the proof.
4. $x \notin \partial E_t$, $x \notin \partial F_s$ for any $t, s \in \mathbb{R}$. Then by Lemma 3.2.2 u, v are constant in some ball around x; thus $x \in C$.

This ends the proof of Theorem 3.1.1. □

3.4 Classification of all solutions

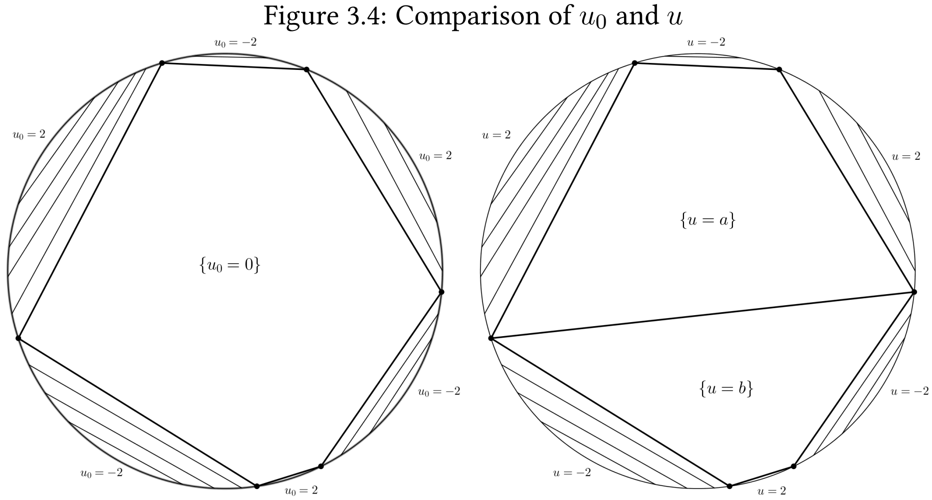
The purpose of this Section is to use Theorem 3.1.1 and the knowledge obtained in Steps 1 and 2 of the proof of Theorem 3.1.1 to form a classification of the solutions to the least gradient problem with boundary data $f \in L^1(\partial\Omega)$. We stress that we do not try to answer any questions about existence of solutions to the least gradient problem and refer to later Chapters. Since Theorem 3.1.1 does not give us any direct information about the structure of solutions, only through comparison with another solution, we assume that at least one solution $u_0 \in BV(\Omega)$ exists and is known.

We start with a two-dimensional toy model. Then we pass to the full classification. However, the presented algorithm to find all solutions is fully applicable only in dimension two; one of the steps is to find all minimal decompositions of the set C, on which u_0 is locally constant, into sets with minimal boundary that satisfy Green's formula. This is equivalent to solving the Plateau problem, in which the spanning set is not homeomorphic to a sphere, but even may fail to be connected (it may have countably many connected components) or simply-connected. Because of that, the reasoning in this Section has two purposes: in dimension 2, the algorithm presented here enables us to find all the solutions; in dimensions 3 to 7, save for situations with additional symmetries, the reasoning below provides a way to determine if a function $u \in BV(\Omega)$ is a solution to the least gradient problem with boundary data f without directly calculating the total variation.

3.4.1 Detailed example of (non)uniqueness

Take $\Omega = B(0,1) \subset \mathbb{R}^2$. Let f be a function with the following properties: it has six discontinuity points $p_1, ..., p_6 \in \partial\Omega$. On each of the arcs $(p_1, p_2), (p_3, p_4)$ and (p_5, p_6) this function is continuous and strictly convex. It has a single minimum with value -2 in each of these arcs and limits equal to -1 at each end of these arcs. Similarly, f is continuous and strictly concave on each of the arcs $(p_2, p_3), (p_4, p_5)$ and (p_6, p_1). It has a single maximum with value 2 in each of these arcs and limits equal to 1 at each end of these arcs. In particular, f has a jump from -1 to 1 at each of the points $p_1, ..., p_6$. It is easy to see that the function u_0 as on the left hand side of Figure 3.4 is a solution of the least gradient problem (for example by proceeding as in the proof of Theorem 2.1.9 as written in [28, Theorem 1.1], i.e. using approximations to the boundary data and the Sternberg-Williams-Ziemer construction).

Figure 3.4: Comparison of u_0 and u



The set C from the statement of Theorem 3.1.1 is the hexagon $H = p_1p_2p_3p_4p_5p_6$. Let $u \in BV(\Omega)$ be a candidate for another solution to the least gradient problem with boundary data f. By Theorem 3.1.1 we have $u = u_0$ in $\Omega \backslash H$. We also know that u is locally constant on H.

Let $H = \bigcup_i H_i$ such that each of the sets H_i is connected and that u is constant and equal t_i on H_i. Then $\partial H_i \subset \partial\{u \geq t_i\} \cup \partial\{u \leq t_i\}$; by the weak maximum principle (Proposition 3.2.6) ∂H_i composes of pairwise disjoint line segments with endpoints in $\partial\Omega$. By Proposition 2.4.4 these line segments cannot intersect the set $\Omega \backslash H$, as they would intersect transversally some line segment of the form $\partial\{u \geq t\}$ for $t \neq t_i$. Thus these line segments have endpoints in the set $\{p_1, ..., p_6\}$. Moreover, analysis as in Step 2 of the proof of Theorem 3.1.1 shows that the sides of the polygon H_i interlace, i.e. belong alternately to $\partial\{u \geq t_i\}$ and $\partial\{u \leq t_i\}$ and satisfy Green's formula.

This means that finding all functions u of least gradient with boundary data h boils down to finding all subpolygons of H which satisfy Green's formula. If there are none (for instance when the hexagon is equilateral), then u is constant on H. After a quick calculation we obtain the value $u(H)$:

$$|Du|(\Omega) = |Du|(\Omega\backslash H) + (\mathcal{H}^1(\overline{p_1p_2}) + \mathcal{H}^1(\overline{p_3p_4}) + \mathcal{H}^1(\overline{p_5p_6}))| - 1 - u(H)| + \\ +(\mathcal{H}^1(\overline{p_2p_3}) + \mathcal{H}^1(\overline{p_4p_5}) + \mathcal{H}^1(\overline{p_6p_1}))|1 - u(H)| + 0$$

and

$$|Du_0|(\Omega) = |Du_0|(\Omega\backslash H) + (\mathcal{H}^1(\overline{p_1p_2}) + \mathcal{H}^1(\overline{p_3p_4}) + \\ +\mathcal{H}^1(\overline{p_5p_6}))| - 1 - 0| + (\mathcal{H}^1(\overline{p_2p_3}) + \mathcal{H}^1(\overline{p_4p_5}) + \mathcal{H}^1(\overline{p_6p_1}))|1 - 0| + 0.$$

Using Green's formula for H and the fact that $u = u_0$ on $\Omega\backslash H$ we easily see that these two numbers are equal, i.e. u is a function of least gradient if and only if $u(H) \in [-1, 1]$.

However, there may exist subpolygons of H which satisfy Green's formula; the only possible case is two trapezoids $H_1 = p_1p_4p_3p_2$ and $H_2 = p_1p_4p_5p_6$ satisfying Green's formula with one common side (without loss of generality the common side is p_1p_4). This situation is presented on Figure 3.4 on the right hand side. Let a be the value on H_1 and b the value on H_2. Suppose that $a \neq b$, so the situation is different from the above. A calculation similar to the one above shows that $a, b \in [-1, 1]$; the only remaining problem is whether a or b is larger. This follows from Step 2 of the proof of Theorem 3.1.1; the sides of H_1 have to interlace, i.e. belong alternately to $\partial\{u \geq a\}$ and $\partial\{u \leq a\}$. Thus, as $p_1p_2 \subset \partial\{u \geq -1\}$, then also $p_1p_2 \subset \partial\{u \geq a\}$; but this implies that $p_1p_4 \subset \partial\{u \leq a\}$, so $a < b$. Quick calculation using Green's formula for H_1 and H_2 shows that a function u such that $u = u_0$ on $\Omega\backslash H$, $u = a$ on H_1, $u = b$ on H_2, $-1 \leq a \leq b \leq 1$ is of least gradient. Thus, we have classified all solutions to the least gradient problem with boundary data f.

3.4.2 Full description

We want to find all functions of least gradient with prescribed boundary data $f \in L^1(\partial\Omega)$. Assume that we know a minimiser $u_0 \in BV(\Omega)$ of the least gradient problem and let

$u \in BV(\Omega)$ be another minimiser. Direct use of Theorem 3.1.1 shows that (up to a choice of representative) $u = u_0$ in $\Omega \backslash C$, where C is a set on which both u and u_0 are locally constant. We want to find all admissible decompositions of C (uniquely defined by u_0) into sets C_i and admissible (constant) values t_i of u on C_i.

Assumption. For simplicity, we will assume that the function f has no level sets of positive measure. At the end of this subsection, under the additional assumption that Ω is strictly convex, we will modify this reasoning to account for such sets.

Motivated by Section 3.4.1, we take a minimal decomposition of C into at most countably many sets C_i which for every i satisfies the following properties: (1) connected components of ∂C_i are area-minimising surfaces; (2) connected components of ∂C_i can be separated into two (at most countable) families $\{S_j\}$ and $\{T_k\}$ which satisfy the interlacing condition as defined in Step 2 of the proof of Theorem 3.1.1; (3) this set and these families satisfy Green's formula. Such decompositions exist, an example is provided by $C_i = \{u_0 = t_i\}$, as we can see from Step 2 of the proof of Theorem 3.1.1. We take a minimal decomposition in the sense that no set can be decomposed further into multiple parts safisfying the same properties as above. We do not claim that such decomposition is unique.

Notation. Given a solution u_0, we take the set C on which it is locally constant, and take a minimal decomposition of C into C_i. We assume that the decomposition into C_i is locally finite so that what follows is well defined. With a slight abuse of notation, we form a graph also denoted by C, whose vertices are the sets C_i. They are connected by an edge if and only if $\mathcal{H}^{N-1}(\partial C_i \cap \partial C_j) > 0$. Additionally, we require that this graph is directed: whenever C_i and C_j are connected by an edge and $t_i \geq t_j$, this edge is directed from C_i to C_j. In particular, if $t_i = t_j$, we draw an arrow in both directions. We say that C_i is a leaf if it shares part of its boundary only with one C_j.

The graph C is not necessarily connected, so in what follows we will consider each connected component separately. Take $\hat{C}$ to be a connected component of the graph C and with the same abuse of notation we denote by $\hat{C}$ also the corresponding set. Since any connected component S of ∂C_i separates Ω into two disjoint open sets, there is at most one path connecting two given vertices, so $\hat{C}$ is a tree. Finally, we will denote the connected components of $\Omega \backslash \hat{C}$ are by U_l. If for some i, l the set $\partial C_i \cap \partial U_l$ is nonempty, by [28, Corollary 3.12] the trace of u_0 on any $\partial C_i \cap \partial U_l$ from U_l is constant and denoted by α_{il}.

The following Proposition provides a necessary and sufficient condition for a given function $u \in BV(\Omega)$ to be a function of least gradient with the same trace as another given function of least gradient u_0. The condition is given separately on every connected component of the graph C. From Theorem 3.1.1 we know that if u is a function of least gradient with the same boundary data as u_0, then $u = u_0$ in $\Omega \backslash C$, so in order to obtain all the solutions, one has to iterate this result over every connected component of the graph C.

Proposition 3.4.1. *Let $\Omega \subset \mathbb{R}^N$, where $2 \leq N \leq 7$ be an open bounded convex set with Lipschitz boundary. Suppose that $f \in L^1(\partial\Omega)$ and there is at least one solution $u_0 \in BV(\Omega)$ to the least gradient problem. In the notation introduced above, $\hat{C}$ is a connected component of the graph C, which is decomposed into C_i and take t_i^0 to be the (constant) values of u_0 on C_i.*

Suppose that $u = u_0$ in $\Omega \backslash \hat{C}$. If Ω is strictly convex or if f has no level sets of positive measure, then u is a solution of the least gradient problem with boundary data f if and only if for all C_i u has constant value t_i on C_i such that the following conditions are satisfied:

(1) In the notation introduced above, the graph for u (restricted to $\hat{C}$) is the following: the arrows from leaves to their neighbours are well defined using the interlacing condition. They, using the same technique, define arrows on all other edges. This graph is the same as for u_0, with a possible exception that some of the arrows in are in both directions in one graph and in one direction in the other graph. Such graphs are possible, as there exists a graph for u_0;
(2) Whenever $\mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) > 0$ and $t_i^0 \geq \alpha_{il}$, then $t_i \geq \alpha_{il}$;
(3) Whenever $\mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) > 0$ and $t_i^0 \leq \alpha_{il}$, then $t_i \leq \alpha_{il}$.

The exception that some of the arrows in are in both directions in one graph and in one direction in the other graph corresponds to the fact that for some solutions we may have equalities $t_i = t_j$ for some neighbouring sets C_i, C_j and strict inequality for other solutions, as we saw in Section 3.4.1.

Proof. In the notation introduced above, fix any minimal decomposition C_i of the set $\hat{C}$. Different decompositions will give us different functions of least gradient. Since u_0 is a function of least gradient, u is a function of least gradient if and only if $|Du|(\Omega) = |Du_0|(\Omega)$. We calculate $|Du|(\Omega)$:

$$|Du|(\Omega) = |Du|(\Omega \backslash \hat{C}) + \sum_{i,l} |\alpha_{il} - t_i| \mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) + \sum_{i>j} |t_i - t_j| \mathcal{H}^{N-1}(\partial C_i \cap \partial C_j).$$

We may write here a sum over all i, j, l because if C_i and C_j or U_l do not share a boundary, the corresponding value is zero. We obtain an analogous result for u_0.

Sufficiency of conditions (1)-(3). To summarise, u satisfies the same inequalities between values of u on C_i, C_j, and U_l as u_0, except for the fact that they may cease to be strict, and whenever $\mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) > 0$ we have inequalities of the form

$$\min_{l: \partial C_i \cap \partial U_l \subset \partial\{u_0 \geq t_i^0\}} \alpha_{il} \leq t_i^0 \leq \min_{l: \partial C_i \cap \partial U_l \subset \partial\{u_0 \leq t_i^0\}} \alpha_{il}.$$

We shall see that every u which satisfies these properties is a function of least gradient. Whenever $\mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) > 0$, denote by s_{il} the function encoding inequalities between t_i and α_{il}: set $s_{il} = -1$ if $t_i > \alpha_{il}$ and $s_{il} = 1$ if the opposite inequality holds. When $t_i = \alpha_{il}$, we set $s_{il} = 0$. Similarly, whenever $\mathcal{H}^{N-1}(\partial C_i \cap \partial C_j) > 0$, we denote by s^{ij} the function encoding inequalities between t_i and t_j: set $s^{ij} = -1$ if $t_i > t_j$ and $s^{ij} = 1$ if the opposite inequality holds. When $t_i = t_j$, we set $s^{ij} = 0$. To prove that u is of least gradient we have to check that $|Du|(\Omega) - |Du_0|(\Omega) = 0$.

$$|Du|(\Omega) - |Du_0|(\Omega) = \sum_{i,l} |\alpha_{il} - t_i| \mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) - \sum_{i,l} |\alpha_{il} - t_i^0| \mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) +$$

$$+ \sum_{i>j} |t_i - t_j| \mathcal{H}^{N-1}(\partial C_i \cap \partial C_j) - \sum_{i>j} |t_i^0 - t_j^0| \mathcal{H}^{N-1}(\partial C_i \cap \partial C_j) =$$

$$= \sum_{i,l} s_{il}(\alpha_{il} - t_i) \mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) - \sum_{i,l} s_{il}(\alpha_{il} - t_i^0) \mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) +$$

$$+ \sum_{i>j} s^{ij}(t_j - t_i) \mathcal{H}^{N-1}(\partial C_i \cap \partial C_j) - \sum_{i>j} s^{ij}(t_j^0 - t_i^0) \mathcal{H}^{N-1}(\partial C_i \cap \partial C_j) =$$

$$= \sum_{i,l} s_{il}(t_i^0 - t_i) \mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) + \sum_{i>j} s^{ij}(t_i^0 - t_i - t_j^0 + t_j) \mathcal{H}^{N-1}(\partial C_i \cap \partial C_j) =$$

$$= \sum_i (\sum_l s_{il}(t_i^0 - t_i) \mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) + \sum_j s^{ij}(t_i^0 - t_i) \mathcal{H}^{N-1}(\partial C_i \cap \partial C_j)) =$$

$$= \sum_i (t_i^0 - t_i)(\sum_l s_{il} \mathcal{H}^{N-1}(\partial C_i \cap \partial U_l) + \sum_j s^{ij} \mathcal{H}^{N-1}(\partial C_i \cap \partial C_j)) = 0,$$

because for every i the last summand is precisely Green's formula for the sides of C_i. Thus, every u satisfying the assumptions above is a function of least gradient.

Necessity of conditions (1)-(3). Let C_i be a leaf, i.e. C_i shares its boundary with only one C_j. Then C_i shares its boundary with at least two sets of the form U_l. On the set C_i the function u_0 has a constant value t_i^0 and u has a constant value t_i. Without loss of generality assume that $\partial C_i \cap \partial U_{l_1} \subset \partial\{u_0 \geq t_i^0\}$; in particular, we have $t_i^0 \geq \alpha_{il_1}$. Using the interlacing condition we have that $\partial C_i \cap \partial U_{l_2} \subset \partial\{u_0 \leq t_i^0\}$; thus $t_i^0 \leq \alpha_{il_2}$.

Suppose that the structure of u is different from the structure of u_0, i.e. $\partial C_i \cap \partial U_{l_1} \subset \partial\{u_0 \leq t_i^0\}$. In particular $t_i \neq t_i^0$. Repeating the reasoning above we obtain that $t_i \leq \alpha_{il_1}$ and $t_i \geq \alpha_{il_2}$. Putting these results together, we obtain

$$t_i \leq \alpha_{il_1} \leq t_i^0 \leq \alpha_{il_2} \leq t_i.$$

Thus $t_i = t_i^0$, contradiction. Thus on every leaf conditions (1)-(3) are necessary. Once we do this for all the leaves, we eliminate all the leaves from the graph and repeat, treating the leaves the same as the sets U_l. Thus conditions (1)-(3) are necessary for every $C_i \subset \hat{C}$.

Relaxing the assumption. Assume Ω to be additionally strictly convex. Suppose that f is constant and equal to t on a set Γ of positive measure on $\partial\Omega$. In two dimensions, if Γ

is an arc with endpoints p_1, p_2, denote by Ω_Γ the set enclosed by Γ and the line segment $\overline{p_1 p_2}$. In the general case we have to remember that $\partial\{u > t\} = \partial\{u \leq t\}$ and $\partial\{u \geq t\}$ is composed of minimal surfaces; thus Γ spans a set consisting of minimal surfaces. Denote by Ω_Γ the set enclosed by these surfaces and $\partial\Omega$.

Then, by Lemma 3.2.2 we observe that on Ω_Γ the value of u has to be constant and equal t. This value is fixed, so from now on we may treat Ω_Γ as one of the sets U_l in the reasoning above. Thus, we do not need to assume that f does not have level sets of positive measure. □

Let us note that Proposition 3.4.1 has algorythmic value in case when $\Omega \subset \mathbb{R}^2$, because the only connected minimal surfaces are line segments, and when the decomposition into C_i is finite. Finally, the following well-known examples serve as an illustration to this result:

Example 3.4.2. Let $\Omega = B(0,1) \subset \mathbb{R}^2$.
(1) f has a single maximum and a single minimum and $\partial\Omega$ can be divided into two arcs, on which f is monotone. Then the solution to the least gradient problem is unique;
(2) f takes only three values: 0 on the arc (p_1, p_2), $\alpha_1 > 0$ on the arc (p_2, p_3), and $\alpha_1 + \alpha_2 > \alpha_1$ on the arc (p_3, p_1) (see [31, Section 3.4]). Then the solution to the least gradient problem is unique and equals α_1 on the curvilinear triangle $p_1 p_2 p_3$ and 0 and α_2 in the respective flaps;
(3) f is the function from the Brothers example, see [48, Example 2.7]. It is given by the formula

$$f(x,y) = \begin{cases} x^2 - y^2 + 1 & |x| > \frac{1}{\sqrt{2}}, \\ x^2 - y^2 - 1 & |x| < \frac{1}{\sqrt{2}}. \end{cases}$$

Then $u \in BV(\Omega)$ is a function of least gradient if and only if

$$u(x,y) = \begin{cases} 2x^2 & |x| > \frac{1}{\sqrt{2}}, \\ \lambda & |x|, |y| < \frac{1}{\sqrt{2}}, \\ -2y^2 & |y| > \frac{1}{\sqrt{2}}, \end{cases}$$

where $\lambda \in [-1, 1]$. We will revisit this example from a different perspective in Chapter 5.

A new type of example is the one presented in Section 3.4.1. There, we witness the phenomenon of breaking of a level set into multiple parts. Of course it can be reversed, i.e. take u_0 to be the function which takes two values on the hexagon H and u the function which takes one value; in that case the two level sets of u_0 merge into a single level set of u. Finally, in the following example we consider a three-dimensional setting with axial symmetry.

Example 3.4.3. Let $\Omega = B(0,1) \subset \mathbb{R}^3$. Take the boundary data to be

$$f(x,y,z) = \begin{cases} 1 & |z| > a, \\ -1 & |z| < a, \end{cases}$$

where the constant a is chosen so that the two circles, which are intersections of Ω and the planes $\{z = \pm a\}$, have the same area as the catenoid spanned by them. Using Proposition 3.4.1 and the axial symmetry which helps us solve the Plateau problem, we prove that

$$u(x,y,z) = \begin{cases} 1 & |z| > a, \\ \lambda & |z| < a, \text{inside the catenoid}, \\ -1 & |z| < a, \text{outside the catenoid}, \end{cases}$$

where $\lambda \in [-1,1]$. Moreover, we may take a closer look at the interlacing condition: the set $\partial\{u \geq \lambda\}$ is the catenoid and the set $\partial\{u \leq \lambda\}$ is the two circles. The two circles do not intersect and the catenoid intersects the circles at the boundary, so the interlacing condition is satisfied. We will revisit this example from a different perspective in Chapter 5.

3.5 Selection criterion for minimisers

The strain-gradient plasticity model, as introduced in [2], is a problem of minimisation of the functional

$$\widetilde{F_1}(u) = \int_\Omega (u^2 + |\nabla u|^2)^{\frac{1}{2}}\, dx,$$

well-defined over $W^{1,1}(\Omega)$. In the literature, for example see [2], this functional is minimised with respect to two contraints: the Dirichlet boundary conditions and a condition on the total mass of the solution.

Here, we want to introduce a parameter ε and examine the behaviour of minimisers for small ε. For Dirichlet boundary data $f \in L^1(\partial\Omega)$, we set

$$W^{1,1}_f(\Omega) = \Big\{ u \in W^{1,1}(\Omega) : \quad Tu = f \Big\}.$$

Then, we define a functional $\widetilde{F_\varepsilon}$ over $L^1(\Omega)$ by the formula

$$\widetilde{F_\varepsilon}(u) = \begin{cases} \int_\Omega (\varepsilon u^2 + |Du|^2)^{\frac{1}{2}}\, dx & u \in W^{1,1}_f(\Omega), \\ +\infty & u \in L^1(\Omega) \backslash W^{1,1}_f(\Omega). \end{cases}$$

As it turns out even for the simplest possible boundary data, this functional may have no minimisers in $L^1(\Omega)$. We may derive its lower semicontinous envelope similarly as it was calculated in [2, Section 7] for $\varepsilon = 1$:

$$F_\varepsilon(u) = \begin{cases} \int_\Omega (\varepsilon u^2 + |\nabla u|^2)^{\frac{1}{2}} dx + \int_\Omega |D^s u| + \int_{\partial\Omega} |Tu - f| d\mathcal{H}^{N-1} & u \in BV(\Omega), \\ +\infty & u \in L^1(\Omega) \backslash BV(\Omega). \end{cases}$$

Here we focus on the relationship between this functional and the functional F, the relaxed functional in the least gradient problem, namely

$$F(u) = \begin{cases} \int_\Omega |Du| + \int_{\partial\Omega} |Tu - f|\, d\mathcal{H}^{N-1} & u \in BV(\Omega), \\ +\infty & u \in L^1(\Omega) \backslash BV(\Omega). \end{cases}$$

Firstly, we prove $\Gamma-$convergence of F_ε (and a similar functional $G_{p,\varepsilon}$) to F and some of its consequences. Secondly, we shall see that minimisers of $G_{p,\varepsilon}$ converge in L^p to minimisers of F which have the smallest norm in L^p; this provides a selection criterion for least gradient functions with prescribed boundary conditions, as in general the solutions for Dirichlet least gradient problem may be not unique. Finally, we shall discuss some stronger modes of convergence of these minimisers. In what follows, we simplify the notation by not writing what is the measure with respect to which we integrate: unless the integrand is a measure which is explicitly written, integrals of functions on Ω are taken with respect to the Lebesgue measure $\mathcal{L}^N$ and integrals of functions on $\partial\Omega$ are taken with respect to the Hausdorff measure $\mathcal{H}^{N-1}$.

Proposition 3.5.1. *With respect to convergence in $L^1(\Omega)$, we have $\Gamma - \lim_{\varepsilon\to 0} F_\varepsilon = F$.*

Proof. We have to check the two conditions in the definition of $\Gamma-$convergence.

(1) We show that for any sequence $u_n \to u$ in $L^1(\Omega)$ and any sequence $\varepsilon_n \to 0$ we have $F(u) \leq \liminf_{n\to\infty} F_{\varepsilon_n}(u_n)$.

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n) = \liminf_{n\to\infty} \int_\Omega (\varepsilon_n u_n^2 + |\nabla u_n|^2)^{\frac{1}{2}} + \int_\Omega |D^s u_n| + \int_{\partial\Omega} |Tu_n - f| \geq$$

$$\geq \liminf_{n\to\infty} \int_\Omega |\nabla u_n| + \int_\Omega |D^s u_n| + \int_{\partial\Omega} |Tu_n - f| = \liminf_{n\to\infty} F(u_n) \geq F(u).$$

The first inequality follows from a pointwise inequality between functions under the integral. The second inequality follows from lower semicontinuity of F.

(2) We show that for any function $u \in L^1(\Omega)$ and any sequence $\varepsilon_n \to 0$ there exists a sequence $u_n \to u$ such that $F(u) \geq \limsup_{n\to\infty} F_{\varepsilon_n}(u_n)$.

If $u \notin BV(\Omega)$, the inequality is obvious. If $u \in BV(\Omega)$, take any sequence u_n converging strictly to u, i.e. $u_n \to u$ in $L^1(\Omega)$ and $\int_\Omega |Du_n| \to \int_\Omega |Du|$. In particular, $\int_\Omega |u_n| \leq M$. Then,

$$\limsup_{n\to\infty} F_{\varepsilon_n}(u_n) = \limsup_{n\to\infty} \int_\Omega (\varepsilon_n u_n^2 + |\nabla u_n|^2)^{\frac{1}{2}} + \int_\Omega |D^s u_n| + \int_{\partial\Omega} |Tu_n - f| \leq$$

$$\leq \limsup_{n\to\infty} \int_\Omega (\sqrt{\varepsilon_n}|u_n| + |\nabla u_n|) + \int_\Omega |D^s u_n| + \int_{\partial\Omega} |Tu_n - f| \leq$$

$$\leq \limsup_{n\to\infty} \sqrt{\varepsilon_n} M + \int_\Omega |Du_n| + \int_{\partial\Omega} |Tu_n - f| = 0 + \limsup_{n\to\infty} F(u_n) = F(u).$$

The first inequality follows from a pointwise inequality between functions under the integral. The second inequality follows the upper bound on L^1 norms of u_n. The limit of $F(u_n)$ equals $F(u)$ because of strict convergence and continuity of trace in the strict topology. □

Remark 3.5.2. Note that in particular we proved that for strict convergence $u_n \to u$ we have $F_{\varepsilon_n}(u_n) \to F(u)$.

From the $\Gamma-$convergence of F_ε to F it follows that if u_n is a minimiser of F_{ε_n}, then every cluster point of the sequence u_n is a minimiser of F. We shall see that we have a common bound in BV norm for minimisers of F_ε for $\varepsilon \leq 1$, so there is a convergent subsequence in $L^1(\Omega)$.

Proposition 3.5.3. *Let u_n be a sequence of minimisers of F_{ε_n}, $\varepsilon_n \to 0$. We may assume that $\varepsilon \leq 1$. Then, there is a convergent subsequence $u_{n_k} \to u$ in $L^1(\Omega)$.*

Proof. Notice that for $f \in L^1(\partial\Omega)$ we have $F_1(v \equiv 0) = \int_\Omega 0 + \int_{\partial\Omega} |f| < \infty$. Then,

$$\int_\Omega |Du_n| \leq F(u_n) \leq F_{\varepsilon_n}(u_n) \leq F_{\varepsilon_n}(v \equiv 0) \leq F_1(v \equiv 0) < \infty.$$

Thus, the total variations of u_n are uniformly bounded. This observation and Dirichlet boundary condition imply a common bound in L^1 norm, which together with the bound on the total variation gives us a bound in the BV norm as well. Take an extension of u_n on some ball $B(0,R)$ such that $\Omega \subset\subset B(0,R)$ defined by the formula

$$\widetilde{u_n}(x) = \begin{cases} u_n(x) & x \in \Omega, \\ 0 & \text{otherwise.} \end{cases}$$

We apply the Poincaré inequality to $\widetilde{u_n}$ (note that $\widetilde{u_n}$ has compact support). Thus,

$$\int_\Omega |u_n| = \int_{B(0,R)} |\widetilde{u_n}| \leq C|D\widetilde{u_n}|(B(0,R)) = C|Du_n|(\Omega) + 0 + C\int_{\partial\Omega} |Tu_n| \leq$$

$$\leq C|Du_n|(\Omega) + C\int_{\partial\Omega} |Tu_n - f| + C\int_{\partial\Omega} |f| = CF(u_n) + C\int_{\partial\Omega} |f| \leq$$

$$\leq CF_1(v \equiv 0) + CF_1(v \equiv 0) = 2CF_1(v \equiv 0) < \infty.$$

We put together the estimates for the L^1 norm and total variation of u_n. It follows that $\|u_n\|_{BV} \leq (2C+1)F_1(v \equiv 0) < \infty$, so it has a convergent subsequence $u_{n_k} \to u$ in $L^1(\Omega)$. □

The following result shows that the convergence guaranteed by Proposition 3.5.3 is sometimes in fact not only in $L^1(\Omega)$, but in strict topology of $BV(\Omega)$.

Proposition 3.5.4. *Let u_n be a sequence of minimisers of F_{ε_n}, $\varepsilon_n \to 0$. Let $u_n \to u$ in $L^1(\Omega)$; in particular u is a minimiser of F. Then:*
(1) $F(u_n) \to F(u)$;
(2) If $Tu = f$, then $u_n \to u$ in the strict topology of $BV(\Omega)$.

Proof. (1) Since u_n are minimisers of F_{ε_n} and u is a minimiser of F, we have

$$F(u) \leq F(u_n) \leq F_{\varepsilon_n}(u_n) \leq F_{\varepsilon_n}(u) \to F(u).$$

(2) Since u_n are minimisers of F_{ε_n}, we have

$$\int_\Omega |Du_n| \leq F_{\varepsilon_n}(u_n) \leq F_{\varepsilon_n}(u) \leq \int_\Omega \sqrt{\varepsilon_n}|u| + \int_\Omega |Du| + \int_{\partial\Omega} 0,$$

so

$$\limsup_{n\to\infty} \int_\Omega |Du_n| \leq \int_\Omega |Du|.$$

By lower semicontinuity of the total variation we obtain the opposite inequality, so

$$\lim_{n\to\infty} \int_\Omega |Du_n| = \int_\Omega |Du|.$$

□

However, looking at the functional F_ε gives us little information about pointwise properties of the approximating sequence u_n. It also gives us convergence to some minimiser of F, while we want our sequence to choose one particular element of $\arg\min F$. To this end, let us define for $1 \leq p < \frac{N}{N-1}$ an auxiliary functional $G_{p,\varepsilon}$:

$$G_{p,\varepsilon}(u) = \begin{cases} (\int_\Omega \sqrt{\varepsilon}|u|^p \, dx)^{\frac{1}{p}} + \int_\Omega |Du| + \int_{\partial\Omega} |Tu - f| \, d\mathcal{H}^{N-1} & u \in BV(\Omega) \\ +\infty & \text{otherwise.} \end{cases}$$

In other words, we have $G_{p,\varepsilon}(u) = \sqrt[2p]{\varepsilon}\|u\|_p + F(u)$. Using the continuous embedding of $BV(\Omega)$ into $L^p(\Omega)$, we see that all the above results hold also for $G_{p,\varepsilon}$ with an analogous proof. Now, we shall see that $G_{p,\varepsilon}$ provides a selection criterion for minimisers of F:

Theorem 3.5.5. *Let $v_n \in \arg\min G_{p,\varepsilon_n}$ and $\varepsilon_n \to 0$. Suppose that $v_n \to v$ in $L^p(\Omega)$. By $\Gamma-$convergence of G_{ε_n} we have $v \in \arg\min F$. Then, v is an element with the smallest L^p norm among minimisers of F.*

Proof. Suppose that u is another minimiser of F, which has smaller L^p norm than v. Let $\delta < \frac{\|v\|_p - \|u\|_p}{2}$. Fix n big enough, so that $|\|v_n\|_p - \|v\|_p| < \delta$. Since v_n are minimisers of G_{ε_n}, we have

$$0 \geq G_{p,\varepsilon_n}(v_n) - G_{p,\varepsilon_n}(u) = \sqrt[2p]{\varepsilon}\|v_n\|_p + F(v_n) - \sqrt[2p]{\varepsilon}\|u\|_p - F(u) \geq$$

$$\geq \sqrt[2p]{\varepsilon}\|v_n\|_p - \sqrt[2p]{\varepsilon}\|u\|_p = \sqrt[2p]{\varepsilon}(\|v_n\|_p - \|v\|_p) + \sqrt[2p]{\varepsilon}(\|v\|_p - \|u\|_p) \geq$$
$$\geq -\sqrt[2p]{\varepsilon}\delta + 2\sqrt[2p]{\varepsilon}\delta > 0,$$

a contradiction. Thus, v_n cannot converge to an element which does not have smallest L^p norm. □

Let us note that compact embedding of $BV(\Omega)$ into $L^p(\Omega)$ implies that the sequence u_n, due to its boundedness in $BV(\Omega)$, is convergent in $L^p(\Omega)$ on some subsequence. As the natural underlying space for $BV(\Omega)$ is $L^1(\Omega)$, it is tempting to consider only $p = 1$; however, we do not know if the minimiser of F with the smallest norm in L^1 is unique, while for $p > 1$ it is unique (see later in Proposition 3.5.9). Furthermore, $G_{p,\varepsilon}$ have unique minimisers for $p > 1$, because they are strictly convex; it does not apply to $p = 1$.

However, convergence in L^1 is quite weak, so a natural question is if some stronger mode of convergence might be at play. The natural candidate is strict convergence; however, for boundary data with a constant sign we may prove a much stronger result.

Proposition 3.5.6. *Let $f \in L^1(\partial\Omega)$ be nonnegative. Let $\varepsilon_1 > \varepsilon_2$. Then any minimiser of G_{p,ε_1} is pointwise smaller than any minimiser of G_{p,ε_2}, i.e. let $u_1 \in \arg\min G_{p,\varepsilon_1}$ and $u_2 \in \arg\min G_{p,\varepsilon_2}$. Then $u_2 \geq u_1$.*

Proof. In the beginning, let us note that since f is nonnegative, u_1 and u_2 are as well: it is enough to compare the value of G_{p,ε_i} on u_i and $\max(u_i, 0)$.

Our starting point is the inequality

$$G_{p,\varepsilon_1}(u_1) + G_{p,\varepsilon_2}(u_2) \leq G_{p,\varepsilon_1}(\min(u_1, u_2)) + G_{p,\varepsilon_2}(\max(u_1, u_2)), \tag{3.5.1}$$

which is automatically fulfilled, because u_1 and u_2 are minimisers of G_{p,ε_1} and G_{p,ε_2} respectively. We expand the left hand side of the above inequality:

$$G_{p,\varepsilon_1}(u_1) + G_{p,\varepsilon_2}(u_2) = \sqrt[2p]{\varepsilon_1}\|u_1\|_p + \int_\Omega |Du_1| +$$
$$+ \int_{\partial\Omega} |Tu_1 - f| + \sqrt[2p]{\varepsilon_2}\|u_2\|_p + \int_\Omega |Du_2| + \int_{\partial\Omega} |Tu_2 - f|$$

We also expand the right hand side:

$$G_{p,\varepsilon_1}(\min(u_1, u_2)) + G_{p,\varepsilon_2}(\max(u_1, u_2)) = \sqrt[2p]{\varepsilon_1}\|\min(u_1, u_2)\|_p +$$
$$+ \int_\Omega |D\min(u_1, u_2)| + \int_{\partial\Omega} |T\min(u_1, u_2) - f| + \sqrt[2p]{\varepsilon_2}\|\max(u_1, u_2)\|_p +$$
$$+ \int_\Omega |D\max(u_1, u_2)| + \int_{\partial\Omega} |T\max(u_1, u_2) - f|.$$

Firstly, let us recall that Lemma A.0.1 states that

$$\int_\Omega |D\max(u, v)| + \int_\Omega |D\min(u, v)| \leq \int_\Omega |Du| + \int_\Omega |Dv|.$$

Secondly, Lemma A.0.2 implies that

$$\int_{\partial\Omega} |T\max(u_1, u_2) - f| + \int_{\partial\Omega} |T\min(u_1, u_2) - f| = \int_{\partial\Omega} |Tu_1 - f| + \int_{\partial\Omega} |Tu_2 - f|$$

because we have pointwise equality $\mathcal{H}^{N-1}$-a.e. Thus, most of summands in inequality (3.5.1) cancel out and it reduces to the following inequality:

$$\sqrt[2p]{\varepsilon_1}\|u_1\|_p + \sqrt[2p]{\varepsilon_2}\|u_2\|_p \leq \sqrt[2p]{\varepsilon_1}\|\min(u_1,u_2)\|_p + \sqrt[2p]{\varepsilon_2}\|\max(u_1,u_2)\|_p.$$

Since u_1, u_2 are nonnegative, we may expand the left hand side in the following way:

$$\sqrt[2p]{\varepsilon_1}\|u_1\|_p + \sqrt[2p]{\varepsilon_2}\|u_2\|_p = \sqrt[2p]{\varepsilon_1}\int_0^\infty pt^{p-1}|\{u_1 > t\}|\,dt +$$

$$+ \sqrt[2p]{\varepsilon_2}\int_0^\infty pt^{p-1}|\{u_2 > t\}|\,dt = \sqrt[2p]{\varepsilon_1}\int_0^\infty pt^{p-1}|\{u_1 > t\}\cap\{u_2 > t\}|\,dt +$$

$$+ \sqrt[2p]{\varepsilon_1}\int_0^\infty pt^{p-1}|\{u_1 > t\}\backslash\{u_2 > t\}|\,dt + \sqrt[2p]{\varepsilon_2}\int_0^\infty pt^{p-1}|\{u_2 > t\}\backslash\{u_1 > t\}|\,dt +$$

$$+ \sqrt[2p]{\varepsilon_2}\int_0^\infty pt^{p-1}|\{u_1 > t\}\cap\{u_2 > t\}|\,dt.$$

We write the right hand side in the following way:

$$\sqrt[2p]{\varepsilon_1}\|\min(u_1,u_2)\|_p + \sqrt[2p]{\varepsilon_2}\|\max(u_1,u_2)\|_p = \sqrt[2p]{\varepsilon_1}\int_0^\infty pt^{p-1}|\{\min(u_1,u_2) > t\}|\,dt +$$

$$+ \sqrt[2p]{\varepsilon_2}\int_0^\infty pt^{p-1}|\{\max(u_1,u_2) > t\}|\,dt = \sqrt[2p]{\varepsilon_1}\int_0^\infty pt^{p-1}|\{u_1 > t\}\cap\{u_2 > t\}|\,dt +$$

$$+ \sqrt[2p]{\varepsilon_2}\int_0^\infty pt^{p-1}|\{u_1 > t\}\backslash\{u_2 > t\}|\,dt + \sqrt[2p]{\varepsilon_2}\int_0^\infty pt^{p-1}|\{u_2 > t\}\backslash\{u_1 > t\}|\,dt +$$

$$+ \sqrt[2p]{\varepsilon_2}\int_0^\infty pt^{p-1}|\{u_1 > t\}\cap\{u_2 > t\}|\,dt.$$

Again, most of the summands cancel out and we are left with

$$(\sqrt[2p]{\varepsilon_2} - \sqrt[2p]{\varepsilon_1})\int_0^\infty pt^{p-1}|\{u_1 > t\}\backslash\{u_2 > t\}|\,dt \geq 0,$$

which implies that for almost every t the Lebesgue measure of the set $\{u_1 > t\}\backslash\{u_2 > t\}$ is zero, so $u_2 \geq u_1$ a.e. □

At this point, let us clearly state a few implications of the above result. Firstly, if ε_n goes monotonically to zero, then for nonnegative boundary data every sequence of minimisers of G_{p,ε_n} is convergent to u without the need of choosing a subsequence. Secondly, we have

Corollary 3.5.7. *In the setting of Proposition 3.5.6, the sequence $F(u_n)$ is nonincreasing and converges to $F(u)$.*

Proof. Suppose that $\varepsilon_1 > \varepsilon_2$ and let us look closer at the inequality $G_{p,\varepsilon_2}(u_2) \leq G_{p,\varepsilon_2}(u_1)$ (which is true by definition of u_2). After expanding both sides we get

$$\sqrt[2p]{\varepsilon_2}\|u_2\|_p + F(u_2) \leq \sqrt[2p]{\varepsilon_2}\|u_1\|_p + F(u_1),$$

so in view of Proposition 3.5.6 this implies that $F(u_1) \geq F(u_2)$. Thus, the sequence $F(u_n)$ is decreasing and by Proposition 3.5.4 it converges to $F(u)$. □

The fact that by Proposition 3.5.6 u_n is an increasing sequence allows us to prove an improved version of Proposition 3.5.4.

