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Limits of saturated ideals of points with applications to secant varieties

PhD dissertation

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I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

The main object of study of this thesis is a class of multigraded Hilbert schemes. Given a smooth projective toric variety X with the Cox ring $S[X]$ we consider the Hilbert function of r points on X in general position, i.e. $h_{r,X}\colon \operatorname{Pic}(X) \to \mathbb{N}$ given by

$$h_{r,X}([D]) = \min\{\dim_{\mathbb{C}} \Gamma(X, \mathcal{O}_X(D)),\ r\}.$$

The multigraded Hilbert scheme $\operatorname{Hilb}^{h_{r,X}}_{S[X]}$ associated with $S[X]$ and $h_{r,X}$ has a distinguished irreducible component $\operatorname{Slip}_{r,X}$ which is the closure of the locus of points corresponding to radical ideals that are saturated. The aim of this dissertation is to find necessary or sufficient conditions for a point of $\operatorname{Hilb}^{h_{r,X}}_{S[X]}$ to belong to $\operatorname{Slip}_{r,X}$. This problem is motivated by the border apolarity lemma established by Buczyńska and Buczyński.

Our main focus is on the case $X = \mathbb{P}^n$. We present three necessary conditions for $[I] \in \operatorname{Hilb}^{h_{r,\mathbb{P}^n}}_{S[\mathbb{P}^n]}$ to be in $\operatorname{Slip}_{r,\mathbb{P}^n}$. The first of them is obtained by bounding the degrees of minimal generators of saturated ideals $J \subseteq S[\mathbb{P}^n]$ such that $[J] \in \operatorname{Hilb}^{h_{r,\mathbb{P}^n}}_{S[\mathbb{P}^n]}$. The second criterion is based on the computation of the Hilbert polynomial of a power of a radical ideal J such that $[J] \in \operatorname{Hilb}^{h_{r,\mathbb{P}^n}}_{S[\mathbb{P}^n]}$ and establishing the bound on the degree from which it agrees with the Hilbert function. The proof of the third necessary condition uses deformation theory and flag multigraded Hilbert schemes. We also present a sufficient condition for $[I] \in \operatorname{Hilb}^{h_{r,\mathbb{P}^2}}_{S[\mathbb{P}^2]}$ to be in $\operatorname{Slip}_{r,\mathbb{P}^2}$.

We consider a morphism with connected fibers $f\colon X \to Y$ between smooth projective toric varieties. We obtain a necessary condition for $[I] \in \operatorname{Hilb}^{h_{r,X}}_{S[X]}$ to be in $\operatorname{Slip}_{r,X}$. Namely, we show that there is a natural morphism $\operatorname{Hilb}^{h_{r,X}}_{S[X]} \to \operatorname{Hilb}^{h_{r,Y}}_{S[Y]}$ and that it maps $\operatorname{Slip}_{r,X}$ onto $\operatorname{Slip}_{r,Y}$. We also prove another necessary condition in the cases that X is a product of $k \geq 2$ projective spaces.

We illustrate the criteria with examples. In particular, we describe all ideals which correspond to points of $\operatorname{Slip}_{r,\mathbb{P}^2}$ for $r \leq 6$. Furthermore, we apply our techniques to obtain some results on wild polynomials.

Keywords: multigraded Hilbert schemes, saturated ideals of points, smooth projective toric varieties, secant varieties, border rank.

AMS MSC 2020 classification: 14C05, 14M25, 14N07.

Streszczenie

Głównym obiektem badań niniejszej rozprawy jest klasa schematów Hilberta z wielogradacją. Dla gładkiej rzutowej rozmaitości torycznej X z pierścieniem Coxa $S[X]$, rozważamy funkcję Hilberta r punktów w położeniu ogólnym na X, tzn. $h_{r,X}\colon \operatorname{Pic}(X) \to \mathbb{N}$ zadaną przez

$$h_{r,X}([D]) = \min\{\dim_{\mathbb{C}} \Gamma(X, \mathcal{O}_X(D)),\ r\}.$$

Schemat Hilberta z wielogradacją $\operatorname{Hilb}^{h_{r,X}}_{S[X]}$ stowarzyszony z $S[X]$ i $h_{r,X}$ ma wyróżnioną składową nieprzywiedlną $\operatorname{Slip}_{r,X}$, która jest domknięciem zbioru punktów odpowiadających ideałom radykalnym i nasyconym. Celem tej rozprawy jest znalezienie kryteriów koniecznych lub wystarczających do tego by punkt $\operatorname{Hilb}^{h_{r,X}}_{S[X]}$ należał do $\operatorname{Slip}_{r,X}$. Motywacja do badania tego problemu pochodzi z lematu o brzegowej abiegunowości udowodnionego przez Buczyńską i Buczyńskiego.

Główny nacisk kładziemy na przypadek $X = \mathbb{P}^n$. Prezentujemy trzy warunki wystarczające do tego by punkt $[I] \in \operatorname{Hilb}^{h_{r,\mathbb{P}^n}}_{S[\mathbb{P}^n]}$ należał do $\operatorname{Slip}_{r,\mathbb{P}^n}$. Pierwszy z nich jest uzyskany poprzez ograniczenie stopni minimalnych generatorów ideałów nasyconych $J \subseteq S[\mathbb{P}^n]$ takich, że $[J] \in \operatorname{Hilb}^{h_{r,\mathbb{P}^n}}_{S[\mathbb{P}^n]}$. Drugie kryterium bazuje na obliczeniu wielomianu Hilberta potęgi ideału radykalnego J takiego, że $[J] \in \operatorname{Hilb}^{h_{r,\mathbb{P}^n}}_{S[\mathbb{P}^n]}$ oraz uzyskaniu ograniczenia na stopień od którego zgadza się on z funkcją Hilberta. Dowód trzeciego kryterium wykorzystuje teorię deformacji i flagowe schematy Hilberta z wielogradacją. Prezentujemy również warunek wystarczający do tego aby $[I] \in \operatorname{Hilb}^{h_{r,\mathbb{P}^2}}_{S[\mathbb{P}^2]}$ należał do $\operatorname{Slip}_{r,\mathbb{P}^2}$.

Rozważamy morfizm o spójnych włóknach $f\colon X \to Y$ pomiędzy gładkimi rzutowymi rozmaitościami torycznymi. Uzyskujemy warunek konieczny do tego aby $[I] \in \operatorname{Hilb}^{h_{r,X}}_{S[X]}$ należał do $\operatorname{Slip}_{r,X}$. Mianowicie pokazujemy, że istnieje naturalny morfizm $\operatorname{Hilb}^{h_{r,X}}_{S[X]} \to \operatorname{Hilb}^{h_{r,Y}}_{S[Y]}$, który odzworowuje $\operatorname{Slip}_{r,X}$ na $\operatorname{Slip}_{r,Y}$. Dowodzimy również innego warunku koniecznego w przypadku gdy X jest produktem $k \geq 2$ przestrzeni rzutowych.

Kryteria ilustrujemy przykładami. W szczególności, opisujemy wszystkie ideały, które odpowiadają punktom $\operatorname{Slip}_{r,\mathbb{P}^2}$ dla $r \leq 6$. Co więcej, wykorzystujemy nasze metody do uzyskania pewnych wyników o dzikich wielomianach.

Słowa kluczowe: schematy Hilberta z wielogradacją, nasycone ideały punktów, gładkie rzutowe rozmaitości toryczne, rozmaitości siecznych, ranga brzegowa.

Klasyfikacja tematyczna AMS 2020: 14C05, 14M25, 14N07.

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The computer algebra system Macaulay2 [42] was helpful for computing examples.

Contents

Chapter 1

Introduction

In real life, it is often the case, that certain objects naturally appear together but from human perspective one of them is more interesting than other ones. Typical examples include fruits and their peels and seeds (people care less about the peel and the seeds than the rest of the fruit). In such situations, a need arises to separate the part that we care about from the part that is redundant or less attractive from our point of view. In many cases the distinction is pretty clear. However, there are also more subtle examples. One of them is the farmland. Here the crops and the weeds grow together and distinguishing between them requires both more attention and certain knowledge.

The main topic of this thesis is about

identifying the "good" inside the set of "all"

in the setting that will be described below.

Problems of similar nature appear commonly in mathematics. For a basic example, consider a finite dimensional real vector space V, the set $\mathrm{End}_{\mathbb{R}}(V)$ of all linear endomorphisms of V and its subset $\mathrm{Aut}_{\mathbb{R}}(V)$ consisting of invertible maps. Given an element $\varphi \in \mathrm{End}_{\mathbb{R}}(V)$ we can check whether it belongs to $\mathrm{Aut}_{\mathbb{R}}(V)$ by computing its determinant.

Another easy to state problem is provided by univariate polynomials with real coefficients. We might be interested in understanding which of them have a real root. There is an easy sufficient condition. Namely, if the degree of the polynomial is odd, then it has a real root. However, there are also more subtle criteria like Sturm's theorem [57, §5.2] which gives the number of distinct real roots of a given polynomial in a given interval.

The main motivational example for this thesis in the realm of algebraic geometry is the Hilbert scheme of r points in the projective n-space over the complex numbers $\mathbb{C}$. Before explaining this example in more detail, we outline the main results about the Hilbert schemes concentrating on Hilbert schemes of points. The Hilbert scheme $\mathcal{H}ilb(\mathbb{P}^n)$ is a scheme parametrizing all closed subschemes of $\mathbb{P}^n$. It was constructed by Grothendieck [43]. It has a decomposition into the disjoint union $\mathcal{H}ilb(\mathbb{P}^n) = \coprod_P \mathcal{H}ilb_P(\mathbb{P}^n)$ where P is the Hilbert polynomial of a closed subscheme of $\mathbb{P}^n$ and $\mathcal{H}ilb_P(\mathbb{P}^n)$ parametrizes all closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial P. More generally, one may consider $\mathcal{H}ilb(X)$ or $\mathcal{H}ilb_P(X)$ for a projective scheme $X \subseteq \mathbb{P}^N$ over $\mathbb{C}$. In 1966 Hartshorne [46] proved that $\mathcal{H}ilb_P(\mathbb{P}^n)$ is connected for every Hilbert polynomial P. If the Hilbert polynomial P is constant and equal to r for some positive integer r, we write $\mathcal{H}ilb_r(\mathbb{P}^n)$ instead of $\mathcal{H}ilb_P(\mathbb{P}^n)$. In 1968 Fogarty [35] showed that $\mathcal{H}ilb_r(X)$ is smooth and irreducible if X is an irreducible smooth surface. On the other hand, $\mathcal{H}ilb_r(\mathbb{P}^n)$ is reducible for $n \geq 3$ and $r \gg 0$.

This was established by Iarrobino [53] in 1972. The question, whether $\mathcal{H}ilb(X)$ is reduced was also addressed. Mumford [68] showed in 1962 that $\mathcal{H}ilb_P(\mathbb{P}^3)$ is not reduced for P a polynomial of degree 1. Jelisiejew [60] showed in 2020 that $\mathcal{H}ilb_r(\mathbb{P}^n)$ is in general non-reduced.

We return to the description of our main motivational example of identifying "good" from "all". For fixed positive integers r, n, the closure of the locus of points of $\mathcal{H}ilb_r(\mathbb{P}^n)$ corresponding to r-tuples of points in $\mathbb{P}^n$ is an irreducible component of $\mathcal{H}ilb_r(\mathbb{P}^n)$. We call this component the smoothable component and we denote it by $\mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$. By the above-mentioned result of Iarrobino, in general, $\mathcal{H}ilb_r(\mathbb{P}^n) \neq \mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$. Thus, if we care only about the set of r-tuples of points of the projective space $\mathbb{P}^n$ together with their limits, we need to have methods of identifying whether a given point of the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^n)$ belongs to $\mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$. This problem was studied by Cartwright, Erman, Velasco and Viray [20]. Similar problem for Gorenstein subschemes was considered by Casnati, Jelisiejew and Notari [21] and Jelisiejew [58].

In this thesis we concentrate on an analogous problem in the setting of multigraded Hilbert schemes "of points in general position". Multigraded Hilbert schemes were introduced by Haiman and Sturmfels [45] in 2004. Let S be a polynomial ring over $\mathbb{C}$, graded by an abelian group A. Given a numerical function $h\colon A \to \mathbb{N}$, there is a corresponding multigraded Hilbert scheme Hilb_S^h parametrizing homogeneous ideals I of S such that S/I has Hilbert function h.

It is important to emphasize, that even when S is a standard $\mathbb{Z}$-graded polynomial ring and h is the Hilbert function of a closed subscheme of a projective space, this leads to an object different than the classical Hilbert scheme, which is specified by Hilbert *polynomial*. In the case of multigraded Hilbert schemes we care about the Hilbert function in all degrees, while in the case of the usual Hilbert scheme we are interested only in the Hilbert function in large degrees.

The concept of a multigraded Hilbert scheme is a common generalization of various notions of Hilbert schemes:

1. $\mathcal{H}ilb_r(\mathbb{A}^n)$ - the Hilbert scheme of r-points in affine n-space;

2. $\mathcal{H}ilb_P(\mathbb{P}^n)$ - the Hilbert scheme parametrizing closed subschemes of $\mathbb{P}^n$ with Hilbert polynomial P;

3. the so-called toric Hilbert schemes whose special cases where studied by Peeva and Stillman [70].

We present some known results about general multigraded Hilbert schemes Hilb_S^h. Maclagan and Smith [64] showed in 2010 that if S is a polynomial ring in two variables, then Hilb_S^h is smooth and irreducible (for any grading of S in any abelian group A and for any Hilbert function $h\colon A \to \mathbb{N}$). Beside this, little is known about general multigraded Hilbert scheme. People usually study one of the three particular cases described above. It turns out that there exists a non-connected toric Hilbert scheme. In 2005 Santos [75] gave such an example for a polynomial ring in 26 variables graded by $\mathbb{Z}^6$. This is in sharp contrast with the Hartshorne's result [46] concerning $\mathcal{H}ilb_P(\mathbb{P}^n)$.

It is worth comparing the above-mentioned facts about the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^n)$ with the results on the multigraded Hilbert schemes. The Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^n)$ is nicely-behaved, i.e. smooth and irreducible, for $n = 2$ [35]. On the other hand, the multigraded Hilbert scheme Hilb_S^h is smooth and irreducible when the polynomial ring S has two variables which in some sense corresponds to the case of the projective line. Similarly, $\mathcal{H}ilb_r(\mathbb{P}^3)$ is in general reducible [53], while Hilb_S^h can already be reducible for a polynomial ring in three variables. In fact, the class of multigraded Hilbert schemes discussed in this thesis provides natural examples of such behavior.

We introduce our main object of study in the case of projective space. Let n be a positive integer and $S = \mathbb{C}[\alpha_0, \dots, \alpha_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$. Let r be a positive integer and $h_{r,n}\colon \mathbb{Z} \to \mathbb{N}$ be the Hilbert function of r-points in general position in $\mathbb{P}^n$:

$$h_{r,n}(a) = \min\{\dim_{\mathbb{C}} S_a,\ r\}.$$

Let $\operatorname{Sip}_{r,n}$ denote the locus of points of $\operatorname{Hilb}_S^{h_{r,n}}$ corresponding to radical ideals. Let $\operatorname{Slip}_{r,n}$ be the closure of $\operatorname{Sip}_{r,n}$ in $\operatorname{Hilb}_S^{h_{r,n}}$. The subset $\operatorname{Slip}_{r,n}$ is an irreducible component [13, Prop. 3.13]. Points of $\operatorname{Hilb}_S^{h_{r,n}}$ that belong to $\operatorname{Slip}_{r,n}$ are the "good" points inside the set of "all" points from the scheme $\operatorname{Hilb}_S^{h_{r,n}}$. This is a consequence of the border apolarity lemma which we discuss below.

The main goal of this thesis is to establish sufficient and necessary conditions for a closed point in $\operatorname{Hilb}_S^{h_{r,n}}$ to be in the irreducible component $\operatorname{Slip}_{r,n}$.

We also consider analogous problem for a smooth projective toric variety X. We study the multigraded Hilbert scheme associated with the Cox ring $S[X]$ of X and the Hilbert function $h_{r,X}\colon \operatorname{Pic}(X) \to \mathbb{N}$ of r-points in general position in X, i.e.

$$h_{r,X}([D]) = \min\{\dim_{\mathbb{C}} S[X]_{[D]}, r\}.$$

See Section 4.1 for relevant definitions. Again, there is a distinguished irreducible component $\operatorname{Slip}_{r,X}$ of $\operatorname{Hilb}_{S[X]}^{h_{r,X}}$ that is the closure of the locus of radical ideals that are saturated with respect to the irrelevant ideal of X. We want to find criteria that identify points in $\operatorname{Hilb}_{S[X]}^{h_{r,X}}$ that belong to $\operatorname{Slip}_{r,X}$.

Significance of the considered problem

It is necessary to explain how does the irreducible component $\operatorname{Slip}_{r,n}$ fit into the above philosophy of identifying "good" from "all". This is based on the border apolarity lemma introduced by Buczyńska and Buczyński [13]. This result shows that there is a connection between border rank of a homogeneous polynomial and the multigraded Hilbert scheme $\operatorname{Hilb}_S^{h_{r,n}}$. Our discussion here is informal. The precise statement of the border apolarity, as well as the formal definitions of the border rank and secant varieties appear in Chapter 2. Suppose that F is a homogeneous polynomial of degree d in the polynomial ring $S^* = \mathbb{C}[x_0, \dots, x_n]$. We say that F has *rank* r if r is the smallest integer such that $F = \sum_{i=1}^{r} \ell_i^d$ for some linear forms $\ell_i \in S_1^*$. We also consider the *border rank* of F which is the smallest integer r such that $[F] \in \mathbb{P}S_d^*$ is in the closure of the set of polynomials with rank at most r. Calculating the border rank of a given polynomial is a classical problem in algebraic geometry and is strongly related to studies of secant varieties of the Veronese variety. The border apolarity lemma says that F has border rank at most r for a positive integer r if and only if there exists a point $[I] \in \operatorname{Slip}_{r,n}$ such that I is apolar to F. Thus,

points from $\operatorname{Slip}_{r,n}$ are the "good" points among "all" points of $\operatorname{Hilb}_S^{h_{r,n}}$ since they serve as witnesses of small border rank.

As a result, the more conditions (both sufficient and necessary) for a point in $\operatorname{Hilb}_S^{h_{r,n}}$ to be in the irreducible component $\operatorname{Slip}_{r,n}$ we have at our disposal, the greater the scope of applicability of the border apolarity lemma.

One potential application of border apolarity is in studying homogeneous polynomials, called wild polynomials, whose border rank is smaller than the smoothable rank (see Section 2.4 for definitions of these ranks). These polynomials are known to exist (see [12] and [52]). They appear naturally in the context of the border apolarity lemma. Indeed, by definition, it is precisely for these polynomials that the apolarity for smoothable rank (depending on the smoothable component $\mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$ of the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^n)$) fails to compute the border rank. For them, it is necessary to consider the multigraded Hilbert scheme $\mathrm{Hilb}_S^{h_{r,n}}$ and its irreducible component $\mathrm{Slip}_{r,n}$.

A crucial motivation for studying $\mathrm{Slip}_{r,X}$ for more general toric varieties than $\mathbb{P}^n$ is the problem of computing the border rank of matrix multiplication tensors. This is a vitally important problem for complexity theory, however it is very complicated. For instance the border rank of matrix multiplication tensor for 3×3 matrices is unknown. See [23] for recent progress on that problem.

Structure of the thesis and main results

In Chapter 2 we present the relevant background from commutative algebra and scheme theory. In particular, we formally define the multigraded Hilbert schemes by their functors of points. We also study the flag multigraded Hilbert schemes and basic notions of deformation theory.

Chapter 3 contains the main results of the thesis in the case of projective space. We present three necessary conditions for a point $[I] \in \mathrm{Hilb}_S^{h_{r,n}}$ to be in the irreducible component $\mathrm{Slip}_{r,n}$. Moreover, we prove a sufficient condition for a point $[I] \in \mathrm{Hilb}_S^{h_{r,2}}$ to be in $\mathrm{Slip}_{r,2}$. We illustrate these criteria with simple examples. Furthermore, we end the chapter with the complete description of points from $\mathrm{Slip}_{r,2}$ for $r \leq 6$. The complexity of these examples is perhaps surprising, especially in view of the fact that the usual Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^2)$ is smooth and irreducible.

We now summarize the main results presented in Chapter 3. We simplify the statements of some of the more technical theorems by considering their special cases, or omitting some parts of the conclusions. Given a polynomial ring $S = \mathbb{C}[\alpha_0, \ldots, \alpha_n]$, we denote by $\mathfrak{m}$ the ideal $(\alpha_0, \ldots, \alpha_n)$.

Proposition 3.1 provides a necessary condition for $[I] \in \mathrm{Hilb}_S^{h_{r,n}}$ to be in $\mathrm{Slip}_{r,n}$. It is based on bounding from the above, the degree in which all saturated ideals corresponding to points of $\mathrm{Hilb}_S^{h_{r,n}}$ are generated.

Proposition 1.1 (Proposition 3.1)**.** *Let r, n be positive integers and $I \subseteq S = \mathbb{C}[\alpha_0, \ldots, \alpha_n]$ be a homogeneous ideal such that S/I has Hilbert function $h_{r,n}$. Let $e = \min\{a \in \mathbb{Z} \mid h_{r,n}(a) = r\}$ and $d \geq e + 2$. If $\dim_{\mathbb{C}} \mathrm{Hom}_S\left(I + \mathfrak{m}^d, S/(I + \mathfrak{m}^d)\right)_0 < rn$, then $[I] \notin \mathrm{Slip}_{r,n}$.*

Theorem 3.5 shows that if $[I] \in \mathrm{Slip}_{r,n}$, then the Hilbert function of S/I^k for any positive integer k is bounded from below by $r \cdot \dim_{\mathbb{C}} S_{k-1}$ for large enough degrees (depending on k). This result is obtained by calculating the Hilbert polynomials of powers of a radical ideal corresponding to a point of $\mathrm{Hilb}_S^{h_{r,n}}$ and establishing a bound from which they agree with the Hilbert functions.

Theorem 1.2 (Theorem 3.5)**.** *Let $n, r \geq 1$ be integers and $I \subseteq S = \mathbb{C}[\alpha_0, \ldots, \alpha_n]$ be a homogeneous ideal such that S/I has Hilbert function $h_{r,n}$. Let $e = \min\{a \in \mathbb{Z} \mid h_{r,n}(a) = r\}$. If $[I] \in \mathrm{Slip}_{r,n}$, then $H_{S/I^k}(d) \geq r \cdot \dim_{\mathbb{C}} S_{k-1}$ for every positive integer k and for every $d \geq ke + k$.*

Theorem 3.12 presents a sufficient condition for $[I] \in \mathrm{Hilb}_S^{h_{r,2}}$ to be in the irreducible component $\mathrm{Slip}_{r,2}$. We show that if the Hilbert function of $S/(I : \mathfrak{m}^\infty)$ differs from $h_{r,2}$ only in one degree, then $[I] \in \mathrm{Slip}_{r,2}$. The proof is obtained by showing that $[I]$ belongs to an irreducible subset of $\mathrm{Hilb}_S^{h_{r,2}}$ which intersects $\mathrm{Slip}_{r,2}$ at a smooth point of $\mathrm{Hilb}_S^{h_{r,2}}$. Furthermore, we comment why natural generalizations of this criterion for $\mathbb{P}^n$ with $n > 2$ fail.

Theorem 1.3 (Theorem 3.12)**.** *Let r be a positive integer and $S = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2]$ be a polynomial ring. Consider a closed point $[I]$ of the multigraded Hilbert scheme $\mathrm{Hilb}_S^{h_{r,2}}$. If $(I : \mathfrak{m}^\infty)_d \neq I_d$ for a unique integer d, then $[I] \in \mathrm{Slip}_{r,2}$.*

Theorem 3.40 is the most technically involved result in this thesis. It is stated in a general setup in which the proof follows by a short argument using deformation theory. Then we discuss some conditions which imply the assumptions of Theorem 3.40. Finally, we present two applications of this theorem, Theorems 3.65 and 3.74. These are the versions of the theorem that we use in the rest of the thesis.

Theorem 1.4 (Theorem 3.40)**.** *Let $r, n \geq 1$ be integers, $S = \mathbb{C}[\alpha_0, \ldots, \alpha_n]$ be a polynomial ring and $[I] \in \mathrm{Hilb}_S^{h_{r,n}}$ be a closed point. Assume that $I \neq (I : \mathfrak{m}^\infty)$ and let d be such that $I_d \neq (I : \mathfrak{m}^\infty)_d$. Let $J = \mathfrak{m}^d \cap (I : \mathfrak{m}^\infty)$ and $K = \mathfrak{m}^d \cap I$. Assume that the following conditions hold:*

1. *the natural map $\mathrm{Hom}_S(J, S/J)_0 \to \mathrm{Hom}_S(K, S/J)_0$ is surjective;*
2. *$[J] \in \mathrm{Hilb}_S^h$ is a smooth point where h is the Hilbert function of S/J;*
3. *the natural map $\mathrm{Hom}_S(K, S/K)_0 \to \mathrm{Hom}_S(K, S/J)_0$ is surjective.*

Then there is no $[I'] \in \mathrm{Slip}_{r,n}$ such that $I'_{\geq d} = I_{\geq d}$. In particular, $[I] \notin \mathrm{Slip}_{r,n}$.

As an application of Theorem 1.4 we obtain the following result.

Theorem 1.5 (Theorem 3.65)**.** *Let $[I] \in \mathrm{Hilb}_S^{h_{r,n}}$ be a closed point corresponding to an ideal I such that $S/\overline{I}$ has Hilbert function $h_{r,1}$. Then there exists $[I'] \in \mathrm{Slip}_{r,n}$ such that $I_{\geq r-2} = I'_{\geq r-2}$ if and only if $(\overline{I}^2)_{r-2} \subseteq I_{r-2}$.*

We end Chapter 3 with the complete set-theoretic description of $\mathrm{Slip}_{r,2}$ for $r \leq 6$. To give some insight into the complexity of this problem we present here a short discussion. For $r \leq 3$, scheme $\mathrm{Hilb}_S^{h_{r,2}}$ is irreducible (see Propositions 3.36, 3.37 and 3.38). However, Corollary 3.78 shows that $\mathrm{Hilb}_S^{h_{4,2}}$ is reducible. In fact, it has two irreducible components. Here, the description of $\mathrm{Slip}_{4,2}$ follows easily from Theorem 3.65. In the next case, $r = 5$, the scheme $\mathrm{Hilb}_S^{h_{5,2}}$ still has only two irreducible components but the description of $\mathrm{Slip}_{5,2}$ obtained in Proposition 3.89 requires some further observations. Finally, $\mathrm{Hilb}_S^{h_{6,2}}$ has four irreducible components. We use all four criteria mentioned above, to obtain the description of $\mathrm{Slip}_{6,2}$ (see Proposition 3.105). Still, the proof is of significant complexity. Since there was no prior systematic study of the component $\mathrm{Slip}_{r,X}$, it was not clear what to expect. By analogy to Fogarty's result [35] on smoothness of $\mathcal{H}ilb_r(\mathbb{P}^2)$, we expected that $\mathrm{Hilb}_{S[\mathbb{P}^2]}^{h_{r,\mathbb{P}^2}}$ should not be too complicated. However, it seems that the proper analogue is rather $\mathcal{H}ilb_r(\mathbb{P}^3)$, where little is known about the smoothable component.

Chapter 4 is concerned with the case of a smooth, projective toric variety X. Here again, one may consider the multigraded Hilbert scheme $\mathrm{Hilb}_{S[X]}^{h_{r,X}}$ where $S[X]$ is the Cox ring of X and $h_{r,X}$ is the Hilbert function of r points in general position on X. Again, there is an irreducible component $\mathrm{Slip}_{r,X}$ which is defined analogously to $\mathrm{Slip}_{r,n}$ considered above. Theorem 4.15 describes a relation between $\mathrm{Slip}_{r,X}$ and $\mathrm{Slip}_{r,Y}$ where $f\colon X \to Y$ is a map of smooth projective toric varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. The proof is based on the possibility of lifting f to a homomorphism of Cox rings of X and Y. This is discussed in Subsection 4.1.3.

Theorem 1.6 (Theorem 4.15)**.** *Let $f\colon X \to Y$ be a morphism between smooth projective toric varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let r be a positive integer and $[I] \in \mathrm{Hilb}^{h_{r,X}}_{S[X]}$ be a closed point. Let $\overline{f}^{\#}\colon S[Y] \to S[X]$ be a lift of f as in Definition 4.2. If $[I] \in \mathrm{Slip}_{r,X}$ then*

$$[(\overline{f}^{\#})^{-1}(I)] \in \mathrm{Slip}_{r,Y}\,.$$

Theorem 4.25 presents another necessary condition for $[I] \in \mathrm{Hilb}^{h_{r,X}}_{S[X]}$ to be in the irreducible component $\mathrm{Slip}_{r,X}$, when X is a product of $k \geq 2$ projective spaces. The proof of the theorem is based on simple properties of Hilbert functions of saturated ideals of points in X.

Theorem 1.7 (Theorem 4.25)**.** *Let $k \geq 2$ and $n_1, \ldots, n_k$ be positive integers. Let $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ and for $i \in \{1, \ldots, k\}$ let $B(\Sigma_i) \subseteq S[X]$ be the extension of the irrelevant ideal of $\mathbb{P}^{n_i}$ under the natural inclusion $S[\mathbb{P}^{n_i}] \to S[X]$. If $[I] \in \mathrm{Slip}_{r,X}$ for some positive integer r, then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{S[X]}\Big(I + B(\Sigma_i)^2, S[X]/\big(I + B(\Sigma_i)^2\big)\Big)_0 \geq r(n_1 + \ldots + n_k)$$

for $i \in \{1, \ldots, k\}$.

In Chapter 5 we present some applications of border apolarity lemma to secant varieties. In Section 5.1 we study polynomials whose border rank is smaller than the smoothable rank (see Section 2.4 for relevant definitions). This is in accordance with the initial motivation for developing criteria for points of $\mathrm{Hilb}^{h_{r,n}}_{S}$ to belong to the irreducible component $\mathrm{Slip}_{r,n}$.

Results from Sections 5.2, 5.3 and 5.4 are contained in [39]. They are about identifying (in special cases) points in the cactus variety that are not in the secant variety. Here, we use the border apolarity lemma without actually needing any insight into the irreducible component $\mathrm{Slip}_{r,n}$. The problem of distinguishing the secant variety from the cactus variety is another illustration of identifying "good" inside the set of "all". Secant varieties are classical objects of study but their equations are in general unknown. Moreover, various classes of known equations have been shown to actually vanish on a larger variety - the cactus variety. See [11], [38] and [61, §10.2].

Open problems

We end this chapter with a short list of natural directions of further investigation.

Given a smooth projective toric variety X and a positive integer r, we may divide closed points of $\mathrm{Hilb}^{h_{r,X}}_{S[X]}$ into four sets depending on whether $[I]$ is in the closure of the locus of saturated ideals and whether the subscheme of X defined by I is smoothable. Then, $\mathrm{Slip}_{r,X}$ consists of points that are in the closure of the locus of saturated ideals and that define smoothable subschemes. However, the following natural problem remains open.

Problem 1.8. Is there a projective toric variety X and a positive integer r such that there exists $[I] \in \mathrm{Hilb}^{h_{r,X}}_{S[X]} \setminus \mathrm{Slip}_{r,X}$ which satisfies conditions:

1. $[I]$ is in the closure of the locus of saturated ideals;

2. the subscheme of X defined by I is smoothable?

If the answer to the above question is negative, this could allow us to split the problem of describing $\mathrm{Slip}_{r,X}$ into two. One of them, which has been studied longer, is describing the smoothable component of the usual Hilbert scheme. The other problem would be studying the closure of the locus of saturated ideals inside $\mathrm{Hilb}^{h_{r,X}}_{S[X]}$.

Another problem is related to the geometry of $\mathrm{Hilb}^{h_{r,\mathbb{P}^n}}_{S[\mathbb{P}^n]}$.

Problem 1.9. Is $\mathrm{Hilb}^{h_{r,\mathbb{P}^n}}_{S[\mathbb{P}^n]}$ ever non-reduced?

We also want to discuss the criterion based on the flag condition for secant varieties [24, Prop. 2.3]. It seems that a natural analogue for $\mathrm{Slip}_{r,X}$ should hold. Namely, we expect that if $[I] \in \mathrm{Slip}_{r,X}$ then there is a flag of ideals $I_r = I \subseteq I_{r-1} \subseteq \ldots \subseteq I_0 = S$ such that $[I_k] \in \mathrm{Slip}_{k,X}$ for every k.

One natural question, especially in view of Problem 1.8 is the following.

Problem 1.10. Let X be a smooth projective toric variety and r be a positive integer. Assume that $[I] \in \mathrm{Hilb}^{h_{r,X}}_{S[X]}$ is in the closure of the locus of saturated ideals. Is there a flag of ideals $I_r = I \subseteq I_{r-1} \subseteq \ldots \subseteq I_0 = S$ such that $[I_k] \in \mathrm{Hilb}^{h_{r,X}}_{S[X]}$ is in the closure of the locus of saturated ideals?

A final general problem that is worth studying is as follows.

Problem 1.11. Is there a homogeneous ideal in three variables whose border rank is strictly smaller than the smoothable rank?

It is known that if F is such a polynomial then $\mathrm{br}(F) > \deg(F) + 1$ (see [11, Prop. 2.5]). We show in Proposition 5.5 that $\mathrm{br}(F) > \deg(F) + 2$.

There are also some natural problems related more closely to our methods. They are less general and thus, not as important. However, we would like to discuss them shortly.

Criterion from Theorem 1.4 is stated in a general version and we describe two situations where its assumptions are fulfilled: Theorems 3.65 and 3.74. It seems that there might be more general setups where Theorem 1.4 could be applied. We intend to investigate this in the future.

In a similar spirit, the description of $\mathrm{Slip}_{6,2}$ is quite lengthy and involved. It is a natural question, whether the methods used there could be abstracted to work in more general situations.

Chapter 2

Background material

In this chapter we collect some definitions and results that will be used in the rest of the thesis. Section 2.1 deals with commutative algebra. Material from Subsections 2.1.1 and 2.1.2 is standard but it was hard to find a reference for some of the results discussed there. We present the proofs for the sake of completeness. Subsections 2.1.3, 2.1.4 and 2.1.5 contain some results that will be used in Chapter 3. These subsections are based on [66]. In Subsection 2.2.1 we present some general results related to scheme theory. Subsections 2.2.2 and 2.2.3 are concerned with multigraded Hilbert schemes and Subsection 2.2.4 recalls the notion of a flag multigraded Hilbert scheme. Section 2.3 deals with deformation theory. We present basic definitions and results that will be used in the subsequent chapters. Section 2.4 is devoted to various notions of ranks and related apolarity lemmas.

Notation

Throughout this chapter $\Bbbk$ is a fixed algebraically closed field. Unless stated otherwise, all polynomial rings over $\Bbbk$ that we shall consider will have standard $\mathbb{Z}$-grading. Therefore, $\mathbb{Z}$-graded modules over these rings will be simply called graded modules.

2.1 Commutative algebra

In this section we present some results from commutative algebra that will be needed for the proofs of the main results.

In Subsection 2.1.1 we study locally free modules of finite rank since these appear in the definition of the functor of points of a multigraded Hilbert scheme. Since these schemes are the main object of our investigation we feel that it is appropriate to recall the notion explicitly.

Subsection 2.1.2 deals with saturated ideals. We present a few results that will be mainly used in Chapter 3.

In Subsection 2.1.3 we study the Hilbert function of a power of a radical ideal defining zero dimensional subscheme of a projective space. The obtained results are key in the proof of Theorem 3.5.

Subsection 2.1.4 is concerned with the computation of the dimension of the vector space $\mathrm{Hom}_T(I, T/I)_{>0}$ for a monomial ideal $I \neq T$ in the homogeneous coordinate ring T of projective line such that $\dim_{\Bbbk} T/I$ is finite. This is related to the tangent space to an appropriate multigraded Hilbert scheme. This observation will be used in the proof of Theorem 3.12.

In Subsection 2.1.5 we study the Ext groups $\mathrm{Ext}^i_S(\Bbbk, M)$ and $\mathrm{Ext}^i_S(M, \Bbbk)$ where S is a polynomial ring and M is a finitely generated graded S-module. These results will be used in Chapter 3.

For the sake of completeness, we give detailed proofs even though some of the results might be well known. On the other hand, we only refer to the basic properties of Gröbner bases and local cohomology as we need them. We believe that the theory of Gröbner bases is already well established, and in any case there are many excellent introductions to the topic, e.g. [27], [30]. The theory of local cohomology is only used as a tool in one of the proofs so it seems to be more natural to just cite the relevant result from [56].

Subsections 2.1.3, 2.1.4 and 2.1.5 are based on [66].

2.1.1 Locally free modules

We recall the definition of a locally free module and study some simple properties of such modules.

Definition 2.1. Given a ring R, we say that a module M is *locally free of finite rank* if, for every prime ideal $\mathfrak{p}$ of R, there is an element $f \in R \setminus \mathfrak{p}$ such that M_f is a finite free R_f-module.

Observe that an R-module M is locally free of finite rank if and only if the corresponding quasicoherent sheaf $\widetilde{M}$ on $\operatorname{Spec} R$ is locally free of finite rank. We shall prove algebraically some properties of locally free modules of finite rank. These results are intuitively clear from the interpretation in terms of sheaves of $\mathcal{O}_{\operatorname{Spec} R}$-modules.

Lemma 2.2. *Let M be an R-module and let $\mathfrak{p}$ be a prime ideal of R. If $f \in R\setminus\mathfrak{p}$ is such that M_f is a free R_f-module of rank r, then $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module of rank r. In particular, if $f' \in R \setminus \mathfrak{p}$ is such that $M_{f'}$ is a free $R_{f'}$-module of rank r', then $r = r'$.*

Proof. We may rewrite $M_\mathfrak{p}$ as

$$M_\mathfrak{p} \cong M \otimes_R R_\mathfrak{p} \cong M \otimes_R (R_f)_{\mathfrak{p}R_f}$$

where the latter isomorphism comes from [6, Prop. 11 §1.2]. This can be further transformed as

$$M \otimes_R (R_f)_{\mathfrak{p}R_f} \cong M \otimes_R (R_f \otimes_{R_f} (R_f)_{\mathfrak{p}R_f}) \cong M_f \otimes_{R_f} (R_f)_{\mathfrak{p}R_f} \cong (M_f)_{\mathfrak{p}R_f} \cong (R_f^r)_{\mathfrak{p}R_f} \cong R_\mathfrak{p}^r$$

where the last isomorphism follows from [6, Prop. 11 §1.2] since localization commutes with direct sums. □

Definition 2.3. We say that an R-module M is *locally free of rank* r if, given any prime ideal $\mathfrak{p}$ of R, there is an element $f \in R \setminus \mathfrak{p}$ such that M_f is a free R_f-module of rank r.

It follows from Lemma 2.2 that an R-module M can be locally free of rank r for a unique integer r. Moreover, if R is a Noetherian ring, $\operatorname{Spec} R$ is connected (equivalently, if R has no non-trivial idempotents) and M is locally free of finite rank, then M is locally free of rank r for some integer r by [47, Ex. II.5.8].

We study how the locally free condition behaves in short exact sequences.

Lemma 2.4. *Let R be a Noetherian ring and let $M \subseteq N \subseteq R^d$ be R-submodules. Assume that R^d/N is a locally free R-module of rank a for a positive integer a. Then, for a positive integer b, the following conditions are equivalent:*

(i) R^d/M is a locally free R-module of rank b;

(ii) N/M is a locally free R-module of rank $b-a$.

Proof. Localization is exact, so it is enough to show that R^d/M is a locally free R-module of finite rank if and only if N/M is a locally free R-module of finite rank. An R-module U is a locally free R-module of finite rank if and only if U is a flat R-module of finite presentation (see [6, Prop. 3 §4.4]). Moreover, since R is a Noetherian ring, it is also equivalent to U being a flat R-module of finite type.

We have an exact sequence of R-modules

$$0 \to N/M \to R^d/M \to R^d/N \to 0. \tag{2.5}$$

Since R^d/M is an R-module of finite type, it is a Noetherian module. Thus, N/M is an R-module of finite type.

The R-module R^d/N is flat by assumption. Hence, using the Tor exact sequence coming from short exact sequence (2.5), we obtain that N/M is R-flat if and only if R^d/M is R-flat. $\square$

2.1.2 Saturation and homogeneous saturated ideals in polynomial ring

Let $S = \Bbbk[\underline{\alpha}] := \Bbbk[\alpha_0, \alpha_1, \ldots, \alpha_n]$ be a polynomial ring. By $\mathfrak{m}$ we shall denote the irrelevant ideal $(\alpha_0, \alpha_1, \ldots, \alpha_n)$. Given an ideal I in S, we write $\overline{I}$ for the saturation of I with respect to $\mathfrak{m}$.

In the proof of Lemma 2.6 we will use the notion of local cohomology. See [56, Ch. 7] for its basic properties. Recall that $H^i_{\mathfrak{m}}(-)$ are the right derived functors of the functor

$$\Gamma_{\mathfrak{m}}(-) \colon \mathfrak{Mod}(S) \to \mathfrak{Mod}(S)$$

defined by

$$\Gamma_{\mathfrak{m}}(M) = \{m \in M \mid \mathfrak{m}^d m = 0 \text{ for some } d \in \mathbb{Z}_{>0}\}.$$

There is a close connection between the zeroth local cohomology group $H^0_{\mathfrak{m}}(S/I)$ and the saturation $\overline{I}$ of I. Namely, $\overline{I}$ is the kernel of the natural map $S \to (S/I)/H^0_{\mathfrak{m}}(S/I)$. Therefore, we are able to use the general results about local cohomology to prove the following lemma. Note that we are interested only in the zeroth local cohomology group.

Lemma 2.6. *Let $R \to T$ be a flat homomorphism of $\Bbbk$-algebras. Let $I \subseteq R[\underline{\alpha}] = \bigoplus_d (S_d \otimes_{\Bbbk} R)$ be a homogeneous ideal saturated with respect to $\mathfrak{m} \otimes_{\Bbbk} R$. Then $I \otimes_R T \subseteq S \otimes_{\Bbbk} T$ is saturated with respect to $\mathfrak{m} \otimes_{\Bbbk} T$.*

Proof. The saturation of I is the kernel of the natural map

$$S \otimes_{\Bbbk} R \to \big((S \otimes_{\Bbbk} R)/I\big)/H^0_{\mathfrak{m}\otimes_{\Bbbk} R}\big((S \otimes_{\Bbbk} R)/I\big).$$

The ideal I is assumed to be saturated. Thus, $H^0_{\mathfrak{m}\otimes_{\Bbbk} R}\big((S \otimes_{\Bbbk} R)/I\big) = 0$. Since T is a flat R-algebra, we have

$$H^0_{\mathfrak{m}\otimes_{\Bbbk} T}\big((S \otimes_{\Bbbk} T)/(I \otimes_R T)\big) \cong \Big(H^0_{\mathfrak{m}\otimes_{\Bbbk} R}\big((S \otimes_{\Bbbk} R)/I\big)\Big) \otimes_R T = 0$$

by [56, Prop. 7.15]. It follows that $I \otimes_R T$ is saturated with respect to $\mathfrak{m} \otimes_{\Bbbk} T$. $\square$

Lemma 2.7 states, that if an initial ideal $\mathrm{in}_<(I)$ of a homogeneous ideal I is saturated, then I is saturated. This is a typical situation. The process of taking the initial ideal usually worsens the properties of the corresponding quotient algebra. As a key example, for a homogeneous ideal $I \subseteq S$ we have an inequality of Betti numbers

$$\beta_{ij}(S/I) \leq \beta_{ij}\big(S/\operatorname{in}_<(I)\big)$$

for all $i, j \in \mathbb{Z}_{\geq 0}$, see [49, Cor. 3.3.3]. Therefore, if $S/\operatorname{in}_<(I)$ has some nice property, then often S/I has the same property. Compare the following lemma to [49, Cor. 3.3.5].

Lemma 2.7. *Let I be an ideal in S and let $<$ be a monomial order. Then $\operatorname{in}_<(\overline{I}) \subseteq \overline{\operatorname{in}_<(I)}$. In particular, if I is a homogeneous ideal and $\operatorname{in}_<(I)$ is a saturated ideal, then I is a saturated ideal.*

Proof. Let $f \in \overline{I}$. There is an integer l such that $\alpha_i^l f \in I$ for $i = 0, 1, \ldots, n$. Therefore, $\alpha_i^l \cdot \operatorname{in}_<(f) \in \operatorname{in}_<(I)$ for $i = 0, 1, \ldots, n$. Consequently, $\operatorname{in}_<(f) \in \overline{\operatorname{in}_<(I)}$.

Now assume that I is a homogeneous ideal such that $\operatorname{in}_<(I)$ is a saturated ideal. We have

$$\operatorname{in}_<(I) \subseteq \operatorname{in}_<(\overline{I}) \subseteq \overline{\operatorname{in}_<(I)} = \operatorname{in}_<(I).$$

It follows that $\operatorname{in}_<(I) = \operatorname{in}_<(\overline{I})$. Thus, I and $\overline{I}$ have the same Hilbert function. As a result, $I = \overline{I}$. □

We will frequently use the following observation.

Lemma 2.8. *Let M be a graded S-module and let I be a homogeneous ideal of S such that $I = \overline{I} \cap \mathfrak{m}^d$ for a positive integer d. Assume that $M_{<d} = 0$ and that there is a positive integer r such that $\mathfrak{m}^r \cdot M = 0$. Then $\operatorname{Hom}_S(M, S/I)_0 = 0$.*

Proof. Let $\varphi \in \operatorname{Hom}_S(M, S/I)_0$ and $x \in M_e$ for some $e \geq d$. Then $\mathfrak{m}^r \cdot \varphi(x) = 0$. Therefore, in the quotient algebra S/I, the element $\varphi(x)$ is represented by an element from $\overline{I}_e = I_e$. Thus, it is zero. □

Next we study some properties of Hilbert functions of saturated ideals.

Lemma 2.9. *Let $I \neq S$ be a homogeneous saturated ideal of S. Then:*

(i) There exists a linear form $f \in S_1$ that is a non-zero divisor on S/I;

(ii) $H_{S/I}(d+1) - H_{S/I}(d) \geq 0$ for all integers d;

(iii) If $H_{S/I}(d) = H_{S/I}(d-1)$ for a positive integer d, then $H_{S/I}(d+1) = H_{S/I}(d)$.

Proof. (i) Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the associated primes of S/I. It is enough to show that $\bigcup_{i=1}^k (\mathfrak{p}_i)_1 \neq S_1$. Suppose that it does not hold. Since $\Bbbk$ is infinite we have $(\mathfrak{p}_i)_1 = S_1$ for some i and therefore, $\mathfrak{m}$ is an associated prime of S/I. This gives a contradiction with the assumption that I is saturated.

(ii) Let $f \in S_1$ be a non-zero divisor on S/I. Then, the map $(S/I)_d \xrightarrow{\cdot f} (S/I)_{d+1}$ is injective for every d.

(iii) Let $f \in S_1$ be a non-zero divisor on S/I. Suppose that $H_{S/I}(d) = H_{S/I}(d-1)$. Then $(S/I)_{d-1} \xrightarrow{\cdot f} (S/I)_d$ is an isomorphism of $\Bbbk$-vector spaces. We claim that also $(S/I)_d \xrightarrow{\cdot f} (S/I)_{d+1}$ is an isomorphism. It is injective since f is a non-zero divisor on S/I. Let $g \in (S/I)_{d+1}$. Then $g = \sum_{i=0}^n \alpha_i h_i$ for some $h_i \in (S/I)_d$. By assumptions, there are $k_0, \dots, k_n \in (S/I)_{d-1}$ such that $h_i = fk_i$ for $i = 0, 1, \dots, n$. It follows that $g = f(\sum_{i=0}^{n} \alpha_i k_i)$.

□

The following corollary of a theorem by Bayer and Stillman enables us to deform a saturated ideal to a saturated ideal with special properties - a Borel-fixed ideal. Recall the notion of a generic initial ideal from [30, §15.9].

Corollary 2.10. *Suppose that $I \neq S$ is a homogeneous saturated ideal of S. Then the generic initial ideal I' of I with respect to the grevlex order with $\alpha_0 > \ldots > \alpha_n$ is a saturated ideal.*

Proof. Since I is a saturated ideal, $\operatorname{depth}(S/I) \geq 1$ by Lemma 2.9(i). Thus, $\operatorname{depth} S/I' \geq 1$ by [49, Cor. 4.3.18]. It follows that I' is saturated. □

The following lemma gives a useful property of saturated Borel-fixed ideals.

Lemma 2.11. *Let $T = \Bbbk[\alpha_0, \dots, \alpha_{n-1}]$. If $I \subseteq S$ is a saturated Borel-fixed ideal then there exists an ideal $\mathfrak{a} \subseteq T$ such that $\mathfrak{a} \cdot S = I$.*

Proof. Let G be the set of minimal monomial generators of I. It is enough to show that there is no element $\prod_{i=0}^{n} \alpha_i^{a_i} \in G$ with $a_n > 0$. Assume that $\prod_{i=0}^{n} \alpha_i^{a_i} \in G$ with $a_n > 0$.

Then we claim that

$$(\prod_{i=0}^{j-1} \alpha_i^{a_i}) \cdot \alpha_j^{\sum_{k=j}^{n} a_k} \in I \text{ for every } 0 \leq j \leq n.$$

This follows from [67, Prop. 2.3] if $\operatorname{char} \Bbbk = 0$ and from [50, Prop. 1.2] if $\operatorname{char} \Bbbk > 0$. Therefore, $\prod_{i=0}^{n-1} \alpha_i^{a_i} \in \overline{I} = I$ since

$$\alpha_j^{\sum_{k=j}^{n} a_k} \cdot (\prod_{i=0}^{n-1} \alpha_i^{a_i}) \in I$$

for $j \in \{0, \dots, n\}$. This shows that g is not a minimal monomial generator and gives a contradiction. □

We will use the following observation which is a special case of Macaulay's theorem [10, Thm. 4.2.10].

Lemma 2.12. *Let $T = \Bbbk[\alpha_0, \alpha_1]$ and $\mathfrak{a} \subseteq T$ be a homogeneous ideal. Then*

$$H_{T/\mathfrak{a}}(d) - H_{T/\mathfrak{a}}(d+1) \geq 0$$

for every d such that $\mathfrak{a}_d \neq 0$.

Proof. If $\mathfrak{a}_d \neq 0$, then $H_{T/\mathfrak{a}}(d) \leq \dim_{\Bbbk} T_d - 1 = d$. It follows from [10, Thm. 4.2.10] that $H_{T/\mathfrak{a}}(d+1) \leq H_{T/\mathfrak{a}}(d)$. □

As a consequence of the above observation we obtain some bounds on the number of minimal homogeneous generators of a homogeneous saturated ideal $\overline{I} \subseteq S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$.

Lemma 2.13. *Let $\overline{I} \subseteq S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ be a homogeneous saturated ideal. Let m be a positive integer such that $\overline{I}_m \neq 0$ and let f be the Hilbert function of $S/\overline{I}$. For every $d \geq m$ we have*

$$\beta_{1,d+1}(S/\overline{I}) \leq 2f(d) - f(d+1) - f(d-1).$$

Proof. Let I' be the generic initial ideal (see [30, §15.9]) of $\overline{I}$ with respect to the grevlex order with $\alpha_0 > \alpha_1 > \alpha_2$. By Corollary 2.10 the ideal I' is saturated. Moreover, $\beta_{1,a}(S/I') \geq \beta_{1,a}(S/\overline{I})$ for every a by [49, Cor. 3.3.3]. Therefore, it is enough to prove the lemma for a saturated Borel-fixed ideal I'. Let $\mathfrak{a} = I' \cap \Bbbk[\alpha_0, \alpha_1]$ and let g be the Hilbert function of $\Bbbk[\alpha_0, \alpha_1]/\mathfrak{a}$. Then $I' = \mathfrak{a} \cdot S$ by Lemma 2.11, so $f(a) - f(a-1) = g(a)$ for every $a \in \mathbb{Z}$.

Let $d \geq m$ be such that $\beta_{1,d+1}(S/I') = \beta_{1,d+1}(\Bbbk[\alpha_0, \alpha_1]/\mathfrak{a}) > 2f(d) - f(d+1) - f(d-1)$. Since $\mathfrak{a}_d \neq 0$, it follows from Lemma 2.12 that $2f(d) - f(d+1) - f(d-1) = g(d) - g(d+1) \geq 0$. Let G be a set of minimal monomial generators of $\mathfrak{a}$ and let G' be obtained from G by deleting $s = g(d) - g(d+1) + 1$ minimal monomial generators of degree $d+1$. Let $\mathfrak{a}'$ be the ideal of $\Bbbk[\alpha_0, \alpha_1]$ generated by monomials from G' and let $R = \Bbbk[\alpha_0, \alpha_1]/\mathfrak{a}'$. Then $\mathfrak{a}'_d = \mathfrak{a}_d \neq 0$, but

$$H_R(d+1) - H_R(d) = g(d+1) + s - g(d) = 1.$$

This contradicts Lemma 2.12. □

2.1.3 Hilbert function of a power of an ideal of points

We keep the notation of Subsection 2.1.2. In particular, $S = \Bbbk[\alpha_0, \ldots, \alpha_n]$ is a polynomial ring. The main result of this subsection is Proposition 2.19. Let I be a radical homogeneous ideal in S such that S/I has constant Hilbert polynomial. In the proposition we compute the Hilbert polynomial of S/I^k (for a positive integer k) and bound the degree from which it agrees with the Hilbert function of S/I^k. This result is crucial in the proof of Theorem 3.5.

We begin with studying the following condition on homogeneous ideals J of S:

$$\text{there exists a positive integer } d \text{ such that } S_d \subseteq J. \tag{$*$}$$

This algebraic condition can be restated geometrically. Namely, a homogeneous ideal J in the polynomial ring S satisfies condition $(*)$ if and only if the corresponding closed subset of the projective space $\operatorname{Proj} S$ is the empty set. See [76, Lem. 1.1] for a proof of this equivalence.

We collect some useful properties of condition $(*)$ in the following lemma. They are probably all well-known. Nevertheless, we could not find a reference for all of them so we present a simple proof.

We stress that in the following lemma, the lower index of I_s does not indicate the degree s part of the homogeneous ideal I, as it usually does in the rest of the thesis.

Lemma 2.14. *Let $m \geq 2$ be an integer and $J, K, I_1, \ldots, I_m$ be homogeneous ideals of S. Then:*

(i) $\overline{J \cap K} = \overline{J} \cap \overline{K}$*;*

(ii) *$J + K$ satisfies condition $(*)$ if and only if $\sqrt{J} + \sqrt{K}$ satisfies condition $(*)$;*

(iii) *If $I_i + I_m$ satisfies condition $(*)$ for $i = 1, 2, \ldots, m-1$, then $I_1 \cdot I_2 \cdot \ldots \cdot I_{m-1} + I_m$ and $I_1 \cap I_2 \cap \ldots \cap I_{m-1} + I_m$ satisfy condition $(*)$;*

(iv) *If $I_1, I_2, \ldots, I_m$ are homogeneous ideals such that $I_i + I_j$ satisfies condition $(*)$ for $1 \leq i < j \leq m$, then $\overline{I_1 \cdot I_2 \cdot \ldots \cdot I_m} = \overline{I_1 \cap I_2 \cap \ldots \cap I_m}$.*

Proof. (i) Let $f \in \overline{J} \cap \overline{K}$. Then, by definition of saturation, there are integers k_1, k_2 such that $\alpha_i^{k_1} f \in J$ and $\alpha_i^{k_2} f \in K$ for $i = 0, 1, \ldots, n$. Therefore, we get $\alpha_i^{\max\{k_1,k_2\}} f \in J \cap K$ for

$i = 0, 1, \dots, n$. Since f was arbitrary, we obtain $\overline{J} \cap \overline{K} \subseteq \overline{J \cap K}$. On the other hand, we have $\overline{J \cap K} \subseteq \overline{J} \cap \overline{K}$ since $J \cap K$ is contained in both J and K.

(ii) An ideal of S satisfies condition $(*)$ if and only if its radical satisfies condition $(*)$. Therefore, it is enough to observe that

$$\sqrt{J+K} = \sqrt{\sqrt{J} + \sqrt{K}}.$$

This follows from the definition of a radical of an ideal, and is well-known [2, Ex. 1.13 v)].

(iii) By [2, Ex. 1.13 iii)] and induction we have

$$\sqrt{I_1 \cdot I_2 \cdot \ldots \cdot I_{m-1}} = \sqrt{I_1 \cap I_2 \cap \ldots \cap I_{m-1}} = \sqrt{I_1} \cap \sqrt{I_2} \cap \ldots \cap \sqrt{I_{m-1}}.$$

Thus, by part (ii) it is enough to show that $\sqrt{I_1} \cap \sqrt{I_2} \cap \ldots \cap \sqrt{I_{m-1}} + \sqrt{I_m}$ satisfies condition $(*)$. By assumptions there is an integer d such that $\alpha_i^d \in I_j + I_m$ for $i = 0, 1, \dots, n$ and for $j = 1, 2, \dots, m-1$. It follows that there are elements $s_{ij} \in \sqrt{I_j}$ and $t_{ij} \in \sqrt{I_m}$ satisfying $\alpha_i^d = s_{ij} + t_{ij}$ for $i = 0, 1, \dots, n$ and for $j = 1, 2, \dots, m-1$. Multiplying these identities for $j \in \{1, 2, \dots, m-1\}$ and fixed i, we obtain

$$\alpha_i^{d(m-1)} = \prod_{j=1}^{m-1} (s_{ij} + t_{ij}) = \prod_{j=1}^{m-1} s_{ij} + \left(\prod_{j=1}^{m-1} (s_{ij} + t_{ij}) - \prod_{j=1}^{m-1} s_{ij} \right). \tag{2.15}$$

We have

$$\prod_{j=1}^{m-1} s_{ij} \in \sqrt{I_1} \cdot \sqrt{I_2} \cdot \ldots \cdot \sqrt{I_{m-1}} \subseteq \sqrt{I_1} \cap \sqrt{I_2} \cap \ldots \cap \sqrt{I_{m-1}}$$

and $\prod_{j=1}^{m-1}(s_{ij} + t_{ij}) - \prod_{j=1}^{m-1} s_{ij} \in \sqrt{I_m}$. Hence, by Equation (2.15)

$$\alpha_i \in \sqrt{\sqrt{I_1} \cap \sqrt{I_2} \cap \ldots \cap \sqrt{I_{m-1}} + \sqrt{I_m}}$$

for $i = 0, 1, \dots, n$. It follows that $\sqrt{I_1} \cap \sqrt{I_2} \cap \ldots \cap \sqrt{I_{m-1}} + \sqrt{I_m}$ satisfies condition $(*)$.

(iv) We prove it by induction on m starting with $m = 2$. The inclusion $\overline{I_1 \cdot I_2} \subseteq \overline{I_1 \cap I_2}$ is a consequence of $I_1 \cdot I_2 \subseteq I_1 \cap I_2$. In order to establish the opposite inclusion, observe that for some positive integer d

$$S_d \cdot (I_1 \cap I_2) \subseteq (I_1 + I_2)(I_1 \cap I_2) \subseteq I_1 \cdot I_2$$

by the assumption that $I_1 + I_2$ satisfies condition $(*)$. It follows that $I_1 \cap I_2 \subseteq \overline{I_1 \cdot I_2}$ and thus, $\overline{I_1 \cap I_2} \subseteq \overline{I_1 \cdot I_2}$.

Let $k \geq 3$ and assume that part (iv) holds for all integers m smaller than k. From part (i) we get $\overline{I_1 \cap I_2 \cap \ldots \cap I_k} = \overline{I_1 \cap I_2 \cap \ldots \cap I_{k-1}} \cap \overline{I_k}$.

Applying the inductive hypothesis for $m = k-1$ we conclude that

$$\overline{I_1 \cap I_2 \cap \ldots \cap I_k} = \overline{I_1 \cap I_2 \cap \ldots \cap I_{k-1}} \cap \overline{I_k} = \overline{I_1 \cdot I_2 \cdot \ldots \cdot I_{k-1}} \cap \overline{I_k}.$$

The ideal $I_1 \cdot I_2 \cdot \ldots \cdot I_{k-1} + I_k$ satisfies condition $(*)$ by part (iii). Therefore, from part (i) and inductive hypothesis for $m = 2$, we obtain

$$\overline{I_1 \cap I_2 \cap \ldots \cap I_k} = \overline{\overline{I_1 \cdot I_2 \cdot \ldots \cdot I_{k-1}} \cap \overline{I_k}} = \overline{(I_1 \cdot I_2 \cdot \ldots \cdot I_{k-1}) \cap I_k} = \overline{I_1 \cdot I_2 \cdot \ldots \cdot I_k},$$

as claimed.

□

The following lemma shows that if two ideals have the same saturation, then their k-th powers for any positive integer k also have the same saturation.

Lemma 2.16. *Let I, J be homogeneous ideals of S and k be a positive integer. Then:*

(i) There is an integer d_0 such that for all integers $d_1, \ldots, d_k \geq d_0$ the map $\bigotimes_{i=1}^{k} I_{d_i} \to I^k_{d_1+\ldots+d_k}$ induced by multiplication is surjective;

(ii) If $\overline{I} = \overline{J}$, then $\overline{I^k} = \overline{J^k}$.

Proof. (i) Consider a minimal set of homogeneous generators of I. We can take d_0 to be the maximum of degrees of elements of this set. This can be expressed in terms of Betti numbers as $d_0 = \max\{j \mid \beta_{1,j}(S/I) \neq 0\}$.

(ii) Let $d_0 = \max\{j \mid \beta_{1,j}(S/I) \neq 0\}$ and $e_0 = \max\{j \mid \beta_{1,j}(S/J) \neq 0\}$. Let r_0 be an integer such that $I_{\geq r_0} = \overline{I}_{\geq r_0}$ and $J_{\geq r_0} = \overline{J}_{\geq r_0}$. Let $s_0 = \max\{d_0, e_0, r_0\}$. Then for all $d_1, \ldots, d_k \geq s_0$ we have

$$I^k_{d_1+\ldots+d_k} = \overline{I}^k_{d_1+\ldots+d_k} = \overline{J}^k_{d_1+\ldots+d_k} = J^k_{d_1+\ldots+d_k}$$

where the first and last equality follow from part (i). As a result, $\overline{I^k} = \overline{J^k}$.

□

Using the above algebraic results we will compute the Hilbert polynomial of a power of a homogeneous radical ideal which defines a closed, zero dimensional subscheme of projective space.

Lemma 2.17. *Let $I \subseteq S$ be a homogeneous radical ideal such that the Hilbert polynomial of the quotient algebra S/I is constant, equal r for some positive integer r. Then for a positive integer k, the Hilbert polynomial of S/I^k is constant equal to $r \cdot \dim_{\Bbbk} S_{k-1}$.*

Proof. Let $P_1, \ldots, P_r$ be the (distinct) points of the support of $\operatorname{Proj} S/I \subseteq \mathbb{P}^n$. Define $\mathfrak{p}_i$ to be the homogeneous prime ideal of S defining P_i. Then $I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_r$. We have $\mathfrak{p}_i + \mathfrak{p}_j = \mathfrak{m}$ for all $1 \leq i < j \leq m$. Therefore, by Lemma 2.14(iv), $\overline{I} = \overline{J}$, where $J = \mathfrak{p}_1 \cdot \ldots \cdot \mathfrak{p}_r$. Hence $\overline{I^k} = \overline{J^k}$ by Lemma 2.16(ii). As a result, it suffices to show that the Hilbert polynomial of S/J^k is $r \cdot \dim_{\Bbbk} S_{k-1}$. Let $K = \mathfrak{p}_1^k \cap \ldots \cap \mathfrak{p}_r^k$. Observe that $\mathfrak{p}_i^k + \mathfrak{p}_j^k$ satisfies condition $(*)$ for every $1 \leq i < j \leq m$ by Lemma 2.14(ii). Therefore, $\overline{K} = \overline{J^k}$ by Lemma 2.14(iv). Thus, it is enough to consider the Hilbert polynomial of S/K. As a set, $\operatorname{Proj} S/K$ is the disjoint union of r-points $P_1, \ldots, P_r$. Consequently, it is enough to show that the degree of $\operatorname{Proj} S/\mathfrak{p}_i^k$ is $\dim_{\Bbbk} S_{k-1}$ for every $i = 1, \ldots, r$. This is clear, since up to a linear change of variables $\mathfrak{p}_i = (\alpha_1, \ldots, \alpha_n)$. □

The following example shows that the assumption in Lemma 2.17 that I is reduced cannot be weakened to the assumption that I is saturated.

Example 2.18. Let $I = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2) \subseteq S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$. It is a saturated ideal and the corresponding subscheme $\operatorname{Proj} S/I \subseteq \mathbb{P}^2$ is zero-dimensional of degree 3. However, $\operatorname{Proj} S/I^2$ has degree 10.

In Lemma 2.17 we have calculated the Hilbert polynomial of S/I^k for a homogeneous radical ideal I defining a zero dimensional closed subscheme of a projective space and a positive integer k. Now we provide an upper bound on the least degree, from which the Hilbert function of S/I^k agrees with the Hilbert polynomial of S/I^k. The proof uses the notion of regularity. We recall its definition in terms of Betti numbers. For a finitely generated graded S-module M, its regularity $\operatorname{reg} M$ is defined to be $\operatorname{reg} M = \max\{j - i \mid \beta_{i,j}(M) \neq 0\}$.

Proposition 2.19. *Let r, k be positive integers and $I \subseteq S$ be a homogeneous radical ideal with the Hilbert polynomial of the quotient algebra S/I equal to r. Define $e = \min\{a \in \mathbb{Z} \mid H_{S/I}(a) = r\}$. Then $H_{S/I^k}(d) = r \cdot \dim_{\Bbbk} S_{k-1}$ for $d \geq ke + k$.*

Proof. The Hilbert polynomial of S/I^k is $r \cdot \dim_{\Bbbk} S_{k-1}$ by Lemma 2.17. Therefore, we are left with establishing the bound on the degree from which the Hilbert function agrees with the Hilbert polynomial. This is related to the regularity. By [31, Thm. 4.2], it is enough to show that

$$ke + k - 1 \geq \operatorname{reg} S/I^k. \tag{2.20}$$

We have $\operatorname{reg} S/I = e$ by [31, Thm. 4.2]. Hence, from the definition of regularity in terms of Betti numbers, we get $\operatorname{reg} I = e + 1$. Thus, $\operatorname{reg} I^k \leq ke + k$ by [22, Thm. 6]. Inequality (2.20) follows. □

Unlike in Lemma 2.17, to obtain the bound on the degree from which the Hilbert function agrees with the Hilbert polynomial, the assumption in Proposition 2.19 that I is radical could be replaced by the weaker assumption that I is saturated. However, we need to control both the value of the Hilbert polynomial and the degree from which the Hilbert function has this value. Therefore, we need to restrict our attention to radical ideals.

2.1.4 Tangent space at extended ideal

In this subsection we consider polynomial ring $T = \Bbbk[\alpha_0, \alpha_1]$. The main result is Proposition 2.22 which computes

$$\dim_{\Bbbk} \operatorname{Hom}_T(I, T/I)_{>0}$$

for a monomial ideal $I \neq T$ of T such that $\dim_{\Bbbk} T/I$ is finite. This will be later used to compute the dimension of the tangent space to the multigraded Hilbert scheme at the point corresponding to the extended ideal $I^{ex} \subseteq \Bbbk[\alpha_0, \alpha_1, \alpha_2]$.

If M, N are graded T-modules and M is finitely generated then the Ext groups $\operatorname{Ext}^i_T(M, N)$ are graded T-modules in a natural way (see [10, §1.5]). For a graded T-module M and an integer d, by $M(d)$ we denote the graded T-module given by $M(d)_e = M_{e+d}$ for all $e \in \mathbb{Z}$.

Let I be a monomial ideal in T such that $\dim_{\Bbbk} T/I = r$ for a positive integer r. We can consider the associated staircase diagram (see [67, §3.1]). We recall its construction. For each pair of non-negative integers (s, t) such that $\alpha_0^s\alpha_1^t \notin I$ put a 1×1 box with sides parallel to coordinate axis and (s, t) as lower left corner of the box. The diagram corresponding to I will be denoted by $\mathcal{D}_I$. The set of boxes of the diagram $\mathcal{D}_I$ (or, the set of monomials outside I) will be denoted by Λ_I. There is a canonical minimal free resolution of T/I (see [67, Prop. 3.1]). The set of minimal monomial generators of I will be denoted by M_I and the generating set of relations (or more precisely the set of their degrees when T is considered with the natural $\mathbb{Z}^2$-grading) used in that resolution will be denoted by R_I.

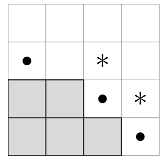

Figure 2.1: Staircase diagram of the ideal $I = (\alpha_0^3, \alpha_0^2\alpha_1, \alpha_1^2)$.

Example 2.21. Figure 2.1 presents the staircase diagram of $I = (\alpha_0^3, \alpha_0^2\alpha_1, \alpha_1^2)$. Filled boxes correspond to monomials outside I (i.e. elements of Λ_I), dots correspond to elements of M_I (i.e. minimal monomial generators of I) and asterisks correspond to elements of R_I (i.e. minimal relations between those generators).

Note that we have $\#\Lambda_I = \dim_{\Bbbk} T/I = r$ and $\#R_I = \#M_I - 1$. We will identify monomials of T with lattice points in $\mathbb{Z}^2$. Given a point $\mathbf{u} = (s,t)$ in $\mathbb{Z}^2$ we will write $|\mathbf{u}|$ for $s+t$. We define three functions from integers to integers:

$$\begin{aligned}
\lambda_I(a) =& H_{T/I}(a) = \#\{\mathbf{u} \in \Lambda_I \mid |\mathbf{u}| = a\},\\
\mu_I(a) =& \beta_{1,a}(T/I) = \#\{\mathbf{u} \in M_I \mid |\mathbf{u}| = a\},\\
\rho_I(a) =& \beta_{2,a}(T/I) = \#\{\mathbf{u} \in R_I \mid |\mathbf{u}| = a\}.
\end{aligned}$$

The goal of this subsection is the proof of the following proposition.

Proposition 2.22. *Let r be a positive integer. Given a monomial ideal I in T with $\dim_{\Bbbk} T/I = r$ we have*

$$\dim_{\Bbbk} \operatorname{Hom}_T(I, T/I)_{>0} = \sum_{\mathbf{u}\in M_I} \sum_{a>|\mathbf{u}|} \lambda_I(a) - \sum_{\mathbf{u}\in R_I} \sum_{a>|\mathbf{u}|} \lambda_I(a). \tag{2.23}$$

Observe that [33, Lem. 3.2] presents a more general formula for $\dim_{\Bbbk} \operatorname{Hom}_T(I, T/I)_{\mathbf{u}}$ where $\mathbf{u} \in \mathbb{Z}^2$ and we consider T with the natural $\mathbb{Z}^2$-grading.

The proof of Proposition 2.22 is based on the following observation.

Lemma 2.24. *Let I be a monomial ideal in T such that T/I is a finite $\Bbbk$-vector space. Then:*

(i) The natural map $T \to \operatorname{Hom}_T(I, T)$ given by $f \mapsto (g \mapsto fg)$ is an isomorphism of graded T-modules.

(ii) $\operatorname{Ext}^1_T(I, T/I)_{>0} = 0$.

Proof. Since $\dim_{\Bbbk} T/I$ is finite, $I = (\alpha_0^{a_0}, \alpha_0^{a_1}\alpha_1^{b_1}, \dots, \alpha_0^{a_{s-1}}\alpha_1^{b_{s-1}}, \alpha_1^{b_s})$ for some positive integers $a_0 > a_1 > \ldots > a_{s-1}$ and $b_1 < b_2 < \ldots < b_s$. Set $a_s = b_0 = 0$.

(i) Let $\varphi\colon I \to T$ be a homomorphism of T-modules. It is enough to show that there exists an element $f \in T$ such that $\varphi(g) = fg$ for every $g \in I$. Define $f_i = \varphi(\alpha_0^{a_i}\alpha_1^{b_i})$ for $i = 0, \dots, s$. Then for each $i \in \{1, \dots, s\}$ we have relations of the form

$$\alpha_1^{b_i - b_{i-1}} f_{i-1} = \varphi(\alpha_0^{a_{i-1}}\alpha_1^{b_i}) = \alpha_0^{a_{i-1}-a_i} f_i. \tag{2.25}$$

From Equation (2.25) for $i = s$ we deduce that $\alpha_0^{a_{s-1}}$ divides f_{s-1}. It follows by induction that $\alpha_0^{a_i}$ divides f_i for all i. Thus, $f_0 = \alpha_0^{a_0} f$ for some $f \in T$. From Equation (2.25) we conclude that $f_i = \alpha_0^{a_i}\alpha_1^{b_i} f$ for each i.

(ii) We start with showing that $\mathrm{Ext}^1_T(I,T)_{>0} = 0$. Consider the canonical minimal graded free resolution

$$\bigoplus_{i=1}^{s} T(-a_{i-1}-b_i) \to \bigoplus_{i=0}^{s} T(-a_i - b_i) \to I \to 0$$

of I (see [67, Prop. 3.1]). Applying the functor $\mathrm{Hom}_T(-,T)$ to the above resolution, we obtain for every integer c a $\Bbbk$-linear map

$$\psi_c\colon \bigoplus_{i=0}^{s} T(a_i+b_i)_c \to \bigoplus_{i=1}^{s} T(a_{i-1}+b_i)_c.$$

We claim that ψ_c is surjective for every $c > 0$. Observe that $\ker\psi_c \cong \mathrm{Hom}_T(I,T)_c \cong T_c$ by part (i). Therefore, the claim is a consequence of the calculation

$$\dim_\Bbbk \bigoplus_{i=0}^{s} T_{a_i+b_i+c} = (s+1)(c+1) + \sum_{i=1}^{s}(a_{i-1}+b_i) = \dim_\Bbbk \bigoplus_{i=1}^{s} T_{a_{i-1}+b_i+c} + \dim_\Bbbk T_c.$$

Since ψ_c is surjective for positive c, it follows that $\mathrm{Ext}^1_T(I,T)_{>0} = 0$.

Now we prove that $\mathrm{Ext}^1_T(I,T/I)_{>0} = 0$. Consider the following part of the long exact sequence of Ext groups obtained from the short exact sequence $0 \to I \to T \to T/I \to 0$ by applying the functor $\mathrm{Hom}_T(I,-)$:

$$\ldots \to \mathrm{Ext}^1_T(I,T)_{>0} \to \mathrm{Ext}^1_T(I,T/I)_{>0} \to \mathrm{Ext}^2_T(I,I)_{>0} \to \ldots.$$

We have shown that $\mathrm{Ext}^1_T(I,T)_{>0} = 0$. Moreover, $\mathrm{Ext}^2_T(I,I)_{>0}$ since I has projective dimension 1. It follows that $\mathrm{Ext}^1_T(I,T/I)_{>0} = 0$.

□

Proof of Proposition 2.22. Consider the canonical minimal free resolution of I

$$0 \to \bigoplus_{a\in\mathbb{Z}} T(-a)^{\rho_I(a)} \to \bigoplus_{b\in\mathbb{Z}} T(-b)^{\mu_I(b)} \to I \to 0.$$

Applying the functor $\mathrm{Hom}_T(-,T/I)_{>0}$ and using Lemma 2.24(ii) we get an exact sequence

$$0 \to \mathrm{Hom}_T(I,T/I)_{>0} \to \bigoplus_{b\in\mathbb{Z}} \mathrm{Hom}_T(T(-b)^{\mu_I(b)},T/I)_{>0} \to \bigoplus_{a\in\mathbb{Z}} \mathrm{Hom}_T(T(-a)^{\rho_I(a)},T/I)_{>0} \to 0.$$

This can be rewritten as

$$0 \to \mathrm{Hom}_T(I,T/I)_{>0} \to \bigoplus_{b\in\mathbb{Z}} (T/I)_{>b}^{\mu_I(b)} \to \bigoplus_{a\in\mathbb{Z}} (T/I)_{>a}^{\rho_I(a)} \to 0.$$

Thus,

$$\dim_\Bbbk \mathrm{Hom}_T(I,T/I)_{>0} = \sum_{\mathbf{u}\in M_I}\sum_{c>|\mathbf{u}|} \dim_\Bbbk (T/I)_c - \sum_{\mathbf{u}\in R_I}\sum_{c>|\mathbf{u}|} \dim_\Bbbk (T/I)_c.$$

This is equivalent to Equation (2.23). $\square$

We end this subsection with an example.

Example 2.26. Let $I = (\alpha_0^3, \alpha_0^2\alpha_1, \alpha_1^6)$. Its staircase diagram is presented in Figure 2.2. For this ideal Equation (2.23) takes the form

$$\dim_{\Bbbk} \mathrm{Hom}_T(I, T/I)_{>0} = \big(2\cdot\sum_{a>3} \lambda_I(a) + \sum_{a>6} \lambda_I(a)\big) - \big(\sum_{a>4} \lambda_I(a) + \sum_{a>8} \lambda_I(a)\big) = (2\cdot 5+0)-(3+0) = 7.$$

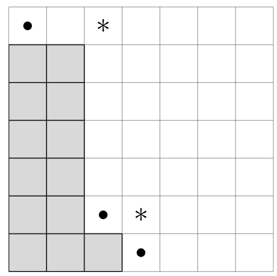

Figure 2.2: Staircase diagram of the ideal $I = (\alpha_0^3, \alpha_0^2\alpha_1, \alpha_1^6)$

2.1.5 Dimensions of Ext groups

In Lemmas 2.27 and 2.28 we present general results about finitely generated modules over polynomial rings. They will be used in Chapter 3.

Lemma 2.27. *Let n be a positive integer and $S = \Bbbk[\alpha_0, \dots, \alpha_n]$ be a polynomial ring. Let M be a finitely generated graded S-module. Then*

$$\sum_{i=0}^{n+1} (-1)^i \dim_{\Bbbk} \mathrm{Ext}^i_S(\Bbbk, M)_e = \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} \dim_{\Bbbk} M_{e+i}$$

for every $e \in \mathbb{Z}$.

Proof. Let $P_\bullet$ be the Koszul resolution of $\Bbbk$. Applying the functor $\mathrm{Hom}_S(-, M)_e$ we obtain a complex

$$\mathrm{Hom}_S(P_\bullet, M)_e,$$

whose cohomology groups are $\mathrm{Ext}^i_S(\Bbbk, M)_e$ for $i = 0, \dots, n+1$. Therefore, by a standard argument (see [80, Ex. 1.6.B]) by splitting the above complex into short exact sequences we get

$$\sum_{i=0}^{n+1} (-1)^i \dim_{\Bbbk} \mathrm{Ext}^i_S(\Bbbk, M)_e = \sum_{i=0}^{n+1} (-1)^i \dim_{\Bbbk} \mathrm{Hom}_S(P_i, M)_e.$$

Since $P_i \cong S(-i)^{\binom{n+1}{i}}$, it follows that

$$\dim_{\Bbbk} \mathrm{Hom}_S(P_i, M)_e = \binom{n+1}{i} \dim_{\Bbbk} M_{e+i}. \qquad \square$$

Lemma 2.28. *Let n be a positive integer and $S = \Bbbk[\alpha_0, \dots, \alpha_n]$ be a polynomial ring. Given a finitely generated graded S-module M and an integer $e \in \mathbb{Z}$ we have $\dim_{\Bbbk} \mathrm{Ext}^i_S(M, \Bbbk)_e = \beta_{i,-e}(M)$.*

Proof. Apply the functor $\operatorname{Hom}_S(-, \Bbbk)$ to a minimal graded free resolution $P_\bullet$ of M. The Ext groups $\operatorname{Ext}^i_S(M, \Bbbk)$ can be computed as cohomology groups of the obtained complex. Since the i-th differential in $P_\bullet$ maps P_i into $\mathfrak{m}P_{i-1}$, the differentials in the complex $\operatorname{Hom}_S(P_\bullet, \Bbbk)$ are zero. Therefore, $\dim_\Bbbk \operatorname{Ext}^i_S(M, \Bbbk)_e = \dim_\Bbbk \operatorname{Hom}_S(P_i, \Bbbk)_e = \beta_{i,-e}(M)$. $\square$

2.2 Scheme theory and multigraded Hilbert schemes

In this section we give the definition of multigraded Hilbert schemes and study the basic properties of these schemes.

In Subsection 2.2.1 we present general results from scheme theory that will be used in the proofs of the main results.

In Subsection 2.2.2 we give a formal definition of the functor of points of a multigraded Hilbert scheme and study its basic properties.

In Subsection 2.2.3 we introduce multigraded Hilbert schemes "of points in general position". This is the main object of investigation in this thesis.

In Subsection 2.2.4 we define the flag multigraded Hilbert scheme by its functor of points and we prove existence of this parameter space using existence of multigraded Hilbert schemes.

2.2.1 Scheme theory

The following lemma gives some conditions under which a morphism of $\Bbbk$ schemes that is bijective on $\Bbbk$-valued points is a homeomorphism.

Lemma 2.29. *Let $f\colon X \to Y$ be a closed morphism of schemes locally of finite type over $\Bbbk$. Assume that f induces a bijection of $\Bbbk$-valued points $X(\Bbbk) \to Y(\Bbbk)$. Then f is a homeomorphism.*

Proof. In both X and Y, closed points are very dense by [40, Prop. 3.35]. Since f induces a bijection of closed points, it is dominant and hence surjective. Moreover, if $f(p) = f(q)$, then $f(\overline{\{p\}}) = \overline{\{f(p)\}} = \overline{\{f(q)\}} = f(\overline{\{q\}})$. As a result, the sets of closed points of X that are contained in $\overline{\{p\}}$ and $\overline{\{q\}}$ are equal. It follows that $\overline{\{p\}} = \overline{\{q\}}$ and therefore, $p = q$. This shows that $f\colon X \to Y$ is a bijective, closed, continuous map and thus, a homeomorphism. $\square$

In describing the intersection of irreducible components of some multigraded Hilbert schemes we shall use the following lemma.

Lemma 2.30. *Let X be a scheme locally of finite type over $\Bbbk$. Let Z_1, Z_2 be irreducible closed subsets of X of dimensions d_1, d_2, respectively. Let $P \in Z_1 \cap Z_2$ be a closed point of the intersection and let $d = \dim_\Bbbk \mathbf{T}_P X$. Then every irreducible component W of $Z_1 \cap Z_2$ such that $P \in W$ satisfies $\dim W \geq d_1 + d_2 - d$.*

Proof. By [79, Tag 0C2G], there exists an open neighborhood U of P in X and a closed immersion $i\colon U \to Y$ where Y is a smooth d-dimensional variety over $\Bbbk$. Let $W_1 = i(|Z_1| \cap |U|)$ and $W_2 = i(|Z_2| \cap |U|)$, where $|\cdot|$ denotes the underlying topological space. These are d_1 and d_2-dimensional irreducible closed subsets of Y, respectively. Therefore, every irreducible component of $W_1 \cap W_2$ has dimension at least $d_1 + d_2 - d$ (see [36, §8.2]). $\square$

2.2.2 Multigraded Hilbert schemes

In this subsection we introduce multigraded Hilbert schemes following [45]. We give the definition in terms of the functor of points $\underline{\mathrm{Hilb}_S^h}\colon \Bbbk-\mathfrak{Alg} \to \mathfrak{Set}$ and then verify that the scheme represents the natural extension to the functor $\underline{\mathrm{Hilb}_S^h}\colon \mathfrak{Sch}_{\Bbbk}^{op} \to \mathfrak{Set}$.

Let n be a positive integer and let $S = \Bbbk[\underline{\alpha}] := \Bbbk[\alpha_0, ..., \alpha_n]$ be the polynomial ring over $\Bbbk$. We identify monomials of S with $\mathbb{N}^{n+1}$. Let A be an abelian group and let $\deg\colon \mathbb{N}^{n+1} \to A$ be a homomorphism of semigroups. We assume that A is generated by $\deg(\alpha_i)$ for $i = 0, ..., n$. We consider S with the A-grading induced by deg

$$S = \bigoplus_{a\in A} S_a \text{ satisfying } S_a \cdot S_b \subseteq S_{a+b},$$

where S_a is the $\Bbbk$-vector space spanned by monomials $x^{\mathbf{u}}$ with $\deg(\mathbf{u}) = a$. Given a $\Bbbk$-algebra R, we write $R[\underline{\alpha}]$ for $S \otimes_{\Bbbk} R$ together with the A-grading given by $R[\underline{\alpha}]_a = S_a \otimes_{\Bbbk} R$.

Definition 2.31. Given a $\Bbbk$-algebra R and a function $h\colon A \to \mathbb{N}$, we say that a homogeneous ideal $I \subseteq R[\underline{\alpha}]$ is *admissible for Hilbert function h* if $R[\underline{\alpha}]_a/I_a$ is a locally free R-module of rank $h(a)$ for every $a \in A$.

We define the functor $\underline{\mathrm{Hilb}_S^h}\colon \Bbbk\text{-}\mathfrak{Alg} \to \mathfrak{Set}$ by

$$R \mapsto \{I \subseteq R[\underline{\alpha}] \mid I \text{ is an admissible ideal for Hilbert function } h\}$$

and given $\varphi\colon R \to R'$ we define

$$\underline{\mathrm{Hilb}_S^h}(\varphi)\colon I \mapsto I \otimes_R R'.$$

The following lemma confirms that the above data define a functor.

Lemma 2.32. *Let R be a $\Bbbk$-algebra and $I \subseteq R[\underline{\alpha}]$ be an admissible ideal for Hilbert function h. If $\varphi\colon R \to R'$ is a homomorphism of $\Bbbk$-algebras, then $I \otimes_R R'$ is an ideal of $R'[\underline{\alpha}]$, admissible for Hilbert function h.*

Proof. Consider the exact sequence of R-modules

$$0 \to I \to R[\underline{\alpha}] \to R[\underline{\alpha}]/I \to 0.$$

Since $R[\underline{\alpha}]_a/I_a$ is a locally free R-module of finite rank it is flat (see [6, Prop. 3 §4.4]). Therefore, by tensoring the above sequence with R' over R we obtain an exact sequence of R' modules

$$0 \to I \otimes_R R' \to R'[\underline{\alpha}] \to R'[\underline{\alpha}]/(I \otimes_R R') \to 0.$$

Since $I \otimes_R R'$ is an R'-submodule of $R'[\underline{\alpha}]$ stable under multiplication by any monomial, it is an ideal. We shall show that $R'[\underline{\alpha}]_a/(I_a \otimes_R R')$ is a locally free R' module of rank $h(a)$ for every $a \in A$. Let $\mathfrak{q} \in \operatorname{Spec} R'$ and let $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Given $a \in A$, there is an element $f \in R \setminus \mathfrak{p}$ such that $(R[\underline{\alpha}]_a/I_a)_f \cong R_f^{h(a)}$. Let $g = \varphi(f)$. We have $g \in R' \setminus \mathfrak{q}$ and we shall show that $\big(R'[\underline{\alpha}]_a/(I_a \otimes_R R')\big)_g \cong (R'_g)^{h(a)}$. Indeed,

$$\big(R'[\underline{\alpha}]_a/(I_a \otimes_R R')\big)_g \cong \big((R[\underline{\alpha}]_a/I_a) \otimes_R R'\big)_g \cong (R[\underline{\alpha}]_a/I_a) \otimes_R R'_g \cong$$

$$\cong (R[\underline{\alpha}]_a/I_a)_f \otimes_{R_f} R'_g \cong R_f^{h(a)} \otimes_{R_f} R'_g \cong (R'_g)^{h(a)}. \qquad \square$$

The following existence statement is the foundation of the theory of multigraded Hilbert schemes.

Theorem 2.33 ([45, Thm. 1.1])**.** *Let $S = \Bbbk[\alpha_0, \ldots, \alpha_n]$ be a polynomial ring graded by an abelian group A. Let $h\colon A \to \mathbb{N}$ be a numerical function. Then there exists a quasiprojective $\Bbbk$-scheme Hilb^h_S representing the functor $\underline{\mathrm{Hilb}}^h_S\colon \Bbbk - \mathfrak{Alg} \to \mathfrak{Set}$.*

Moreover, under additional assumptions on the grading $\deg\colon \mathbb{N}^{n+1} \to A$, the scheme Hilb^h_S is projective.

Definition 2.34. Let $S = \Bbbk[\alpha_0, \ldots, \alpha_n]$ be a polynomial ring and A be an abelian group. The grading of S given by a semigroup homomorphism $\deg\colon \mathbb{N}^{n+1} \to A$ is called *positive* if $S_0 = \Bbbk$.

In the cases that are studied in this thesis, the grading is positive.

Example 2.35. (a) The homogeneous coordinate ring $S[\mathbb{P}^n] = \Bbbk[\alpha_0, \ldots, \alpha_n]$ of projective space $\mathbb{P}^n$ is $\mathbb{Z}$-graded by $\deg(\alpha_i) = 1$. This grading is positive.

(b) More generally, let X be a smooth projective toric variety over the field of complex numbers. Then its Cox ring $S[X]$ is a $\mathrm{Pic}(X)$-graded polynomial ring and $S[X]_0 = \Gamma(X, \mathcal{O}_X) = \mathbb{C}$. See Chapter 4 for more about smooth projective toric varieties and their Cox rings.

The main consequence of the assumption that S is positively graded, is the following result.

Theorem 2.36 ([45, Cor. 1.2.])**.** *Let $S = \Bbbk[\alpha_0, \ldots, \alpha_n]$ be a polynomial ring graded by an abelian group A. If the grading is positive then for every function $h\colon A \to \mathbb{N}$, the multigraded Hilbert scheme Hilb^h_S is projective over $\Bbbk$.*

Moreover, there is also an algebraic consequence of this restriction on the grading.

Theorem 2.37 ([67, Thm. 8.6])**.** *Let $S = \Bbbk[\alpha_0, \ldots, \alpha_n]$ be a polynomial ring graded by an abelian group A. Then the following conditions are equivalent:*

1. *$S_0 = \Bbbk$;*
2. *S_a is a finite dimensional $\Bbbk$-vector space for every $a \in A$.*

Now we discuss the extension of the functor $\underline{\mathrm{Hilb}}^h_S$ to the category $\mathfrak{Sch}^{op}_{\Bbbk}$. The definition comes from [67, §18.5]. Then we check that the extended functor is indeed the functor of points of Hilb^h_S.

Definition 2.38. Let X be a $\Bbbk$-scheme and $h\colon A \to \mathbb{N}$ be a function. We say that a closed subscheme $Z \subseteq \mathbb{A}^{n+1}_X$ is an *admissible family* for the function h if for every affine open subscheme $\operatorname{Spec} R \subseteq X$, the pullback of the ideal sheaf of Z to $\mathbb{A}^{n+1}_R$ corresponds to an ideal of $R[\underline{\alpha}]$ that is admissible for the function h.

As is often the case, this condition can be checked on any affine open covering.

Lemma 2.39. *Let $h\colon A \to \mathbb{N}$ be a function. Let X be a $\Bbbk$-scheme with an affine open covering $X = \bigcup_{j \in J} U_j$ where $U_j = \operatorname{Spec} R_j$. Let $Z \subseteq \mathbb{A}^{n+1}_X$ be a closed subscheme such that the pullback of its ideal sheaf to $\mathbb{A}^{n+1}_{R_j}$ is an admissible ideal of $R_j[\underline{\alpha}]$ for the function h for every $j \in J$. Then Z is an admissible family over X for the function h.*

Proof. Let $V = \operatorname{Spec} R$ be an affine open subscheme of X and let I be the ideal in $R[\underline{\alpha}]$ corresponding to the pullback of the ideal sheaf of Z. Then I is an admissible ideal for the function h if and only if for each a in A, the degree a part of $\widetilde{R[\underline{\alpha}]/I}$ is a locally free sheaf of rank $h(a)$. This can be checked on any affine open covering of $\mathbb{A}^{n+1}_R$. Given a point $P \in \mathbb{A}^{n+1}_R$ there is an affine open subscheme $U = \operatorname{Spec} T \subseteq V \cap U_j$ for some $j \in J$ that is a distinguished open subscheme of U_j and such that $P \in \mathbb{A}^{n+1}_T$. It is enough to show that the pullback of the ideal sheaf of Z to $\mathbb{A}^{n+1}_T$ corresponds to an admissible ideal. Since this is true for the pullback to $\mathbb{A}^{n+1}_{R_j}$, the claim follows from Lemma 2.32. $\square$

We define the natural extension of the multigraded Hilbert scheme functor to the category $\mathfrak{Sch}_{\Bbbk}^{op}$, opposite of the category of $\Bbbk$-schemes. For now we will denote this functor by $\widetilde{\underline{\mathrm{Hilb}}^h_S}$.

Given a $\Bbbk$-scheme X, let

$$\widetilde{\underline{\mathrm{Hilb}}^h_S}(X) = \{\textit{admissible families } Z \subseteq \mathbb{A}^{n+1}_X \textit{ for the function } h\}$$

and for a morphism $f\colon X \to Y$ of $\Bbbk$-schemes let

$$\widetilde{\underline{\mathrm{Hilb}}^h_S}(f)\colon Z \mapsto (f \times \mathrm{id}_{\mathbb{A}^{n+1}})^{-1}(Z).$$

The following lemma checks that this functor is well-defined.

Lemma 2.40. *Let $f\colon X \to Y$ be a morphism of $\Bbbk$-schemes. If $Z \subseteq \mathbb{A}^{n+1}_Y$ is an admissible family over Y, then the scheme theoretic inverse image*

$$(f \times \mathrm{id}_{\mathbb{A}^{n+1}})^{-1}(Z) \subseteq \mathbb{A}^{n+1}_X$$

is an admissible family over X.

Proof. To simplify notation, let $Z' = (f \times \mathrm{id}_{\mathbb{A}^{n+1}})^{-1}(Z)$. We may choose affine open subschemes $\operatorname{Spec} T \subseteq X$ and $\operatorname{Spec} R \subseteq Y$ such that $f(\operatorname{Spec} T) \subseteq \operatorname{Spec} R$. By Lemma 2.39, it is enough to show that the pullback of Z' to $\mathbb{A}^{n+1}_T$, denoted by Z'_T, corresponds to an admissible ideal of $T[\underline{\alpha}]$. Let Z_R denote the pullback of Z to $\mathbb{A}^{n+1}_R$. We have the following diagrams with all inner squares pullback diagrams:

$$\begin{tikzcd}
\mathbb{A}^{n+1}_T \times_{\mathbb{A}^{n+1}_R} Z_R \arrow[r] \arrow[d] & Z_R \arrow[r] \arrow[d] & Z \arrow[d] \\
\mathbb{A}^{n+1}_T \arrow[r] \arrow[d] & \mathbb{A}^{n+1}_R \arrow[r] \arrow[d] & \mathbb{A}^{n+1}_Y \arrow[d] \\
\operatorname{Spec} T \arrow[r] & \operatorname{Spec} R \arrow[r] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Z'_T \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & Z \arrow[d] \\
\mathbb{A}^{n+1}_T \arrow[r] \arrow[d] & \mathbb{A}^{n+1}_X \arrow[r] \arrow[d] & \mathbb{A}^{n+1}_Y \arrow[d] \\
\operatorname{Spec} T \arrow[r] & X \arrow[r] & Y.
\end{tikzcd}$$

Since Z is an admissible ideal over Y, Z_R corresponds to an admissible ideal of $R[\underline{\alpha}]$. Thus, $\mathbb{A}^{n+1}_T \times_{\mathbb{A}^{n+1}_R} Z_R$ corresponds to an admissible ideal of $T[\underline{\alpha}]$ by Lemma 2.32. Since the bottom arrows from $\operatorname{Spec} T$ to Y in both diagrams are the same, so are the middle arrows $\mathbb{A}^{n+1}_T \to \mathbb{A}^{n+1}_Y$. Therefore, $Z'_T = \mathbb{A}^{n+1}_T \times_{\mathbb{A}^{n+1}_R} Z_R$ as a closed subscheme of $\mathbb{A}^{n+1}_T$ which finishes the proof. $\square$

As expected, the multigraded Hilbert scheme represents this extended functor.

Lemma 2.41. *Let $S = \Bbbk[\alpha_0, \ldots, \alpha_n]$ be a polynomial ring graded by an abelian group A. Let $h\colon A \to \mathbb{N}$ be a function. Then the functor $\underline{\widetilde{\mathrm{Hilb}^h_S}}\colon \mathfrak{Sch}_{\Bbbk}^{op} \to \mathfrak{Set}$ defined above is the functor of points of the multigraded Hilbert scheme Hilb^h_S.*

Proof. Let $h_{\mathrm{Hilb}^h_S}$ be the functor of points of Hilb^h_S. Let Y be a $\Bbbk$-scheme and cover it by affine open subschemes $\{V_{i_s}\}_{i_s \in I}$. Moreover, cover each intersection $V_{i_s} \cap V_{i_t}$ by affine open subschemes $V_{i_s i_t k}$ for $k \in I_{i_s i_t}$. Since data of an admissible family over Y is affine local on Y, we have an equalizer diagram of sets

$$\underline{\widetilde{\mathrm{Hilb}^h_S}}(Y) \to \prod_{i_s \in I} \underline{\widetilde{\mathrm{Hilb}^h_S}}(V_{i_s}) \rightrightarrows \prod_{i_s \in I} \prod_{i_t \in I} \prod_{k \in I_{i_s i_t}} \underline{\widetilde{\mathrm{Hilb}^h_S}}(V_{i_s i_t k}).$$

See [74, page 225] for the definition and universal property of equalizer. Since $h_{\mathrm{Hilb}^h_S}$ is representable, we also have an equalizer diagram of sets

$$h_{\mathrm{Hilb}^h_S}(Y) \to \prod_{i_s \in I} h_{\mathrm{Hilb}^h_S}(V_{i_s}) \rightrightarrows \prod_{i_s \in I} \prod_{i_t \in I} \prod_{k \in I_{i_s i_t}} h_{\mathrm{Hilb}^h_S}(V_{i_s i_t k}).$$

Since $\underline{\widetilde{\mathrm{Hilb}^h_S}}$ and $h_{\mathrm{Hilb}^h_S}$ are isomorphic when restricted to the category of affine $\Bbbk$-schemes, the middle and right terms of the above sequences are naturally isomorphic. Thus, we have an isomorphism $\underline{\widetilde{\mathrm{Hilb}^h_S}}(Y) \to h_{\mathrm{Hilb}^h_S}(Y)$ by the universal property of equalizer.