Corollary 3.5.8. *Take an increasing sequence $u_n \to u$ in $L^p(\Omega)$ as given by Proposition 3.5.6. Suppose that $Tu \leq f$. Then $u_n \to u$ in the strict topology of $BV(\Omega)$.*

Proof. We proceed similarly to the proof of Proposition 3.5.4:

$$\int_\Omega |Du_n| + \int_{\partial\Omega} |Tu_n - f| = F(u_n) \leq G_{p,\varepsilon_n}(u_n) \leq G_{p,\varepsilon_n}(u) \to$$

$$\to F(u) = \int_\Omega |Du| + \int_{\partial\Omega} |Tu - f|,$$

so

$$\limsup_{n\to\infty} \left(\int_\Omega |Du_n| + \int_{\partial\Omega} |Tu_n - f| \right) \leq \int_\Omega |Du| + \int_{\partial\Omega} |Tu - f|.$$

By lower semicontinuity of the functional F we get the opposite inequality:

$$\liminf_{n\to\infty} \left(\int_\Omega |Du_n| + \int_{\partial\Omega} |Tu_n - f| \right) \geq \int_\Omega |Du| + \int_{\partial\Omega} |Tu - f|.$$

This implies that the sequence $\int_\Omega |Du_n| + \int_{\partial\Omega} |Tu_n - f|$ has a limit and

$$\lim_{n\to\infty} \left(\int_\Omega |Du_n| + \int_{\partial\Omega} |Tu_n - f| \right) = \int_\Omega |Du| + \int_{\partial\Omega} |Tu - f|.$$

Since u_n is monotone increasing, we have $Tu_n \leq Tu \leq f$. In particular, also the sequence $\int_{\partial\Omega} |Tu_n - f|$ is monotone increasing, so it has a limit, and we have

$$\int_{\partial\Omega} |Tu - f| \leq \lim_{n\to\infty} \int_{\partial\Omega} |Tu_n - f|,$$

hence $\int_\Omega |Du_n|$ also has a limit and

$$\lim_{n\to\infty} \int_\Omega |Du_n| - \int_\Omega |Du| = \int_{\partial\Omega} |Tu - f| - \lim_{n\to\infty} \int_{\partial\Omega} |Tu_n - f| \leq 0.$$

This coupled with the lower semicontinuity of the total variation gives us

$$\lim_{n\to\infty} \int_\Omega |Du_n| = \int_\Omega |Du|.$$

□

As it was mentioned above, we are going to take advantage of the fact that for $1 < p < \frac{N}{N-1}$ there is a unique minimiser of F in $L^p(\Omega)$. This will give us convergence on the whole sequence of minimisers of G_{p,ε_n}. Moreover, it turns out that Theorem 3.1.1 helps us to estabilish a similar claim for minimisers of F which attain the trace f also for $p = 1$.

Proposition 3.5.9. *Let X be the set of minimisers of F. Then X is a compact convex set in $L^p(\Omega)$, where $1 \leq p < \frac{N}{N-1}$. In particular it has a unique element of the smallest $p-$norm for $1 < p < \frac{N}{N-1}$.*

Proof. Since F is convex, the arithmetic mean of minimisers is also a minimiser, so X is convex. Since F is lower semicontinuous, the set of minimisers is closed in $L^1(\Omega)$ (because the limit of minimisers attains the same value of F), so it is closed in $BV(\Omega)$ and by continuity of the embedding into $L^p(\Omega)$ for $1 \leq p \leq \frac{N}{N-1}$ it is closed in $L^p(\Omega)$ for $1 \leq p \leq \frac{N}{N-1}$. It is a bounded set in every $L^p(\Omega)$ for $1 \leq p \leq \frac{N}{N-1}$, because it is bounded in $BV(\Omega)$: firstly, if u is a minimiser of F, then $\int_\Omega |Du| \leq F(u) = m$, the minimal value of F (note that also $\int_{\partial\Omega} |Tu - f| \leq m$). Secondly, let us extend u by 0 on some ball $B(0, r)$ including Ω. From the Poincaré inequality we have

$$\|u\|_1 \leq C(\int_\Omega |Du| + \int_{\partial\Omega} |Tu|) \leq C(\int_\Omega |Du| + \int_{\partial\Omega} |Tu - f| + \int_{\partial\Omega} |f|) \leq$$

$$\leq C(2m + \int_{\partial\Omega} |f|) = M,$$

so X is a bounded set in $BV(\Omega)$. Thus X is bounded and closed in L^p. For $1 \leq p < \frac{N}{N-1}$ it is compact in L^p, so for $1 < p < \frac{N}{N-1}$ it has a unique element of the smallest norm. □

Corollary 3.5.10. *Set $1 < p < \frac{N}{N-1}$ and let $u \in BV(\Omega)$ be the element of the smallest $p-$norm of X. Then u_n, the minimisers of G_{p,ε_n}, converge to u not only on some subsequence, but on the whole sequence. It is a consequence of the fact that in metric spaces (and $BV(\Omega)$ endowed with strict topology is metrisable) if we can from every subsequence x_{n_l} extract a subsubsequence $x_{n_{l_m}} \to x$, then $x_n \to x$. It provides a selection criterion for elements of X.*

Corollary 3.5.11. *In a slightly different case, where X is the set of minimisers of F with trace f (the boundary condition is met in the trace sense), Proposition 3.5.9 also holds. This is a consequence of Theorem 3.1.1.*

Proof. The proof of convexity, boundedness and compactness does not change. We only have to prove that X is closed in $L^p(\Omega)$ (it is enough to prove closedness in $L^1(\Omega)$).

Take a sequence u_n of least gradient functions with trace f which converges to u in $L^1(\Omega)$. By Miranda's theorem (Theorem 2.1.4) u is a function of least gradient. Now, by Theorem 3.1.1 the functions u_m and u_n differ only on some set C_{nm}, on which both functions are locally constant. As both functions have the same trace, we have $\mathcal{H}^1(\overline{C_{nm}} \cap \partial\Omega) = 0$. If we denote $C = \bigcup_{n=1}^{\infty} C_{nm}$, then

$$\mathcal{H}^{N-1}(\overline{C} \cap \partial\Omega) \leq \sum_{n,m} \mathcal{H}^{N-1}(\overline{C_{nm}} \cap \partial\Omega) = 0.$$

On $\Omega \backslash C$ the sequence u_n is constant, so on this set $u = u_n$. As $\mathcal{H}^{N-1}(\overline{C} \cap \partial\Omega) = 0$, the boundary of $\Omega \backslash C$ is the whole $\partial\Omega$. This means that $Tu = Tu_n = f$, so X is a closed set. □

Let us conclude this Section with noticing that Theorem 3.1.1 together with the analysis in Section 3.4 implies that the element of X with the smallest $1-$norm is the same as the element of X with the smallest $p-$norm, in particular it is unique. Thus, the functional $G_{p,\varepsilon}$ produces a selection criterion for elements of X also for $p = 1$.

Chapter 4

Least gradient problem with respect to a non-strictly convex norm

4.1 Introduction

We are interested in the issue of existence and uniqueness of minimisers to the least gradient problem in the following setting:

$$\min \left\{ \int_\Omega \phi(Du), \quad u \in BV(\Omega), \quad u|_{\partial\Omega} = f \right\}, \tag{aLGP2}$$

where we assume that ϕ is a norm without any additional regularity assumptions. Throughout this Chapter we assume that $\Omega \subset \mathbb{R}^2$. Moreover, since for discontinuous boundary data there can be multiple solutions or no solutions at all even in the isotropic case, see [48] and [64] respectively, in this Chapter we assume continuity of the boundary data $f \in C(\partial\Omega)$; we will examine the discontinuous case in Chapter 5.

As described in Chapter 2, in the classical least gradient problem, when ϕ is the Euclidean norm, existence, regularity, and uniqueness of minimisers depend on the geometry of the set Ω. Here the situation is slightly more complicated, because we have additionally the interplay between the shapes of Ω and the unit ball in the anisotropic norm $B_\phi(0,1)$; our goal is to explore this relationship. Recall that we say that ϕ is strictly convex if the unit ball $B_\phi(0,1)$ is strictly convex. We divide our analysis into two stages:

(1) Suppose that ϕ is strictly convex. Then, regardless of the regularity of ϕ, we are able to prove existence and uniqueness of minimisers for strictly convex Ω and obtain regularity estimates in terms of the modulus of continuity of the boundary data, which are independent on the choice of ϕ.

(2) Suppose that the unit ball $B_\phi(0,1)$ has flat facets. Then, under stronger assumptions on Ω, we use the regularity estimates from the strictly convex case to prove existence of a single minimiser with the same regularity; we also obtain existence of minimisers, which are continuous up to the boundary, for domains that are only strictly convex. However, we lose uniqueness of minimisers and the additional minimisers may have regularity no better than $BV(\Omega) \cap L^\infty(\Omega)$ even for smooth boundary data.

In this Chapter, we focus on the case when the boundary data are continuous. The passage from existence of minimisers for continuous boundary data to existence of minimisers with BV boundary data has been discussed for example in [28]; here, we give another argument for this in Theorem 4.5.1. On the other hand, if the boundary data have infinite total variation, then even in the isotropic case existence of minimisers may fail (see [64]) and the trace space of least gradient functions is not fully understood; hence, in the anisotropic setting it is most important to understand what happens for continuous boundary data.

Let us stress that in the anisotropic least gradient problem known results on uniqueness of minimisers for continuous boundary data depend not only on the geometry of Ω, but also on the regularity of ϕ. For instance, the uniqueness proof in [36] is based on a maximum principle and requires uniform convexity and a condition slightly weaker than $W^{3,\infty}$ regularity of ϕ away from $\{\xi = 0\}$; for a precise assumption, see [36, Theorem 1.2], which additionally covers the case when ϕ may depend also on location. In the course of this Chapter, we are going to relax the assumptions on the regularity of ϕ in the case when it is a fixed norm in order to be able to deal with non-strictly convex norms.

This Chapter is organised as follows: in Section 4.2 we investigate the functional

$$F_\phi(v) = \int_\Omega |Dv|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)\, |Tv - f| \, d\mathcal{H}^{N-1}$$

and (without the restrictions as to the form of ϕ or the dimension) we prove that if metric integrands $\phi_n \to \phi$ in $C(\overline{\Omega} \times \partial B(0,1))$, then the functionals F_{ϕ_n} $\Gamma-$converge to F_ϕ. In particular, it provides a stability result in the spirit of Miranda's theorem (Theorem 2.1.4), which states that a sequence of (isotropic) least gradient functions converges to a least gradient function.

In Section 4.3 we explore the results of Jerrard, Nachman and Tamasan, [36], and extend the framework under which they are valid. In [36] the two most important assumptions are the *barrier condition*, see Definition 2.3.4, which is essential in existence proofs, and uniform convexity and quite strong regularity of the metric integrand ϕ, which is used in uniqueness proofs. Here, we prove existence and uniqueness of minimisers for strictly convex ϕ regardless of the regularity of ϕ; see Theorem 4.3.3. Moreover, we prove a regularity estimate for the minimiser depending only on Ω and regularity of the boundary data, see Proposition 4.3.14; it does not depend on the regularity of ϕ.

In Section 4.4 we show that only some results can be extended to the case when ϕ is not strictly convex. In Theorem 4.4.1 we prove existence of a minimiser, which has the same regularity as if ϕ was strictly convex, provided that Ω is uniformly convex; in Theorem 4.4.2 we prove existence of a minimiser for strictly convex Ω. However, we lose uniqueness of minimisers and not all minimisers reflect the same regularity and we also include examples (see Proposition 4.4.8) such that the solution u has regularity no better than $BV(\Omega)$ even if the boundary data are smooth.

Finally, in Section 4.5 we discuss which results in the isotropic case can be extended to the anisotropic case due to the newly estabilished results. Moreover, in the various stages of the reasoning, we interrupt it to focus solely on the geometry underneath these results: we discuss where stronger modes of convexity come into play and which sets Ω satisfy the barrier condition.

This Chapter is based on the article [26], of which I am the sole author.

4.2 $\Gamma-$convergence

Recall that the functional defined in (2.3.1) is a relaxation of the total variation functional with respect to Dirichlet boundary data (see [46]):

$$F_\phi(v) = \int_\Omega |Dv|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)\, |Tv - f|\, d\mathcal{H}^{N-1}.$$

Now, we state the main result in this Section. The main idea behind it is extending Miranda's theorem, see [50], which states that a sequence of least gradient functions convergent in $L^1(\Omega)$ converges to a least gradient function. The Theorem below allows us to consider ϕ_n-least gradient functions, where the anisotropic norm ϕ_n is not fixed and changes with n. This enables us to prove existence results in the anisotropic least gradient problem, when ϕ is not strictly convex, as we need to approximate the anisotropic norm using strictly convex metric integrands.

Theorem 4.2.1. *Let ϕ and ϕ_n be metric integrands such that $\phi_n \to \phi$ in $C(\overline{\Omega} \times \partial B(0,1))$. Then the sequence of functionals F_{ϕ_n} $\Gamma-$converges (with respect to the L^1 convergence) to the functional F_ϕ.*

Proof. (1) We show that for any sequence u_n such that $u_n \to u$ in $L^1(\Omega)$ the first property of $\Gamma-$convergence holds, i.e. we have $F_\phi(u) \leq \liminf_{n\to\infty} F_{\phi_n}(u_n)$.

Denote by F_{l_2} the functional F_ψ, where ψ is the isotropic norm. The inequality is obvious if $\liminf_{n\to\infty} F_{\phi_n}(u_n) = +\infty$; assume that this number is finite and take the subsequence, still denoted by u_n, such that this limit is achieved. In particular, the (new) sequence u_n is bounded in $BV(\Omega)$ and thus $F_{l_2}(u_n)$ is bounded. Assume for now that $F_\phi(u) < \infty$. Then,

$$\liminf_{n\to\infty}(F_{\phi_n}(u_n) - F_\phi(u)) \geq \liminf_{n\to\infty}(F_{\phi_n}(u_n) - F_\phi(u_n)) + \liminf_{n\to\infty}(F_\phi(u_n) - F_\phi(u))$$

and by the lower semicontinuity of F_ϕ the second summand is nonnegative. Hence,

$$\liminf_{n\to\infty} F_{\phi_n}(u_n) \geq \liminf_{n\to\infty}(F_{\phi_n}(u_n) - F_\phi(u_n)) = 0,$$

because we have

$$|F_{\phi_n}(u_n) - F_\phi(u_n)| =$$

$$= \int_\Omega (\phi_n(x,\nu^{u_n}) - \phi(x,\nu^{u_n}))\,|Du_n| + \int_{\partial\Omega} (\phi_n(x,\nu^\Omega) - \phi(x,\nu^\Omega))\,|Tu_n - f|\,d\mathcal{H}^{N-1} \leq$$

$$\leq \sup_{\overline{\Omega}\times\partial B(0,1)} |\phi_n - \phi| (\int_\Omega |Du_n| + \int_{\partial\Omega} |Tu_n - f|\,d\mathcal{H}^{N-1}) =$$

$$= \sup_{\overline{\Omega}\times\partial B(0,1)} |\phi_n - \phi|\,F_{l_2}(u_n) \leq \sup_{\overline{\Omega}\times\partial B(0,1)} |\phi_n - \phi|\,M \to 0.$$

If $F_\phi(u) = \infty$, then $u \notin BV(\Omega)$. By the lower semicontinuity of the total variation for every approximating sequence $u_n \to u$ we also have $\liminf_{n\to\infty} |Du_n| = \infty$. Since the sequence ϕ_n converges uniformly, ϕ_n-norms are uniformly equivalent to the Euclidean norm, so

$$\liminf_{n\to\infty} F_{\phi_n}(u_n) \geq \liminf_{n\to\infty} \int_\Omega |Du_n|_{\phi_n} \geq \liminf_{n\to\infty} C \int_\Omega |Du_n| = \infty.$$

(2) We show that for any function $u \in L^1(\Omega)$ there exists a sequence $u_n \to u$ such that $F_\phi(u) \geq \limsup_{n\to\infty} F_{\phi_n}(u_n)$. In fact, it is enough to consider the constant sequence $u_n = u$.

If $u \notin BV(\Omega)$, the inequality is obvious. If $u \in BV(\Omega)$, then all the integrals in the definitions of $F_\phi, F_{\phi_n}, F_{l_2}$ are convergent and we have

$$|F_{\phi_n}(u) - F_\phi(u)| =$$

$$= \int_\Omega (\phi_n(x,\nu^u) - \phi(x,\nu^u))\,|Du| + \int_{\partial\Omega} (\phi_n(x,\nu^\Omega) - \phi(x,\nu^\Omega))\,|Tu - f|\,d\mathcal{H}^{N-1} \leq$$

$$\leq \sup_{\partial B(0,1)} |\phi_n - \phi| (\int_\Omega |Du| + \int_{\partial\Omega} |Tu - f|\,d\mathcal{H}^{N-1}) = \sup_{\partial B(0,1)} |\phi_n - \phi|\,F_{l_2}(u) \to 0.$$

□

The assumption that $\phi_n \to \phi$ in $C(\overline{\Omega}\times\partial B(0,1))$ is quite natural in this context: since metric integrands are $1-$homogenous in the second variable, it is sufficient to check convergence only on the unit sphere. Furthermore, as the following Example shows, we may not relax the assumption concerning uniform convergence in Ω; however, as we can see in Proposition 4.2.3, some form of uniform convergence in the second variable is guaranteed.

Example 4.2.2. Let $\Omega = [-1,1]$, with boundary data given by $f(-1) = 0$ and $f(1) = 1$. Let $\phi_n(x,p) = a_n(x)||p||_{l_2}$, where $a_n \in C^{1,1}(\Omega)$ such that $a_n(x) \in [\frac{1}{2}, 1]$, $\min a_n = a_n(\frac{1}{n}) = \frac{1}{2}$, and $a_n(x) = 1$ for $x \in [-1,0] \cup [\frac{2}{n}, 1]$. We note that $a_n \to a$ pointwise and in every $L^q([-1,1])$, $q < \infty$, where $a \equiv 1$; thus $\phi_n \to \phi$ pointwise and in $L^q(\overline{\Omega} \times \partial B(0,1))$, where $\phi(x,p) = \|p\|_{l_2}$.

Let $u_n = \chi_{[\frac{1}{n},1]}$. We have that $u_n \to u = \chi_{[0,1]}$ in $L^1([-1,1])$. We notice that both u_n and u have trace f. Thus

$$F_\phi(u) = \int_{[-1,1]} a(x)|Du| = \int_{[-1,1]} |Du| = 1 >$$

$$> \frac{1}{2} = a_n(\frac{1}{n}) = \int_{[-1,1]} a_n(x)|Du_n| = F_{\phi_n}(u_n).$$

In particular, the first condition in the definition of $\Gamma-$convergence is not satisfied. Thus, at least in dimension one, we need to assume uniform convergence in $\overline{\Omega}$ to prove that F_{ϕ_n} $\Gamma-$converges to F_ϕ.

Proposition 4.2.3. *Let ϕ_n, ϕ be metric integrands such that $\phi_n(x,p) = \phi_n(p)$ and suppose that $\phi_n \to \phi$ pointwise. Then, $\phi_n \to \phi$ in $C(\partial B(0,1))$.*

Proof. In a finite dimensional space any norm is equivalent to the isotropic norm. In particular, each ϕ_n is continuous and attains its supremum on $\partial B(0,1)$. Since ϕ_n is convex and $1-$homogenous, we have

$$|\phi_n(x) - \phi_n(y)| \leq \phi_n(x-y) = \phi_n(\frac{x-y}{|x-y|})|x-y| \leq (\sup_{\partial B(0,1)} \phi_n)|x-y|,$$

so ϕ_n is Lipschitz continuous with constant $\sup_{\partial B(0,1)} \phi_n$. We have two possibilities:

1. $\sup_n(\sup_{\partial B(0,1)} \phi_n) \leq M$. Take any subsequence ϕ_{n_k}. By Arzela-Ascoli theorem it has a convergent subsequence $\phi_{n_{k_l}}$; by our assumption we have that $\phi_{n_{k_l}} \to \phi$ uniformly. Since $C(\partial B(0,1))$ is a metric space, we have that $\phi_n \to \phi$ in $C(\partial B(0,1))$.

2. $\sup_n(\sup_{\partial B(0,1)} \phi_n) = +\infty$. Let x_n be the point where ϕ_n attains its supremum on $\partial B(0,1)$. Take $\{q_k\}$ to be the $\varepsilon-$net on $\partial B(0,1)$ for $\varepsilon = \frac{1}{2}$. Fix n and let q_k be such that $|q_k - x_n| \leq \frac{1}{2}$. By the Lipschitz continuity of ϕ_n

$$\phi_n(q_k) \geq \phi_n(x_n) - (\sup_{\partial B(0,1)} \phi_n)|q_k - x_n| \geq \frac{1}{2}\phi_n(x_n).$$

Since the set $\{q_k\}$ is finite, for some k the sequence $\phi_n(q_k)$ is unbounded. But this is impossible, as $\phi_n(q_k) \to \phi(q_k)$. □

4.3 Strictly convex norm ϕ

From now on, we introduce the following notation:

Notation. Until the end of this Chapter, let $\Omega \subset \mathbb{R}^2$ be an open bounded set with Lipschitz boundary. We denote by ϕ (or ϕ_n) a metric integrand on $\overline{\Omega} \times \mathbb{R}^2$ which depends only on the second variable (a fixed norm on $\mathbb{R}^2$), i.e. $\phi(x, \xi) = \phi(\xi)$. Recall that we say that ϕ is strictly convex whenever the unit ball $B_\phi(0, 1)$ is strictly convex. Moreover, we will denote by l_p the metric integrand defined by the formula $\phi(x, \xi) = \|\xi\|_p$.

When necessary, we will additionally assume some form of convexity of Ω. Our reasoning is divided into two main parts: in this Section, we explore the case when ϕ is strictly convex and in the next Section we explore the case when the unit ball $B_\phi(0, 1)$ has flat parts of the boundary.

This Section is divided into two subsections. Firstly, we explore how do the $\phi-$minimal sets look like and use this knowledge to infer that for continuous boundary data the minimisers exist and are unique if ϕ is strictly convex. In the second part, we will prove that minimisers inherit some of the regularity of the boundary data. As in the isotropic least gradient problem (see [65]), existence and regularity of minimisers require respectively strict convexity and uniform convexity (in the sense of Definition 4.3.8) of Ω; however, let us underline that these results will not depend on the regularity of ϕ.

4.3.1 Existence and uniqueness of minimisers

At the beginning of this Section, we would like to recall the following result shown in [44, Remark 20.4] and the consequences it has for existence and uniqueness of minimisers in the anisotropic least gradient problem.

Proposition 4.3.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is open and that $E \subset \mathbb{R}^N$ is a set of finite perimeter. Let $H \subset \mathbb{R}^N$ be a halfspace. Whenever $E \backslash H \subset\subset \Omega$, we have*

$$P_\phi(E, \Omega) \geq P_\phi(E \cap H, \Omega).$$

Moreover, if ϕ is strictly convex and $|E \backslash H| > 0$, this inequality is strict. □

In other words, projection onto a halfplane decreases the ϕ-perimeter. In particular, line segments always are $\phi-$minimal surfaces regardless of the regularity or convexity of ϕ; moreover, it implies that for strictly convex ϕ there are no other connected $\phi-$minimal surfaces; this is not true for non-strictly convex ϕ and this case will be discussed separately in Section 4.4.

Corollary 4.3.2. *Suppose that ϕ is strictly convex and let E be a minimal set with respect to ϕ. Then $\partial E = \bigcup_{i=1}^{\infty} L_i$, where L_i is a family of line segments, pairwise disjoint in $\overline{\Omega}$.*

Proof. We begin by pointing out that a density estimate for local perimeter minimisers proved in [44, Theorem 16.14] works also in the anisotropic case (up to a change of constants). In particular, the topological boundary of E coincides with its essential boundary $\partial_m E$ (and has finite $\mathcal{H}^1$-measure). But Proposition 4.3.1 implies that every connected component of $\partial_m E$ is a line segment. Because the triangle inequality is strict, these line segments are disjoint in $\overline{\Omega}$. ☐

This leads directly to uniqueness of solutions for continuous boundary data. If ϕ is strictly convex, as we know that the only connected ϕ-minimal surfaces are line segments, the proof of [65, Theorem 4.1] holds with minimal changes and we have the solution to problem (aLGP2) with continuous boundary data is unique.

Now, we turn to the issue of existence of solutions to problem (aLGP2). To this end, recall the barrier condition (Definition 2.3.4). By [36, Theorem 1.1] it is a sufficient condition for existence of solutions for continuous boundary data. However, in dimension two, Proposition 4.3.1 implies that the class of sets that satisfy the barrier condition are precisely the open bounded strictly convex sets (regardless of the choice of ϕ, provided it is strictly convex). Thus, if $\Omega \subset \mathbb{R}^2$ is an open bounded strictly convex set and ϕ is strictly convex, then there exists a solution to the least gradient problem. We summarise the above discussion in

Theorem 4.3.3. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded strictly convex set. Suppose that ϕ is a fixed strictly convex norm on $\mathbb{R}^2$. Let $f \in C(\partial\Omega)$. Then there exists a unique solution to Problem* (aLGP2). ☐

It is important to note that the unique solution given by the above Theorem does not necessarily coincide with the Euclidean solution. We recall an example given in [28, Example 5.19].

Example 4.3.4. Let $\Omega \subset \mathbb{R}^2$ be an open, bounded, strictly convex set such that points $q_1 = (1, 0), q_2 = (11, 0), q_3 = (0, 11)$ and $q_4 = (0, 1)$ lie on $\partial\Omega$. Take $f \in C(\partial\Omega)$ such that $f > 0$ on the arcs $q_1q_2 \subset \partial\Omega$ and $q_3q_4 \subset \partial\Omega$ and $f < 0$ on the two other arcs. We are interested in the zero superlevel set for $p \in (1, \infty)$. We easily see that $\partial\{u \geq 0\}$ is the union of two line segments; either $\partial\{u \geq 0\} = \overline{q_1q_2} \cup \overline{q_3q_4}$ or $\partial\{u \geq 0\} = \overline{q_2q_3} \cup \overline{q_4q_1}$.

We compare the sums of lengths of these line segments with respect to l_p norms. The sum of lengths of $\overline{q_1q_2}$ and $\overline{q_3q_4}$ is independent of p and equals 20. The sum of lengths of $\overline{q_2q_3}$ and $\overline{q_4q_1}$ equals $(1^p + 1^p)^{1/p} + (11^p + 11^p)^{1/p} = 12 \cdot 2^{1/p}$. Let $p_0 = \frac{\log(2)}{\log(\frac{5}{3})}$. Then for $p > p_0$ it is smaller than 20, so the trapezoid $q_1q_2q_3q_4 \subset \{u \geq 0\}$; if $p < p_0$, then it is greater than 20, so $q_1q_2q_3q_4 \subset \{u < 0\}$. Finally, for $p = p_0$ the trapezoid $q_1q_2q_3q_4$ is a zero level set of positive measure; this is the only p with this property. In particular, solutions to the anisotropic least gradient problem for the same boundary data can vary with p.

Remark 4.3.5. While the existence proof is based on [36, Theorem 1.1], let us note that the above result is substantially different from the uniqueness theorem [36, Theorem 1.2]. That result requires ϕ to satisfy two additional conditions: firstly, that ϕ has regularity somewhat stronger than $W^{2,\infty}$ outside the origin; what is more relevant in our case, the other assumption states that there exists $C > 0$ such that

$$\sum_{i,j=1}^{n} \phi_{\xi_i \xi_j}(x,\xi) p^i p^j \geq C|p - (p\cdot \xi)\xi|^2$$

for every $p \in \mathbb{R}^n$ and every $\xi \in S^{n-1}$. This is a convexity assumption stronger than strict convexity of ϕ. Thus the results proved in this Section, and even more so in the next Section, are independent from the results in [36]: the case when ϕ is not strictly convex is not covered at all by [36, Theorem 1.2], while in the case when ϕ is strictly convex we are able to prove uniqueness of minimisers to Problem (aLGP2) regardless of the regularity or stronger modes of convexity of ϕ. However, we note that in our setting we only allow ϕ to depend on the direction of the derivative and not on location.

We also recall two useful results, proved in [28, Proposition 3.5] and [31, Lemma 3.8] respectively in the isotropic case.

Proposition 4.3.6. *Suppose that Ω is convex, ϕ is strictly convex and suppose that $u \in BV(\Omega)$ is a function of $\phi-$least gradient. Then, for every $t \in \mathbb{R}$ we have $\partial\{u > t\} = \bigcup_{i=1}^{\infty} L_{t,i}$, where $L_{t,i}$ are line segments with ends on $\partial\Omega$ and this union is locally finite in Ω. Furthermore, $\overline{L_{t,i}}$ are pairwise disjoint in $\overline{\Omega}$.*

The proof in the isotropic case relies only on the regularity of $\phi-$minimal sets and the fact that the triangle inequality is strict. Both of these facts are true if ϕ is strictly convex, so the Proposition remains true.

Lemma 4.3.7. *Suppose that Ω is strictly convex, ϕ is strictly convex and suppose that u is a minimiser of Problem* (aLGP2)*. Let $f \in C(\partial\Omega)$. Then, for every $t \in \mathbb{R}$ we have*

$$\partial\{u \geq t\} \cap \partial\Omega \subset f^{-1}(t).$$

The proof in the isotropic case relies only on the regularity of $\phi-$minimal sets and a blow-up argument, which is applied at regular points of $\partial\Omega$ and does not depend on ϕ. Thus, the Lemma remains true in the anisotropic case.

4.3.2 Regularity of minimisers

We briefly recall the regularity results from [65] concerning the isotropic case. The authors assume that $\partial\Omega$ is of class C^2 and that Ω is uniformly convex, i.e. the mean curvature of $\partial\Omega$ is positive. Then, if the boundary data f is of class $C^{0,\alpha}(\partial\Omega)$, where $\alpha \in (0,1]$, then the corresponding minimiser to the least gradient problem u is in the class $C^{0,\alpha/2}(\Omega)$. A similar result is obtained if the mean curvature can vanish at isolated points and has polynomial growth. As the (two-dimensional) examples provided by the authors show, the above results are optimal.

Our goal is to extend these results to the anisotropic case. This issue has been recently explored in [19]; the authors use a different approach, going through the optimal transport theory and using the equivalence proved in [31], and are able to prove regularity estimates for $W^{1,p}$ boundary data (with $p \leq 2$). Here, we discuss the issue of regularity of minimisers when the boundary data are not weakly differentiable, expressed only in terms of the modulus of continuity of the boundary data. Moreover, as we use a different approach, we may replace the regularity assumptions on the $\partial\Omega$ by weaker ones. However, [65, Example 5.8] shows that we cannot get rid of some form of uniform convexity altogether. In this Chapter, we will assume Ω to have Lipschitz boundary and satisfy the following definition of uniform convexity (which agrees with the classical definition for C^2 sets, see Proposition 4.3.9):

Definition 4.3.8. We say that the set an open bounded convex set Ω is uniformly convex, if the following condition is satisfied: let $P = \{y \geq ax^2\}$, where $a > 0$. Let $x_0 \in \partial\Omega$ and let l be a supporting line at x_0. Then there exists P', an isometric image of P, tangent to l at x_0 such that $\overline{\Omega} \subset P'$ and $\partial P' \cap \overline{\Omega} = \{x_0\}$.

Similarly, we will say that an open bounded set Ω is $\beta-$uniformly convex, if for some $\beta > 0$ the Definition above is satisfied with $\widetilde{P} = \{y \geq ax^{\beta+2}\}$ in place of P.

Proposition 4.3.9. *If $\partial\Omega \in C^2$ and its curvature is positive, then Ω is uniformly convex in the sense of Definition 4.3.8.*

Proof. As $\partial\Omega$ is compact, the mean curvature has a positive lower bound c. Take any $x \in \partial\Omega$ and let l be a line tangent to $\partial\Omega$ at x. We choose the coordinate system so that $x = (0,0)$ and $l = \{y = 0\}$. As $\partial\Omega$ is strictly convex, it is a union of two graphs of convex functions. Let g be one of these functions and let $g(0) = 0$. Then, by the formula for the curvature of a graph, we have

$$g'' = (1 + (g')^2)^{3/2} k \geq k \geq c,$$

so we have $g(0) = 0$, $g'(0) = 0$ and $g'' \geq c$. Thus $g(x) \geq \frac{c}{2}x^2$. As for the second function, its graph lies above the graph of g, so also above the parabola $y = \frac{c}{2}x^2$. As the coefficient does not depend on $x \in \partial\Omega$, Ω is uniformly convex in the sense of Definition 4.3.8. □

With essentially the same proof we obtain

Corollary 4.3.10. *If $\partial\Omega \in C^2$ and for every $x_0 \in \partial\Omega$ its curvature satisfies a bound $k(x) \geq a|x - x_0|^\beta$ in some neighbourhood of x_0, then Ω is $\beta-$uniformly convex.* □

Remark 4.3.11. The condition in Definition 4.3.8 seems as if it was hard to check for any given set Ω. However, it is sufficient to check it for every x_0 for at most two supporting lines: without loss of generality assume that $x_0 = (0, 0)$. The set of supporting lines $y = ax$, parametrised by the coefficient a, is closed and convex, so it is an interval $[a_1, a_2]$. Take the supporting parabolas P_1 and P_2 corresponding to lines $y = a_1 x$ and $y = a_2 x$. Take the parabola P corresponding to a line $y = ax$, where $a \in (a_1, a_2)$. Then $P_1 \cap P_2 \subset P$ and in particular $\Omega \subset P_1 \cap P_2 \subset P$; this can be easily seen in the polar form of the equation for the parabola. Thus, if the condition from the definition of uniform convexity is satisfied for two extreme supporting lines at x_0, it is satisfied for all supporting lines at x_0.

In particular, an important class of sets Ω uniformly convex in the sense of Definition 4.3.8 are strictly convex sets such that $\partial\Omega$ is C^2 except for finitely many corners and the curvature of Ω is bounded from below (on the set where $\partial\Omega$ is C^2).

We turn our attention to the regularity of minimisers to Problem (aLGP2). Firstly, we use a variant of a result from [36] to prove that any minimiser is continuous up to the boundary of Ω. We recall that [36, Theorem 1.3], which asserts the continuity of solutions up to the boundary, follows from the following comparison principle:

Proposition 4.3.12. ([36, Theorem 4.6]) *Let $\Omega \subset \mathbb{R}^2$ be an bounded convex set. If Ω satisfies the barrier condition, $E_1, E_2 \subset \mathbb{R}^2$ are $\phi-$area minimising in Ω and*

$$E_1 \backslash \Omega \subset\subset E_2 \backslash \Omega,$$

then $E_1 \subset\subset E_2$.

While this result is originally stated without restrictions as to the dimension, but for a metric integrand satisfying some additional regularity properties, as we have a special form of ϕ, i.e. it depends only on the direction of derivative of u, the above Proposition follows directly from Proposition 4.3.2. Thus, we obtain

Corollary 4.3.13. *Suppose that $\Omega \subset \mathbb{R}^2$ is an open bounded convex set. Let ϕ be a fixed strictly convex norm on $\mathbb{R}^2$ and let u be a solution of Problem* (aLGP2) *with boundary data f. Then, $u \in C(\overline{\Omega})$.*

We recall that an increasing function $\omega : [0, \infty] \to [0, \infty]$ such that $\omega(0^+) = 0$ is a modulus of continuity of a continuous function f, if $|f(x) - f(y)| \leq \omega(|x - y|)$. Since $\partial\Omega$ is compact, any function $f \in C(\partial\Omega)$ is uniformly continuous, so f admits a modulus of continuity. The next Theorem is our main regularity result; we present it for uniformly convex sets for the sake of clarity and then show how to pass to $\beta-$uniformly convex sets.

Theorem 4.3.14. *Suppose that $\Omega \subset \mathbb{R}^2$ is uniformly convex (in the sense of Definition 4.3.8) and ϕ is a fixed strictly convex norm on $\mathbb{R}^2$. Let $f \in C(\partial\Omega)$ and take ω to be a modulus of continuity of f. Let u be the solution of Problem (aLGP2) with boundary data f. Then, $u \in C(\overline{\Omega})$ and it is continuous with modulus of continuity*

$$\overline{\omega}(|x-y|) = \omega(c(\Omega)|x-y|^{1/2}).$$

Proof. The proof will follow in three steps. Firstly, we prove the statement in a special geometric situation and then gradually reduce the general case to the special case.

Step 1. Let $p, q \in \Omega$. Suppose that $p \in \partial E_t$ and $q \in \partial E_s$. Let l_p be a line passing through p such that the connected component of ∂E_t containing p lies inside l_p (and similarly we define l_q). Suppose that l_p and l_q are parallel and the line pq is perpendicular to l_p (and l_q). Let $x_0 \in \partial\Omega$ be a point such that there is a supporting line l at x_0 parallel to l_p (and l_q); there are two such points, without loss of generality l_p is closer to x_0 then l_q.

We change coordinates so that $x_0 = (0,0)$, $l = \{y = 0\}$ and x_0 is the lowest point of $\overline{\Omega}$. Take a supporting parabola $P = \{y = ax^2\}$ at x_0 as in Definition 4.3.8. Let $p' \in \partial\Omega \cap l_p$ and $p'' \in \partial P \cap l_p$. Similarly we define q' and q'' and we require that p' and q' (and p'' and q'') lie on the same side with respect to the vertical line pq; without loss of generality they lie on the left of pq. The situation is presented on Figure 4.1.

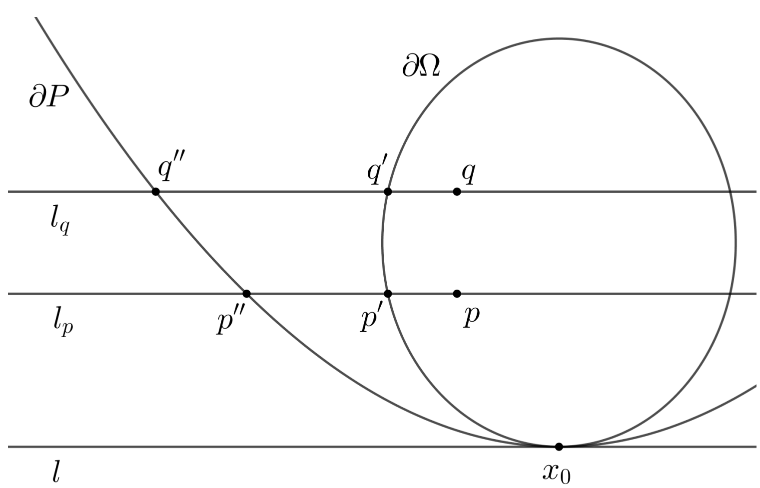


Figure 4.1: Application of the uniform convexity of Ω

Since $u \in C(\overline{\Omega})$, we have

$$|u(q) - u(p)| = |u(q') - u(p')| = |f(q') - f(p')| \leq \omega(|q' - p'|).$$

Now, we will see from the geometrical situation that $|q' - p'| \leq |q'' - p''|$. Denote by (p_x, p_y) the coordinates of p and similarly for other points. Let $r = (r_x, r_y)$ be the (unique) point with a minimal value of the first coefficient among all points of $\partial\Omega$. For now, assume that r does not lie on the open arc $p'q'$ not containing x_0, so that this arc is a graph.

We may assume that $q'_x \leq p'_x$; otherwise we choose x_0 by the topmost point of $\partial\Omega$ and proceed in the same way. Take a supporting line at p'. Take a parabola P', isometric to P such as in Definition 4.3.8. Then, by Definition 4.3.8, $|q' - p'|$ is smaller than $|\widetilde{q'} - p'|$, where $\widetilde{q'}$ lies on the intersection of P' and l_q. As P' is an isometric image of P, the curvature of P' at p' is bounded from below by the curvature of the parabola $P + p'$ at p'; hence, while $|q''_y - p''_y| = |q'_y - p'_y|$, we have $|q''_x - p''_x| > |q'_x - p'_x|$.