We claim that this isomorphism is natural with respect to $f\colon X \to Y$ so that it defines a natural isomorphism of functors between $\underline{\widetilde{\mathrm{Hilb}^h_S}}$ and $h_{\mathrm{Hilb}^h_S}$. Indeed, we may choose an affine open covering $\{U_{j_a}\}_{j_a \in J}$ of X refining the open covering of X by preimages of V_{i_s}'s for $i_s \in I$. Let $\gamma\colon J \to I$ be the corresponding map of indexing sets, such that $U_{j_a} \subseteq f^{-1}(V_{\gamma(j_a)})$ for every $j_a \in J$. Furthermore, we can cover each $U_{j_a} \cap U_{j_b}$ with $j_a, j_b \in J$ by affine open subsets $U_{j_a j_b l}$ for $l \in J_{j_a j_b}$ refining the open covering of $U_{j_a} \cap U_{j_b}$ by preimages of $V_{\gamma(j_a)\gamma(j_b)k}$ for $k \in I_{\gamma(j_a)\gamma(j_b)}$. Then we have a commutative diagram

$$\begin{array}{ccccc}
\underline{\widetilde{\mathrm{Hilb}^h_S}}(Y) & \longrightarrow & \prod_{i_s \in I} \underline{\widetilde{\mathrm{Hilb}^h_S}}(V_{i_s}) & \rightrightarrows & \prod_{i_s \in I} \prod_{i_t \in I} \prod_{k \in I_{i_s i_t}} \underline{\widetilde{\mathrm{Hilb}^h_S}}(V_{i_s i_t k}) \\
\downarrow{\scriptstyle \underline{\widetilde{\mathrm{Hilb}^h_S}}(f)} & & \downarrow & & \downarrow \\
\underline{\widetilde{\mathrm{Hilb}^h_S}}(X) & \longrightarrow & \prod_{j_a \in J} \underline{\widetilde{\mathrm{Hilb}^h_S}}(U_{j_a}) & \rightrightarrows & \prod_{j_a \in J} \prod_{j_b \in J} \prod_{l \in J_{j_a j_b}} \underline{\widetilde{\mathrm{Hilb}^h_S}}(U_{j_a j_b l})
\end{array}$$

and a similar one for $h_{\mathrm{Hilb}^h_S}$. Since the isomorphisms

$$\alpha\colon \underline{\widetilde{\mathrm{Hilb}^h_S}}(Y) \to h_{\mathrm{Hilb}^h_S}(Y) \text{ and } \beta\colon \underline{\widetilde{\mathrm{Hilb}^h_S}}(X) \to h_{\mathrm{Hilb}^h_S}(X)$$

are induced from the universal property of equalizer and isomorphism of restricted functors, it follows from the universal property of equalizer that $\beta \circ \underline{\widetilde{\mathrm{Hilb}^h_S}}(f) = h_{\mathrm{Hilb}^h_S}(f) \circ \alpha$ which finishes the proof of naturality. $\square$

We end this subsection with two technical results. The first of them is concerned with the fact that smoothness of $[I] \in \mathrm{Hilb}^h_S$ "does not depend" on I_1. The following lemma makes it precise.

Lemma 2.42. *Let $m \leq n$ be positive integers and let $I \subseteq S[\mathbb{P}^m] = \Bbbk[\alpha_0, \ldots, \alpha_m]$ be a homogeneous ideal. Denote the Hilbert function of $S[\mathbb{P}^m]/I$ by h. Let $I' = I + (\alpha_{m+1}, \ldots, \alpha_n) \subseteq S[\mathbb{P}^n] =$*

$\Bbbk[\alpha_0, \dots, \alpha_m, \dots, \alpha_n]$. *Then* $[I] \in \mathrm{Hilb}^h_{S[\mathbb{P}^m]}$ *is a smooth point if and only if* $[I'] \in \mathrm{Hilb}^h_{S[\mathbb{P}^n]}$ *is a smooth point.*

Proof. Let $d = h(1)$. Then $d \leq m+1$ and we can consider $S[\mathbb{P}^{d-1}] = \Bbbk[\alpha_0, \dots, \alpha_{d-1}]$ as a subring of $S[\mathbb{P}^m]$. Up to a linear change of variables in $S[\mathbb{P}^m]$ we may assume that $I = I'' + (\alpha_d, \dots, \alpha_m)$ for an ideal $I'' \subseteq S[\mathbb{P}^{d-1}]$ such that $S[\mathbb{P}^{d-1}]/I''$ has Hilbert function h.

The scheme $\mathrm{Hilb}^h_{S[\mathbb{P}^n]}$ is a $\mathrm{Hilb}^h_{S[\mathbb{P}^{d-1}]}$-bundle over $\mathrm{Gr}(n+1-d, S[\mathbb{P}^n]_1)$ and $\mathrm{Hilb}^h_{S[\mathbb{P}^m]}$ is a $\mathrm{Hilb}^h_{S[\mathbb{P}^{d-1}]}$-bundle over $\mathrm{Gr}(m+1-d, S[\mathbb{P}^m]_1)$ by [20, Prop. 3.1]. Therefore,

$$[I] \text{ is a smooth point } \Leftrightarrow [I''] \text{ is a smooth point } \Leftrightarrow [I'] \text{ is a smooth point.} \qquad \square$$

We compute the fiber of a natural map of multigraded Hilbert schemes, associated with the homogeneous coordinate ring of a projective space, given by restricting ideals to high degree.

Lemma 2.43. *Let n be a positive integer and I be a homogeneous ideal of $S = S[\mathbb{P}^n]$. Let $m = \min\{a \in \mathbb{Z} \mid I_a \neq 0\}$ and g be the Hilbert function of S/I. Let $d > m$ be a positive integer and define $h\colon \mathbb{Z} \to \mathbb{Z}$ by*

$$h(a) = \begin{cases} \dim_\Bbbk S_a & \text{for } a < d \\ g(a) & \text{for } a \geq d. \end{cases}$$

Then there is a natural map $\pi\colon \mathrm{Hilb}^g_S \to \mathrm{Hilb}^h_S$ given on closed points by $[J] \mapsto [J \cap \mathfrak{m}^d]$. Let $[K]$ be a closed point of Hilb^h_S such that $H_{S/\overline{K}}(a) = g(a)$ for every $a \geq m+1$. Then the fiber of π over $[K]$ is the Grassmannian

$$\mathrm{Gr}(\dim_\Bbbk S_m - g(m), \overline{K}_m).$$

Proof. The point $[K] \in \mathrm{Hilb}^h_S$ gives a natural morphism

$$\operatorname{Spec} \Bbbk = \operatorname{Spec} k([K]) \to \mathrm{Hilb}^h_S.$$

Its functor of points $\Bbbk - \mathfrak{Alg} \to \mathfrak{Set}$ is given by

$$R \mapsto \{K \otimes_\Bbbk R\}.$$

The scheme theoretic fiber over $[K]$ is the fiber product

$$\mathrm{Hilb}^g_S \times_{\mathrm{Hilb}^h_S} \operatorname{Spec} k([K]).$$

Therefore, its functor of points $\Bbbk - \mathfrak{Alg} \to \mathfrak{Set}$ is the fiber product of the corresponding functors, i.e. it is given by

$$R \mapsto \{J \in \underline{\mathrm{Hilb}^g_S}(R) \mid J \cap (\mathfrak{m} \otimes_\Bbbk R)^d = K \otimes_\Bbbk R\}.$$

Since all schemes considered in this proof are of finite type over $\operatorname{Spec} \Bbbk$, they can be recovered from their functors of points restricted to the subcategory $\Bbbk - \mathfrak{f.g.Alg}$ of finitely generated $\Bbbk$-algebras. In what follows we restrict to this subcategory.

The ideal $\overline{K} \otimes_\Bbbk R$ is saturated with respect to $\mathfrak{m} \otimes_\Bbbk R$ by Lemma 2.6. Therefore, by definition of m and the assumption that $H_{S/\overline{K}}(a) = g(a)$ for every $a \geq m+1$, the functor of points of the fiber is naturally isomorphic to the functor $\Bbbk - \mathfrak{f.g.Alg} \to \mathfrak{Set}$ defined by

$$R \mapsto \{J_m \subseteq \overline{K}_m \otimes_{\Bbbk} R \subseteq S_m \otimes_{\Bbbk} R \,|\, J_m \text{ is an } R\text{-submodule of } \overline{K}_m \otimes_{\Bbbk} R \text{ and } (S_m \otimes_{\Bbbk} R)/J_m \text{ is a locally free } R\text{-module of rank } g(m)\}.$$

By Lemma 2.4, this functor coincides with the following functor $\Bbbk - \mathfrak{f.g.Alg} \to \mathfrak{Set}$:

$$R \mapsto \{J_m \subseteq \overline{K}_m \otimes_{\Bbbk} R \,|\, J_m \text{ is an } R\text{-submodule of } \overline{K}_m \otimes_{\Bbbk} R \text{ and } (\overline{K}_m \otimes_{\Bbbk} R)/J_m \text{ is a locally free } R\text{-module of rank } g(m) + \dim_{\Bbbk} \overline{K}_m - \dim_{\Bbbk} S_m\}.$$

This is the functor of points of $\mathrm{Gr}(\dim_{\Bbbk} S_m - g(m), \overline{K}_m)$. □

2.2.3 Multigraded Hilbert schemes of points in general position and Slip

In this subsection we introduce the main object of study of this thesis in the case of projective space. We will define the multigraded Hilbert scheme $\mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$ of r points in general position on projective n-space. The more general definition for a smooth projective toric variety appears in Subsection 4.1.5. For positive integers r, n, the scheme $\mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$ has a distinguished irreducible component called $\mathrm{Slip}_{r,n}$ which plays a key role in the border apolarity lemma - Proposition 2.91.

Fix a positive integer n and let $S = S[\mathbb{P}^n] := \Bbbk[\underline{\alpha}] = \Bbbk[\alpha_0, \ldots, \alpha_n]$ be the homogeneous coordinate ring of $\mathbb{P}^n$ with its standard $\mathbb{Z}$-grading given by $\deg(\alpha_i) = 1$ for $i = 0, \ldots, n$. Let r be a positive integer and let $h_{r,n} = h_{r,\mathbb{P}^n} : \mathbb{Z} \to \mathbb{N}$ be given by

$$h_{r,n}(a) = \min\{r, \dim_{\Bbbk} S_a\}.$$

This is the Hilbert function of r points in $\mathbb{P}^n$ in general position. We shall study the multigraded Hilbert scheme $\mathrm{Hilb}^{h_{r,n}}_S$.

This scheme has a natural morphism into the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^n)$ parametrizing zero-dimensional, length r subschemes of the projective n-space. We will describe this in some details. We start with the following observation.

Lemma 2.44 ([45, Lem. 4.1])**.** *Let $f_{r,n} : \mathbb{Z} \to \mathbb{N}$ be defined by*

$$f_{r,n}(a) = \begin{cases} \dim_{\Bbbk} S_a & \text{for } a < r, \\ r & \text{for } a \geq r. \end{cases}$$

Then $\mathcal{H}ilb_r(\mathbb{P}^n)$ is isomorphic to $\mathrm{Hilb}^{f_{r,n}}_S$ where on closed points, the isomorphism identifies $[I] \in \mathrm{Hilb}^{f_{r,n}}_S$ with $[\mathrm{Proj}(S/I)] \in \mathcal{H}ilb_r(\mathbb{P}^n)$.

Consider a morphism of functors $\underline{\mathrm{Hilb}^{h_{r,n}}_S} \to \underline{\mathrm{Hilb}^{f_{r,n}}_S}$ defined for a $\Bbbk$-algebra R by

$$S \otimes_{\Bbbk} R \supseteq I \mapsto I \cap (\mathfrak{m} \otimes_{\Bbbk} R)^r,$$

where $\mathfrak{m} = (\alpha_0, \ldots, \alpha_n)$ is the irrelevant ideal of S. Using the identification from Lemma 2.44 this gives a morphism of schemes $\varphi_{r,n} : \mathrm{Hilb}^{h_{r,n}}_S \to \mathcal{H}ilb_r(\mathbb{P}^n)$ that on closed points sends an ideal to the subscheme defined by this ideal. Unless stated otherwise, we shall identify $\mathcal{H}ilb_r(\mathbb{P}^n)$ with $\mathrm{Hilb}^{f_{r,n}}_S$.

Following [13] we define $\mathrm{Sip}_{r,n}$ to be the set of closed points of $\mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$ corresponding to saturated ideals of r distinct points. The next result is fundamental. Its proof for $\Bbbk = \mathbb{C}$ appears in [13, Prop. 3.13] for an arbitrary smooth projective toric variety.

Proposition 2.45. *Let r, n be positive integers. Then the closure of $\mathrm{Sip}_{r,n}$ in $\mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$ is an irreducible component.*

Proof. Let $\mathscr{U} \subseteq \mathcal{H}ilb_r(\mathbb{P}^n) \times \mathbb{P}^n$ be the universal family over $\mathcal{H}ilb_r(\mathbb{P}^n)$ and let $\pi\colon \mathscr{U} \to \mathcal{H}ilb_r(\mathbb{P}^n)$ denote the natural morphism.

Let U be the locus of points in $\mathcal{H}ilb_r(\mathbb{P}^n)$ corresponding to smooth subschemes of $\mathbb{P}^n$ and let V be the locus of points in $\mathcal{H}ilb_r(\mathbb{P}^n)$ corresponding to subschemes with Hilbert function $h_{r,n}$. We claim that U, V are open. For a non-negative integer d, let $V_d \subseteq \mathcal{H}ilb_r(\mathbb{P}^n)$ be the locus of points corresponding to subschemes $R \subseteq \mathbb{P}^n$ such that $\dim_\Bbbk H^0(\mathbb{P}^n, \mathcal{I}_R(d)) \leq \dim_\Bbbk S_d - h_{r,n}(d)$ where $\mathcal{I}_R$ is the ideal sheaf of R. The subset V_d is open by [47, Thm. III.12.8]. Furthermore, $V_d = \mathcal{H}ilb_r(\mathbb{P}^n)$ for $d \geq r$ by Gotzmann's regularity theorem [10, Thm. 4.3.2] and [31, Thm. 4.2]. Therefore, $V = \bigcap_{d=0}^{r-1} V_d$ is open.

Let $W \subseteq \mathscr{U}$ be the locus of points x such that the fiber of π over $\pi(x)$ is smooth. This is an open subset of $\mathscr{U}$ by [44, Thm. 12.1.6]. Therefore, its image U under π is open since π is flat and locally of finite presentation and thus, open by [40, Thm. 14.33].

It follows that $\mathrm{Sip}_{r,n} = \varphi_{r,n}^{-1}(U \cap V)$ is open. Furthermore, it is homeomorphic to $U \cap V$ by Lemma 2.29. In particular, it is irreducible since $U \cap V$ is a non-empty open subset of the smoothable component of $\mathcal{H}ilb_r(\mathbb{P}^n)$. It follows that the closure of $\mathrm{Sip}_{r,n}$ is an irreducible component of $\mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$. ∎

The irreducible component $\overline{\mathrm{Sip}_{r,n}}$ will be denoted by $\mathrm{Slip}_{r,n}$.

We end this subsection with a remark about a relation between the irreducible component $\mathrm{Slip}_{r,n}$ and the smoothable component $\mathcal{H}ilb^{sm}_r(\mathbb{P}^n)$ of the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^n)$.

Remark 2.46. Let $\varphi_{r,\mathbb{P}^n} = \varphi_{r,n}\colon \mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]} \to \mathcal{H}ilb_r(\mathbb{P}^n)$ be the map considered above. Namely, on closed points it is given by

$$[I] \mapsto [\operatorname{Proj} S/I].$$

Since both schemes are projective, it is a closed map. It sends $\mathrm{Sip}_{r,n}$ onto an open subset of the locus of reduced subschemes. It follows that $\varphi_{r,n}(\mathrm{Slip}_{r,n}) = \mathcal{H}ilb^{sm}_r(\mathbb{P}^n)$ set-theoretically. In particular, if $\mathcal{H}ilb_r(\mathbb{P}^n)$ is irreducible then for every closed point $[I] \in \mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$ there is a point $[I'] \in \mathrm{Slip}_{r,n}$ with $\overline{I} = \overline{I'}$. As a special case, if I is saturated, $\mathcal{H}ilb_r(\mathbb{P}^n)$ is irreducible and $[I] \in \mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$ then $[I] \in \mathrm{Slip}_{r,n}$.

2.2.4 Flag multigraded Hilbert schemes

Let $S = \Bbbk[\alpha_0, \ldots, \alpha_n]$ be a polynomial ring. Assume that S is graded by an abelian group A. Let $f_1, f_2\colon A \to \mathbb{N}$ be numerical functions. Then there are multigraded Hilbert schemes $\mathrm{Hilb}^{f_1}_S$ and $\mathrm{Hilb}^{f_2}_S$. The goal of this subsection is the construction of the scheme $\mathrm{Hilb}^{f_1,f_2}_S$ parametrizing pairs K, J of homogeneous ideals such that $K \subseteq J$ and S/K, S/J have Hilbert functions f_1 and f_2, respectively. The idea is to show that the condition that $K \subseteq J$ defines a closed subscheme of the product $\mathrm{Hilb}^{f_1}_S \times \mathrm{Hilb}^{f_2}_S$. The rest of this subsection makes this intuition precise.

Let $\underline{\mathrm{Hilb}^{f_1}_S}$ and $\underline{\mathrm{Hilb}^{f_2}_S}$ be the functors of points of $\mathrm{Hilb}^{f_1}_S$ and $\mathrm{Hilb}^{f_2}_S$, respectively. We start with defining the functor $\underline{\mathrm{Hilb}^{f_1,f_2}_S}\colon \Bbbk - \mathfrak{Alg} \to \mathfrak{Set}$ that $\mathrm{Hilb}^{f_1,f_2}_S$ should represent. It will be a subfunctor of the product functor $\underline{\mathrm{Hilb}_S^{f_1}} \times \underline{\mathrm{Hilb}_S^{f_2}}$. Given $R \in \Bbbk - \mathfrak{Alg}$, let

$$\underline{\mathrm{Hilb}_S^{f_1,f_2}}(R) = \{(K,J) \in \underline{\mathrm{Hilb}_S^{f_1}}(R) \times \underline{\mathrm{Hilb}_S^{f_2}}(R) \mid K \subseteq J\}.$$

We can now state the main result of this subsection.

Proposition 2.47. *Let $S = \Bbbk[\alpha_0, \ldots, \alpha_n]$ be a polynomial ring graded by an abelian group A. Let $f_1, f_2 \colon A \to \mathbb{N}$ be numerical functions. Then the functor $\underline{\mathrm{Hilb}_S^{f_1,f_2}} \colon \Bbbk - \mathfrak{Alg} \to \mathfrak{Set}$ is represented by a closed subscheme of $\mathrm{Hilb}_S^{f_1} \times \mathrm{Hilb}_S^{f_2}$. In particular, it is a projective scheme if the grading of S is positive.*

Before proving Proposition 2.47, we recall some results related to representable functors. See [32, Cha. VI] or [40, Cha. 8] for good introductions to this topic.

Definition 2.48. A functor $F \colon \Bbbk - \mathfrak{Alg} \to \mathfrak{Set}$ is a *sheaf in the Zariski topology* if for every $\Bbbk$-algebra R and for every open covering $\operatorname{Spec} R = \bigcup U_i$ by distinguished open subschemes $U_i = \operatorname{Spec} R_{f_i}$ we have an equalizer sequence

$$F(R) \to \prod_i F(R_{f_i}) \rightrightarrows \prod_{i,j} F(R_{f_i f_j}).$$

If a functor $F \colon \Bbbk - \mathfrak{Alg} \to \mathfrak{Set}$ is represented by a $\Bbbk$-scheme, then F is a sheaf in the Zariski topology [32, Thm. VI-14].

Following [45] we introduce the following notion.

Definition 2.49. Let R be a $\Bbbk$-algebra and let $\mathcal{C}$ be a condition on R-algebras. We say that the condition $\mathcal{C}$ is *closed* if there exists an ideal $\mathfrak{a} \subseteq R$ such that an R-algebra $\phi \colon R \to T$ satisfies condition $\mathcal{C}$ if and only if $\phi(\mathfrak{a}) = 0$.

Let $F \colon \Bbbk - \mathfrak{Alg} \to \mathfrak{Set}$ be a sheaf in the Zariski topology and assume that it is a subfunctor of the functor of points h_X of a $\Bbbk$-scheme X. Given a $\Bbbk$-algebra R and an element $\lambda \in h_X(R)$ we say that an R-algebra $\phi \colon R \to T$ satisfies condition $V_{R,\lambda}$ if $h_X(\phi)(\lambda) \in F(T) \subseteq h_X(T)$.

In the following remark we show that the ideal $\mathfrak{a}$ from the definition of a closed condition is uniquely determined by the condition.

Remark 2.50. Let R be a $\Bbbk$-algebra and let $\mathcal{C}$ be a closed condition on R-algebras. Then the ideal $\mathfrak{a} \subseteq R$ such that R-algebra $\phi \colon R \to T$ satisfies condition $\mathcal{C}$ if and only if $\phi(\mathfrak{a}) = 0$ is uniquely determined by $\mathcal{C}$. Indeed, if $\mathfrak{b}$ has analogous property then considering R-algebras $R \to R/\mathfrak{a}$ and $R \to R/\mathfrak{b}$ we conclude that $\mathfrak{a} = \mathfrak{b}$.

In order to show that $\underline{\mathrm{Hilb}_S^{f_1,f_2}}$ is represented by a closed subscheme of $\mathrm{Hilb}_S^{f_1} \times \mathrm{Hilb}_S^{f_2}$ we will use the following result.

Proposition 2.51 ([45, Prop. 2.9])**.** *Let $F \colon \Bbbk - \mathfrak{Alg} \to \mathfrak{Set}$ be a sheaf in the Zariski topology and assume that it is a subfunctor of the functor of points h_X of a $\Bbbk$-scheme X. Assume that for every $\Bbbk$-algebra R and for every $\lambda \in h_X(R)$ the condition $V_{R,\lambda}$ on R-algebras is closed. Then F is represented by a closed subscheme of X.*

Being a closed condition can be checked affine locally. Lemma 2.52 makes this precise.

Lemma 2.52. *Let $F\colon \Bbbk - \mathfrak{Alg} \to \mathfrak{Set}$ be a sheaf in the Zariski topology and assume that it is a subfunctor of the functor of points h_X of a $\Bbbk$-scheme X.*

Let R be a $\Bbbk$-algebra and $\lambda \in h_X(R)$. Suppose that there is a covering $\operatorname{Spec} R = \bigcup_{i\in I} \operatorname{Spec} R_{g_i}$ of $\operatorname{Spec} R$ by distinguished affine open subsets. Let $\lambda_i = h_X(\tau_i)(\lambda)$ where $\tau_i\colon R \to R_{g_i}$ is the localization map. If the condition $V_{R_{g_i},\lambda_i}$ on R_{g_i}-algebras is closed for every $i \in I$, then the condition $V_{R,\lambda}$ on R-algebras is closed.

Proof. Let $\tau_{ii'}\colon R_{g_i} \to R_{g_i g_{i'}}$ be the localization map for every $i, i' \in I$. For $i \in I$, let $\mathfrak{a}_i \subseteq R_{g_i}$ be the ideal such that an R_{g_i}-algebra $\psi\colon R_{g_i} \to U$ satisfies condition $V_{R_{g_i},\lambda_i}$ if and only if $\psi(\mathfrak{a}_i) = 0$. Then by Remark 2.50 we have $\mathfrak{a}_i R_{g_i g_{i'}} = \mathfrak{a}_{i'} R_{g_i g_{i'}}$ since both ideals show that the condition $V_{R_{g_i g_{i'}}, h_X(\tau_{ii'})(\lambda_i)}$ is closed.

It follows that there is an ideal $\mathfrak{a} \subseteq R$ such that $\mathfrak{a}R_{g_i} = \mathfrak{a}_i$ for $i \in I$. We claim that an R-algebra $\phi\colon R \to T$ satisfies condition $V_{R,\lambda}$ if and only if $\phi(\mathfrak{a}) = 0$.

Let $h_i = \phi(g_i)$ for $i \in I$ and $\sigma_i\colon T \to T_{h_i}$, $\sigma_{ii'}\colon T_{h_i} \to T_{h_i h_{i'}}$ be the localization maps. Since F is a sheaf in the Zariski topology, $h_X(\phi)(\lambda) \in F(T)$ if and only if

$$h_X(\sigma_i \circ \phi)(\lambda) \in F(T_{h_i}) \tag{2.53}$$

for every $i \in I$ and

$$F(\sigma_{ii'})\Big(h_X(\sigma_i \circ \phi)(\lambda)\Big) = F(\sigma_{i'i})\Big(h_X(\sigma_{i'} \circ \phi)(\lambda)\Big) \tag{2.54}$$

for every $i, i' \in I$.

First we show that Equation (2.54) always holds. Indeed, F is a subfunctor of h_X. As a result, we can replace $F(\sigma_{ii'})$ by $h_X(\sigma_{ii'})$ and $F(\sigma_{i'i})$ by $h_X(\sigma_{i'i})$. Since $\sigma_{ii'} \circ \sigma_i \circ \phi = \sigma_{i'i} \circ \sigma_{i'} \circ \phi$ the claimed equality follows from the fact that h_X is a functor.

Therefore, we need to show that Equation (2.53) is satisfied if and only if $\phi(\mathfrak{a}) = 0$. For every $i \in I$, there is an induced map $\phi_i\colon R_{g_i} \to T_{h_i}$ such that $\phi_i \circ \tau_i = \sigma_i \circ \phi$. Thus, $h_X(\sigma_i \circ \phi)(\lambda) = h_X(\phi_i) \circ h_X(\tau_i)(\lambda) = h_X(\phi_i)(\lambda_i)$. It follows that Equation (2.53) is satisfied if and only if $\phi_i(\mathfrak{a}_i) = 0$ for every $i \in I$.

We have

$$\phi(\mathfrak{a}) = 0 \Leftrightarrow \sigma_i \circ \phi(\mathfrak{a}) = 0 \text{ for all } i \in I \Leftrightarrow \phi_i \circ \tau_i(\mathfrak{a}) = 0 \text{ for all } i \in I \Leftrightarrow \phi_i(\mathfrak{a}_i) = 0 \text{ for all } i \in I.$$

This implies that Equation (2.53) is satisfied if and only if $\phi(\mathfrak{a}) = 0$ and thus, finishes the proof. □

We start with checking that the functor $\underline{\mathrm{Hilb}_S^{f_1,f_2}}$ is a sheaf in the Zariski topology. In fact, it will be convenient to prove this in slightly greater generality. Let $D \subseteq A$ and define the subfunctor $\underline{\mathrm{Hilb}_{S,D}^{f_1,f_2}}$ of the product functor $\underline{\mathrm{Hilb}_S^{f_1}} \times \underline{\mathrm{Hilb}_S^{f_2}}$ by

$$\underline{\mathrm{Hilb}_{S,D}^{f_1,f_2}}(R) = \{(K,J) \in \underline{\mathrm{Hilb}_S^{f_1}}(R) \times \underline{\mathrm{Hilb}_S^{f_2}}(R) \mid K_a \subseteq J_a \text{ for every } a \in D\}.$$

Lemma 2.55. *In the above notation, $\underline{\mathrm{Hilb}_{S,D}^{f_1,f_2}}$ is a sheaf in the Zariski topology. In particular, $\underline{\mathrm{Hilb}_S^{f_1,f_2}}$ is a sheaf in the Zariski topology.*

Proof. Let R be a $\Bbbk$-algebra. Consider a covering of $\operatorname{Spec} R$ with distinguished affine open subschemes $\{\operatorname{Spec} R_{g_i}\}_{i\in I}$. We need to show that we have an equalizer sequence

$$\underline{\mathrm{Hilb}^{f_1,f_2}_{S,D}}(R) \to \prod_{i\in I}\underline{\mathrm{Hilb}^{f_1,f_2}_{S,D}}(R_{g_i}) \rightrightarrows \prod_{i,i'\in I}\underline{\mathrm{Hilb}^{f_1,f_2}_{S,D}}(R_{g_ig_{i'}}).$$

For $i,i'\in I$, let $\tau_{ii'}\colon R_{g_i}\to R_{g_ig_{i'}}$ and $\tau_i\colon R\to R_{g_i}$ be the localization maps. Let $(K_i,J_i)\in \underline{\mathrm{Hilb}^{f_1,f_2}_{S,D}}(R_{g_i})$ for $i\in I$ be such that

$$\underline{\mathrm{Hilb}^{f_1,f_2}_{S,D}}(\tau_{ii'})(K_i,J_i) = \underline{\mathrm{Hilb}^{f_1,f_2}_{S,D}}(\tau_{i'i})(K_{i'},J_{i'})$$

for all $i,i'\in I$. We need to show that there exists a unique element $(K,J)\in\underline{\mathrm{Hilb}^{f_1,f_2}_{S,D}}(R)$ such that $\underline{\mathrm{Hilb}^{f_1,f_2}_{S,D}}(\tau_i)(K,J)=(K_i,J_i)$ for every $i\in I$.

Since $\underline{\mathrm{Hilb}^{f_1,f_2}_{S,D}}$ is a subfunctor of the representable functor $\underline{\mathrm{Hilb}^{f_1}_{S}}\times\underline{\mathrm{Hilb}^{f_2}_{S}}$, it follows that there is a unique element $(K,J)\in\underline{\mathrm{Hilb}^{f_1}_{S}}(R)\times\underline{\mathrm{Hilb}^{f_2}_{S}}(R)$ such that $\underline{\mathrm{Hilb}^{f_1}_{S}}(\tau_i)(K)=K_i$ and $\underline{\mathrm{Hilb}^{f_2}_{S}}(\tau_i)(J)=J_i$ for $i\in I$. We are left with showing that $K_a\subseteq J_a$ for every $a\in D$.

Let π be the natural map $\bigoplus_{a\in D}K_a\to\bigoplus_{a\in D}(R[\underline{\alpha}]/J)_a$ and denote $\ker\pi\to\bigoplus_{a\in D}K_a$ by θ. We claim that θ is an isomorphism. Indeed, θ_{g_i} is an isomorphism for all $i\in I$ since $(K_i)_a\subseteq(J_i)_a$ for every $a\in D$. Subschemes $\operatorname{Spec}R_{g_i}$ cover $\operatorname{Spec}R$. It follows that $\theta_{\mathfrak{p}}$ is an isomorphism for all $\mathfrak{p}\in\operatorname{Spec}R$ and thus, θ is an isomorphism. $\square$

Now we can give a proof of the existence of flag multigraded Hilbert schemes.

Proof of Proposition 2.47. The functor $\underline{\mathrm{Hilb}^{f_1,f_2}_{S}}$ is a sheaf in the Zariski topology by Lemma 2.55. Therefore, by Proposition 2.51, it is enough to show that the following holds. Let R be a $\Bbbk$-algebra and $(K,J)\in\underline{\mathrm{Hilb}^{f_1}_{S}}(R)\times\underline{\mathrm{Hilb}^{f_2}_{S}}(R)$. There exists an ideal $\mathfrak{a}\subseteq R$ such that $K\otimes_R T\subseteq J\otimes_R T$ for an R-algebra $\phi\colon R\to T$ if and only if $\phi(\mathfrak{a})=0$.

We start with the following reduction. Given $a\in A$ consider the functor $\underline{\mathrm{Hilb}^{f_1,f_2}_{S,\{a\}}}$. We will show that there exists an ideal $\mathfrak{b}_a\subseteq R$ such that $K_a\otimes_R T\subseteq J_a\otimes_R T$ for an R-algebra $\phi\colon R\to T$ if and only if $\phi(\mathfrak{b}_a)=0$. Then we take $\mathfrak{a}=\sum_{a\in A}\mathfrak{b}_a$.

Moreover, by Lemma 2.52 by replacing R by its localization, we may assume that $(R[\underline{\alpha}]/J)_a$ is a free R-module. Let $(R[\underline{\alpha}]/J)_a=\oplus_{k=1}^{f_2(a)}R\cdot e_{a,k}$. Let $\pi\colon K_a\to(R[\underline{\alpha}]/J)_a$ be the natural map. Let $\mathcal{C}$ be the condition on R-algebras such that $\phi\colon R\to T$ satisfies condition $\mathcal{C}$ if and only if $\pi\otimes_R\mathrm{id}_T$ is the zero map. We need to show that condition $\mathcal{C}$ is closed. Let $(b_{a,i})_{i\in I_a}$ be a set of generators of K_a as an R-module. Let $\pi(b_{a,i})=\sum_{k=1}^{f_2(a)}c_{a,i,k}e_{a,k}$. Then $\pi\otimes_R\mathrm{id}_T$ is the zero map if and only if $\phi(c_{a,i,k})=0$ for every $i\in I_a$ and $k\in\{1,\dots,f_2(a)\}$. The ideal $\mathfrak{b}_a=(c_{a,i,k})_{i\in I_a,k\in\{1,\dots,f_2(a)\}}$ shows that condition $\mathcal{C}$ is closed. $\square$

2.3 Deformation theory

In this section we recall some definitions and results from deformation theory. Subsection 2.3.1 introduces a small amount of general theory that we will need. Our main reference is [34].

In Subsection 2.3.2 we study the tangent-obstruction theory of multigraded Hilbert schemes and flag multigraded Hilbert schemes.

2.3.1 Definitions and basic results

Let $\mathfrak{Art}/\Bbbk$ be the category of local Artin $\Bbbk$-algebras with residue field $\Bbbk$. Observe that every morphism of $\Bbbk$-algebras from $\mathfrak{Art}/\Bbbk$ is a local morphism of local rings.

The main objects of study of deformation theory are deformation functors. We will consider the class of deformation functors coming from $\Bbbk$-points of $\Bbbk$-schemes.

Definition 2.56. Let X be a $\Bbbk$-scheme and $h_X\colon \mathfrak{Sch}_{\Bbbk}^{op} \to \mathfrak{Set}$ be its functor of points. Let $x \in h_X(\operatorname{Spec}\Bbbk)$. Then we have the corresponding *deformation functor*

$$D_{X,x}\colon \mathfrak{Art}/\Bbbk \to \mathfrak{Set}$$

defined by $D_{X,x}(A) = \{\lambda \in h_X(\operatorname{Spec} A) \mid h_X(\pi_A^{\#})(\lambda) = x\}$, where $\pi_A\colon A \to A/\mathfrak{m}_A$ is the natural map to the residue field of A and $\pi_A^{\#}$ is the corresponding map of affine schemes. A *morphism of deformation functors* $D_{X,x} \to D_{Y,y}$ is a natural transformation of functors.

A morphism of $\Bbbk$-schemes determines morphisms of deformation functors.

Example 2.57. Let $f\colon X \to Y$ be a morphism of $\Bbbk$-schemes and let $x \in X$, $y = f(x)$ be $\Bbbk$-points. Then the natural transformation $h_X \to h_Y$ of functors of points corresponding to f, induces a morphism of deformation functors $D_{X,x} \to D_{Y,y}$.

Suppose we have a $\Bbbk$-scheme X and a $\Bbbk$-point $x \in X$. A key question is whether given a surjection $\pi\colon B \to A$ of algebras in $\mathfrak{Art}/\Bbbk$ and an element of $\lambda \in D_{X,x}(A)$ we can lift it to an element of $D_{X,x}(B)$. The simplest situation is when the kernel of π is killed by the maximal ideal of B.

Definition 2.58. A *small extension* is a short exact sequence

$$0 \to M \to B \to A \to 0$$

where A, B are in $\mathfrak{Art}/\Bbbk$ and $\mathfrak{m}_B \cdot M = 0$. Here $\mathfrak{m}_B$ is the maximal ideal of B.

Definition 2.59. A deformation functor $D_{X,x}$ has a *tangent-obstruction theory* if there are finite dimensional $\Bbbk$-vector spaces $\mathbf{T}_{X,x}$ (called the *tangent space*) and $\mathrm{Ob}_{X,x}$ (called the *obstruction space*) such that:

1. For all small extensions $0 \to M \to B \to A \to 0$ there exists an exact sequence of sets

   $$\mathbf{T}_{X,x} \otimes_{\Bbbk} M \to D_{X,x}(B) \to D_{X,x}(A) \xrightarrow{ob_{B\to A}} \mathrm{Ob}_{X,x} \otimes_{\Bbbk} M;$$

   (Exactness at $D_{X,x}(A)$ means that an element of $D_{X,x}(A)$ lifts to $D_{X,x}(B)$ if and only if its image in $\mathrm{Ob}_{X,x} \otimes_{\Bbbk} M$ is zero. Exactness at $D_{X,x}(B)$ means that there is a transitive action of $\mathbf{T}_{X,x} \otimes_{\Bbbk} M$ on each fiber of $D_{X,x}(B) \to D_{X,x}(A)$).

2. If $A = \Bbbk$ then the sequence becomes

   $$0 \to \mathbf{T}_{X,x} \otimes_{\Bbbk} M \to D_{X,x}(B) \to D_{X,x}(\Bbbk) \xrightarrow{ob_{B\to \Bbbk}} \mathrm{Ob}_{X,x} \otimes_{\Bbbk} M;$$

3. Suppose we have a commutative diagram whose rows are small extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \varphi_M} & & \big\downarrow{\scriptstyle \varphi_B} & & \big\downarrow{\scriptstyle \varphi_A} & & \\
0 & \longrightarrow & M' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & 0.
\end{array}
$$

Then, there is a commutative diagram

$$
\begin{array}{ccccccc}
\mathbf{T}_{X,x} \otimes_{\Bbbk} M & \longrightarrow & D_{X,x}(B) & \longrightarrow & D_{X,x}(A) & \xrightarrow{ob_{B\to A}} & \mathrm{Ob}_{X,x} \otimes_{\Bbbk} M \\
\big\downarrow{\scriptstyle \mathrm{id}_{\mathbf{T}_{X,x}} \otimes_{\Bbbk} \varphi_M} & & \big\downarrow{\scriptstyle D_{X,x}(\varphi_B)} & & \big\downarrow{\scriptstyle D_{X,x}(\varphi_A)} & & \big\downarrow{\scriptstyle \mathrm{id}_{\mathrm{Ob}_{X,x}} \otimes_{\Bbbk} \varphi_M} \\
\mathbf{T}_{X,x} \otimes_{\Bbbk} M' & \longrightarrow & D_{X,x}(B') & \longrightarrow & D_{X,x}(A') & \xrightarrow{ob_{B'\to A'}} & \mathrm{Ob}_{X,x} \otimes_{\Bbbk} M'.
\end{array}
$$

The tangent space $\mathbf{T}_{X,x}$ is uniquely determined and agrees with the usual definition of tangent space to scheme X at point x.

Proposition 2.60 ([34, Prop. 6.1.23]). *Let $D_{X,x}$ be a deformation functor with tangent obstruction theory $(\mathbf{T}_{X,x}, \mathrm{Ob}_{X,x})$. Then $\mathbf{T}_{X,x} = D_{X,x}(\Bbbk[\varepsilon]/(\varepsilon^2))$ is the tangent space to X at x.*

On the other hand, the obstruction space $\mathrm{Ob}_{X,x}$ is not uniquely determined by the deformation functor $D_{X,x}$. In fact, given a tangent-obstruction theory $(\mathbf{T}_{X,x}, \mathrm{Ob}_{X,x})$ and an injective $\Bbbk$-linear map $\iota\colon \mathrm{Ob}_{X,x} \to \mathrm{Ob}'_{X,x}$ to a $\Bbbk$-vector space $\mathrm{Ob}'_{X,x}$, there is a tangent-obstruction theory for $D_{X,x}$ with obstruction space $\mathrm{Ob}'_{X,x}$ and obstruction maps $ob'_{B\to A} = (\iota \otimes_{\Bbbk} \mathrm{id}_M) \circ ob_{B\to A}$ for a small extension $0 \to M \to B \to A \to 0$.

We will need the following notion of a map of tangent-obstruction theories.

Definition 2.61. Let $\eta\colon D_{X,x} \to D_{Y,y}$ be a morphism of deformation functors with tangent-obstruction theories $\mathbf{T}_{X,x}, \mathrm{Ob}_{X,x}$ and $\mathbf{T}_{Y,y}, \mathrm{Ob}_{Y,y}$, respectively. A map of tangent-obstruction theories is a pair of linear maps $\mathbf{T}_\eta\colon \mathbf{T}_{X,x} \to \mathbf{T}_{Y,y}$ and $\mathrm{Ob}_\eta\colon \mathrm{Ob}_{X,x} \to \mathrm{Ob}_{Y,y}$ such that for every small extension $0 \to M \to B \to A \to 0$ there is a commutative diagram

$$
\begin{array}{ccccccc}
\mathbf{T}_{X,x} \otimes_{\Bbbk} M & \longrightarrow & D_{X,x}(B) & \longrightarrow & D_{X,x}(A) & \xrightarrow{ob_{B\to A}} & \mathrm{Ob}_{X,x} \otimes_{\Bbbk} M \\
\big\downarrow{\scriptstyle \mathbf{T}_\eta \otimes_{\Bbbk} \mathrm{id}_M} & & \big\downarrow{\scriptstyle \eta(B)} & & \big\downarrow{\scriptstyle \eta(A)} & & \big\downarrow{\scriptstyle \mathrm{Ob}_\eta \otimes_{\Bbbk} \mathrm{id}_M} \\
\mathbf{T}_{Y,y} \otimes_{\Bbbk} M & \longrightarrow & D_{Y,y}(B) & \longrightarrow & D_{Y,y}(A) & \xrightarrow{ob_{B\to A}} & \mathrm{Ob}_{Y,y} \otimes_{\Bbbk} M.
\end{array}
\tag{2.62}
$$

The commutativity of the left square means that $\eta(B)$ is equivariant with respect to the natural actions of $\mathbf{T}_{X,x} \otimes M$ and $\mathbf{T}_{Y,y} \otimes M$.

Proposition 2.60 showed that the tangent space of a deformation functor is uniquely defined. Similarly, for a natural transformation of deformation functors $\eta\colon D_{X,x} \to D_{Y,y}$, if we set $\mathbf{T}_\eta = \eta\big(\Bbbk[\varepsilon]/(\varepsilon)^2\big)$, then the left square of diagram (2.62) commutes.

The main result from deformation theory that we will use is the smoothness criterion [34, Rmk. 6.3.2]. In order to state it, we recall the definition of a smooth morphism of deformation functors.

Definition 2.63. Let $f\colon X \to Y$ be a morphism of $\Bbbk$-schemes locally of finite type. Let $x, y = f(x)$ be $\Bbbk$-points. A morphism of deformation functors $D_{X,x} \to D_{Y,y}$ is called *smooth* if for every small extension $0 \to M \to B \to A \to 0$, the natural map $D_{X,x}(B) \to D_{Y,y}(B) \times_{D_{Y,y}(A)} D_{X,x}(A)$ is surjective.

Theorem 2.64 ([34, Rmk. 6.3.2])**.** *Let $f\colon X \to Y$ be a morphism of $\Bbbk$-schemes locally of finite type. Let $x, y = f(x)$ be $\Bbbk$-points. Then f is smooth at x if and only if the morphism of deformation functors $D_{X,x} \to D_{Y,y}$ is smooth.*

We will use the following special cases of Theorem 2.64.

Corollary 2.65. *Let $f\colon X \to Y$ be a morphism of $\Bbbk$-schemes locally of finite type. Let $x, y = f(x)$ be $\Bbbk$-points. Then:*

(i) The point x is a smooth point of X if and only if $D_{X,x}(B) \to D_{X,x}(A)$ is surjective for every small extension $0 \to M \to B \to A \to 0$.

(ii) Assume that deformation functors $D_{X,x}$ and $D_{Y,y}$ have tangent-obstruction theories. Let $\eta\colon D_{X,x} \to D_{Y,y}$ be the morphism of deformation functors defined by f. Suppose that there is a map of tangent-obstruction theories $(\mathbf{T}_\eta, \mathrm{Ob}_\eta)$ which is surjective on tangent spaces and injective on obstruction spaces. Then f is smooth at x.

Proof. (i) This follows from the fact that if $Y = \operatorname{Spec} \Bbbk$ then $D_{Y,y}(A)$ is a singleton for every $A \in \mathfrak{Art}/\Bbbk$.

(ii) This follows from chasing the diagram (2.62).

□

We end this subsection with two lemmas from deformation theory that will be used in Chapter 3.

Lemma 2.66. *Assume that $f\colon X \to Y$ is a morphism of $\Bbbk$-schemes locally of finite type. Let $x, y = f(x)$ be closed points. Assume that functors $D_{X,x}$, $D_{Y,y}$ have tangent-obstruction theories with obstruction spaces $\mathrm{Ob}_{X,x}$ and $\mathrm{Ob}_{Y,y}$, respectively. If y is a smooth point and f induces a map of obstruction theories which is injective on obstruction spaces then x is a smooth point.*

Proof. By Corollary 2.65(i) it is enough to show that for every small extension $0 \to M \to B \to A \to 0$, the map $D_{X,x}(B) \to D_{X,x}(A)$ is surjective. Consider the commutative diagram of sets

$$\begin{tikzcd}
D_{X,x}(B) \arrow[r] \arrow[d] & D_{X,x}(A) \arrow[r] \arrow[d] & \mathrm{Ob}_{X,x} \otimes M \arrow[d] \\
D_{Y,y}(B) \arrow[r] & D_{Y,y}(A) \arrow[r] & \mathrm{Ob}_{Y,y} \otimes M
\end{tikzcd}$$

whose rows are the exact sequences from Definition 2.59. Since y is a smooth point, the lower left map is surjective and thus, the lower right map takes every element of $D_{Y,y}(A)$ to 0. By assumption the map on obstruction spaces is injective. It follows that every element of $D_{X,x}(A)$ is mapped to 0 by the upper right map. Hence the upper left map is surjective, as claimed. □

Before presenting the final lemma of this subsection, we introduce some notation. Let $g'\colon X \to X'$ and $g''\colon X \to X''$ be morphisms of $\Bbbk$-schemes locally of finite type. Let $x \in X$ and $x' = g'(x)$, $x'' = g''(x)$ be $\Bbbk$-points.

Assume that there are tangent-obstruction theories for $D_{X',x'}$ with obstruction space $\mathrm{Ob}_{X',x'}$ and for $D_{X'',x''}$ with obstruction space $\mathrm{Ob}_{X'',x''}$. Assume that there is a $\Bbbk$-vector space L and $\Bbbk$-linear maps $\alpha'\colon \mathrm{Ob}_{X',x'} \to L$ and $\alpha''\colon \mathrm{Ob}_{X'',x''} \to L$ such that there is a tangent-obstruction theory for $D_{X,x}$ with obstruction space $\mathrm{Ob}_{X,x}$ given by the pullback

$$\begin{tikzcd}
\mathrm{Ob}_{X,x} \arrow[r, "\beta''"] \arrow[d, "\beta'"] & \mathrm{Ob}_{X'',x''} \arrow[d, "\alpha''"] \\
\mathrm{Ob}_{X',x'} \arrow[r, "\alpha'"] & L.
\end{tikzcd}$$

Moreover, assume that β', β'' determine maps of tangent-obstruction theories.

Lemma 2.67. *In the above notation, assume additionally that x' is a smooth point of X'. Then there is a tangent-obstruction theory for $D_{X,x}$ with obstruction space* $\ker\alpha''$. *Moreover, the canonical injection*

$$\iota\colon \ker\alpha'' \to \mathrm{Ob}_{X'',x''}.$$

induces a map of tangent-obstruction theories.

Proof. By the universal property of fiber product, there is a map $\gamma\colon \ker\alpha'' \to \mathrm{Ob}_{X,x}$ such that

$$\beta'' \circ \gamma = \iota \tag{2.68}$$

and

$$\beta' \circ \gamma = 0. \tag{2.69}$$

Let $\eta'\colon D_{X,x} \to D_{X',x'}$ and $\eta''\colon D_{X,x} \to D_{X'',x''}$ be the natural transformations induced by g', g'', respectively. Given a small extension $0 \to M \to B \to A \to 0$, we denote the obstruction maps $D_{X,x}(A) \to \mathrm{Ob}_{X,x} \otimes M$ by $ob_{X,x;B\to A}$. We do similarly with $D_{X',x'}$ and $D_{X'',x''}$.