Now, we remove the assumption that r does not lie on the open arc $p'q'$ not containing x_0. Let r'' be a point on ∂P corresponding to r, i.e. $r''_y = r_y$. By the previous paragraph, we may estimate $|q'_x - r_x|$ and $|r_x - p'_x|$ from above, so

$$|q'_x - p'_x| \leq |q'_x - r_x| + |r_x - p'_x| \leq |q''_x - r''_x| + |r''_x - p''_x| = |q''_x - p''_x|.$$

Furthermore, since ∂P is a parabola, we see that $|q'' - p''|^2 \leq C(\Omega)|q - p|$. Given two Borel sets A, B, we write in short $d(A, B) = \text{dist}(A, B)$ and calculate

$$|q'' - p''|^2 = (q''_y - p''_y)^2 + (q''_x - p''_x)^2 = d(l_p, l_q)^2 + \Big(\sqrt{\frac{d(l_q, l)}{a}} - \sqrt{\frac{d(l_p, l)}{a}}\Big)^2 =$$

$$= d(l_p, l_q)^2 + \frac{d(l_q, l)}{a} + \frac{d(l_p, l)}{a} - \frac{2}{a}\sqrt{d(l_q, l)d(l_p, l)} = d(l_p, l_q)^2 + \frac{d(l_q, l_p)}{a} +$$

$$+ \frac{2}{a}\Big(d(l_p, l) - \sqrt{d(l_q, l)d(l_p, l)}\Big) \leq d(l_p, l_q)\Big(d(l_p, l_q) + \frac{1}{a}\Big) + 0 \leq$$

$$\leq \Big(\text{diam}\,\Omega + \frac{1}{a}\Big)d(l_p, l_q) = c(\Omega)|q - p|.$$

Returning to the level of moduli of continuity, we obtain that

$$|u(q) - u(p)| \leq \omega(|q' - p'|) \leq \omega(|q'' - p''|) \leq \omega(c(\Omega)|q - p|^{1/2}) = \overline{\omega}(|q - p|).$$

Step 2. Let $p, q \in \Omega$ and suppose that $p \in \partial E_t$ and $q \in \partial E_s$. We prove that it is sufficient to assume that p and q are as assumed in Step 1. We use the notation as in Step 1, except now l_p is not parallel to l_q (so l is parallel only to l_p and not to l_q). Since $u \in C(\overline{\Omega})$, l_p and l_q intersect somewhere outside of $\overline{\Omega}$. Let p' and q' be on this side of the line pq so that q'_y is smaller. Now, we draw a line l'_q parallel to l_p passing through q'. By Step 2 we have

$$|u(q) - u(p)| = |f(q') - f(p')| \leq \overline{\omega}(d(l_p, l'_q)) \leq \overline{\omega}(|q - p|).$$

Step 3. Finally, we prove that it is sufficient to assume that p and q are as assumed in Step 2. Firstly, we notice that if p lies on $\partial\{u > t\}$ instead of ∂E_t, then the proofs in Steps 1 and

2 remain unchanged. Let $p, q \in \Omega$. If $u(p) = u(q)$, then there is nothing to prove; without loss of generality $u(p) > u(q)$. Let $u(p) = t$ and suppose that $p \notin \partial E_t$ and $p \notin \partial\{u > t\}$; hence p lies on a level set U of u of a positive measure. Then, we may replace p with $\widetilde{p}$, which lies on ∂U, so on ∂E_t or $\partial\{u > t\}$ and is closer to q than p. A similar analysis applies to q. □

Let us stress the fact that in the Proposition above the constant $c(\Omega)$ depends only on Ω and not on the choice of the norm ϕ (and is explicitly given in the calculation in Step 1). If the boundary is C^2, then it depends only on the diameter and the lower bound c on curvature of Ω: by Step 1 and Proposition 4.3.9 we have $c(\Omega) = \text{diam}\,\Omega + \frac{1}{a} = \text{diam}\,\Omega + \frac{2}{c}$.

Corollary 4.3.15. *Under the assumptions of Theorem 4.3.14, suppose that $f \in C^{0,\alpha}(\partial\Omega)$. Then $u \in C^{0,\alpha/2}(\overline{\Omega})$.* □

Remark 4.3.16. The reasoning from the proof of Theorem 4.3.14 adapts well to other environments, such as when Ω is a polygon (under certain admissibility conditions on the boundary data); for instance, by using parabolas supported at corners of the polygon, we may provide an alternative proof of [31, Lemma 4.2]. Furthermore, notice that in the case of strictly convex Ω we do not need a supporting parabola for each of the points $x \in \partial\Omega$, but for at least one supporting line in each direction; therefore, Theorem 4.3.14 is also valid when only some sufficiently large part of $\partial\Omega$ is uniformly convex (however, in the case when some parts of $\partial\Omega$ are not uniformly convex, the modulus of continuity of the boundary data may grow very quickly and we may not have Hölder regularity of minimisers for smooth boundary data, see [65, Example 5.8]).

Now, we will formulate a similar result for $\beta-$uniformly convex sets.

Proposition 4.3.17. *Suppose that Ω is $\beta-$uniformly convex and ϕ is strictly convex. Suppose that $f \in C(\partial\Omega)$ and take ω to be a modulus of continuity of f. Let u be the solution of Problem* (aLGP2) *with boundary data f. Then $u \in C(\overline{\Omega})$ and it is continuous with modulus of continuity*

$$\overline{\omega}(|x-y|) = \omega(c(\Omega)|x-y|^{1/(\beta+2)}).$$

Proof. The proof remains unchanged except for the final calculation of distances in Step 1, where we obtain

$$|q'' - p''|^2 = (q''_y - p''_y)^2 + (q''_x - p''_x)^2 = d(l_p, l_q)^2 + ((\frac{d(l_q, l)}{a})^{\frac{1}{\beta+2}} - (\frac{d(l_p, l)}{a})^{\frac{1}{\beta+2}})^2 =$$

$$= d(l_p, l_q)^2 + (\frac{d(l_q, l)}{a})^{\frac{2}{\beta+2}} + (\frac{d(l_p, l)}{a})^{\frac{2}{\beta+2}} - 2(\frac{d(l_q, l) d(l_p, l)}{a^2})^{\frac{1}{\beta+2}} \leq$$

$$\leq d(l_p, l_q)^2 + (\frac{d(l_q, l_p)}{a})^{\frac{2}{\beta+2}} + (\frac{d(l_p, l)}{a})^{\frac{2}{\beta+2}} + (\frac{d(l_p, l)}{a})^{\frac{2}{\beta+2}} - 2(\frac{d(l_q, l) d(l_p, l)}{a^2})^{\frac{1}{\beta+2}} \leq$$

$$\leq d(l_p, l_q)^{\frac{2}{\beta+2}} (d(l_p, l_q)^{2-\frac{2}{\beta+2}} + (\frac{1}{a})^{\frac{2}{\beta+2}}) + 0 \leq$$

$$\leq ((\text{diam}\,\Omega)^{2-\frac{2}{\beta+2}} + (\frac{1}{a})^{\frac{2}{\beta+2}}) d(l_p, l_q)^{\frac{2}{\beta+2}} = c(\Omega)|q-p|^{\frac{2}{\beta+2}}.$$

□

Corollary 4.3.18. *Under the assumptions of Proposition 4.3.17, suppose that $f \in C^{0,\alpha}(\partial\Omega)$. Then $u \in C^{0,\alpha/(\beta+2)}(\overline{\Omega})$.* □

We conclude with a comparison of the above results with the results in [65, Section 5]. We see that in dimension two we obtained the same regularity estimates as in the isotropic case. As the (counter)examples in [65] show, these results are optimal. Moreover, this line of reasoning enables us to prove them with weaker assumptions concerning the regularity of $\partial\Omega$.

4.4 Non-strictly convex norm ϕ

When ϕ is a non-strictly convex norm on the plane, then the unit ball $B_\phi(0,1)$ has flat facets. Hence, as we prove near the end of this Section in Proposition 4.4.9, no open set Ω with C^1 boundary satisfies the barrier condition. In particular, existence of minimisers is not guaranteed and we have to prove it using another means. Furthermore, line segments are not the only connected $\phi-$minimal surfaces (we give a family of ϕ-minimal surfaces sufficient for our reasoning in Lemma 4.4.5); thus, we may not use the argument from [65] to conclude uniqueness of minimisers. Throughout this Section, $B_\phi(0,1)$ is convex but not strictly convex and I always denotes a line segment in $\partial B_\phi(0,1)$.

This Section is organised as follows. Firstly, we see that if Ω is uniformly convex (or $\beta-$uniformly convex), then there exists a solution to Problem (aLGP2) and it satisfies the regularity estimates as proved for strictly convex ϕ in Theorem 4.3.14. Furthermore, we prove that if Ω is only strictly convex, there still exists a minimiser of Problem (aLGP2), which is continuous up to the boundary. Secondly, we will show that because line segments are not the only $\phi-$minimal surfaces, minimisers to Problem (aLGP2) may fail to be unique even for smooth boundary data (however, there still may exist boundary data for which minimisers are unique; see [28, Example 5.15]). Thirdly, we show that there exist such boundary data $f \in C^\infty(\partial\Omega)$ such that some minimisers of Problem (aLGP2) have regularity no better than $BV(\Omega) \cap L^\infty(\Omega)$. Finally, we will see why the barrier condition is not satisfied for sets with C^1 boundary.

Theorem 4.4.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded uniformly convex set. Suppose that ϕ is a fixed norm on $\mathbb{R}^2$. Let $f \in C(\partial\Omega)$. Then, there exists a solution $u \in C(\overline{\Omega})$ to Problem (aLGP2).*

Additionally, if ω is a modulus of continuity of f, then $\overline{\omega}$ defined as in Theorem 4.3.14 is a modulus of continuity of u.

Proof. Let ω be a modulus of continuity of f. Take ϕ to be any norm on $\mathbb{R}^2$. Then, $\phi_n = \phi + \frac{1}{n} l_2$ is strictly convex. By Theorem 4.3.3 there exists a solution $u_n \in C(\overline{\Omega})$ to Problem (aLGP2) with boundary data f with respect to the anisotropic norm ϕ_n.

By Theorem 4.3.14 the solution u_n is continuous on $\overline{\Omega}$ with modulus of continuity $\overline{\omega}$, which depends only on the geometry of Ω and not on the norm ϕ_n; thus the sequence u_n has the same modulus of continuity, so it is equicontinuous. Also, the sequence u_n is uniformly bounded from below and above by the maximum and minimum of f. By Arzela-Ascoli theorem the sequence u_n admits a subsequence which converges uniformly on the compact set $\overline{\Omega}$.

We obtain that $u_{n_k} \to u$ uniformly on $\overline{\Omega}$. In particular, this convergence is in $L^1(\Omega)$, hence $u \in BV(\Omega)$, so also $u \in BV_\phi(\Omega)$. As the convergence is uniform, we have $Tu = u|_{\partial\Omega} = f$. By Theorem 4.2.1 u is a function of $\phi-$least gradient (note that we may not use the usual Miranda's theorem, since we change the anisotropic norm). Finally, we observe that as each u_n admitted the same modulus of continuity $\overline{\omega}$, u is uniformly continuous with the same modulus of continuity. □

An analogous proof shows that the above result also holds if Ω is only $\beta-$uniformly convex. In the next result, we show that if Ω is only strictly convex, we still obtain existence of minimisers for continuous boundary data; in this case the minimisers are continuous up to the boundary, which is the same regularity as given for strictly convex ϕ by Theorem 4.3.3.

Theorem 4.4.2. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded strictly convex set. Let ϕ be a fixed norm on $\mathbb{R}^2$. Let $f \in C(\partial\Omega)$. Then there exists at least one solution to Problem (aLGP2).*

Proof. 1. Due to Theorem 4.3.3 we only have to prove the result if ϕ is not strictly convex. Then, $\phi_n = \phi + \frac{1}{n} l_2$ is strictly convex. By Theorem 4.3.3 there exists a solution $u_n \in C(\overline{\Omega})$ to Problem (aLGP2) with boundary data f with respect to the anisotropic norm ϕ_n. Furthermore, the family u_n is uniformly bounded in $BV(\Omega)$, as

$$\int_\Omega |u_n| \, dx + \int_\Omega |Du_n| \leq \int_\Omega \sup_{\partial\Omega} |f| \, dx + C \int_\Omega |Du_n|_\phi + C \int_\Omega \frac{1}{n} |Du_n| \leq$$

$$\leq |\Omega| \sup_{\partial\Omega} |f| + C F_{\phi_n}(u_n) \leq |\Omega| \sup_{\partial\Omega} |f| + C F_{\phi_n}(v \equiv 0) \leq$$

$$\leq |\Omega| \sup_{\partial\Omega} |f| + C \int_{\partial\Omega} (\sup_{\partial B(0,1)} \phi + \frac{1}{n}) |f| \, d\mathcal{H}^1 \leq M.$$

In particular, u_n admits a subsequence (still denoted by u_n) convergent in $L^1(\Omega)$. By Theorem 4.2.1 u is a minimiser of the functional F_ϕ; if we prove additionally that $Tu = f$, then u is a solution to Problem (aLGP2).

2. We recall that if the trace of u equals f, then the set $\mathcal{Z}$ of such $x \in \partial\Omega$ that

$$\fint_{B(x,r)\Omega} |u(y) - f(x)| \, dy \to 0,$$

when $r \to 0$ is of $\mathcal{H}^1-$full measure (see [20, Theorem 5.3.2]). Fix $x \in \mathcal{Z}$.

3. Take arbitrary $\varepsilon > 0$. As $f \in C(\partial\Omega)$, there exists a neighbourhood of x in $\partial\Omega$ such that

$$f(x) - \varepsilon \leq f(y) \leq f(x) + \varepsilon \qquad \text{in } B(x, \delta_1) \cap \partial\Omega.$$

As Ω is strictly convex, for sufficiently small δ_1 the set $B(x, \delta_1) \cap \partial\Omega$ consists of two points p_1, p_2 and the line segment $\overline{p_1 p_2}$ lies inside Ω. Denote by Δ the open set bounded by an arc of $\partial\Omega$ containing x and the line segment $\overline{p_1 p_2}$. Let us take a ball $B(x, \delta_2)$ such that $B(x, \delta_2) \cap \Omega \subset \Delta$. Then for every n we have

$$f(x) - \varepsilon \leq u_n(y) \leq f(x) + \varepsilon \qquad \text{in } B(x, \delta_2) \cap \Omega;$$

suppose otherwise, i.e. that for some $y \in B(x, \delta_2) \cap \Omega$ we have $y \in \partial\{u_n \geq t\}$, where $t > f(x) + \varepsilon$. Take the connected component S of $\partial\{u_n \geq t\}$ containing y. By Lemma 4.3.7 we have $S \cap \partial\Omega = \{q_1, q_2\} \subset f^{-1}(t)$. As $y \in B(x, \delta_2) \cap \Omega \subset \Delta$, at least one of points q_1, q_2 lies on $\partial\Omega \cap \partial\Delta$; on the other hand, by Step 2 we have $u_n(y) \leq f(x) + \varepsilon < t$ on $\partial\Omega \cap \partial\Delta$, contradiction. The case when $t < f(x) - \varepsilon$ is handled similarly.

4. Since $u_n \to u$ in $L^1(\Omega)$, on some subsequence (still denoted u_n) we have convergence almost everywhere; hence

$$f(x) - \varepsilon \leq u(y) \leq f(x) + \varepsilon \qquad \text{for a.e. } y \in B(x, \delta_2) \cap \Omega.$$

5. Now, suppose that $Tu(x) = a > f(x)$. As ε was arbitrary, we choose it to be small enough to satisfy $a > f(x) + \varepsilon$. Then, for $r < \delta_2(\varepsilon)$ we have

$$\fint_{B(x,r)\cap\Omega} |u(y) - a| \, dy = \fint_{B(x,r)\cap\Omega} |u(y) - (f(x) + \varepsilon) + (f(x) + \varepsilon - a)| \, dy =$$

$$= \fint_{B(x,r)\cap\Omega} |u(y) - (f(x) + \varepsilon)| \, dy + \fint_{B(x,r)\cap\Omega} |(f(x) + \varepsilon - a)| \, dy \geq 0 + |(f(x) + \varepsilon - a)|.$$

Since $x \in \mathcal{Z}$, the mean integral should vanish in the limit $r \to 0$, contradiction. A similar argument covers the case when $Tu(x) < f(x)$. Thus $Tu(x) = f(x)$ almost everywhere with respect to $\mathcal{H}^1$, so $Tu = f$. $\square$

The following Proposition gives us the best regularity we can have without assuming some form of uniform convexity of Ω. This is the same regularity as proved in [65] for the isotropic case. However, this only concerns the one solution obtained as a limit of minimisers of approximate problems; other solutions may have weaker regularity, see Proposition 4.4.8.

Proposition 4.4.3. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded strictly convex set. Suppose that ϕ is a fixed norm on $\mathbb{R}^2$. Let $f \in C(\partial\Omega)$. Then, the solution to Problem (aLGP2) constructed in Theorem 4.4.2 belongs to the class $C(\overline{\Omega})$.*

Proof. 1. Fix ϕ_n and u_n as in the proof of Theorem 4.4.2. Firstly, we will prove that the solution $u \in BV(\Omega)$ is in fact continuous at each point of Ω. To this end, fix $x \in \Omega$ and let $r < \frac{1}{2}\text{dist}(x, \partial\Omega)$. We intend to prove that there exists a common modulus of continuity for all minimisers u_n in the ball $B(x,r)$; we will more or less follow the outline of the proof of Theorem 4.3.14.

2. Fix $n \in \mathbb{N}$. Denote by E_t the superlevel set of u_n, i.e. $E_t = \{u_n \geq t\}$. We argue as in the proof of Theorem 4.3.14 that it is sufficient to consider points $p, q \in B(x,r)$ such that $p \in \partial E_t$ and $q \in \partial E_s$. Furthermore, we may assume that the lines l_p and l_q which contain connected components of ∂E_t and ∂E_s which contain p and q respectively are parallel. Since u_n is constant and equal to t along each connected component of ∂E_t (and the same for s), we may assume that p and q lie on a line perpendicular to l_p and l_q. The situation is nearly the same as on Figure 4.1; the only difference is that there is no parabola P'. We define x_0, l, p' and q' as in Theorem 4.3.14. Finally, we may assume that p and q lie directly above x_0.

3. Similarly to Theorem 4.3.14, we need to estimate $|p-q|$ in terms of $|p'-q'|$. To this end, consider the line segment $\overline{x_0 q'}$ and take $\widetilde{p} = l_p \cap \overline{x_0 q'} \in \Omega$; it lies on the same side of the vertical line pq as p' and q'. We use Thales' theorem to see that

$$\frac{|p-q|}{|q-x_0|} = \frac{|q'-\widetilde{p}|}{|q'-x_0|}$$

$$|p-q| = \frac{|q-x_0|}{|q'-x_0|}|q'-\widetilde{p}| \geq \frac{\text{dist}(x,\partial\Omega)}{2\,\text{diam}\,\Omega}|q'-p'|,$$

where $|q-x_0| \geq \frac{1}{2}\text{dist}(x,\partial\Omega)$ because $q \in B(x,r)$, $|q'-x_0|$ is smaller than diam Ω and $|q'-\widetilde{p}| \geq |q'-p'|$ because of the convexity of Ω. Hence, at the level of moduli of continuity, if ω is the modulus of continuity of the boundary data, then

$$\widetilde{\omega}(\rho) = \omega\left(\frac{\text{diam}\,\Omega}{2\,\text{dist}(x,\partial\Omega)}\rho\right)$$

is the modulus of continuity of u_n inside the ball $B(x,r)$. We notice that it blows up near the boundary of Ω.

4. By the weak maximum principle, see Proposition 4.3.6, the sequence u_n is uniformly bounded (in Ω) by $\max_{\partial\Omega}|f|$. By the previous point it is equicontinuous in $B(x,r)$, because all functions u_n admit the same modulus of continuity. Hence it converges (on a subsequence) uniformly to a continuous function in the ball $B(x,r)$; as it converges in L^1 and (on a further subsequence) almost everywhere to u, by uniqueness of the limit u is continuous in $B(x,r)$. We repeat this argument for all points in Ω and see that $u \in C(\Omega)$.

5. Now, fix $x \in \partial\Omega$. We proceed as in steps 3-4 of the proof of Theorem 4.4.2. We recall the result of step 4:

$$f(x) - \varepsilon \leq u(y) \leq f(x) + \varepsilon \qquad \text{for a.e. } y \in B(x,\delta_2) \cap \Omega.$$

Hence, as for each ε we can find an appropriate $r = \delta_2$, we see that

$$\lim_{r \to 0} \text{esssup}_{\Omega \cap B(x,r)} |u(y) - f(x)| = 0;$$

as u is continuous inside Ω, we may replace the essential supremum with a supremum. Hence, $u \in C(\overline{\Omega})$. □

We turn our attention to the issue of uniqueness of solutions. The key idea here is that we may perturb the level sets of a solution as long, as a positive multiple of the normal vector ν lies in a flat part of the unit sphere $\partial B_\phi(0,1)$, i.e. $\alpha\nu \in I \subset \partial B_\phi(0,1)$.

Remark 4.4.4. We notice that if $\alpha\nu \in \text{int } I$, then there exists a neighbourhood $N \subset S^1$ of ν_0 such that for each $\nu \in N$ a positive multiple of ν, namely $(\nu_0 \cdot \nu)^{-1}\alpha\nu$, lies in I.

We use this observation to explicitly construct a family of $\phi-$minimal surfaces other than a line segment that we will use later to prove existence of solutions with regularity no better than $BV(\Omega) \cap L^\infty(\Omega)$.

Lemma 4.4.5. *Suppose that there is a line segment $I \subset \partial B_\phi(0,1)$. Let $p_1, p_2 \in \partial\Omega$, take ν_0 to be a vector normal to $\overline{p_1 p_2}$ and suppose that $\alpha\nu_0 \in$ int I. Let $E \subset \Omega$ be an open set such that its boundary is the line segment $p_1 p_2$. Let $F \subset \Omega$ be an open set such that its boundary is a (finite) polygonal chain $p_1 q_1 ... q_n p_2$ such that the normal vector to each of the line segments in this polygonal chain lies in N. Then*

$$P_\phi(F,\Omega) = P_\phi(E,\Omega).$$

In particular, E is not the only $\phi-$minimal set relative to its boundary data.

Proof. Rename the points so that $q_0 = p_1$ and $q_{n+1} = p_2$. Take the polygonal chain $q_0 q_1 ... q_n q_{n+1}$ as in the assumption of the Lemma. It is enough to show that the set F bounded by this polygonal chain has the same anisotropic perimeter as the set F' bounded by the

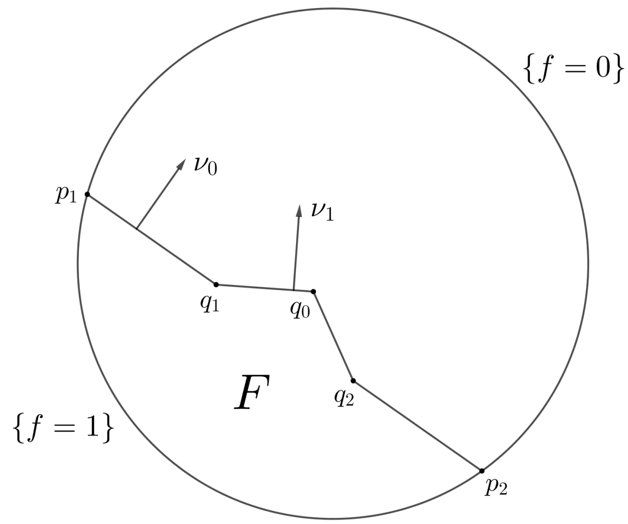


Figure 4.2: The construction of the competitor F

polygonal chain $q_0q_2...q_np_{n+1}$; then, we use this result to expand our line segment into a polygonal chain using finitely many steps without changing the perimeter.

We calculate the perimeter of F. As ∂F is piecewise C^1 (and the measure $D\chi_F$ has no atoms), we write:

$$P_\phi(F,\Omega) = \int_{\partial F} \phi(\nu(x))\, d\mathcal{H}^1 = \sum_{i=0}^{n} \phi(\nu_{\overline{q_iq_{i+1}}})\, \mathcal{H}^1(\overline{q_iq_{i+1}})$$

and

$$P_\phi(F',\Omega) = \int_{\partial F'} \phi(\nu(x))\, d\mathcal{H}^1 = \phi(\nu_{\overline{q_0q_2}})\, \mathcal{H}^1(\overline{q_0q_2}) + \sum_{i=2}^{n} \phi(\nu_{\overline{q_iq_{i+1}}})\, \mathcal{H}^1(\overline{q_iq_{i+1}}).$$

Since we assumed that $\alpha_i\nu_{\overline{q_iq_{i+1}}} \in I$ for every i, we see that also some positive multiple α of $\nu_{\overline{q_0q_2}}$ belongs to I. Thus $(\nu_{\overline{q_0q_2}}\cdot \nu_{\overline{q_0q_1}})^{-1}\alpha\nu_{q_0q_1}$ and $(\nu_{q_0q_2}\cdot \nu_{\overline{q_1q_2}})^{-1}\alpha\nu_{\overline{q_0q_2}}$ belong to I. In this case, we have $\phi(\nu_{\overline{q_0q_1}}) = (\nu_{\overline{q_0q_2}}\cdot \nu_{\overline{q_0q_1}})\phi(\nu_{\overline{q_0q_2}})$. Similarly, $\phi(\nu_{\overline{q_1q_2}}) = (\nu_{\overline{q_0q_2}}\cdot \nu_{\overline{q_1q_2}})\phi(\nu_{\overline{q_0q_2}})$. Hence, we compare the two expressions and obtain

$$P_\phi(F,\Omega) - P_\phi(F',\Omega) = \phi(\nu_{\overline{q_0q_1}})\mathcal{H}^1(\overline{q_0q_1}) + \phi(\nu_{\overline{q_1q_2}})\mathcal{H}^1(\overline{q_1q_2}) - \phi(\nu_{\overline{q_0q_2}})\mathcal{H}^1(\overline{q_0q_2}) =$$

$$= \phi(\nu_{\overline{q_0q_2}})((\nu_{\overline{q_0q_2}}\cdot \nu_{\overline{q_0q_1}})\mathcal{H}^1(\overline{q_0q_1}) + (\nu_{\overline{q_0q_2}}\cdot \nu_{\overline{q_1q_2}})\mathcal{H}^1(\overline{q_1q_2}) - \mathcal{H}^1(\overline{q_0q_2})) = 0.$$

Thus the anisotropic perimeter of the sets F and F' bounded by the polygonal chains $q_0q_1q_2...q_{n+1}$ and $q_0q_2...q_{n+1}$ respectively are the same, so inductively (with finitely many steps) we obtain that the anisotropic perimeter is the same as the anisotropic perimeter of E. □

Corollary 4.4.6. *Let F be a set such that its boundary is a piecewise C^1 curve from p_1 to p_2 such that the normal vector to ∂F at each point lies in N. Then, since we can approximate* *it in the strict topology by sets whose boundaries are the polygonal chains as above, F is a $\phi-$minimal set.* $\square$

Proposition 4.4.7. *Let $\Omega = B(0,1) \subset \mathbb{R}^2$ and suppose that ϕ is not strictly convex. Then there exist boundary data $f \in C^\infty(\partial\Omega)$ such that the solution to Problem (aLGP2) is not unique.*

Proof. Let I and ν_0 be as above. Let $u \in C^\infty(\overline{\Omega})$ be such that it takes values in the interval $[0,1]$, all its level sets are line segments perpendicular to ν_0 and u is decreasing in the direction of ν_0. Now, let $f = u|_{\partial\Omega}$. The preimage of every $t \in (0,1)$ consists of two points p_1, p_2 and the isotropic solution is such that the $t-$level set is the line segment p_1p_2.

We want to perturb the line segment p_1p_2 into a polygonal chain ∂F consisting of four line segments, $\overline{p_1q_1}, \overline{q_1q_0}, \overline{q_0q_2}, \overline{q_2p_2}$ such that q_1 and q_2 lie on the line segment $\overline{p_1p_2}$ and such that the vector ν_1 normal to q_1q_0 is sufficiently close to ν_0 without changing the anisotropic perimeter. The length of the first and last of there line segments is l_0 and the length of the two in the middle is l_1: this way the vector ν_2 normal to $\overline{q_0q_2}$ is such that $\nu_1 + \nu_2$ is parallel to ν_0, i.e. the triangle $\Delta q_0q_1q_2$ is an isosceles triangle. The sides of this triangle have lengths l_1, l_1 and $2(\nu_0 \cdot \nu_1)l_1$. By Lemma 4.4.5 this polygonal chain has the same anisotropic perimeter as the original line segments, provided that all the normal vectors are such that a positive multiple of them lies in I. Hence, Theorem 2.3.3 implies that such $\widetilde{u}$ is another solution of Problem (aLGP2). This construction is summarised on Figure 4.3. $\square$

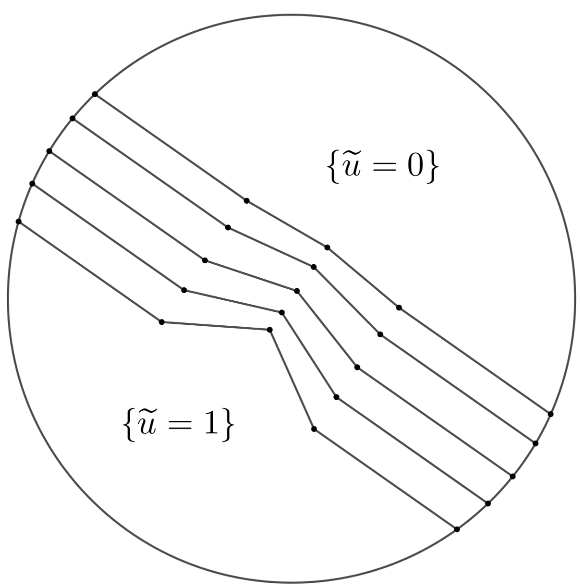


Figure 4.3: The construction of the competitor $\widetilde{u}$

We would like to remark that this phenomenon of nonuniqueness does not contradict the structure results from [53], which say that all solutions are determined by a single vector field; here, as the norm is not strictly convex, the vector field only determines the angle of the boundaries of superlevel sets to coordinate axes, as we can see in Lemma 4.4.5. In particular, it allows multiple solutions for continuous boundary data.

Now, we want to inspect the issue of regularity of the solutions to Problem (aLGP2) when ϕ is not strictly convex. We present a series of examples, in which given a non-strictly convex norm ϕ and an open bounded convex set Ω we construct a boundary datum $f \in C(\partial\Omega)$ such that there exists a solution that has regularity no better than $BV(\Omega) \cap L^\infty(\Omega)$; if $\partial\Omega$ is smooth, then f is also smooth. We recall that $SBV(\Omega)$ is the space of functions in $BV(\Omega)$ such that their distributional derivative are measures which admit a decomposition into an absolutely continuous part and jump part. In other words, functions in $SBV(\Omega)$ are such that their distributional derivatives have no Cantor part.

Proposition 4.4.8. *Suppose that Ω is convex and ϕ is not strictly convex. Then, there exists a boundary datum $f \in C(\partial\Omega)$, for which there exist solutions u_1, u_2 to Problem* (aLGP2) *such that $u_1 \notin W^{1,1}(\Omega)$ and $u_2 \notin SBV(\Omega)$.*

Proof. (1) Let $I \subset \partial B_\phi(0,1)$ be a line segment and let $\alpha\nu_0 \in \text{int } I$. Let $f \in C(\partial\Omega)$ be given by the formula

$$f(x,y) = \nu_0 \cdot \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then the function u_0 given by the same formula inside Ω has the prescribed trace f and all its superlevel sets are line segments; by Proposition 4.3.1 they are $\phi-$minimal. Thus by Theorem 2.3.3 u_0 is a function of $\phi-$least gradient.

We are going to modify the function u_0 on a compact subset of Ω, so that the trace remains unchanged. We will modify carefully the level sets of u_0. To this end, we are going to use Proposition 4.4.5 to construct as superlevel sets of u_1 another sets with the same perimeter and trace.

Take any line l perpendicular to ν_0 intersecting Ω. Choose two points $q_1, q_2 \in l$. Then, take two lines m, m' perpendicular to l, intersecting l inside Ω, such that the points q_1, q_2 lie between m and m'. Now, we choose two other lines l', l'' parallel to l such that l lies between l' and l'' and such that they are sufficiently close to l, so the following condition is fulfilled: let $p_1 = l' \cap m, p_2 = l' \cap m', p_3 = l'' \cap m, p_4 = l'' \cap m'$. Then a positive multiple of the normal vector to each of the line segments $\overline{p_i q_j}$ belongs to I.

We construct each of the level sets of u_1 in the following way: outside the region bounded by l and l' we define it to equal u_0. Now, fix any line k parallel to l lying between l' and l''. Then the level set is a polygonal chain along this line until the point $k \cap m$, then it is a line segment from this point to q_1, then the line segment $\overline{q_1 q_2}$, then again from q_0 to $k \cap m'$ and then again along k. This construction is shown on Figure 4.4.

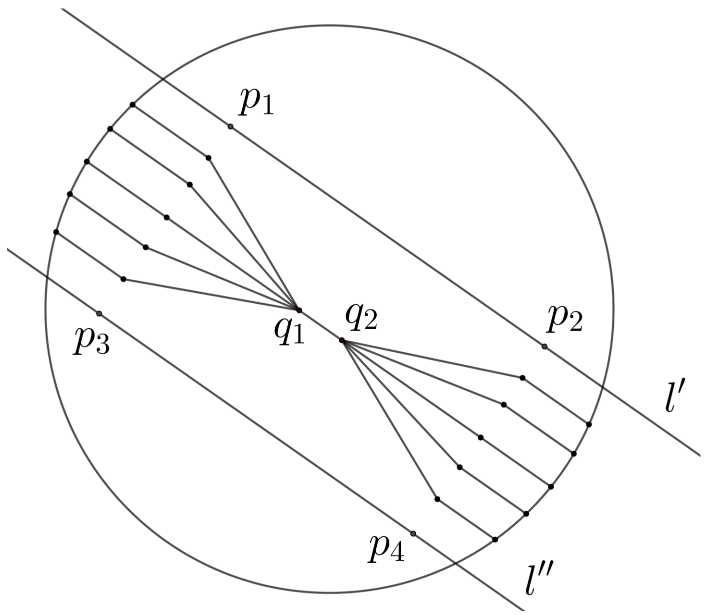


Figure 4.4: The construction of another minimiser u_1

By Lemma 4.4.5, the $\phi-$perimeter of each of the level sets of the function u_1 is the same as the $\phi-$perimeter of each of the level sets of u_0, as the normal vector to each of the line segments lies in I. By Theorem 2.3.3, u_1 is a function of $\phi-$least gradient. We notice that the constructed minimiser has a constant jump along the line segment $\overline{q_1 q_2}$; therefore, it fails the ACL characterisation and does not belong to $W^{1,1}(\Omega)$.

(2) Again, we are going to modify u_0 on a compact subset of Ω and the trace remains unchanged. Take any line l perpendicular to ν_0 intersecting Ω. Take four lines m_1, m_2, m_3, m_4 perpendicular to l intersecting l inside Ω (in this order along l). Now, take a parallel line l' sufficiently close to l, so that the following condition is satisfied: let $\{p_i\}$ be all eight possible intersections between the lines l or l' and the lines m_k. Then, a positive multiple of the normal vector to the each of the line segments $\overline{p_i p_j}$, where $\overline{p_i p_j}$ does not lie on m_k for some k, belongs to I.

We construct each of the level sets of u_2 in the following way: without loss of generality, we may assume that $u_0 \equiv 0$ on l and $u_0 \equiv 1$ on l'. Let g be the Cantor stairs function. The level sets of u_2 for $t \notin (0,1)$ are the same as for u_0. Now, fix $t \in (0,1)$. Let l_t denote the line parallel to l corresponding to value t of the minimiser u_0. Then, the level set $\{u = t\}$ is as follows: firstly along l_t up to the intersection with m_1; then the line segment $[l_t \cap m_1, l_{\frac{t+g(t)}{2}} \cap m_2]$; then along $l_{\frac{t+g(t)}{2}}$ up to the intersection with m_3; then the line segment $[l_{\frac{t+g(t)}{2}} \cap m_3, l_t \cap m_4]$; finally, again along l_t. This construction is shown on Figure 4.5.

Again, by Lemma 4.4.5 and Theorem 2.3.3 u_2 is a function of $\phi-$least gradient. We notice that in the rectangle bounded by the lines l, l'', m_2, m_3 the derivative of the constructed minimiser is a continuous measure, but it is not absolutely continuous; thus, $u_2 \notin SBV(\Omega)$.

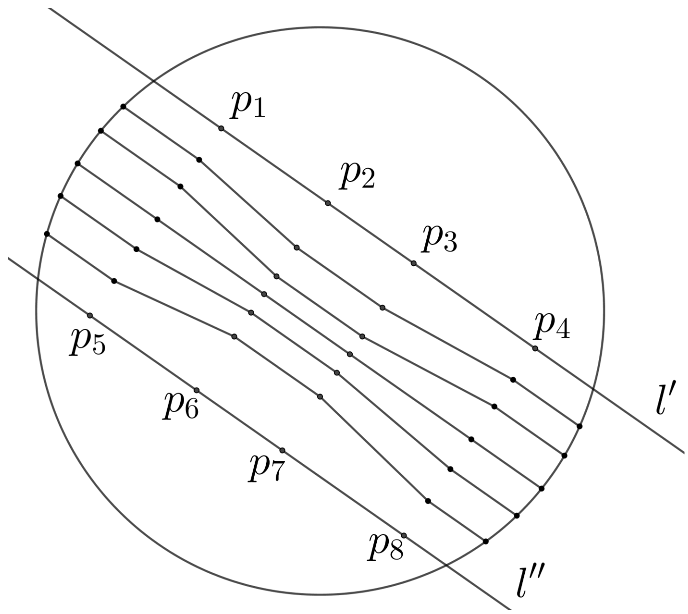


Figure 4.5: The construction of another minimiser u_2

$\square$

Finally, we turn our attention to the barrier condition. The next Proposition justifies the need for Theorems 4.4.1 and 4.4.2, as we cannot apply the existing theory from [36] to obtain existence of minimisers to Problem (aLGP2).

Proposition 4.4.9. *Suppose that Ω has C^1 boundary and ϕ is not strictly convex. Then Ω does not satisfy the barrier condition.*

Proof. By Proposition 4.3.1 a line segment is a $\phi-$minimal surface. Therefore if Ω is not strictly convex, it does not satisfy the barrier condition.

Now, take Ω to be a strictly convex set. As ϕ is not strictly convex, there exists a line segment $I \subset \partial B_\phi(0,1)$. Fix ν_0 such that $\alpha\nu_0 \in \text{int } I$. Take $x_0 \in \partial\Omega$ such that the normal vector to $\partial\Omega$ at x_0 has direction ν_0. Take ε small enough, so that a positive multiple of the normal vector ν at every $x \in \partial\Omega \cap B(x_0,\varepsilon)$ lies in I.

Provided ε is small enough, the set $\partial\Omega \cap \partial B(x_0,\varepsilon)$ consists of two points x_1, x_2. Take V to be the open connected set bounded by $\partial\Omega$ and the two line segments $[x_1, x_0]$ and $[x_2, x_0]$. Then $V \backslash B(x_0,\varepsilon) = \Omega \backslash B(x_0,\varepsilon)$ and by Lemma 4.4.5 V is a minimal set. However, $\partial V \cap \partial\Omega \cap \partial B(x_0,\varepsilon) = \{x_0\} \neq \emptyset$, so the barrier condition is not satisfied. $\square$

However, if $\partial\Omega$ has corners, it is still possible that the barrier condition is satisfied. This depends on the balance between the width of an angle corresponding to a flat part of $\partial B_\phi(0,1)$ and the width of the corner. For instance, it is easy to see that for l^1 anisotropy no open set satisfies the barrier condition; the next Example shows that if the flat part of $\partial B_\phi(0,1)$ is small enough, the barrier condition may hold for properly chosen Ω.

Example 4.4.10. Let ϕ be an anisotropic norm such that $\partial B_\phi(0,1)$ has only two flat parts on the boundary corresponding to angles $(\frac{1\pi}{8}, \frac{3\pi}{8})$ and $(\frac{9\pi}{8}, \frac{11\pi}{8})$ in the polar coordinates on the plane. Let Ω be an open strictly convex set, symmetric with respect to the line $y = x$, such that it is C^∞ except for two corners at $\pm(1,1)$, such that the angle of incidence of $\partial\Omega$ at $\pm(1,1)$ to the line $y = x$ is $\frac{\pi}{16}$ and the angle of incidence of $\partial\Omega$ to any line of the form $y = x + a$ is bounded from above by $\frac{\pi}{16}$. Then the barrier condition is satisfied, as any $\phi-$minimal surface with an angle of incidence to $y = x + a$ smaller than $\frac{\pi}{8}$ is a line segment.

The proof of Proposition 4.4.9 fails, as we cannot take a $x \in \partial\Omega$ such that in its neighbourhood the normal vector to $\partial\Omega$ has direction corresponding to a flat part on the boundary; here, there are only two isolated points, $\pm(1,1)$, with such normal vectors.

Since the solutions to Problem (aLGP2) may be not unique if ϕ is not strictly convex, we may ask if the solution with minimal L^1 norm exists and has any additional regularity (this issue has been discussed for the isotropic case with discontinuous boundary data in Chapter 3). However, using a technique similar as in the previous Proposition, we may prove that existence of such solutions fails for sets with C^1 boundary.

Proposition 4.4.11. *Suppose that $\Omega \subset \mathbb{R}^2$ has C^1 boundary. Suppose that ϕ is not strictly convex. Then there exist C^∞ boundary data such that there is no minimiser of Problem (1) with minimal L^1 norm.*

Proof. As before, we may assume that Ω is strictly convex and $\nu_0, x_0, \varepsilon, x_1$ and x_2 are as in the proof of the previous Proposition. Take $f \in C^\infty(\partial\Omega)$ satisfying the following conditions: $f = 0$ on $\partial\Omega \backslash B(x_0, \varepsilon)$; $f(x_0) = 1$; f is strictly monotone on the arcs (x_1, x_0) and (x_0, x_2) on $\partial\Omega$; f is one-dimensional in the direction of ν_0, i.e. $f(x,y) = \widetilde{f}(\nu_0 \cdot (x,y))$.

Define the functions u_n in the following way: let $y_n \in (x_0 + \nu_0\mathbb{R}) \cap \Omega \cap B(x, \varepsilon)$ and $y_n \to x_0$. For $y \in [y_n, x_0]$ we fix $u_n(y) = t$, where $y = (1-t)y_n + t x_0$. Denote by x_1^t and x_2^t the two elements of $f^{-1}(t)$ for $t \in (0,1)$. Then, we fix $u_n = t$ on a shifted boundary of Ω, namely $\partial\Omega + (y_n - x_0)$ in a smaller ball $B(x_0, \varepsilon')$ and $u_n = t$ on the line segments $[x_1^t, y_1^t]$, $[y_2^t, x_2^t]$, where y_1^t and y_2^t are the two points of $(\partial_\Omega + (y_n - x_0)) \cap \partial B(x_0, \varepsilon')$. This sequence converges to 0 everywhere in Ω (as $y_n \to x_0$) and by Lemma 4.4.5 each u_n is a function of least gradient. By construction also $Tu_n = f$. However, $\|u_n\|_{L^1(\Omega)} \to 0$ and $u \equiv 0$ is not a minimiser of Problem (aLGP2), as it does not satisfy the boundary condition. Thus the minimum L^1 norm among minimisers is not attained. □

The above construction was given to show that for every non-strictly convex ϕ there exist nonzero boundary data such that the infimum of the L^1 norms of the solutions equals 0. An example illustrating this phenomenon for the l_1 anisotropy is shown on Figure 4.6.

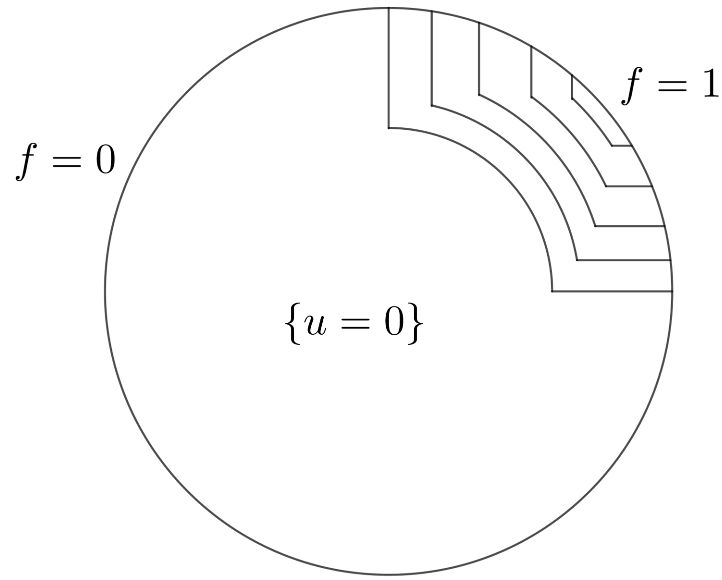


Figure 4.6: Minimisers with arbitrarily small l_1 norm

4.5 Conclusion

In this final Section we want to discuss what implications do the results from previous Sections have for the validity of several results from the isotropic case also in the anisotropic case. We will recall these results and shortly discuss the methods used in their proofs to see that they also hold in the anisotropic case with properly modified proofs.

The first result concerns the existence of minimisers of Problem (aLGP2) if the boundary data lie in the class $BV(\partial\Omega)$. It is an anisotropic version of Theorem 2.1.9; we will later upgrade it in Proposition 5.3.7 using a different approach.

Theorem 4.5.1. *Let Ω be an open bounded strictly convex set with C^1 boundary. Let ϕ be a fixed norm on $\mathbb{R}^2$. Then for every $f \in BV(\partial\Omega)$ there exists a solution to Problem* (aLGP2).

Proof. For now, assume that ϕ is strictly convex. The proof of Theorem 4.5.1 adapts very well to this case; we approximate the boundary data f with smooth functions f_n and construct the corresponding minimisers u_n. We prove a common bound in $BV(\Omega)$ for the sequence u_n and prove that it has a subsequence convergent to u, which is a minimiser of the functional

F_ϕ (defined in Section 4.2). The argument that the trace of u equals f is based on the weak maximum principle (Proposition 4.3.6) and the resulting geometric analysis.

This reasoning does not work for non-strictly convex norm ϕ. In this case, we will use a second layer of approximations; let $\phi_k = \phi + \frac{1}{k} l_2$ and denote by v_k the mimimisers of Problem (aLGP2) for ϕ_k and boundary data f obtained in the previous step. Their convergence to a minimiser v of the functional F_ϕ is guaranteed by Theorem 4.2.1. We prove that $Tv = f$ precisely as in Theorem 4.4.2; we see that the proof holds at points of continuity of f and, as the set of such points is of full measure, it suffices to prove the estimate for the mean integral defining the trace at the points of continuity of f. □

The second ([28, Theorem 1.2]) and third (Theorem 3.1.1) result concern the structure of minimisers for arbitrary boundary data. While the second result gives additional regularity of the solutions, the third result can also be understood as a uniqueness-type result. For discontinuous boundary data the minimisers need not be unique, see [46, Example 2.7], but this result shows that the minimisers may differ only on level sets of positive Lebesgue measure. As we can see in Proposition 4.4.8, these results do not hold if ϕ is not strictly convex.

Theorem 4.5.2. *Let Ω be an open bounded convex set. Suppose that ϕ is strictly convex. Let u be a minimiser of Problem* (aLGP2). *Then $u_j = u_c + u_j$, where u_c is continuous, u_j has only jump-type derivative and this decomposition is unique up to an additive constant.*

Theorem 4.5.3. *Let Ω be an open bounded convex set. Suppose that ϕ is strictly convex. Let u, v be two precise representatives of minimisers of Problem* (aLGP2). *Then $u = v$ on $\Omega \backslash (C \cup \mathcal{N})$, where both u and v are locally constant on C and $\mathcal{N}$ has Hausdorff dimension at most* 1.

The proof of these results involves mostly regularity theory for minimal sets, a (strong) maximum principle for minimal surfaces (inside Ω) and a weak maximum principle, i.e. the fact that two connected components of $\partial\{u > t\}$ cannot intersect on the boundary of Ω; as the regularity theory is irrelevant due to Proposition 4.3.1 and the last result is adaptable to the anisotropic case, these results hold also in the anisotropic case.

Finally, it is worth noting that in the isotropic case, if Ω is only convex, we may still obtain existence of minimisers, if f satisfies some additional admissibility conditions, see [58]; in principle, due to results from Section 4.3, the methods used there are also adaptable in the anisotropic case if ϕ is strictly convex. However, justifying this for suitable anisotropic version of the admissibility conditions goes beyond the scope of this thesis. We will later introduce a more efficient method to deal with the admissibility conditions in the planar case in Chapter 7, where we consider Ω which is an annulus.

Chapter 5

Existence of minimisers in the least gradient problem for general boundary data

5.1 Introduction

As already mentioned in Chapter 2, in [65] the authors established that for continuous boundary data, under a set of conditions on an open bounded set $\Omega \subset \mathbb{R}^N$ slightly weaker than strict convexity, there exists a unique solution to Problem (LGP) and it is continuous up to the boundary (see Theorem 2.1.5). However, if we relax some of these conditions, there arise additional problems:

(1) The first possible difficulty concerns discontinuous boundary data. Let us recall two results valid for $\Omega \subset \mathbb{R}^2$: as Theorem 2.1.6 shows, if the boundary data is discontinuous on a set of positive measure, there may be no minimisers to Problem (LGP). On the other hand, as proved in Theorem 2.1.9, for boundary data $f \in BV(\partial\Omega)$ there exists a minimiser to Problem (LGP); notice that in this case the set of discontinuities is at most countable.

(2) The second possible difficulty concerns unbounded sets $\Omega \subset \mathbb{R}^N$. Even if the set Ω is strictly convex, the construction from [65] fails, as it involves minimisation of perimeter in the class of sets which need not admit even one set with finite perimeter. Therefore, we need to work with approximations to both the set Ω and the boundary data f in order to prove existence of minimisers.

(3) Finally, if we relax the assumptions concerning strict convexity of Ω, the situation be-

comes much different - if Ω was convex with a flat part on the boundary, there exist continuous boundary data, for which there is no minimiser to Problem (LGP). Then, we need a different approach, involving finding a set of admissibility conditions sufficient for existence of minimisers. This issue is explored for instance in [58] and in Chapter 7.

The purpose of this Chapter is twofold: firstly, we prove existence of minimisers for boundary data that may be discontinuous on a set of $\mathcal{H}^{N-1}-$measure zero. Primarily, we are interested in the case when the boundary data $f \in BV(\partial\Omega)$ and the jump set of f is small enough (in dimension two, this condition is automatically satisfied, because BV boundary data are continuous except at countably many points). However, let us emphasise that this approach allows for boundary data $f \in L^1(\partial\Omega)\backslash BV(\partial\Omega)$; hence, it is an extension of known results both in terms of the dimension of the domain and the class of admissible boundary data. Furthermore, we explore this problem in an anisotropic setting, where we encounter additional difficulties concerning our regularity and convexity assumptions on ϕ and $\partial\Omega$ due to the use of the barrier condition and comparison principles. These two cases will require slightly different methods; in the former, we will exploit the translation invariance and projections, while in the latter we will rely primarily on the comparison principle. We will address this issue in Section 5.3.

Secondly, we use a technique developed in Section 5.4 to find the appropriate function space on $\partial\Omega$, for which the least gradient problem (and its anisotropic versions) are well-posed for unbounded domains. For geometric reasons arising even in the bounded domain case, we assume Ω to be strictly convex and not equal to $\mathbb{R}^N$. We want to deal with two kinds of phenomena: the regularity of boundary data and the shape of the domain. The main existence result, Theorem 5.4.2, concerns existence of minimisers for continuous boundary data and general strictly convex domains Ω. Later, we impose additional constraints on the regularity of boundary data and the shape of the domain to obtain uniqueness and additional regularity of minimisers. We also show that if we relax these constraints, we lose uniqueness of minimisers. The main regularity result is Proposition 5.4.5, which concerns continuity of least gradient functions for continuous boundary data; its proof relies on the maximum principle for minimal surfaces. We will address this issue in Section 5.4.

This Chapter is based on the article [29], of which I am the sole author and which has been accepted for publication in the Indiana University Mathematics Journal.

5.2 Preliminaries

Suppose that ϕ does not depend on the first variable (in other words - it is a norm). Recall that we say that ϕ is strictly convex if the unit ball in this norm, $B_\phi(0,1)$, is strictly convex.

If ϕ is strictly convex, then the barrier condition is something between convexity and strict convexity of Ω. To see that the barrier condition implies convexity of Ω, it is enough to note that a hyperplane is always a $\phi-$minimal surface. The fact that strict convexity entails the barrier condition is proved in the following Proposition. In fact, in dimension two the barrier condition is equivalent to strict convexity, see Section 4.3.

Proposition 5.2.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded strictly convex set. Let ϕ be a strictly convex norm. Then Ω satisfies the barrier condition with respect to ϕ.*

Proof. Suppose otherwise and let $x_0 \in \partial\Omega$ be such that the condition from Definition 2.3.4 is not satisfied at x_0. Let V minimise $P_\phi(\cdot, \mathbb{R}^N)$ in

$$\{W \subset \Omega : W \backslash B(x_0, \varepsilon) = \Omega \backslash B(x_0, \varepsilon)\}$$

and let $x \in \partial V \cap \partial\Omega \cap B(x_0, \varepsilon)$. Let H be a supporting hyperplane at x such that the normal vector ν to H points inside Ω. Let us denote by H_+ the halfspace with boundary H containing Ω. Then, let us shift the hyperplane a little bit along ν, so that $(H + t\nu) \cap \Omega \subset B(x_0, \varepsilon)$. Then the set $V' = V \cap (H_+ + t\nu)$ satisfies $V' \backslash B(x_0, \varepsilon) = \Omega \backslash B(x_0, \varepsilon)$. Furthermore, unless $V' = V$, V' has strictly smaller $\phi-$perimeter than V. To see this, we notice that if $|V \backslash V'| > 0$, then part of ∂V around x is projected onto a part of the hyperplane $H + t\nu$ and (since ϕ is strictly convex) projection strictly decreases the $\phi-$perimeter, see Proposition 4.3.1. Hence $x \notin \partial V \cap \partial\Omega \cap B(x_0, \varepsilon)$ for all $x \in B(x_0, \varepsilon) \cap \partial\Omega$ and the barrier condition holds. □

Now, we write a very simple observation:

Lemma 5.2.2. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Let $u \in BV_\phi(\Omega)$ be a function of $\phi-$least gradient in Ω. Then*

$$\int_\Omega |Du|_\phi \leq \int_{\partial\Omega} \phi(x, \nu^\Omega)\, |Tu|\, d\mathcal{H}^{N-1},$$

where ν^Ω is the $\mathcal{H}^{N-1}$-a.e. well-defined outer normal to $\partial\Omega$ at x.

Proof. We recall that the relaxed functional F_ϕ for the anisotropic least gradient problem with boundary data f is given by equation (2.3.1). Since u is a function of $\phi-$least gradient with boundary data Tu, we have

$$\int_\Omega |Du|_\phi = \int_\Omega |Du|_\phi + \int_{\partial\Omega} \phi(x, \nu^\Omega)\, |Tu - Tu|\, d\mathcal{H}^{N-1} = F_\phi(u) \leq$$

$$\leq F_\phi(v \equiv 0) = \int_\Omega 0\, dx + \int_{\partial\Omega} \phi(x, \nu^\Omega)\, |0 - Tu|\, d\mathcal{H}^{N-1} =$$

$$= \int_{\partial\Omega} \phi(x, \nu^\Omega)\, |Tu|\, d\mathcal{H}^{N-1}.$$

□

We need another simple observation, based on the Poincaré inequality:

Lemma 5.2.3. *Let Ω be an open bounded set with Lipschitz boundary which lies in a strip of width d and let $u \in BV(\Omega)$. Then*

$$\|u\|_{L^1(\Omega)} \leq C(d)(\int_\Omega |Du| + \int_{\partial\Omega} |Tu| \, d\mathcal{H}^{N-1}).$$

Proof. Let us extend the function u by 0 on $\mathbb{R}^N \backslash \Omega$. The extension $\widetilde{u}$ is a function with compact support in $\mathbb{R}^N$ and by the Poincaré inequality for $\widetilde{u}$ we have

$$\|u\|_{L^1(\Omega)} = \|\widetilde{u}\|_{L^1(\mathbb{R}^N)} \leq C(d) \int_{\mathbb{R}^N} |D\widetilde{u}| =$$

$$= C(d)(\int_\Omega |Du| + \int_{\partial\Omega} |Tu| \, d\mathcal{H}^{N-1}) + \int_{\mathbb{R}^N \backslash \overline{\Omega}} 0 \, dx,$$

where $C(d)$ is the constant in the Poincaré inequality which depends only on the width d of the strip containing Ω. □

Lemma 5.2.4. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Suppose that ϕ is a metric integrand and let f_n be a uniformly bounded sequence in $L^1(\partial\Omega)$. Finally, let $u_n \in BV_\phi(\Omega)$ be functions of $\phi-$least gradient with traces f_n. Then u_n has a convergent subsequence $u_{n_k} \rightarrow u$ in $L^1(\Omega)$.*

The above Lemma generalises [28, Proposition 4.1] concerning the isotropic case. Furthermore, if $f_n \rightarrow f$ in $L^1(\partial\Omega)$, it does not imply that $Tu = f$, because the trace operator is not continuous with respect to convergence in $L^1(\Omega)$.

Proof. Recall that $BV_\phi(\Omega) = BV(\Omega)$ as sets. To start, we estimate the L^1 norm of u by using the Poincaré inequality. By Lemma 5.2.3 we have

$$\|u_n\|_{L^1(\Omega)} \leq C(\Omega)(\int_\Omega |Du_n| + \int_{\partial\Omega} |Tu_n| \, d\mathcal{H}^{N-1}).$$

We estimate the (isotropic) total variations of u_n using Lemma 5.2.2 and the equivalence of norms between $BV_\phi(\Omega)$ and $BV(\Omega)$:

$$\int_\Omega |Du_n| \leq \lambda^{-1} \int_\Omega |Du_n|_\phi \leq \lambda^{-1} \int_{\partial\Omega} \phi(x, \nu^\Omega) \, |Tu_n| \, d\mathcal{H}^{N-1} \leq$$

$$\leq \lambda^{-1} \Lambda \int_{\partial\Omega} |Tu_n| \, d\mathcal{H}^{N-1}.$$

We bring these two estimates together and get

$$\|u_n\|_{BV(\Omega)} = \|u_n\|_{L^1(\Omega)} + \int_\Omega |Du_n| \leq$$

$$\le (C(\Omega)+1)(\int_\Omega |Du_n| + \int_{\partial\Omega} |Tu_n|\, d\mathcal{H}^{N-1}) \le$$

$$\le (C(\Omega)+1)(\lambda^{-1}\Lambda+1)\int_{\partial\Omega} |Tu_n|\, d\mathcal{H}^{N-1} =$$

$$= (C(\Omega)+1)(\lambda^{-1}\Lambda+1)\int_{\partial\Omega} |f_n|\, d\mathcal{H}^{N-1}.$$

As the sequence f_n is uniformly bounded in $L^1(\partial\Omega)$, the sequence u_n is uniformly bounded in $BV(\Omega)$, hence it admits a convergent subsequence in $L^1(\Omega)$. □

5.3 Results for discontinuous boundary data

This section is devoted to proving existence of minimisers for bounded sets, but with fairly general boundary data. We will consider two different cases. Firstly, we will consider the case when ϕ is a norm with strictly convex unit ball, without any regularity assumptions on ϕ. Secondly, we will assume that the metric integrand ϕ may depend on location, but has to satisfy the regularity hypothesis (H). In particular, both approaches cover the isotropic least gradient problem.

5.3.1 Existence theorems

The main results of this Section, Theorem 5.3.1 and Theorem 5.3.6, concern the case when the discontinuity set of the boundary data f is small - the precise assumption is that its $\mathcal{H}^{N-1}-$measure is zero. These results are motivated by Theorem 2.1.9, which states that in two dimensions, if Ω is strictly convex and $f \in BV(\partial\Omega)$, then there exists a minimiser to Problem (LGP). The two Theorems extend this result to higher dimensions, while generalising it to anisotropic cases as well.

Theorem 5.3.1. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded strictly convex set. Let ϕ be a strictly convex norm on $\mathbb{R}^N$. Suppose that $f \in L^1(\partial\Omega)$ is a function such that $\mathcal{H}^{N-1}$-almost all points of $\partial\Omega$ are continuity points of f. Then, there exists a minimiser to the Problem (aLGP) with boundary data f.*

Proof. 1. We want to define a sequence of approximations f_n which is continuous, converges almost uniformly to f and which locally preserves the L^∞ bounds of f. Mollification has all of the above properties, but it need not be defined if $\partial\Omega$ is not a Lie group, see Remark 5.3.3; therefore we have to construct a similar operator.

Let ρ_ε be a standard mollification kernel on $\mathbb{R}^{N-1}$ with the diameter of support equal to ε. As Ω is strictly convex and bounded, its boundary is compact and locally it is a graph of a Lipschitz function. Hence, it is a topological manifold, equipped with an atlas of finitely many (due to compactness) bi-Lipschitz maps $\phi_i : U_i \to \partial\Omega$ for $i = 1, ..., I$, where $U_i \subset \mathbb{R}^{N-1}$ is open (the inverse of each ϕ_i is a projection, so also a Lipschitz map). We denote $V_i = \phi_i(U_i)$; the sets V_i form an open cover of $\partial\Omega$. Let φ_i be a continuous partition of unity subject to the cover V_i.

We define $f_n \in C(\partial\Omega)$ in the following way: in the domain of a map ϕ_i, we pull back f with a map ϕ_i to $\mathbb{R}^{N-1}$. We mollify the pullback with a kernel $\rho_{\frac{1}{n}}$ and go back to $\partial\Omega$. In other words, we write

$$f_n(x) = \sum_{i=1}^{I} \varphi_i(x) g_{n,i}(\phi_i^{-1}(x)),$$

where

$$g_{n,i} = (\varphi_i \circ \phi_i)(f \circ \phi_i) * \rho_{\frac{1}{n}} \in C_c^\infty(U_i),$$

and whenever $\phi_i^{-1}(x)$ is not defined, then φ_i equals zero and nothing changes in the sum. Then f_n is a continuous function and the value of f_n at x depends only on the values of f in a ball $B(x, r(n))$, where $r(n) \to 0$ as $n \to \infty$; the precise form of $r(n)$ depends on the Lipschitz constants of maps ϕ_i.

2. Since Ω is strictly convex, by Proposition 5.2.1 it satisfies the barrier condition with respect to ϕ. Now, since $f_n \in C(\partial\Omega)$, by Theorem 2.3.5 there exist solutions $u_n \in BV(\Omega)$ of the anisotropic least gradient problem in Ω with boundary data f_n. By Lemma 5.2.4 and an anisotropic version of Theorem 2.1.4 they converge on a subsequence to a function u of $\phi-$least gradient in Ω; we only have to ensure that the trace is correct.

3. The set of points on $\partial\Omega$ where the trace of u is well-defined by the mean value property is of $\mathcal{H}^{N-1}-$full measure. Furthermore, by our assumption $\mathcal{H}^{N-1}-$almost all points of $\partial\Omega$ are continuity points of f. We denote the set (of $\mathcal{H}^{N-1}-$full measure) where the two above points hold by $\mathcal{Z} \subset \partial\Omega$.

4. Fix $x_0 \in \mathcal{Z}$; in particular x_0 is a point of continuity of f. Fix any $\delta > 0$. Then, there exists a ball $B(x_0, r)$ such that for all $x \in B(x_0, r) \cap \partial\Omega$ we have

$$f(x_0) - \delta \leq f(x) \leq f(x_0) + \delta.$$

Since the construction of the approximating sequence f_n involves mollification, we have for sufficiently large n (so that the mollification kernel has support with diameter such that $r(n) < \frac{r}{2}$)

$$f(x_0) - \delta \leq f_n(x) \leq f(x_0) + \delta$$

on a smaller set $B(x_0, \frac{r}{2}) \cap \partial\Omega$.

5. We introduce the following notation: let H be a supporting hyperplane at x_0. We choose it from the set of all supporting hyperplanes so that the normal ν to H at x_0 points inside Ω. Let H_- be the halfspace bounded by H which does not contain Ω. Take $s > 0$ small enough, so that $(H + s\nu) \cap \Omega \subset\subset B(x_0, \frac{r}{2})$. We want to estimate the value of u_n in the set $\Omega' = (H_- + s\nu) \cap \Omega$.

6. We will see that in the set Ω' for sufficiently large n all the functions u_n satisfy the inequalities

$$f(x_0) - \delta \leq u_n(x) \leq f(x_0) + \delta.$$

Let $E_t^n = \{u_n \geq t\}$. We recall our convention of choosing representatives of BV functions so that sets of bounded perimeter consist of points of positive density. Now, suppose that the inequalities do not hold; without loss of generality for some $x' \in \Omega'$ we have $u_n(x') = t > f(x_0) + \delta$. In view of $Tu_n = f_n$ for f_n satisfying $f(x_0) - \delta \leq f_n(x) \leq f(x_0) + \delta$, by our choice of representative of u the set ∂E_t^n intersects Ω'.

7. By the barrier condition the set E_t^n does not intersect $\partial\Omega' \cap \partial\Omega$. By our convention of choosing representatives, E_t^n has positive distance to the closed set $\partial\Omega' \cap \partial\Omega$ and the function $\chi_{E_t^n \setminus \Omega'}$ has the same trace as the function $\chi_{E_t^n}$; however, by Proposition 4.3.1 the former function has strictly smaller $\phi-$perimeter, as any parts of E_t^n intersecting Ω' are projected onto the hyperplane $H + s\nu$. This contradicts Theorem 2.3.3 (an anisotropic version of Theorem 2.1.3), hence $u_n(x) \in [f(x_0) - \delta, f(x_0) + \delta]$. Let us stress that this reasoning is only valid because the norm ϕ is strictly convex.

8. Now, we see that the trace of u at x_0 equals $f(x_0)$. Passing to the pointwise limit almost everywhere in the inequality from the Step 6 (possibly passing to a subsequence), we obtain that in $(B(x_0, \rho) \cap \Omega) \subset \Omega'$ we have

$$f(x_0) - \delta \leq u(x) \leq f(x_0) + \delta.$$

As for arbitrary $\delta > 0$ there exists a ball $B(x_0, \rho)$ for which the above inequality is satisfied, we see that

$$\lim_{\rho \to 0} \text{ ess sup}_{y \in B(x_0,\rho) \cap \Omega} |u(y) - f(x_0)| = 0,$$

so $Tu(x_0) = f(x_0)$. This equality holds for $\mathcal{H}^{N-1}$-almost all $x_0 \in \partial\Omega$, so $Tu = f$. □

Let us note that the method used here is in principle similar to the one used in the proof of Theorem 4.4.2. Now, in the following remarks we will discuss the necessity of the assumptions in the above Theorem. There are two types of assumptions in play: strict convexity of Ω and strict convexity of the norm ϕ.

Remark 5.3.2. The assumption that the set Ω is strictly convex is quite natural for two reasons. Firstly, by Proposition 5.2.1 it implies the barrier condition for any norm ϕ with strictly convex unit ball. Secondly, when the set is not strictly convex, another problem is choosing the appropriate hyperplane in Step 5 - so that the set Ω' is indeed contained in some ball; for instance, in three dimensions, if $\partial\Omega$ contains a line segment, then the set Ω' would contain a neighbourhood of this line segment and not be contained in any $B(x_0, r)$ for small r.

Remark 5.3.3. If $\partial\Omega$ is a smooth manifold with group structure, the first part of the proof becomes simpler - we can just take the approximating sequence to be defined using convolution with a mollification kernel: $f_n = f * \rho_{\frac{1}{n}}$. Moreover, in Step 5 we take simply a tangent hyperplane instead of choosing a proper supporting hyperplane.

Remark 5.3.4. Let us highlight what does and does not work when ϕ is not strictly convex. As we can see in Proposition 4.4.9, no set with C^1 boundary satisfies the barrier condition. Moreover, unless $\Omega \subset \mathbb{R}^2$ we do not know if there exist minimisers for continuous boundary data. Finally, the projection on a hyperplane does not necessarily decrease the anisotropic total variation. We will further discuss the two-dimensional case in Proposition 5.3.7.

Now, we turn our attention to the case when ϕ is a metric integrand and not necessarily a norm; however, we will assume high regularity of ϕ in order to use a comparison principle. We begin by a lemma which generalises a very simple observation from the isotropic case:

Lemma 5.3.5. *Let ϕ be a metric integrand satisfying hypothesis (H). Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary which satisfies the barrier condition. Let $g \in C(\partial\Omega)$. If g is constant in some neighbourhood of x_0 in $\partial\Omega$, then the minimiser v with boundary data g is constant in some neighbourhood of x_0 in Ω.*

Proof. Fix $\varepsilon > 0$ sufficiently small as in the definition of the barrier condition. Firstly, let us take V_α to be a family of sets from the definition of the barrier condition, i.e. for each $\alpha \in \mathcal{A}$ the set V_α minimises $P_\phi(\cdot, \mathbb{R}^N)$ in

$$\{W \subset \Omega : \quad W \backslash B(x_0, \varepsilon) = \Omega \backslash B(x_0, \varepsilon)\}. \tag{*}$$

Consider the set $V = \bigcup_{\alpha \in \mathcal{A}} V_\alpha$. It satisfies $V \backslash B(x_0, \varepsilon) = \Omega \backslash B(x_0, \varepsilon)$ and by lower semicontinuity of the $\phi-$total variation it minimises the perimeter in the class (*). By the barrier condition $\partial V \cap \partial\Omega \cap B(x_0, \varepsilon) = \emptyset$; but as $\partial\Omega$ is Lipschitz and ϕ satisfies (H), the distance from x_0 to V is positive, hence $V \cap B(x_0, r) = \emptyset$ for some $r = r(\varepsilon) > 0$.

Now, suppose that g is constant in $B(x_0, \varepsilon) \cap \partial\Omega$ with constant value t_0. Then, take $V_t = \{v > t\}$ with $t > t_0$. The set V_t falls into the class (*) and by the continuity of g its closure does not contain x_0. Hence, there is a positive distance $r = r(\varepsilon)$ separating x_0 and V_t for all $t > t_0$, so $v \leq t_0$ in $B(x_0, r(\varepsilon)) \cap \Omega$. Using a similar argument with $\widetilde{V_t} = \{v < t\}$ with $t < t_0$ we obtain that $v \geq t_0$ in $B(x_0, r(\varepsilon)) \cap \Omega$. Hence v is constant with value t_0 in some neighbourhood of x_0 in Ω. □

Theorem 5.3.6. *Let ϕ be a metric integrand satisfying hypothesis (H) and let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary which satisfies the barrier condition. Suppose that*

$f \in L^1(\partial\Omega)$ is a function such that $\mathcal{H}^{N-1}$-almost all points of $\partial\Omega$ are continuity points of f. Then, there exists a minimiser to the Problem (aLGP) with boundary data f.

Proof. 1. We define f_n as in Step 1 of the proof of Proposition 5.3.1. In that construction we only utilised Lipschitz regularity and compactness of $\partial\Omega$.

2. As $f_n \in C(\partial\Omega)$ and Ω satisfies the barrier condition, there exist solutions $u_n \in BV(\Omega)$ of the Problem (aLGP) for all $n \geq 1$. By Lemma 5.2.4 they converge on a subsequence to a function u of $\phi-$least gradient in Ω; we only have to ensure that the trace is correct.

3. The set of points on $\partial\Omega$ where the trace of u is well-defined by the mean value property is of $\mathcal{H}^{N-1}-$full measure. Furthermore, by our assumption $\mathcal{H}^{N-1}-$almost all points of $\partial\Omega$ are continuity points of f. We denote the set (of $\mathcal{H}^{N-1}-$full measure) where the two above points hold by $\mathcal{Z} \subset \partial\Omega$.

4. Fix $x_0 \in \mathcal{Z}$; in particular x_0 is a point of continuity of f. Fix any $\delta > 0$. Then there exists a ball $B(x_0, r)$ such that for all $x \in B(x_0, r) \cap \partial\Omega$ we have

$$f(x_0) - \delta \leq f(x) \leq f(x_0) + \delta.$$

As the sequence f_n is constructed via mollification, for sufficiently large n (so that the mollification kernel has support with diameter smaller than $\frac{r}{2}$)

$$f(x_0) - \delta \leq f_n(x) \leq f(x_0) + \delta$$

on a smaller set $B(x_0, \frac{r}{2}) \cap \partial\Omega$.

5. We will use the comparison principle to obtain a uniform bound on the sequence u_n in some neighbourhood of x_0. To this end, let us define $f_n^\pm \in C(\partial\Omega)$ as follows:

$$f_n^+ = \max(f_n, f(x_0) + \delta), \qquad f_n^- = \min(f_n, f(x_0) - \delta).$$

By definition $f_n^- \leq f_n \leq f_n^+$ on $\partial\Omega$ and $f_n^\pm$ are constant in $B(x_0, \frac{r}{2}) \cap \partial\Omega$ and equal to $f(x_0) \pm \delta$ respectively. Since Ω satisfies the barrier condition, by Theorem 2.3.5 there exist minimisers $u_n^\pm$ for boundary data $f_n^\pm$. We assumed that ϕ satisfies hypothesis (H), hence by Theorem 2.3.6 we have

$$u_n^- \leq u_n \leq u_n^+ \qquad \text{in } \Omega.$$

6. By Lemma 5.3.5 the barrier condition implies that if boundary data g is constant in a neighbourhood of x_0 in $\partial\Omega$, then the minimiser v with boundary data g is constant in a neighbourhood of x_0 in Ω. We apply this to the sequence u_n and obtain a ball $B(x_0, r')$ such that $u_n^\pm = f(x_0) \pm \delta$ in $\Omega \cap B(x_0, r')$. We notice that the radius r' is independent of n and depends only on the original radius r.

7. By the comparison principle (Theorem 2.3.6) all the functions u_n satisfy the inequality

$$f(x_0) - \delta \leq u_n(x) \leq f(x_0) + \delta$$

in $B(x_0, r') \cap \Omega$. Now, we see that the trace of u at x_0 equals f. Passing to the pointwise limit almost everywhere in the inequality from the Step 6 (possibly passing to a subsequence), we obtain that in a ball $B(x_0, r') \cap \Omega$ we have

$$f(x_0) - \delta \leq u(x) \leq f(x_0) + \delta.$$

As for arbitrary $\delta > 0$ there exists a ball $B(x_0, \rho)$ for which the above inequality is satisfied, we see that

$$\lim_{\rho \to 0} \text{ ess sup}_{y \in B(x_0,\rho) \cap \Omega} |u(y) - f(x_0)| = 0,$$

so $Tu(x_0) = f(x_0)$. This equality holds for $\mathcal{H}^{N-1}$-almost all $x_0 \in \partial\Omega$, so $Tu = f$. □

We conclude this Section with an existence result for non-strictly convex norms. We will assume that $\Omega \subset \mathbb{R}^2$.

Proposition 5.3.7. *Let ϕ be any norm on $\mathbb{R}^2$ and let $\Omega \subset \mathbb{R}^2$ be a bounded strictly convex set with Lipschitz boundary. Suppose that $f \in L^1(\partial\Omega)$ is a function such that $\mathcal{H}^1$-almost all points of $\partial\Omega$ are continuity points of f. Then there exists a minimiser to the Problem (aLGP) with boundary data f.*

Proof. 1. The case when ϕ is strictly convex is already covered; it suffices to consider the case when $\partial B_\phi(0,1)$ has flat parts on the boundary. Define $\phi_n = \phi + \frac{1}{n} l_2$, where l_2 is the Euclidean norm. Then ϕ_n is strictly convex and by Theorem 5.3.1 there exists a solution $u_n \in BV(\Omega)$ to Problem (aLGP) with boundary data f with respect to the anisotropic norm ϕ_n. In particular, $Tu_n = f$. Furthermore, the family u_n is uniformly bounded in $BV(\Omega)$: we use Lemmata 5.2.3 and 5.2.2 respectively in the first and second inequality below to obtain

$$\|u_n\|_{BV(\Omega)} = \int_\Omega |u_n| \, dx + \int_\Omega |Du_n| \leq C(\Omega)(\int_\Omega |Du_n| + \int_{\partial\Omega} |Tu_n| \, d\mathcal{H}^1) \leq$$

$$\leq C(\Omega, \phi)(\int_{\partial\Omega} \phi_n(\nu)|Tu_n| \, d\mathcal{H}^1 + \int_{\partial\Omega} |Tu_n| \, d\mathcal{H}^1) \leq$$

$$\leq C(\Omega, \phi) \int_{\partial\Omega} (1 + \sup_{\partial B(0,1)} \phi + \frac{1}{n})|f| \, d\mathcal{H}^1 \leq M.$$

Hence, u_n admits a subsequence (still denoted by u_n) convergent in $L^1(\Omega)$. By an anisotropic variant of Miranda's theorem, when the anisotropic norm changes with n (Theorem 4.2.1), u is a function of $\phi-$least gradient. If we prove that additionally $Tu = f$, then u is a solution to Problem (aLGP).

2. Fix any continuity point $x_0 \in \partial\Omega$. Take arbitrary $\varepsilon > 0$. Then, there exists a neighbourhood of x_0 in $\partial\Omega$ such that

$$f(x_0) - \varepsilon \leq f(y) \leq f(x_0) + \varepsilon \qquad \text{for } y \in B(x_0, \delta_1) \cap \partial\Omega.$$

Since Ω is strictly convex, for sufficiently small δ_1 the set $\partial(B(x_0, \delta_1) \cap \partial\Omega)$ consists of two points p_1, p_2 and the line segment $\overline{p_1 p_2}$ lies inside Ω. Denote by Δ the open set bounded by an arc of $\partial\Omega$ containing x and the line segment $\overline{p_1 p_2}$. Let us take a ball $B(x_0, \delta_2)$ such that $B(x_0, \delta_2) \cap \Omega \subset \Delta$. We argue as in Step 7 of the proof of Theorem 5.3.1 that for every n we have

$$f(x_0) - \varepsilon \leq u_n(y) \leq f(x_0) + \varepsilon \qquad \text{for } y \in B(x_0, \delta_2) \cap \Omega;$$

3. Since $u_n \to u$ in $L^1(\Omega)$, on some subsequence (still denoted u_n) we have convergence almost everywhere; hence

$$f(x_0) - \varepsilon \leq u(y) \leq f(x_0) + \varepsilon \qquad \text{for a.e. } y \in B(x_0, \delta_2) \cap \Omega,$$

in other words

$$\text{ess sup}_{y \in B(x_0, \delta_2) \cap \Omega} |u(y) - f(x_0)| \leq \varepsilon.$$

Since for arbitrary $\varepsilon > 0$ there exists a ball $B(x_0, \delta_2)$ for which the above inequality is satisfied, we see that

$$\lim_{\rho \to 0} \ \text{ess sup}_{y \in B(x_0, \rho) \cap \Omega} |u(y) - f(x_0)| = 0,$$

so $Tu(x_0) = f(x_0)$. This equality holds for $\mathcal{H}^{N-1}$-almost all $x_0 \in \partial\Omega$, so $Tu = f$. □

5.3.2 Examples

Now, we illustrate the results of this Section with a few examples. The first example gives yet another proof of an already known result, namely Theorem 2.1.9; see also [19, Proposition 5.1]. This result serves as a toy problem and motivation for Theorem 5.3.1. Moreover, it is generalised to the anisotropic setting.

Example 5.3.8. Let $\Omega \subset \mathbb{R}^2$ be a strictly convex set. Suppose that $f \in BV(\partial\Omega)$. Then the discontinuity set of f is at most countable, hence it has $\mathcal{H}^1-$measure zero and by Theorem 5.3.1 there exists a minimiser to Problem (LGP). Furthermore, this reasoning extends to norms other than the Euclidean norm and to metric integrands satisfying (H), provided that Ω satisfies the barrier condition.

It is important to note that in dimension two the class of boundary data for which we have existence of minimisers is much larger than $BV(\partial\Omega)$; for instance, it contains all functions such that the set of discontinuity points is countable, even if the sum of jumps is infinite.

Example 5.3.9. Let $\Omega = B(0,1) \subset \mathbb{R}^2$. Take the boundary data to equal

$$f(x,y) = \sum_{i=0}^{\infty} (-1)^i \chi_{(3^{-i-1},3^{-i})}(x)\chi_{(0,1)}(y).$$

The boundary data is continuous except at countably many points and the sum of jumps is infinite, hence $f \notin BV(\partial\Omega)$. However, by Theorem 5.3.1 there exists a minimiser u to the least gradient problem with boundary data f. The connected components of $\partial\{u \geq t\}$ for $t \in (-1,1)$ have an accumulation point at $p = (0,1)$; this phenomenon was not possible for BV boundary data.

However, there exist functions in $L^\infty(\partial\Omega)$ not covered by Theorem 5.3.1, for instance characteristic functions of fat Cantor sets; in particular, Theorem 5.3.1 does not apply for the counterexample introduced in [64].

Example 5.3.10. Let $\Omega = B(0,1) \subset \mathbb{R}^2$. Take $f = \chi_C$ to be the characteristic function of a certain fat Cantor set constructed in [64]. This function is discontinuous on C, which has positive $\mathcal{H}^1-$measure, so Theorem 5.3.1 does not guarantee existence of a minimiser to Problem (LGP); indeed, the authors of [64] prove that there is no minimiser.

Another issue is the uniqueness of minimisers. A well-known example, attributed to John Brothers (see [48, Example 2.7]), shows that if the boundary data is discontinuous, then even in the isotropic case we may not expect uniqueness of minimisers.

Example 5.3.11. Let $\Omega = B(0,1) \subset \mathbb{R}^2$. Take boundary data given by the formula

$$h(x,y) = \begin{cases} x^2 - y^2 + 1 & \text{if } |x| > \frac{1}{\sqrt{2}} \\ x^2 - y^2 - 1 & \text{if } |x| < \frac{1}{\sqrt{2}}. \end{cases}$$

Fix any $\lambda \in [-1,1]$. Then $u \in BV(\Omega)$ is a function of least gradient if and only if

$$u(x,y) = \begin{cases} 2x^2 & \text{if } |x| > \frac{1}{\sqrt{2}} \\ \lambda & \text{if } |x| < \frac{1}{\sqrt{2}}, |y| < \frac{1}{\sqrt{2}} \\ -2y^2 & \text{if } |y| > \frac{1}{\sqrt{2}}. \end{cases}$$

The structure of level sets is precisely the same as for boundary data $h_0(x,y) = x^2 - y^2$, the only difference being that the values of the minimiser u_0 for boundary data h_0 are shifted in two opposite directions in different subsets of Ω. It turns out that in the square where the function u_0 is constant we can choose freely the value of the shift.

In general, the nonuniqueness is related to the formation of level sets of u of positive Lebesgue measure, as in Theorem 3.1.1. The boundary data in this example has only four discontinuity points - it is the smallest number of discontinuity points for which we may lose uniqueness (the situation with fewer discontinuity points is considered for instance in [31, Corollary 3.2]).

Finally, we abandon $\mathbb{R}^2$ and use Theorem 5.3.1 to obtain existence of minimisers in a higher dimension. Moreover, the method used in Theorem 5.3.1 is constructive and enables us to directly find the minimiser (provided that we can directly compute the result of Sternberg-Williams-Ziemer construction for the approximation).