Step 1: Definition of obstruction maps

Fix a small extension $0 \to M \to B \to A \to 0$. We will construct a map $D_{X,x}(A) \to \ker\alpha'' \otimes M$. Consider the commutative diagram:

$$\begin{tikzcd}
D_{X,x}(A) \arrow[r, "ob_{X,x;B\to A}"] & \mathrm{Ob}_{X,x} \otimes M \arrow[r, "\beta'\otimes\mathrm{id}_M"] \arrow[d, "\beta''\otimes\mathrm{id}_M"] & \mathrm{Ob}_{X',x'} \otimes M \arrow[d, "\alpha'\otimes\mathrm{id}_M"] \\
\ker\alpha'' \otimes M \arrow[r, "\iota\otimes\mathrm{id}_M"] & \mathrm{Ob}_{X'',x''} \otimes M \arrow[r, "\alpha''\otimes\mathrm{id}_M"] & L \otimes M.
\end{tikzcd}$$

We have

$$(\beta' \otimes \mathrm{id}_M) \circ ob_{X,x;B\to A} = ob_{X',x';B\to A} \circ \eta'(A) = 0, \tag{2.70}$$

where the first equality follows from the fact that β' defines a map of tangent-obstruction theories and the second is a consequence of smoothness of x' (see Corollary 2.65(i)).

It follows that the image of $(\beta'' \otimes \mathrm{id}_M) \circ ob_{X,x;B\to A}$ is contained in $\ker\alpha'' \otimes M$. Thus, there is a map $ob_{B\to A}\colon D_{X,x}(A) \to \ker\alpha'' \otimes M$ such that

$$(\iota \otimes \mathrm{id}_M) \circ ob_{B\to A} = (\beta'' \otimes \mathrm{id}_M) \circ ob_{X,x;B\to A}. \tag{2.71}$$

Step 2: Factorization of obstruction maps

We claim that

$$(\gamma \otimes \mathrm{id}_M) \circ ob_{B\to A} = ob_{X,x;B\to A} \tag{2.72}$$

for every small extension $0 \to M \to B \to A \to 0$. By assumptions we have a pullback diagram

$$\begin{array}{ccc}
\mathrm{Ob}_{X,x} \otimes M & \xrightarrow{\beta' \otimes \mathrm{id}_M} & \mathrm{Ob}_{X'x'} \otimes M \\
\Big\downarrow{\scriptstyle \beta'' \otimes \mathrm{id}_M} & & \Big\downarrow{\scriptstyle \alpha' \otimes \mathrm{id}_M} \\
\mathrm{Ob}_{X'',x''} \otimes M & \xrightarrow{\alpha'' \otimes \mathrm{id}_M} & L \otimes M.
\end{array}$$

It remains a pullback in the category $\mathfrak{Set}$ of sets. Therefore, in order to show Equation (2.72) it is enough to observe that

$$(\beta'' \otimes \mathrm{id}_M) \circ (\gamma \otimes \mathrm{id}_M) \circ ob_{B \to A} \overset{(2.68)}{=} (\iota \otimes \mathrm{id}_M) \circ ob_{B \to A} \overset{(2.71)}{=} (\beta'' \otimes \mathrm{id}_M) \circ ob_{X,x;B \to A}$$

and

$$(\beta' \otimes \mathrm{id}_M) \circ (\gamma \otimes \mathrm{id}_M) \circ ob_{B \to A} \overset{(2.69)}{=} 0 \overset{(2.70)}{=} (\beta' \otimes \mathrm{id}_M) \circ ob_{X,x;B \to A}.$$

Step 3: Verification of axiom 1 of tangent-obstruction theory

We verify that for every small extension $0 \to M \to B \to A \to 0$ an element $\lambda \in D_{X,x}(A)$ lifts to $D_{X,x}(B)$ if and only if $ob_{B \to A}(\lambda) = 0$.

The map $\gamma \otimes \mathrm{id}_M$ is injective. Therefore, by Equation (2.72) we get $ob_{B \to A}(\lambda) = 0$ if and only if $ob_{X,x;B \to A}(\lambda) = 0$. This is equivalent to the existence of a lift of λ since the map $ob_{X,x;B \to A}$ is a part of the data of tangent-obstruction theory of $D_{X,x}$.

Step 4: Verification of axiom 3 of tangent-obstruction theory

Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & B & \longrightarrow & A & \longrightarrow & 0 \\
& & \Big\downarrow{\scriptstyle \varphi_M} & & \Big\downarrow{\scriptstyle \varphi_B} & & \Big\downarrow{\scriptstyle \varphi_A} & & \\
0 & \longrightarrow & M' & \longrightarrow & B' & \longrightarrow & A' & \longrightarrow & 0
\end{array}$$

be a commutative diagram whose rows are small extensions. We need to verify that

$$(\mathrm{id}_{\ker \alpha''} \otimes \varphi_M) \circ ob_{B \to A} = ob_{B' \to A'} \circ D_{X,x}(\varphi_A).$$

Since $(\gamma \otimes \mathrm{id}_{M'})$ is injective it is enough to observe that

$$\begin{aligned}
(\gamma \otimes \mathrm{id}_{M'}) \circ (\mathrm{id}_{\ker \alpha''} \otimes \varphi_M) \circ ob_{B \to A} &= (\mathrm{id}_{\mathrm{Ob}_{X,x}} \otimes \varphi_M) \circ (\gamma \otimes \mathrm{id}_M) \circ ob_{B \to A} \\
&\overset{(2.72)}{=} (\mathrm{id}_{\mathrm{Ob}_{X,x}} \otimes \varphi_M) \circ ob_{X,x;B \to A} = ob_{X,x;B' \to A'} \circ D_{X,x}(\varphi_A) \\
&\overset{(2.72)}{=} (\gamma \otimes \mathrm{id}_{M'}) \circ ob_{B' \to A'} \circ D_{X,x}(\varphi_A).
\end{aligned}$$

The third equality follows from the fact that the maps $ob_{X,x;B \to A}$ and $ob_{X,x;B' \to A'}$ are part of the data of a tangent-obstruction theory.

Step 5: Map of tangent-obstruction theories

Finally, we verify that $\iota \colon \ker \alpha'' \to \mathrm{Ob}_{X'',x''}$ defines a map of tangent-obstruction theories. Let $0 \to M \to B \to A \to 0$ be a small extension. Then

$$(\iota \otimes \mathrm{id}_M) \circ ob_{B \to A} \overset{(2.71)}{=} (\beta'' \otimes \mathrm{id}_M) \circ ob_{X,x;B \to A} = ob_{X'',x'';B \to A} \circ \eta''(A)$$

where the second equality follows from the fact that β'' induces a map of tangent-obstruction theories. $\square$

2.3.2 Tangent-obstruction theory of multigraded Hilbert schemes

Let $S = \Bbbk[\alpha_0, \alpha_1, \dots, \alpha_n]$ be a polynomial ring graded by an abelian group $\mathbf{A}$. Note that in this subsection we denote the grading group by $\mathbf{A}$ instead of A to avoid confusion with small extensions that will be used in the proofs. Let $h\colon \mathbf{A} \to \mathbb{N}$ be a numerical function. In this subsection we study a tangent-obstruction theory of Hilb^h_S.

Remark 2.73. Suppose that $0 \to M \to B \to A \to 0$ is a small extension and $\widetilde{J} \in D_{\mathrm{Hilb}^h_S,[J]}(A)$. Then a homogeneous ideal $\widetilde{J}' \subseteq S \otimes_\Bbbk B$ is a lift of $\widetilde{J}$ if and only if:

1. $(S \otimes_\Bbbk B)/\widetilde{J}'$ is B-flat;
2. $\widetilde{J}' \otimes_B A \cong \widetilde{J}$.

Indeed, since B is an Artin local ring, it follows from [79, Lemma 051G] that condition 1. implies that $(S_a \otimes_\Bbbk B)/\widetilde{J}'_a$ is a free B-module for every $a \in \mathbf{A}$. Thus, it is locally free of rank $h(a)$ since $((S_a \otimes_\Bbbk B)/\widetilde{J}'_a) \otimes_B A$ is a locally free A-module of rank $h(a)$.

Theorem 2.74. *In the above notation, let $[J]$ be a closed point of Hilb^h_S. Then the deformation functor $D_{\mathrm{Hilb}^h_S,[J]}$ has a tangent-obstruction theory with tangent space $\mathbf{T}_{\mathrm{Hilb}^h_S,[J]} = \mathrm{Hom}_S(J, S/J)_0$ and obstruction space $\mathrm{Ob}_{\mathrm{Hilb}^h_S,[J]} = \mathrm{Ext}^1_S(J, S/J)_0$.*

Proof. For tangent space, see [45, Prop. 1.6]. We only sketch the construction of obstruction maps. For details, we refer to [34, Thm. 6.4.5] where the ungraded case is considered.

Let $0 \to M \to B \to A \to 0$ be a small extension. Let $\widetilde{J} \in D_{\mathrm{Hilb}^h_S,[J]}(A)$. We study the lifts of $\widetilde{J}$ to $\widetilde{J}' \in D_{\mathrm{Hilb}^h_S,[J]}(B)$.

Consider the following commutative diagram with exact row and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & \widetilde{J} \otimes_A M \arrow[d] \arrow[dr, "\alpha"] & & \widetilde{J} \arrow[d] & \\
0 \arrow[r] & S \otimes_\Bbbk M \arrow[r] \arrow[d] & S \otimes_\Bbbk B \arrow[r] \arrow[dr, "\beta"] & S \otimes_\Bbbk A \arrow[r] \arrow[d] & 0 \\
 & \big((S \otimes_\Bbbk A)/\widetilde{J}\big) \otimes_A M \arrow[d] & & (S \otimes_\Bbbk A)/\widetilde{J} \arrow[d] & \\
 & 0 & & 0 &
\end{tikzcd}.$$

By a diagram chase, we have a short exact sequence of $S \otimes_\Bbbk B$-modules

$$0 \to \big((S \otimes_\Bbbk A)/\widetilde{J}\big) \otimes_A M \to \ker\beta/\operatorname{im}\alpha \to \widetilde{J} \to 0.$$

Furthermore, $M \cdot (\ker\beta) \subseteq \operatorname{im}\alpha$. Thus, the above short exact sequence is a sequence of $S \otimes_\Bbbk A$-modules. Finally, since $\mathfrak{m}_A M = 0$, we have

$$\big((S \otimes_\Bbbk A)/\widetilde{J}\big) \otimes_A M \cong \big((S \otimes_\Bbbk A)/\widetilde{J}\big) \otimes_A \Bbbk) \otimes_\Bbbk M \cong S/J \otimes_\Bbbk M.$$

Therefore, we have associated with $\widetilde{J} \in D_{\mathrm{Hilb}^h_S,[J]}(A)$ a short exact sequence of $S \otimes_{\Bbbk} A$-modules

$$0 \to S/J \otimes_{\Bbbk} M \to \ker\beta/\operatorname{im}\alpha \to \widetilde{J} \to 0. \tag{2.75}$$

We can consider the corresponding class $ob \in \operatorname{Ext}^1_{S\otimes_{\Bbbk} A}(\widetilde{J}, S/J \otimes_{\Bbbk} M)$. Since all morphisms were of degree 0, in fact $ob \in \operatorname{Ext}^1_{S\otimes_{\Bbbk} A}(\widetilde{J}, S/J \otimes_{\Bbbk} M)_0$.

Lifts of $\widetilde{J} \in D_{\mathrm{Hilb}^h_S,[J]}(A)$ to $\widetilde{J}' \in D_{\mathrm{Hilb}^h_S,[J]}(B)$ are in 1-1 correspondence with splittings $\xi\colon \widetilde{J} \to \ker\beta/\operatorname{im}\alpha$ of the short exact sequence (2.75) which are homogeneous of degree 0.

Finally, we have natural isomorphisms:

$$\operatorname{Ext}^1_{S\otimes_{\Bbbk} A}(\widetilde{J}, S/J \otimes_{\Bbbk} M)_0 \cong \operatorname{Ext}^1_S(\widetilde{J} \otimes_A \Bbbk, S/J \otimes_{\Bbbk} M)_0 \cong \operatorname{Ext}^1_S(J, S/J)_0 \otimes_{\Bbbk} M. \tag{2.76}$$

□

We will also use the following description of tangent-obstruction theory for flag multigraded Hilbert schemes.

Theorem 2.77. *In the above notation, let $k\colon \mathbf{A} \to \mathbb{N}$ be another numerical function. Let $[K \subseteq J]$ be a closed point of $\mathrm{Hilb}^{k,h}_S$ and assume that the natural map*

$$\operatorname{Hom}_S(K, S/K)_0 \to \operatorname{Hom}_S(K, S/J)_0$$

is surjective. Then there is a tangent-obstruction theory for the deformation functor $D_{\mathrm{Hilb}^{k,h}_S,[K\subseteq J]}$ with tangent and obstruction spaces given by pullbacks

$$\begin{tikzcd} \mathbf{T}_{\mathrm{Hilb}^{k,h}_S,[K\subseteq J]} \arrow[r, "\mathbf{T}_{\pi_1}"] \arrow[d, "\mathbf{T}_{\pi_2}"] & \operatorname{Hom}_S(K, S/K)_0 \arrow[d] \\ \operatorname{Hom}_S(J, S/J)_0 \arrow[r] & \operatorname{Hom}_S(K, S/J)_0 \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Ob}_{\mathrm{Hilb}^{k,h}_S,[K\subseteq J]} \arrow[r, "\mathrm{Ob}_{\pi_1}"] \arrow[d, "\mathrm{Ob}_{\pi_2}"] & \operatorname{Ext}^1_S(K, S/K)_0 \arrow[d] \\ \operatorname{Ext}^1_S(J, S/J)_0 \arrow[r] & \operatorname{Ext}^1_S(K, S/J)_0. \end{tikzcd}$$

Moreover, the maps $\mathbf{T}_{\pi_1}, \mathrm{Ob}_{\pi_1}$ and $\mathbf{T}_{\pi_2}, \mathrm{Ob}_{\pi_2}$ induce maps of tangent-obstruction theories.

Proof. Proof is analogous to the proof of [59, Thm 4.10]. Therefore, we only sketch the proof.

For tangent space, observe that the bijection from [45, Prop. 1.6]

$$\mathbf{T}_{\mathrm{Hilb}^h_S,[J]} = D_{\mathrm{Hilb}^h_S,[J]}(\Bbbk[\varepsilon]/(\varepsilon^2)) \leftrightarrow \operatorname{Hom}_S(J, S/J)_0$$

is given explicitly by

$$\operatorname{Hom}_S(J, S/J)_0 \ni \varphi \mapsto \widetilde{J} = \{x + \varepsilon y \mid x \in J, y \in S \text{ such that } y + J = \varphi(x)\} \in D_{\mathrm{Hilb}^h_S,[J]}(\Bbbk[\varepsilon]/(\varepsilon^2))\}.$$

See also [48, Prop. 2.3] for the proof of the analogous result in the ungraded case.

Therefore, an element of $D_{\mathrm{Hilb}^{k,h}_S,[K\subseteq J]}(\Bbbk[\varepsilon]/(\varepsilon^2))$ is a pair of homomorphisms

$$\varphi \in \operatorname{Hom}_S(J, S/J)_0 \text{ and } \psi \in \operatorname{Hom}_S(K, S/K)_0$$

corresponding to ideals $\widetilde{J}, \widetilde{K} \subseteq \Bbbk[\varepsilon]/(\varepsilon^2) \otimes_{\Bbbk} S$. However, we need to consider only those pairs φ, ψ for which $\widetilde{K} \subseteq \widetilde{J}$. Therefore, given $x \in K$ and $y \in S$ such that $y + K = \psi(x)$ we want $\varphi(x) = y + J$. This means precisely, that the images of φ and ψ in $\operatorname{Hom}_S(K, S/J)_0$ agree.

Now we proceed to the construction of obstruction maps. Let $0 \to M \to B \to A \to 0$ be a small extension and let $\{\widetilde{K}, \widetilde{J}\} \in D_{\mathrm{Hilb}_S^{k,h}}(A)$. Then there are obstructions $ob_J \in \operatorname{Ext}^1_S(J, S/J)_0 \otimes_{\Bbbk} M$ for lifting $\widetilde{J}$ to an element $\widetilde{J'} \in D_{\mathrm{Hilb}_S^h}(B)$ and $ob_K \in \operatorname{Ext}^1_S(K, S/K)_0 \otimes_{\Bbbk} M$ for lifting $\widetilde{K}$ to $\widetilde{K'} \in D_{\mathrm{Hilb}_S^k}(B)$.

Let

$$0 \to S/K \otimes_{\Bbbk} M \xrightarrow{\iota_K} \ker \beta_K / \operatorname{im} \beta_K \to \widetilde{K} \to 0 \tag{2.78}$$

and

$$0 \to S/J \otimes_{\Bbbk} M \to \ker \beta_J / \operatorname{im} \beta_J \to \widetilde{J} \to 0$$

be the extensions defining ob_K and ob_J, respectively (see the proof of Theorem 2.74). Then, the images of ob_K and ob_J in $\operatorname{Ext}^1_S(K, S/J)_0 \otimes_{\Bbbk} M \cong \operatorname{Ext}^1_{S \otimes_{\Bbbk} A}(\widetilde{K}, S/J \otimes_{\Bbbk} M)_0$ agree since they coincide with the class of the extension

$$0 \to S/J \otimes_{\Bbbk} M \to \ker \beta_K / \operatorname{im} \alpha_J \to \widetilde{K} \to 0.$$

Therefore, we have a well defined map $ob_{B \to A} \colon D_{\mathrm{Hilb}_S^{k,h}}(A) \to \mathrm{Ob}_{\mathrm{Hilb}_S^{k,h}, [K \subseteq J]} \otimes_{\Bbbk} M$ where $\mathrm{Ob}_{\mathrm{Hilb}_S^{k,h}, [K \subseteq J]}$ is given by the pullback as in the statement. If $\{\widetilde{K}, \widetilde{J}\} \in D_{\mathrm{Hilb}_S^{k,h}}(A)$ extends to $\{\widetilde{K'}, \widetilde{J'}\} \in D_{\mathrm{Hilb}_S^{k,h}}(B)$ then in particular $\widetilde{J}$ and $\widetilde{K}$ lift to $\widetilde{J'}$ and $\widetilde{K'}$. Thus, $ob_J = 0$ and $ob_K = 0$.

Conversely, assume that $ob_J = 0$ and $ob_K = 0$. Then there are $\widetilde{J'} \in D_{\mathrm{Hilb}_S^h}(B), \widetilde{K'} \in D_{\mathrm{Hilb}_S^k}(B)$ lifting $\widetilde{J}$ and $\widetilde{K}$, respectively. However, there is no reason to expect that $\widetilde{K'} \subseteq \widetilde{J'}$.

Let $f \colon \widetilde{K'} \to (S \otimes_{\Bbbk} B)/\widetilde{J'}$ be the natural map. We will modify $\widetilde{K'}$ if necessary so that f becomes zero. Consider the following commutative diagram:

$$\begin{tikzcd}
 & K \otimes_{\Bbbk} M \arrow[rr] \arrow[dl] \arrow[dd, "a"] & & S \otimes_{\Bbbk} M \arrow[rr] \arrow[dl, equal] \arrow[dd] & & S/K \otimes_{\Bbbk} M \arrow[dl] \arrow[dd] \\
J \otimes_{\Bbbk} M \arrow[rr] \arrow[dd] & & S \otimes_{\Bbbk} M \arrow[rr] \arrow[dd] & & S/J \otimes_{\Bbbk} M \arrow[dd] & \\
 & \widetilde{K'} \arrow[rr] \arrow[dd] & & S \otimes_{\Bbbk} B \arrow[rr] \arrow[dl, equal] \arrow[dd] & & (S \otimes_{\Bbbk} B)/\widetilde{K'} \arrow[dd] \\
\widetilde{J'} \arrow[rr] \arrow[dd] & & S \otimes_{\Bbbk} B \arrow[rr] \arrow[dd] & & (S \otimes_{\Bbbk} B)/\widetilde{J'} \arrow[dd, "b"] & \\
 & \widetilde{K} \arrow[rr] \arrow[dl] & & S \otimes_{\Bbbk} A \arrow[rr] \arrow[dl, equal] & & (S \otimes_{\Bbbk} A)/\widetilde{K} \arrow[dl] \\
\widetilde{J} \arrow[rr] & & S \otimes_{\Bbbk} A \arrow[rr] & & (S \otimes_{\Bbbk} A)/\widetilde{J} & .
\end{tikzcd}$$

By a diagram chase, we have $f \circ a = 0 = b \circ f$. Therefore, f is induced by a map $f' \colon \widetilde{K} \to S/J \otimes_{\Bbbk} M$.

We have a commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_{S\otimes_{\Bbbk} A}(\widetilde{K}, S/K \otimes_{\Bbbk} M)_0 & \longrightarrow & \operatorname{Hom}_{S\otimes_{\Bbbk} A}(\widetilde{K}, S/J \otimes_{\Bbbk} M)_0 \\ \big\downarrow \cong & & \big\downarrow \cong \\ \operatorname{Hom}_S(K, S/K)_0 \otimes_{\Bbbk} M & \longrightarrow & \operatorname{Hom}_S(K, S/J)_0 \otimes_{\Bbbk} M. \end{array}$$

By assumptions, the lower horizontal map is surjective. Therefore, so is the upper horizontal map. Thus, there is a map $g\colon \widetilde{K} \to S/K \otimes_{\Bbbk} M$ homogeneous of degree 0 such that it maps to f' under the upper horizontal map.

The ideal $\widetilde{K'}$ is defined by a splitting $\xi_K\colon \widetilde{K} \to \ker \beta_K / \operatorname{im} \alpha_K$ of the short exact sequence (2.78). Then $\xi_K - \iota_K \circ g\colon \widetilde{K} \to \ker \beta_K / \operatorname{im} \alpha_K$ is another splitting of the short exact sequence (2.78) so it defines a lift $\widetilde{K''} \in D_{\mathrm{Hilb}^k_S,[K]}(B)$ of $\widetilde{K}$. Then it can be checked by a diagram chase that the natural map $\widetilde{K''} \to (S \otimes_{\Bbbk} B)/\widetilde{J'}$ is zero. Thus $\widetilde{K''} \subseteq \widetilde{J'}$ is a lift of $\widetilde{K} \subseteq \widetilde{J}$. □

Let S have standard $\mathbb{Z}$-grading. Let f be the Hilbert function of S/I where I is a homogeneous ideal of S. Let d be a positive integer and $g\colon \mathbb{Z} \to \mathbb{Z}$ be defined by

$$g(a) = \begin{cases} f(a) & \text{for } a \geq d \\ \dim_{\Bbbk} S_a & \text{otherwise.} \end{cases}$$

Lemma 2.79. *In the above notation, let $\pi\colon \mathrm{Hilb}^f_S \to \mathrm{Hilb}^g_S$ be the natural map given on closed points by $[I] \mapsto [I \cap \mathfrak{m}^d]$. Let $[I] \in \mathrm{Hilb}^f_S$ be a closed point. Let $\eta\colon D_{\mathrm{Hilb}^f_S,[I]} \to D_{\mathrm{Hilb}^g_S,[I\cap\mathfrak{m}^d]}$ be the morphism of deformation functors determined by π. Then the natural maps*

$$\operatorname{Hom}_S(I, S/I)_0 \to \operatorname{Hom}_S(I \cap \mathfrak{m}^d, S/I)_0 \xrightarrow{\cong} \operatorname{Hom}_S(I \cap \mathfrak{m}^d, S/I \cap \mathfrak{m}^d)_0$$

and

$$\operatorname{Ext}^1_S(I, S/I)_0 \to \operatorname{Ext}^1_S(I \cap \mathfrak{m}^d, S/I)_0 \xrightarrow{\cong} \operatorname{Ext}^1_S(I \cap \mathfrak{m}^d, S/I \cap \mathfrak{m}^d)_0$$

from exact sequence of Ext *groups, define a morphism of tangent-obstruction theories as defined in Theorem 2.74.*

Proof. Observe that both isomorphisms in the statement follow from the fact that $I \cap \mathfrak{m}^d$ has minimal generators of degree at least d, so the degree 0 parts of the Ext groups do not depend on $(S/I)_{<d}$. The claimed description of the map on tangent spaces follows from the bijection

$$\mathbf{T}_{[I]} \mathrm{Hilb}^f_S \leftrightarrow D_{\mathrm{Hilb}^f_S}(\Bbbk[\varepsilon]/(\varepsilon^2))$$

from [45, Prop. 1.6] which has been recalled at the beginning of the proof of Theorem 2.77.

Now we concentrate on obstruction spaces. Let $0 \to M \to B \to A \to 0$ be a small extension, $\widetilde{I} \in D_{\mathrm{Hilb}^f_S,[I]}(A)$ and $\widetilde{I}_{\geq d} \in D_{\mathrm{Hilb}^g_S,[I\cap\mathfrak{m}^d]}(A)$ be its image under $\eta(A)$.

Let

$$0 \to S/I \otimes_{\Bbbk} M \to \ker \beta / \operatorname{im} \alpha \to \widetilde{I} \to 0$$

and

$$0 \to S/I_{\geq d} \otimes_{\Bbbk} M \to \ker \beta' / \operatorname{im} \alpha' \to \widetilde{I}_{\geq d} \to 0$$

be as in Equation (2.75).

Let $ob_I \in \operatorname{Ext}^1_S(I, S/I)_0 \otimes_{\Bbbk} M$ and $ob_{I\cap\mathfrak{m}^d} \in \operatorname{Ext}^1_S(I \cap \mathfrak{m}^d, S/(I \cap \mathfrak{m}^d))_0 \otimes_{\Bbbk} M$ be obstructions corresponding to $\widetilde{I}$ and $\widetilde{I}_{\geq d}$, respectively.

Then, as in the proof of Theorem 2.77 we consider the extension

$$0 \to S/I \otimes_{\Bbbk} M \to \ker \beta' / \operatorname{im} \alpha \to \widetilde{I}_{\geq d} \to 0. \tag{2.80}$$

We see, that the images of ob_I and $ob_{I\cap\mathfrak{m}^d}$ in $\operatorname{Ext}^1_S(I \cap \mathfrak{m}^d, S/I)_0 \otimes_{\Bbbk} M$ agree since they coincide with the class of the extension given by Equation (2.80). Therefore, if we consider $D_{\mathrm{Hilb}^g_S,[I\cap\mathfrak{m}^d]}$ with obstruction space $\operatorname{Ext}^1_S(I\cap\mathfrak{m}^d, S/I)_0$ via the isomorphism $\operatorname{Ext}^1_S(I\cap\mathfrak{m}^d, S/I)_0 \cong \operatorname{Ext}^1_S(I \cap \mathfrak{m}^d, S/I \cap \mathfrak{m}^d)_0$, then the natural map of Ext groups as in the statement induces a map of obstruction theories. □

2.4 Ranks and apolarity lemmas

In this section we recall various notions of rank and corresponding versions of apolarity lemma. Moreover, we define secant and cactus varieties. The apolarity lemma for border rank from [13] reflects a connection between ideals in $\mathrm{Slip}_{r,n}$ and the condition that a homogeneous polynomial in $n+1$ variables has border rank at most r. This is the main motivation to study the irreducible component $\mathrm{Slip}_{r,n}$. Observe that the connection from [13] works for more general toric varieties.

We present the theory of ranks of homogeneous subspaces instead of the more standard version of ranks of homogeneous polynomials. This is due to the fact that we shall need the general version in Chapter 5.

2.4.1 Apolarity action

Let n be a positive integer and $S = \Bbbk[\alpha_0, \dots, \alpha_n]$ be a polynomial ring. Let $S^* = \Bbbk_{dp}[x_0, \dots, x_n]$ be the graded dual ring. That is, as a $\mathbb{Z}$-graded $\Bbbk$-vector space

$$S^* = \bigoplus_{a\in\mathbb{Z}} \operatorname{Hom}_{\Bbbk}(S_a, \Bbbk)$$

and the ring structure is the divided power structure. See [55, Appendix A] for basic properties of divided power ring S^*. For $\mathbf{u} = (u_0, \dots, u_n) \in \mathbb{N}^{n+1}$ we define

$$\alpha^{\mathbf{u}} = \prod_{i=0}^{n} \alpha_i^{u_i}.$$

For an integer $a \in \mathbb{Z}$, the vector space S_a has a monomial basis, i.e.

$$S_a = \bigoplus_{\mathbf{u}\in\mathbb{N}^{n+1}, |\mathbf{u}|=a} \Bbbk\alpha^{\mathbf{u}}$$

where $|\mathbf{u}| = \sum_{i=0}^{n} u_i$. Let $\{x^{[\mathbf{u}]} \mid \mathbf{u} \in \mathbb{N}^{n+1}, |\mathbf{u}| = a\}$ be the dual basis of S^*_a. We define the multiplication in S^* on divided power monomials by

$$x^{[\mathbf{u}]} \cdot x^{[\mathbf{v}]} = \frac{(\mathbf{u}+\mathbf{v})!}{\mathbf{u}!\mathbf{v}!} x^{[\mathbf{u}+\mathbf{v}]},$$

where $\mathbf{w}! = w_0! \cdot \ldots \cdot w_n!$ for $\mathbf{w} \in \mathbb{N}^{n+1}$. Then we extend it by linearity to define a $\Bbbk$-algebra structure on S^*.

There is a natural action of S on S^* denoted by $\lrcorner$ given on homogeneous elements $\theta \in S_a$ and $F \in S_b^*$ by

$$(\theta \lrcorner F)(\xi) = F(\theta\xi) \text{ for every } \xi \in S_{b-a}.$$

Using the monomial bases, the action of S on S^* can be written in the form

$$\alpha^{\mathbf{u}} \lrcorner x^{[\mathbf{v}]} = \begin{cases} x^{[\mathbf{v}-\mathbf{u}]} & \text{if } v_k \geq u_k \text{ for } k = 0, \dots, n \\ 0 & \text{otherwise.} \end{cases}$$

If $\Bbbk$ has characteristic zero then $x^{[\mathbf{v}]} = \frac{x^{\mathbf{v}}}{\mathbf{v}!}$ and S^* is a polynomial ring.

Let d be a positive integer and $W \subseteq S^*_{\leq d}$ be a vector subspace. Using the action of S on S^*, we can associate with W a $\Bbbk$-algebra $\operatorname{Apolar}(W)$.

Definition 2.81. Let d be a positive integer and $W \subseteq S^*_{\leq d}$ be a non-zero linear subspace. The *annihilator* of W is the ideal

$$\operatorname{Ann}(W) = \{\theta \in S \mid \theta \lrcorner f = 0 \text{ for every } f \in W\}.$$

The *apolar algebra* of W is $\operatorname{Apolar}(W) = S/\operatorname{Ann}(W)$. If $W = \langle f \rangle$ for some $f \in S^*_{\leq d}$, we write $\operatorname{Ann}(f)$ and $\operatorname{Apolar}(f)$ instead of $\operatorname{Ann}(\langle f \rangle)$ and $\operatorname{Apolar}(\langle f \rangle)$.

Annihilator $\operatorname{Ann}(W)$ of $W \subseteq S^*_{\leq d}$ plays a key role in apolarity lemmas which connect the properties of $\operatorname{Ann}(W)$ and various notions of ranks of W.

2.4.2 (Border) rank, smoothable rank and (border) cactus rank

We keep the notation of Subsection 2.4.1. Let d be a positive integer. We recall various notions of rank of a subspace V of S_d^*. It is important to realize that the process of generalizing the definition of rank was not as straight-forward as it may seem from the short presentation that we give.

We start with introducing the rank of V. If $V = \langle F \rangle$ is one-dimensional, this is a classical notion that goes back to works of Sylvester. The general case was studied among others by Terracini [78] and Bronowski [8].

Definition 2.82. Let d be a positive integer and $V \subseteq S_d^*$ be a non-zero linear subspace. We define *rank* of V to be

$$\mathrm{r}(V) = \min\{r \in \mathbb{Z} \mid \mathbb{P}V \subseteq \langle L_1^{[d]}, \dots, L_r^{[d]} \rangle \text{ for some } L_i \in S_1^*\},$$

where $\langle - \rangle$ denotes the projective linear span.

It is interesting to describe, for a positive integer k, the locus of points $[V] \in \operatorname{Gr}(k, S_d^*)$ with rank at most r. However, a more natural geometric object is the Zariski closure of this locus. Recall that $\nu_d \colon \mathbb{P}S_1^* \to \mathbb{P}S_d^*$ given by $[L] \mapsto [L^{[d]}]$ is the Veronese embedding.

Definition 2.83. Let d, k be positive integers. The *r-th Grassmann secant variety* is

$$\sigma_{r,k}\big(\nu_d(\mathbb{P}S_1^*)\big) = \overline{\{[V] \in \operatorname{Gr}(k, S_d^*) \mid \mathrm{r}(V) \leq r\}}.$$

If $k = 1$, we write $\sigma_r\big(\nu_d(\mathbb{P}S_1^*)\big)$ instead of $\sigma_{r,1}\big(\nu_d(\mathbb{P}S_1^*)\big)$ and call it the *r-th secant variety.*

With Grassmann secant varieties already defined, it is natural to introduce another variant of rank.

Definition 2.84. Let d, k be positive integers and $V \subseteq S_d^*$ be a k-dimensional linear subspace. Then the *border rank* of V is defined to be

$$\mathrm{br}(V) = \min\{r \in \mathbb{Z} \mid [V] \in \sigma_{r,k}\big(\nu_d(\mathbb{P}S_1^*)\big)\}.$$

The ranks and borders ranks of monomials have been studied in [63]. See also [62], [11] and [38] for some results concerning equations of secant varieties.

The definition of rank of $[V] \in \mathrm{Gr}(k, S_d^*)$ can be restated as follows

$$\begin{aligned}\mathrm{r}(V) = \min\{r \in \mathbb{Z} \mid \ &\text{there exists a smooth zero-dimensional subscheme} \\ &R \subseteq \mathbb{P}S_1^* \text{ of length } r \text{ such that } \mathbb{P}V \subseteq \langle \nu_d(R) \rangle\}.\end{aligned}$$

One can consider different variants of this definition. The condition that R is smooth could be replaced by the condition that it is a limit of smooth schemes or it can be even skipped completely. These lead to the notions of smoothable rank and cactus rank. Let $\mathcal{H}ilb_r^{sm}(\mathbb{P}S_1^*)$ be the smoothable component, i.e. the closure of the locus of points of $\mathcal{H}ilb_r(\mathbb{P}S_1^*)$ corresponding to smooth subschemes.

Definition 2.85. Let d, k be positive integers and $[V] \in \mathrm{Gr}(k, S_d^*)$. The *smoothable rank* of V is defined to be

$$\mathrm{sr}(V) = \min\{r \in \mathbb{Z} \mid \mathbb{P}V \subseteq \langle \nu_d(R) \rangle \text{ for some } [R] \in \mathcal{H}ilb_r^{sm}(\mathbb{P}S_1^*)\}.$$

We will be interested only in the smoothable rank of a non-zero homogeneous polynomial $F \in S_d^*$. We have $\mathrm{br}(F) \leq \mathrm{sr}(F)$. Following [12] we make a definition capturing the cases where the inequality is strict.

Definition 2.86. Let d be a positive integer and $F \in S_d^*$ be a non-zero polynomial. Then F is *wild* if $\mathrm{br}(F) < \mathrm{sr}(F)$.

Considering the whole Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}S_1^*)$ instead of its smoothable component leads to the definition of cactus rank.

Definition 2.87. Let d, k be a positive integers and $[V] \in \mathrm{Gr}(k, S_d^*)$. The *cactus rank* of V is defined to be

$$\mathrm{cr}(V) = \min\{r \in \mathbb{Z} \mid \mathbb{P}V \subseteq \langle \nu_d(R) \rangle \text{ for some } [R] \in \mathcal{H}ilb_r(\mathbb{P}S_1^*)\}.$$

Similarly, as in the case of rank, we can consider the Zariski closure in $\mathrm{Gr}(k, S_d^*)$ of those $[V]$ which have cactus rank at most r.

Definition 2.88. Let d, k be positive integers. The *r-th Grassmann cactus variety* is

$$\kappa_{r,k}\big(\nu_d(\mathbb{P}S_1^*)\big) = \overline{\{[V] \in \mathrm{Gr}(k, S_d^*) \mid \mathrm{cr}(V) \leq r\}}.$$

If $k = 1$, we write $\kappa_r\big(\nu_d(\mathbb{P}S_1^*)\big)$ instead of $\kappa_{r,1}\big(\nu_d(\mathbb{P}S_1^*)\big)$ and call it the *r-th cactus variety.*

Cactus varieties have been introduced in [11]. The name might be slightly confusing since in general cactus varieties are not irreducible. An example when the cactus variety is reducible is presented in [39, Thm. 1.4, 1.5]. The cactus rank has been studied for instance in [73] and [4].

Finally, we define the border cactus rank in a way analogous to the definition of border rank.

Definition 2.89. Let d, k be positive integers and $V \subseteq S_d^*$ be a k-dimensional linear subspace. Then the *border cactus rank* of V is defined to be

$$\operatorname{bcr}(V) = \min\{r \in \mathbb{Z} \mid [V] \in \kappa_{r,k}\big(\nu_d(\mathbb{P}S_1^*)\big)\}.$$

2.4.3 Apolarity lemmas

The ranks of a subspace $V \subseteq S_d^*$ can be computed by apolarity lemmas. We state only the versions of apolarity lemma that we shall use. We start with the one related to the cactus rank.

Proposition 2.90 (Cactus apolarity lemma)**.** *Let d be a positive integer and $V \subseteq S_d^*$ be a non-zero subspace. Then $\operatorname{cr}(V) \leq r$ for a positive integer r if and only if there is a saturated, homogeneous ideal $I \subseteq \operatorname{Ann}(V)$ such that S/I has Hilbert polynomial r.*

For a proof, see [77, Thm. 4.7].

Next, we deal with the border rank. The version for a polynomial $F \in S_d^*$ is a special case of a recent result by Buczyńska and Buczyński [13]. It is the main motivation for studying the irreducible component $\operatorname{Slip}_{r,n}$ of the multigraded Hilbert scheme $\operatorname{Hilb}_S^{h_{r,n}}$. The following version for subspaces is presented in [39, Prop. 2.3] and follows from the proof of [14, Thm. 1.3].

Proposition 2.91 (Border apolarity lemma)**.** *Let d be a positive integer and $V \subseteq S_d^*$ be a non-zero subspace. Then $\operatorname{br}(V) \leq r$ for a positive integer r, if and only if there exists a homogeneous ideal $I \subseteq \operatorname{Ann}(V)$ such that $[I] \in \operatorname{Slip}_{r,n} \subseteq \operatorname{Hilb}_S^{h_{r,n}}$.*

Finally, we present a weak version of apolarity lemma for border cactus rank. Following [39] we introduce the following definition.

Definition 2.92. For positive integers r, n, a function $h\colon \mathbb{Z} \to \mathbb{Z}$ is called an $(r, n+1)$-*standard Hilbert function* if it satisfies the following conditions:

(i) $h(d) \leq h(d+1)$ for all $d \in \mathbb{Z}$;

(ii) if $h(d) = h(d+1)$ for some $d \geq 0$, then $h(e) = r$ for all $e \geq d$;

(iii) $0 \leq h(d) \leq h_{r,n}(d)$ for all $d \in \mathbb{Z}$.

Proposition 2.93 (Weak border cactus apolarity lemma)**.** *Let d be a positive integer and $V \subseteq S_d^*$ be a non-zero subspace. If $\operatorname{bcr}(V) \leq r$ for some positive integer r, then there exists a homogeneous ideal $I \subseteq \operatorname{Ann}(V)$ such that S/I has an $(r, n+1)$-standard Hilbert function.*

See [14, Thm. 1.1] for a proof.

Chapter 3

Criteria for projective space

In this chapter we present conditions for a point $[I]$ in the multigraded Hilbert scheme $\mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$ to be in the irreducible component $\mathrm{Slip}_{r,n}$.

Section 3.1 contains a necessary condition based on bounding the degrees of minimal generators of saturated ideals with Hilbert function $h_{r,n}$.

The criterion from Section 3.2 is based on the properties of the Hilbert function of a power of a radical ideal with Hilbert function $h_{r,n}$ that were established in Proposition 2.19.

In Section 3.3 we show that the locus of points of Hilb^{f}_{S} corresponding to saturated ideals is smooth and irreducible when $n = 2$ and f is the Hilbert function of a zero dimensional subscheme of $\mathbb{P}^2$. We also show that in $\mathcal{H}ilb_r(\mathbb{P}^2)$ the locus of points corresponding to subschemes with fixed Hilbert function is irreducible. In characteristic 0 this has been shown by Gotzmann [41].

In Section 3.4 we present a sufficient condition for $[I]$ as above to be in $\mathrm{Slip}_{r,n}$ for $n = 2$.

Sections 3.5 and 3.7 contain some examples. In particular, in Section 3.7 we present the full set-theoretic description of $\mathrm{Slip}_{r,2}$ for $r = 4, 5, 6$.

Section 3.6 presents a necessary condition (Theorem 3.40). This criterion has three technical assumptions, one is based on smoothness of the Hilbert scheme at a prescribed point while two are about surjectivity of some maps of spaces of homomorphisms. In Subsections 3.6.1, 3.6.2, 3.6.3 we study some cases in which the assumptions of Theorem 3.40 are fulfilled. In Subsections 3.6.4 and 3.6.5 we present some nice applications of Theorem 3.40.

The main results of this chapter are criteria for $[I] \in \mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$ to be in $\mathrm{Slip}_{r,n}$:

- Proposition 3.1 which is an example of a small tangent space argument;
- Theorem 3.5 which shows that if $[I] \in \mathrm{Slip}_{r,n}$ then H_{S/I^k} is large enough in large degrees;
- Theorem 3.12 which is a necessary condition in the case $n = 2$. It states that if the Hilbert function of $S/\overline{I}$ differ from $h_{r,2}$ in exactly one degree then $[I] \in \mathrm{Slip}_{r,n}$;
- Theorem 3.40 and its applications Theorems 3.65 and 3.74.

Notation

Throughout this chapter, r and n are positive integers and $S = S[\mathbb{P}^n] = \Bbbk[\alpha_0, \ldots, \alpha_n]$ is a polynomial ring over a fixed algebraically closed field $\Bbbk$. Recall that $h_{r,n} \colon \mathbb{Z} \to \mathbb{Z}$ is defined by

$$h_{r,n}(a) = \min\{\dim_{\Bbbk} S_a, r\}.$$

3.1 Criterion based on degrees of minimal generators

Our first criterion is obtained by bounding the degrees of minimal generators of saturated ideals of points. The proof is an illustration of a small tangent space method: if the dimension of the tangent space to a $\Bbbk$-scheme X at a point x is smaller than the dimension of an irreducible closed subset Y, then $x \notin Y$. See [54] for a classical application of this argument.

Proposition 3.1. *Let $I \subseteq S$ be a homogeneous ideal such that S/I has Hilbert function $h_{r,n}$. Let $e = \min\{a \in \mathbb{Z} \mid h_{r,n}(a) = r\}$ and $d \geq e+2$. If $\dim_\Bbbk \operatorname{Hom}_S\left(I + \mathfrak{m}^d, S/(I+\mathfrak{m}^d)\right)_0 < rn$, then $[I] \notin \operatorname{Slip}_{r,n}$.*

Proof. Define $g\colon \mathbb{Z} \to \mathbb{Z}$ by

$$g(a) = \begin{cases} h_{r,n}(a) & \text{if } a < d \\ 0 & \text{otherwise.} \end{cases}$$

Let $\pi\colon \operatorname{Hilb}_S^{h_{r,n}} \to \operatorname{Hilb}_S^g$ be the morphism defined on closed points by $[J] \mapsto [J + \mathfrak{m}^d]$. We shall show that

$$\text{if } J,\ J' \text{ are saturated ideals and } \pi([J]) = \pi([J']), \text{ then } J = J'. \tag{3.2}$$

It is enough to show that every saturated ideal J of S such that S/J has Hilbert function $h_{r,n}$ is generated in degrees at most $d-1$. Since J is saturated, $\operatorname{depth} S/J \geq 1$ by Lemma 2.9(i). The quotient algebra S/J has Krull dimension 1. It follows that S/J is Cohen-Macaulay. Furthermore, by the Auslander-Buchsbaum Theorem [69, Thm. 15.3] the projective dimension of S/J is n. Therefore, $\operatorname{reg}(S/J) = e$ by [31, Thm. 4.2]. Consequently, $\beta_{1,a}(S/J) = 0$ for $a \geq e+2$. Thus, J is generated in degrees at most $e + 1 \leq d - 1$.

The irreducible component $\operatorname{Slip}_{r,n}$ has dimension rn. Therefore, by (3.2) the irreducible closed subset $\pi(\operatorname{Slip}_{r,n})$ is also of dimension rn. Consequently, if $[I] \in \operatorname{Slip}_{r,n}$ then

$$rn \leq \dim_\Bbbk \mathbf{T}_{[I+\mathfrak{m}^d]} \operatorname{Hilb}_S^g = \dim_\Bbbk \operatorname{Hom}_S\left(I + \mathfrak{m}^d, S/(I+\mathfrak{m}^d)\right)_0,$$

where the equality follows from Theorem 2.74. ☐

Example 3.3. Let $S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ and consider $I = (\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_1^6)$. Then $[I] \in \operatorname{Hilb}_S^{h_{6,2}}$. We claim that $[I] \notin \operatorname{Slip}_{6,2}$. Let $J = I + (\alpha_0, \alpha_1, \alpha_2)^5$. Then

$$\dim_\Bbbk \operatorname{Hom}_S(J, S/J)_0 = 8 < 12.$$

The claim follows from Proposition 3.1.

3.2 Criterion based on a power of ideal

The criterion presented in this section is based on Proposition 2.19. There we computed the Hilbert polynomial of a power of a homogeneous radical ideal defining a zero dimensional subscheme of projective space. Moreover, we bounded the degree from which this Hilbert polynomial agrees with Hilbert function. Now, using semicontinuity, we obtain a criterion for $[I] \in \operatorname{Hilb}_S^{h_{r,n}}$ to be in $\operatorname{Slip}_{r,n}$.

In Subsection 2.2.3 we introduced subsets $\operatorname{Sip}_{r,n}$ and $\operatorname{Slip}_{r,n}$ of the multigraded Hilbert scheme $\operatorname{Hilb}_S^{h_{r,n}}$. Here we generalize this for functions $h\colon \mathbb{Z} \to \mathbb{Z}$ more general than $h_{r,n}$.

Definition 3.4. Assume that $h\colon \mathbb{Z} \to \mathbb{Z}$ is the Hilbert function of a zero-dimensional closed subscheme of $\mathbb{P}^n$. We denote by $\mathrm{Sip}_{h,n}$ the locus of closed points of Hilb^h_S corresponding to radical ideals. Moreover, let $\mathrm{Slip}_{h,n}$ be the closure of $\mathrm{Sip}_{h,n}$ in Hilb^h_S.

The following theorem provides a necessary condition for a closed point of Hilb^h_S to be in $\mathrm{Slip}_{h,n}$, where $h\colon \mathbb{Z} \to \mathbb{Z}$ is as in Definition 3.4.

Theorem 3.5. *Let $h\colon \mathbb{Z} \to \mathbb{Z}$ be the Hilbert function of a zero-dimensional length r closed subscheme of $\mathbb{P}^n$. Define $e = \min\{a \in \mathbb{Z} \mid h(a) = r\}$ and let $[I] \in \mathrm{Hilb}^h_S$ be a closed point. If $[I] \in \mathrm{Slip}_{h,n}$, then $H_{S/I^k}(d) \geq r \cdot \dim_{\Bbbk} S_{k-1}$ for every positive integer k and for every $d \geq ke + k$.*

Proof. Let $\mathscr{J}$ be the universal ideal sheaf on $\mathrm{Hilb}^h_S \times \mathbb{A}^{n+1}$. Consider the quotient $\mathscr{P}$ of

$$\mathcal{O}_{\mathrm{Hilb}^h_S \times \mathbb{A}^{n+1}} \cong \mathcal{O}_{\mathrm{Hilb}^h_S}[\alpha_0, \ldots, \alpha_n]$$

by $\mathscr{J}^k$ and let $\mathscr{Q}$ be the pushforward of $\mathscr{P}$ under the projection morphism $\pi\colon \mathrm{Hilb}^h_S \times \mathbb{A}^{n+1} \to \mathrm{Hilb}^h_S$. Then $\mathscr{Q} = \bigoplus_d \mathscr{Q}_d$ is a quasi-coherent sheaf on Hilb^h_S with $\mathscr{Q}_d$ a coherent sheaf for every $d \in \mathbb{Z}$. Therefore, for every $d \in \mathbb{Z}$, the rank function $\varphi_d\colon \mathrm{Hilb}^h_S \to \mathbb{Z}$ given by $\varphi_d(x) = \dim_{\kappa(x)}(\mathscr{Q}_d)_x \otimes_{\mathcal{O}_{\mathrm{Hilb}^h_S,x}} \kappa(x)$ is upper semicontinuous (see [47, Ex. II.5.8]). We claim that for a closed point $P = [K] \in \mathrm{Hilb}^h_S$ we have $\varphi_d(P) = H_{S/K^k}(d)$.