Example 5.3.12. Let $\Omega = B(0,1) \subset \mathbb{R}^3$. Take the boundary data to be

$$f(x,y,z) = \begin{cases} 1 & \text{if } |z| > a, \\ -1 & \text{if } |z| < a, \end{cases}$$

where $a \in (0,1)$. The boundary data is continuous everywhere except at the two circles which are intersections of $\partial\Omega$ and the planes $\{z = \pm a\}$. Hence, the discontinuity set has $\mathcal{H}^2-$measure zero and there exists a minimiser to Problem (LGP).

Using an approximation as in the proof of Theorem 5.3.1 and the axial symmetry, we can obtain the structure of minimisers, which differs with a. If a is chosen so that the area of the two discs which are intersections of Ω and $\{z = \pm a\}$ is smaller than the area of the part of the catenoid spanned by them, we have

$$u(x,y,z) = \begin{cases} 1 & \text{if } |z| > a, \\ -1 & \text{if } |z| < a. \end{cases}$$

If the area of the discs is larger, then

$$u(x,y,z) = \begin{cases} 1 & \text{if } |z| > a, \\ 1 & \text{if } |z| < a, \text{ inside the catenoid,} \\ -1 & \text{if } |z| < a, \text{ outside the catenoid.} \end{cases}$$

Finally, if the area of the discs is equal to the area of the part of catenoid spanned by them, mimimisers are no longer unique and

$$u(x,y,z) = \begin{cases} 1 & \text{if } |z| > a, \\ \lambda & \text{if } |z| < a, \text{ inside the catenoid,} \\ -1 & \text{if } |z| < a, \text{ outside the catenoid,} \end{cases}$$

where $\lambda \in [-1,1]$. By Theorem 3.1.1 these are all possible minimisers. Moreover, we observe that we can obtain all the minimisers in the last case using slightly different approximations, i.e. if we take the mollification kernel ρ to be asymmetric.

5.4 Results for unbounded domains

In this Section, we consider the case when the domain Ω is unbounded. As in the least gradient problem for bounded domains, our main interest is to find what conditions do we need to impose on the domain and the boundary data in order to obtain existence and uniqueness of minimisers. Of course we will need to modify our notion of a solution; as we will see later, the solutions we construct need not lie in $BV(\Omega)$, but rather in $BV_{loc}(\Omega)$. Moreover, we have two kinds of additional difficulties: the regularity of boundary data and the shape of the domain.

For clarity, in this Section we will present the reasoning in the setting of the isotropic least gradient problem; we will remark when an analogous reasoning works also in the anisotropic case. Most notably the difference will appear with respect to the barrier condition and arguments concerning uniqueness. Following Miranda, see [50], we introduce the following definition of least gradient functions in an unbounded set Ω:

Definition 5.4.1. We say that $u \in BV_{loc}(\Omega)$ is a function of least gradient in Ω if for every function $g \in BV(\Omega)$ with compact support $K \subset \Omega$ we have

$$\int_K |Du| \leq \int_K |D(u+g)|.$$

Given $f \in L^1_{loc}(\partial\Omega)$, we say that u solves the least gradient problem on Ω with respect to boundary data f, if both of the following conditions hold:

$$u \text{ is a least gradient function in } \Omega \quad and \tag{uLGP}$$

$$\text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in \partial\Omega \text{ we have } \fint_{B(x,r)\cap\Omega} |u(y) - f(x)| \, dy \to 0 \text{ as } r \to 0.$$

For bounded domains this is equivalent to Problem (LGP); however, here we cannot minimise the total variation, as the total variation turns out to be finite only in the most restrictive cases. Furthermore, the trace condition is understood pointwise, $\mathcal{H}^{N-1}$-a.e. on the boundary of Ω. As the boundary of Ω is unbounded, continuous functions need not be bounded; this is our first additional difficulty:

A. Regularity of boundary data. In this Chapter, we are mostly interested in the case of continuous boundary data. However, for unbounded domains the Sternberg-Williams-Ziemer construction (see [65]) does not work - the construction relies on two auxiliary problems, involving minimisation of perimeter and maximisation of area, which do not need to have solutions if the domain is unbounded. For this reason, we are going to consider multiple function spaces on $\partial\Omega$ and make distinctions between them when it comes to existence, uniqueness and regularity of minimisers. We consider the following spaces:

1. $L^1(\partial\Omega) \cap C_0(\partial\Omega)$. This is the most natural function space arising in this problem - for instance, we may regard the trace as an operator on Ω and not merely as a pointwise property that holds almost everywhere.

2. $C_0(\partial\Omega)$. This class arises naturally if we try to approximate the boundary data with continuous boundary data with compact support. In this case, we will be able to utilise a variant of the Sternberg-Williams-Ziemer construction to obtain uniqueness of minimisers.

3. $C_b(\partial\Omega)$ and $C(\partial\Omega)$. In this class there appear some interesting phenomena concerning the shape of superlevel sets, such as creation of ”shock waves” that extend to infinity. In particular, this leads to nonuniqueness of minimisers for a wide class of domains.

4. The case when the data are continuous almost everywhere. In particular, this covers the case $BV(\partial\Omega)$ for $\Omega \subset \mathbb{R}^2$. As we know from the bounded domain case, we cannot expect much more than existence of minimisers - this case combined with the results from Section 5.3 is a corollary to the previous ones.

We cannot expect much less regularity; in the case when Ω is a disk, see [64] for an example of a function that is $L^\infty(\partial\Omega)$, is discontinuous on a set of positive measure, and there is no minimiser. The example adapts well to the unbounded case.

B. Shape of domains. The second kind of assumptions concerns the shape of domains. As in the bounded case, we have to assume a condition similar to strict convexity of the domain. We will see that apart from this, the shape of the domain makes no difference in the existence proof; however, the shape of the domain may influence the regularity of the resulting minimiser. We are interested in three kinds of domains:

1. Strictly convex unbounded domains such that $\Omega \neq \mathbb{R}^N$. In particular, Ω lies in a halfspace. In this class, we are able to obtain existence of minimisers and uniqueness of minimisers for data in $C_0(\partial\Omega)$.

2. Domains with special features: firstly, domains which are in a sense ”one-dimensional”, i.e. domains that are unbounded only in one direction. In particular, these domains lie in a strip, so we have a uniform Poincaré inequality for any open subset $\Omega' \subset \Omega$ with Lipschitz boundary. Secondly, we will consider domains which contain a cone; these domains will be crucial to the phenomenon of nonuniqueness.

3. Finally, we want to consider boundary values at infinity. This is motivated mostly by the case when the domain Ω contains a cone. For simplicity, we restrict ourselves to the whole $\mathbb{R}^N$ and consider a radial compactification of $\mathbb{R}^N$ defined by adding a point in each direction; the resulting space is denoted by $\mathbb{R}^N$ with added $\partial B(0,1)$ at infinity and the boundary data are given as a function $f \in L^1(\partial B(0,1))$.

5.4.1 Existence of minimisers

The Theorem below proves existence of minimisers for the least gradient problem on an unbounded domain in full generality. Later, we will consider what modifications can we make to the proof below in order to obtain some additional regularity or uniqueness of minimisers.

Theorem 5.4.2. *Let $\Omega \subset \mathbb{R}^N$ be a strictly convex set and $\Omega \neq \mathbb{R}^N$. Let $f \in C(\partial\Omega)$. Then there exists a minimiser $u \in BV_{loc}(\Omega)$ of Problem* (uLGP) *with boundary data f.*

Proof. 1. We begin by noting that as Ω is a convex set which is not equal to $\mathbb{R}^N$, it is contained in a halfspace; it suffices to fix any $x \in \partial\Omega$ and consider any supporting hyperplane H with inward normal vector ν. Then Ω lies entirely on one side of this hyperplane, which is a halfspace we denote by H_+. The other halfspace H_- is disjoint with Ω. We notice that the shifted halfspaces of the form $H_- + t\nu$ for $t > 0$ intersect Ω and their union is $\mathbb{R}^N$, so they cover the whole Ω.

2. Now, we introduce both approximating sets Ω_n and the approximate boundary data $f_n \in C(\partial\Omega_n)$. Choose M_n to be a sequence of positive numbers such that $M_n \geq M_{n-1} + 2$. We set f_n to be a continuous function with compact support such that $f_n = f$ in $\partial\Omega \cap (H_- + M_{n-1}\nu)$, $f_n = 0$ in $\partial\Omega\backslash(H_- + (M_{n-1} + 1)\nu)$ and (by a variant of Tietze extension theorem) as a continuous function with values in the line segment $[-f(x), f(x)]$ in $\partial\Omega \cap ((H_- + (M_{n-1} + 1)\nu)\backslash(H_- + M_{n-1}\nu))$. In particular, the sequence f_n converges to f locally uniformly and the sequence $|f_n|$ is monotone and converges locally uniformly to $|f|$.

3. Let Ω_n be an increasing sequence of strictly convex sets such that $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$ and such that $\Omega_n \cap (H_- + M_n\nu) = \Omega \cap (H_- + M_n\nu)$. The construction is shown in Figure 5.1; the shaded set is Ω_1. We may consider f_n to be a function on $\partial\Omega_n$ via a simple identification: let $\widetilde{f_n} \in C(\partial\Omega_n)$ be defined as $\widetilde{f_n} = f_n$ on $\partial\Omega_n \cap \partial\Omega$ and $\widetilde{f_n} = 0$ on $\partial\Omega_n\backslash\partial\Omega$. Using the Sternberg-Williams-Ziemer construction we obtain a minimiser $u_n \in BV(\Omega_n)$ of the least gradient problem in Ω_n with boundary data $\widetilde{f_n}$. By [31, Proposition 4.1] the restriction of u_n to Ω_m with $m \leq n$ also lies in $BV(\Omega_m)$ and is a function of least gradient.

4. For every m, we need to show a uniform bound in $BV(\Omega_m)$ for the sequence of approximations in order to obtain existence of a limit function in the topology of $L^1(\Omega_m)$. Let $n > m$ and consider the restrictions $u_n|_{\Omega_m}$. Using Lemmata 5.2.2 and 5.2.3, we calculate (the integrals on $\partial\Omega$, $\partial\Omega_m$ or their subsets are taken with respect to the Hausdorff measure of codimension one):

$$\|u_n\|_{BV(\Omega_m)} = \int_{\Omega_m} |u_n|\, dx + \int_{\Omega_m} |Du_n| \leq (C(\Omega_m) + 1)(\int_{\Omega_m} |Du_n| + \int_{\partial\Omega_m} |Tu_n|) \leq$$

$$\leq 2(C(\Omega_m) + 1) \int_{\partial\Omega_m} |Tu_n| = 2C(\Omega_m)(\int_{\partial\Omega_m \cap \partial\Omega} |Tu_n| + \int_{\partial\Omega_m \backslash \partial\Omega} |Tu_n|) \leq$$

$$= 2(C(\Omega_m) + 1)(\int_{\partial\Omega_m \cap \partial\Omega} |f| + \int_{\partial\Omega_m \backslash \partial\Omega} \sup_{\Omega_m} |u_n|) \leq \widetilde{C}(\Omega_m) \sup_{\partial\Omega \cap (H_- + M_{n+1}\nu)} |f|.$$

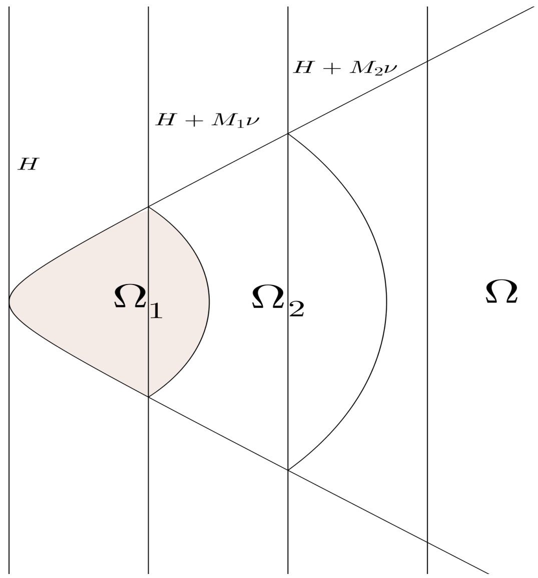


Figure 5.1: The construction of the approximating sets Ω_n

Hence the sequence $u_n|_{\Omega_m}$ is bounded in $BV(\Omega_m)$ and admits a convergent subsequence in $L^1(\Omega_m)$ and almost everywhere. Using a diagonalisation argument, we obtain existence of a subsequence u_{n_k} (extended by zero outside Ω_{n_k}) convergent to $u \in BV_{loc}(\Omega)$ in $L^1_{loc}(\Omega)$ and almost everywhere. By Theorem 2.1.4 u is a function of least gradient in Ω.

5. We have to ensure that $Tu = f$. We proceed as in Section 4.4 or as in the proof of Theorem 5.3.1; since f is continuous, we fix any point $x_0 \in \partial\Omega$ such that the mean integral condition in the definition of trace is satisfied. For every $\delta > 0$ we find a ball $B(x_0, r(\delta))$ such that every u_n satisfies

$$f(x_0) - \delta \leq u_n(x) \leq f(x_0) + \delta \qquad \text{in } B(x_0, r(\delta)) \cap \Omega.$$

Hence, as in the proof of Theorem 5.3.1, u satisfies the same inequalities in $\Omega \cap B(x_0, r'(\delta))$ and so $Tu(x_0) = f(x_0)$. Since this holds for almost all $x_0 \in \partial\Omega$, $Tu = f$. □

Corollary 5.4.3. *Let $\Omega \subset \mathbb{R}^N$ be a strictly convex set and $\Omega \neq \mathbb{R}^N$. Let $f \in L^\infty_{loc}(\partial\Omega)$ such that $\mathcal{H}^{N-1}$-almost all points of $\partial\Omega$ are continuity points of f. Then there exists a minimiser $u \in BV_{loc}(\Omega)$ of Problem* (uLGP) *with boundary data f.*

Proof. The proof of Theorem 5.4.2 requires only a few modifications. We choose the sequence Ω_n in the same way and construct the approximating sequence f_n in a simpler way: we extend f by zero on $\partial\Omega_n \backslash \partial\Omega$. The resulting function $f_n \in L^\infty(\partial\Omega_n)$ is such that $\mathcal{H}^{N-1}$-almost all points of $\partial\Omega_n$ are continuity points of f. By Theorem 5.3.1 there exists a minimiser to Problem (LGP) $u_n \in BV(\Omega_n)$. Now, we notice that the uniform estimates in Step 4 depend only on the local L^∞ bounds on f and not on its continuity; finally, we repeat Step 5 only for continuity points of f. □

Remark 5.4.4. The results above hold with minor changes also for norms other than the Euclidean norm; since strictly convex sets satisfy the barrier condition, we may perform the approximation as above. However, when ϕ depends on location, the greatest difficulty is to ensure that the barrier condition is satisfied; this is not immediate and depends on the exact form of ϕ. If $\partial\Omega \in C^2$, this boils down to solving a degenerate elliptic equation, see [36, Lemma 3.2]. Moreover, we need condition (H) to hold in order to use a comparison principle as in the proof of Theorem 5.3.6 to prove that the trace of the minimiser is correct.

5.4.2 Regularity

In this subsection, we discuss the main regularity features of minimisers in the least gradient problem on unbounded domains. The main result in this subsection, Proposition 5.4.5, concerns continuity of minimisers. While minimisers to Problem (LGP) for continuous boundary data are continuous up to the boundary if the domain Ω is bounded, it is not necessarily obvious here - our approximation procedure provides no estimates which enable us to prove that the convergence $u_n \to u$ is locally uniform. Furthermore, this approach would only prove continuity of the one minimiser obtained using the approximation procedure; as we will see in Example 5.4.15, minimisers may fail to be unique. Proposition 5.4.5 goes around both of these problems. In order to prove it, we will rely on the maximum principle for minimal surfaces (Proposition 2.4.4). Now, we state our main regularity result for least gradient functions on unbounded domains.

Proposition 5.4.5. *Let $\Omega \subset \mathbb{R}^N$ be a strictly convex set and $\Omega \neq \mathbb{R}^N$. Let $f \in C(\partial\Omega)$. Suppose that $u \in BV_{loc}(\Omega)$ is a minimiser of Problem* (uLGP) *with boundary data f. Then $u \in C(\overline{\Omega})$.*

For a function $u \in L^1_{loc}(\Omega)$ we recall the notion of lower and upper approximate limits $u^\wedge(x)$ and $u^\vee(x)$:

$$u^\wedge(x) = \sup\{t \in \mathbb{R} : \lim_{r\to 0^+} \frac{\mathcal{L}^n(\{u \geq t\} \cap B(x,r))}{\mathcal{L}^n(B(x,r))} = 1\},$$

$$u^\vee(x) = \inf\{t \in \mathbb{R} : \lim_{r\to 0^+} \frac{\mathcal{L}^n(\{u \leq t\} \cap B(x,r))}{\mathcal{L}^n(B(x,r))} = 1\}.$$

We say that $x \in S_u$, the approximate discontinuity set of u, if $u^\wedge(x)$ and $u^\vee(x)$ do not coincide (see for instance [3, Chapter 3], [33, Section 2]).

Proof. Denote $E_t = \{u \geq t\}$. Let $x \in \Omega$ be a point such that u is not continuous at x. By [33, Theorem 4.1] we have $x \in S_u$. By the definition of S_u, $x \in \partial E_t$ for each $t \in (u^\wedge(x), u^\vee(x))$. As for $s > t$ we have $E_s \subset E_t$, by Proposition 2.4.4 the connected components of ∂E_t for $t \in (u^\wedge(x), u^\vee(x))$ passing through x agree; we will denote this surface by S.

We will see that $S \cap \partial\Omega = \emptyset$. Suppose the contrary and take $z_0 \in S \cap \partial\Omega$. Fix $\varepsilon > 0$. Then, by continuity of f, in a neighbourhood of z_0 we have

$$f(z_0) - \varepsilon \leq f(z) \leq f(z_0) + \varepsilon.$$

Using the same argument as in Steps 6 and 7 of the proof of Theorem 5.3.1, we obtain that for $y \in B(z_0, r(\varepsilon)) \cap \Omega$ we have

$$f(z_0) - \varepsilon \leq u(y) \leq f(z_0) + \varepsilon.$$

However, since $z_0 \in S \cap \partial\Omega$, in any neighbourhood of z_0 in Ω there are values of u greater or equal to $u^\vee(x)$ and smaller or equal to $u^\wedge(x)$. This contradicts the estimates above for sufficiently small ε.

Since $S \cap \partial\Omega = \emptyset$, S has empty boundary. Since S is area-minimising, by representing it as an integral current and appealing to the general theory or area-minimising integral currents we can see that there exists a set E of locally finite perimeter such that $\partial E = S$; see for instance [63, Theorem 27.6]. This implies that S cannot be bounded: otherwise we would modify E_t with compactly supported variation by taking either $E_t \cup E$ or $E_t \backslash E$ and obtain a set with smaller perimeter, contradiction with minimality of E_t for $t \in (u^\wedge(x), u^\vee(x))$ given by Theorem 2.1.3. Hence S is unbounded; but then we could replace S by a truncated S (in the notation of the proof of Theorem 5.4.2, using a projection of S onto a plane $H + s\nu$) and obtain a surface with smaller area. Hence for $t \in (u^\wedge(x), u^\vee(x))$ the set E_t is not minimal, which contradicts Theorem 2.1.3.

Now, take $x \in \partial\Omega$. Then, as u is continuous inside Ω, the essential supremum is the same as supremum and we see that

$$\lim_{r \to 0} \text{ess sup}_{y \in B(x,r) \cap \Omega} |u(y) - f(x)| = \lim_{r \to 0} \sup_{y \in B(x,r) \cap \Omega} |u(y) - f(x)| = 0,$$

hence u is continuous at x. □

A careful inspection of the proof above yields a generalisation of [65, Lemma 3.3]:

Corollary 5.4.6. *Let $\Omega \subset \mathbb{R}^N$ be a strictly convex set and $\Omega \neq \mathbb{R}^N$. Suppose that $u \in BV_{loc}(\Omega)$ is a least gradient function with trace $f \in L^1_{loc}(\partial\Omega)$. Then for each $t \in \mathbb{R}$ we have an inclusion*

$$\partial\{u \geq t\} \cap \partial\Omega \subset f^{-1}(t) \cup D,$$

where D is the set of discontinuity points of f. □

Remark 5.4.7. In the anisotropic setting, the assumptions concerning ϕ in the continuity proof are more restrictive than in the existence proof. The main problem is that we use Proposition 2.4.4. Hence, Proposition 5.4.5 is valid in the anisotropic case in two settings: firstly, when $\Omega \subset \mathbb{R}^2$ and ϕ is a strictly convex norm, the connected components of $\phi-$area-minimising sets are line segments and we do not need to use the maximum principle. Secondly, when an anisotropic version of the maximum principle holds, for instance for weighted least gradient problem, where $\phi(x, Du) = a(x)|Du|$ with $a \in C^2(\overline{\Omega})$, see [71, Theorem 3.1].

However, contrary to the results in the bounded domain case, we cannot expect Hölder continuity of minimisers. This is due to the fact that $\partial\Omega$ becomes asymptotically flat as $|x| \to \infty$; it means that the regularity of minimisers at infinity is the same as near a point where mean curvature vanishes and its growth rate is slower than polynomial. For the fact that in the neighbourhood of such points we may lose Hölder continuity, see [65, Remark 5.8].

Now, we will see that integrability of the minimiser and its total variation can only happen under very special circumstances, both in terms of the regularity of boundary data and the shape of the domain.

Proposition 5.4.8. *Let $\Omega \subset \mathbb{R}^N$ be a domain that is unbounded only in one direction and such that its cross-sectional area is uniformly bounded. Suppose that $u \in BV_{loc}(\Omega)$ is a minimiser to Problem* (uLGP) *with boundary data $f \in L^1(\partial\Omega) \cap L^\infty(\partial\Omega)$. Then $u \in BV(\Omega)$.*

Proof. We are going to utilise the Poincaré inequality. Firstly, let us see that the Poincaré inequality holds in Ω for each Ω_m, which is Ω cut at level m by a hyperplane Π_m perpendicular to the direction in which the domain is unbounded, uniformly in m - the constant in the inequality depends only on the width d of the strip. Let us denote $\Gamma_m = \Pi_m \cap \Omega$; moreover, let A denote the cross-sectional area of Ω, namely $A = \sup_m \mathcal{H}^{N-1}(\Gamma_m)$.

Now, we use Lemma 5.2.3 to estimate (the integrals on $\partial\Omega$, $\partial\Omega_m$ and their subsets are taken with respect to the Hausdorff measure of codimension one)

$$\int_{\Omega_m} |u| \, dx \leq C(d)(\int_{\Omega_m} |Du| + \int_{\partial\Omega_m} |Tu|)$$

and so

$$\|u\|_{BV(\Omega_m)} \leq C(d) \int_{\partial\Omega_m} |Tu| + (C(d)+1) \int_{\Omega_m} |Du| \leq (2C(d)+1) \int_{\partial\Omega_m} |Tu| \leq$$

$$\leq (2C(d)+1)(\int_{\partial\Omega} |f| + \int_{\Gamma_m} |Tu|) \leq (2C(d)+1)(\|f\|_{L^1(\partial\Omega)} + A\|f\|_{L^\infty(\partial\Omega)}).$$

Hence the BV norm of u is uniformly bounded in each Ω_m by quantities which only depend on the width d, the cross-sectional area A, the supremum of $|f|$ and the L^1 norm of f. Thus

$$\|u\|_{BV(\Omega)} \leq \widetilde{C}(d)(\|f\|_{L^1(\partial\Omega)} + \|f\|_{L^\infty(\partial\Omega)}).$$

We point out that this proof did not require continuity of f, only the boundedness. □

If $f \notin L^1(\partial\Omega)$ or Ω is not one-dimensional, then we cannot hope that the minimiser u (if it exists) is in $BV(\Omega)$; let us see two simple examples:

Example 5.4.9. Let $\Omega \subset \mathbb{R}^2$ be defined as

$$\Omega = \{(x,y) : \, x > 0, \, e^{-x} - 1 \leq y \leq -e^{-x} + 1\}.$$

Now, we will construct the boundary data $f \in C_0(\partial\Omega)$. Let ρ be a standard mollifier on $\mathbb{R}$ with support in $(-1,1)$. Now, let f be defined by the formula

$$f(x,y) = \sum_{n=1}^{\infty} \frac{1}{n} \rho(x - n^2).$$

Then the minimiser to the least gradient problem given by Theorem 5.4.2 is

$$u(x,y) = \sum_{n=1}^{\infty} \frac{1}{n} \rho(x - n^2),$$

i.e. all level lines are vertical. However, $u \notin L^1(\Omega)$ and the total variation of u is infinite.

Example 5.4.10. Let $\Omega \subset \mathbb{R}^2$ be defined as

$$\Omega = \{(x,y) : \, x > 0, \, |y| \leq x^3\}.$$

Let $f(x,y) = \frac{1}{(x+1)^2}$. Then the minimiser in the least gradient problem exists and again all level lines are vertical and equals $u(x,y) = \frac{1}{(x+1)^2}$; however, $u \notin L^1(\Omega)$ and the total variation of u is infinite.

Finally, we will briefly discuss a new phenomenon that happens when $f \notin C_0(\partial\Omega)$ and is the reason behind the lack of uniqueness: formation of level sets which do not connect points in $\partial\Omega$, but instead escape to infinity. In particular, if u is a solution of Problem (uLGP), then there may exist connected components of $\partial\{u \geq t\}$ with infinite area. However, we will see that in dimension two there is at most one such connected component.

Example 5.4.11. Let $\Omega \subset \mathbb{R}^2$ be defined as

$$\Omega = \{(x,y) : \, x > 0, \, e^{-x} - 1 \leq y \leq -e^{-x} + 1\}.$$

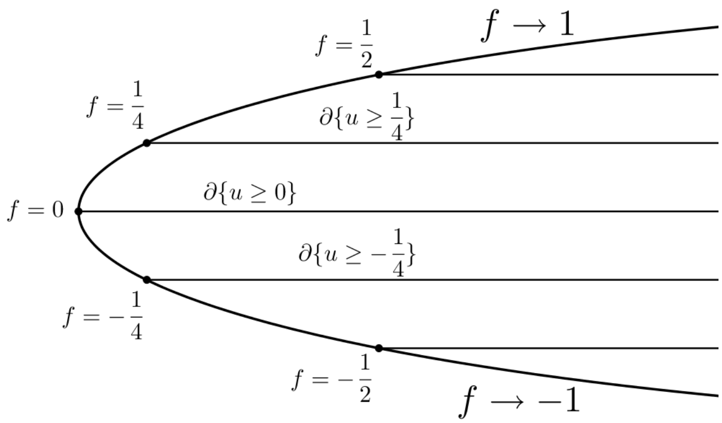


Figure 5.2: The level lines escape to infinity

We take the boundary data $f(x, y) = y$. Then on the lower branch of $\partial\Omega$ we have $f(x, y) \to -1$ as (x, y) tends to infinity and on the upper branch of $\partial\Omega$ we have $f(x, y) \to 1$ as (x, y) tends to infinity. Therefore for all $t \in (-1, 1)$ the set $\partial\{u \geq t\}$ is a halfline, going in the horizontal direction to the right, starting at a point of the form (x, t). The situation is presented in Figure 5.2.

Proposition 5.4.12. *Let $\Omega \subset \mathbb{R}^2$ be an open unbounded strictly convex set. Let $u \in BV_{loc}(\Omega)$ be a least gradient function. Then, for each $t \in \mathbb{R}$ there is at most one unbounded connected component of $\partial\{u \geq t\}$.*

Proof. Suppose that $\partial\{u \geq t\}$ has at least two unbounded connected components; in dimension two these are halflines with starting point on $\partial\Omega$. If these halflines are parallel, then we can replace them with a U-shaped polygonal chain consisting of two halflines and a line segment, locally reducing the total variation; the situation is presented on the left hand side of Figure 5.3. If we choose points p_2, q_2 sufficiently far from $\partial\Omega$, then the line segment $\overline{p_2 q_2}$ is shorter than the line segments $\overline{p_1 p_2}$ and $\overline{q_1 q_2}$ and hence the function $\chi_{\{u \geq t\}}$ was not a function of least gradient, which contradicts Theorem 2.1.3.

If these halflines are not parallel, they intersect in a point $r \notin \Omega$. Then we can replace them by another U-shaped polygonal chain if the line segment is far enough from r; the situation is presented on the right hand side of Figure 5.3. By the triangle inequality, the line segment $\overline{p_2 q_2}$ is shorter than the union of the line segments $\overline{p_2 r}$ and $\overline{r q_2}$. If we choose points p_2, q_2 sufficiently far from $\partial\Omega$, then the line segment $\overline{p_2 q_2}$ is also shorter than the union of the line segments $\overline{p_1 p_2}$ and $\overline{q_1 q_2}$. Hence the function $\chi_{\{u \geq t\}}$ was not a function of least gradient, which contradicts Theorem 2.1.3. $\square$

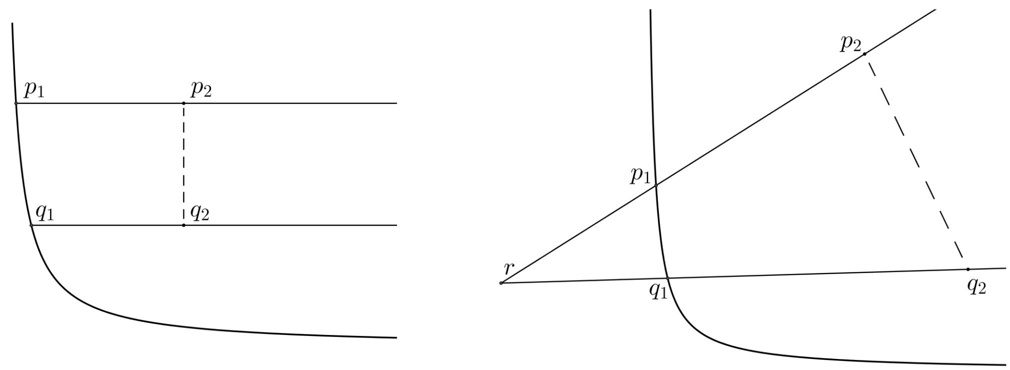


Figure 5.3: There is at most one level line escaping to infinity

5.4.3 Uniqueness of minimisers

It turns out that the natural space for uniqueness of minimisers is the space $C_0(\partial\Omega)$. In this space, we can infer the uniqueness of minimisers from the uniqueness of minimisers in bounded domains in a similar way as in the existence proof, only using a more careful approximation. If the boundary data is less regular than $C_0(\partial\Omega)$, we may construct several minimisers even for very simple boundary data.

Theorem 5.4.13. *Let $\Omega \subset \mathbb{R}^N$ be an unbounded strictly convex set and $\Omega \neq \mathbb{R}^N$. Let $f \in C_0(\partial\Omega)$. Then there exists a unique minimiser $u \in BV_{loc}(\Omega) \cap C(\overline{\Omega})$ of Problem* (uLGP) *with boundary data f.*

Proof. 1. We recall a bit of the Sternberg-Williams-Ziemer construction of minimisers for continuous boundary data and a bounded strictly convex set Ω' (see [65]). Let $g \in C(\partial\Omega')$. We take its extension $G \in C(\mathbb{R}^N \backslash \Omega') \cap BV(\mathbb{R}^N \backslash \Omega')$ with compact support and denote $L_t = \{G \geq t\}$. For almost all t, the superlevel sets $E_t = \{u \geq t\}$ are solutions of the problem

$$\min\{P(E, \mathbb{R}^N) : E \backslash \overline{\Omega'} = L_t \backslash \overline{\Omega'}\},$$

$$\max\{|E| : \; E \text{ is a solution of the above}\}.$$

The result does not depend on the choice of the extension G. In particular, the sets E_t are defined uniquely.

2. As in the proof of Theorem 5.4.2, fix any $x \in \partial\Omega$ and consider any supporting hyperplane H with inward normal vector ν. Let us take the halfspace H_-, which is disjoint with Ω and whose boundary is H. Again, the shifted halfspaces of the form $H_- + s\nu$ for $s > 0$ intersect Ω and their union is $\mathbb{R}^N$, so they cover the whole Ω. As $f \in C_0(\partial\Omega)$, for every n there exists M_n so that

$$|f(x)| \leq \frac{1}{n} \qquad \text{in } \partial\Omega \backslash (H_- + M_n \nu).$$

We may additionally require that $M_n \geq M_{n-1} + 2$. Let Ω_n be an increasing sequence of strictly convex sets such that $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$ and such that $\Omega_n \cap (H_- + M_n \nu) = \Omega \cap (H_- + M_n \nu)$. By the definition of M_n, for $t > \frac{1}{n}$ we have an inclusion

$$\{f \geq t\} \subset \partial\Omega \cap (H_- + M_n \nu).$$

3. By Proposition 5.4.5 the minimisers to Problem (uLGP) are continuous up to the boundary. Suppose that $u \in BV_{loc}(\Omega) \cap C(\overline{\Omega})$ and $v \in BV_{loc}(\Omega) \cap C(\overline{\Omega})$ are two minimisers to the least gradient problem on Ω. Using an argument with projections as in Step 7 of the proof of Theorem 5.3.1, we obtain that for $t > \frac{1}{n}$

$$\{u \geq t\} \subset \Omega \cap (H_- + M_n \nu) \qquad \text{and} \qquad \{v \geq t\} \subset \Omega \cap (H_- + M_n \nu).$$

Let us consider the restrictions of u and v to $\overline{\Omega_n}$. Both functions are continuous, hence u solves the least gradient problem on a bounded strictly convex domain Ω_n with boundary data

$$\widetilde{u} = \begin{cases} f & \text{on } \partial\Omega \cap \partial\Omega_n, \\ u & \text{on } \partial\Omega_n \backslash \partial\Omega. \end{cases}$$

Analogously v solves the least gradient problem on Ω_n for boundary data

$$\widetilde{v} = \begin{cases} f & \text{on } \partial\Omega \cap \partial\Omega_n, \\ v & \text{on } \partial\Omega_n \backslash \partial\Omega. \end{cases}$$

4. As $\widetilde{u}$ and $\widetilde{v}$ agree on $\partial\Omega \cap \partial\Omega_n$, we can choose their extensions $\widetilde{U}, \widetilde{V} \in C(\mathbb{R}^N \backslash \Omega_n) \cap BV(\mathbb{R}^N \backslash \Omega_n)$ to agree in a neighbourhood of $\partial\Omega \cap (H_- + M_n \nu)$. Moreover, by continuity the functions $\widetilde{U}, \widetilde{V}$ are less or equal to $t > \frac{1}{n}$ in a neighbourhood of $\partial\Omega \cap \partial\Omega_n$.

5. Pick t such that both $\{u \geq t\}$ and $\{v \geq t\}$ solve the problem from the Sternberg-Williams-Ziemer construction for boundary data $\widetilde{u}$ and $\widetilde{v}$ respectively on $\partial\Omega_n$. The set $\{u \geq t\}$ is determined only by the set $\{\widetilde{U} \geq t\}$. However, it agrees with the set $\{\widetilde{V} \geq t\}$ in a neighbourhood of Ω_n, as $\widetilde{U} = \widetilde{V}$ in a neighbourhood of $\partial\Omega \cap (H_- + M_n \nu)$ and both sets are empty in a neighbourhood of $(\partial\Omega_n \backslash \partial\Omega) \subset \Omega \backslash (H_- + M_n \nu)$. By the uniqueness of the set resulting from the Sternberg-Williams-Ziemer procedure we have that $\{u \geq t\} = \{v \geq t\}$. Hence, for almost all $t > \frac{1}{n}$ we have $\{u \geq t\} = \{v \geq t\}$. We proceed similarly for $t < -\frac{1}{n}$. Hence almost all superlevel sets of u and v are uniquely determined and equal, so $u = v$ almost everywhere. □

Remark 5.4.14. Clearly, the Proposition above holds also in the case when $f \in C(\partial\Omega)$ and f has a finite limit f_0 as $|x| \to \infty$. Furthermore, it holds also if the limit is infinite; if $f \to +\infty$ as $|x| \to \infty$, then we have to choose the sequence M_n in Step 1 so that

$$f(x) \geq n \qquad \text{in } \partial\Omega \backslash (H_- + M_n \nu)$$

and continue the proof as above.

Example 5.4.15. Let $\Omega \subset \mathbb{R}^2$ be defined as

$$\Omega = \{(x, y) : \, x > 0, \, y > 0, \, xy > 1\}.$$

Let the boundary data equal $f(x,y) = e^{-x}$. The boundary data is monotone, $f \in C_b(\partial\Omega)$ and it has finite limits at infinity.

Then the functions $u_1, u_2 : \Omega \to \mathbb{R}$ defined by $u_1(x,y) = e^{-x}$ and $u_2(x,y) = e^{-\frac{1}{y}}$ with boundary data f are functions of least gradient. The first one has all level lines vertical and the second one has all level lines horizontal. Moreover, each value is attained only on one half-line; if we take any strictly convex $\Omega' \subset \Omega$, then u_i restricted to $\partial\Omega'$ has only one minimum and one maximum. Hence both are functions of least gradient and the minimisers to Problem (uLGP) are not unique. This situation is presented in Figure 5.4. We note that we could obtain an uncountable family of minimisers by choosing the angles of incidence of the level lines. In particular, even though f is monotone, we can obtain a least gradient function having level sets of positive (infinite) measure if we set for $x_0 \in \mathbb{R}_+$

$$u_3 = \begin{cases} e^{-x} & \text{if } |x| < x_0, \\ e^{-\frac{1}{y}} & \text{if } |y| < \frac{1}{x_0}, \\ e^{-x_0} & \text{otherwise.} \end{cases}$$

Also note that not all level lines have to be in the same direction; for instance, consider the function $u_4 = e^{-\sqrt{\frac{x}{y}}}$. It is of the form $u(x,y) = g(y/x)$, hence it is also a function of least gradient. These cases are shown in Figure 5.5.

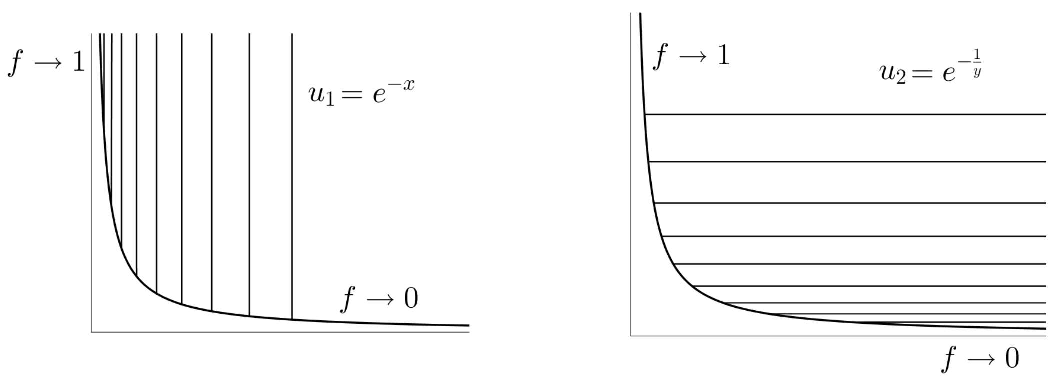


Figure 5.4: Nonuniqueness of minimisers for data in $C_b(\partial\Omega)$

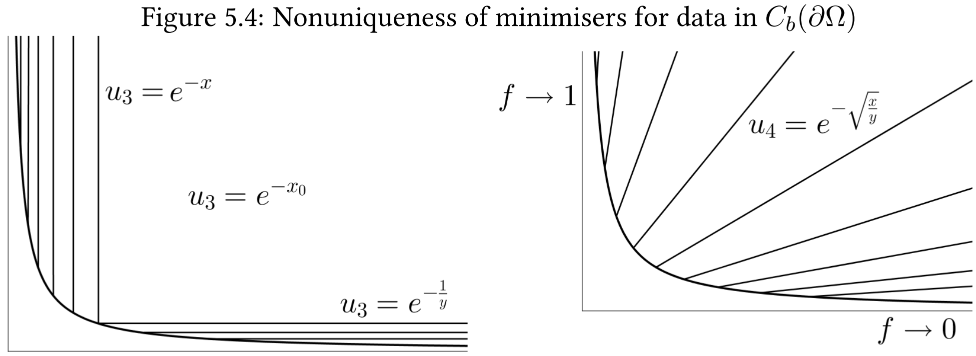


Figure 5.5: Nonuniqueness of minimisers for data in $C_b(\partial\Omega)$

The reason why uniqueness fails in the example above is that the set Ω is not unbounded in only one direction, but it contains a cone. Therefore the level lines could "choose" a direction in which the solution propagates; this is the primary motivation for the considerations in the next subsection. Due to this phenomenon, the above example implies that in the unbounded case the structure theorems for least gradient functions, such as Theorem 3.1.1 and [53, Theorem 1.1], cannot hold - there exist multiple continuous functions which solve the least gradient problem and their derivatives do not agree on a set of positive Lebesgue measure.

Remark 5.4.16. As we rely in the uniqueness proof on the Sternberg-Williams-Ziemer construction, it generalises to the anisotropic setting in the same cases as in the continuity proof - when $\Omega \subset \mathbb{R}^2$ and ϕ is a strictly convex norm, and for weighted least gradient problem, when $\phi(x, Du) = a(x)|Du|$ with $a \in C^2(\overline{\Omega})$. The reason is that uniqueness provided by the Sternberg-Williams-Ziemer construction stems from the maximum principle for minimal surfaces, which has been established in the case of weighted least gradient problem in [71].