This can be checked affine locally, so we can replace Hilb^h_S by an affine open subset $U = \operatorname{Spec} A$ containing $[K]$. Let J be the ideal in $A[\alpha_0, \ldots, \alpha_n]$ defining the restriction of $\mathscr{J}$ to $\pi^{-1}(U)$. Let $[K]$ in U correspond to the maximal ideal $\mathfrak{n}$ of A. In what follows, we will consider $\Bbbk$ with A-module structure given by $A \to A_{\mathfrak{n}}/\mathfrak{n}A_{\mathfrak{n}} \cong \Bbbk$. By the definition of universal ideal sheaf we have $(A[\alpha_0, \ldots, \alpha_n]/J) \otimes_A \Bbbk \cong S/K$. Therefore, from the universal property of kernel, there is an induced map $J \otimes_A \Bbbk \to K$ fitting into the commutative diagram

$$\begin{tikzcd}
 & J \otimes_A \Bbbk \arrow[r] \arrow[d, dotted] & S \arrow[r] \arrow[d, equal] & S/(J \otimes_A \Bbbk) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & S \arrow[r] & S/K \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. It follows from snake lemma that the map $J \otimes_A \Bbbk \to K$ is surjective. Hence also the map $J^k \otimes_A \Bbbk \to K^k$ is surjective. The snake lemma applied to the diagram

$$\begin{tikzcd}
 & J^k \otimes_A \Bbbk \arrow[r] \arrow[d] & S \arrow[r] \arrow[d, equal] & S/(J^k \otimes_A \Bbbk) \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & K^k \arrow[r] & S \arrow[r] & S/K^k \arrow[r] & 0
\end{tikzcd}$$

implies that the dotted arrow induced by the universal property of cokernel is injective. Since it is clearly surjective, it is an isomorphism. Thus

$$\varphi_d([K]) = \dim_{\Bbbk} \left(S/(J^k \otimes_A \Bbbk)\right)_d = \dim_{\Bbbk}(S/K^k)_d = H_{S/K^k}(d)$$

and the claim of the theorem follows from Proposition 2.19. $\square$

Example 3.6. Let $S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ and consider the ideal

$$I' = (\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_2^6).$$

Then $[I'] \in \mathrm{Hilb}_S^{h_{6,2}}$ and we claim that $[I'] \notin \mathrm{Slip}_{6,2}$. We have $H_{S/(I')^2}(9) = 17 < 18$. Thus, the claim follows from Theorem 3.5.

Observe that $\dim_{\Bbbk} \mathrm{Hom}_S(I' + \mathfrak{m}^d, S/(I' + \mathfrak{m}^d)) \geq 12$ for $d \geq 4$, so the criterion from Proposition 3.1 cannot be applied to deduce that $[I'] \notin \mathrm{Slip}_{6,2}$.

Consider again the ideal $I = (\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_1^6)$ from Example 3.3. Observe that $H_{S/I^2}(d) \geq 18$ for $d \geq 6$. Hence we cannot use the criterion from Theorem 3.5 (with $k = 2$) to deduce that $[I] \notin \mathrm{Slip}_{6,2}$. We summarize this in the following table which will be extended after we discuss more criteria.

Ideal	Proposition 3.1	Theorem 3.5
$(\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_1^6) \subseteq \Bbbk[\alpha_0, \alpha_1, \alpha_2]$	✓	?
$(\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_2^6) \subseteq \Bbbk[\alpha_0, \alpha_1, \alpha_2]$	?	✓

Here, for both ideals, the corresponding point of $\mathrm{Hilb}_S^{h_{6,2}}$ is outside the irreducible component $\mathrm{Slip}_{6,2}$. The check mark sign indicates that a given criterion shows that a given point is not in $\mathrm{Slip}_{6,2}$. The question mark shows that a given criterion is inconclusive.

3.3 Smoothness and irreducibility of the locus of saturated ideals of points in projective plane

Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be the Hilbert function of a zero-dimensional subscheme of $\mathbb{P}^2$ of length r. Let V_f be the locally closed subset of $\mathcal{H}ilb_r(\mathbb{P}^2)$ whose points correspond to the subschemes of $\mathbb{P}^2$ with Hilbert function f. If the field $\Bbbk$ has characteristic 0 then V_f with the reduced structure is smooth and irreducible [41].

In this section we show that in fact V_f is irreducible for any characteristic of $\Bbbk$. Furthermore, let $E_f \subseteq \mathrm{Hilb}_S^f$ be the open subset whose points correspond to saturated ideals. We show that E_f is irreducible and smooth.

In this section $S = \Bbbk[\alpha_0, \alpha_1, \ldots, \alpha_n]$ and $T = \Bbbk[\alpha_0, \alpha_1, \ldots, \alpha_{n-1}]$ for a positive integer n. We will eventually restrict our attention to the case $n = 2$, but we do not make this assumption when the proofs work more generally.

We shall need the following result on the behavior of Ext groups under flat base change.

Lemma 3.7. *Let $R \to S$ be a flat ring homomorphism with R a Noetherian ring. Then for every finitely generated R-module M and any R-module N, the natural map*

$$\mathrm{Ext}_R^i(M, N) \otimes_R S \to \mathrm{Ext}_S^i(M \otimes_R S, N \otimes_R S)$$

is an isomorphism for all integers i.

Proof. Since R is Noetherian and M is a finitely generated R-module, there exists a projective resolution $P_\bullet$ of M by finitely generated free R-modules. The R-module S is flat. Hence we obtain a projective resolution $P_\bullet \otimes_R S$ of $M \otimes_R S$.

Finally, since M is finitely presented and S is a flat R-module, the natural map

$$\mathrm{Hom}_R(P_\bullet, N) \otimes_R S \to \mathrm{Hom}_S(P_\bullet \otimes_R S, N \otimes_R S)$$

is an isomorphism of chain complexes of S-modules by [30, Prop. 2.10]. Therefore, we have natural isomorphisms of S-modules

$$\begin{aligned}&\operatorname{Ext}^i_R(M,N)\otimes_R S = H^i\big(\operatorname{Hom}_R(P_\bullet,N)\big)\otimes_R S = H^i\big(\operatorname{Hom}_R(P_\bullet,N)\otimes_R S\big)\\ &= H^i\big(\operatorname{Hom}_S(P_\bullet\otimes_R S, N\otimes_R S)\big) = \operatorname{Ext}^i_S(M\otimes_R S, N\otimes_R S).\end{aligned}$$ □

The following lemma gives a condition under which $E_f \subseteq \operatorname{Hilb}^f_S$ is smooth. Observe that this condition is satisfied for $n=2$ since $\mathcal{H}ilb_r(\mathbb{P}^2)$ is smooth.

Lemma 3.8. *Let f be the Hilbert function of a zero-dimensional subscheme of $\mathbb{P}^n$ of length r. Let $E_f \subseteq \operatorname{Hilb}^f_S$ be the open subset whose points correspond to saturated ideals. Assume that $[\operatorname{Proj} S/I]$ is a smooth point of $\mathcal{H}ilb_r(\mathbb{P}^n)$ for every point $[I]\in E_f$ such that I is a saturated Borel-fixed ideal. Then E_f is smooth.*

Proof. Let $[J]\in \operatorname{Hilb}^f_S$ be a closed point such that J is a saturated ideal. Let J' be the generic initial ideal (see [30, §15.9]) of J' with respect to the grevlex order with $\alpha_0>\alpha_1>\ldots>\alpha_n$. It is enough to show that $[J']\in\operatorname{Hilb}^f_S$ is a smooth point.

The ideal J' is saturated by Corollary 2.10. Moreover, it is Borel-fixed by [30, Thm. 15.20]. Thus, $[\operatorname{Proj} S/J']$ is a smooth point of $\mathcal{H}ilb_r(\mathbb{P}^n)$ by assumptions.

Therefore, by Lemma 2.66, it is enough to show that the natural map of deformation functors $D_{\operatorname{Hilb}^f_S,[J']} \to D_{\mathcal{H}ilb_r(\mathbb{P}^n),[\operatorname{Proj} S/J']}$ induced by the map $\operatorname{Hilb}^f_S \to \mathcal{H}ilb_r(\mathbb{P}^n)$ admits a map of tangent-obstruction theories that is injective on obstruction spaces. By Lemmas 2.44 and 2.79 there is a map of tangent-obstruction theories which on obstruction spaces is the natural map

$$\operatorname{Ext}^1_S(J', S/J')_0 \to \operatorname{Ext}^1_S(J'\cap\mathfrak{m}^r, S/J')_0$$

from the exact sequence of Ext groups. It suffices to show that

$$\operatorname{Ext}^1_S(J'/J'\cap\mathfrak{m}^r, S/J')_0 \to \operatorname{Ext}^1_S(J', S/J')_0 \tag{3.9}$$

is the zero map. Since J' is a saturated and Borel-fixed ideal, it is an extended ideal from the polynomial ring $T=\Bbbk[\alpha_0,\ldots,\alpha_{n-1}]$ by Lemma 2.11. Therefore, by Lemma 3.7 we get $\operatorname{Ext}^1_S(J',S/J')\cong \operatorname{Ext}^1_T(\mathfrak{a},T/\mathfrak{a})\otimes_T S$ where $\mathfrak{a}=T\cap J'$. In particular, multiplication by α_n is injective on $\operatorname{Ext}^1_S(J',S/J')$. On the other hand, since multiplication by α_n^r is zero on $J'/J'\cap\mathfrak{m}^r$, it is zero on $\operatorname{Ext}^1_S(J'/J'\cap\mathfrak{m}^r, S/J')$ as well. It follows that the map from Equation (3.9) is the zero map. □

Finally we present the main results of this section.

Proposition 3.10. *Let $n=2$ so that $S=\Bbbk[\alpha_0,\alpha_1,\alpha_2]$ and $T=\Bbbk[\alpha_0,\alpha_1]$. Let f be the Hilbert function of a zero-dimensional length r subscheme of $\mathbb{P}^2$. Let E_f be the open subset of Hilb^f_S whose points correspond to saturated ideals. Then E_f is smooth and irreducible.*

Proof. Smoothness of E_f follows from Lemma 3.8 since the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^2)$ is smooth for every positive integer r.

Now we show that E_f is connected. Since it is also smooth this would finish the proof. Given a point $[I]\in E_f$ we may connect it to the point $[I']$ corresponding to the generic initial ideal I' of I with respect to the grevlex order with $\alpha_0>\alpha_1>\alpha_2$. Then I' is saturated by Corollary 2.10

and it is Borel-fixed by [30, Thm. 15.20]. Therefore, it is enough to find a connected subset of E_f that contains all points corresponding to Borel-fixed saturated ideals.

Let $g\colon \mathbb{Z} \to \mathbb{Z}$ be given by $g(a) = f(a) - f(a-1)$ for every $a \in \mathbb{Z}$. We have a natural map $\mathrm{Hilb}^g_T \to \mathrm{Hilb}^f_S$. The scheme Hilb^g_T is irreducible by [64]. Therefore, its image Z in Hilb^f_S is irreducible. By construction, Z is contained in E_f. Furthermore, it contains all saturated Borel-fixed ideals by Lemma 2.11. □

Proposition 3.11. *Let $n = 2$ so that $S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$. Let f be the Hilbert function of a zero-dimensional length r subscheme of $\mathbb{P}^2$. Let V_f be the locally closed subset of $\mathcal{H}ilb_r(\mathbb{P}^2)$ whose points correspond to the subschemes with Hilbert function f. Then V_f is irreducible.*

Proof. As before, let E_f be the open subset of Hilb^f_S whose points correspond to the saturated ideals. The natural map $\mathrm{Hilb}^f_S \to \mathcal{H}ilb_r(\mathbb{P}^n)$ induces a map $E_f \to V_f$ which is bijective on $\Bbbk$-points. Therefore, V_f is homeomorphic with E_f by Lemma 2.29. It follows from Proposition 3.10 that V_f is irreducible. □

3.4 Sufficient condition for projective plane

In this section we assume that $n = 2$ so $S = S[\mathbb{P}^2] = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ is the homogeneous coordinate ring of projective plane. We show that a closed points $[I] \in \mathrm{Hilb}^{h_{r,2}}_S$ for which $S/\overline{I}$ has Hilbert function differing from $h_{r,2}$ in only one degree is in $\mathrm{Slip}_{r,2}$.

For ease of reference we write this condition more precisely. We will consider functions $f\colon \mathbb{Z} \to \mathbb{Z}$ that satisfy the following condition:

$$\text{there exist } e, d \in \mathbb{Z}_{>0} \text{ such that } f(a) = \begin{cases} h_{r,2}(a) & \text{if } a \neq e \\ d & \text{if } a = e. \end{cases} \tag{$\star$}$$

The main result of this section is the following theorem which gives a sufficient condition for a closed point $[I] \in \mathrm{Hilb}^{h_{r,2}}_S$ to be in $\mathrm{Slip}_{r,2}$.

Theorem 3.12. *Let $[I]$ be a closed point of the multigraded Hilbert scheme $\mathrm{Hilb}^{h_{r,2}}_S$. If the Hilbert function of $S/\overline{I}$ satisfies condition ($\star$), then $[I] \in \mathrm{Slip}_{r,2}$.*

Before proving Theorem 3.12 we need a few lemmas. The first one will enable us to consider a more restrictive condition ($\star\star$) instead of condition ($\star$).

Lemma 3.13. *Let $[I] \in \mathrm{Hilb}^{h_{r,2}}_S$. If the Hilbert function f of $S/\overline{I}$ satisfies condition ($\star$) for some integers d, e then $[I] \in \mathrm{Slip}_{r,2}$ unless the following holds:*

$$f \text{ satisfies condition } (\star) \text{ and } \dim_{\Bbbk} S_{e-1} < d < r < \dim_{\Bbbk} S_e. \tag{$\star\star$}$$

Proof. Assume that $[I] \in \mathrm{Hilb}^{h_{r,2}}_S \setminus \mathrm{Slip}_{r,2}$ and the Hilbert function of $S/\overline{I}$ satisfies condition ($\star$) for some integers d, e.

Suppose that $d = r$. Then $f = h_{r,2}$, so $[I] \in \mathrm{Slip}_{r,2}$ by Remark 2.46 since $\mathcal{H}ilb_r(\mathbb{P}^2)$ is irreducible. Thus, by Lemma 2.9(ii), we may assume that $d < r$. Moreover, if $\dim_{\Bbbk} S_e \leq r$, then $[I] \in \mathrm{Slip}_{r,2}$ since in that case $[I] = [\overline{I} \cap \mathfrak{m}^{e+1}]$ is the unique closed point of the fiber over $\varphi_{r,\mathbb{P}^2}([I])$ of the natural map $\varphi_{r,\mathbb{P}^2}\colon \mathrm{Hilb}^{h_{r,2}}_S \to \mathcal{H}ilb_r(\mathbb{P}^2)$ from Remark 2.46. Therefore, we may assume that $r < \dim_{\Bbbk} S_e$. We claim that it is enough to consider the case that $f(e-1) = h_{r,2}(e-1) =$

$\dim_{\Bbbk} S_{e-1}$. Indeed, otherwise $f(e-1) = h_{r,2}(e-1) = r$ and this contradicts Lemma 2.9(ii) since $f(e) = d < r = f(e-1)$. Using Lemma 2.9(ii) we obtain $\dim_{\Bbbk} S_{e-1} \leq d$ and moreover by Lemma 2.9(iii) this inequality is strict, since $r = f(e+1) > d$. $\square$

For a fixed positive integer r, let

$$\Omega_r = \{f\colon \mathbb{Z} \to \mathbb{Z} \,|\, f \text{ satisfies condition } (\star\star) \text{ and there exists } [I] \in \mathrm{Hilb}_S^{h_{r,2}} \text{ such that } S/\overline{I} \text{ has Hilbert function } f\}.$$

By virtue of the following lemma, in order to obtain a proof of Theorem 3.12, it is enough to find a point $[I_f] \in \mathrm{Slip}_{r,2}$, for every $f \in \Omega_r$ that satisfies the properties:

1. $H_{S/\overline{I_f}} = f$;
2. $[I_f]$ is a smooth point of $\mathrm{Hilb}_S^{h_{r,2}}$.

Lemma 3.14. *Let $f \in \Omega_r$ for some positive integer r. Then the locus of closed points $[I]$ of $\mathrm{Hilb}_S^{h_{r,2}}$ such that $S/\overline{I}$ has Hilbert function f is irreducible.*

Proof. Denote this locus by U_f. Let $V_f \subseteq \mathcal{H}ilb_r(\mathbb{P}^2)$ be the locally closed subset whose closed points correspond to the closed subschemes of $\mathbb{P}^2$ with Hilbert function f. By definition, U_f is the set of closed points of the preimage of V_f under the map

$$\varphi_{r,\mathbb{P}^2}\colon \mathrm{Hilb}_S^{h_{r,2}} \to \mathcal{H}ilb_r(\mathbb{P}^2).$$

The locus V_f is irreducible by Proposition 3.11. Furthermore, $h_{r,2}(a) = f(a) = \dim_{\Bbbk} S_a$ for every $a < e$ and $h_{r,n}(a) = f(a)$ for every $a > e$. Therefore, U_f is irreducible by [80, 11.4.C] and Lemmas 2.43 and 2.44. $\square$

Fix a positive integer r and a function $f \in \Omega_r$, or equivalently, a pair of integers d, e corresponding to f. To simplify the notation let $s := \dim_{\Bbbk} S_{e-1}$ and we define $A_i = \alpha_0^i\alpha_1^{e-i}$ for $0 \leq i \leq e$, $B_i = \alpha_0^i\alpha_1^{e+1-i}$ for $0 \leq i \leq e+1$ and $C_i = \alpha_0^i\alpha_1^{e+2-i}$ for $0 \leq i \leq e+2$ to make it easier to distinguish between the generators of different degrees. We define the ideals

$$J_f = (A_e, A_{e-1}, \ldots, A_{d-s}, B_{d-s-1}, B_{d-s-2}, \ldots, B_{r-d}, C_{r-d-1}, C_{r-d-2}, \ldots, C_0) \tag{3.15}$$

and

$$\begin{aligned} I_f = (&A_e, \ldots, A_{r-s}, A_{r-s-1}\alpha_1, A_{r-s-1}\alpha_2, \ldots, A_{d-s}\alpha_1, A_{d-s}\alpha_2, \\ &B_{d-s-1}, \ldots, B_{r-d}, C_{r-d-1}, \ldots, C_0). \end{aligned} \tag{3.16}$$

Note that $A_i\alpha_1 = B_i$ but we have written I_f in the form as above since it will be more convenient in the proof of the following lemma.

Lemma 3.17. *In the above notation, the Hilbert function of S/I_f is $h_{r,2}$ and the Hilbert function of $S/\overline{I_f}$ is f. Moreover, J_f is the saturation of I_f.*

Proof. Let $T = \Bbbk[\alpha_0, \alpha_1]$. We start with showing that J_f is a saturated ideal and that the Hilbert function of S/J_f is f. Indeed, J_f is an extension of the ideal $\mathfrak{a}_f = J_f \cap T$ in T so it is saturated with respect to α_2 and thus, is saturated with respect to $\mathfrak{m}$. Moreover, $H_{S/J_f}(a) = \sum_{b=0}^{a} H_{T/\mathfrak{a}_f}(b)$ and the latter sum can be computed from the staircase diagram of $\mathfrak{a}_f$.

It follows from the generators of I_f and J_f displayed above, that the saturation of I_f contains J_f. Therefore, $\overline{I_f} = J_f$ since J_f is saturated. The quotient algebra S/J_f has Hilbert function f. Furthermore, $(J_f)_{>e} = (I_f)_{>e}$ and $(J_f)_{<e} = (I_f)_{<e} = 0$. Therefore, it follows from Equations (3.15) and (3.16) that the Hilbert function of S/I_f is $h_{r,2}$. $\square$

We will now find a saturated ideal K_f such that the initial ideal of K_f with respect to an appropriate monomial order is I_f. Let

$$\begin{aligned} K_f = (&A_e, \ldots, A_{r-s}, A_{r-s-1}\alpha_1, A_{r-s-1}\alpha_2 + B_{r-d-1}, \ldots, A_{d-s}\alpha_1, A_{d-s}\alpha_2 + B_0, \\ &B_{d-s-1}, \ldots, B_{r-d}, C_{r-d-1}, \ldots, C_0). \end{aligned} \tag{3.18}$$

Lemma 3.19. *In the above notation, the initial ideal of K_f with respect to the lex order $>$ with $\alpha_2 > \alpha_1 > \alpha_0$ is I_f. In particular, S/K_f has Hilbert function $h_{r,2}$.*

Proof. All S-polynomials of the generators displayed in Equation (3.18) belong to the ideal $\mathfrak{b} := (C_{e+2}, \ldots, C_0)$. Let $\mathfrak{c} := (A_e, \ldots A_{r-s}, B_{r-s-1}, \ldots, B_{r-d}, C_{r-d-1}, \ldots C_0)$. We have $\mathfrak{b} \subseteq \mathfrak{c} \subseteq K_f$. It follows that the set of generators from Equation (3.18) satisfies the Büchberger criterion (see [27, Thm. 6 in Ch. 2 §6]). Hence it is a Gröbner basis. In particular, the initial ideal $\mathrm{in}_<(K_f)$ is I_f so S/K_f has Hilbert function $h_{r,2}$ by Lemma 3.17. $\square$

Next, we verify that K_f is a saturated ideal.

Lemma 3.20. *In the above notation, K_f is a saturated ideal. In particular, $[I_f] \in \mathrm{Slip}_{r,2}$.*

Proof. Let $>$ be the lex order with $\alpha_2 > \alpha_1 > \alpha_0$. From Lemmas 2.7, 3.17 and 3.19 we obtain

$$\mathrm{in}_<(\overline{K_f}) \subseteq \overline{\mathrm{in}_<(K_f)} = \overline{I_f} = J_f. \tag{3.21}$$

Suppose that $\overline{K_f} \neq K_f$. Then $I_f = \mathrm{in}_<(K_f) \subsetneq \mathrm{in}_<(\overline{K_f}) \subseteq J_f$. Since I_f, J_f differ only in degree e, it follows that there is an element $g \in S_e \cap \overline{K_f}$ such that $\mathrm{in}_<(g)$ does not belong to the set of monomial generators of I_f of degree e. However, $\mathrm{in}_<(g) \in (J_f)_e$ by Equation (3.21). Therefore, by the choice of the monomial order and Equation (3.15) we get that $g = \sum_{i=d-s}^{e} a_i A_i$ for some $a_i \in \Bbbk$. We assumed that $\mathrm{in}_<(g) \notin I_f$. Thus, by Equation (3.16), we have $a_i \neq 0$ for some $i \in \{d-s, \ldots, r-s-1\}$. Furthermore, we may assume that $a_i = 0$ for $i = r-s, \ldots, e$ by Equation (3.18). Multiplying g by α_2^2 and using the generators of K_f given in Equation (3.18) we obtain

$$g' := -\alpha_2^2 g + \sum_{i=d-s}^{r-s-1} a_i\alpha_2(A_i\alpha_2 + B_{i+s-d}) = \sum_{i=d-s}^{r-s-1} a_i B_{i+s-d}\alpha_2 \in \overline{K_f}.$$

We claim that it is not possible. By Equation (3.21), it is enough to show that no monomial of the form $B_j\alpha_2$ for $j \in \{0, \ldots, r-d-1\}$ is in J_f. This is clear since monomials of degree $e+2$ in J_f that are divisible by α_2 are also divisible by α_0^{r-d}.

Since K_f is saturated, it follows from Remark 2.46 that $[K_f] \in \mathrm{Slip}_{r,2}$. Therefore, $[I_f] \in \mathrm{Slip}_{r,2}$ by Lemma 3.19. $\square$

Finally, we will show that $[I_f]$ is a smooth point of $\mathrm{Hilb}_S^{h_{r,2}}$. It is enough to show that

$$\dim_\Bbbk \mathrm{Hom}_S(I_f, S/I_f)_0 = \dim_\Bbbk \mathbf{T}_{[I_f]} \mathrm{Hilb}_S^{h_{r,2}} \leq \dim \mathrm{Slip}_{r,2} = 2r,$$

where the first equality follows from Theorem 2.74. Lemmas 3.22, 3.24, 3.25 and 3.31 are devoted to this calculation.

Lemma 3.22. *In the above notation,* $\dim_{\Bbbk} \mathrm{Hom}_S(J_f/I_f, S/I_f)_0 = (r-d)^2$.

Proof. Since $J_f/I_f \cong \Bbbk(-e)^{\oplus(r-d)}$ we have

$$\begin{aligned}&\dim_{\Bbbk} \mathrm{Hom}_S(J_f/I_f, J_f/I_f)_0 = (r-d)^2 \dim_{\Bbbk} \mathrm{Hom}_S(\Bbbk(-e), \Bbbk(-e))_0 \\ &= (r-d)^2 \dim_{\Bbbk} \mathrm{Hom}_S(\Bbbk, \Bbbk)_0 = (r-d)^2.\end{aligned}$$

Since $(J_f\colon \alpha_2) = J_f$ and $\alpha_2 \cdot J_f/I_f = 0$ it follows that $\mathrm{Hom}_S(J_f/I_f, S/J_f)_0 = 0$. Therefore,

$$\dim_{\Bbbk} \mathrm{Hom}_S(J_f/I_f, S/I_f)_0 = \dim_{\Bbbk} \mathrm{Hom}_S(J_f/I_f, J_f/I_f)_0 = (r-d)^2$$

from the long exact sequence obtained by applying $\mathrm{Hom}_S(J_f/I_f, -)_0$ to the short exact sequence

$$0 \to J_f/I_f \to S/I_f \to S/J_f \to 0. \tag{3.23}$$

□

Lemma 3.24. *In the above notation,* $\dim_{\Bbbk} \mathrm{Ext}^1_S(J_f/I_f, S/I_f)_0 = (r-d)^2$.

Proof. We claim that $\dim_{\Bbbk} \mathrm{Ext}^i_S(J_f/I_f, J_f/I_f)_0 = 0$ for $i = 1, 2$. It is enough to show that

$$\dim_{\Bbbk} \mathrm{Ext}^i_S(\Bbbk, \Bbbk)_0 = 0$$

since $\mathrm{Ext}^i_S(J_f/I_f, J_f/I_f)_0 = (\mathrm{Ext}^i_S(\Bbbk, \Bbbk)_0)^{\oplus(r-d)^2}$. Therefore, the claim follows from Lemma 2.28.

Applying the functor $\mathrm{Hom}_S(J_f/I_f, -)_0$ to the short exact sequence (3.23) we obtain

$$\dim_{\Bbbk} \mathrm{Ext}^1_S(J_f/I_f, S/I_f)_0 = \dim_{\Bbbk} \mathrm{Ext}^1_S(J_f/I_f, S/J_f)_0.$$

We have $\mathrm{Ext}^1_S(J_f/I_f, S/J_f)_0 \cong (\mathrm{Ext}^1_S(\Bbbk, S/J_f)_e)^{\oplus(r-d)}$ so it is enough to compute the dimension of $\mathrm{Ext}^1_S(\Bbbk, S/J_f)_e$ as a $\Bbbk$-vector space.

Applying the functor $\mathrm{Hom}_S(-, S/J_f)_e$ to the Koszul resolution of $\Bbbk$ we obtain the following complex:

$$(S/J_f)_e \xrightarrow{\begin{bmatrix}\alpha_0\\ \alpha_1\\ \alpha_2\end{bmatrix}} (S/J_f)^{\oplus 3}_{e+1} \xrightarrow{\begin{bmatrix}-\alpha_1 & \alpha_0 & 0\\ -\alpha_2 & 0 & \alpha_0\\ 0 & -\alpha_2 & \alpha_1\end{bmatrix}} (S/J_f)^{\oplus 3}_{e+2} \xrightarrow{\begin{bmatrix}\alpha_2 & -\alpha_1 & \alpha_0\end{bmatrix}} (S/J_f)_{e+3}.$$

We need to show that the cohomology at $(S/J_f)^{\oplus 3}_{e+1}$ is an $(r-d)$-dimensional $\Bbbk$-vector space. We will denote the map $(S/J_f)_e \to (S/J_f)^{\oplus 3}_{e+1}$ by d_0 and the map $(S/J_f)^{\oplus 3}_{e+1} \to (S/J_f)^{\oplus 3}_{e+2}$ by d_1. Let $h_0, h_1, h_2 \in S_{e+1}$ be such that $d_1(\overline{h_0}, \overline{h_1}, \overline{h_2}) = 0$, where $\overline{h_i}$ is the class of h_i in the quotient ring S/J_f. Let $h_2 = \alpha_2 h_2'(\alpha_0, \alpha_1, \alpha_2) + h_2''(\alpha_0, \alpha_1)$. Then $(-\alpha_1 h_0 + \alpha_0 h_1, \alpha_2(-h_0 + \alpha_0 h_2') + \alpha_0 h_2'', \alpha_2(-h_1 + \alpha_1 h_2') + \alpha_1 h_2'') \subseteq J_f$. The ideal J_f is monomial. Therefore, since $(J_f\colon \alpha_2) = J_f$ and h_2'' does not depend on α_2, we get $(h_0 - \alpha_0 h_2', h_1 - \alpha_1 h_2') \subseteq J_f$. Thus, $(\overline{h_0}, \overline{h_1}, \overline{h_2}) = (0, 0, \overline{h_2''}) + d_0(\overline{h_2'})$. We claim that $(0, 0, \overline{h_2''}) \in \ker d_1$ for every degree $e+1$ homogeneous polynomial $h_2'' \in \Bbbk[\alpha_0, \alpha_1]$. Indeed, we have $\alpha_0 h_1'', \alpha_1 h_1'' \in J_f$ as $(J_f)_{e+2} \cap \Bbbk[\alpha_0, \alpha_1] = \Bbbk[\alpha_0, \alpha_1]_{e+2}$. It follows that

$$\dim_{\Bbbk} \mathrm{Ext}^1_S(\Bbbk, S/J_f)_e = \dim_{\Bbbk} \Bbbk[\alpha_0, \alpha_1]_{e+1}/(J_f \cap \Bbbk[\alpha_0, \alpha_1])_{e+1} = (r-d).$$

□

Lemma 3.25. *In the above notation,* $\dim_{\Bbbk} \operatorname{Hom}_S(J_f, S/I_f)_0 = 2r$.

Proof. Observe that $\operatorname{Ext}^1_S(J_f, J_f/I_f)_0 \cong (\operatorname{Ext}^1_S(J_f, \Bbbk)_{-e})^{\oplus(r-d)} = 0$ is zero by Lemma 2.28. Indeed, $(J_f)_{e-1} = 0$ so $\beta_{1,e}(J_f) = \beta_{2,e}(S/J_f) = 0$.

Therefore, applying the functor $\operatorname{Hom}_S(J_f, -)_0$ to the short exact sequence (3.23) we obtain a short exact sequence

$$0 \to \operatorname{Hom}_S(J_f, J_f/I_f)_0 \to \operatorname{Hom}_S(J_f, S/I_f)_0 \to \operatorname{Hom}_S(J_f, S/J_f)_0 \to 0. \tag{3.26}$$

Since $(J_f)_{\leq e-1} = 0$ we have

$$\begin{aligned}\dim_{\Bbbk} \operatorname{Hom}_S(J_f, J_f/I_f)_0 = (r-d)\cdot \dim_{\Bbbk} \operatorname{Hom}_S\left(J_f, \Bbbk(-e)\right)_0 &= (r-d)\cdot \dim_{\Bbbk}(J_f)_e \\ &= (r-d)(\dim_{\Bbbk} S_e - d).\end{aligned} \tag{3.27}$$

Finally, let $\mathfrak{a}_f = J_f \cap T$, where $T = \Bbbk[\alpha_0, \alpha_1]$. Then $J_f = \mathfrak{a}_f \cdot S$, which implies that

$$\dim_{\Bbbk} \operatorname{Hom}_S(J_f, S/J_f)_0 = \sum_{i\leq 0} \dim_{\Bbbk} \operatorname{Hom}_T(\mathfrak{a}_f, T/\mathfrak{a}_f)_i. \tag{3.28}$$

Since $\operatorname{Spec}(T/\mathfrak{a}_f)$ corresponds to a point of the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{A}^2)$ which is smooth and $2r$-dimensional we get

$$\sum_{i\leq 0} \dim_{\Bbbk} \operatorname{Hom}_T(\mathfrak{a}_f, T/\mathfrak{a}_f)_i = 2r - \sum_{i>0} \dim_{\Bbbk} \operatorname{Hom}_T(\mathfrak{a}_f, T/\mathfrak{a}_f)_i \tag{3.29}$$

by [48, Prop. 2.3]. By Equation (3.15) the minimal generators of $\mathfrak{a}_f$ appear in degrees $e, e+1, e+2$. Furthermore, $H_{T/\mathfrak{a}_f}(e+2) = 0$. Therefore, by Proposition 2.22 we have

$$\sum_{i>0} \dim_{\Bbbk} \operatorname{Hom}_T(\mathfrak{a}_f, T/\mathfrak{a}_f)_i = \beta_{1,e}(T/\mathfrak{a}_f)\cdot H_{T/\mathfrak{a}_f}(e+1) = (e+1-d+s)\cdot(r-d) = (\dim_{\Bbbk} S_e - d)(r-d). \tag{3.30}$$

The exact sequence (3.26) and Equations (3.27), (3.28), (3.29) and (3.30) imply that

$$\dim_{\Bbbk} \operatorname{Hom}_S(J_f, S/I_f)_0 = 2r.$$

□

Lemma 3.31. *In the above notation,* $\dim_{\Bbbk} \operatorname{Hom}_S(I_f, S/I_f)_0 \leq 2r$.

Proof. From the long exact sequence obtained by applying the functor $\operatorname{Hom}_S(-, S/I_f)_0$ to the short exact sequence $0 \to I_f \to J_f \to J_f/I_f \to 0$ we get

$$\begin{aligned}\dim_{\Bbbk} \operatorname{Hom}_S(I_f, S/I_f)_0 \leq \dim_{\Bbbk} \operatorname{Hom}_S(J_f, S/I_f)_0 &+ \dim_{\Bbbk} \operatorname{Ext}^1_S(J_f/I_f, S/I_f)_0 \\ &- \dim_{\Bbbk} \operatorname{Hom}_S(J_f/I_f, S/I_f)_0.\end{aligned}$$

Using Lemmas 3.22, 3.24 and 3.25 we conclude that $\dim_{\Bbbk} \operatorname{Hom}_S(I_f, S/I_f)_0 \leq 2r$. □

We summarize the above considerations to obtain a proof of Theorem 3.12.

Proof of Theorem 3.12. Let f be the Hilbert function of $S/\overline{I}$. By Lemma 3.13 we may assume that f satisfies condition $(\star\star)$, i.e. that $f \in \Omega_r$. Let U_f be the locus of those closed points $[I']$ of $\operatorname{Hilb}^{h_{r,2}}_S$ for which $S/\overline{I'}$ has Hilbert function f. We shall show that $U_f \subseteq \operatorname{Slip}_{r,2}$. Locus U_f is irreducible by Lemma 3.14. We claim that it is enough to find a point $[I''] \in \mathrm{Slip}_{r,2} \cap U_f$ such that $\dim_{\Bbbk} \mathbf{T}_{[I'']} \mathrm{Hilb}_S^{h_{r,2}} = 2r$. Indeed, then by Lemma 2.30 every irreducible component of the intersection $\overline{U_f} \cap \mathrm{Slip}_{r,2}$ passing through $[I'']$ has dimension at least $\dim U_f + \dim \mathrm{Slip}_{r,2} - 2r = \dim U_f$. It follows that $\overline{U_f} \subseteq \mathrm{Slip}_{r,2}$.

We claim that we may take $I'' = I_f$ as defined by Equation (3.16). We have $[I_f] \in U_f \cap \mathrm{Slip}_{r,2}$ by Lemmas 3.17 and 3.20. Moreover, $[I_f]$ is a smooth point of $\mathrm{Hilb}_S^{h_{r,2}}$ by Lemma 3.31 and Theorem 2.74. □

We illustrate Theorem 3.12 with the following example.

Example 3.32. Let $S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ be a polynomial ring and $J = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0\alpha_2^2, \alpha_1^4)$. Then $[J] \in \mathrm{Hilb}_S^{h_{5,2}}$ and we claim that $[J] \in \mathrm{Slip}_{5,2}$. The Hilbert function of S/J satisfies condition $(\star)$. Thus, the claim follows from Theorem 3.12.

3.4.1 The analogue of the sufficient condition does not hold in general for projective space

For fixed positive integers r, n, condition $(\star)$ can be generalized as follows:

$$\text{there exist } e, d \in \mathbb{Z}_{>0} \text{ such that } f(a) = \begin{cases} h_{r,n}(a) & \text{if } a \neq e \\ d & \text{if } a = e. \end{cases} \qquad (\star\star\star)$$

For $n \geq 3$ and r large enough, the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^n)$ is reducible (see [53]). Therefore, it cannot be expected that a naive analogue of Theorem 3.12 holds in $\mathbb{P}^n$. The following remark gives a counterexample.

Remark 3.33. There are non-smoothable closed subschemes of $\mathbb{P}^6$ with Hilbert function

$$(1, 7, 13, 14, 14 \ldots)$$

(see [21], [58]). Let R be such a subscheme and $I = I(R)$ be its homogeneous ideal. Choose a 14-dimensional subspace V of I_2 and construct an ideal $J = V \oplus I_{\geq 3}$. Then $[J] \in \mathrm{Hilb}_{S[\mathbb{P}^6]}^{h_{14,6}}$ and the Hilbert function of $S[\mathbb{P}^6]/J$ satisfies condition $(\star\star\star)$ (with $r = 14, n = 6$). However, $[J] \notin \mathrm{Slip}_{14,6}$.

The above remark may suggest, that existence of more irreducible components of $\mathcal{H}ilb_r(\mathbb{P}^n)$ is the only obstacle. The following example shows that the Hilbert function of S/I may satisfy condition $(\star\star\star)$ for $[I] \in \mathrm{Hilb}_{S[\mathbb{P}^n]}^{h_{r,n}}$ that is not in the closure of the locus of points corresponding to saturated ideals.

Remark 3.34. Consider the ideal

$$I = (\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_2^4) \in S[\mathbb{P}^3] = \Bbbk[\alpha_0, \ldots, \alpha_3].$$

Then $[I] \in \mathrm{Hilb}_{S[\mathbb{P}^3]}^{h_{6,3}}$ and the Hilbert function of S/I satisfies condition $(\star\star\star)$ with $r = 6, n = 3$. Suppose that $[I]$ is in the closure of the locus of points corresponding to saturated ideals. Since, $\mathcal{H}ilb_6(\mathbb{P}^3)$ is irreducible (see [20, Thm. 1.1]) it follows that $[I] \in \mathrm{Slip}_{6,3}$. This contradicts Theorem 3.5 since $H_{S/I^2}(6) = 23 < r(n+1) = 24$.

3.5 Points on projective space – examples, part I

3.5.1 Initial cases

Proposition 3.35. *Let r be a positive integer. Then* $\mathrm{Hilb}^{h_{r,1}}_{S[\mathbb{P}^1]} \cong \mathbb{P}^r$. *In particular,* $\mathrm{Hilb}^{h_{r,1}}_{S[\mathbb{P}^1]} = \mathrm{Slip}_{r,1}$.

Proof. We have $h_{r,1} = f_{r,1}$ where $f_{r,1}$ is as in Lemma 2.44. It follows that $\mathrm{Hilb}^{h_{r,1}}_{S[\mathbb{P}^1]} \cong \mathcal{H}ilb_r(\mathbb{P}^1)$. The latter scheme is isomorphic to $\mathbb{P}^r$ (see [34, pages 111,112]). □

Proposition 3.36. *Let n be a positive integer. Then* $\mathrm{Hilb}^{h_{1,n}}_{S[\mathbb{P}^n]} \cong \mathbb{P}^n$. *In particular,* $\mathrm{Hilb}^{h_{1,n}}_{S[\mathbb{P}^n]} = \mathrm{Slip}_{1,n}$.

Proof. We have $h_{1,n} = f_{1,n}$ where $f_{1,n}$ is as in Lemma 2.44. It follows that $\mathrm{Hilb}^{h_{1,n}}_{S[\mathbb{P}^n]} \cong \mathcal{H}ilb_1(\mathbb{P}^n)$. The latter scheme is isomorphic with $\mathbb{P}^n$ (see [34, Ex. 7.3.1]). □

Proposition 3.37. *Let n be a positive integer. Then* $\mathrm{Hilb}^{h_{2,n}}_{S[\mathbb{P}^n]}$ *is a $\mathbb{P}^2$-bundle over*

$$\mathrm{Gr}(n-1, S[\mathbb{P}^n]_1).$$

In particular, $\mathrm{Hilb}^{h_{2,n}}_{S[\mathbb{P}^n]} = \mathrm{Slip}_{2,n}$.

Proof. It follows from [20, Prop. 3.1] that the Hilbert scheme $\mathrm{Hilb}^{h_{2,n}}_{S[\mathbb{P}^n]}$ is a $\mathrm{Hilb}^{h_{2,1}}_{S[\mathbb{P}^1]}$-bundle over $\mathrm{Gr}(n-1, S[\mathbb{P}^n]_1)$. Furthermore, $\mathrm{Hilb}^{h_{2,1}}_{S[\mathbb{P}^1]} \cong \mathbb{P}^2$ by Proposition 3.35. □

Proposition 3.38. *Let n be a positive integer. Then* $\mathrm{Hilb}^{h_{3,n}}_{S[\mathbb{P}^n]}$ *is irreducible. In particular,* $\mathrm{Slip}_{3,n} = \mathrm{Hilb}^{h_{3,n}}_{S[\mathbb{P}^n]}$.

Proof. By [20, Prop. 3.1] we may restrict to the case that $n = 1$ or $n = 2$. In the first case, the claim follows from Proposition 3.35.

Assume that $n = 2$. Let $[I]$ be a closed point of $\mathrm{Hilb}^{h_{3,2}}_{S[\mathbb{P}^2]}$. If $I = \overline{I}$, then $[I] \in \mathrm{Slip}_{3,2}$ by Remark 2.46. On the other hand, if $\overline{I} \neq I$, then $S/\overline{I}$ has Hilbert function $h_{3,1}$ by Lemma 2.9(ii), (iii). Therefore, $[I] \in \mathrm{Slip}_{3,2}$ by Theorem 3.12. □

3.5.2 Example of a singular point in the interior of Slip for projective plane

Since $\mathcal{H}ilb_r(\mathbb{P}^2)$ is smooth and $\mathrm{Slip}_{r,2}$ is related to $\mathcal{H}ilb_r(\mathbb{P}^2)$ by the natural morphism

$$\varphi_{r,\mathbb{P}^2} \colon \mathrm{Hilb}^{h_{r,2}}_{S[\mathbb{P}^2]} \to \mathcal{H}ilb_r(\mathbb{P}^2)$$

from Remark 2.46, it could be expected that the only singular points of $\mathrm{Hilb}^{h_{r,2}}_{S[\mathbb{P}^2]}$ in $\mathrm{Slip}_{r,2}$ are the points that lie in another irreducible component. We apply Theorem 3.12 to give an example of a singular point in the interior of $\mathrm{Slip}_{8,2}$.

We start with introducing some notation. Let Θ_8 be the set of all functions $f \colon \mathbb{Z} \to \mathbb{Z}$ such that f is the Hilbert function of a saturated homogeneous ideal of S defining a zero-dimensional closed subscheme of $\mathbb{P}^2$ of length 8. Given $f \in \Theta_8$, let V_f be the locally closed subset of $\mathcal{H}ilb_r(\mathbb{P}^2)$ defined by those closed points that correspond to subschemes of $\mathbb{P}^2$ with Hilbert function f. These sets with varying $f \in \Theta_8$ form a stratification of $\mathcal{H}ilb_r(\mathbb{P}^2)$ by locally closed irreducible subsets (see Proposition 3.11). Let U_f be the set-theoretic inverse image of V_f under $\varphi_{r,\mathbb{P}^2}$. In particular,

$\mathrm{Slip}_{8,2} = \overline{U_{h_{8,2}}}$. Also, we say that $f \le g$ for $f, g\colon \mathbb{Z} \to \mathbb{Z}$ if for every $a \in \mathbb{Z}$ we have $f(a) \le g(a)$. This gives a partial order on Θ_8.

Let $f_1, f_2\colon \mathbb{Z} \to \mathbb{Z}$ be given by

$$f_1(a) = \begin{cases} \dim_{\Bbbk} S_a & \text{for } a < 3 \\ 7 & \text{for } a = 3 \\ 8 & \text{for } a > 3 \end{cases} \quad \text{and} \quad f_2(a) = \begin{cases} \dim_{\Bbbk} S_a & \text{for } a < 2 \\ 5 & \text{for } a = 2 \\ 7 & \text{for } a = 3 \\ 8 & \text{for } a > 3 \end{cases}$$

or, in brief form, $f_1 = (1, 3, 6, 7, 8, 8, \ldots)$ and $f_2 = (1, 3, 5, 7, 8, 8, \ldots)$.

Let $I = (\alpha_1^2\alpha_2, \alpha_0\alpha_1^2 + \alpha_1\alpha_2^2, \alpha_1^4, \alpha_1\alpha_2^3, \alpha_2^5)$. Then the Hilbert function of $S/\overline{I}$ is f_1 so $[I] \in \mathrm{Slip}_{8,2}$ by Theorem 3.12. Hence also its initial ideal with respect to lex order ($\alpha_0 > \alpha_1 > \alpha_2$), i.e. $I' = (\alpha_0\alpha_1^2, \alpha_1^2\alpha_2, \alpha_1^4, \alpha_1\alpha_2^3, \alpha_2^5)$, is in $\mathrm{Slip}_{8,2}$. Since $\dim_{\Bbbk} \mathbf{T}_{[I']} \mathrm{Hilb}^{h_{8,2}}_{S[\mathbb{P}^2]} = 16 = \dim \mathrm{Slip}_{8,2}$, it follows that every irreducible closed subset of $\mathrm{Hilb}^{h_{8,2}}_{S[\mathbb{P}^2]}$ passing through $[I']$ is contained in $\mathrm{Slip}_{8,2}$. In particular

$$J = (\alpha_1^3, \alpha_1^2\alpha_2, \alpha_1^2\alpha_0^2, \alpha_1\alpha_2^3, \alpha_2^5)$$

belongs to $\mathrm{Slip}_{8,2}$ since $[I'], [J]$ lie in U_{f_2} which is irreducible by Lemma 2.43.

We have $\dim_{\Bbbk} \mathbf{T}_{[J]} \mathrm{Hilb}^{h_{8,2}}_{S[\mathbb{P}^2]} = 17 > \dim \mathrm{Slip}_{8,2}$. Let Z be an irreducible component of $\mathrm{Hilb}^{h_{8,2}}_{S[\mathbb{P}^2]}$ containing $[J]$. We will show that $Z = \mathrm{Slip}_{8,2}$. Let η be the generic point of Z, if $\eta \notin U_{h_{8,2}} \cup U_{f_1} \cup U_{f_2}$ then $[J] \notin \overline{\{\eta\}}$ since U_{f_2} is open in $\mathrm{Hilb}^{h_{8,2}}_{S[\mathbb{P}^2]} \setminus (U_{h_{8,2}} \cup U_{f_1})$ as f_2 is the greatest element of $\Theta_8 \setminus \{h_{8,2}, f_1\}$. Therefore, $Z = \overline{\{\eta\}} \subseteq \overline{U_{f_1}} \cup \overline{U_{f_2}} \cup \mathrm{Slip}_{8,2}$. As shown above, we have $U_{f_1} \cup U_{f_2} \subseteq \mathrm{Slip}_{8,2}$. Hence $Z = \mathrm{Slip}_{8,2}$.

We have established the following result.

Proposition 3.39. *Let $J = (\alpha_1^3, \alpha_1^2\alpha_2, \alpha_1^2\alpha_0^2, \alpha_1\alpha_2^3, \alpha_2^5)$. Then $[J] \in \mathrm{Hilb}^{h_{8,2}}_S$ is a singular point that belongs to the irreducible component $\mathrm{Slip}_{8,2}$ and no other.*

3.6 Criterion based on smoothness

In this section we give another criterion for a closed point $[I] \in \mathrm{Hilb}^{h_{r,n}}_S$ to be in $\mathrm{Slip}_{r,n}$. The criterion is based on smoothness of a point in a certain related multigraded Hilbert scheme. It is first stated in a general form (Theorem 3.40) but later we will impose additional assumptions to guarantee that conditions 1.-3. from the theorem are fulfilled.

Subsections 3.6.1 - 3.6.3 are concerned with describing some situations in which assumptions 1.-3. are fulfilled (each subsection deals with one assumption). The main results of these subsections are:

- Proposition 3.45 which implies condition 1. of Theorem 3.40
- Proposition 3.53 which implies condition 2. of Theorem 3.40
- Lemma 3.56 which implies condition 3. of Theorem 3.40.

Moreover, in Subsections 3.6.4 and 3.6.5 we present two applications of Theorem 3.40: Theorems 3.65 and 3.74. In the proof of the first of them we use Proposition 3.45 and Lemma 3.56. The proof of the second result, i.e. Theorem 3.74 is based on Propositions 3.45, 3.53 and Lemma 3.56.

Notation

In this section $[I] \in \mathrm{Hilb}_S^{h_{r,n}}$ is a closed point corresponding to an ideal that is not saturated. By d we denote a positive integer such that $I_d \neq \overline{I}_d$. We define $J = \overline{I} \cap \mathfrak{m}^d$ and $K = I \cap \mathfrak{m}^d$.

Now we present the main result of this section.

Theorem 3.40. *Assume that the following conditions hold:*

1. *the natural map* $\mathrm{Hom}_S(J, S/J)_0 \to \mathrm{Hom}_S(K, S/J)_0$ *is surjective;*
2. $[J] \in \mathrm{Hilb}_S^h$ *is a smooth point where h is the Hilbert function of S/J;*
3. *the natural map* $\mathrm{Hom}_S(K, S/K)_0 \to \mathrm{Hom}_S(K, S/J)_0$ *is surjective.*

Then there is no $[I'] \in \mathrm{Slip}_{r,n}$ *such that* $I'_{\geq d} = I_{\geq d}$. *In particular,* $[I] \notin \mathrm{Slip}_{r,n}$.

Proof. Let k be the Hilbert function of S/K. Consider the multigraded flag Hilbert scheme $\mathrm{Hilb}_S^{k,h}$ (see Subsection 2.2.4) and natural morphisms $\pi_k \colon \mathrm{Hilb}_S^{k,h} \to \mathrm{Hilb}_S^k$ and $\pi_h \colon \mathrm{Hilb}_S^{k,h} \to \mathrm{Hilb}_S^h$.

We first show that π_k induces isomorphism on tangent spaces $\mathbf{T}_{[K\subseteq J]} \mathrm{Hilb}_S^{k,h} \to \mathbf{T}_{[K]} \mathrm{Hilb}_S^k$. This map on tangent spaces is the upper horizontal map in the pullback diagram

$$\begin{array}{ccc} \mathbf{T}_{[K\subseteq J]} \mathrm{Hilb}_S^{k,h} & \longrightarrow & \mathrm{Hom}_S(K, S/K)_0 \\ \big\downarrow & & \big\downarrow \\ \mathrm{Hom}_S(J, S/J)_0 & \longrightarrow & \mathrm{Hom}_S(K, S/J)_0 \end{array}$$

in which the right vertical and the lower horizontal maps are natural maps of Hom groups (see Theorem 2.77). By assumption 1., the lower horizontal map is surjective. Moreover, $\mathrm{Hom}_S(J/K, S/J)_0 = 0$ by Lemma 2.8 since $J = \overline{J}_{\geq d}$ and $\mathfrak{m}^r \cdot J/K = 0$. Therefore, the lower horizontal map is bijective. Thus, so is the upper horizontal map since the diagram is a pullback.

Now we show that the natural transformation $D_{\mathrm{Hilb}_S^{k,h},[K\subseteq J]} \to D_{\mathrm{Hilb}_S^k,[K]}$ of deformation functors induced by π_k admits a map of tangent-obstruction theories which is injective on obstruction spaces. By Theorem 2.74, there are tangent-obstruction theories for $D_{\mathrm{Hilb}_S^h,[J]}$ and $D_{\mathrm{Hilb}_S^k,[K]}$ with obstruction spaces $\mathrm{Ext}_S^1(J, S/J)_0$ and $\mathrm{Ext}_S^1(K, S/K)_0$, respectively. Moreover, by Theorem 2.77 there is a tangent-obstruction theory for $D_{\mathrm{Hilb}_S^{k,h},[K\subseteq J]}$ with obstruction space given by the pullback diagram

$$\begin{array}{ccc} \mathrm{Ob}_{[K\subseteq J]} \mathrm{Hilb}_S^{k,h} & \xrightarrow{\beta} & \mathrm{Ext}_S^1(K, S/K)_0 \\ \big\downarrow{\scriptstyle \alpha} & & \big\downarrow \\ \mathrm{Ext}_S^1(J, S/J)_0 & \longrightarrow & \mathrm{Ext}_S^1(K, S/J)_0, \end{array}$$

where the lower horizontal and the right vertical maps are maps from long exact sequences of Ext groups. Furthermore, α and β induce maps of tangent-obstruction theories. Here we have used assumption 3. Since $[J] \in \mathrm{Hilb}_S^h$ is a smooth point, we can use Lemma 2.67, to change the tangent-obstruction theory of $D_{\mathrm{Hilb}_S^{k,h},[K\subseteq J]}$ so that $D_{\mathrm{Hilb}_S^{k,h},[K\subseteq J]} \to D_{\mathrm{Hilb}_S^k,[K]}$ admits a map of tangent-obstruction theories which is injective on obstruction spaces.

It follows from Corollary 2.65 that the map π_k is étale at $[K \subseteq J]$. In particular, there is an open subset U of $\mathrm{Hilb}_S^{k,h}$ containing $[K \subseteq J]$ that is mapped onto an open subset V of Hilb_S^k containing $[K]$. If there is a point $[I'] \in \mathrm{Slip}_{r,n}$ such that $I_{\geq d} = I'_{\geq d}$, then there is a saturated ideal I'' such that $[I''] \in \mathrm{Hilb}_S^{h_{r,n}}$ and $[I'' \cap \mathfrak{m}^d] \in V$. Therefore, there is an ideal $[J''] \in \mathrm{Hilb}_S^h$ such that $I'' \cap \mathfrak{m}^d \subseteq J''$. This gives a contradiction since $I'' = \overline{J''}$ and $I''_d \subsetneq J''_d$. $\square$

Remark 3.41. It seems that assumption 2. of the above theorem is both the most restrictive and potentially the hardest to check in practice. On the other hand, if $n = 2$ and

$$d = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\}$$

then condition 2. always holds (see Proposition 3.10) since the usual Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^2)$ is smooth. Furthermore, for $d = \max\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\}$ condition 3. is satisfied (see Lemma 3.56). Even in the case of $\mathbb{P}^2$ and $d = \max\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\}$ it is not clear, in how general setups can we expect condition 2. of Theorem 3.40 to hold. We present one specific situation when this holds in Proposition 3.53.

In the following subsections we will study some situations in which conditions 1.-3. of Theorem 3.40 hold.

3.6.1 About condition 1.

The main result of this subsection is Proposition 3.45 which describes a situation in which condition 1. of Theorem 3.40 holds.

We keep the notation of Theorem 3.40. Let $R = S/\overline{I}$ and pick a linear form $L \in S_1$ that is a non-zero divisor on R. This is possible by Lemma 2.9(i). By a linear change of variables, we may and will assume that $L = \alpha_0$.

We start with the following simple observation.

Lemma 3.42. *Let $\mathfrak{b}$ be a homogeneous ideal of S that is generated in degrees at most d for a positive integer d. Let $\mathfrak{a} = \frac{\mathfrak{b}+\overline{I}}{\overline{I}} \subseteq R$. If $\dim_{\Bbbk} \mathfrak{a}_d = \dim_{\Bbbk} \mathfrak{a}_{d+1}$ then*

$$\mathfrak{a}_{a'} = \alpha_0^{a'-a}\mathfrak{a}_a$$

for every $a' \geq a \geq d$.

Proof. Let $\iota\colon R \to R$ be the multiplication by α_0. This induces an injective map $\mathfrak{a}_a \to \mathfrak{a}_{a+1}$ for every integer a. It is enough to show that for $a \geq d$ this map is surjective. We prove this by induction. The case $a = d$ follows from the assumption that $\dim_{\Bbbk} \mathfrak{a}_d = \dim_{\Bbbk} \mathfrak{a}_{d+1}$. Let $a_0 > d$ and suppose that $\mathfrak{a}_{a_0} = \alpha_0\mathfrak{a}_{a_0-1}$. Let $g \in \mathfrak{a}_{a_0+1}$. Since $\mathfrak{b}$ is generated in degrees at most d we obtain

$$g = \sum_{i=0}^{n} \alpha_i f_i \text{ for some } f_i \in \mathfrak{a}_{a_0}.$$

By induction there are $h_i \in \mathfrak{a}_{a_0-1}$ such that $f_i = \alpha_0 h_i$ for $i = 0, 1, \ldots, n$. Therefore, $g = \alpha_0(\sum_{i=0}^{n} \alpha_i h_i)$. $\square$

Notation 3.43. Let $d = \max\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\}$. Let $\mathfrak{b} \subseteq S$ be a homogeneous ideal that is generated in degrees at most d. Let $\mathfrak{a} = \frac{\mathfrak{b}+\overline{I}}{\overline{I}} \subseteq S/\overline{I} = R$. Recall that $J = \mathfrak{m}^d \cap \overline{I}$ and $K = \mathfrak{m}^d \cap I$. In particular, $J/K \cong \Bbbk(-d)^s$ where $s = H_{S/K}(d) - H_{S/J}(d)$ and $(S/J)_{d+1} = (S/K)_{d+1} = R_{d+1}$. Assume that $\dim_{\Bbbk} \mathfrak{a}_d = \dim_{\Bbbk} \mathfrak{a}_{d+1}$, so that we can apply Lemma 3.42 for $\mathfrak{a}$.

The following lemma will be used in the proof of Proposition 3.45 to extend a homomorphism $\varphi \in \mathrm{Hom}_S(K, S/J)_0$ to an element $\psi \in \mathrm{Hom}_S(J, S/J)_0$ under some additional assumptions.

Lemma 3.44. *With Notation 3.43, let $F \in J_d$ and assume that it is of the form $F = fg$ for some homogeneous $f \in \mathfrak{b}$ and $g \in \overline{I}$. Let $\varphi \in \operatorname{Hom}_S(K, S/J)_0$. Then, there is $h \in (S/J)_d$ such that $\varphi(\alpha_0 F) = \alpha_0 h$.*

Proof. Observe that $\alpha_0 F \in \overline{I}_{d+1} = K_{d+1}$, so $\varphi(\alpha_0 F)$ is a well-defined element of R_{d+1}. Let $a = \deg(f)$. Then

$$\alpha_0^a \varphi(\alpha_0 F) = \alpha_0^a \varphi(\alpha_0 f g) = f \varphi(\alpha_0^{a+1} g).$$

Hence $\alpha_0^a \varphi(\alpha_0 F) \in \mathfrak{a}_{d+a+1}$. Therefore, by Lemma 3.42 there is $h \in \mathfrak{a}_d \subseteq R_d = (S/J)_d$ such that $\alpha_0^a \varphi(\alpha_0 F) = \alpha_0^{a+1} h$. Since α_0 is a non-zero divisor on R, it follows that $\varphi(\alpha_0 F) = \alpha_0 h$. $\square$

Now we can present the main result of this subsection.

Proposition 3.45. *With Notation 3.43, let $F_1, \ldots, F_s \in J_d$ be elements whose classes form a basis of J_d/K_d. Assume that $F_i = f_i g_i$ for some homogeneous $f_i \in \mathfrak{b}$ and $g_i \in \overline{I}$. Let $\varphi \in \operatorname{Hom}_S(K, S/J)_0$. Then there exists $\psi \in \operatorname{Hom}_S(J, S/J)_0$ such that $\psi|_K = \varphi$. Thus, condition 1. from Theorem 3.40 is fulfilled.*

Proof. Let $\{p_1, \ldots, p_t\}$ be a minimal set of homogeneous generators of J containing $\{F_1, \ldots, F_s\}$. We may and will assume that $p_i \in K$ if $p_i \notin \{F_1, \ldots, F_s\}$. By Lemma 3.44, there are $h_i \in (S/J)_d$ such that

$$\alpha_0 h_i = \varphi(\alpha_0 F_i) \text{ for } i \in \{1, 2, \ldots, s\}. \tag{3.46}$$

We define ψ on generators $\{p_1, \ldots, p_t\}$ of J by

$$\psi(p_i) = \begin{cases} \varphi(p_i) & \text{if } p_i \in K \\ h_j & \text{if } p_i = F_j \text{ for some } j \in \{1, 2, \ldots, s\}. \end{cases}$$

We claim that ψ is a well-defined element of $\operatorname{Hom}_S(J, S/J)_0$. Indeed, let

$$\{p_1, \ldots, p_t\} = \{F_1, \ldots, F_s, Q_{s+1}, \ldots, Q_t\}$$

and assume that $G_i, H_i \in S$ are such that

$$\sum_{i=1}^{s} G_i F_i + \sum_{i=s+1}^{t} H_i Q_i = 0.$$

We need to show that

$$\sum_{i=1}^{s} G_i \psi(F_i) + \sum_{i=s+1}^{t} H_i \psi(Q_i) = \sum_{i=1}^{s} G_i h_i + \sum_{i=s+1}^{t} H_i \varphi(Q_i) = 0.$$

Since α_0 is a non-zero divisor on R, it is enough to observe that

$$\begin{aligned} \alpha_0 \Big(\sum_{i=1}^{s} G_i h_i + \sum_{i=s+1}^{t} H_i \varphi(Q_i) \Big) &\overset{(3.46)}{=} \sum_{i=1}^{s} G_i \varphi(\alpha_0 F_i) + \alpha_0 \sum_{i=s+1}^{t} H_i \varphi(Q_i) \\ &= \varphi\Bigg(\alpha_0 \Big(\sum_{i=1}^{s} G_i F_i + \sum_{i=s+1}^{t} H_i Q_i \Big) \Bigg) = \varphi(0) = 0. \end{aligned}$$

$\square$

3.6.2 About condition 2.

In this subsection we show in Proposition 3.53 that if $n = 2$ and under some additional assumptions, condition 2. from Theorem 3.40 is fulfilled.

In the notation of Theorem 3.40, assume that $d = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\}$. We claim that we have $d = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq 0\}$ and therefore, that $J = \overline{I}$. Let $d' = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq 0\}$. If $\overline{I}_a = 0$ then $I_a = \overline{I}_a$ so $d' \leq d$. On the other hand, if $d' < d$ then $I_{d'} = \overline{I}_{d'} \neq 0$. Thus, $H_{S/I}(a) = H_{S/\overline{I}}(a) = r$ for every $a \geq d'$ by Lemma 2.9(ii) and the definition of $h_{r,n}$. It follows that $I_{\geq d'} = \overline{I}_{\geq d'}$ which contradicts the definition of d and proves the claim.

The smoothness of $\mathcal{H}ilb_r(\mathbb{P}^2)$ and its consequence, Proposition 3.10, play a key role in our approach to condition 2. of Theorem 3.40. However, Proposition 3.45 requires $d = \max\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\}$ while Proposition 3.10 corresponds to the case $d = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\}$, when $[J] = [\overline{I} \cap \mathfrak{m}^d] = [\overline{I}]$. Therefore, we would like to show that, under some additional assumptions, the condition that $[\overline{I} \cap \mathfrak{m}^d]$ is a smooth point of Hilb_S^h holds also for $d = \max\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\}$. This will be achieved in Proposition 3.53.

Lemma 3.47. *Let $I \subseteq S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ be a homogeneous ideal. Then*

$$\dim_{\Bbbk} \operatorname{Ext}^2_S(\Bbbk, S/I)_a = \beta_{1,a+3}(S/I)$$

for every $a \geq 0$.

Proof. Consider the short exact sequence

$$0 \to I \to S \to S/I \to 0.$$

Applying the functor $\operatorname{Hom}_S(\Bbbk, -)_a$ to the above short exact sequence we obtain the exact sequence

$$\ldots \to \operatorname{Ext}^2_S(\Bbbk, S)_a \to \operatorname{Ext}^2_S(\Bbbk, S/I)_a \to \operatorname{Ext}^3_S(\Bbbk, I)_a \to \operatorname{Ext}^3_S(\Bbbk, S)_a \to \ldots. \tag{3.48}$$

We claim that $\operatorname{Ext}^i_S(\Bbbk, S)_a = 0$ for $i = 2, 3$.

Consider the Koszul complex

$$0 \to S(-3) \xrightarrow{\begin{bmatrix} \alpha_2 \\ -\alpha_1 \\ \alpha_0 \end{bmatrix}} S(-2)^{\oplus 3} \xrightarrow{\begin{bmatrix} -\alpha_1 & -\alpha_2 & 0 \\ \alpha_0 & 0 & -\alpha_2 \\ 0 & \alpha_0 & \alpha_1 \end{bmatrix}} S(-1)^{\oplus 3} \xrightarrow{\begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 \end{bmatrix}} S \to 0. \tag{3.49}$$

The Ext groups $\operatorname{Ext}^i_S(\Bbbk, S)_a = 0$ for $i = 2, 3$ can be computed as the cohomology groups at $S_{a+2}^{\oplus 3}$ and S_{a+3} of the complex

$$0 \to S_a \xrightarrow{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix}} S_{a+1}^{\oplus 3} \xrightarrow{\begin{bmatrix} -\alpha_1 & \alpha_0 & 0 \\ -\alpha_2 & 0 & \alpha_0 \\ 0 & -\alpha_2 & \alpha_1 \end{bmatrix}} S_{a+2}^{\oplus 3} \xrightarrow{\begin{bmatrix} \alpha_2 & -\alpha_1 & \alpha_0 \end{bmatrix}} S_{a+3} \to 0$$

obtained from the Koszul complex (3.49) by applying the functor $\operatorname{Hom}_S(-, S)_a$. These groups are trivial.

Thus,

$$\dim_{\Bbbk} \operatorname{Ext}^2_S(\Bbbk, S/I)_a = \dim_{\Bbbk} \operatorname{Ext}^3_S(\Bbbk, I)_a$$

by exact sequence (3.48). Applying the functor $\operatorname{Hom}_S(-, S/I)_a$ to the Koszul complex (3.49) we get that $\operatorname{Ext}^3_S(\Bbbk, I)_a$ is the cokernel of the map

$$I_{a+2}^{\oplus 3} \xrightarrow{\begin{bmatrix}\alpha_2 & -\alpha_1 & \alpha_0\end{bmatrix}} I_{a+3}.$$

Hence $\dim_{\Bbbk} \operatorname{Ext}^3_S(\Bbbk, I)_a = \beta_{1,a+3}(S/I)$. □

Lemma 3.50. *Let f be the Hilbert function of a zero-dimensional length r subscheme of $\mathbb{P}^n$. Let $\overline{I}$ be a saturated ideal of S such that $[\overline{I}] \in \operatorname{Hilb}^f_S$. Let $m = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq 0\}$ and let $d \geq m$ be a positive integer. Let h be the Hilbert function of $S/\overline{I} \cap \mathfrak{m}^d$. Assume that $[\overline{I}]$ is a smooth point and that the natural map $\operatorname{Hom}_S(\overline{I} \cap \mathfrak{m}^a, S/\overline{I})_0 \to \operatorname{Hom}_S(\overline{I} \cap \mathfrak{m}^{a+1}, S/\overline{I})_0$ is bijective for every $m \leq a < d$. Then $[\overline{I} \cap \mathfrak{m}^d] \in \operatorname{Hilb}^h_S$ is a smooth point. Moreover, $\dim_{\Bbbk} \mathbf{T}_{[\overline{I}\cap\mathfrak{m}^d]} \operatorname{Hilb}^h_S = \dim W$ where $W \subseteq \operatorname{Hilb}^h_S$ is the locally closed subset whose closed points correspond to ideals defining subschemes of $\mathbb{P}^n$ with Hilbert function f.*

Proof. For an integer $m \leq a \leq d$ let g_a be the Hilbert function of $S/\overline{I}\cap\mathfrak{m}^a$. In particular, $g_m = f$ and $g_d = h$. Let W_a be the locally closed subset of $\operatorname{Hilb}^{g_a}_S$ whose closed points correspond to ideals defining subschemes of $\mathbb{P}^n$ with Hilbert function f. In particular, $W_d = W$. For $m \leq a \leq d-1$ let $\pi_a \colon \operatorname{Hilb}^{g_a}_S \to \operatorname{Hilb}^{g_{a+1}}_S$ be the natural map given on closed points by $[I'] \mapsto [I' \cap \mathfrak{m}^{a+1}]$. The map π_a induces a homeomorphism $W_a \cong W_{a+1}$ by Lemma 2.29.

We assumed that the natural map $\operatorname{Hom}_S(\overline{I}\cap\mathfrak{m}^a, S/\overline{I})_0 \to \operatorname{Hom}_S(\overline{I}\cap\mathfrak{m}^{a+1}, S/\overline{I})_0$ is bijective for every $m \leq a \leq d-1$. Therefore, by Lemma 2.79 the map π_a induces an isomorphism of tangent spaces

$$\mathbf{T}_{[\overline{I}\cap\mathfrak{m}^a]} \operatorname{Hilb}^{g_a}_S \cong \mathbf{T}_{[\overline{I}\cap\mathfrak{m}^{a+1}]} \operatorname{Hilb}^{g_{a+1}}_S \tag{3.51}$$

for every $m \leq a \leq d-1$.

Now we show that $[\overline{I} \cap \mathfrak{m}^d] \in \operatorname{Hilb}^h_S$ is a smooth point. Observe that $W_m \subseteq \operatorname{Hilb}^f_S$ is open and $[\overline{I} \cap \mathfrak{m}^m] = [\overline{I}] \in \operatorname{Hilb}^f_S$ is a smooth point by assumption. Therefore,

$$\dim_{\Bbbk} \mathbf{T}_{[\overline{I}\cap\mathfrak{m}^d]} \operatorname{Hilb}^h_S = \dim_{\Bbbk} \mathbf{T}_{[\overline{I}]} \operatorname{Hilb}^f_S = \dim W_m = \dim W_d = \dim W \tag{3.52}$$

where the first equality follows from Equation (3.51), the second from the fact that $[\overline{I}]$ is a smooth point and $W_m \subseteq \operatorname{Hilb}^f_S$ is open and the third equality follows from the homeomorphism $W_m \cong W_d$. Equation (3.52) implies that $[\overline{I}\cap\mathfrak{m}^d]$ is a smooth point of Hilb^h_S since $[\overline{I}\cap\mathfrak{m}^d] \in W$. □

Finally, we present the main result of this subsection.

Proposition 3.53. *In the notation of Theorem 3.40, assume that $n = 2$. Let*

$$m = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\} = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq 0\}.$$

Let f be the Hilbert function of $S/\overline{I}$ and assume that

$$f(a) - 3f(a+1) + 3f(a+2) - f(a+3) = \beta_{1,a+3}(S/\overline{I}) \tag{3.54}$$

for every $m \leq a \leq d-1$. Then $[J]$ is smooth in Hilb^h_S, i.e. condition 2. from Theorem 3.40 is *fulfilled. Moreover,* $\dim_{\Bbbk} \operatorname{Hom}_S(J, S/J)_0 = \dim W$ *where* $W \subseteq \operatorname{Hilb}_S^h$ *is the locally closed subset whose closed points correspond to ideals defining subschemes of* $\mathbb{P}^2$ *with Hilbert function* f*.*

Proof. The point $[\overline{I} \cap \mathfrak{m}^m] = [\overline{I}] \in \operatorname{Hilb}_S^f$ is a smooth point by Proposition 3.10. Therefore, by Lemma 3.50 it is enough to show that the natural map

$$\operatorname{Hom}_S(\overline{I} \cap \mathfrak{m}^a, S/\overline{I})_0 \to \operatorname{Hom}_S(\overline{I} \cap \mathfrak{m}^{a+1}, S/\overline{I})_0 \tag{3.55}$$

coming from the exact sequence of Hom groups is bijective for every $m \leq a < d$. Fix $m \leq a \leq d-1$ and let $Q = \overline{I} \cap \mathfrak{m}^a / \overline{I} \cap \mathfrak{m}^{a+1}$. Since $\mathfrak{m} \cdot Q = 0$ and $(\overline{I} \colon \mathfrak{m}) = \overline{I}$, we conclude by Lemma 2.8 that $\operatorname{Hom}_S(Q, S/\overline{I})_0 = 0$. Thus, the map from Equation (3.55) is injective.

We have $Q = \Bbbk(-a)^s$ for some integer s. Therefore, by Lemma 2.27 we have

$$s\Big(f(a) - 3f(a+1) + 3f(a+2) - f(a+3)\Big) = \sum_{i=0}^{3} (-1)^i \dim_{\Bbbk} \operatorname{Ext}_S^i(Q, S/\overline{I})_0.$$

Since $\dim_{\Bbbk} \operatorname{Hom}_S(Q, S/\overline{I})_0 = 0$, it follows from Equation (3.54) and Lemma 3.47 that

$$\dim_{\Bbbk} \operatorname{Ext}_S^1(Q, S/\overline{I})_0 + \dim_{\Bbbk} \operatorname{Ext}_S^3(Q, S/\overline{I})_0 = 0.$$

In particular, $\operatorname{Ext}_S^1(Q, S/\overline{I})_0 = 0$. Thus, the map from Equation (3.55) is surjective. □

3.6.3 About condition 3.

Lemma 3.56. *In the notation of Theorem 3.40, assume that* $d = \max\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\}$*. Then the natural map* $\operatorname{Hom}_S(K, S/K)_0 \to \operatorname{Hom}_S(K, S/J)_0$ *is surjective, i.e. condition 3. from Theorem 3.40 is fulfilled.*

Proof. It is enough to establish that $\operatorname{Ext}_S^1(K, J/K)_0 = 0$. Let $P_\bullet$ be a minimal graded free resolution of K. Then $\operatorname{Ext}_S^1(K, J/K)_0$ is a subquotient of $\operatorname{Hom}_S(P_1, J/K)_0$ so it is enough to show that the latter group is trivial. This holds, since the minimal generators of P_1 are of degree at least $d+1$ and $(J/K)_{\geq d+1} = 0$. □

3.6.4 Application one: subschemes contained in a line

In this subsection we consider ideals defining subschemes contained in a line. The statement of Theorem 3.65 coincides with [66, Thm. 2.8] but we provide a new proof in the setting of Theorem 3.40.

We start with a lemma. It is stated in a general version since we will also use it in Subsection 3.6.5. We introduce some notation. Let f be the Hilbert function of a zero-dimensional, length r subscheme of $\operatorname{Proj} S$. Assume that $f \neq h_{r,n}$. Let

$$e = \max\{a \in \mathbb{Z} \mid f(a) \neq h_{r,n}(a)\}.$$

Let $h \colon \mathbb{Z} \to \mathbb{Z}$ be defined by

$$h(a) = \begin{cases} \dim_{\Bbbk} S_a & \text{for } a < e; \\ f(a) & \text{for } a \geq e \end{cases}$$

and $k\colon \mathbb{Z} \to \mathbb{Z}$ be defined by

$$k(a) = \begin{cases} \dim_{\Bbbk} S_a & \text{for } a < e; \\ h_{r,n}(a) & \text{for } a \geq e. \end{cases}$$

Let $\pi\colon \operatorname{Hilb}_S^{h_{r,n}} \to \operatorname{Hilb}_S^k$, $\pi_h\colon \operatorname{Hilb}_S^h \to \mathcal{H}ilb_r(\mathbb{P}^n)$ and $\pi_k\colon \operatorname{Hilb}_S^k \to \mathcal{H}ilb_r(\mathbb{P}^n)$ be the natural morphisms.

Let V be the locally closed subset of $\mathcal{H}ilb_r(\mathbb{P}^n)$ consisting of points corresponding to subschemes with Hilbert function f. Let W be the set-theoretic inverse image of V under π_h and let W' be the set-theoretic inverse image of V under π_k.

Lemma 3.57. *In the above notation, assume that:*

1. $f(e) = r - 1$*;*
2. $V \subseteq \mathcal{H}ilb_r(\mathbb{P}^n)$ *is irreducible;*
3. *there exists an irreducible closed* $(rn-1)$*-dimensional subset* $U \subseteq W'$ *such that* $\pi(\operatorname{Slip}_{r,n}) \cap W' \subseteq U$*, set-theoretically;*
4. $\dim_{\Bbbk} \operatorname{Hom}_S(J, S/J)_0 = \dim W$ *for every* $[J] \in W \subseteq \operatorname{Hilb}_S^h$*;*
5. $\dim_{\Bbbk} \operatorname{Ext}_S^1(\Bbbk, S/J)_e \leq 1$ *for every* $[J] \in W \subseteq \operatorname{Hilb}_S^h$*.*

Then $\pi(\operatorname{Slip}_{r,n}) \cap W' = U$*, set-theoretically.*

Proof. From assumption 1. and $h_{r,n}(e) \neq f(e)$ we get $h_{r,n}(e) = r$. Let $N = \dim V$. It follows from Lemmas 2.43, 2.44, [80, 11.4.C] and assumption 2. that W, W' are irreducible and their dimensions are $\dim W = N$ and $\dim W' = N + (\dim_{\Bbbk} S_e - (r-1) - 1) = N + \dim_{\Bbbk} S_e - r$.

We have $\dim \pi(\operatorname{Slip}_{r,n}) = rn$. We claim that $V \cap \mathcal{H}ilb_r^{sm}(\mathbb{P}^n) \neq \emptyset$. Let $[R] \in V$ and let I' be the generic initial ideal of the ideal $I(R)$ with respect to the grevlex order with $\alpha_0 > \alpha_1 > \ldots > \alpha_n$. Then $[\operatorname{Proj} S/I'] \in V$ by Corollary 2.10 and $[\operatorname{Proj} S/I'] \in \mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$ by [20, Prop. 4.15] since I' is an extended ideal from $\Bbbk[\alpha_0, \ldots, \alpha_{n-1}]$ by Lemma 2.11. Since $\pi_k \circ \pi(\operatorname{Slip}_{r,n}) = \mathcal{H}ilb_r^{sm}(\mathbb{P}^n)$ and $V \cap \mathcal{H}ilb_r^{sm}(\mathbb{P}^n) \neq \emptyset$, we have $\pi(\operatorname{Slip}_{r,n}) \cap W' \neq \emptyset$.

Now we will show that

$$\dim_{\Bbbk} \operatorname{Hom}_S(K, S/K)_0 \leq N + \dim_{\Bbbk} S_e - r + 1 = \dim W' + 1 \tag{3.58}$$

for every $[K] \in W' \subseteq \operatorname{Hilb}_S^k$.

Let $[K] \in W'$ and $J = \overline{K} \cap \mathfrak{m}^e$. Consider the exact sequences

$$0 \to \operatorname{Hom}_S(J/K, S/J)_0 \to \operatorname{Hom}_S(J, S/J)_0 \to \operatorname{Hom}_S(K, S/J)_0 \to \operatorname{Ext}_S^1(J/K, S/J)_0 \tag{3.59}$$

and

$$0 \to \operatorname{Hom}_S(K, J/K)_0 \to \operatorname{Hom}_S(K, S/K)_0 \to \operatorname{Hom}_S(K, S/J)_0 \to \operatorname{Ext}_S^1(K, J/K)_0. \tag{3.60}$$

By assumption 1., $J/K = \Bbbk(-e)$. It follows from Lemma 2.28 that

$$\dim_{\Bbbk} \operatorname{Hom}_S(K, J/K)_0 = \beta_{1,e}(S/K) = \dim_{\Bbbk} S_e - r \tag{3.61}$$

and

$$\dim_{\Bbbk} \operatorname{Ext}^1_S(K, J/K)_0 = 0. \tag{3.62}$$

Thus, by Equations (3.60) and (3.62) we get

$$\dim_{\Bbbk} \operatorname{Hom}_S(K, S/K)_0 = \dim_{\Bbbk} \operatorname{Hom}_S(K, J/K)_0 + \dim_{\Bbbk} \operatorname{Hom}_S(K, S/J)_0. \tag{3.63}$$

Moreover, $\operatorname{Hom}_S(J/K, S/J)_0 = 0$ by Lemma 2.8. Therefore, it follows from Equations (3.59) and (3.63) that

$$\dim_{\Bbbk} \operatorname{Hom}_S(K, S/K)_0 \leq \dim_{\Bbbk} \operatorname{Hom}_S(K, J/K)_0 + \dim_{\Bbbk} \operatorname{Hom}_S(J, S/J)_0 + \dim_{\Bbbk} \operatorname{Ext}^1_S(\Bbbk, S/J)_e.$$

Equation (3.58) follows from Equation (3.61) and assumptions 4., 5.

Let $[K] \in W' \cap \pi(\mathrm{Slip}_{r,n})$ and let Z be an irreducible component of $\overline{W'} \cap \pi(\mathrm{Slip}_{r,n})$ passing through $[K]$. By Theorem 2.74, Lemma 2.30 and Equation (3.58) we get that

$$\dim Z \geq \dim \pi(\mathrm{Slip}_{r,n}) + \dim W' - (\dim W' + 1) = rn - 1.$$

Moreover, W' is open in $\overline{W'}$ since W' is locally closed. Therefore, the generic point of Z belongs to W'. As a result, $Z \subseteq \overline{U}$ by assumption 3. and in fact $Z = \overline{U}$ since $\dim Z \geq rn - 1 = \dim \overline{U}$. Since U is closed in W' we get $U = W' \cap Z \subseteq W' \cap \pi(\mathrm{Slip}_{r,n})$ and therefore, $W' \cap \pi(\mathrm{Slip}_{r,n}) = U$ by assumption 3. $\square$

Lemma 3.64. *Let J be a homogeneous ideal of S such that S/J has Hilbert function $h_{r,1}$. Then*

$$\dim_{\Bbbk} \operatorname{Ext}^1_S(\Bbbk, S/J)_a = 0$$

for $1 \leq a < r - 2$ and

$$\dim_{\Bbbk} \operatorname{Ext}^1_S(\Bbbk, S/J)_{r-2} = 1.$$

Proof. Up to a linear change of variables, we may assume that $J = \big(\alpha_0, \ldots, \alpha_{n-2}, \theta_r(\alpha_{n-1}, \alpha_n)\big)$. The Ext groups from the statement are the middle cohomology groups of the complex

$$(S/J)_a \xrightarrow{\delta_0} (S/J)^{n+1}_{a+1} \xrightarrow{\delta_1} (S/J)^{\binom{n+1}{2}}_{a+2}$$

obtained from the Koszul complex. Here

$$\delta_0([f]) = ([\alpha_i f])_{i=0,\ldots,n} \text{ and } \delta_1([f_i]_{i=0,\ldots,n}) = ([\alpha_i f_j - \alpha_j f_i])_{0 \leq i < j \leq n}.$$

Let $a \in \{1, \ldots, r-3\}$ and assume that $\delta_1([f_i]_{i=0,\ldots,n}) = 0$. Observe that $[\alpha_j f_i] = 0$ for every $i \in \{0, \ldots, n\}$ and $j \in \{0, \ldots, n-2\}$. Furthermore, multiplications by α_{n-1} or α_n give injective maps $(S/J)_a \to (S/J)_{a+1}$. It follows that $[f_j] = 0$ for $j \leq n-2$. Moreover, we have $[\alpha_n f_{n-1}] = [\alpha_{n-1} f_n]$. There are unique representatives f_{n-1}, f_n of the classes $[f_{n-1}]$ and $[f_n]$ in S/J which are polynomials in variables α_{n-1}, α_n. Since $J_{a+2} = (\alpha_0, \ldots, \alpha_{n-2})_{a+2}$, it follows that there is a polynomial g in variables α_{n-1}, α_n such that $g\alpha_{n-1} = f_{n-1}$ and $g\alpha_n = f_n$. Hence $\delta_0([g]) = ([f_i])_{i=0,\ldots,n}$. Consequently, $\operatorname{Ext}^1_S(\Bbbk, S/J)_a = 0$.

Now assume that $a = r - 2$. As in the previous case, we get $[f_i] = 0$ for $i \leq n-2$. Lift $[f_{n-1}], [f_n]$ to unique representatives which are polynomials in α_{n-1}, α_n. Now from the equation

$[\alpha_n f_{n-1}] = [\alpha_{n-1} f_n]$ we deduce that there is a constant c such that

$$\alpha_n f_{n-1} = \alpha_{n-1} f_n + c\theta_r.$$

The space

$$\Theta = \{(f_{n-1}, f_n) \in (\Bbbk[\alpha_{n-1}, \alpha_n]_{r-1})^2 \mid \alpha_n f_{n-1} - \alpha_{n-1} f_n \text{ is divisible by } \theta_r\}$$

is r-dimensional. It follows that

$$\dim_{\Bbbk} \operatorname{Ext}^1_S(\Bbbk, S/J)_{r-2} = r - \dim_{\Bbbk} \delta_0\Big((S/J)_{r-2}\Big) = r - (r-1) = 1. \qquad \square$$

Finally, we can present the criterion for ideals defining subschemes contained in a line.

Theorem 3.65. *Let $[I] \in \operatorname{Hilb}^{h_{r,n}}_S$ be a closed point corresponding to an ideal I such that $S/\overline{I}$ has Hilbert function $h_{r,1}$. Then there exists $[I'] \in \operatorname{Slip}_{r,n}$ such that $I_{\geq r-2} = I'_{\geq r-2}$ if and only if $(\overline{I}^2)_{r-2} \subseteq I_{r-2}$.*

Proof. We may assume that $n \geq 2$ and $r \geq 4$ since otherwise both conditions are trivially satisfied. Indeed, if $n = 1$ or $r < 4$ then $\operatorname{Hilb}^{h_{r,n}}_S$ is irreducible (see Propositions 3.35 - 3.38) so we may take $I' = I$. On the other hand, if $n = 1$ then $I = \overline{I}$ while if $r \leq 3$ then $(\overline{I}^2)_{r-2} = 0$.

We use the notation of the beginning of this subsection with $f = h_{r,1}$. Let U be the locus of those points $[K]$ from W' that satisfy

$$(\overline{K}^2)_{r-2} \subseteq K_{r-2}.$$

In this notation, we need to prove that $\pi(\operatorname{Slip}_{r,n}) \cap W' = U$, set-theoretically.

We start with showing that $\pi(\operatorname{Slip}_{r,n}) \cap W' \subseteq U$. Let $[I''] \in \operatorname{Hilb}^{h_{r,n}}_S$ be such that $[I'' \cap \mathfrak{m}^{r-2}] \in W' \setminus U$. We will use Theorem 3.40 with $d = r-2$ and $I = I''$ to show that $[I'' \cap \mathfrak{m}^{r-2}] \notin \pi(\operatorname{Slip}_{r,n})$. By Lemma 3.56 condition 3. of Theorem 3.40 holds. Recall Notation 3.43. Let $R = S/\overline{I''}$ and let $\mathfrak{b} = ((\overline{I''})_1) \subseteq S$. Then $(0) = \mathfrak{a} = \frac{\mathfrak{b} + \overline{I''}}{\overline{I''}} \subseteq R$. In particular, $\dim_{\Bbbk} \mathfrak{a}_{r-2} = \dim_{\Bbbk} \mathfrak{a}_{r-1} = 0$. Furthermore, since $[I'' \cap \mathfrak{m}^{r-2}] \notin U$ we have $(\overline{I''})_{r-2} = I''_{r-2} + \operatorname{lin}\{F\}$ for some $F \in ((\overline{I''})^2)_{r-2} = ((\overline{I''}_1) \cdot \overline{I''})_{r-2} = (\mathfrak{b} \cdot \overline{I''})_{r-2}$. Thus, by Proposition 3.45 condition 1. of Theorem 3.40 is fulfilled. We are left with proving that $[\overline{I''} \cap \mathfrak{m}^{r-2}]$ is a smooth point of Hilb^h_S. Note that the following proof of this fact uses only the assumption that $S/\overline{I''}$ has Hilbert function $h_{r,1}$.

By Lemma 2.42 and Proposition 3.35, $[\overline{I''}] = [\overline{I''} \cap \mathfrak{m}]$ is a smooth point. Therefore, by Lemma 3.50 it is enough to show that the natural map

$$\operatorname{Hom}_S(\overline{I''} \cap \mathfrak{m}^a, S/\overline{I''})_0 \to \operatorname{Hom}_S(\overline{I''} \cap \mathfrak{m}^{a+1}, S/\overline{I''})_0$$

is bijective for every $1 \leq a < r-2$. Let $1 \leq a \leq r-3$ and $Q = (\overline{I''} \cap \mathfrak{m}^a)/(\overline{I''} \cap \mathfrak{m}^{a+1})$. Then, we have an exact sequence

$$0 \to \operatorname{Hom}_S(Q, S/\overline{I''})_0 \to \operatorname{Hom}_S(\overline{I''} \cap \mathfrak{m}^a, S/\overline{I''})_0 \to \operatorname{Hom}_S(\overline{I''} \cap \mathfrak{m}^{a+1}, S/\overline{I''})_0 \to \operatorname{Ext}^1_S(Q, S/\overline{I''})_0.$$

Since $\operatorname{Hom}_S(Q, S/\overline{I''})_0 = 0$ by Lemma 2.8, it follows from Lemma 3.64 that

$$\operatorname{Hom}_S(\overline{I''} \cap \mathfrak{m}^a, S/\overline{I''})_0 \cong \operatorname{Hom}_S(\overline{I''} \cap \mathfrak{m}^{a+1}, S/\overline{I''})_0. \tag{3.66}$$

Thus, $[\overline{I''} \cap \mathfrak{m}^{r-2}] \in \mathrm{Hilb}^h_S$ is a smooth point by Lemma 3.50. By Theorem 3.40 we conclude that $[I'' \cap \mathfrak{m}^{r-2}] \notin \pi(\mathrm{Slip}_{r,n})$. Hence

$$\pi(\mathrm{Slip}_{r,n}) \cap W' \subseteq U.$$

Now we shall show the opposite inclusion using Lemma 3.57. Assumption 1. is satisfied. Moreover, assumption 4. holds by Lemma 3.50 and Equation (3.66). We have shown above that $\pi(\mathrm{Slip}_{r,n}) \cap W' \subseteq U$. Assumption 5. is fulfilled by Lemma 3.64. In our case, V is homeomorphic to a $\mathrm{Hilb}^{h_{r,1}}_{S[\mathbb{P}^1]}$-bundle over $\mathrm{Gr}(n-1, S_1)$ by [20, Prop. 3.1]. In particular, V is irreducible of dimension $2(n-1)+r$.

We need to show that U is irreducible of dimension $rn-1$. Consider the natural map $U \to V$. The fiber over every closed point is irreducible of dimension $(n-1)(r-2)-1$. Indeed, up to a linear change of variables we may assume that we have $[\operatorname{Proj} S/\overline{I}] \in V$ with $\overline{I} = (\alpha_0, \ldots, \alpha_{n-2}, \theta_r(\alpha_{n-1}, \alpha_n))$. Then the fiber over $[\operatorname{Proj} S/\overline{I}]$ is the set of codimension one subspaces of

$$\mathrm{lin}\{\alpha_i \alpha_{n-1}^a \alpha_n^b \mid i \in \{0, \ldots, n-2\}, a+b = r-3, a, b \geq 0\}.$$

It follows from [80, 11.4.C] that U is irreducible of dimension

$$(n-1)(r-2) - 1 + 2(n-1) + r = rn - 1. \qquad \square$$

We end this subsection with two examples of applications of Theorem 3.65. In the first of them, we show that a certain point does not belong to $\mathrm{Slip}_{r,n}$.

Example 3.67. Let $S = \Bbbk[\alpha_0, \ldots, \alpha_3]$ be a polynomial ring and let

$$I'' = (\alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_0^3, \alpha_2^4).$$

Then $[I''] \in \mathrm{Hilb}^{h_{4,3}}_S$. We claim that $[I''] \notin \mathrm{Slip}_{4,3}$. Indeed, $\overline{I''} = (\alpha_0, \alpha_1, \alpha_2^4)$ but $\alpha_0^2 \in (\overline{I''}^2)_2 \setminus I''_2$. Thus, the claim follows from Theorem 3.65.

Observe that $H_{S/(I'')^2}(6) = 15 < 16$. Thus, we could have deduced that $[I''] \notin \mathrm{Slip}_{4,3}$ from Theorem 3.5. In fact, the proof of Theorem 3.65 presented in [66, Thm. 2.8] is based on the criterion from Theorem 3.5. Here we presented another proof that fits into the bigger picture (Theorem 3.40).

On the other hand $\dim_{\Bbbk} \mathrm{Hom}_S(I'' + \mathfrak{m}^d, S/(I'' + \mathfrak{m}^d))_0 \geq 12$ for every $d \geq 3$. It follows that Proposition 3.1 cannot be used to deduce that $[I''] \notin \mathrm{Slip}_{4,3}$.

Consider again ideals I from Example 3.3 and I' from Example 3.6. We have $(\overline{I}^2)_4 = (\overline{I'}^2)_4 = (\alpha_0^2)_4$. Moreover, $(\alpha_0^2)_4 \subseteq I_4$ and $(\alpha_0^2)_4 \subseteq I'_4$. Thus, Theorem 3.65 cannot be used to deduce that $[I] \notin \mathrm{Slip}_{6,2}$ or $[I'] \notin \mathrm{Slip}_{6,2}$.

The following table summarizes Examples 3.3, 3.6 and 3.67.

Ideal	Prop. 3.1	Thm. 3.5	Thm. 3.65
$(\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_1^6) \subseteq \Bbbk[\alpha_0, \alpha_1, \alpha_2]$	✓	?	?
$(\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_2^6) \subseteq \Bbbk[\alpha_0, \alpha_1, \alpha_2]$	?	✓	?
$(\alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_0^3, \alpha_2^4) \subseteq \Bbbk[\alpha_0, \ldots, \alpha_3]$	?	✓	✓

In the second example, we use Theorem 3.65 to show that a given point belongs to $\mathrm{Slip}_{r,n}$.

Example 3.68. Let $S = \Bbbk[\alpha_0, \dots, \alpha_3]$ be a polynomial ring and

$$J' = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2, \alpha_1\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_3^2, \alpha_2^4).$$

Then $[J'] \in \mathrm{Hilb}_S^{h_{4,3}}$ and we claim that $[J'] \in \mathrm{Slip}_{4,3}$. We have $\overline{J'} = (\alpha_0, \alpha_1, \alpha_2^4)$ and $(\overline{J'}^2)_2 \subseteq J_2$. Therefore, by Theorem 3.65 there exists $[K] \in \mathrm{Slip}_{4,3}$ such that $K_{\geq 2} = J'_{\geq 2}$. Since $K_1 = J'_1 = 0$, we conclude that $[J'] = [K] \in \mathrm{Slip}_{4,3}$.

3.6.5 Application two: constant growth on projective plane

In this subsection $n = 2$, so $S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$. Let $[I] \in \mathrm{Hilb}_S^{h_{r,2}}$ and let f be the Hilbert function of $S/\overline{I}$. Let $m = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq 0\}$. Assume that there exist positive integers t and $e > m$ such that f is given by

$$f(a) = \begin{cases} \dim_\Bbbk S_a & \text{for } a < m \\ r - (e + 1 - a)t & \text{for } a \in \{m, m+1, \dots, e\} \\ r & \text{for } a \geq e + 1 \end{cases} \tag{3.69}$$

Observe that for $r \geq 4$ the function $h_{r,1}$ is of the above form with $m = 1, e = r - 2$ and $t = 1$. Another example of such Hilbert function is $(1, 3, 5, 7, 9, 11, 11, \dots)$. Here $m = 2, e = 4, t = 2$.

Thus, in this subsection we consider more general Hilbert functions of $S/\overline{I}$ than in Subsection 3.6.4 but we require that $n = 2$.

The goal of this subsection is Theorem 3.74. There, we give a necessary condition for $[I]$ to be in $\mathrm{Slip}_{r,2}$.

In the following lemma, we show that $[\overline{I} \cap \mathfrak{m}^e]$ is a smooth point. Thus, we verify condition 2. from Theorem 3.40.

Lemma 3.70. *In the above notation we have*

$$\beta_{1,a}(S/\overline{I}) = 0 \text{ for } e + 2 \neq a \geq m + 2, \tag{3.71}$$

and

$$\beta_{1,e+2}(S/\overline{I}) = t. \tag{3.72}$$

As a result, $[\overline{I} \cap \mathfrak{m}^e] \in \mathrm{Hilb}_S^h$ is a smooth point where h is the Hilbert function of $S/(\overline{I} \cap \mathfrak{m}^e)$. Moreover, $\dim_\Bbbk \mathrm{Hom}_S\left(\overline{I} \cap \mathfrak{m}^e, S/(\overline{I} \cap \mathfrak{m}^e)\right)_0 = \dim W$ where $W \subseteq \mathrm{Hilb}_S^h$ is the locally closed subset whose closed points correspond to the ideals defining subschemes of $\mathbb{P}^2$ with Hilbert function f.

Proof. Recall that the Hilbert function f of $S/\overline{I}$ satisfies Equation (3.69). Let $e+2 \neq a \geq m+2$. Then, by Lemma 2.13 we obtain $\beta_{1,a}(S/\overline{I}) \leq 2f(a-1) - f(a) - f(a-2) = 0$. Similarly, we get $\beta_{1,e+2}(S/\overline{I}) \leq 2f(e+1) - f(e+2) - f(e) = t$.