5.4.4 The case when $\Omega = \mathbb{R}^N$

When we consider unbounded domains, in previous subsections we only discussed boundary conditions on $\partial\Omega$, disregarding the limit behaviour. In this subsection, we are interested in the case when we impose the limit behaviour at infinity. We consider the following problem: given a function $f \in L^1(\partial B(0,1))$, we want to find a least gradient function $u \in BV_{loc}(\mathbb{R}^N)$ on $\mathbb{R}^N$ such that for $\mathcal{H}^{N-1}$-a.e. $x \in \partial B(0,1)$

$$u(tx) \to f(x) \qquad \text{as } t \to \infty.$$

Let us first focus on $\mathbb{R}^2$.

Proposition 5.4.17. *Let $\Omega = \mathbb{R}^2$. Then, all the least gradient functions in $\mathbb{R}^2$ are one-dimensional. In particular, the only limit values of u at infinity are of the form $f = a\chi_{\Gamma_1} + b\chi_{\Gamma_2}$, where Γ_1, Γ_2 denote any two non-intersecting half-circles in $\partial B(0,1)$, i.e. $\Gamma_1 \cup \Gamma_2 = \partial B(0,1)$.*

Proof. Suppose that $u \in BV_{loc}(\mathbb{R}^2)$ is a function of least gradient. Fix any $t \in \mathbb{R}$. By Theorem 2.1.3 the function $\chi_{\{u \geq t\}}$ is also a function of least gradient. Using the regularity theory for minimal sets proved by Giusti in [25], we obtain that each connected component of $\partial\{u \geq t\}$ is in fact a smooth minimal surface. In two dimensions, it means that $\partial\{u \geq t\}$ is a union of at most countably many parallel lines. However, we easily see that this union contains only one element - if it contained more than one element, we could replace any two of them with two U-shaped polygonal chains, which would have locally smaller total variation (analogously to what is presented in Figure 5.3).

Hence for each $t \in \mathbb{R}$ we have that $\partial\{u \geq t\}$ is either empty of $\partial\{u \geq t\} = l_t$. Furthermore, as for $t > s$ we have $\{u \geq t\} \subset \{u \geq s\}$, all of these lines are parallel to a line l passing through the origin. Then u is a function of one variable z, defined along the line l' passing through the origin and perpendicular to l. The orientation of the line l' is chosen so that z is growing in the direction of angles from the interval $[\frac{\pi}{2}, \frac{3\pi}{2})$.

Let us denote the angle of incidence of l by α_0. Let $u(z) \to a$ as $z \to \infty$ and $u(z) \to b$ as $z \to -\infty$. Then, as for $\alpha \neq \alpha_0$ and $\alpha \neq \alpha_0 + \pi$ any ray starting from the origin has to intersect all the lines l_t, the limit value of u at infinity is

$$f(x,y) = f(\alpha) = \begin{cases} a & \text{if } \alpha \in (\alpha_0, \alpha_0 + \pi), \\ b & \text{if } \alpha \in (\alpha_0 + \pi, \alpha_0 + 2\pi). \end{cases}$$

Note that here, with a slight abuse of notation, at most one of a, b may be equal to $+\infty$ and at most one of a, b may be equal to $-\infty$. □

For a discussion in $\mathbb{R}^3$ we will need an additional result. It comes from the classical theory of minimal surfaces and is called the strong halfspace theorem, proved by Hoffman and Meeks in [35].

Theorem 5.4.18. ([35, Theorem 2]) *Two proper, possibly branched, connected minimal surfaces in $\mathbb{R}^3$ must intersect, unless they are parallel planes.*

Proposition 5.4.19. *Let $\Omega = \mathbb{R}^3$. Then, all the least gradient functions in $\mathbb{R}^3$ are either one-dimensional or have a single jump along a smooth minimal surface S. In particular, the only limit values of u at infinity are of the form $f = a\chi_{\Gamma_1} + b\chi_{\Gamma_2}$, where Γ_1, Γ_2 denote any two non-intersecting half-spheres in $\partial B(0,1)$, i.e. $\Gamma_1 \cup \Gamma_2 = \partial B(0,1)$.*

Proof. Suppose that $u \in BV_{loc}(\mathbb{R}^3)$ is a function of least gradient. Fix any $t \in \mathbb{R}$. As in the proof of the previous Proposition, we obtain that each connected component of $\partial\{u \geq t\}$ is a smooth properly embedded minimal surface. By Theorem 5.4.18 there is only one connected component of $\partial\{u \geq t\}$ unless it is a union of parallel planes; however, we argue as in the proof of the previous Proposition that in that case $\partial\{u \geq t\}$ is a single plane.

Now, consider $t > s$. As $\{u \geq t\} \subset \{u \geq s\}$, if $\partial\{u \geq t\} \cap \partial\{u \geq s\}$, then Proposition 2.4.4 combined with the previous observation shows that these sets agree. Therefore, if for some $t \in \mathbb{R}$ the surface $S = \partial\{u \geq t\}$ is not a plane, then for any $s \in \mathbb{R}$, by Theorem 5.4.18 $\partial\{u \geq s\}$ intersects S and therefore agrees with it; hence the function u has a single jump across S and is locally constant in $\mathbb{R}^3 \backslash S$.

Now, we discuss the limit behaviour at infinity. If all sets of the form $\partial\{u \geq t\}$ are planes, then they are parallel to a plane Π passing through the origin and we proceed as in the proof of the previous Proposition to get that f is a function with two values which are obtained on two disjoint halfspheres. If the function u has only two values and a single jump across a smooth properly embedded minimal surface S, then recall that S admits a limit tangent plane at infinity (for instance in the sense of [49, Definition 6.1]) denoted by Π (by definition, it passes through the origin); let us take any ray starting from the origin such that its direction does not lie in Π. Then the value of u along that ray stabilises. Thus, the limit value of u at infinity is

$$f(x,y,z) = \begin{cases} a & \text{in a halfsphere above } \Pi \\ b & \text{in a halfsphere below } \Pi. \end{cases}$$

Again, with a slight abuse of notation at most one of a, b may be equal to $+\infty$ and at most one of a, b may be equal to $-\infty$. □

In higher dimensions the situation is much less clear. There are two main reasons for this: firstly, we do not have the halfspace theorem and therefore there may exist continuous least gradient functions which are not one-dimensional. Moreover, in dimensions eight and above there exist least gradient functions such that $\partial\{u \geq t\}$ may have singularities. To illustrate this, we recall the construction developed in [9].

Example 5.4.20. (1) Let $\Omega = \mathbb{R}^8$. Let $C \subset \mathbb{R}^8$ denote the interior of the Simons' cone, namely

$$C = \{x_1^2 + x_2^2 + x_3^2 + x_4^2 \geq x_5^2 + x_6^2 + x_7^2 + x_8^2\}.$$

Then $u = \chi_C$ is a function of least gradient. However, the limit values of u at infinity equal $f = \chi_{C \cap \partial B(0,1)}$, which is not constant on halfspheres. Moreover, the authors of [9] construct (in the proof of Theorem A) a continuous function F of least gradient such that the Simons' cone is its zero level set. In particular, this function is not one-dimensional neither has any jumps.

(2) Let $N \geq 9$. Then, as was also shown in [9], the answer to the Bernstein problem is positive and there exist entire complete analytic minimal graphs in $\mathbb{R}^N$ which are not hyperplanes. Hence, we can construct a function of least gradient such that its level sets are translations of a single Bernstein graph; in particular, this function is not one-dimensional and we can ensure that is has no jumps.

Overall, these results suggest that the formulation of the Dirichlet problem with boundary conditions at infinity is not the proper way to introduce the least gradient problem on unbounded domains, due to the fact that the problem in this formulation does not have any solutions in low dimensions, unless the prescribed data has a very specific form. Hence, the more proper approach is to restrict ourselves to strictly convex sets Ω which are not equal to the whole of $\mathbb{R}^N$ and consider the Dirichlet boundary data only on $\partial\Omega$. Furthermore, there is no need to impose boundary conditions at infinity for such sets, because the limit behaviour is regulated by the Dirichlet data on $\partial\Omega$.

Chapter 6

L^p regularity of least gradient functions

6.1 Introduction

In the least gradient problem, even in the general context of metric measure spaces, see [33, Section 5], we have a maximum principle: L^∞ bound on the boundary data gives an immediate L^∞ bound on the solution. However, under suitable regularity assumptions on ϕ, the results in Chapter 5 yield existence of solutions to Problem (aLGP) also for unbounded boundary data, provided that their discontinuity set has $\mathcal{H}^{N-1}$-measure zero. In this case, the direct method gives no regularity estimates for the solutions.

This chapter is devoted to proving L^p estimates on $\phi-$least gradient functions in terms of the integrability of their boundary data. It is organised as follows: Section 6.2 is devoted to proving the main result of this chapter, i.e. Theorem 6.2.2, which concerns $L^{\frac{Np}{N-1}}$ regularity of solutions to the anisotropic least gradient problem for boundary data which lie in $L^p(\partial\Omega)$, using an argument based on the isoperimetric inequality. Moreover, in Example 6.2.5 we see that the exponent $\frac{Np}{N-1}$ is optimal.

Let us stress that in this Chapter we do not discuss existence or uniqueness of minimisers; here, given a minimiser of Problem (aLGP), we prove an estimate of its $L^{\frac{Np}{N-1}}$ norm. For this reason, we only assume Ω to be an open bounded set with Lipschitz boundary; we do not impose geometric assumptions on Ω sufficient to obtain existence of minimisers. However, we have an indirect assumption that the set Ω and the function f support at least one solution to the anisotropic least gradient problem.

In Section 6.3 we prove that solutions to the anisotropic least gradient problem are locally bounded. This is done in two settings: firstly, in $\mathbb{R}^2$ in the anisotropic case, using a characterisation of one-dimensional integral currents; secondly, in the isotropic least gradient problem in any dimension, using the monotonicity formula for area-minimising boundaries.

This Chapter is based on the article [30], of which I am the sole author and which has been accepted for publication in the Proceedings of the American Mathematical Society.

6.2 L^p regularity

In this Section, we prove $L^{\frac{Np}{N-1}}$ regularity of least gradient functions for L^p boundary data. The exponent we obtain is consistent with the exponent in the inclusion $BV(\Omega) \subset L^p(\Omega)$ for $p \leq \frac{N}{N-1}$; at the end of the Section, we provide an example that this estimate is optimal. The following Theorem is valid without any regularity assumptions on the metric integrand ϕ. The first result in this Section is an estimate on the Lebesgue measure of a superlevel set of a function of ϕ-least gradient. Then, we will prove that this estimate implies Theorem 6.2.2, which is the main result in this Section.

Lemma 6.2.1. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary. Let $u \in BV(\Omega)$ be a ϕ-least gradient function such that $Tu = f$. Then for almost all $t \in \mathbb{R}$ we have*

$$|\{u \geq t\}| \leq C(\phi, N)(\mathcal{H}^{N-1}(\{f \geq t\}))^{\frac{N}{N-1}}.$$

Proof. Denote $E_t = \{u \geq t\}$. Recall the isoperimetric inequality: if $E \subset \mathbb{R}^N$ is a bounded set of finite perimeter, then (see for instance [20, Theorem 5.6.2])

$$|E|^{\frac{N-1}{N}} \leq C_N P(E, \mathbb{R}^N).$$

We want to use the isoperimetric inequality to estimate the Lebesgue measure of the set E_t. To this end, as E_t is defined as a superlevel set of u and hence a subset of Ω, we firstly have to estimate $P(E, \mathbb{R}^N)$. We recall that (see for instance [20, Theorem 5.4.1]) if Ω is an open bounded set with Lipschitz boundary and $u_1 \in BV(\Omega)$ and $u_2 \in BV(\mathbb{R}^N \backslash \overline{\Omega})$, then the extension $\widetilde{u} = u_1 \chi_\Omega + u_2 \chi_{\mathbb{R}^N \backslash \overline{\Omega}}$ lies in $BV(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |D\widetilde{u}| = \int_\Omega |Du_1| + \int_{\mathbb{R}^N \backslash \overline{\Omega}} |Du_2| + \int_{\partial\Omega} |Tu_1 - Tu_2| \, d\mathcal{H}^{N-1}.$$

We use this result with $u_1 = \chi_{E_t}$ and $u_2 = 0$ to estimate $P(E_t, \mathbb{R}^N)$. For almost all t, so that the statements of Theorem 2.3.3 and Lemma A.0.4 hold, we calculate

$$P(E_t, \mathbb{R}^N) = P(E_t, \Omega) + 0 + \int_{\partial\Omega} |T\chi_{E_t}| \, d\mathcal{H}^{N-1} = P(E_t, \Omega) + \mathcal{H}^{N-1}(\{f \geq t\}),$$

where the last equality follows from Lemma A.0.4. Now, we recall that by Theorem 2.3.3 χ_{E_t} is a function of ϕ-least gradient for almost all $t \in \mathbb{R}$. By Lemma 5.2.2

$$P(E_t, \Omega) \leq \lambda^{-1} P_\phi(E_t, \Omega) = \lambda^{-1} \int_\Omega |D\chi_{E_t}|_\phi \leq \lambda^{-1} \int_{\partial\Omega} \phi(x, \nu^\Omega) \, |T\chi_{E_t}| \, d\mathcal{H}^{N-1} \leq$$

$$\leq \lambda^{-1} \Lambda \int_{\partial\Omega} |T\chi_{E_t}| \, d\mathcal{H}^{N-1} = \lambda^{-1} \Lambda \mathcal{H}^{N-1}(\{f \geq t\}),$$

where in the last equality we use Lemma A.0.4. Hence,

$$P(E_t, \mathbb{R}^N) \leq (\lambda^{-1}\Lambda + 1)\mathcal{H}^{N-1}(\{f \geq t\})$$

and by isoperimetric inequality we obtain

$$|E_t|^{\frac{N-1}{N}} \leq C_N(\lambda^{-1}\Lambda + 1)\mathcal{H}^{N-1}(\{f \geq t\}).$$

We take both sides of this inequality to the power $\frac{N}{N-1}$ to obtain the desired inequality with $C(\phi, N) = (C_N(\lambda^{-1}\Lambda + 1))^{\frac{N}{N-1}}$. □

Theorem 6.2.2. *Suppose that $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary. Suppose that $1 \leq p < \infty$. Let $u \in BV(\Omega)$ be a ϕ-least gradient function such that*

$$Tu = f \in L^p(\partial\Omega).$$

Then, $u \in L^{\frac{Np}{N-1}}(\Omega)$.

Proof. Denote $q = \frac{Np}{N-1}$. Let us decompose u into a positive and negative part, i.e. $u = u_+ - u_-$, where $u_+ = \max(u, 0)$ and $u_- = \max(-u, 0)$. Let $f = f_+ - f_-$ be an analogous decomposition for f. We will prove that $u_+ \in L^q(\Omega)$ and at the end remark how to modify this proof to show that also $u_- \in L^q(\Omega)$.

Firstly, we recall that for any measure space (X, μ) we have an inclusion

$$L^p(X, \mu) \subset L^p_w(X, \mu),$$

where $L^p_w(X, \mu)$ denotes the weak Lebesgue space, and that the seminorm $\|g\|_{L^p_w(X,\mu)}$ is bounded by the norm $\|g\|_{L^p(X,\mu)}$. In other words, for all $t > 0$

$$\mu(\{|g| \geq t\}) \leq \frac{\|g\|^p_{L^p(X,\mu)}}{t^p}.$$

We apply this to $(X, \mu) = (\partial\Omega, \mathcal{H}^{N-1})$, $g = f_+ \in L^p(\partial\Omega)$ and take both sides of the inequality to power $\frac{1}{N-1}$ to obtain that for all $t > 0$

$$(\mathcal{H}^{N-1}(\{f_+ \geq t\}))^{\frac{1}{N-1}} \leq \frac{\|f_+\|^{\frac{p}{N-1}}_{L^p(\partial\Omega)}}{t^{\frac{p}{N-1}}}. \tag{*}$$

Denote $E_t = \{u \geq t\}$. We immediately see that for $t > 0$ we have $\{u_+ \geq t\} = E_t$; hence, by Theorem 2.3.3 almost every superlevel set of u_+ is area-minimising. We calculate

$$\int_\Omega (u_+)^q \, dx = q \int_0^\infty t^{q-1} |\{u_+ \geq t\}| \, dt = q \int_0^\infty t^{q-1} |E_t| \, dt.$$

Now, we use Lemma 6.2.1 to estimate the last integral.

$$q \int_0^\infty t^{q-1} |E_t| \, dt \overset{(6.2.1)}{\leq} q \int_0^\infty t^{q-1} C(\phi, N) (\mathcal{H}^{N-1}(\{f \geq t\}))^{\frac{N}{N-1}} \, dt =$$

$$= q \int_0^\infty t^{q-1} C(\phi, N) (\mathcal{H}^{N-1}(\{f_+ \geq t\}))^{\frac{N}{N-1}} \, dt =$$

$$= C(\phi, N) \, q \int_0^\infty t^{q-1} (\mathcal{H}^{N-1}(\{f_+ \geq t\}))^{\frac{1}{N-1}} \mathcal{H}^{N-1}(\{f_+ \geq t\}) \, dt \overset{(*)}{\leq}$$

$$\overset{(*)}{\leq} C(\phi, N) \, q \int_0^\infty t^{q-1} \frac{\|f_+\|_{L^p(\partial\Omega)}^{\frac{p}{N-1}}}{t^{\frac{p}{N-1}}} \mathcal{H}^{N-1}(\{f_+ \geq t\}) \, dt =$$

$$= C(\phi, N) \|f_+\|_{L^p(\partial\Omega)}^{\frac{p}{N-1}} \, q \int_0^\infty t^{q-1-\frac{p}{N-1}} \mathcal{H}^{N-1}(\{f_+ \geq t\}) \, dt =$$

$$= C(\phi, N) \|f_+\|_{L^p(\partial\Omega)}^{\frac{p}{N-1}} \frac{q}{p} p \int_0^\infty t^{p-1} \mathcal{H}^{N-1}(\{f_+ \geq t\}) \, dt =$$

$$= \frac{N}{N-1} C(\phi, N) \|f_+\|_{L^p(\partial\Omega)}^{\frac{p}{N-1}} \int_{\partial\Omega} (f_+)^p \, d\mathcal{H}^{N-1}.$$

We combine the above estimates to obtain

$$\|u_+\|_{L^q(\Omega)}^q \leq \frac{N}{N-1} C(\phi, N) \|f_+\|_{L^p(\partial\Omega)}^{\frac{p}{N-1}+p}.$$

We take both sides to power $\frac{1}{q}$ and obtain

$$\|u_+\|_{L^q(\Omega)} \leq \left(\frac{N}{N-1} C(\phi, N)\right)^{\frac{N-1}{Np}} \|f_+\|_{L^p(\partial\Omega)}.$$

Hence, if $f_+ \in L^p(\partial\Omega)$, then $u_+ \in L^q(\Omega)$. Now, we make a similar calculation for u_-: we take $\widetilde{E_t} = \{u \leq t\}$ for $t < 0$. We easily see that $\{u_- \geq -t\} = \widetilde{E_t}$ and we proceed as above, except for the fact that we use a Lemma 6.2.1 for $-u$ in place of u to estimate the measure of $\widetilde{E_t}$. Finally, as $u_+ \in L^q(\Omega)$ and $u_- \in L^q(\Omega)$, we have that $u \in L^q(\Omega)$. $\square$

Remark 6.2.3. Since the proof of Theorem 6.2.2 comes in two parts in which we estimate separately the $L^{\frac{Np}{N-1}}$ norm of $u_\pm$ in terms of the L^p norm of $f_\pm$, a following variant of the Theorem holds: in the notation of Theorem 6.2.2, let $f_\pm \in L^p(\partial\Omega)$. Then $u_\pm \in L^{\frac{Np}{N-1}}(\Omega)$.

Remark 6.2.4. Notice that the estimate on the norm of u does not depend on Ω, only on the dimension (both directly and via the constant in the isoperimetric inequality) and the bounds on the metric integrand ϕ. Moreover, we see that if we let $p \to \infty$, we obtain exactly the maximum principle for least gradient functions (see for instance [33, Theorem 5.1]):

$$\|u_+\|_{L^\infty(\Omega)} \leq \|f_+\|_{L^\infty(\partial\Omega)}.$$

Finally, we present an example showing that the exponent $\frac{Np}{N-1}$ in Theorem 6.2.2 is optimal.

Example 6.2.5. Let $\Omega = \{(x, y) : |x - 1| + |y| \leq 1\} \subset \mathbb{R}^2$. Take $f(x, y) = g(x)$, where $g \in L^1((0, 2)) \cap C((0, 2))$ is a decreasing function such that $g(x) = 1$ on $[1, 2)$. Then the function $u(x, y) = g(x)$, i.e. such that all level lines are vertical, is a function of least gradient with trace f.

Now, we look at the measure of superlevel sets of u. For all $t > 1$, $\{u \geq t\}$ is a triangle with vertices $(0, 0)$, $(g^{-1}(t), g^{-1}(t))$ and $(g^{-1}(t), -g^{-1}(t))$, so

$$|\{u \geq t\}| = (g^{-1}(t))^2.$$

Let $p \geq 1$. We use this estimate to calculate

$$\int_\Omega u^p \, dx = p \int_0^\infty t^{p-1} |\{u \geq t\}| \, dt = p \int_0^\infty t^{p-1} (g^{-1}(t))^2 \, dt \geq p \int_1^\infty t^{p-1} (g^{-1}(t))^2 \, dt.$$

Now, we fix a function g_n defined by the formula $g_n(x) = x^{-1+\frac{1}{n}}$. We see that g_n is continuous, strictly decreasing, and that $g(x) = 1$ on $[1, 2)$. We put g_n in the calculation above and obtain

$$\int_\Omega (u_n)^p \, dx \geq p \int_1^\infty t^{p-1} t^{-\frac{2n}{n-1}} \, dt = p \int_1^\infty t^{p-\frac{2n}{n-1}-1} \, dt,$$

and the last integral is finite if and only if $p < \frac{2n}{n-1}$. We pass with $n \to \infty$ and see that the statement of Theorem 6.2.2 can only hold for $p \leq 2$, which is precisely the exponent given by Theorem 6.2.2.

6.3 L^∞_{loc} regularity

When the boundary data f lie in $L^\infty(\partial\Omega)$, then the maximum principle as in Remark 6.2.4 implies that any solution u to Problem (aLGP) lies in $L^\infty(\Omega)$. Conversely, if $f \notin L^\infty(\partial\Omega)$, then $u \notin L^\infty(\Omega)$, as the trace of a bounded function cannot be unbounded. However, it turns out that u may blow up only near the boundary of Ω.

This Section contains three versions of the result stating that ϕ-least gradient functions are locally bounded. Firstly, we prove this result on a toy model: we assume that $\Omega \subset \mathbb{R}^2$ and that ϕ is the Euclidean norm. Then, in Proposition 6.3.2 we prove this in $\Omega \subset \mathbb{R}^2$ for any metric integrand ϕ, using a characterisation of one-dimensional integral currents. Finally, in Theorem 6.3.3 we prove this in any dimension for the isotropic least gradient problem, using the monotonicity formula for area-minimising boundaries.

Proposition 6.3.1. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with Lipschitz boundary. Suppose that u is a least gradient function. Then, $u \in L^\infty_{loc}(\Omega)$.*

Proof. Denote $E_t = \{u \geq t\}$. Let $\Omega' \subset\subset \Omega$ be open with Lipschitz boundary and suppose that $u \notin L^\infty(\Omega')$. Without loss of generality u is unbounded from above. In particular, for each $t > 0$ we have $|E_t \cap \Omega'| > 0$. Since $u \in L^1(\Omega)$, for sufficiently large $t \geq M$ we have $|E_t \cap \Omega'| \neq |\Omega'|$, hence $\partial E_t \cap \Omega' \neq \emptyset$.

Since by Theorem 2.1.3 each connected component of ∂E_t is a line segment with ends on $\partial\Omega$, the connected component of E_t passing through Ω' has length equal at least to $\text{dist}(\partial\Omega, \partial\Omega')$. By the co-area formula

$$\int_\Omega |Du| = \int_\mathbb{R} P(E_t, \Omega)\, dt \geq \int_M^\infty P(E_t, \Omega)\, dt \geq \int_M^\infty \text{dist}(\partial\Omega, \partial\Omega')\, dt = +\infty,$$

in contradiction with $u \in BV(\Omega)$. □

Proposition 6.3.2. *Let $\Omega \subset \mathbb{R}^2$ be an open bounded set with Lipschitz boundary. Suppose that ϕ is a metric integrand and that u is a ϕ-least gradient function. Then, $u \in L^\infty_{loc}(\Omega)$.*

Proof. Let $A \subset \Omega$ be a set of finite perimeter. Since $\Omega \subset \mathbb{R}^2$, the measure $D\chi_A$ is a one-dimensional integral current. By [21, Section 4.2.25], each one-dimensional integral current may be decomposed into a (possibly infinite) sum of indecomposable integral currents. Each such current T is an oriented simple curve with finite length, i.e. its support is parametrised by a function $h : \mathbb{R} \to \mathbb{R}^N$ with $\text{Lip}(h) \leq 1$ and $f_\#((0, \mathbb{M}(T))) = T$.

Let $\Omega' \subset\subset \Omega$ be open with Lipschitz boundary and suppose that $u \notin L^\infty(\Omega')$. Without loss of generality u is unbounded from above. As in the proof of the previous Proposition, for each $t > 0$ we have $|E_t \cap \Omega'| > 0$. Since $u \in L^1(\Omega)$, for sufficiently large $t \geq M$ we have $|E_t \cap \Omega'| \neq |\Omega'|$, hence $\partial E_t \cap \Omega' \neq \emptyset$.

Let E_t be as above, hence it is a $\phi-$minimal set. Then $\partial^* E_t$, the reduced boundary of E_t, can be represented (up to a set of $\mathcal{H}^1$-measure zero) as a possibly infinite union of Lipschitz curves. We have

$$\partial^* E_t \cup S = \bigcup_i \Gamma_i,$$

where Γ_i are Lipschitz curves and $\mathcal{H}^1(S) = 0$. As E_t is a ϕ-minimal set, none of these curves are closed loops. Without loss of generality, assume that $x \in \Gamma_i$ for some $i \in \mathbb{N}$; if

$x \in S$, then (with our convention of representing sets of finite perimeter) we could replace it by a point in some Γ_i arbitrarily close to x. Now, we notice that the Euclidean length of a Lipschitz curve connecting x and a point in $\partial\Omega$ is at least $\text{dist}(\partial\Omega, \partial\Omega')$ and estimate

$$P_\phi(E_t, \Omega) = \int_{\partial^* F} \phi(x, \nu(x)) \, d\mathcal{H}^1 = \sum_{i=1}^{\infty} \int_{\Gamma_i} \phi(x, \nu(x)) \, d\mathcal{H}^1 \geq$$

$$= \int_{\Gamma_i} \phi(x, \nu(x)) \, d\mathcal{H}^1 \geq \int_{\Gamma_i} \lambda |\nu(x)| \, d\mathcal{H}^1 = \lambda \mathcal{H}^1(\Gamma_i) \geq \lambda \, \text{dist}(\partial\Omega, \partial\Omega'),$$

hence we have a uniform bound from below. By the co-area formula

$$\int_\Omega |Du| = \int_{\mathbb{R}} P(E_t, \Omega) \, dt \geq \int_M^\infty P(E_t, \Omega) \, dt \geq \int_M^\infty \text{dist}(\partial\Omega, \partial\Omega') \, dt = +\infty,$$

in contradiction with $u \in BV(\Omega)$. □

Theorem 6.3.3. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary. Suppose that u is a least gradient function. Then, $u \in L^\infty_{loc}(\Omega)$.*

Proof. We start as in the previous Propositions: denote $E_t = \{u \geq t\}$, let $\Omega' \subset\subset \Omega$ be open with Lipschitz boundary and suppose that u is unbounded from above. In particular, for sufficiently large $t \geq M$ we have $\partial E_t \cap \Omega' \neq \emptyset$.

As previously, we intend to use the co-area formula and we need to estimate from below the perimeter of E_t. To this end, we will use Proposition 2.4.2, i.e. the monotonicity formula. We recall that since E_t are area-minimising, we have $P(E_t, \Omega) = \mathcal{H}^{N-1}(\partial E_t)$. Now, let us fix $x \in \partial E_t \cap \Omega'$. Then, in the notation of Proposition 2.4.2, the density Θ satisfies $\Theta(D\chi_{E_t}, x) \geq 1$, hence $f(x,r) \geq 1$ for $r < \text{dist}(x, \partial\Omega)$. We set $r = \frac{\text{dist}(\partial\Omega, \partial\Omega')}{2}$ and calculate

$$\int_\Omega |Du| = \int_{\mathbb{R}} P(E_t, \Omega) \, dt \geq \int_M^\infty P(E_t, \Omega) \, dt =$$

$$= \int_M^\infty \mathcal{H}^{N-1}(\partial E_t) \, dt \geq \int_M^\infty \mathcal{H}^{N-1}(\partial E_t \cap B(x_t, \frac{\text{dist}(\partial\Omega, \partial\Omega')}{2})) \, dt \geq$$

$$\geq \int_M^\infty \omega_{N-1} (\frac{\text{dist}(\partial\Omega, \partial\Omega')}{2})^{N-1} \, dt = +\infty,$$

in contradiction with $u \in BV(\Omega)$. □

In fact, this proof leads to an explicit bound on the essential range on u on Ω', which depends on $\|Tu\|_{L^1(\partial\Omega)}$ and $\text{dist}(\partial\Omega, \partial\Omega')$. Before we state Corollary 6.3.4, let us notice that

$$\text{ess} \sup_{\Omega'} u = \sup_{\{t : \partial E_t \cap \Omega' \neq \emptyset\}} t$$

and

$$\text{ess} \inf_{\Omega'} u = \inf_{\{t : \partial E_t \cap \Omega' \neq \emptyset\}} t.$$

Corollary 6.3.4. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with Lipschitz boundary and $\Omega' \subset\subset \Omega$ be an open bounded set with Lipschitz boundary. Suppose that u is a least gradient function with trace $Tu = f \in L^1(\partial\Omega)$. Then,*

$$\textit{ess} \sup_{\Omega'} u - \textit{ess} \inf_{\Omega'} u \leq \frac{C(N)\|f\|_{L^1(\partial\Omega)}}{(\textit{dist}(\partial\Omega, \partial\Omega'))^{N-1}}.$$

The left hand side of the above inequality describes the width of the essential range of u on Ω'.

Proof. By Lemma 5.2.2 we have $\int_\Omega |Du| \leq \int_{\partial\Omega} |Tu| \, d\mathcal{H}^{N-1} = \|f\|_{L^1(\partial\Omega)}$. We make a similar calculation as in the proof of Theorem 6.3.3 and we see that

$$\|f\|_{L^1(\partial\Omega)} \geq \int_\Omega |Du| = \int_{\mathbb{R}} P(E_t, \Omega) \, dt \geq \int_{\text{ess inf}_{\Omega'} u}^{\text{ess sup}_{\Omega'} u} P(E_t, \Omega) \, dt \geq$$

$$\geq \int_{\text{ess inf}_{\Omega'} u}^{\text{ess sup}_{\Omega'} u} \omega_{N-1} (\frac{\text{dist}(\partial\Omega, \partial\Omega')}{2})^{N-1} \, dt =$$

$$= \frac{\omega_{N-1}}{2^{N-1}} (\text{ess} \sup_{\Omega'} u - \text{ess} \inf_{\Omega'} u)(\text{dist}(\partial\Omega, \partial\Omega'))^{N-1},$$

from which follows the desired inequality with constant $C(N) = \frac{2^{N-1}}{\omega_{N-1}}$. □

Chapter 7

Least gradient problem on annuli

7.1 Introduction

In this Chapter, we are interested in the study of the planar least gradient problem in an unusual setting. As we saw in previous Chapters, this problem is typically considered under the assumption of strict convexity of Ω; in this Chapter, we aim to relax this assumption and provide an analysis of the least gradient problem on an annulus.

As we have seen in Example 2.1.8, if the domain is not strictly convex, then there may be no solution to the least gradient problem. The first attempt to prove existence of minimisers when the domain is not strictly convex has been made in [31] in a special case of a rectangle. Later, in [58] the authors considered a domain Ω which is convex, but not strictly convex; this led the authors to prove existence and uniqueness of solutions to Problem (LGP) under some admissibility conditions on the behaviour of boundary data on the flat parts of $\partial\Omega$. In this Chapter, we will follow a similar approach and provide a set of admissibility conditions under which we will prove existence and uniqueness of solutions.

We will approach this problem using a link between the least gradient problem and the optimal transport problem. For a convex domain Ω, the authors of [19, 31] prove that the problem (LGP) is equivalent to the Beckmann problem (BP) introduced in [6] with source and target measures located on the boundary $\partial\Omega$, which is in turn related to the optimal transport problem with Euclidean cost. In other words, the problem (LGP) is equivalent to:

$$\min\left\{ \int_{\overline{\Omega}} |v| \, : \, v \in \mathcal{M}(\overline{\Omega};\mathbb{R}^2),\ \nabla\cdot v = 0 \ \text{ and } \ v\cdot\nu = g \text{ on } \partial\Omega \right\}. \tag{BP}$$

The equivalence between the least gradient problem (LGP) and the Beckmann problem (BP) is formally given by noticing that if $u \in BV(\Omega)$ with $Tu = f$, then $v := R_{-\frac{\pi}{2}} Du$ is an admissible flow in (BP) with $g = \partial_\tau f$, where $\partial_\tau f$ denotes the tangential derivative of f. On the other hand, given a vector field v such that $\nabla \cdot v = 0$ and $v \cdot \nu = g$ on $\partial\Omega$, there is a function u such that $v = R_{-\frac{\pi}{2}} Du$. Furthermore, if $|v|$ gives zero mass to the boundary (i.e. $|v|(\partial\Omega) = 0$), then $Tu = f$. In other words, there is a one-to-one correspondence between vector measures Du which are derivatives which are minimisers of the relaxed total variation functional F defined by (2.1.1) (considered as measures on $\overline{\Omega}$, so that we also include the part of the derivative of u which is on the boundary, i.e. the possible jump from $u_{|\partial\Omega}$ to f) and vector measures v in (BP). In particular, this implies that if v is an optimal flow for the Beckmann problem (BP) such that $|v|$ gives zero mass to the boundary, then a minimiser u of the functional F defined by (2.1.1) turns out to be a solution for (LGP).

In addition, it is well known that on bounded convex domains the Beckmann problem (BP) is equivalent to the Monge-Kantorovich optimal transportation problem (see, for instance, [39, 52, 60]):

$$\min \left\{ \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\gamma : \gamma \in \mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\ (\Pi_x)_\# \gamma = g^+ \text{ and } (\Pi_y)_\# \gamma = g^- \right\}, \quad (7.1.1)$$

where g^+ and g^- are the positive and negative parts of g. Now, let us go back to the equivalence between Problems (LGP) and (BP). As soon as we prove uniqueness of the optimal flow v for (BP), we immediately get uniqueness of the solution u (if it exists) for (LGP). In addition, the L^p summability of the minimal flow v implies $W^{1,p}$ regularity for the solution u of the least gradient problem (LGP). We will achieve both goals using optimal transport methods. However, we have to prove this properties for solutions of (BP) directly, as our assumptions about the masses $g_\pm$ are not classical - instead of a condition such as $g_\pm \in L^p(\Omega)$, the masses $g_\pm$ are concentrated on $\partial\Omega$. Such situation has been already considered in case when Ω is strictly convex in [19].

In this Chapter, we consider the least gradient problem (LGP) on an annulus, so the domain Ω is not convex; even its boundary is not connected. To be more precise, let $\Omega_\pm$ be two bounded strictly convex domains such that $\Omega_- \subset\subset \Omega_+$. Then, we consider the planar least gradient problem on an annulus $\Omega = \Omega_+ \backslash \overline{\Omega_-}$. Since the annulus is not strictly convex (even convex), we do not have any general results concerning existence of minimisers for (LGP). Again, we may point out very easy boundary data such that the corresponding least gradient problem (LGP) has no minimiser - suppose that $f|_{\partial\Omega_+} \equiv 1$ and $f|_{\partial\Omega_-} \equiv 0$. However, the tangent derivative of f equals zero and there exists a (zero) solution of the Beckmann problem. Hence, we will not consider the least gradient problem, but rather prove equivalence between the Beckmann problem (BP) and the following problem

$$\min \left\{ \int_\Omega |Du| : u \in BV(\Omega),\ \partial_\tau(Tu) = g \right\} \quad (7.1.2)$$

and then pass from this problem to the usual least gradient problem (LGP) for an admissible function $\widetilde{f} \in BV(\partial\Omega)$ such that $\partial_\tau \widetilde{f} = g$. In other words, we allow for certain vertical shifts of the values of g on each of the connected components of $\partial\Omega$. Since the fundamental group of an annulus is nontrivial, it is not obvious that from a divergence-free vector field

v we may recover a function u such that $v = R_{-\frac{\pi}{2}} Du$ and the method introduced in [31] applies; another difficulty is that if the domain Ω is not convex, it is not obvious if we have the equivalence between (BP) and (7.1.1) (see [19, 31]). We deal with these issues in Section 7.3.

As the non-connectedness of the boundary plays a role, we do not aim to prove a general result concerning existence and uniqueness of minimisers, but rather a set of quite general sufficient conditions that imply existence of a solution for (7.1.2) (it may be hard to find a set of conditions which is both necessary and sufficient to get existence of minimisers for (7.1.2); see also the work of Rybka and Sabra [58] which concerns the case where the domain is convex). Then, under the same structural hypotheses, we pass from a solution to problem (7.1.2) to a solution of the usual least gradient problem (LGP). We will address these issues in Section 7.4.

Another problem is the regularity of least gradient functions. As we are using techniques derived from optimal transport, we want to extend the results proved in [19] concerning $W^{1,p}$ regularity of least gradient functions. However, these require uniform convexity of Ω. Under the structural hypotheses introduced in Section 7.4, we work around this problem and prove L^p summability of the transport density, which translates to $W^{1,p}$ regularity of solutions to the least gradient problem. We will address these issues in Section 7.5.

Finally, in Section 7.6, we discuss the limits and possible extensions to the approach presented in this Chapter. We focus on two issues: the first one is the optimality of our structural assumptions and possible extensions to general Lipschitz domains; the second one is validity of our results for strictly convex norms on $\mathbb{R}^2$ other than the Euclidean norm.

This Chapter is based on the article [17], which is a result of my collaboration with Samer Dweik; now, he is from the University of British Columbia. The authors of [17] are grateful to prof. Piotr Rybka for suggesting that optimal transport methods could be well suited for solving the least gradient problem on nonconvex domains.

7.2 Preliminaries

In this Section, we study what is the structure of traces of least gradient functions on an annulus Ω. Here and in the whole Chapter we will use the following notation.

Definition 7.2.1. We say that $\Omega \subset \mathbb{R}^2$ is an annulus, if $\Omega = \Omega_+ \backslash \overline{\Omega}_-$, where $\Omega_\pm$ are open bounded strictly convex subsets of $\mathbb{R}^2$ such that $\Omega_- \subset\subset \Omega_+$. Let $f \in L^1(\partial\Omega)$. Then, we will denote by $f_\pm$ the restrictions $f_\pm = f_{|\partial\Omega_\pm} \in L^1(\partial\Omega_\pm)$ and denote by $T_\pm$ the trace operator $T : BV(\Omega) \to L^1(\partial\Omega) = L^1(\partial\Omega_+) \oplus L^1(\partial\Omega_-)$ composed with a projection onto $L^1(\partial\Omega_\pm)$.

Moreover, if $f_\pm \in BV(\partial\Omega_\pm)$, we will denote by $g_\pm = \partial_\tau f_\pm \in \mathcal{M}(\partial\Omega_\pm)$ its tangential derivative and decompose it into a positive part $g_\pm^+$ and negative part $g_\pm^-$.

This Chapter is devoted to the study of least gradient functions on annuli, nevertheless in the following results we will clearly state if they are valid only for annuli, only for Lipschitz domains, or for general open sets.

7.2.1 Traces of least gradient functions on annuli

In [65], the authors have shown existence and uniqueness of solutions to the least gradient problem for continuous boundary data and strictly convex Ω (or, to be more precise, the authors assume that $\partial\Omega$ has non-negative mean curvature and is not locally area-minimising; in dimension two, these conditions are equivalent to strict convexity). The proof of existence is constructive and its main idea is reversing Theorem 2.1.3 in order to construct almost all level sets of the solution. However, the authors provide counterexamples if the domain fails to be strictly convex.

In this subsection, we look at least gradient functions defined on annuli. We are particularly interested in their traces - the domain is not strictly convex, so not all continuous traces will be admissible. In particular, we will see why restriction of boundary data to the class $BV(\partial\Omega)$ in our analysis is reasonable. Moreover, the results presented in this subsection are of independent interest as interior regularity results for least gradient functions on strictly convex domains.

On an annulus Ω, Theorem 2.1.3 gives us an important restriction on the shape of superlevel sets E_t. As connected components of ∂E_t are line segments which lie entirely inside Ω, each of these line segments which starts at a point of $\partial\Omega_-$ has to end at a point from $\partial\Omega_+$; however, the converse is not necessarily true. In view of Theorem 7.3.4 (see below), for boundary data $f \in BV(\partial\Omega)$, we may think of each connected component of ∂E_t as a transport ray in a corresponding transport problem. In this formulation, this observation means that there is no transport between points of $\partial\Omega_-$, but there may be transport between points of $\partial\Omega_+$.