We shall show that $\beta_{1,e+2}(S/\overline{I}) \geq t$. Since $\overline{I}$ is saturated, $\dim_\Bbbk \mathrm{Hom}_S(\Bbbk, S/\overline{I}) = 0$. Therefore, by Lemma 2.27 we get

$$\begin{aligned}
&- \dim_\Bbbk \mathrm{Ext}_S^1(\Bbbk, S/\overline{I})_{e-1} + \dim_\Bbbk \mathrm{Ext}_S^2(\Bbbk, S/\overline{I})_{e-1} - \dim_\Bbbk \mathrm{Ext}_S^3(\Bbbk, S/\overline{I})_{e-1} \\
&= H_{S/\overline{I}}(e-1) - 3H_{S/\overline{I}}(e) + 3H_{S/\overline{I}}(e+1) - H_{S/\overline{I}}(e+2) \\
&= (r - 2t) - 3(r - t) + 3r - r = t.
\end{aligned}$$

It follows from Lemma 3.47 that $\beta_{1,e+2}(S/\overline{I}) \geq t$.

Having calculated the Betti numbers, we proceed to proving the second part of the lemma. By Proposition 3.53 it is enough to show that

$$H_{S/\overline{I}}(a) - 3H_{S/\overline{I}}(a+1) + 3H_{S/\overline{I}}(a+2) - H_{S/\overline{I}}(a+3) = \beta_{1,a+3}(S/\overline{I}) \tag{3.73}$$

for every $a \in \{m, m+1, \dots, e-1\}$. By direct calculation, the left-hand side of Equation (3.73) equals zero for $a \in \{m, \dots, e-2\}$ and equals t for $a = e-1$. The same is true for the right-hand side by Equations (3.71) and (3.72). $\square$

We present the main result of this subsection.

Theorem 3.74. *In the notation of Theorem 3.40 assume additionally that $n = 2$. Let $m = \min\{a \in \mathbb{Z} \mid \overline{I}_a \neq 0\}$ and $e = \max\{a \in \mathbb{Z} \mid \overline{I}_a \neq I_a\}$. Assume that $e > m$ and that there exists a positive integer t such that $S/\overline{I}$ has Hilbert function f as in Equation* (3.69). *Then*

(i) There exists $\theta \in S_t$ such that $\overline{I}_e \subseteq (\theta)_e$;

(ii) Let θ be as in part (i). If $\overline{I}_e = I_e + (\theta \cdot \overline{I})_e$ then there is no $[I'] \in \mathrm{Slip}_{r,2}$ such that $I'_{\geq e} = I_{\geq e}$. In particular, $[I] \notin \mathrm{Slip}_{r,2}$;

(iii) If $t = 1$ and $\theta \in S_1$ is as in part (i), then there exists $[I'] \in \mathrm{Slip}_{r,2}$ such that $I'_{\geq e} = I_{\geq e}$ if and only if $(\theta \cdot \overline{I})_e \subseteq I_e$.

Proof. (i) By Lemma 2.9(i) there is an element $L \in S_1$ that is a non-zero divisor on $S/\overline{I}$. Moreover, we can take for L any general linear form. Let $T = S/(L)$ and

$$\mathfrak{a} = \big(\overline{I} + (L)\big)/(L) \subseteq T.$$

Then $H_{S/\overline{I}}(a) - H_{S/\overline{I}}(a-1) = H_{T/\mathfrak{a}}(a)$ for any positive integer a. In particular, $\mathfrak{a}_m \neq 0$. Therefore, $t = H_{T/\mathfrak{a}}(m+1) \leq H_{T/\mathfrak{a}}(m) \leq m$, where the first inequality follows from Lemma 2.12 and the second from the fact that $\mathfrak{a}_m \neq 0$. We get $t = H_{T/\mathfrak{a}}(e+1) = H_{T/\mathfrak{a}}(e) < e$. Hence $\mathfrak{a}$ has maximal growth in degree e. By [5, Lem. 1.4], we get that $I' = (\overline{I}_{\leq e})$ is a saturated ideal. Moreover, $\beta_{1,e+1}(S/\overline{I}) = 0$ by Lemma 3.70. Thus, S/I' has Hilbert polynomial $P(a) = at + (r - e - 1)$ by Gotzmann's persistence theorem [10, Thm. 4.3.3] applied to $T/(\mathfrak{a}_{\leq e})$. Since P is of degree 1 and its leading coefficient is t it follows from [47, Prop. I.7.6] that the subscheme of $\mathbb{P}^2$ defined by I' contains a curve of degree t. Hence there exists $\theta \in S_t$ such that $\overline{I}_e \subseteq (\theta)_e$.

(ii) We want to use Theorem 3.40 with $d = e$. Assumption 3. of the theorem is fulfilled by Lemma 3.56. Lemma 3.70 implies that assumption 2. is satisfied.

Finally, we address assumption 1. Recall Notation 3.43. Let $\mathfrak{b} = (\theta)$ and $\mathfrak{a} = \frac{(\theta)+\overline{I}}{\overline{I}} \subseteq S/\overline{I}$. We assumed that $\overline{I}_e = I_e + (\theta \cdot \overline{I})_e$. Therefore, by Proposition 3.45 it is enough to show that $\dim_{\Bbbk} \mathfrak{a}_e = \dim_{\Bbbk} \mathfrak{a}_{e+1}$. By part (i) and Equation (3.71) we get $\overline{I}_{\leq e+1} \subseteq (\theta)$. It follows that

$$\begin{aligned}
&\dim_{\Bbbk} \mathfrak{a}_{e+1} - \dim_{\Bbbk} \mathfrak{a}_e = (\dim_{\Bbbk}(\theta)_{e+1} - \dim_{\Bbbk}(\theta)_e) - (\dim_{\Bbbk} \overline{I}_{e+1} - \dim_{\Bbbk} \overline{I}_e) \\
&= (\dim_{\Bbbk} S_{e-t+1} - \dim_{\Bbbk} S_{e-t}) - \big(\dim_{\Bbbk} S_{e+1} - H_{S/\overline{I}}(e+1) - \dim_{\Bbbk} S_e + H_{S/\overline{I}}(e)\big) \\
&= (e+2-t) - (e+2-t) = 0.
\end{aligned}$$

(iii) Assume that there exists $[I'] \in \mathrm{Slip}_{r,2}$ such that $I'_{\geq e} = I_{\geq e}$. We shall show that $(\theta \cdot \overline{I})_e \subseteq I_e$. It follows form $t = 1$ that I_e is of codimension 1 in $\overline{I}_e$. Therefore, if $(\theta \cdot \overline{I})_e \not\subseteq I_e$ then $(\theta \cdot \overline{I})_e + I_e = \overline{I}_e$. Thus, we obtain a contradiction with part (ii).

We proceed to the proof of the other implication. Let h, k, V, W, W' be defined as in the beginning of Subsection 3.6.4 with $n = 2$ and f being the Hilbert function of $S/\overline{I}$.

Let U be the locus of those points $[K]$ of W' for which

$$(\theta \cdot \overline{K})_e \subseteq K_e$$

where $\theta \in S_1$ is the common divisor of $\overline{K}_e$. We need to show that $U \subseteq \pi(\mathrm{Slip}_{r,2}) \cap W'$ set-theoretically. We shall use Lemma 3.57. Assumption 1. is clear and assumption 2. follows from Proposition 3.11. Assumption 4. is a consequence of Lemma 3.70. Part (ii) implies that $\pi(\mathrm{Slip}_{r,2}) \cap W' \subseteq U$, set-theoretically. Therefore, we need to show that

(a) U is irreducible of dimension $2r - 1$;

(b) $\mathrm{Ext}^1_S(\Bbbk, S/J)_e \leq 1$ for every $[J] \in W$.

We start with (a). By [80, 11.4.C] the subset U is an irreducible subset of Hilb^k_S of dimension

$$\dim V + \Big(\dim_{\Bbbk} S_e - r - \big(\dim_{\Bbbk} S_{e-1} - (r-2) \big) \Big) = \dim V + e - 1.$$

Indeed, the fiber over a closed point $[\mathrm{Proj}\, S/I'] \in V$ of the natural map $U \to V$ is the set of those codimension 1 subspaces of $(\overline{I'})_e$ that contain $(\theta \cdot \overline{I'})_e$ where $\theta \in S_1$ is the common divisor of $(\overline{I'})_e$. Now, it suffices to establish the equality $\dim V = 2r - e$.

Let $E_f \subseteq \mathrm{Hilb}^f_S$ be the open subset whose points correspond to saturated ideals. Then E_f is homeomorphic with V by Lemma 2.29 applied to the natural map $\mathrm{Hilb}^f_S \to \mathcal{H}ilb_r(\mathbb{P}^2)$.

Let $I'' = (\alpha_0^m, \alpha_0^{m-1}\alpha_1, \ldots, \alpha_0^s\alpha_1^{m-s}, \alpha_0^{s-1}\alpha_1^{m-s+2}, \ldots, \alpha_0\alpha_1^m, \alpha_1^{e+2}) \subseteq S$ where $m + 1 - s = \dim_{\Bbbk} S_m - f(m)$. Then S/I'' has Hilbert function f and I'' is saturated. Therefore, by Propositon 3.10 and Theorem 2.74, it is enough to show that

$$\dim_{\Bbbk} \mathrm{Hom}_S(I'', S/I'')_0 = 2r - e.$$

Let $T = \Bbbk[\alpha_0, \alpha_1]$ and $\mathfrak{a} = T \cap I''$. We have

$$\dim_{\Bbbk} \mathrm{Hom}_S(I'', S/I'')_0 = \dim_{\Bbbk} \mathrm{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{\leq 0} = 2r - \dim_{\Bbbk} \mathrm{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{>0},$$

where the last equality follows from the fact that $[\mathrm{Spec}\, T/\mathfrak{a}]$ is a point of the smooth $2r$-dimensional scheme $\mathcal{H}ilb_r(\mathbb{A}^2)$ and [48, Prop. 2.3]. By Proposition 2.22 the dimension of $\mathrm{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{>0}$ can be computed from the staircase diagram of $\mathfrak{a}$.

Observe that $\mathfrak{a}$ can have minimal generators only in degrees $m, m+1, e+2$. Furthermore, $\beta_{2,a}(T/\mathfrak{a})$ can be non-zero only for $a \in \{m+1, m+2, e+3\}$. Let $A = (m+1-s) = \beta_{1,m}(T/\mathfrak{a})$. Then $\beta_{1,m+1}(T/\mathfrak{a}) = m - A$, $\beta_{2,m+1}(T/\mathfrak{a}) = A - 1$ and $\beta_{2,m+2}(T/\mathfrak{a}) = m - A$. Therefore, by Proposition 2.22 we get

$$\begin{aligned}\operatorname{Hom}_T(\mathfrak{a}, T/\mathfrak{a})_{>0} &= \beta_{1,m}(T/\mathfrak{a})(e-m+1) + \beta_{1,m+1}(T/\mathfrak{a})(e-m) \\ &\quad -\beta_{2,m+1}(T/\mathfrak{a})(e-m) - \beta_{2,m+2}(T/\mathfrak{a})(e-m-1) \\ = A(e-m+1) + (m-A)(e-m) &- [(A-1)(e-m) + (m-A)(e-m-1)] = e.\end{aligned}$$

This concludes the proof of (a).

Let $[J] \in W$. Using Lemmas 3.47 and 3.70 we get

$$\dim_{\Bbbk} \operatorname{Ext}^2_S(\Bbbk, S/J)_e = \dim_{\Bbbk} \operatorname{Ext}^2_S(\Bbbk, S/\overline{J})_e = \beta_{1,e+3}(S/\overline{J}) = 0.$$

Furthermore, Lemma 2.27 and Equation (3.69) (with $t = 1$) imply that

$$\begin{aligned}&-\dim_{\Bbbk} \operatorname{Ext}^1_S(\Bbbk, S/J)_e + \dim_{\Bbbk} \operatorname{Ext}^2_S(\Bbbk, S/J)_e - \dim_{\Bbbk} \operatorname{Ext}^3_S(\Bbbk, S/J)_e \\ &\quad = H_{S/J}(e) - 3H_{S/J}(e+1) + 3H_{S/J}(e+2) - H_{S/J}(e+3) = -1.\end{aligned}$$

It follows that $\dim_{\Bbbk} \operatorname{Ext}^1_S(\Bbbk, S/J)_e \leq 1$. Thus, (b) holds.

□

We end this subsection with two more examples. In the first, we shall use Theorem 3.74 to show that a certain point is outside of $\operatorname{Slip}_{6,2}$.

Example 3.75. Let $S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ be a polynomial ring and let

$$I''' = (\alpha_0^2\alpha_1, \alpha_0^2\alpha_2, \alpha_0\alpha_1^2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_1^5).$$

Then $[I'''] \in \operatorname{Hilb}_S^{h_{6,2}}$ and we claim that $[I'''] \notin \operatorname{Slip}_{6,2}$. We have $\overline{I'''} = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^5)$. Thus, the Hilbert function of $S/\overline{I'''}$ is as in Equation 3.69 with $r = 6, m = 2, e = 3$ and $t = 1$. We have $(\overline{I'''})_3 \subseteq (\alpha_0)_3$ but $\alpha_0^3 \in (\alpha_0 \cdot \overline{I'''})_3 \setminus I'''_3$. It follows from Theorem 3.74 that $[I'''] \notin \operatorname{Slip}_{6,2}$.

On the other hand, criteria from Proposition 3.1 and Theorem 3.5 (with $k = 2$) do not show that $[I'''] \notin \operatorname{Slip}_{6,2}$. Furthermore, criterion from Theorem 3.65 cannot be applied to $[I''']$ since $H_{S/\overline{I'''}}(1) \neq 2$.

We summarize Examples 3.3, 3.6, 3.67 and 3.75 in the following table. Here ideals from first, second and fourth rows are in $\Bbbk[\alpha_0, \alpha_1, \alpha_2]$, while ideal in the third row is in the ring $\Bbbk[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$. The symbol NA means that a certain criterion cannot be applied to a given point since the assumptions are not fulfilled. As before, the question mark means that a given necessary condition is satisfied by the given point, i.e. the criterion is inconclusive.

Ideal	Prop. 3.1	Thm. 3.5	Thm. 3.65	Thm. 3.74
$(\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_1^6)$	✓	?	?	?
$(\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_2^6)$	?	✓	?	?
$(\alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_2, \alpha_1\alpha_3, \alpha_0^3, \alpha_2^4)$	?	✓	✓	NA
$(\alpha_0^2\alpha_1, \alpha_0^2\alpha_2, \alpha_0\alpha_1^2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_1^5)$	?	?	NA	✓

In the second example, we use Theorem 3.74(iii) to deduce that a certain point belongs to the irreducible component $\operatorname{Slip}_{6,2}$.

Example 3.76. Let $S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ and $J'' = (\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^3, \alpha_1^5)$. Then $[J''] \in \mathrm{Hilb}_S^{h_{6,2}}$ and we claim that $[J''] \in \mathrm{Slip}_{6,2}$. The Hilbert function of $S/\overline{J''}$ is of the form given by Equation 3.69 with $m = 2, e = 3$, $t = 1$ and $r = 6$. Furthermore, $(\overline{J''})_{\leq 3} \subseteq (\alpha_0)_3$. We have $(\alpha_0 \cdot \overline{J''})_3 = (\alpha_0^2\alpha_1, \alpha_0^2\alpha_2)_3 \subseteq J''_3$. Therefore, by Theorem 3.74(iii), there exists $[K] \in \mathrm{Slip}_{6,2}$ with $K_{\geq 3} = J''_{\geq 3}$. Since $K_{\leq 2} = J''_{\leq 2} = 0$, we conclude that $[J''] = [K] \in \mathrm{Slip}_{6,2}$.

We summarize Examples 3.32, 3.68 and 3.76 in the following table. Here the ideal in the second row is in $\Bbbk[\alpha_0, \dots, \alpha_3]$ while the other two ideals are in $\Bbbk[\alpha_0, \alpha_1, \alpha_2]$. As before, NA means that a certain criterion cannot be applied to the given ideal.

Ideal	Theorem 3.12	Theorem 3.65	Theorem 3.74
$(\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0\alpha_2^2, \alpha_1^4)$	✓	NA	NA
$(\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2, \alpha_1\alpha_2, \alpha_0\alpha_3, \alpha_1\alpha_3^2, \alpha_2^4)$	NA	✓	NA
$(\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^3, \alpha_1^5)$	NA	NA	✓

Observe that we cannot use Theorem 3.74(iii) for J since, in the notation of that theorem, we have $m = e = 2$.

3.7 Points on projective space – examples, part II

From Fogarty's result [35] on $\mathcal{H}ilb_r(\mathbb{P}^2)$ it may seem that $\mathrm{Hilb}_{S[\mathbb{P}^2]}^{h_{r,2}}$ should be smooth, or at least not too complicated. We show that this is not the case. Speculating a bit, we may say that Fogarty's result concerns the case of codepth two, while we work in nonsaturated setting, hence in codepth three. Thus, the correct parallel would be $\mathcal{H}ilb_r(\mathbb{P}^3)$, where almost nothing is known about the principal component.

3.7.1 4 points on projective space

In this subsection, we describe the closed points of $\mathrm{Slip}_{4,n}$ for a positive integer n.

Proposition 3.77. *Let $I \subseteq S[\mathbb{P}^n]$ be a homogeneous ideal such that $S[\mathbb{P}^n]/I$ has Hilbert function $h_{4,n}$. Then $[I] \in \mathrm{Slip}_{4,n}$ if and only if $(\overline{I}^2)_2 \subseteq I_2$.*

Proof. Condition $(\overline{I}^2)_2 \subseteq I_2$ holds trivially for $[I] \in \mathrm{Hilb}_{S[\mathbb{P}^n]}^{h_{4,n}}$ if I is saturated. On the other hand, $\mathcal{H}ilb_4(\mathbb{P}^n)$ is irreducible by [20, Thm. 1.1]. Thus, by Remark 2.46, it is enough to consider ideals that are not saturated. Furthermore, by [20, Prop. 3.1], we may assume that $n \leq 3$.

If $n = 1$, then every closed point $[I] \in \mathrm{Hilb}_{S[\mathbb{P}^1]}^{h_{4,1}}$ corresponds to a saturated ideal.

If $n = 2$ and $[I] \in \mathrm{Hilb}_{S[\mathbb{P}^2]}^{h_{4,2}}$ with $I \neq \overline{I}$, then $S/\overline{I}$ has Hilbert function $h_{4,1}$. This follows from Lemma 2.9. Therefore, by Theorem 3.65

$$(\overline{I}^2)_2 \subseteq I_2 \Leftrightarrow \text{ there exists } [J] \in \mathrm{Slip}_{4,2} \text{ such that } J_{\geq 2} = I_{\geq 2} \Leftrightarrow [I] \in \mathrm{Slip}_{4,2}\,.$$

The latter equivalence follows from the fact that $J_{\leq 1} = I_{\leq 1} = 0$.

Assume that $n = 3$ and $[I] \in \mathrm{Hilb}_{S[\mathbb{P}^3]}^{h_{4,3}}$ is such that $I \neq \overline{I}$. Then $S/\overline{I}$ has Hilbert function $h_{4,2}$ or $h_{4,1}$. In the first case the condition $(\overline{I}^2)_2 \subseteq I_2$ holds. We claim that $[I] \in \mathrm{Slip}_{4,3}$. Indeed, $[\mathrm{Proj}\, S/I] \in \mathcal{H}ilb_4^{sm}(\mathbb{P}^3) = \mathcal{H}ilb_4(\mathbb{P}^3)$, so there exists an ideal $[J] \in \mathrm{Slip}_{4,3}$ with $\overline{J} = \overline{I}$. However, $J_1 = I_1 = 0$ and $J_{\geq 2} = \overline{I}_{\geq 2} = I_{\geq 2}$, so $[I] = [J] \in \mathrm{Slip}_{4,3}$.

Finally, assume that $[I] \in \mathrm{Hilb}^{h_{4,3}}_{S[\mathbb{P}^3]}$ is such that $S/\overline{I}$ has Hilbert function $h_{4,1}$. Then, as in the case $n = 2$, by Theorem 3.65 we get

$$(\overline{I}^2)_2 \subseteq I_2 \Leftrightarrow \text{ there exists } [J] \in \mathrm{Slip}_{4,3} \text{ such that } J_{\geq 2} = I_{\geq 2} \Leftrightarrow [I] \in \mathrm{Slip}_{4,3}\,. \qquad \square$$

As a corollary, we obtain an example of a reducible multigraded Hilbert scheme.

Corollary 3.78. *The scheme* $\mathrm{Hilb}^{h_{4,2}}_{S[\mathbb{P}^2]}$ *is reducible. In fact,* $[(\alpha_0\alpha_1, \alpha_0\alpha_2, \alpha_0^3, \alpha_1^4)] \notin \mathrm{Slip}_{4,2}$.

We stress the fact, that there is a point outside of $\mathrm{Slip}_{4,2}$ that corresponds to a *monomial* ideal.

Remark 3.79. The comment after [13, Cor. 6.3] puts forward a conjecture that conditions (i) and (iii) of [13, Cor. 6.3] imply condition (iv). Corollary 3.78 shows that this is not true.

3.7.2 5 points on projective plane

In this subsection, we describe the closed points of $\mathrm{Slip}_{5,2}$ (see Proposition 3.89). Let $S = S[\mathbb{P}^2] = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ and $[I]$ be a closed point of $\mathrm{Hilb}^{h_{5,2}}_S$. By Lemma 2.9, the Hilbert function of $S/\overline{I}$ is one of the three: $h_{5,2}, g, h_{5,1}$, where $g\colon \mathbb{Z} \to \mathbb{Z}$ is given by

$$g(a) = \begin{cases} \dim_{\Bbbk} S_a & \text{for } a \leq 0; \\ a + 2 & \text{for } a = 1, 2; \\ 5 & \text{for } a \geq 3; \end{cases}$$

or informally, $g = (1, 3, 4, 5, 5, \ldots)$.

We start with points corresponding to saturated ideals.

Lemma 3.80. *Let* $[I] \in \mathrm{Hilb}^{h_{5,2}}_S$ *be a closed point such that* $\overline{I} = I$*. Then* $[I] \in \mathrm{Slip}_{5,2}$.

Proof. This follows from Remark 2.46 since $\mathcal{H}ilb_5(\mathbb{P}^2)$ is irreducible. $\square$

The case when $S/\overline{I}$ has Hilbert function g is also easy.

Lemma 3.81. *Let* $[I] \in \mathrm{Hilb}^{h_{5,2}}_S$ *be a closed point such that* $S/\overline{I}$ *has Hilbert function* g*. Then* $[I] \in \mathrm{Slip}_{5,2}$.

Proof. Observe that $g(a) \neq h_{5,2}(a)$ if and only if $a = 2$. Thus, the claim follows from Theorem 3.12. $\square$

Finally, we study those points $[I] \in \mathrm{Hilb}^{h_{5,2}}_S$ for which $S/\overline{I}$ has Hilbert function $h_{5,1}$. We introduce some more notation. Let $h\colon \mathbb{Z} \to \mathbb{Z}$ be given by $h = (1, 3, 6, 5, 5, \ldots)$ or more formally,

$$h(a) = \begin{cases} \dim_{\Bbbk} S_a & \text{for } a \leq 2; \\ 5 & \text{for } a \geq 3. \end{cases}$$

Let $\pi\colon \mathrm{Hilb}^{h_{5,2}}_S \to \mathrm{Hilb}^{h}_S$ and $\pi'\colon \mathrm{Hilb}^{h}_S \to \mathcal{H}ilb_5(\mathbb{P}^2)$ be the natural morphisms. Let $V \subseteq \mathcal{H}ilb_r(\mathbb{P}^2)$ be the closed subset whose closed points correspond to the subschemes with Hilbert function $h_{5,1}$.

Let $W \subseteq \mathrm{Hilb}_S^h$ be the set-theoretic inverse image of V under π' and let U be the set-theoretic inverse image of W under π. We show that these subsets are irreducible and we calculate their dimensions. We start with V, but we state it in greater generality since we need this also in Subsection 3.7.3.

Lemma 3.82. *Let r be a positive integer. Let V be the closed subset of $\mathcal{H}ilb_r(\mathbb{P}^2)$, whose closed points correspond to subschemes with Hilbert function $h_{r,1}$. Then V is irreducible and $r+2$ dimensional.*

Proof. The scheme $\mathrm{Hilb}_S^{h_{r,1}}$ is irreducible, smooth and $r+2$-dimensional by Proposition 3.35 and [20, Prop. 3.1]. Therefore, the natural morphism $\mathrm{Hilb}_S^{h_{r,1}} \to \mathcal{H}ilb_r(\mathbb{P}^2)$ factors through V (with reduced subscheme structure). It follows from Lemma 2.29 that V is homeomorphic to $\mathrm{Hilb}_S^{h_{r,1}}$. In particular, it is irreducible and $r+2$-dimensional. □

Lemma 3.83. *The subset $U \subseteq \mathrm{Hilb}_S^{h_{5,2}}$ is irreducible and 11-dimensional.*

Proof. By Lemma 3.82, the locus V is irreducible and 7-dimensional. Thus, in order to show that U is irreducible and 11-dimensional, it is enough to show that the fiber of $\pi' \circ \pi$ over every closed point of V is irreducible and 4-dimensional (see [80, 11.4.C]). Choose a closed point $[\mathrm{Proj}\, S/\overline{I}] \in V$. Denote the fiber of $\pi' \circ \pi$ over this point by X. Then we have a pullback diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathrm{Fl}(3,5,\overline{I}_3) \\ \downarrow & & \downarrow \\ \mathrm{Gr}(1,\overline{I}_2) & \longrightarrow & \mathrm{Gr}(3,\overline{I}_3), \end{array}$$

where $\mathrm{Fl}(3,5,\overline{I}_3)$ is the flag variety parametrizing pairs of linear subspaces $A_3 \subseteq A_5$ of $\overline{I}_3$ such that $\dim A_i = i$ for $i = 3,5$. The lower horizontal morphism maps $\mathrm{lin}\{\ell\}$ to $\mathrm{lin}\{\alpha_0\ell, \alpha_1\ell, \alpha_2\ell\}$. Since the fibers of the right vertical map are irreducible and of dimension 2, it follows that X is irreducible and of dimension $\dim X = \dim \mathrm{Gr}(1,\overline{I}_2) + 2 = 4$. □

Let U' be the subset of U whose closed points $[I]$ satisfy $(\overline{I}^2)_3 \subseteq I_3$.

Lemma 3.84. *The subset U' of U is closed.*

Proof. Consider the diagram

$$\begin{array}{ccccc} & & & & \mathrm{Hilb}_S^{h_{5,2}} \\ & & & & \downarrow{\scriptstyle d} \\ & & \mathrm{Fl}(3,5,S_3) & \xrightarrow{c} & \mathrm{Gr}(5,S_3) \\ & & \downarrow{\scriptstyle b} & & \\ \mathrm{Gr}(1,S_1) & \xrightarrow{a} & \mathrm{Gr}(3,S_3) & & , \end{array}$$

where $a(\mathrm{lin}(\ell)) = \mathrm{lin}\{\alpha_0\ell^2, \alpha_1\ell^2, \alpha_2\ell^2\}$ and b,c,d are the natural maps. Then $U' = U \cap d^{-1}(c(b^{-1}(a(\mathrm{Gr}(1,S_1)))))$, so it is closed. □

Let $W' \subseteq \mathrm{Hilb}_S^h$ be the set-theoretic image $\pi(U')$. It is closed by Lemma 3.84 since π is a morphism of projective schemes by Theorem 2.36.

Lemma 3.85. *The subset $W' \subseteq \mathrm{Hilb}_S^h$ is irreducible and 9-dimensional.*

Proof. Consider the natural map $W' \to V$. By Lemma 3.82 and [80, 11.4.C], it is enough to show that the fibers are irreducible and 2-dimensional. Let $[\operatorname{Proj} S/\overline{I}] \in V$. We may assume that $\overline{I} = \big(\alpha_0, \theta_5(\alpha_1, \alpha_2)\big)$. We have $(\alpha_0^2)_3 \subseteq I_3$ for every $[I] \in W'$ that is in the fiber over $[\operatorname{Proj} S/\overline{I}]$. Thus, the fiber of $W' \to V$ is $\operatorname{Gr}(2, \operatorname{lin}\{\alpha_0\alpha_1^2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^2\})$. $\square$

Let
$$W_i' = \{[K] \in W' \mid \dim_{\Bbbk}(K \colon \mathfrak{m})_2 = i\}$$
for $i = 1, 2$. Observe that $W' = W_1' \cup W_2'$. Indeed, let $[K] \in W'$ and assume that $\overline{K}_1 = (\alpha_0)_1$. Then $\alpha_0^2 \in (K \colon \mathfrak{m})_2$ by the definition of W'. On the other hand, if $(K \colon \mathfrak{m})_2 \geq 3$ then $(\alpha_0)_2 \subseteq (K \colon \mathfrak{m})_2$. It follows that $(\alpha_0)_3 \subseteq K_3$. This contradicts the assumption that $H_{S/K}(3) = 5$.

Lemma 3.86. *The closed subset W_2' of W' is irreducible and 8-dimensional.*

Proof. We have a natural map $W_2' \to V$. By Lemma 3.82 and [80, 11.4.C], it is enough to show that its fibers are irreducible and 1-dimensional. Consider the point $[\operatorname{Proj} S/(\alpha_0, \theta_5(\alpha_1, \alpha_2))] \in V$. Fiber over this point is

$$\{[K_3] \in \operatorname{Gr}(5, (\alpha_0)_3) \mid (\alpha_0^2)_3 \subseteq K_3 \text{ and } \dim_{\Bbbk}((K_3) \colon \mathfrak{m})_2 = 2\}.$$

We have $\alpha_0^2 \in ((K_3) \colon \mathfrak{m})_2$ for every $[K_3]$ in the fiber. Therefore, the fiber is $\mathbb{P}^1$ corresponding to the choice of $[\ell] \in \mathbb{P}(\operatorname{lin}\{\alpha_1, \alpha_2\})$ such that $K_3 = \operatorname{lin}\{\alpha_0^3, \alpha_0^2\alpha_1, \alpha_0^2\alpha_2, \alpha_0\alpha_1\ell, \alpha_0\alpha_2\ell\}$. $\square$

Let
$$Z_1 = \{[I] \in U' \mid \dim_{\Bbbk}(I_{\geq 3} \colon \mathfrak{m})_2 = 1\}$$
and
$$Z_2 = \{[I] \in U' \mid \dim_{\Bbbk}(I_{\geq 3} \colon \mathfrak{m})_2 = 2\}.$$

Lemma 3.87. *The closed subsets $\overline{Z_1}, Z_2$ are irreducible and 9-dimensional. Moreover, $U' = Z_1 \cup Z_2$ set-theoretically.*

Proof. By definition we have $Z_i = \pi^{-1}(W_i')$ set-theoretically. Moreover, Z_1 is homeomorphic to W_1'. We claim that the fiber of $Z_2 \to W_2'$ over every closed point is irreducible and 1-dimensional. Indeed, the fiber over $[K]$ is $\mathbb{P}^1$ corresponding to the choice of a non-zero element of $(K \colon \mathfrak{m})_2$.

Therefore, $\overline{Z_1}, Z_2$ are irreducible and 9-dimensional by Lemmas 3.85, 3.86 and [80, 11.4.C]. $\square$

Now we can describe the set-theoretic intersection of $\operatorname{Slip}_{5,2}$ with U.

Lemma 3.88. *Set-theoretically we have $U \cap \operatorname{Slip}_{5,2} = Z_1 \cup Z_2$.*

Proof. Containment $U \cap \operatorname{Slip}_{5,2} \subseteq Z_1 \cup Z_2$ follows from Theorem 3.65. Moreover, for every $[I] \in Z_1$, the only ideal $[J] \in \operatorname{Hilb}_S^{h_{5,2}}$ with $J_{\geq 3} = I_{\geq 3}$ is I. Thus, $Z_1 \subseteq \operatorname{Slip}_{5,2}$ by Theorem 3.65.

We shall show that $Z_2 \subseteq \operatorname{Slip}_{5,2}$. Let $I = (\alpha_0\alpha_1, \alpha_0^3, \alpha_0^2\alpha_2, \alpha_0\alpha_2^3 + \alpha_1^4)$. We have $[I] \in \operatorname{Hilb}_S^{h_{5,2}}$. Moreover, the Hilbert function of $S/\overline{I}$ is g. Thus, $[I] \in \operatorname{Slip}_{5,2}$ by Lemma 3.81. Consider the initial ideal I' of I with respect to the grevlex order with $\alpha_1 < \alpha_2 < \alpha_0$. Then $I' = (\alpha_0\alpha_1, \alpha_0^3, \alpha_0^2\alpha_2, \alpha_0\alpha_2^3, \alpha_1^5)$. We have $[I'] \in \operatorname{Slip}_{5,2} \cap Z_2$. Furthermore, $\dim_{\Bbbk} \operatorname{Hom}_S(I', S/I')_0 = 12 = \dim U + 1$ (see Lemma 3.83). It follows from Theorem 2.74 and Lemma 2.30 that every irreducible component of $\operatorname{Slip}_{5,2} \cap U$ passing through $[I']$ is at least 9-dimensional. This intersection is contained in $Z_1 \cup Z_2$ and this is a union of two irreducible 9-dimensional subsets. Hence it suffices to show that $[I'] \notin \overline{Z_1}$. Consider the projection $\mathrm{Hilb}_S^{h_{5,2}} \to \mathrm{Gr}(1, S_2)$. Then the image of Z_1 is

$$\{\mathrm{lin}\{\ell^2\} \mid \ell \in S_1 \setminus \{0\}\},$$

i.e. it is the image of the second Veronese embedding of $\mathbb{P}S_1$. On the other hand, $\alpha_0\alpha_1 \in I'_2$ is not a power of a linear form. Thus, $[I'] \notin \overline{Z_1}$. $\square$

We summarize the above results in the following proposition which describes $\mathrm{Slip}_{5,2}$.

Proposition 3.89. *Let $S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ and $[I] \in \mathrm{Hilb}_S^{h_{5,2}}$ be a closed point. Then $[I] \in \mathrm{Slip}_{5,2}$ if and only if $(\overline{I}^2)_3 \subseteq I_3$.*

Proof. If $S/\overline{I}$ has Hilbert function $h_{5,2}$ or g, then $[I] \in \mathrm{Slip}_{5,2}$ by Lemma 3.80 or Lemma 3.81. On the other hand, in both this cases $(\overline{I}^2)_3 \subseteq I_3$ holds since $(\overline{I}^2)_3 = 0$.

Assume that $S/\overline{I}$ has Hilbert function $h_{5,1}$. Then $(\overline{I}^2)_3 \subseteq I_3$ if and only if $[I] \in Z_1 \cup Z_2$ (see Lemma 3.87). This is equivalent to $[I] \in \mathrm{Slip}_{5,2}$ by Lemma 3.88. $\square$

3.7.3 6 points on projective plane

The main result of this subsection is the description of $\mathrm{Slip}_{6,2}$ (see Proposition 3.105). Let $S = \Bbbk[\alpha_0, \alpha_1, \alpha_2]$ and let $[I] \in \mathrm{Hilb}_S^{h_{6,2}}$ be a closed point. Then the Hilbert function of $S/\overline{I}$ is one of the four: $h_{6,2}, f, g, h_{6,1}$ where $f \colon \mathbb{Z} \to \mathbb{Z}$ is given by

$$f(a) = \begin{cases} 0 & \text{for } a < 0; \\ 2a+1 & \text{for } a \in \{0,1,2\}; \\ 6 & \text{for } a \geq 3 \end{cases}$$

(or, in a brief form, $f = (1, 3, 5, 6, 6, \ldots)$) and $g \colon \mathbb{Z} \to \mathbb{Z}$ is given by

$$g(a) = \begin{cases} \dim_{\Bbbk} S_a & \text{for } a \leq 1; \\ a+2 & \text{for } a \in \{2,3\}; \\ 6 & \text{for } a \geq 4 \end{cases}$$

(or, $g = (1, 3, 4, 5, 6, 6, \ldots)$).

We start with the points corresponding to saturated ideals.

Lemma 3.90. *Let $[I] \in \mathrm{Hilb}_S^{h_{6,2}}$ be a closed point such that $I = \overline{I}$. Then $[I] \in \mathrm{Slip}_{6,2}$.*

Proof. This follows from Remark 2.46. $\square$

The case when $S/\overline{I}$ has Hilbert function f is also simple.

Lemma 3.91. *Let $[I] \in \mathrm{Hilb}_S^{h_{6,2}}$ be a closed point such that $S/\overline{I}$ has Hilbert function f. Then $[I] \in \mathrm{Slip}_{6,2}$.*

Proof. Observe that $f(a) \neq h_{6,2}(a)$ if and only if $a = 2$. Thus, the claim follows from Theorem 3.12. $\square$

Next we consider points $[I]$ corresponding to ideals such that $S/\overline{I}$ has Hilbert function g.

Lemma 3.92. *Let $[I] \in \mathrm{Hilb}_S^{h_{6,2}}$ be a closed point such that $S/\overline{I}$ has Hilbert function g. Then there is a linear form $\theta \in S_1$ such that $\overline{I}_3 \subseteq (\theta)_3$. We have $[I] \in \mathrm{Slip}_{6,2}$ if and only if $(\theta \cdot \overline{I})_3 \subseteq I_3$.*

Proof. The existence of θ as in the statement follows from Theorem 3.74(i). Moreover, by Theorem 3.74(iii) there exists $[J] \in \mathrm{Slip}_{6,2}$ such that $J_{\geq 3} = I_{\geq 3}$ if and only if $(\theta \cdot \overline{I})_3 \subseteq I_3$. We claim that necessarily $[J] = [I]$.

We have $h_{6,2}(a) = \dim_{\Bbbk} S_a$ for every $a \leq 2$. Therefore, if $[J] \in \mathrm{Hilb}_S^{h_{6,2}}$ is such that $J_{\geq 3} = I_{\geq 3}$, then $J = I$. □

In order to study the case when $S/\overline{I}$ has Hilbert function $h_{6,1}$, we introduce some notation. Let U_g be a locally closed subset of $\mathrm{Hilb}_S^{h_{6,2}}$, whose closed points correspond to ideals I for which $S/\overline{I}$ has Hilbert function g. Similarly, let U be the closed subset corresponding to Hilbert function $h_{6,1}$. Let $h\colon \mathbb{Z} \to \mathbb{Z}$ be given by $h = (1, 3, 6, 10, 6, 6, \ldots)$, or more formally,

$$h(a) = \begin{cases} \dim_{\Bbbk} S_a & \text{for } a \leq 3; \\ 6 & \text{for } a > 3. \end{cases}$$

We have the natural morphisms $\pi\colon \mathrm{Hilb}_S^{h_{6,2}} \to \mathrm{Hilb}_S^{h}$ and $\pi'\colon \mathrm{Hilb}_S^{h} \to \mathcal{H}ilb_6(\mathbb{P}^2)$.

We start with showing that U_g is irreducible and we compute its dimension.

Lemma 3.93. *The locus U_g is irreducible and 13-dimensional.*

Proof. Consider the locally closed subset V_g (with reduced subscheme structure) of $\mathcal{H}ilb_6(\mathbb{P}^2)$ whose closed points correspond to subschemes with Hilbert function g. This locus is irreducible by Proposition 3.11. We claim that V_g has dimension 9. Let $\pi_g\colon \mathrm{Hilb}_S^{g} \to \mathcal{H}ilb_6(\mathbb{P}^2)$ be the natural map. Let $E_g \subseteq \mathrm{Hilb}_S^{g}$ be the open subset of points corresponding to saturated ideals. Then E_g is smooth by Proposition 3.10. Therefore, $\pi_g\colon E_g \to \mathcal{H}ilb_6(\mathbb{P}^2)$ factors through V_g. It follows from Lemma 2.29 that V_g is homeomorphic to E_g. Consider the point $[I] \in E_g$ with $I = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^5)$. We have $\dim_{\Bbbk} \mathrm{Hom}_S(I, S/I)_0 = 9$. Therefore, by Theorem 2.74 we have $\dim_{\Bbbk} \mathbf{T}_{[I]} \mathrm{Hilb}_S^{g} = 9$. Thus, $\dim V_g = \dim E_g = 9$ by Proposition 3.10.

By definition, U_g is the inverse image of V_g under $\pi' \circ \pi$. By [80, 11.4.C], it is enough to show that the fiber over each point is irreducible and 4-dimensional. This follows from Lemma 2.43. □

Let $U_i = \{[I] \in U \mid \dim_{\Bbbk}(I_{\geq 4} \colon \mathfrak{m})_3 = i\}$. We claim that $U = U_4 \cup U_5$. Indeed, let $[I] \in U$ and denote $(I_{\geq 4} \colon \mathfrak{m})$ by K. We have $I_3 \subseteq K_3 \subsetneq \overline{I}_3$. The latter inclusion is proper since otherwise we obtain $\overline{I}_4 \subseteq I_4$ which contradicts the assumption $H_{S/I}(4) = 6$.

We claim that U_4, U_5 are irreducible. In order to prove this, we introduce more notation. Let $V \subseteq \mathcal{H}ilb_6(\mathbb{P}^2)$ be the closed subset of points corresponding to the subschemes with Hilbert function $h_{6,1}$. Then U is the set-theoretic inverse image of V under $\pi' \circ \pi$. Let $W \subseteq \mathrm{Hilb}_S^{h}$ be the set-theoretic inverse image of V under π'.

For an integer $0 \leq i \leq 10$, let $W_i = \{[K] \in W \mid \dim_{\Bbbk}(K \colon \mathfrak{m})_3 = i\}$ and $W_{\geq i} = \bigcup_{j \geq i} W_j$. Observe that $W_{\geq i}$ is closed for every i. Furthermore, $W_5 = W_{\geq 5} \cap \pi(U)$ and $W_4 \cup W_5 = W_{\geq 4} \cap \pi(U)$ are closed in Hilb_S^{h}.

We shall show that W_4, W_5 are irreducible. Moreover, we compute their dimensions. We have natural maps $W_5 \to V$ and $W_4 \cup W_5 \to V$ and we want to study their fibers over a closed point $[\mathrm{Proj}\, S/\overline{I}] \in V$.

Recall the notion of the dual ring from Subsection 2.4.1.

Lemma 3.94. *Let $\overline{I} = (\alpha_0, \theta_6(\alpha_1, \alpha_2))$ for some $\theta_6 \in \Bbbk[\alpha_1, \alpha_2]_6 \setminus \{0\}$. Then the fiber of $\pi'\colon \mathrm{Hilb}^h_S \to \mathcal{H}ilb_6(\mathbb{P}^2)$ over $[\operatorname{Proj} S/\overline{I}]$ is isomorphic with $\mathbb{P}^9$. Moreover, the points of the fiber are in the natural correspondence with points $[F]$ in $\mathbb{P}S^*_3$ where $S^* = \Bbbk_{dp}[x_0, x_1, x_2]$ is the dual ring of S.*

Proof. By Lemma 2.43 this fiber is isomorphic to the Grassmannian of codimension 1 subspaces of $\overline{I}_4$. Dually, a point of the fiber corresponds to a choice of an element $[G] \in \mathbb{P}(S^*_4/(\overline{I}_4)^{\perp})$ where $(\overline{I}_4)^{\perp}$ is the set of element of S^*_4 which are annihilated by $\overline{I}_4 = (\alpha_0)_4$. Therefore, we may take $[G] = [x_0 F]$ for some $F \in S^*_3$. □

Given $F \in S^*_3$ we denote by $\mathcal{C}at_F(1, 2; 3)$ the catalecticant matrix (see [55, §1.1]). This is a 3×6-matrix of coefficients of $\alpha_i \lrcorner F$ (for $i = 0, 1, 2$) in the basis of S^*_2 given by divided power monomials.

In the notation of Lemma 3.94, the condition that a point of the fiber corresponding to $[F] \in \mathbb{P}S^*_3$ is in W_s (for $s \in \{4, 5\}$) is equivalent to the condition that

$$\dim_{\Bbbk} \operatorname{lin}\{\alpha_0 \lrcorner x_0 F, \alpha_1 \lrcorner x_0 F, \alpha_2 \lrcorner x_0 F, x_1^{[3]}, x_1^{[2]} x_2, x_1 x_2^{[2]}, x_2^{[3]}\} = 10 - s.$$

Moreover, $\alpha_i \lrcorner (x_0 F) = x_0(\alpha_i \lrcorner F)$ for $i = 0, 1, 2$. Therefore, the point of the fiber corresponding to $[F] \in \mathbb{P}S^*_3$ is in W_s if and only if the catalecticant matrix $\mathcal{C}at_F(1, 2; 3)$ has rank $6 - s$, that is either 2 or 1.

Lemma 3.95. *The locus W_5 is irreducible and* 10*-dimensional.*

Proof. The locus V is irreducible and 8-dimensional by Lemma 3.82. Since W_5 is closed in Hilb^h_S, it is enough to show that points of the fiber of $\pi'\colon \mathrm{Hilb}^h_S \to \mathcal{H}ilb_6(\mathbb{P}^2)$ belonging to W_5 form an irreducible subset of dimension 2 (see [80, 11.4.C]). In fact, we claim that the locus of these points inside $\mathbb{P}S^*_3$ coincides with $\nu_3(\mathbb{P}S^*_1)$. This follows from [71, Cor. 3.5] since this locus is given by the ideal generated by the 2×2-minors of the generic catalecticant matrix $\mathcal{C}at(1, 2; 3)$. □

The case of W_4 is analogous.

Lemma 3.96. *The locus W_4 is irreducible and* 13*-dimensional.*

Proof. The locus V is irreducible and 8-dimensional by Lemma 3.82. It is enough to show that the closed subset $W_4 \cup W_5 \subseteq \mathrm{Hilb}^h_S$ is irreducible and 13-dimensional. Thus, by [80, 11.4.C] it suffices to show that the fiber over every closed point is irreducible and 5-dimensional. The fiber is given by the ideal generated by the 3×3-minors of the generic catalecticant matrix $\mathcal{C}at(1, 2; 3)$. This coincides set-theoretically with the 2-nd secant variety $\sigma_2(\nu_3(\mathbb{P}S^*_1))$ by [55, Thm. 4.5A]. Furthermore, it is irreducible and 5-dimensional by [55, Prop. 1.23]. □

Remark 3.97. Observe that if $\operatorname{char} \Bbbk = 0$, then in fact in the proof of Lemma 3.96 the set-theoretical equality of the fiber with $\sigma_2(\nu_d(\mathbb{P}S^*_1))$ can be strengthened to the equality of their defining ideals. See [72].

Now we show that U_4, U_5 are irreducible and we compute their dimensions.

Lemma 3.98. *In the above notation, U_4 and U_5 are irreducible. Moreover, $\dim U_4 = 13$ and $\dim U_5 = 14$.*

Proof. Let $i \in \{4,5\}$, $[I] \in U_i$ and $[K] = \pi([I]) \in W_i$. Then I and K differ only in degree 3. Furthermore, $\dim_{\Bbbk} I_3 = 4$ and $I_3 \subseteq (K : \mathfrak{m})_3$.

Therefore, the natural map $U_4 \to W_4$ is bijective on closed points. Thus, U_4 is irreducible and 13-dimensional by Lemmas 2.29 and 3.96.

Let $[K] \in W_5$. The fiber of the map $U_5 \to W_5$ over $[K]$ is irreducible and 4-dimensional. Indeed, it corresponds to the choice of a 4-dimensional subspace of the 5-dimensional linear space $(K : \mathfrak{m})_3$. Thus, W_5 is irreducible and of dimension 14 by Lemma 3.95 and [80, 11.4.C]. □

Let $U_i' = \{[I] \in U_i \mid (\overline{I}^2)_4 \subseteq I_4\}$ for $i = 4, 5$.

Lemma 3.99. *Let $[I] \in U_i$ for $i = 4$ or $i = 5$. Then, there is a point $[J] \in \mathrm{Slip}_{6,2}$ such that $I_{\geq 4} = J_{\geq 4}$ if and only if $[I] \in U_i'$. In particular, $U_4 \cap \mathrm{Slip}_{6,2} = U_4'$, set-theoretically.*

Proof. The first part of the lemma follows from Theorem 3.65. If $[I] \in U_4'$ and $[J] \in \mathrm{Hilb}_S^{h_{6,2}}$ are such that $I_{\geq 4} = J_{\geq 4}$ then we claim that $I = J$. Since $I_{\leq 2} = J_{\leq 2} = 0$ it is enough to show that $I_3 = J_3$. However, by the definition of U_4 we get

$$I_3 = (I_{\geq 4} : \mathfrak{m})_3 = (J_{\geq 4} : \mathfrak{m}) = J_3.$$

Thus, $U_4' \subseteq \mathrm{Slip}_{6,2}$, set-theoretically. □

Let $U_5'' = \{[I] \in U_5' \mid (\overline{I} \cdot \overline{(I_{\leq 4})})_3 \subseteq I_3\}$. We claim that $U_5' \cap \mathrm{Slip}_{6,2} = U_5''$ set-theoretically. We start with describing $\overline{(I_{\leq 4})}$ for $[I] \in U_5'$.