We start with proving that the total variation of f restricted to $\partial\Omega_-$ is finite. Then, we will show that f has some additional structure resulting from the topology of Ω. Apart from their value as regularity results for least gradient functions on annuli, they serve as a justification for the choice of assumptions under which we prove existence of minimisers in Section 7.4.

Lemma 7.2.2. *Let $\Omega \subset \mathbb{R}^2$ be an annulus with C^1 boundary. Suppose that $u \in BV(\Omega)$ is a least gradient function with trace $f \in L^\infty(\partial\Omega)$. Then $f_{|\partial\Omega_-} \in BV(\partial\Omega_-)$.*

Proof. Let us denote by $P(E, U)$ the perimeter of a set E with respect to an open set U. We begin by noticing that by Lemma 5.2.2 all minimal sets in Ω have perimeter less or equal to $P(\Omega, \mathbb{R}^2)$. By Theorem 2.1.3, $\{u \geq t\}$ is a minimal set for every t, i.e. its characteristic function is a function of least gradient; furthermore, by Lemma A.0.4 for almost all $t \in \mathbb{R}$, the trace of $\chi_{\{u \geq t\}}$ equals $\chi_{\{f \geq t\}}$. From now on, we consider only such t.

Since Ω is a convex subset of the plane, $\partial\Omega$ is homeomorphic to a circle; we consider the one-dimensional BV space on $\partial\Omega$. For the equivalence between the one-dimensional definitions of BV spaces on lines, see for instance [20]; this equivalence extends to C^1 one-dimensional boundaries, see for instance [28]. By the co-area formula for $f_{|\partial\Omega_-}$, we have

$$|Df|(\partial\Omega_-) = \int_{\mathbb{R}} P(\{f \geq t\}, \partial\Omega_-)\, dt.$$

Suppose that $f_{|\partial\Omega_-} \notin BV(\partial\Omega_-)$. Then the left hand side is infinite. Hence, the integrand on the right hand side is unbounded and, for any $M > 0$, we can find $t \in \mathbb{R}$ so that $P(\{f \geq t\}, \partial\Omega_-) \geq M$. Since $\partial\Omega_-$ is one-dimensional, if $P(\{f \geq t\}, \partial\Omega_-)$ is finite, it is a natural number (for the characterisation of the BV space in one dimension, see for instance [20, Chapter 5.10]). Take a minimal set E_t with trace $\chi_{\{f \geq t\}}$; then, at each of the points from $\partial^*\{f \geq t\}$, the reduced boundary of $\{f \geq t\}$, there is a line segment from $\partial^*\{u \geq t\}$ which ends at this point. However, no such line segment may connect two points from $\partial\Omega_-$; hence each of these line segments goes from $\partial\Omega_-$ to $\partial\Omega_+$. Then,

$$P(\Omega, \mathbb{R}^2) \geq M \,\mathrm{dist}(\partial\Omega_-, \partial\Omega_+).$$

However, M was arbitrary and $P(E_t, \Omega)$ is bounded, which yields to a contradiction. Hence $f_{|\partial\Omega_-} \in BV(\partial\Omega_-)$. □

However, the structure of Ω imposes even stricter conditions on the structure of $f_{|\partial\Omega_-}$. The following results serve as motivations for admissibility conditions (H1)-(H4) in Section 7.4; they do not enter the proof of equivalence between the least gradient problem and the optimal transport one (see Section 7.3) and so, we will use this equivalence to prove them.

First, since no line segment $l \subset \partial E_t$ may have both ends on $\partial\Omega_-$, the total variation of f on $\partial\Omega_-$ is smaller than the total variation of f on $\partial\Omega_+$.

Lemma 7.2.3. *Let $\Omega \subset \mathbb{R}^2$ be an annulus. Suppose that $u \in BV(\Omega)$ is a least gradient function with trace $f \in BV(\partial\Omega)$. Then $TV(f_-) \leq TV(f_+)$.*

Proof. Set $g = \partial_\tau f$. We will use Proposition 7.3.1, which was proved as a step in the proof of [31, Theorem 2.1]. It states that a rotation of the gradient of a BV function is an admissible vector field in (BP), i.e. the Beckmann problem. In particular, boundaries of superlevel sets correspond to transport rays.

Divide the derivative g into four parts: g_-^+, g_-^-, g_+^+ and g_+^-. Since u is a least gradient function, there is no boundary of a superlevel set which connects two points from $\partial\Omega_-$. Hence, there can be no transport from $\partial\Omega_-$ to $\partial\Omega_-$, so g_-^+ is transported to g_+^-; similarly, a part of g_+^+ is transported to g_-^-. Summing up these inequalities, we obtain $TV(f_-) \leq TV(f_+)$. □

Moreover, we have the following:

Proposition 7.2.4. *Let $\Omega \subset \mathbb{R}^2$ be an annulus. Suppose that $u \in BV(\Omega)$ is a least gradient function with trace $f \in BV(\partial\Omega)$ and set $g = \partial_\tau f$. Then, we have supp$(g_-^+) \cap$ supp$(g_-^-) = \{p_1, ..., p_k\}$. Moreover, for each $i = 1, ..., k$, the point p_i lies on a line segment $l_i \subset \partial E_t$ with both ends on $\partial\Omega_+$.*

Proof. Suppose that $p \in \text{supp}(g_-^+) \cap \text{supp}(g_-^-)$. For every n, consider the set $V_n = \partial\Omega_- \cap B(p, \frac{1}{n})$. Inside any V_n, pick two points p_n^+, p_n^- such that $p_n^+ \in \text{supp}(g_-^+)$ and $p_n^- \in \text{supp}(g_-^-)$. So, there are two corresponding points $q_n^\pm \in \partial\Omega_+$ such that $[q_n^+, p_n^-]$ and $[p_n^+, q_n^-]$ are two transport rays. From the cyclical monotonicity property of the optimal transport plan for (7.1.1) (see, for instance, [60, Chapter 1] or Lemma 7.4.2), we have

$$|p_n^+ - q_n^-| + |p_n^- - q_n^+| \leq |p_n^+ - p_n^-| + |q_n^+ - q_n^-|.$$

Equivalently, in the setting of least gradient functions, there is such $t \in \mathbb{R}$ that two connected components of ∂E_t are of the form $[q_n^+, p_n^-]$ and $[p_n^+, q_n^-]$. Then, the inequality above follows from the minimality of the set E_t. We notice that up to a subsequence, we have $q_n^\pm \to q^\pm$, where $q^\pm \in \partial\Omega_+$. Then, passing to the limit when $n \to \infty$, we get

$$|p - q^+| + |p - q^-| \leq |q^+ - q^-|,$$

which implies that q^+, p and q^- are collinear. But, this is possible only for finitely many points $p \in \partial\Omega_-$, thanks to the fact that the transport rays cannot intersect at an interior point. □

In particular, if $\text{supp}(g_-^+) \cap \text{supp}(g_-^-) \neq \emptyset$, this requires a very special configuration of the boundary values - if $f \in C(\partial\Omega)$, then necessarily $p, q^\pm \in f^{-1}(t)$ and the line segment $\overline{q^+q^-}$ lies on a supporting line to $\partial\Omega_-$ at p; see the following example:

Example 7.2.5. Let $\Omega = B(0,2)\backslash\overline{B(0,1)}$. Take the boundary data equal to $f_-(x,y) = y$, for every $(x,y) \in \partial B(0,1)$, and

$$f_+(x,y) = \begin{cases} -1 & y < -1, \\ y & y \in [-1,1], \\ 1 & y > 1. \end{cases}$$

Then, it is easy to see that the solution to the least gradient problem exists and equals

$$u(x,y) = \begin{cases} -1 & y < -1, \\ y & y \in [-1,1], \\ 1 & y > 1. \end{cases}$$

Here, we see that $p_\pm \in \text{supp}(g_-^+) \cap \text{supp}(g_-^-) = \{(0, \pm 1)\}$, $q_\pm^+ = (-\sqrt{3}, \pm 1)$ and $q_\pm^- = (\sqrt{3}, \pm 1)$.

Proposition 7.2.6. *Let $\Omega \subset \mathbb{R}^2$ be an annulus. Suppose that $u \in BV(\Omega)$ is a least gradient function with trace $f \in BV(\partial\Omega)$. Then, $f_- \in BV(\partial\Omega_-)$ changes monotonicity finitely many times.*

Proof. Set $g = \partial_\tau f$. There are two possibilities so that f_- changes monotonicity: either at a point $p \in \text{supp}(g_-^+) \cap \text{supp}(g_-^-)$ or there is a flat part where f_- is constant between $\text{supp}(g_-^+)$ and $\text{supp}(g_-^-)$. By Proposition 7.2.4, the first variant can happen only finitely many times. We will argue by contradiction and assume that there are countably many flat parts F_k^- of f_-.

Fix any $\varepsilon > 0$. As there are countably many flat parts of f_- and $\mathcal{H}^1(\partial\Omega_-)$ is finite, countably many of them have length smaller than ε. Now, take a flat part F^- such that $\mathcal{H}^1(F^-) < \varepsilon$ and $\partial F^- = \{p^+, p^-\}$, where $p^\pm \in \text{supp}(g_-^\pm)$. Now, we make a similar argument as in the proof of Proposition 7.2.4: consider the sets $V_n^\pm = \partial\Omega_- \cap B(p^\pm, \frac{1}{n})$. Then, inside any $V_n^\pm$, there is a point $p_n^\pm$ such that there is a transport ray coming out of $p_n^\pm$ to a point $q_n^\mp$ in $\partial\Omega_+$. Yet, we have

$$|p_n^+ - q_n^-| + |p_n^- - q_n^+| \leq |p_n^+ - p_n^-| + |q_n^+ - q_n^-|.$$

Now, passing to the limit when $n \to \infty$, we have $p_n^\pm \to p^\pm$ and $q_n^\pm \to q^\pm$ where $q^\pm \in \partial\Omega_+$, and then

$$|p^+ - q^-| + |p^- - q^+| \leq |p^+ - p^-| + |q^+ - q^-| \leq \varepsilon + |q^+ - q^-|.$$

Yet, we have $\delta := \text{dist}(\partial\Omega_+, \partial\Omega_-) > 0$. Then, this means that there are two sequences $q_k^\pm \in \partial\Omega_+$ such that the curves that connect q_k^+ to q_k^- on $\partial\Omega_+$ are disjoint (thanks to the fact that the transport rays cannot intersect) and

$$\delta \leq |q_k^+ - q_k^-|,$$

which is a contradiction since $\mathcal{H}^1(\partial\Omega_+) < +\infty$. Finally, this means that there are only finitely many flat parts of f_-, so f_- changes monotonicity only finitely many times. □

The following Lemma, which follows from the strict convexity of $\partial\Omega_+$, will play a part in the proof to come (it is proved using a blow-up of $\partial\Omega_+$).

Lemma 7.2.7. ([31, Lemma 3.8]) *Let $u \in BV(\Omega)$ be a least gradient function with trace $f \in C(\partial\Omega)$. Then, we have $\partial\{u \geq t\} \cap \partial\Omega_+ \subset f_+^{-1}(t)$.*

The final issue concerns the images of the inner and outer boundary part under the boundary data f. This is important in view of the equivalence proved in Theorem 7.3.4; under the structural hypotheses (H1)-(H4) introduced in Section 7.4, it enables us to find precisely the boundary data for which we have found a solution to the least gradient problem.

Lemma 7.2.8. *Let $\Omega \subset \mathbb{R}^2$ be an annulus. Suppose that $u \in BV(\Omega)$ is a least gradient function with trace $f \in C(\partial\Omega)$. Then, $f(\partial\Omega_-) \subset f(\partial\Omega_+)$.*

Proof. Since $\partial\Omega_\pm$ are compact and connected while f is continuous, then the images $f(\partial\Omega_-)$ and $f(\partial\Omega_+)$ are intervals. Suppose that the inclusion does not hold; then choose $t \in f(\partial\Omega_-)\backslash f(\partial\Omega_+)$. Without loss of generality, assume that t is greater than any element from $f(\partial\Omega_+)$. Consider the set $\partial\{u \geq t\}$; if it is empty, then $\{u \geq t\} = \Omega$, which violates the trace condition on $\partial\Omega_+$. If it is not empty, Lemma 7.2.7 implies that the set $\partial E_t \cap \partial\Omega_+$ is empty; hence there is a line segment in ∂E_t which has both ends in $\partial\Omega_-$, which is a contradiction. □

7.3 On the equivalence between the least gradient problem and the optimal transport

The aim of this Section is to study the equivalences between the least gradient problem (7.1.2), the Beckmann problem (BP) and the classical Monge-Kantorovich problem (7.1.1). Throughout this Section, $\Omega \subset \mathbb{R}^2$ is assumed to be an annulus in the sense of Definition 7.2.1. Firstly, we show a relationship between solutions to the following problems:

$$\min\left\{ \int_{\overline{\Omega}} |v| \,:\, v \in \mathcal{M}(\overline{\Omega}; \mathbb{R}^2),\ \nabla \cdot v = g \right\} \tag{7.3.1}$$

and

$$\min\left\{ \int_{\Omega} |Du| \,:\, u \in BV(\Omega),\, \partial_\tau(Tu) = g \right\}, \tag{7.3.2}$$

where $\partial_\tau(Tu) = g$ is equivalent to saying that $Tu = f$ on $\partial\Omega$ for some f such that $g = \partial_\tau f$, up to adding a constant on each connected component of $\partial\Omega$. The divergence condition in (7.3.1) is understood in the distributional sense: for every $\phi \in C^1(\overline{\Omega})$, we have

$$\int_{\overline{\Omega}} \nabla\phi \, dv = \int_{\partial\Omega} \phi \, dg.$$

In other words, we have $\nabla \cdot v = 0$ in Ω and $v \cdot \nu_{|\partial\Omega} = g$. Moreover, the boundary condition in (7.3.2) is understood in the sense of traces. Furthermore, as g is a tangential derivative of a BV function on the closed sets $\partial\Omega_\pm$, it will be subject to a mass balance condition, i.e.

$$g_+(\partial\Omega_+) = g_-(\partial\Omega_-) = 0.$$

It is important to stress that while problem (7.3.1) is the usual Beckmann problem (also called the free material design problem), problem (7.3.2) is not the usual least gradient problem (i.e., the one with constraint $Tu = f$). Here, we minimise $\int_\Omega |Du|$ over a wider range of boundary data. Since $\partial\Omega$ is not connected, if we shift f by a constant on any of the connected components of $\partial\Omega$, we change the boundary value in (7.3.2), but it remains the same in (7.3.1); hence, the formulation of (7.3.2) involves minimisation over the set of all f such that $g = \partial_\tau f$, i.e. g is the tangential derivative of f. Clearly, if $u \in BV(\Omega)$ solves (7.3.2), then it also solves the standard least gradient problem with boundary data Tu. We will come back to this issue at the end of Section 7.4.

The main idea, coming from [31], is to take an admissible function u in (7.3.2) and use its rotated gradient $v = R_{-\frac{\pi}{2}} \nabla u$. Here, R_α is the rotation with angle α around the origin and we follow the notation in [31], so that τ is the tangent such that (ν, τ) is positively oriented. An opposite convention for τ would correspond to taking rotations with angle $+\frac{\pi}{2}$; this is the notation used in [19].

In dimension two, a rotation of a gradient by $-\frac{\pi}{2}$ is a divergence-free field in Ω and rotation by $-\frac{\pi}{2}$ interchanges the normal and tangent components at the boundary, so v is an admissible vector field in (7.3.1). This fact was shown as a step in the proof of [31, Theorem 2.1] and is formalised in the following Proposition; we present the proof for completeness.

Proposition 7.3.1. *Suppose that $\Omega \subset \mathbb{R}^2$ is an open bounded set with Lipschitz boundary and let $u \in BV(\Omega)$ with trace $Tu = f$. Then, $v = R_{-\frac{\pi}{2}} \nabla u$ is a vector-valued measure such that $\nabla \cdot v = g$, where $g = \partial_\tau f$. In particular, it is an admissible function in* (7.3.1).

Proof. Let $u_n \in C^\infty(\mathbb{R}^2) \cap BV(\mathbb{R}^2)$ be a sequence with the following properties: it converges to u in strict topology of $BV(\Omega)$, i.e. $u_n \to u$ in $L^1(\Omega)$ and $\int_\Omega |\nabla u_n| \, dx \to \int_\Omega |Du|$, and the support of u_n lies in a $\frac{1}{n}$-neighbourhood of Ω. We notice that the rotated gradients of u_n have zero divergence inside Ω, as for smooth functions

$$\mathrm{div}(R_{-\frac{\pi}{2}} \nabla u_n) = \mathrm{div}((u_n)_{x_2}, -(u_n)_{x_1}) = (u_n)_{x_2 x_1} - (u_n)_{x_1 x_2} = 0.$$

We check the definition of distributional divergence. Since u_n is smooth for every n, we have $R_{-\frac{\pi}{2}} \nabla u_n \cdot \nu = \partial_\tau (T u_n)$; so, for any $\phi \in C^1(\overline{\Omega})$ we have

$$\int_\Omega (R_{-\frac{\pi}{2}} \nabla u_n) \cdot \nabla \phi \, dx = \int_{\partial\Omega} (R_{-\frac{\pi}{2}} \nabla u_n) \cdot \nu \, \phi \, d\mathcal{H}^1 =$$

$$= \int_{\partial\Omega} \partial_\tau (T u_n) \, \phi \, d\mathcal{H}^1 = - \int_{\partial\Omega} T u_n \, \partial_\tau \phi \, d\mathcal{H}^1.$$

We pass to the limit. On the left hand side, notice that since the sequence u_n converges strictly, the measures $\nabla u_n \, d\mathcal{L}^2$ converge weakly* to Du; in particular, $\nabla \cdot \big(R_{-\frac{\pi}{2}} Du \big) = 0$ in Ω. On the right hand side, recall that the trace operator is continuous with respect to the strict topology. Hence

$$\int_\Omega \nabla \phi \, d(R_{-\frac{\pi}{2}} Du) = - \int_{\partial\Omega} Tu \, \partial_\tau \phi \, d\mathcal{H}^1 = \int_{\partial\Omega} \phi \, d(\partial_\tau (Tu)) \ \text{ for all } \ \phi \in C^1(\overline{\Omega}),$$

so in the distributional sense we have

$$\nabla \cdot (R_{-\frac{\pi}{2}} Du) = R_{-\frac{\pi}{2}} Du \cdot \nu|_{\partial\Omega} = \partial_\tau (Tu) = g.$$

□

We point out that while the part of the proof of [31, Theorem 2.1] which corresponds to Proposition 7.3.1 does not require g to be a measure, merely a continuous functional over $\mathrm{Lip}(\partial\Omega)$, in this Chapter we require g to be a measure supported on $\partial\Omega$ in order to obtain a converse result.

On the other hand, the authors of [31] proved that if the domain Ω is strictly convex, a vector field $v \in L^1(\Omega, \mathbb{R}^2)$ admissible in (7.3.1) produces a function $u \in W^{1,1}(\Omega)$ admissible in (7.3.2). However, their proof involves definition of u as an integral of a certain 1-form; in our setting, it fails due to the fact that Ω is not simply-connected and the integral may depend on the choice of a path. In the next Proposition, we use the result in the convex case to resolve this problem.

Proposition 7.3.2. *Let $\Omega \subset \mathbb{R}^2$ be an annulus. Suppose that $v \in L^1(\Omega, \mathbb{R}^2)$ is such that $\nabla \cdot v = g$ in $\mathbb{R}^2$ as distributions, where $g \in \mathcal{M}(\partial\Omega)$ is a measure such that $g(\partial\Omega_\pm) = 0$. Then, there exists $u \in W^{1,1}(\Omega)$ such that $v = R_{-\frac{\pi}{2}} \nabla u$. In particular,*

$$\int_\Omega |v| \, dx = \int_\Omega |\nabla u| \, dx.$$

Moreover, if $Tu = f$ then $g = \partial_\tau f$.

Proof. 1. Denote $g_\pm = g|_{\partial\Omega_\pm}$. Let $v \in L^1(\Omega, \mathbb{R}^2)$ be such that $\nabla \cdot v = g$ in $\mathbb{R}^2$ as distributions. To be precise, if we take $\widetilde{v}$ to be a vector field in $L^1(\mathbb{R}^2, \mathbb{R}^2)$ defined as

$$\widetilde{v} = \begin{cases} v & \text{in } \Omega, \\ 0 & \text{in } \mathbb{R}^2 \backslash \Omega, \end{cases}$$

then $\nabla \cdot \widetilde{v} = g$ as distributions in $\mathbb{R}^2$. We want to extend this vector field in a different way so that $\nabla \cdot v = g_+$. To this end, let $f_- \in BV(\partial\Omega_-)$ be such that $g_- = \partial_\tau f_-$ and take any $w \in W^{1,1}(\Omega_-)$ such that $T_{\partial\Omega_-} w = f_-$.

2. Now, we take the rotated gradient of w. Then, $v' = R_{-\frac{\pi}{2}} \nabla w \in L^1(\Omega_-, \mathbb{R}^2)$ is a vector field such that $\nabla \cdot v' = -g_-$ (the minus sign comes from the fact that the orientation of $\partial\Omega_-$ as a boundary of Ω_- is opposite to its orientation as a part of the boundary of Ω). Let $\widetilde{v'}$ be an extension of v' by 0 to the whole of $\mathbb{R}^2$ as above, i.e.

$$\widetilde{v'} = \begin{cases} v' & \text{in } \Omega_-, \\ 0 & \text{in } \mathbb{R}^2 \backslash \Omega_-. \end{cases}$$

So, we have $\nabla \cdot (\widetilde{v} + \widetilde{v'}) = g_+$. Moreover, $(\widetilde{v} + \widetilde{v'})|_\Omega = v$.

3. Now, we use [31, Proposition 2.1] on Ω_+ and obtain that there exists a function $\widetilde{u} \in W^{1,1}(\Omega_+)$ such that $\widetilde{v} + \widetilde{v'} = R_{-\frac{\pi}{2}} \nabla \widetilde{u}$ on Ω_+. In particular, we have

$$\int_{\Omega_+} |\widetilde{v} + \widetilde{v'}| \, dx = \int_{\Omega_+} |\nabla \widetilde{u}| \, dx.$$

Moreover, $\partial_\tau (T_{\partial\Omega_+} \widetilde{u}) = g_+$. By applying again [31, Proposition 2.1] but this time on Ω_-, we obtain also that $\partial_\tau (T_{\partial\Omega_-} \widetilde{u}) = g_-$. Now, set $u = \widetilde{u}|_\Omega \in W^{1,1}(\Omega)$. So, we see easily that

$$\int_\Omega |v| \, dx = \int_\Omega |\nabla u| \, dx.$$

Finally, the trace of u is correct: clearly, $\partial_\tau (T_{\partial\Omega_+} u) = \partial_\tau (T_{\partial\Omega_+} \widetilde{u}) = g_+$. Moveover, as $|\nabla \widetilde{u}|(\partial\Omega_-) = 0$, the trace of $\widetilde{u}$ on $\partial\Omega_-$ from both sides coincides and, we have $T_{\partial\Omega_-} u = T_{\partial\Omega_-} \widetilde{u} = f_-$. □

When v is merely a measure, we can employ a similar trick. However, in order for the trace of the obtained function u to be correct, we need one additional component: v has to give no mass to the boundary. Then the sequence of approximations constructed in the proof of [31, Proposition 2.1] converges in the strict topology. Since the trace operator is continuous with respect to strict topology, an analogue of [31, Proposition 2.1] is valid under this additional assumption (see also the discussion in [19]). So, we have the following:

Proposition 7.3.3. *Let $\Omega \subset \mathbb{R}^2$ be an annulus. Suppose that $v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ is such that $|v|(\partial\Omega) = 0$ and $\nabla \cdot v = g$ as distributions, where $g \in \mathcal{M}(\partial\Omega)$ is a measure such that $g(\partial\Omega_\pm) = 0$. Then, there exists $u \in BV(\Omega)$ such that $v = R_{-\frac{\pi}{2}} Du$. In particular,*

$$\int_{\overline{\Omega}} |v| = \int_\Omega |Du|.$$

Moreover, if $Tu = f$ then $g = \partial_\tau f$.

Now, we are ready to prove the equivalence of problems (7.3.1) and (7.3.2). This boils down to two distinct problems: proving that the infima of these problems are equal and to constructing solutions of one problem from the other one.

Theorem 7.3.4. *Let $\Omega \subset \mathbb{R}^2$ be an annulus. We have* $\inf$ (7.3.1) $=$ $\inf$ (7.3.2)*. Moreover, from each solution $u \in BV(\Omega)$ of* (7.3.2)*, one can construct a solution to* (7.3.1)*. In the other direction, from each solution $v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ of* (7.3.1)*, one can construct a solution to* (7.3.2)*, provided that $|v|(\partial\Omega) = 0$.*

Proof. Suppose that $v_n \in L^1(\Omega, \mathbb{R}^2)$ is a minimising sequence in (7.3.1). By Proposition 7.3.2, for each n there exists $u_n \in W^{1,1}(\Omega)$ admissible in (7.3.2) such that $v_n = R_{-\frac{\pi}{2}} \nabla u_n$. Hence

$$\inf (7.3.1) \longleftarrow \int_\Omega |v_n| = \int_\Omega |\nabla u_n| \, dx \geq \inf (7.3.2).$$

Conversely, suppose that $u_n \in BV(\Omega)$ is a minimising sequence in (7.3.2). By Proposition 7.3.1, the vector fields $v_n = R_{-\frac{\pi}{2}} Du_n$ are admissible in (7.3.1). Hence

$$\inf (7.3.2) \longleftarrow \int_\Omega |Du_n| = \int_\Omega |v_n| \geq \inf (7.3.1).$$

Hence, the two infima are equal. Now, we turn to the issue of constructing solutions of one problem from the other one.

Let $u \in BV(\Omega)$ be a minimiser of (7.3.2). Let $v = R_{-\frac{\pi}{2}} Du$; by Proposition 7.3.1, it is an admissible vector field in (7.3.1). Moreover, we have

$$\inf (7.3.2) = \int_\Omega |Du| = \int_\Omega |v| \geq \inf (7.3.1).$$

Hence, v is a minimiser of (7.3.1).

Finally, let $v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ be a minimiser of (7.3.1) such that $|v|(\partial\Omega) = 0$. By Proposition 7.3.3, there exists $u \in BV(\Omega)$ admissible in (7.3.2) and $v = R_{-\frac{\pi}{2}} Du$. Yet, one has

$$\inf (7.3.1) = \int_\Omega |v| = \int_\Omega |Du| \geq \inf (7.3.2),$$

which implies that this function u is, in fact, a minimiser for the problem (7.3.2). □

In particular, a solution $v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ of the Beckmann problem which satisfies $|v|(\partial\Omega) = 0$ generates a function $u \in BV(\Omega)$ which solves the least gradient problem for boundary data $f = Tu$. If the solution to the Beckmann problem is unique, so is the boundary data f for which we can construct the solution of the least gradient problem, up to adding the same constants on both connected components of $\partial\Omega$. We cannot solve the least gradient problem for a shifted value of f if we added two different constants on $\partial\Omega_\pm$.

Example 7.3.5. Let $\Omega = B(0,2) \backslash \overline{B(0,1)}$. Consider $g \equiv 0 \in \mathcal{M}(\partial\Omega)$ to be the boundary data in the Beckmann problem. Functions $f \in BV(\partial\Omega)$ such that $g = \partial_\tau f$ are of the form

$$f = \begin{cases} c_- & \text{on } \partial B(0,1), \\ c_+ & \text{on } \partial B(0,2). \end{cases}$$

We consider such boundary data in the least gradient problem. The solution to the Beckmann problem is unique and equals $v \equiv 0$ in $\overline{\Omega}$. Then, Theorem 7.3.4 gives us a constant solution to problem (7.3.2). It is a solution to the least gradient problem with $c_+ = c_-$. However, for $c_+ \neq c_-$, the least gradient problem admits no solution.

On the other hand, it is also possible to show some form of equivalence between the Beckmann problem (BP) and the Monge-Kantorovich one (7.1.1), in the case when the domain Ω

is an annulus. A key assumption will be that all transport rays lie inside the domain Ω. We recall (see [60, 69]) that the Kantorovich problem

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,d\gamma:\ \gamma\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\ (\Pi_x)_\#\gamma=g^+\text{ and }(\Pi_y)_\#\gamma=g^-\right\}\tag{7.3.3}$$

admits a dual formulation:

$$\sup\left\{\int_{\overline{\Omega}}\phi\,d(g^+-g^-):\ \phi\in\mathrm{Lip}_1(\Omega)\right\}.\tag{7.3.4}$$

In fact, we have

$$\min\left\{\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,d\gamma:\ \gamma\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega}),\ (\Pi_x)_\#\gamma=g^+\text{ and }(\Pi_y)_\#\gamma=g^-\right\}=$$

$$=\min_{\gamma\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})}\left\{\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|\,d\gamma+\right.$$

$$\left.+\sup_{\phi^\pm\in C(\overline{\Omega})}\left\{\int_{\overline{\Omega}}\phi^+\,dg^+-\int_{\overline{\Omega}}\phi^-\,dg^--\int_{\overline{\Omega}\times\overline{\Omega}}\left[\phi^+(x)-\phi^-(y)\right]\,d\gamma\right\}\right\}=$$

$$=\min_{\gamma\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})}\left\{\sup_{\phi^\pm\in C(\overline{\Omega})}\left\{\int_{\overline{\Omega}\times\overline{\Omega}}\left[|x-y|-(\phi^+(x)-\phi^-(y))\right]\,d\gamma(x,y)+\right.\right.$$

$$\left.\left.+\int_{\overline{\Omega}}\phi^+\,dg^+-\int_{\overline{\Omega}}\phi^-\,dg^-\right\}\right\}=$$

(by a formal inf-sup exchange for which a full justification can be found for instance in [60])

$$=\sup_{\phi^\pm\in C(\overline{\Omega})}\left\{\min_{\gamma\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})}\left\{\int_{\overline{\Omega}\times\overline{\Omega}}\left[|x-y|-(\phi^+(x)-\phi^-(y))\right]\,d\gamma(x,y)\right\}+\right.$$

$$\left.+\int_{\overline{\Omega}}\phi^+\,dg^+-\int_{\overline{\Omega}}\phi^-\,dg^-\right\}.$$

We notice that

$$\min_{\gamma\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})}\left\{\int_{\overline{\Omega}\times\overline{\Omega}}\left[|x-y|-(\phi^+(x)-\phi^-(y))\right]\,d\gamma(x,y)\right\}=$$

$$=\begin{cases}0 & \text{if }\ \phi^+(x)-\phi^-(y)\le|x-y|,\\ -\infty & \text{else.}\end{cases}$$

We plug it in the previous calculation and obtain

$$\min\,(7.3.3)=\sup\left\{\int_{\overline{\Omega}}\phi^+\,dg^+-\int_{\overline{\Omega}}\phi^-\,dg^-\ :\ \phi^\pm\in C(\overline{\Omega}),\ \phi^+(x)-\phi^-(y)\le|x-y|\right\}.$$

But now, it is clear that we can assume $\phi^+(x) := \min\{|x-y| + \phi^-(y) \,:\, y \in \overline{\Omega}\}$, for every $x \in \overline{\Omega}$, and so, $\phi^- = \phi^+$. From this duality result $\min(7.3.3) = \sup(7.3.4)$, we infer that optimal γ and ϕ satisfy the following equality:

$$\int_{\overline{\Omega}\times\overline{\Omega}} [|x-y| - (\phi(x) - \phi(y))] \; d\gamma(x,y) = 0,$$

which implies that

$$\phi(x) - \phi(y) = |x-y| \quad \text{on} \quad \text{supp}(\gamma).$$

Let us introduce the following:

Definition 7.3.6. By a transport ray we call any maximal line segment $[x,y]$ satisfying $\phi(x) - \phi(y) = |x-y|$.

Following this definition, we see that an optimal transport plan γ has to move the mass along the transport rays. Now, we prove equivalence between the Kantorovich problem (7.3.3) and the Beckmann one (7.3.1).

Proposition 7.3.7. *Suppose that all the transport rays between g^+ and g^- are inside the annulus Ω. Let γ be an optimal transport plan for* (7.3.3) *and let us define the vectorial measure w_γ via its action on continuous funtions as follows:*

$$\langle w_\gamma, \xi\rangle := \int_{\overline{\Omega}\times\overline{\Omega}} \int_0^1 \xi((1-t)x + ty)\cdot(y-x)\, dt\, d\gamma(x,y), \quad \textit{for all} \;\; \xi \in C(\overline{\Omega}, \mathbb{R}^2).$$

Then, w_γ solves (7.3.1)*. Moreover, we have* $\min(7.3.1) = \sup(7.3.4) = \min(7.3.3)$.

Proof. First, we see easily that w_γ is admissible in (7.3.1) (this follows immediately by taking as a test function $\xi = \nabla\phi$). On the other hand, we have

$$|w_\gamma|(\overline{\Omega}) \le \int_{\overline{\Omega}\times\overline{\Omega}} |x-y|\, d\gamma = \min(7.3.3) = \sup(7.3.4).$$

Let v be an admissible flow in (7.3.1) and let ϕ be a C^1 function such that $|\nabla\phi| \le 1$. Then, one has

$$\int_{\overline{\Omega}} \phi\, d(g^+ - g^-) = \int_{\overline{\Omega}} \nabla\phi \cdot dv \le \int_{\overline{\Omega}} |v|.$$

This implies that

$$\sup(7.3.4) \le \min(7.3.1).$$

Consequently, we get that w_γ is a solution for (7.3.1) and we have

$$\min(7.3.1) = \sup(7.3.4) = \min(7.3.3).$$

□

In addition, following [60, Chapter 4] and using Proposition 7.3.7, we are able to prove that every solution w for the Beckmann problem (7.3.1) is of the form $w = w_\gamma$ for some optimal transport plan γ for (7.3.3).

On the other hand, one can associate with w_γ a scalar positive measure σ_γ (which is called *transport density* and again is defined via its action on continuous functions):

$$\langle \sigma_\gamma, \varphi \rangle := \int_{\overline{\Omega}\times\overline{\Omega}} \int_0^1 \varphi((1-t)x+ty)|x-y|\,dt\,d\gamma(x,y), \quad \text{for all} \quad \varphi \in C(\overline{\Omega}).$$

Moreover, it is not difficult to see that if ϕ is a Kantorovich potential, between g^+ and g^-, then we have the following:

$$w_\gamma = -\sigma_\gamma \nabla\phi.$$

In this way, we get existence of a solution for the least gradient problem (7.3.2) as soon as the transport density σ_γ gives zero mass to the boundary $\partial\Omega$ (i.e., $\sigma_\gamma(\partial\Omega) = 0$). Moreover, we get uniqueness of the solution u for (7.3.2) if σ_γ does not depend on the choice of γ. We note that the classical assumption for uniqueness of the transport density σ_γ is not satisfied here: g is supported on the boundary, so it is singular with respect to the Lebesgue measure $\mathcal{L}^2$, when the classical assumptions require that either g^+ or g^- is in $L^1(\Omega)$. Yet, we will show uniqueness of σ under some assumptions on Ω, g^+ and g^-.

7.4 Least gradient problem: existence and uniqueness

In this section, we will prove that on an annulus $\Omega \subset \mathbb{R}^2$, under some admissibility assumptions on the boundary datum f, the least gradient problem

$$\min\left\{ \int_\Omega |Du| : \, u \in BV(\Omega), \, Tu = f \right\} \tag{7.4.1}$$

has a solution. We recall that we need to restrict to dimension 2 because only in this framework we can use rotated gradients, and they have zero divergence. In addition, we will assume that $f \in BV(\partial\Omega)$ since, in this way, one has the equivalence between the Beckmann problem (7.3.1) and a version of the least gradient problem (7.3.2). We start with proving existence of a solution to the Beckmann problem which gives no mass to the boundary and then pass through problem (7.3.2) to the least gradient problem (7.4.1).

First of all, let us introduce our admissibility conditions. These are formally conditions on a Dirichlet datum f in the least gradient problem; however, as they do not depend on the exact values of f, only on its structure and total variation, we may think of them equivalently as conditions on its tangential derivative $g = \partial_\tau f$. The conditions are as follows:

(H1) $f \in BV(\partial\Omega)$.

(H2) $\partial\Omega_\pm$ can be decomposed into parts $(\chi_i^\pm)_i$, $(\Gamma_i^\pm)_i$ and $(F_i^\pm)_i$ such that :

• On each $\chi_i^\pm$ (resp. $\Gamma_i^\pm$) the boundary datum f is increasing (resp. decreasing) with $TV(f \llcorner \chi_i^+) = TV(f \llcorner \chi_i^-)$ and $TV(f \llcorner \Gamma_i^+) = TV(f \llcorner \Gamma_i^-)$.

• Between each two curves χ_i^- and Γ_i^-, there is a curve F_i^-, on which f_+ is constant. We call F_i^- a flat part.

• Between each two curves χ_i^+ and Γ_i^+, there is a curve F_i^+. For every i, we require that F_i^+ is a flat part or F_i^+ is a union of two sets, F_i^{++} and F_i^{+-}, where f is increasing on F_i^{++} and decreasing on F_i^{+-} with $TV(f_{|F_i^{++}}) = TV(f_{|F_i^{+-}})$. In particular, we have $[\partial_\tau f_+](F_i^+) = 0$.

In addition, we want to add a condition on f which will be necessary to guarantee that all the transport rays are inside $\overline{\Omega}$. Before that, we need to introduce the following:

Definition 7.4.1. Let $\Gamma^\pm$ be two arcs on $\partial\Omega_\pm$. Then, we say that Γ^+ is visible from Γ^- if the following holds:

$$\text{for all } \ x \in \Gamma^+ \text{ and } \ y \in \Gamma^-, \ \text{ we have } [x, y] \subset \overline{\Omega}.$$

So, we formulate our visibility condition in the following way:

(H3) For every i, χ_i^+ is visible from χ_i^-, Γ_i^+ is visible from Γ_i^- and F_i^{++} is visible from F_i^{+-}.

The second condition that we need so that all the transport rays lie inside $\overline{\Omega}$ is an inequality linking the locations of $\chi_i^\pm$, $\Gamma_i^\pm$ and $F_i^{+\pm}$. Denote by $d_M(\Gamma, \Gamma')$ the maximal distance between two arcs Γ and Γ' (in other words, the diameter of the set $\Gamma \cup \Gamma'$) and by $\text{dist}(\Gamma, \Gamma')$ the minimal distance between them. Set $\Lambda^+ = \bigcup_i \chi_i^+ \cup \Gamma_i^- \cup F_i^{++}$ and $\Lambda^- = \bigcup_i \chi_i^- \cup \Gamma_i^+ \cup F_i^{+-}$. Then, we assume

(H4) For every i, we have the following:

• $d_M(\chi_i^+, \chi_i^-) + d_M(\Lambda^+ \backslash \chi_i^+, \Lambda^- \backslash \chi_i^-) < \text{dist}(\chi_i^+, \Lambda^- \backslash \chi_i^-) + \text{dist}(\chi_i^-, \Lambda^+ \backslash \chi_i^+)$,

• $d_M(\Gamma_i^+, \Gamma_i^-) + d_M(\Lambda^+ \backslash \Gamma_i^-, \Lambda^- \backslash \Gamma_i^+) < \text{dist}(\Gamma_i^+, \Lambda^+ \backslash \Gamma_i^-) + \text{dist}(\Gamma_i^-, \Lambda^- \backslash \Gamma_i^+)$,

• $d_M(F_i^{++}, F_i^{+-}) + d_M(\Lambda^+ \backslash F_i^{++}, \Lambda^- \backslash F_i^{+-}) < \text{dist}(F_i^{++}, \Lambda^- \backslash F_i^{+-}) + \text{dist}(F_i^{+-}, \Lambda^+ \backslash F_i^{++})$.

The conditions above imply the following:

Lemma 7.4.2. *Set $g = \partial_\tau f$. Under assumptions (H1)-(H4), all the transport rays between g^+ and g^- lie inside the annulus Ω. More precisely, any transport ray R is of the form $[x, y]$ with $x \in \chi_i^+$ and $y \in \chi_i^-$, $x \in \Gamma_i^-$ and $y \in \Gamma_i^+$ or $x,\, y \in F_i^+$, for some i.*

Proof. Let $R := [x, y]$ be a transport ray. Since $x \in \text{supp}(g^+)$, then $x \in \chi_i^+$, Γ_i^- or F_i^{++}, for some i. Suppose that $x \in \chi_i^+$ and $y \notin \chi_i^-$; the other cases can be treated in a similar way. Since $TV(f_{|\chi_i^+}) = TV(f_{|\chi_i^-})$, there exists a transport ray $R' := [x', y']$ with $y' \in \chi_i^-$ and $x' \in \Lambda^+ \backslash \chi_i^+$. In particular, we have (x, y), $(x', y') \in \text{supp}(\gamma)$, where γ is an optimal transport plan for (7.3.3). Let ϕ be a Kantorovich potential between g^+ and g^-. Then, we have (this is the so-called *cyclical monotonicity property*):

$$|x - y| + |x' - y'| = \phi(x) - \phi(y) + \phi(x') - \phi(y') \leq |x - y'| + |x' - y|.$$

Yet,

$$|x - y'| + |x' - y| \leq d_M(\chi_i^+, \chi_i^-) + d_M(\Lambda^+ \backslash \chi_i^+, \Lambda^- \backslash \chi_i^-)$$

and

$$\text{dist}(\chi_i^+, \Lambda^- \backslash \chi_i^-) + \text{dist}(\chi_i^-, \Lambda^+ \backslash \chi_i^+) \leq |x - y| + |x' - y'|.$$

This contradicts the assumption (H4). □

We also want to study the uniqueness of the solution of (7.3.1). For this aim, we will prove the uniqueness of the optimal transport plan γ in (7.3.3). More precisely, we have (the proof is essentially based on some arguments used in [19, Proposition 2.5]):

Proposition 7.4.3. *Under the assumptions (H1)-(H4), there is a unique optimal transport plan γ for* (7.3.3)*, between g^+ and g^-, which will be induced by a transport map S, provided that g^+ is atomless.*

Proof. Let γ be an optimal transport plan between g^+ and g^-. Let D be the set of points whose belong to several transport rays. Fix $x \in \Lambda^+ \cap D$ and let $R_x^\pm$ be two different transport rays starting from x. By Lemma 7.4.2 all the transport rays between g^+ and g^- lie inside $\overline{\Omega}$.

There are three possibilities for x: $x \in \chi_i^+$, Γ_i^- or F_i^{++}. Moreover, thanks to Lemma 7.4.2, if $x \in \chi_i^+$, then both transport rays R_x^+ and R_x^- should end on χ_i^-. Similarly, if $x \in \Gamma_i^-$, then both transport rays R_x^+ and R_x^- should end on Γ_i^+. Finally, if $x \in F_i^{++}$, then both transport rays R_x^+ and R_x^- should end on F_i^{+-}. Let $\Delta_x \subset \Omega$ be the region delimited by R_x^+, R_x^- and $\partial\Omega$.

Then, we see easily that for every $x \in D$ the sets $\{\Delta_x\}_{x \in D}$ must be disjoint. This implies that the set D is at most countable. Since we assumed that g^+ is atomless, we have that $g^+(D) = 0$. In addition, taking into account that $\Omega_\pm$ are strictly convex, we have that for g^+-almost every $x \notin D$ there is a unique transport ray R_x starting from x and this ray

R_x intersects $\text{supp}(g^-)$ at exactly one point $S(x)$. This implies that $\gamma = (Id, S)_{\#}g^+$. Yet, this is sufficient to infer that γ is the unique optimal transport plan for (7.3.3) since, if γ' is another optimal transport plan then $\gamma'' = (\gamma + \gamma')/2$ is also optimal for (7.3.3), which is not possible as γ'' must be induced by a transport map. □

Now, we are ready to state our main result concerning the Beckmann problem.

Theorem 7.4.4. *Assume that $\Omega \subset \mathbb{R}^2$ is an annulus. Let $g = \partial_\tau f$, where f satisfies the admissibility conditions (H1)-(H4). Then, the Beckmann problem* (7.3.1) *admits at least one solution $v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ such that $|v|(\partial\Omega) = 0$. Moreover, if g^+ is atomless (for instance if we assume that f is continuous), then the solution is unique.*

Proof. Let g be the tangential derivative of the boundary datum f, i.e. $g = \partial_\tau f$, where f satisfies the admissibility conditions (H1)-(H4). Let γ be an optimal transport plan for (7.3.3). By Lemma 7.4.2 and Proposition 7.3.7, one can construct a minimiser v_γ for (7.3.1). Now, we only need to show that $|v_\gamma|(\partial\Omega) = 0$. Recalling the construction of v_γ, we have

$$|v_\gamma|(\partial\Omega) = \int_{\overline{\Omega}\times\overline{\Omega}} \mathcal{H}^1(\partial\Omega \cap [x,y])\, d\gamma(x,y).$$

Since $\Omega_\pm$ are strictly convex, we infer that $|v_\gamma|(\partial\Omega) = 0$. For uniqueness, it is enough to see that by Proposition 7.4.3 there is a unique optimal transport plan γ for (7.3.3). Since every solution v for (7.3.1) is of the form $v = v_\gamma$ for some optimal transport plan γ, v is the unique solution for (7.3.1). □

Now, we want to go back to the least gradient problem. The first step is to construct a solution to the auxiliary problem (7.3.2) and translate it to a solution of the usual least gradient problem (LGP) for some fixed boundary data f.

Theorem 7.4.5. *Suppose that $\Omega \subset \mathbb{R}^2$ is an annulus and that $g = \partial_\tau f$, where f satisfies the admissibility conditions (H1)-(H4). Then there exists a solution to problem* (7.3.2)*. Moreover, there exists $\widetilde{f} \in BV(\partial\Omega)$ such that $g = \partial_\tau \widetilde{f}$ such that there exists a solution to the least gradient problem* (7.4.1) *with boundary data $\widetilde{f}$. If $f \in C(\partial\Omega)$, then the solutions to both problems are unique.*

Proof. By Theorem 7.4.4, there exists a solution $v \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^2)$ to the Beckmann problem (7.3.1) with $|v|(\partial\Omega) = 0$ (in addition, this solution is unique as soon as g^+ is atomless). Then, by Theorem 7.3.4, there exists a function $u \in BV(\Omega)$ which is a solution of the auxiliary problem (7.3.2) (which is also unique if f is continuous, as then g is atomless). Let $\widetilde{f} = Tu$. Since the infimum in (7.3.2) is taken with respect to all possible traces with tangential derivative g, i.e. functions of the form $\widetilde{f} + \lambda_- \chi_{\partial\Omega_-} + \lambda_+ \chi_{\partial\Omega_+}$, so in particular u is a solution to the least gradient problem (7.4.1) with boundary data $\widetilde{f}$. □

In other words, what happens in the above Theorem is that when we use Theorem 7.3.4, we have no control on the vertical shifts of the boundary data by a constant on each connected component of $\partial\Omega$. Therefore, we are able to prove existence of a solution to the least gradient problem with some boundary data $\widetilde{f}$ (which differs from the original function f by a constant on each connected component of $\partial\Omega$), but without calculating directly the minimiser u in the auxiliary problem (7.3.2) it may be hard to compute $\widetilde{f}$. However, under an additional constraint on the total variation, the following proposition enables us to identify the boundary data $\widetilde{f}$ given by the previous theorem without having to first calculate the solution u of problem (7.3.2).

Proposition 7.4.6. *Suppose that f satisfies assumptions (H1)-(H4) and additionally*

$$|g_-|(\partial\Omega_-) = |g_+|(\partial\Omega^+).$$

Then, there is a unique $f \in BV(\partial\Omega)$ such that $g = \partial_\tau f$ and there exists a solution to the least gradient problem with this boundary data f. Moreover, if g is atomless, then the solution is unique.

Proof. Fix $x_\pm = \chi_1^\pm \cap F_1^\pm$. For $t_\pm \in \partial\Omega_\pm$, we set $f_\pm(t_\pm) = \int_{x_\pm}^{t_\pm} g$ (the integral is taken so that the tangent vector points in the counterclockwise direction). Then, $g = \partial_\tau f$ and, by assumption (H2) and the equality of masses, $f_\pm$ change by the same value on each $\chi_i^\pm$, $\Gamma_i^\pm$ and is constant on each $F_i^\pm$. Then, f is the only function with tangential derivative g such that $f_-(\partial\Omega_-) \subset f_+(\partial\Omega_+)$. Now, Theorems 7.3.4 and 7.4.4 give us existence of a least gradient function $u \in BV(\Omega)$ which solves (7.3.2). However, due to Lemma 7.2.7 traces of least gradient functions satisfy $f_-(\partial\Omega_-) \subset f_+(\partial\Omega_+)$; hence $Tu = f$. Uniqueness is guaranteed by Theorem 7.4.4. □

Finally, we illustrate the results in this Section with the following Example:

Example 7.4.7. Let $\Omega = B(0,2)\backslash\overline{B(0,1)}$. Let $f_- \in C(\partial\Omega_-) \cap BV(\partial\Omega_-)$ be defined as follows:

$$f_-(x,y) = \begin{cases} 1 & \text{if } \ y > \frac{1}{2}, \\ 0 & \text{if } \ y < -\frac{1}{2}, \\ y + \frac{1}{2} & \text{else.} \end{cases}$$

Similarly, let $f_+ \in C(\partial\Omega_+) \cap BV(\partial\Omega_+)$ be defined as follows:

$$f_+(x,y) = \begin{cases} 1 & \text{if } \ y > 1, \\ 0 & \text{if } \ y < -1, \\ \frac{1}{2}(y+1) & \text{else.} \end{cases}$$

We check the admissibility conditions (H1)-(H4). By definition, $f \in BV(\partial\Omega)$. Moreover, we can decompose $\partial\Omega$ as in (H2): in the notation introduced at the beginning of Section 4, let us call the arc where $f_\pm$ is increasing $\chi^\pm$ and the arc where $f_\pm$ is decreasing $\Gamma^\pm$ (we drop the index i as there is only one such arc). We call the remaining arcs, on which f is constant, $F_1^\pm$ (with $f \equiv 1$ on $F_1^\pm$) and $F_2^\pm$ (with $f \equiv 0$ on $F_2^\pm$). In addition, $TV(f \llcorner \chi^-) = 1 = TV(f \llcorner \chi^+)$ and $TV(f \llcorner \Gamma^-) = 1 = TV(f \llcorner \Gamma^+)$. The situation is presented on Figure 7.1.

As for the visibility condition (H3), we check that the tangent line to the inner circle $\partial B(0,1)$ at $(\frac{\sqrt{3}}{2}, \pm\frac{1}{2})$ crosses the outer circle $\partial B(0,2)$ at $(\sqrt{3}, \mp 1)$, hence χ^- is visible from χ^+; similarly, Γ^- is visible from Γ^+.

Finally, we look at condition (H4). The idea behind it is such that the transport should take place between $\chi^\pm$ and between $\Gamma^\pm$, so that transport rays lie inside Ω. Now, fix four points which are ends of two transport rays (of which we may think as points in the preimage $f^{-1}(t)$) $p_\pm \in \chi^\pm$ and $q_\pm \in \Gamma^\pm$. The visibility conditions enforce that the transport rays between these points are $p_- p_+$ and $q_- q_+$; we have to make sure that it is in fact the shortest connection possible between these four points. In other words, we have to check that

$$|p_- - p_+| + |q_- - q_+| < |p_- - q_-| + |p_+ - q_+|,$$

as we can exclude the connection between points in χ^- and Γ^+, because both sets lie in the support of f^-. First, we see that

$$|p_- - p_+| + |q_- - q_+| \leq d_M(\chi^+, \chi^-) + d_M(\Gamma^+, \Gamma^-)$$

and

$$\text{dist}(\chi^-, \Gamma^-) + \text{dist}(\chi^+, \Gamma^+) \leq |p_- - q_-| + |p_+ - q_+|.$$

Yet, we have $d_M(\chi^+, \chi^-) = d_M(\Gamma^+, \Gamma^-) = \text{dist}(\chi^-, \Gamma^-) = \sqrt{3}$ and $\text{dist}(\chi^+, \Gamma^+) = 2\sqrt{3}$. Hence,

$$d_M(\chi^+, \chi^-) + d_M(\Gamma^+, \Gamma^-) = 2\sqrt{3} < 3\sqrt{3} = \text{dist}(\chi^-, \Gamma^-) + \text{dist}(\chi^+, \Gamma^+)$$

and (H4) holds. Hence, by Theorem 7.4.4 there exists a solution to the Beckmann problem with boundary data $g = \partial_\tau f$ and, by Proposition 7.4.6, there exists a unique solution to the least gradient problem with boundary data f.

7.5 $W^{1,p}$ regularity of the solution to the least gradient problem

The aim of this section is to study the $W^{1,p}$ regularity of the solution u of the least gradient problem (7.3.2) in the case where the domain Ω is an annulus. Recalling the relationship between the solution u of the least gradient problem (7.3.2) and the minimiser v of the Beckmann problem (7.3.1), which was given by a rotation of the derivative, $v = R_{-\frac{\pi}{2}} Du$, we see that the study of the $W^{1,p}$ regularity of u is equivalent to the study of L^p summability of the transport density $\sigma = |v|$. The difficulty here is that the measures g^+ and g^- are

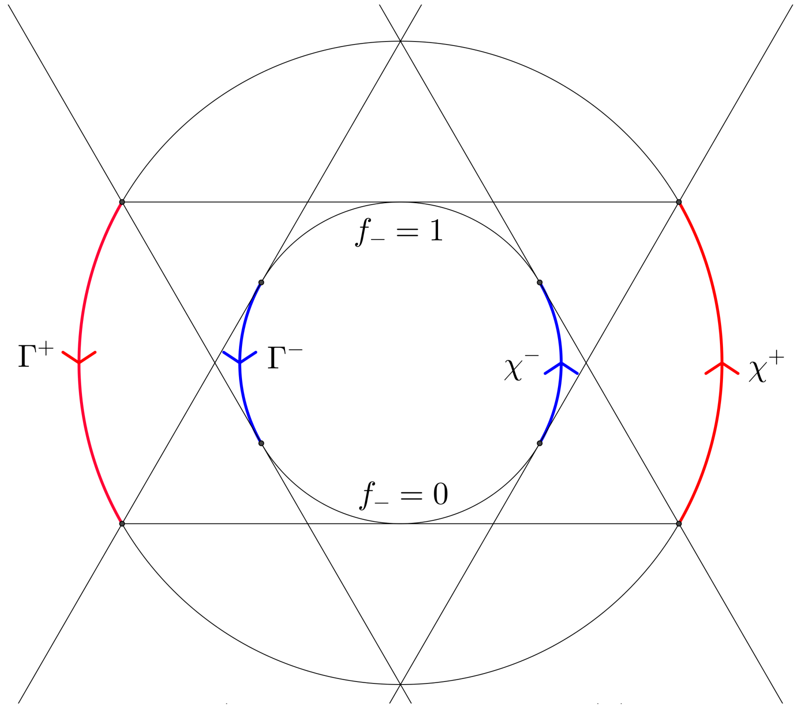


Figure 7.1: Visibility conditions in practice

concentrated on the boundary $\partial\Omega$, so they are not absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^2$. As a consequence, we cannot use the classical results on L^p summability of the transport density between two L^p densities g^+ and g^- on Ω, see for instance [13–15, 59].

In the case of uniformly convex domains, $W^{1,p}$ regularity of solutions to (LGP) has been studied in [19]. The authors proved the following statement (a partial result in this direction concerning L^p estimates on the transport density has also been proved in [18]):

$$f \in W^{1,p}(\partial\Omega) \Rightarrow u \in W^{1,p}(\Omega), \text{ for every } p \leq 2. \tag{7.5.1}$$

In addition, they introduce a counterexample to the $W^{1,p}$ regularity of u for $p > 2$. More precisely, it is possible to construct a Lipschitz function f on the boundary so that the corresponding solution u of the least gradient problem is not in $W^{1,2+\varepsilon}(\Omega)$ for any $\varepsilon > 0$. In our case, the problem is that the domain is an annulus, so we cannot use the results of [19] to obtain L^p summability on the transport density σ (or equivalently, $W^{1,p}$ regularity for the solution u of (7.4.1)) and we need to to study the L^p summability of the transport density on an annulus separately. We need an additional structural assumption, somewhat stronger than (H3), which we will use in the proof in place of uniform convexity of $\partial\Omega$:

(H5) $\exists\, c > 0 \;\forall\, x \in \chi_i^-$ (resp. Γ_i^-), $y \in \chi_i^+$ (resp. Γ_i^+), we have $(y - x) \cdot \nu(x) \geq c$,

where $\nu(x)$ is the outward normal to $\partial\Omega_-$ at x. Under (H5), we will show that if F_i^+ is a flat part (i.e. f_+ is constant on F_i^+), then the following statement holds:

$$f \in W^{1,p}(\partial\Omega) \Rightarrow u \in W^{1,p}(\Omega), \text{ for every } p \in [1, \infty].$$

On the other hand, if any F_i^+ is not a flat part, the $W^{1,p}$ estimates for $p > 2$ on the solution u of (7.3.2) fail. The reason is that since there is transport of some mass between subsets of $\partial\Omega^+$, we are in the setting of [19] and we may use an adaptation of the counterexample presented there. In this case, one can only prove (as in [19]) the following:

$$f \in W^{1,p}(\partial\Omega) \Rightarrow u \in W^{1,p}(\Omega), \text{ for every } p \leq 2.$$

In all that follows, Ω will be an annulus in the sense of Definition 7.2.1. Assume there exists a unique optimal transport plan γ between g^+ and g^- (for instance the one given by Theorem 7.4.4). In order to study the L^p summability of the transport density σ in the case where the domain Ω is an annulus, we will use a similar technique as in the proof of [19, Proposition 3.1]. The main result is the following Theorem:

Theorem 7.5.1. *Under assumptions (H1)-(H5) and the assumption that F_i^+ is a flat part for each i, for all $p \in [1,\infty]$ the transport density σ belongs to $L^p(\Omega)$ as soon as $g \in L^p(\partial\Omega)$. Moreover, if there is some i such that F_i^+ is not a flat part, then the same result holds for every $p \leq 2$ as soon as the exterior domain Ω_+ is uniformly convex.*

Proof. Firstly, let us suppose that F_i^+ is a flat part for every i. Assume for now that the target measure g^- is atomic with $(x_{i,j}^{\pm})_{\{1\leq j\leq n\}}$ being its atoms. Here, for every i and for all $j \in \{1,...,n\}$ we have $x_{i,j}^+ \in \Gamma_i^+$ and $x_{i,j}^- \in \chi_i^-$. We denote

$$\Omega_{i,j}^+ = \left\{ y \in \overline{\Omega} : \ y = (1-t)x + tx_{i,j}^+ \text{ with } x \in \Gamma_i^- \text{ and } t \in [0,1] \right\}.$$

Similarly, we denote by

$$\Omega_{i,j}^- = \left\{ y \in \overline{\Omega} : \ y = (1-t)x + tx_{i,j}^- \text{ with } x \in \chi_i^+ \text{ and } t \in [0,1] \right\}.$$

We recall that the sets $\Omega_{i,j}^{\pm}$ are pairwise disjoint subsets of Ω (up to a set of zero Lebesgue measure). Let us decompose the transport density σ into two parts $\sigma = \sigma^+ + \sigma^-$, where σ^+ and σ^- are defined via their action on a continuous function ϕ as follows:

$$\langle \sigma^+, \phi \rangle := \int_{\overline{\Omega}\times\overline{\Omega}} \int_0^{\frac{1}{2}} \phi((1-t)x + ty)|x-y| \, dt \, d\gamma(x,y), \quad \text{for all } \ \phi \in C(\overline{\Omega}),$$

and

$$\langle \sigma^-, \phi \rangle := \int_{\overline{\Omega}\times\overline{\Omega}} \int_{\frac{1}{2}}^1 \phi((1-t)x + ty)|x-y| \, dt \, d\gamma(x,y), \quad \text{for all } \ \phi \in C(\overline{\Omega}).$$

Now, we will prove L^p estimates for σ^+ (the estimates for σ^- will be similar). By Proposition 7.4.3, there is a unique optimal transport map S from g^+ to g^-, so

$$\langle \sigma^+, \phi \rangle := \int_{\overline{\Omega}} \int_0^{\frac{1}{2}} \phi((1-t)x + tS(x))|x - S(x)| \, dt \, dg^+(x), \quad \text{for all } \ \phi \in C(\overline{\Omega}).$$

Since $\Omega = \bigcup_{i,j} \Omega_{i,j}^{\pm}$ and for every $x \in \Omega_{i,j}^{\pm}$ we have $S(x) = x_{i,j}^{\pm}$, we see that

$$\langle \sigma^+, \phi \rangle := \sum_{i,j} \int_{\Omega_{i,j}^{\pm}} \int_0^{\frac{1}{2}} \phi((1-t)x + tx_{i,j}^{\pm})|x - x_{i,j}^{\pm}|\, dt\, dg^+(x), \quad \text{for all} \quad \phi \in C(\overline{\Omega}).$$

Set $\Gamma_{i,j}^- = \Omega_{i,j}^+ \cap \Gamma_i^-$ and $\chi_{i,j}^+ = \Omega_{i,j}^- \cap \chi_i^+$, for all i, j. For all $\phi \in C(\overline{\Omega})$, we have

$$\langle \sigma^+, \phi \rangle =$$

$$= \sum_{i,j} \int_{\Gamma_{i,j}^-} \int_0^{\frac{1}{2}} \phi((1-t)x+tx_{i,j}^+)|x-x_{i,j}^+|dtdg^+ + \int_{\chi_{i,j}^+} \int_0^{\frac{1}{2}} \phi((1-t)x+tx_{i,j}^-)|x-x_{i,j}^-|dtdg^+.$$

We decompose the measure σ^+ into parts which are supported on $\Omega_{i,j}^{\pm}$. We define $\sigma_{i,j}^{+\pm}$ as follows:

$$\langle \sigma_{i,j}^{++}, \phi \rangle := \int_{\Gamma_{i,j}^-} \int_0^{\frac{1}{2}} \phi((1-t)x + tx_{i,j}^+)|x - x_{i,j}^+|\, dt\, dg^+, \ \text{ for all } \ \phi \in C(\overline{\Omega})$$

and

$$\langle \sigma_{i,j}^{+-}, \phi \rangle := \int_{\chi_{i,j}^+} \int_0^{\frac{1}{2}} \phi((1-t)x + tx_{i,j}^-)|x - x_{i,j}^-|\, dt\, dg^+, \ \text{ for all } \ \phi \in C(\overline{\Omega}).$$

In this way, we have

$$\sigma^+ = \sum_{i,j} \sigma_{i,j}^{++} + \sigma_{i,j}^{+-}.$$

We will prove that $\sigma_{i,j}^{+\pm}$ is in $L^p(\Omega)$ for all i, j. Moreover, we will glue these bounds to a common bound on σ^+ in $L^p(\Omega)$. Fix i, j and consider $\sigma_{i,j}^{++}$. Set $y = (1-t)x + tx_{i,j}^+$, for every $x \in \Gamma_{i,j}^-$ and $t \in [0, 1/2]$. Then, one has

$$\langle \sigma_{i,j}^{++}, \phi \rangle := \int_{\Omega_{i,j}^{+,\frac{1}{2}}} \phi(y) \frac{|y - x_{i,j}^+|}{1-t} g^+\left(\frac{y - tx_{i,j}^+}{1-t}\right) J_{i,j}^+(y)^{-1}\, dy, \ \text{ for all } \ \phi \in C(\overline{\Omega}),$$

where

$$J_{i,j}^+(y) := \det(D_{(t,x)}y)$$

and

$$\Omega_{i,j}^{+,\frac{1}{2}} = \{(1-t)x + tx_{i,j}^+ \,:\, x \in \Gamma_{i,j}^-,\, 0 \le t \le 1/2\}.$$

An easy estimate for $J_{i,j}^+$ gives that

$$J_{i,j}^+(y) = (1-t)(x_{i,j}^+ - x) \cdot \nu(x),$$

where $\nu(x)$ is the outward normal vector to $\partial\Omega_-$ at x. Since $x \in \Gamma_{i,j}^-$ and $x_{i,j}^+ \in \Gamma_{i,j}^+$, by (H5) we have that

$$(x_{i,j}^+ - x) \cdot \nu(x) \ge c.$$

Consequently, we get

$$\sigma_{i,j}^{++}(y) \leq \frac{C}{1-t} g^+\left(\frac{y - t x_{i,j}^+}{1-t}\right) \text{ for a.e. } y \in \Omega_{i,j}^{+,\frac{1}{2}}.$$

Then,

$$||\sigma_{i,j}^{++}||^p_{L^p(\Omega_{i,j}^+)} \leq \int_{\Omega_{i,j}^+} \frac{C^p}{(1-t)^p} g^+\left(\frac{y - t x_{i,j}^+}{1-t}\right)^p dy \leq \int_{\Gamma_{i,j}^-} \int_0^{\frac{1}{2}} \frac{C^p}{(1-t)^{p-1}} g^+(x)^p \, dt \, dx$$

$$\leq \left(\int_0^{\frac{1}{2}} \frac{C^p}{(1-t)^{p-1}} \, dt\right) \int_{\Gamma_{i,j}^-} g^+(x)^p \, dx.$$

Similarly, we obtain

$$||\sigma_{i,j}^{+-}||^p_{L^p(\Omega_{i,j}^-)} \leq \left(\int_0^{\frac{1}{2}} \frac{C^p}{(1-t)^{p-1}} \, dt\right) \int_{\chi_{i,j}^+} g^+(x)^p \, dx.$$

We glue together the estimates on the sets $\Omega_{i,j}^{\pm}$ and obtain

$$||\sigma^+||^p_{L^p(\Omega)} = \sum_{i,j} ||\sigma_{i,j}^{++}||^p_{L^p(\Omega_{i,j}^+)} + ||\sigma_{i,j}^{+-}||^p_{L^p(\Omega_{i,j}^-)}$$

$$\leq \left(\int_0^{\frac{1}{2}} \frac{C^p}{(1-t)^{p-1}} \, dt\right) \sum_{i,j} \left(\int_{\chi_{i,j}^+} g^+(x)^p \, dx + \int_{\Gamma_{i,j}^-} g^+(x)^p \, dx\right) \leq C^p \int_{\partial\Omega} g^+(x)^p \, dx.$$

Passing to the limit when $n \to \infty$, we infer that as soon as $g^+ \in L^p(\partial\Omega)$, the positive part σ^+ of the density of transport between g^+ and g^- is in $L^p(\Omega)$. Moreover, σ^+ satisfies the following estimate:

$$||\sigma^+||^p_{L^p(\Omega)} \leq C^p \int_{\partial\Omega} g^+(x)^p \, dx.$$

We proceed similarly for g^- and obtain

$$||\sigma^-||^p_{L^p(\Omega)} \leq C^p \int_{\partial\Omega} g^-(x)^p \, dx,$$

so

$$||\sigma||_{L^p(\Omega)} \leq C||g||_{L^p(\partial\Omega)} \text{ for every } p \in [1, \infty].$$

On the other hand, if there is some F_i^+ which is not flat, then we can decompose the transport density σ into three parts: σ^{+-}, σ^{-+} and σ^{++}, where σ^{+-} denotes the transport density between $g^+_{|\,\partial\Omega^+ \setminus \cup_i F_i^{++}}$ and $g^-_{|\partial\Omega^-}$, σ^{-+} denotes the transport density between $g^+_{|\partial\Omega^-}$ and $g^-_{|\,\partial\Omega^+ \setminus \cup_i F_i^{+-}}$ and σ^{++} denotes the transport density between $g^+_{|\cup_i F_i^{++}}$ and $g^-_{|\cup_i F_i^{+-}}$. We have already seen that the two transport densities σ^{+-} and σ^{-+} are both in $L^p(\Omega)$ provided that $g \in L^p(\partial\Omega)$. Yet, if Ω_+ is uniformly convex, then from [19] we have that $\sigma^{++} \in L^p(\Omega)$ as soon as $g \in L^p(\partial\Omega)$ with $p \leq 2$. This completes the proof. □

Finally, by the equivalence between a version of the least gradient problem and the optimal transport problem we get the following:

Corollary 7.5.2. *Under assumptions (H1)-(H5), together with the assumption that F_i^+ is a flat part for each i, for all $p \in [1, \infty]$ the solution u of Problem* (7.3.2) *belongs to $W^{1,p}(\Omega)$ as soon as $f \in W^{1,p}(\partial\Omega)$. On the other hand, if there is some i such that F_i^+ is not a flat part, then the same result holds for every $p \leq 2$ as soon as Ω_+ is uniformly convex.*

7.6 Conclusions

In this Section, we will present a few examples and closing remarks to show both the limits of the approach presented in Sections 7.3 and 7.4 and the possible extensions of these results. In particular, we will see that while assumptions (H1)-(H4) are not optimal, they are close to optimal.

The first example concerns assumption (H1). We required the measure g to be finite in order to use optimal transport techniques, which translates to the assumption $f \in BV(\partial\Omega)$ in the least gradient problem. However, in the setting of the least gradient problem alone we do not have to assume $f_+ \in BV(\partial\Omega_+)$ and the solutions might still exist.

Example 7.6.1. Let $\Omega = B(0,2)\backslash\overline{B(0,1)}$. Let $h : [1,2] \to [1,2]$ be an arbitrary continuous function with infinite total variation and such that $h(1) = 1$. Let $u_0 \in C \cap BV(B(0,2))$ be a solution to the least gradient problem on $B(0,2)$ with boundary data

$$f_0(x,y) = \begin{cases} 1 & y < 1 \\ h(y) & y \geq 1 \end{cases}$$

given by Theorem 2.1.5. Take the boundary data f equal to $f_-(x,y) = y$ and

$$f_+(x,y) = \begin{cases} y & y < 1 \\ h(y) & y \geq 1. \end{cases}$$

Then, even though condition (H1) is violated, the solution to the least gradient problem exists and equals

$$u(x,y) = \begin{cases} y & y < 1 \\ u_0(x,y) & y \geq 1. \end{cases}$$

The second example concerns assumption (H2). It requires the boundary data on $\partial\Omega_+$ and $\partial\Omega_-$ to have the same number of monotonicity intervals and determines the total variation on these intervals. By Lemma 7.2.3, we already know that $TV(f_-) \leq TV(f_+)$; let us see what can happen if the inequality is strict.

Example 7.6.2. Let $\Omega = B(0,2)\backslash\overline{B(0,1)}$ and set boundary data to equal $f_-(x,y) = 0$ and $f_+(x,y) = y$. Then the solution to the least gradient problem does not exist.

The third example also concerns assumption (H2). It shows that intervals of monotonicity do not have to be separated by flat parts in order for a solution to exist. However, as we can see from Proposition 7.2.4, this requires a very special configuration of the boundary data.

Example 7.6.3. Let $\Omega = B(0,2)\backslash\overline{B(0,1)}$ and set boundary data to equal $f(x,y) = y$. Then the solution to the least gradient problem exists even though condition (H2) is violated.

The fourth example concerns assumption (H3). It shows that if the intervals of monotonicity of the boundary data g are not visible from one another, then the solution might not exist.

Example 7.6.4. Let $\Omega = B(0,M)\backslash\overline{B(0,M-\varepsilon)}$, where $M > 0$ is a large constant and $\varepsilon > 0$ is small enough. Set boundary data to equal

$$f_-(x,y) = \begin{cases} 0 & x < 0, \\ x & x \in [0,1], \\ 1 & x > 1, \end{cases}$$

and

$$f_+(x,y) = \begin{cases} 0 & y < 0, \\ y & y \in [0,1], \\ 1 & y > 1. \end{cases}$$

Then, the visibility condition (H3) fails and if $u \in BV(\Omega)$, it is not possible that each connected component of $\{u \geq t\}$ for $t \in (0,1)$ is a line segment, as it would lie outside $\overline{\Omega}$. Hence, there is no solution to the least gradient problem.

The second remark concerns an anisotropic version of the least gradient problem. Suppose that ϕ is a strictly convex norm on $\mathbb{R}^2$ and consider the anisotropic least gradient problem with respect to ϕ. Since the only connected minimal surfaces with respect to ϕ are line segments, in light of the analysis performed in [19] for strictly convex domains Ω we still have a one-to-one correspondence between gradients of BV functions and vector-valued measures with zero divergence. This leads to the following:

Remark 7.6.5. Suppose that ϕ is a strictly convex norm on $\mathbb{R}^2$. Then there is a one-to-one correspondence between minimisers of the following problems:

$$\min\left\{ \int_{\overline{\Omega}} \phi(R_{-\frac{\pi}{2}}v) \, : \, v \in \mathcal{M}(\overline{\Omega};\mathbb{R}^2), \; \nabla\cdot v = g \right\} \tag{7.6.1}$$

and

$$\min\left\{ \int_{\Omega} \phi(Du) \, : \, u \in BV(\Omega), \, \partial_\tau(Tu) = g \right\}. \tag{7.6.2}$$

Moreover, both problems admit solutions under the assumptions (H1)-(H4) from Section 4 with ϕ replacing the Euclidean norm in condition (H4), and Theorem 7.5.1 also remains true in the anisotropic setting.

Finally, let us note that the analysis undertaken in this Chapter could also be used to study the least gradient problem on general Lipschitz domains $\Omega \subset \mathbb{R}^2$, regardless of its homotopy type. In the following Remark, we highlight some results in this Chapter that could be easily retrieved for general Lipschitz domains.

Remark 7.6.6. Let $\Omega = \Omega^0 \backslash \bigcup_{i=1}^N \overline{\Omega_i} \subset \mathbb{R}^2$, where Ω^0 is an open bounded set with Lipschitz boundary and $\Omega_i \subset\subset \Omega^0$ are pairwise disjoint open bounded convex sets. Then, the distances between Ω^0 and Ω_i, as well as distances between Ω_i and Ω_j are bounded from below, hence we can reproduce the proof of Lemma 7.2.2 and show that on every connected component $\partial\Omega_i$ of the boundary, except from the outer component $\partial\Omega^0$, the trace of a least gradient function has bounded variation. Hence, the assumption that $f \in BV(\partial\Omega)$ is sensible, and under this assumption we may prove equivalence between problems (7.3.1) and (7.3.2).

However, if the homotopy type of the domain is highly nontrivial, or if Ω_i is not strictly convex, the admissibility conditions introduced in Section 4 and required for existence and uniqueness of minimisers would become much more complicated; the same applies to the discussion about $W^{1,p}$ regularity of the least gradient functions in Section 5. Therefore, we have restricted our reasoning to an annulus for clarity.

Appendices

Appendix A

Pointwise properties of BV functions

In this Chapter, we list a few results concerning pointwise properties of BV functions, which are relatively easy, but their proofs are difficult to locate in the literature. We prove these results here for completeness. For more information regarding basic BV theory, see [3], [20] or [70].

The first two Lemmas concern minima of two BV functions.

Lemma A.0.1. *Suppose that* $u, v \in BV(\Omega)$. *Then also* $\min(u,v), \max(u,v) \in BV(\Omega)$ *and the following inequality holds:*

$$\int_\Omega |D\max(u,v)| + \int_\Omega |D\min(u,v)| \leq \int_\Omega |Du| + \int_\Omega |Dv|.$$

Proof. By [3, Proposition 3.35] we have for any sets A, B of finite perimeter

$$P(A \cup B, \Omega) + P(A \cap B, \Omega) \leq P(A, \Omega) + P(B, \Omega).$$

Let us plug into this inequality $A = E_t = \{u \geq t\}$ and $B = F_t = \{v \geq t\}$. Observe that $E_t \cup F_t = \{\max(u,v) \geq t\}$ and $E_t \cap F_t = \{\min(u,v) \geq t\}$. Thus, for almost every t (such that E_t and F_t have finite perimeter) we have

$$P(\{\max(u,v) \geq t\}, \Omega) + P(\{\min(u,v) \geq t\}, \Omega) \leq$$
$$\leq P(\{u \geq t\}, \Omega) + P(\{v \geq t\}, \Omega).$$

Integration with respect to t and the co-area formula give the result. □

Lemma A.0.2. *Let* $u, v \in BV(\Omega)$. *Then*

$$T\min(u,v) = \min(Tu, Tv) \qquad \mathcal{H}^{N-1} - a.e. \text{ on } \partial\Omega.$$

In particular, if $Tu = Tv = h$, *then* $T\min(u,v) = h$. *Analogous result holds for* $\max(u,v)$.

Proof. One inequality is obvious: the trace is a positive operator, so the inequality

$$\min(u, v) \le u$$

implies that $T\min(u, v) \le Tu$. Similarly,

$$T\min(u, v) \le Tv,$$

so $T\min(u, v) \le \min(Tu, Tv)$.

For the opposite inequality, recall that for any $w \in BV(\Omega)$ on a set of full $\mathcal{H}^{N-1}$ measure we have

$$\fint_{B(x,r)\cap\Omega} |w(y) - Tw(x)|\, dy \to 0.$$

Observe that this implies (by reverse triangle inequality for L^1 norm)

$$0 \leftarrow \fint_{B(x,r)\cap\Omega} |w(y) - Tw(x)|\, dy \ge \left| \fint_{B(x,r)\cap\Omega} |w(y)|\, dy - |Tw(x)| \right|,$$

so

$$\fint_{B(x,r)\cap\Omega} |w(y)|\, dy \to |Tw(x)|.$$

Now, note that for every $s \in \mathbb{R}$, by linearity the trace of $w - s$ equals $Tw - s$; thus, we have

$$\fint_{B(x,r)\cap\Omega} |w(y) - s|\, dy \to |Tw(x) - s|.$$

Let $\mathcal{Z} \subset \partial\Omega$ denote the set of full measure such that for every $x \in \mathcal{Z}$ the property above holds for $w = u, v, \min(u, v)$. Fix $x \in \mathcal{Z}$. Then we have

$$\fint_{B(x,r)\cap\Omega} |u(y)|\, dy \to |a|, \qquad \fint_{B(x,r)\cap\Omega} |v(y)|\, dy \to |b|,$$

$$\fint_{B(x,r)\cap\Omega} |\min(u, v)(y)|\, dy \to |c|.$$

Without loss of generality assume that $a \ge b$. We want to prove that $c \ge \min(a, b) = b$. We argue by contradiction: assume that $a \ge b > c$. Let us shift the functions u, v by $s = a$; we obtain

$$\fint_{B(x,r)\cap\Omega} |u(y) - a|\, dy \to |a - a| = 0, \qquad \fint_{B(x,r)\cap\Omega} |v(y) - a|\, dy \to |b - a|,$$

$$\fint_{B(x,r)\cap\Omega} |\min(u - a, v - a)(y)|\, dy \to |c - a|.$$

Since $|\min(u - a, v - a)| \le |u - a| + |v - a|$, we have

$$|c - a| \le \fint_{B(x,r)\cap\Omega} |\min(u - a, v - a)(y)|\, dy \le \fint_{B(x,r)\cap\Omega} |u(y) - a|\, dy +$$

$$\fint_{B(x,r)\cap\Omega} |v(y) - a|\, dy \to 0 + |b - a|,$$

but in the beginning we assumed that $a \geq b > c$, in particular $|c-a| > |b-a|$, contradiction. Thus, $T\min(u, v)(x) \geq \min(Tu(x), Tv(x))$ for every $x \in \mathcal{Z}$. Since it is a set of $\mathcal{H}^{N-1}$-full measure, the Lemma is proved. □

We bring together Lemmata A.0.1 and A.0.2 to notice that

Corollary A.0.3. *If $u, v \in BV(\Omega)$ are solutions to the least gradient problem with boundary data $h \in L^1(\Omega)$ in the sense of traces, then so are $\min(u, v)$ and $\max(u, v)$.* □

The last Lemma concerns the relationship between the superlevel sets of a trace of a BV function and the traces of characteristic functions of superlevel sets of this function.

Lemma A.0.4. *Let $u \in BV(\Omega)$ and $Tu = f$. For all except countably many $t \in \mathbb{R}$ we have*

$$T\chi_{\{u\geq t\}} = \chi_{\{f\geq t\}}.$$

Proof. Denote $E_t = \{u \geq t\}$. Fix $t \in \mathbb{R}$ such that $\mathcal{H}^{N-1}(\{f = t\}) = 0$; this happens for all but countably many t. For $\mathcal{H}^{N-1}$-almost every $x \in \partial\Omega$ we have

$$\fint_{\Omega\cap B(x,r)} |u(y) - f(x)|\, dy \to 0 \qquad \text{as } r \to 0$$

and

$$\fint_{\Omega\cap B(x,r)} |\chi_{E_t}(y) - T\chi_{E_t}(x)|\, dy \to 0 \qquad \text{as } r \to 0.$$

We denote the set of such points by $\mathcal{Z}$. By our assumption on t, the set $\mathcal{Z} \cap \{f \neq t\}$ is of full measure. Now, fix $x \in \mathcal{Z} \cap \{f \neq t\}$.

There are two possibilities: either $x \in \{f > t\}$ or $x \in \{f < t\}$. Without loss of generality assume that $f(x) = s > t$. Suppose that $T\chi_{E_t}(x) \neq 1 = \chi_{\{f>t\}}(x)$. Then on a subsequence $r_n \to 0$ we have

$$\fint_{\Omega\cap B(x,r_n)} |\chi_{E_t}(y) - 1|\, dy \geq c.$$

We rewrite this as

$$c \leq \fint_{\Omega\cap B(x,r_n)} |\chi_{E_t}(y) - 1|\, dy = \fint_{\Omega\cap B(x,r_n)} |\chi_{\Omega\backslash E_t}(y)|\, dy = \frac{|(\Omega \cap B(x, r_n))\backslash E_t|}{|\Omega \cap B(x, r_n)|}.$$

Now, we see that this leads to a contradiction with $Tu(x) = f(x) = s$. We calculate

$$\fint_{\Omega\cap B(x,r_n)} |u(y) - f(x)|\, dy = \frac{1}{|\Omega \cap B(x, r_n)|} \int_{\Omega\cap B(x,r_n)} |u(y) - s|\, dy \geq$$

$$\geq \frac{1}{|\Omega \cap B(x, r_n)|} \int_{(\Omega \cap B(x,r_n)) \setminus E_t} |u(y) - s| \, dy \geq$$

$$\geq \frac{1}{|\Omega \cap B(x, r_n)|} \int_{(\Omega \cap B(x,r_n)) \setminus E_t} |t - s| \, dy =$$

$$= \frac{|(\Omega \cap B(x, r_n)) \setminus E_t|}{|\Omega \cap B(x, r_n)|} |t - s| \geq c(t - s) > 0,$$

hence there exists a sequence $r_n \to 0$ such that the mean integral condition defining the trace of u at x is not satisfied, contradiction. Thus, $T\chi_{E_t}(x) = 1 = \chi_{\{f > t\}}(x)$. The other case is handled similarly. $\square$

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