Lemma 3.100. *Let $[I] \in U_5'$. Then, up to a linear change of variables, $\overline{(I_{\leq 4})} = (\alpha_0^2, \alpha_0\alpha_1)$.*

Proof. Up to a linear change of variables, we may assume that $\overline{I} = (\alpha_0, \theta_6(\alpha_1, \alpha_2))$ for some non-zero $\theta \in \Bbbk[\alpha_1, \alpha_2]_6$. Let $[F] \in \mathbb{P}S_3^*$ be the point corresponding to the point $[I_{\geq 4}]$ in the fiber of $\pi' \colon \mathrm{Hilb}_S^h \to \mathcal{H}ilb_6(\mathbb{P}^2)$ over $[\mathrm{Proj}\, S/\overline{I}]$ (see Lemma 3.94).

Since $[I] \in U_5'$ we have $\dim_{\Bbbk}(I_{\geq 4} : \mathfrak{m})_3 = 5$ and $[F] \in \mathbb{P}(\Bbbk_{dp}[x_1, x_2]_3)$. As a result,

$$\dim_{\Bbbk} \mathrm{lin}\{\alpha_1 \lrcorner F, \alpha_2 \lrcorner F\} = 1.$$

Therefore, we may assume by a linear change of variables in $\Bbbk[\alpha_1, \alpha_2]$ that $F = x_2^{[3]}$.

Then,

$$I_4 = \mathrm{Ann}(\mathrm{lin}\{x_0 x_2^{[3]}, x_1^{[4]}, x_1^{[3]} x_2, x_1^{[2]} x_2^{[2]}, x_1 x_2^{[3]}, x_2^{[4]}\})_4.$$

It follows that $(\alpha_0^2, \alpha_0\alpha_1) \subseteq \overline{(I_{\leq 4})}$. On the other hand, $\overline{(I_{\leq 4})} \subseteq (\alpha_0)$. Thus, if $(\alpha_0^2, \alpha_0\alpha_1) \subsetneq \overline{(I_{\leq 4})}$ then $\alpha_0\alpha_2^N \in \overline{(I_{\leq 4})}$ for some N. This is impossible since $\alpha_0\alpha_2^N \notin (I_{\leq 4})$ for every positive integer N. □

Let $W_5' = \{[I] \in W_5 \mid (\overline{I}^2)_4 \subseteq I_4\}$.

Lemma 3.101. *In the above notation, W_5' is irreducible and 9-dimensional.*

Proof. Consider the natural morphism $W_5' \to V$. The target is irreducible and 8-dimensional by Lemma 3.82. By [80, 11.4.C], it is enough to show that the fibers are irreducible and 1-dimensional. Let $T^* = \Bbbk_{dp}[x_1, x_2] \subseteq S^* = \Bbbk_{dp}[x_0, x_1, x_2]$. In the notation of Lemma 3.94 we now have $[F] \in \mathbb{P}T_3^* \subseteq \mathbb{P}S_3^*$ since we have to choose a codimension one subspace of $(\alpha_0)_4$ containing $(\alpha_0^2)_4$. Therefore, the fiber inside $\mathbb{P}T_3^*$ is set-theoretically given by the vanishing of 2×2-minors of the generic catalecticant matrix $\mathcal{C}at(1, 2; 2)$. Thus, it is $\nu_3(\mathbb{P}T_1^*)$ by [71, Cor. 3.5]. □

Now we show that U_5'' is irreducible and we compute its dimension.

Lemma 3.102. *In the above notation, U_5'' is irreducible and 11-dimensional.*

Proof. Consider the natural map $U_5'' \to W_5'$. Let $[K] \in W_5'$ be a closed point with $\overline{(K_{\leq 4})} = (\alpha_0^2, \alpha_0\alpha_1)$ (see Lemma 3.100). Then the fiber over $[K]$ is isomorphic to $\mathbb{P}^2$ corresponding to a choice of a two-dimensional subspace of $\operatorname{lin}\{\alpha_0\alpha_1^2, \alpha_0\alpha_2^2, \alpha_0\alpha_1\alpha_2\}$. Thus, U_5'' is irreducible and 11 dimensional by Lemma 3.101 and [80, 11.4.C] ☐

The key technical step is the following lemma.

Lemma 3.103. *The locus $U_5' \setminus U_5''$ is disjoint from* $\mathrm{Slip}_{6,2}$.

We defer the proof of this lemma until the end of the subsection. We use it to describe the set-theoretic intersection $U_5 \cap \mathrm{Slip}_{6,2}$.

Lemma 3.104. *In the above notation, $U_5 \cap \mathrm{Slip}_{6,2} = U_5''$ set-theoretically.*

Proof. Consider the point $[I] \in \mathrm{Hilb}_S^{h_{6,2}}$ where $I = (\alpha_0^3, \alpha_0^2\alpha_1, \alpha_0\alpha_1^2 + \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \alpha_2^6)$. Then $[I] \in U_5''$. We claim that it also belongs to $\mathrm{Slip}_{6,2}$. Let $J = (\alpha_0\alpha_1\alpha_2 + \alpha_2^3, \alpha_0\alpha_1^2 + \alpha_0^2\alpha_2 + \alpha_1\alpha_2^2, \alpha_0^2\alpha_1 + \alpha_0\alpha_2^2, \alpha_0^3)$. Then $[J] \in \mathrm{Hilb}_S^{h_{6,2}}$ and J is a saturated ideal. It follows that $[J] \in \mathrm{Slip}_{6,2}$ by Lemma 3.90. The initial ideal of J with respect to the weight vector $(3, 2, 1)$ is I.

Now we use Lemma 2.30 to conclude that $U_5 \cap \mathrm{Slip}_{6,2} = U_5''$ set-theoretically. Indeed, we have $U_5 \cap \mathrm{Slip}_{6,2} \subseteq U_5''$ by Lemmas 3.99 and 3.103. Moreover, $[I] \in U_5 \cap \mathrm{Slip}_{6,2}$ and $\dim_{\Bbbk} \mathrm{Hom}_S(I, S/I)_0 = 15 = \dim U_5 + 1$. It follows from Theorem 2.74 and Lemma 2.30 that the intersection contains an irreducible subset of dimension 11. Therefore, the intersection is U_5'' by Lemma 3.102. ☐

The following proposition summarizes the above considerations.

Proposition 3.105. *Let $[I] \in \mathrm{Hilb}_S^{h_{6,2}}$. Then $[I] \in \mathrm{Slip}_{6,2}$ if and only if one of the following holds:*

1. *The ideal I is saturated.*

2. *The algebra $S/\overline{I}$ has Hilbert function $f = (1, 3, 5, 6, 6, \ldots)$.*

3. *The algebra $S/\overline{I}$ has Hilbert function $g = (1, 3, 4, 5, 6, 6, \ldots)$ and $(\theta \cdot \overline{I})_3 \subseteq I_3$ where θ is the common linear divisor of two quadratic generators of $\overline{I}$.*

4. *The algebra $S/\overline{I}$ has Hilbert function $h_{6,1} = (1, 2, 3, 4, 5, 6, 6, \ldots)$ and*

$$((\overline{I_{\leq d+1}}) \cdot \overline{I})_d \subseteq I_d$$

for $d = 3$ and $d = 4$.

Proof. The cases 1,2,3 follow from Lemmas 3.90, 3.91 and 3.92.

Observe that if $[I] \in U$ then $I_5 = (\overline{I}_1)_5 = \overline{I}_5$ and therefore

$$(\overline{I}^2)_4 = ((\overline{I_{\leq 5}}) \cdot \overline{I})_4.$$

Hence in case 4, if $[I] \in U_5$ the claim follows from Lemma 3.104.

Assume that $[I] \in U_4$. By Lemma 3.99 we have $[I] \in \mathrm{Slip}_{6,2}$ if and only if

$$(\overline{I}^2)_4 = ((\overline{(I_{\leq 5})} \cdot \overline{I})_4 \subseteq I_4. \tag{3.106}$$

We need to show that if $[I] \in U'_4$, then $((\overline{(I_{\leq 4})} \cdot \overline{I})_3 \subseteq I_3$. Let $f \in ((\overline{(I_{\leq 4})} \cdot \overline{I})_3$ and $i \in \{0, 1, 2\}$. Then $\alpha_i f \in (\overline{I}^2)_4$. From Equation (3.106) we get $\alpha_i f \in I_4$. It follows that $f \in (I_{\geq 4} : \mathfrak{m})_3 = I_3$ where the last equality follows from the definition of U_4. $\square$

We are left with proving Lemma 3.103. Assume that $[I] \in U'_5 \setminus U''_5$ with $\overline{I} = \big(\alpha_0, \theta_6(\alpha_1, \alpha_2)\big)$. By Lemma 3.100 we may assume that $\overline{(I_{\leq 4})} = (\alpha_0^2, \alpha_0\alpha_1)$. It follows that

$$I = (W) + (\alpha_0^2, \alpha_0\alpha_1)_{\geq 4} + (\alpha_0)_{\geq 5} + (\theta_6) \tag{3.107}$$

where $[W] \in \mathrm{Gr}(4, \mathrm{lin}\{\alpha_0^3, \alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2\})$ is such that $\mathrm{lin}\{\alpha_0^3, \alpha_0^2\alpha_1\}$ is not contained in W.

Assume that $[I] \in \mathrm{Slip}_{6,2}$ is as in Equation (3.107) with $\alpha_0^3 \notin W$. Then by taking the initial ideal with respect to lex order with $\alpha_2 > \alpha_1 > \alpha_0$ we obtain an ideal of the form

$$I' = \big(\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \theta'_6(\alpha_1, \alpha_2)\big) \tag{3.108}$$

such that $[I'] \in \mathrm{Slip}_{6,2}$.

On the other hand, if $[I] \in \mathrm{Slip}_{6,2}$ is as in Equation (3.107) with $\alpha_0^3 \in W$ but $\alpha_0^2\alpha_1 \notin W$, then by taking the initial ideal we get a point $[I']$ in $\mathrm{Slip}_{6,2}$ of the form

$$I' = \big(\alpha_0^3, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0\alpha_2^4, \theta'_6(\alpha_1, \alpha_2)\big). \tag{3.109}$$

We claim that if I' is of the form as in Equation (3.108) or (3.109) then $[I'] \notin \mathrm{Slip}_{6,2}$.

Lemma 3.110. *There is no point $[I'] \in \mathrm{Slip}_{6,2}$ with I' as in Equation* (3.109).

Proof. Let $J = I' + \mathfrak{m}^5$. Note that it does not depend on θ'_6 so it is the same for every I' as in Equation (3.109). Then $\dim_{\Bbbk} \mathrm{Hom}_S(J, S/J)_0 = 8$. Thus, $[I'] \notin \mathrm{Slip}_{6,2}$ by Proposition 3.1. $\square$

We shall show that there is no point $[I'] \in \mathrm{Slip}_{6,2}$ with I' as in Equation (3.108). First, we introduce some more multigraded Hilbert schemes. Let $\overline{h_{6,2}} \colon \mathbb{Z} \to \mathbb{Z}$ be defined by

$$\overline{h_{6,2}}(a) = \begin{cases} h_{6,2}(a) & \text{for } a \leq 4; \\ 0 & \text{for } a > 4 \end{cases}$$

or, more briefly, by $\overline{h_{6,2}} = (1, 3, 6, 6, 6, 0, 0, \ldots)$. Define $\overline{f} \colon \mathbb{Z} \to \mathbb{Z}$ by $\overline{f} = (1, 3, 5, 6, 6, 0, 0, \ldots)$ or, more formally, by

$$\overline{f}(a) = \begin{cases} f(a) & \text{for } a \leq 4; \\ 0 & \text{for } a > 4. \end{cases}$$

Finally, let $k \colon \mathbb{Z} \to \mathbb{Z}$ be defined by

$$k(a) = \begin{cases} \dim_{\Bbbk} S_a & \text{for } a \leq 1; \\ a + 3 & \text{for } a \in \{2, 3, 4\}; \\ 0 & \text{for } a > 4 \end{cases}$$

or, in a brief form by $k = (1,3,5,6,7,0,0,\ldots)$.

We have natural maps $\mathrm{Hilb}_S^{k,\overline{f}} \to \mathrm{Hilb}_S^{\overline{f}} \to \mathrm{Hilb}_S^{\overline{h_{6,2}}}$. Let Z be the set-theoretic image of $\mathrm{Hilb}_S^{k,\overline{f}}$ in $\mathrm{Hilb}_S^{\overline{h_{6,2}}}$. We claim that it is irreducible and 12-dimensional.

Lemma 3.111. *In the above notation, Z is irreducible closed and 12-dimensional.*

Proof. Observe that both morphisms $\mathrm{Hilb}_S^{k,\overline{f}} \to \mathrm{Hilb}_S^{\overline{f}}$ and $\mathrm{Hilb}_S^{\overline{f}} \to \mathrm{Hilb}_S^{\overline{h_{6,2}}}$ are closed by Theorem 2.36. We claim that they are both injective on closed points. We start with $\mathrm{Hilb}_S^{\overline{f}} \to \mathrm{Hilb}_S^{\overline{h_{6,2}}}$. Suppose that there are two points $[I] \neq [I'] \in \mathrm{Hilb}_S^{\overline{f}}$ such that $I \cap \mathfrak{m}^3 = I' \cap \mathfrak{m}^3$. Let $J = (I_2 \oplus I_2') \oplus I_{\geq 3}$. Then S/J has Hilbert function $(1,3,4,6,6,0,0,\ldots)$. This contradicts the Macaulay's bound [10, Thm. 4.2.10].

Now we show that $\mathrm{Hilb}_S^{k,\overline{f}} \to \mathrm{Hilb}_S^{\overline{f}}$ is injective on closed points. Let $[I' \subseteq I''] \in \mathrm{Hilb}_S^{k,\overline{f}}$. Then $(I'_{\leq 3}) = (\theta_2, \theta_3)$ for some generators $\theta_i \in S_i$. Furthermore, I' has no minimal generator of degree 4 since there is no homogeneous ideal J of S such that S/J has Hilbert function $(1,3,5,6,8,\ldots)$ by Macaulay's bound [10, Thm. 4.2.10]. Therefore, we have $I' = (I''_{\leq 3}) + \mathfrak{m}^5$. As a result, $\mathrm{Hilb}_S^{k,\overline{f}} \to \mathrm{Hilb}_S^{\overline{f}}$ is injective on closed points.

It follows from the above considerations that it suffices to show that the flag multigraded Hilbert scheme $\mathrm{Hilb}_S^{k,\overline{f}}$ is irreducible and 12-dimensional.

Let $\pi_k \colon \mathrm{Hilb}_S^{k,\overline{f}} \to \mathrm{Hilb}_S^{k}$ be the natural projection. The fiber over a closed point $[K] \in \mathrm{Hilb}_S^{k}$ is irreducible and 6-dimensional corresponding to the choice of a 9-dimensional subspace of S_4 containing the 8-dimensional subspace K_4. By [80, 11.4.C] it is enough to show that Hilb_S^{k} is irreducible and 6-dimensional.

Let X be the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{Fl}(3,4,S_2) \arrow[d] \\ \mathrm{Gr}(1,S_1) \arrow[r] & \mathrm{Gr}(3,S_2) \end{tikzcd}$$

where the lower horizontal map takes $[\ell]$ to $[\mathrm{lin}\{\alpha_0\ell, \alpha_1\ell, \alpha_2\ell\}]$. Then X is irreducible and of dimension 4. Moreover, if $[I] \in \mathrm{Hilb}_S^{k}$ then the generators of I of degree 2 and 3 have a common linear factor since $H_{S/I}(4) = 7 > 6$. Therefore, we have a parametrization $X \times \mathrm{Gr}(1,S_1) \to \mathrm{Hilb}_S^{k}$ given by

$$\big(([\ell],[\ell S_1 \subseteq V]),[\ell']\big) \mapsto (\ell\ell') + (V\ell') + \mathfrak{m}^5.$$

Thus, Hilb_S^{k} is irreducible and of dimension 6. □

Let $p\colon \mathrm{Hilb}_S^{h_{6,2}} \to \mathrm{Hilb}_S^{\overline{h_{6,2}}}$ be the natural map given by $[I] \mapsto [I + \mathfrak{m}^5]$. Let

$$K = (\alpha_0^2\alpha_1, \alpha_0\alpha_1^2, \alpha_0^2\alpha_2, \alpha_0\alpha_1\alpha_2, \alpha_0^4) + \mathfrak{m}^5.$$

For every ideal I' as in Equation (3.108), we have $p([I']) = [K]$. Thus, it is enough to show that there is no point $[J] \in \mathrm{Slip}_{6,2}$ with $p([J]) = [K]$.

Lemma 3.112. *In the above notation, if $[K] \in p(\mathrm{Slip}_{6,2})$, there is an irreducible 10-dimensional subset Z' of the set-theoretic intersection $p(\mathrm{Slip}_{6,2}) \cap Z$. Furthermore, $[K] \in Z'$.*

Proof. Let $K' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2) + \mathfrak{m}^5$ and $K'' = K' + (\alpha_0^4 + \alpha_1^4)$. Then $[K' \subseteq K''] \in \mathrm{Hilb}_S^{k,\overline{f}}$. Therefore, $[K'' \cap \mathfrak{m}^3 + \mathfrak{m}^5] \in Z$. However, the initial ideal of $K'' \cap \mathfrak{m}^3 + \mathfrak{m}^5$ with respect to the lex order with $\alpha_0 > \alpha_1 > \alpha_2$ is K. Thus, $[K] \in p(\mathrm{Slip}_{6,2}) \cap Z$. Moreover, $\dim_{\Bbbk} \mathbf{T}_{[K]} \mathrm{Hilb}_S^{\overline{h_{6,2}}} = \dim_{\Bbbk} \mathrm{Hom}_S(K, S/K)_0 = 14$, where the first equality follows from Theorem 2.74. Therefore, the claim follows from Lemmas 2.30 and 3.111 since $\dim p(\mathrm{Slip}_{6,2}) = \dim \mathrm{Slip}_{6,2} = 12$. $\square$

Suppose that $[K] \in p(\mathrm{Slip}_{6,2})$. There exists an irreducible closed subset $Z'' \subseteq \mathrm{Slip}_{6,2}$ such that $p(Z'') = Z'$, where Z' is as in Lemma 3.112. Since $p(Z'') \subseteq Z$, it follows that Z'' is disjoint from the set of saturated ideals. Moreover, $p(U_g), p(U_4)$ and $p(U_5)$ are of dimension less than 10 (see Lemmas 3.93 and 3.98). It follows that Z'' is contained in the closure of the locus U_f where

$$U_f = \{[I] \in \mathrm{Hilb}_S^{h_{6,2}} \mid S/\overline{I} \text{ has Hilbert function } f = (1, 3, 5, 6, 6, \ldots)\}.$$

Let $q \colon \mathrm{Hilb}_S^f \to \mathrm{Hilb}_S^{h_{6,2}}$ be the natural map. It is a closed map by Theorem 2.36 and it is injective on closed points corresponding to saturated ideals. Thus, Z'' is contained in the image of the closure of the locus of points of Hilb_S^f corresponding to saturated ideals.

It follows that there is an ideal $[I''] \in \mathrm{Hilb}_S^f$ that satisfies $I''_{\leq 4} = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4)_{\leq 4}$ and I'' is a limit of saturated ideals.

By taking an initial ideal, we get that at least one of the following ideals corresponding to points of Hilb_S^f is a limit of saturated ideals:

1. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_1^5)$;
2. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_1^4\alpha_2)$;
3. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_1^3\alpha_2^2)$;
4. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_1^2\alpha_2^3)$;
5. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_1\alpha_2^4)$;
6. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^6)$;
7. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^5\alpha_2)$;
8. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^4\alpha_2^2)$;
9. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^3\alpha_2^3)$;
10. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^2\alpha_2^4)$;
11. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_1^1\alpha_2^5)$;
12. $I''' = (\alpha_0\alpha_1, \alpha_0^2\alpha_2, \alpha_0^4, \alpha_0\alpha_2^4, \alpha_2^6)$.

We claim that this is impossible. Cases 1.-5. can be excluded since then $[I''' \cap \mathfrak{m}^3] \in \mathrm{Slip}_{6,2} \cap U_g$ but $I''' \cap \mathfrak{m}^3$ is not of the form from Lemma 3.92. Furthermore, if I''' is one of the ideals 6.-12., then $H_{S/(I''')^2}(8) = 17 < 18$. Therefore, by Theorem 3.5, I''' is not in the closure of the locus of radical ideals. Thus, it is also not in the closure of the locus of saturated ideals since a general saturated ideal of S such that the quotient algebra has Hilbert function f is radical.

To summarize, we have arrived at a contradiction after assuming that $[K] \in p(\mathrm{Slip}_{6,2})$. This shows in particular, that if I' is as in Equation (3.108) then $[I'] \notin \mathrm{Slip}_{6,2}$. Together with Lemma 3.110, this finishes the proof of Lemma 3.103 and thus, of Proposition 3.105.

Chapter 4

Criteria for smooth projective toric varieties

In this chapter we work in the category of schemes over the complex numbers. We will consider smooth projective toric varieties and the corresponding multigraded Hilbert schemes. The main motivation is to study the case of the product of projective spaces. However, secant varieties of more general toric varieties have also been studied [25], [37]. Therefore, we present our results in their natural generality.

In Section 4.1 we recall the basic notions of the theory of toric varieties. We mainly follow [28]. In Section 4.2 we consider a morphism with connected fibers $f\colon X \to Y$ between smooth projective toric varieties. We present a necessary condition for an ideal I in the Cox ring of X to be in the irreducible component $\mathrm{Slip}_{r,X}$. In Sections 4.3 and 4.4 we present two particular cases of that criterion. In Section 4.3 we assume that X is the blowing up of Y at the closure of a torus orbit. In Section 4.4 we assume that X is a projective toric bundle over Y. In particular, the criterion from this section is applicable to the case of the product of projective spaces. In Section 4.5 we obtain another necessary condition in the case that X is the product of projective spaces. Finally, in Section 4.6 we present two examples of reducible multigraded Hilbert schemes corresponding to two points on a toric surface.

The main technical tool used in this chapter is the possibility to lift a morphism between smooth projective toric varieties to a homomorphism of their Cox rings (see Subsection 4.1.3). This and similar problems have been extensively studied. In particular, in [9] there are general results that could shorten our presentation. This is true, for example for Lemmas 4.4, 4.10, 4.11 and Proposition 4.21. However, since in the generality that we require, most of those results can be presented from scratch, we decided to do so.

4.1 Toric varieties

In Subsections 4.1.1-4.1.3 we recall some basic definitions and results related to toric varieties. This is mainly to fix the notation. Therefore, we will omit most of the proofs, reffering the reader to [28]. Our notation follows closely the one used there. In Subsection 4.1.3 we recall the main technical tool - lifting a morphism between smooth projective toric varieties to a morphism of their Cox rings.

Subsection 4.1.4 is concerned with morphisms $f\colon X \to Y$ between smooth projective toric varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. This property will be assumed in Theorem 4.15 which is one of the main results of this chapter.

In Subsection 4.1.5 we finally give the definition of a multigraded Hilbert scheme of points in general position for a smooth projective toric variety. This is the generality in which it was introduced in [13].

Note that the results from this section are stated for smooth projective toric varieties. Some of them are still true for more general toric varieties.

4.1.1 Fans and toric varieties

By a *toric variety* we shall mean a normal variety X over the field of complex numbers such that X contains an algebraic torus $\mathbb{T} \cong (\mathbb{C}^*)^n$ as an open subset and the action of $\mathbb{T}$ on itself extends to an action of $\mathbb{T}$ on X.

Given an algebraic torus $\mathbb{T} \cong (\mathbb{C}^*)^n$, we will denote by M the character lattice of $\mathbb{T}$, i.e. $M = \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ is the set of algebraic group homomorphisms from $\mathbb{T}$ to the one-dimensional torus. Then $M \cong \mathbb{Z}^n$ is a lattice and the dual lattice $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ will be denoted by N. For $m \in M$ the corresponding character is $\chi^m \colon \mathbb{T} \to \mathbb{C}^*$. We will denote by $\langle\ ,\ \rangle$ the natural pairing $M \times N \to \mathbb{Z}$ and its extension to the $\mathbb{R}$-vector spaces $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ and $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$.

Toric varieties are obtained by gluing affine toric varieties corresponding to certain combinatorial objects in the $\mathbb{R}$-vector space $N_{\mathbb{R}}$. We explain in more details how to obtain affine toric varieties. A subset $\sigma \subseteq N_{\mathbb{R}}$ is called a *(rational polyhedral) cone* if there is a finite set of elements $\mathbf{u}_1, \ldots, \mathbf{u}_k \in N$ such that $\sigma = \{\sum_{i=1}^{k} \lambda_i \mathbf{u}_i \mid \lambda_i \geq 0\}$. Since we will not consider more general cones, we will omit the phrase "rational polyhedral". A cone $\sigma \subseteq N_{\mathbb{R}}$ is *strongly convex* if σ does not contain a positive dimensional vector subspace of $N_{\mathbb{R}}$. Given a cone $\sigma \subseteq N_{\mathbb{R}}$, we can consider the semigroup $S_\sigma = \sigma^\vee \cap M$ where $\sigma^\vee = \{m \in M_{\mathbb{R}} \mid \langle m, \mathbf{u}\rangle \geq 0 \text{ for all } \mathbf{u} \in \sigma\}$ is the dual cone of σ. If σ is strongly convex, then the spectrum of the semigroup algebra $\mathbb{C}[S_\sigma]$ is an n-dimensional affine toric variety and will be denoted by U_σ. As shown in [28, Thm. 1.3.5], all affine toric varieties are of this form.

Given a cone $\sigma \in N_{\mathbb{R}}$, its *face* is the intersection of σ with an affine hyperplane

$$H_m = \{\mathbf{u} \in N_{\mathbb{R}} \mid \langle m, \mathbf{u}\rangle = 0\}$$

in $N_{\mathbb{R}}$ for some $m \in M_{\mathbb{R}}$ such that $\langle m, \mathbf{u}\rangle \geq 0$ for every $\mathbf{u} \in \sigma$. A *fan* is a finite collection Σ of strongly convex cones in $N_{\mathbb{R}}$ satisfying conditions:

1. Each face of each cone in Σ is an element of Σ.

2. Every two cones in Σ intersect along a common face.

Given such combinatorial data there is a toric variety X_Σ obtained by gluing the affine toric varieties $\{U_\sigma \mid \sigma \in \Sigma\}$ (see [28, Thm. 3.1.5]). Moreover, every toric variety with torus $\mathbb{T}$ comes from the above construction for a fan in $N_{\mathbb{R}}$ where N is the lattice dual to the character lattice of the torus (see [28, Cor. 3.1.8]). For a toric variety X we will denote by Σ_X a fan such that $X_{\Sigma_X} = X$. Note that Σ_X is not uniquely determined by X since we can apply any $\mathbb{Z}$-linear automorphism of $N_{\mathbb{R}}$ to a fan Σ and obtain the same abstract toric variety. In Sections 4.3 and 4.4 we will consider a morphism of toric varieties $X \to Y$. Starting from an arbitrary choice of a fan Σ_Y corresponding to Y we will describe a fan Σ_X corresponding to X that will be convenient.

4.1.2 Picard groups and Cox rings

Given a smooth projective toric variety X and a corresponding fan Σ_X in $N_{\mathbb{R}}$ the Picard group $\operatorname{Pic}(X)$ of X can be calculated using the combinatorial data of the one-dimensional cones in Σ_X (see [28, Thm. 4.1.3]). We recall this here. Let $\Sigma_X(1)$ be the set of one-dimensional cones in Σ_X, i.e. cones whose linear span is a one-dimensional real vector subspace of $N_{\mathbb{R}}$. The torus invariant prime divisors on X are in bijective correspondence with elements of $\Sigma_X(1)$. Given $\rho \in \Sigma_X(1)$ we denote by $\mathbf{u}_\rho$ the ray generator of ρ (i.e. the unique generator of the semigroup $\rho \cap N$) and the corresponding divisor by D_ρ.

Let $e_1, \ldots, e_n$ a $\mathbb{Z}$-basis of M. Then $\operatorname{Pic}(X)$ is generated by classes of $[D_\rho]$ for $\rho \in \Sigma_X(1)$. Moreover, these generators are subject to relations

$$0 = [\operatorname{div}(\chi^{e_i})] = \sum_{\rho \in \Sigma_X(1)} \langle e_i, \mathbf{u}_\rho \rangle [D_\rho]$$

for $i = 1, \ldots, n$.

Let X be a smooth projective toric variety associated with a fan $\Sigma_X \subseteq N_{\mathbb{R}}$. Then there is a corresponding polynomial ring $S[X]$ graded by the Picard group $\operatorname{Pic}(X)$. This ring is called the *Cox ring* of X. We have

$$S[X] = \mathbb{C}[\alpha_\rho \mid \rho \in \Sigma_X(1)] \text{ and } \deg(\alpha_\rho) = [D_\rho].$$

By [28, Prop. 5.3.7] we have $S[X]_{[D]} \cong \Gamma(X, \mathcal{O}_X(D))$ for $[D] \in \operatorname{Pic}(X)$.

Remark 4.1. The construction of a Cox ring can be carried out for more general varieties, see [1]. Then it does not have to be a polynomial ring. Moreover, unlike for toric varieties, the construction requires some choices so we speak of a Cox ring of X instead of the Cox ring of X.

4.1.3 Irrelevant ideals and the quotient construction

One of the main tools in this chapter is Theorem 4.3 which states that a morphism $f \colon X \to Y$ between smooth projective toric varieties can be lifted to a graded homomorphism $\overline{f}^{\#} \colon S[Y] \to S[X]$ of their Cox rings.

We start with recalling the quotient construction of a smooth projective toric variety X presented in [28, Thm. 5.1.11]. Let $X = X_{\Sigma_X}$ for a fan $\Sigma_X \subseteq N_{\mathbb{R}}$. Given a cone $\sigma \in \Sigma_X$ we denote by $\sigma(1)$ the set of 1-dimensional faces of σ. Let $S[X] = \mathbb{C}[\alpha_\rho \mid \rho \in \Sigma_X(1)]$ be the Cox ring of X. For $\sigma \in \Sigma_X(1)$ we define $\alpha^{\widehat{\sigma}}$ to be $\prod_{\rho \in \Sigma_X(1) \setminus \sigma(1)} \alpha_\rho$. The *irrelevant ideal* of X is

$$B(\Sigma_X) = (\alpha^{\widehat{\sigma}})_{\sigma \in \Sigma_X} \subseteq S[X].$$

Observe that it is enough to take generators corresponding to maximal cones of Σ_X. We denote the affine space $\operatorname{Spec} S[X]$ by $\overline{X}$ and the open subset $\overline{X} \setminus V\big(B(\Sigma_X)\big)$ by $\widehat{X}$.

By [28, Prop. 4.2.5] we have $\operatorname{Pic}(X) \cong \mathbb{Z}^k$ for some integer k. Therefore, $H_X = \operatorname{Spec} \mathbb{C}[\operatorname{Pic}(X)]$ is a torus. Since $S[X]$ is $\operatorname{Pic}(X)$-graded, there is a natural action of the torus H_X on $\overline{X}$. Then X is the geometric quotient by the induced action of H_X on $\widehat{X}$. We denote the open immersion $\widehat{X} \to \overline{X}$ by i_X and the quotient $\widehat{X} \to X$ by π_X.

Definition 4.2. Suppose that $f \colon X \to Y$ is a morphism between smooth projective toric varieties and let $f^* \colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ be the pullback map. Suppose that there exists a $\mathbb{C}$-algebra homomorphism $\overline{f}^{\#}\colon S[Y] \to S[X]$ such that:

1. $\overline{f}^{\#}(S[Y]_{[D]}) \subseteq S[X]_{f^*([D])}$ for every $[D] \in \mathrm{Pic}(Y)$;
2. the corresponding morphism $\overline{f}\colon \overline{X} \to \overline{Y}$ restricts to a morphism $\widehat{f}\colon \widehat{X} \to \widehat{Y}$;
3. $\pi_Y \circ \widehat{f} = f \circ \pi_X$.

Then we call $\overline{f}^{\#}$ a *lift* of f.

If $\overline{f}^{\#}$ is a lift of a morphism $f\colon X \to Y$ between smooth projective toric varieties, we have a commutative diagram

$$\begin{tikzcd} \overline{X} \arrow[r, "\overline{f}"] & \overline{Y} \\ \widehat{X} \arrow[u, hook, "i_X"] \arrow[r, "\widehat{f}"] \arrow[d, two heads, "\pi_X"] & \widehat{Y} \arrow[u, hook, "i_Y"] \arrow[d, two heads, "\pi_Y"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

Observe that once conditions 1.-2. from Definition 4.2 are satisfied, there is a unique morphism $f'\colon X \to Y$ such that $f' \circ \pi_X = \pi_Y \circ \widehat{f}$. Indeed, $\pi_Y \circ \widehat{f}$ is constant on H_X-orbits and π_X is a categorical quotient (see [28, Thm 5.1.11]). Condition 3. says, that $f' = f$.

The possibility of lifting a morphism $f\colon X \to Y$ to a homomorphism $\overline{f}^{\#}\colon S[Y] \to S[X]$ has been studied in various settings. The version suitable for our needs is considered in [26]. The case of rational maps of toric varieties using multi-valued maps of Cox rings is studied in [9]. Analogous quotient construction holds for the so called Mori dreams spaces. These are varieties admitting a Cox ring that is a finitely generated $\mathbb{C}$-algebra. Lifting of rational maps of Mori dreams spaces is discussed in [18] and the case of a regular map can be found in [51].

Now we can state the key existence theorem.

Theorem 4.3 ([26, Thm. 3.2])**.** *Let $f\colon X \to Y$ be a morphism between smooth projective toric varieties. Then there exists a lift $\overline{f}^{\#}$ of f.*

Let X, Y be smooth projective toric varieties corresponding to fans $\Sigma_X \subseteq (N_X)_{\mathbb{R}}$ and $\Sigma_Y \subseteq (N_Y)_{\mathbb{R}}$, respectively. Assume that $f\colon X \to Y$ is a *toric* morphism, i.e. it maps the torus $\mathbb{T}_X$ of X into the torus $\mathbb{T}_Y$ of Y and the restricted map $\mathbb{T}_X \to \mathbb{T}_Y$ is a group homomorphism. Such morphisms correspond to $\mathbb{Z}$-linear maps $\phi\colon N_X \to N_Y$ such that for every cone $\sigma \in \Sigma_X$, there is a cone $\sigma' \in \Sigma_Y$ satisfying $\phi_{\mathbb{R}}(\sigma) \subseteq \sigma'$ (see [28, Thm. 3.3.4]). Here $\phi_{\mathbb{R}} = \phi \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{R}}$. We say that ϕ is *compatible with the fans Σ_X and Σ_Y*.

In the following lemma we study the condition under which a homomorphism of graded rings $\overline{f}^{\#}\colon S[Y] \to S[X]$ is a lift of the given toric morphism $f\colon X \to Y$.

Lemma 4.4. *Let $f\colon X \to Y$ be a toric morphism between smooth projective toric varieties. Let $S[X] = \mathbb{C}[\alpha_\rho \mid \rho \in \Sigma_X(1)]$ and $S[Y] = \mathbb{C}[\beta_\rho \mid \rho \in \Sigma_Y(1)]$ be the Cox rings of X, Y, respectively. Let $\phi\colon N_X \to N_Y$ be the map corresponding to f.*

Assume that we are given a homomorphism of rings $\overline{f}^{\#}\colon S[Y] \to S[X]$ satisfying conditions 1. and 2. from Definition 4.2. Then $\overline{f}^{\#}$ is a lift of f, if and only if

$$\prod_{\rho \in \Sigma_Y(1)} \left(\overline{f}^{\#}(\beta_\rho)\right)^{\langle m, \mathbf{u}_\rho \rangle} = \prod_{\rho \in \Sigma_X(1)} \alpha_\rho^{\langle \psi(m), \mathbf{u}_\rho \rangle} \tag{4.5}$$

for every $m \in M_Y$, where $\psi\colon M_Y \to M_X$ is the dual map of $\phi\colon N_X \to N_Y$.

Proof. Let $f'\colon X \to Y$ be the morphism induced by $\overline{f}$. It is enough to show that f and f' define the same morphism $U_\sigma \to U_{\sigma'}$ of affine toric varieties for every pair of cones $\sigma \in \Sigma_X$ and $\sigma' \in \Sigma_Y$ such that $\phi_{\mathbb{R}}(\sigma) \subseteq \sigma'$.

Recall that $\beta^{\widehat{\sigma'}} = \prod_{\rho \in \Sigma_Y(1) \setminus \sigma'(1)} \beta_\rho$ and $\alpha^{\widehat{\sigma}} = \prod_{\rho \in \Sigma_X(1) \setminus \sigma(1)} \alpha_\rho$. We have an isomorphism $\mathbb{C}[(\sigma')^\vee \cap M_Y] \cong (S[Y]_{\beta^{\widehat{\sigma'}}})_0$ given by $\chi^m \mapsto \prod_{\rho \in \Sigma_Y(1)} \beta_\rho^{\langle m, \mathbf{u}_\rho \rangle}$ (see the proof of [28, 5.1.11]). There is a similar isomorphism $\mathbb{C}[\sigma^\vee \cap M_X] \cong (S[X]_{\alpha^{\widehat{\sigma}}})_0$.

The map $U_\sigma \to U_{\sigma'}$ induced by f corresponds to the homomorphism $\mathbb{C}[(\sigma')^\vee \cap M_Y] \to \mathbb{C}[\sigma^\vee \cap M_X]$ given by $\chi^m \mapsto \chi^{\psi(m)}$. On the other hand, the map $U_\sigma \to U_{\sigma'}$ induced by f' corresponds to the map $(\overline{f}^{\#}_{\beta^{\widehat{\sigma'}}})_0\colon (S[Y]_{\beta^{\widehat{\sigma'}}})_0 \to (S[X]_{\alpha^{\widehat{\sigma}}})_0$. Therefore, f and f' induce the same map $U_\sigma \to U_{\sigma'}$ if and only if Equation (4.5) holds. Indeed, this is equivalent to the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{C}[(\sigma')^\vee \cap M_Y] \arrow[r, "\chi^m \mapsto \chi^{\psi(m)}"] \arrow[d, "\chi^m \mapsto \prod_{\rho \in \Sigma_Y(1)} \beta_\rho^{\langle m, \mathbf{u}_\rho \rangle}"'] & \mathbb{C}[\sigma^\vee \cap M_X] \arrow[d, "\chi^m \mapsto \prod_{\rho \in \Sigma_X(1)} \alpha_\rho^{\langle m, \mathbf{u}_\rho \rangle}"] \\
(S[Y]_{\beta^{\widehat{\sigma'}}})_0 \arrow[r, "(\overline{f}^{\#}_{\beta^{\widehat{\sigma'}}})_0"] & (S[X]_{\alpha^{\widehat{\sigma}}})_0.
\end{tikzcd}$$

□

The fact that a morphism $\overline{f}\colon \overline{X} \to \overline{Y}$ restricts to a morphism $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ has the following algebraic consequence.

Lemma 4.6. *Let $f\colon X \to Y$ be a morphism between smooth projective toric varieties. Assume that $\overline{f}^{\#}\colon S[Y] \to S[X]$ is a homomorphism of $\mathbb{C}$-algebras satisfying condition 2. from Definition 4.2. Suppose that $I \subseteq S[X]$ is a homogeneous ideal which is saturated with respect to $B(\Sigma_X)$. Then $(\overline{f}^{\#})^{-1}(I)$ is saturated with respect to $B(\Sigma_Y)$.*

Proof. By assumption that $\overline{f}\colon \overline{X} \to \overline{Y}$ restricts to a morphism $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ we conclude that

$$\overline{f}^{-1}\Big(V\big(B(\Sigma_Y)\big)\Big) \subseteq V\big(B(\Sigma_X)\big). \tag{4.7}$$

By [6, Prop. 3 §6.2] we have $\overline{f}^{-1}\Big(V\big(B(\Sigma_Y)\big)\Big) = V\Big(\overline{f}^{\#}\big(B(\Sigma_Y)\big)\Big)$. Therefore, from Equation (4.7) we get

$$B(\Sigma_X) \subseteq \sqrt{\overline{f}^{\#}\big(B(\Sigma_Y)\big) \cdot S[X]}.$$

Since $S[X]$ is a Noetherian ring, there is a positive integer k such that

$$B(\Sigma_X)^k \subseteq \overline{f}^{\#}\big(B(\Sigma_Y)\big) \cdot S[X]. \tag{4.8}$$

Let $J = (\overline{f}^{\#})^{-1}(I)$. Take $F \in \big(J\colon B(\Sigma_Y)\big)$. We need to show that $F \in J$, or equivalently, that $\overline{f}^{\#}(F) \in I$. Since I is saturated with respect to $B(\Sigma_X)$, it is enough to show that $\overline{f}^{\#}(F) \in (I\colon B(\Sigma_X)^k)$. We have

$$\overline{f}^{\#}(F) \cdot B(\Sigma_X)^k \overset{(4.8)}{\subseteq} \overline{f}^{\#}(F) \cdot \overline{f}^{\#}\big(B(\Sigma_Y)\big) \cdot S[X] \subseteq \overline{f}^{\#}\big(F \cdot B(\Sigma_Y)\big) \cdot S[X] \subseteq \overline{f}^{\#}(J) \cdot S[X] \subseteq I,$$

where the penultimate containment follows from the choice of F and the ultimate one is by the definition of J. □

4.1.4 Morphism with connected fibers

In this subsection X, Y are smooth projective toric varieties. We consider a morphism $f\colon X \to Y$ such that the natural map $f\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. We lift f to a homomorphism $\overline{f}^{\#}\colon S[Y] \to S[X]$ as in Definition 4.2.

In what follows we do not assume that f is a toric morphism. However, if f happens to be a toric morphism, then the condition $f_*\mathcal{O}_X \cong \mathcal{O}_Y$ has an equivalent combinatorial reformulation (see [29, Prop. 2.1]).

Lemma 4.9. *In the above notation, let $f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ be the pullback map. Then*

$$\dim_{\mathbb{C}} S[X]_{f^*([D])} = \dim_{\mathbb{C}} S[Y]_{[D]}$$

for every $[D] \in \operatorname{Pic}(Y)$.

Proof. We have

$$\begin{aligned} S[X]_{f^*([D])} &\cong H^0(X, f^*\mathcal{O}_Y(D)) \cong H^0(Y, f_*(f^*\mathcal{O}_Y(D))) \\ &\cong H^0(Y, f_*\mathcal{O}_X \otimes \mathcal{O}_Y(D)) \cong H^0(Y, \mathcal{O}_Y(D)) \cong S[Y]_{[D]}. \end{aligned}$$

The middle equality follows from projection formula [47, Ex. II.5.1]. The first and the last equality follow from [28, Prop. 5.3.7]. □

The following lemma will be used in the proof of Theorem 4.15 - one of the main results of this chapter.

Lemma 4.10. *In the above notation we have:*

(i) the pullback map $f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ is injective;

(ii) the map $\overline{f}^{\#}$ induces an isomorphism of the $\mathbb{C}$-vector spaces $S[Y]_{[D]} \to S[X]_{f^*([D])}$ for every $[D] \in \operatorname{Pic} Y$.

Proof. (i) Since $f_*\mathcal{O}_X \to \mathcal{O}_Y$, it follows from projection formula (see [47, Ex. II.5.1]) that $f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ is injective.

(ii) The pullback $f^*\colon \operatorname{Pic}(Y) \to \operatorname{Pic}(X)$ is injective by part (i). Hence the corresponding map of algebraic tori $H_X = \operatorname{Spec} \mathbb{C}[\operatorname{Pic}(X)] \to \operatorname{Spec} \mathbb{C}[\operatorname{Pic}(Y)] = H_Y$ is dominant. Thus, it is surjective by [28, Prop. 1.1.1].

Since f is a projective morphism such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$, it is surjective. We claim that $\overline{f}$ is dominant. It is enough to show that $\widehat{f}\colon \widehat{X} \to \widehat{Y}$ is surjective. Let $\widehat{y} \in \widehat{Y}$. Since f, π_X are surjective, there is a point $\widehat{x}$ such that $f \circ \pi_X(\widehat{x}) = \pi_Y(\widehat{y})$. Thus, there is an element $t \in H_Y$ such that $t \cdot \big(\widehat{f}(\widehat{x})\big) = \widehat{y}$. Using the fact that the map of tori is surjective and $\widehat{f}$ is equivariant, we conclude that there is $t' \in H_X$ such that $\widehat{f}(t' \cdot \widehat{x}) = \widehat{y}$.

This shows $\overline{f}$ is dominant and hence $\overline{f}^{\#}$ is injective. In particular, it induces injections $S[Y]_{[D]} \to S[X]_{f^*([D])}$ for every $[D] \in \operatorname{Pic}(Y)$. These maps are surjective by Lemma 4.9. □

Given a finite set of points $\{p_1, \ldots, p_r\} \in X$, we denote by $I(\{p_1, \ldots, p_r\})$ the unique $B(\Sigma_X)$-saturated homogeneous ideal of $S[X]$ defining this set of points as a reduced subscheme of X.

Lemma 4.11. *In the above notation, we have*

$$(\overline{f}^{\#})^{-1}(I(\{p_1,\ldots,p_r\})) = I(\{f(p_1),\ldots,f(p_r)\}).$$

Proof. Let $R \subseteq X$ be the (reduced) subscheme $\{p_1,\ldots,p_r\}$. Let $i\colon R \to X$ be the closed immersion. Let R' be the scheme-theoretic image of R. Since R is reduced and f is closed, R' is the (reduced) subscheme $\{f(p_1),\ldots,f(p_r)\}$.

The scheme theoretic image R' of R is defined by the ideal sheaf $\ker(\mathcal{O}_Y \to f_*\mathcal{O}_X \to f_*i_*\mathcal{O}_R)$. By assumption $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. Therefore, the ideal sheaf of R' is $\ker(f_*\mathcal{O}_X \to f_*i_*\mathcal{O}_R)$. Moreover, f_* is left-exact, so the ideal sheaf of R' is the pushforward under f of the ideal sheaf of R.

The ideal $I(\{p_1,\ldots,p_r\})$ is saturated with respect to $B(\Sigma_X)$. Hence, the subscheme of Y corresponding to $(\overline{f}^{\#})^{-1}(I(\{p_1,\ldots,p_r\})) \subseteq S[Y]$ is R' by [65, Thm. 3.5] and Lemma 4.10. By Lemma 4.6, the ideal $(\overline{f}^{\#})^{-1}(I(\{p_1,\ldots,p_r\}))$ is saturated with respect to $B(\Sigma_Y)$. Thus,

$$(\overline{f}^{\#})^{-1}(I(\{p_1,\ldots,p_r\})) = I(\{f(p_1),\ldots,f(p_r)\}),$$

as claimed. $\square$

4.1.5 Multigraded Hilbert schemes

Let Y be a smooth projective toric variety with Cox ring $S[Y]$. A natural generalization of the function $h_{r,\mathbb{P}^n}\colon \mathbb{Z} \to \mathbb{Z}$ studied in Chapters 2 and 3 is the function $h_{r,Y}\colon \operatorname{Pic}(Y) \to \mathbb{Z}$ defined by

$$h_{r,Y}([D]) = \min\{\dim_{\mathbb{C}} H^0(Y,\mathcal{O}_Y(D)), r\} = \min\{\dim_{\mathbb{C}} S[Y]_{[D]}, r\},$$

where the latter equality follows from [28, Prop. 5.3.7]. Observe that in the case of the projective space we have implicitly used the identifications $\operatorname{Pic}(\mathbb{P}^n) = \mathbb{Z}[H]$ where H is a hyperplane divisor and $H^0(\mathbb{P}^n,\mathcal{O}_{\mathbb{P}^n}(dH)) \cong \mathbb{C}[\alpha_0,\ldots,\alpha_n]_d$.

Let $\mathrm{Hilb}^{h_{r,Y}}_{S[Y]}$ be the corresponding multigraded Hilbert scheme (see Subsection 2.2.2). We shall denote by $\mathrm{Sip}_{r,Y}$ the subset of $\mathrm{Hilb}^{h_{r,Y}}_{S[Y]}$ whose closed points correspond to $B(\Sigma_Y)$-saturated, homogeneous ideals of r-tuples of disjoint points in Y. Let $\mathrm{Slip}_{r,Y}$ be the closure of $\mathrm{Sip}_{r,Y}$ in $\mathrm{Hilb}^{h_{r,Y}}_{S[Y]}$. By [13, Prop. 3.13] it is an irreducible component of $\mathrm{Hilb}^{h_{r,Y}}_{S[Y]}$.

We will construct natural morphisms from r-tuples of distinct points of Y (respectively, r-tuples of distinct points of Y in general position) to $\mathcal{H}ilb_r(Y)$ (respectively, $\mathrm{Hilb}^{h_{r,Y}}_{S[Y]}$). Let $Y^r_{dis} = \{(p_1,\ldots,p_r) \mid p_i \neq p_j \text{ for } i \neq j\}$ be the set of r-tuples of distinct points of Y. This is an open subset of Y^r so it has a natural scheme structure.

Recall that, given a point $(p_1,\ldots,p_r) \in Y^r_{dis}$, we denote by $I(\{p_1,\ldots,p_r\})$ the unique $B(\Sigma_Y)$-saturated homogeneous ideal defining this set of points as a reduced subscheme of Y. Let

$$Y^r_{gen} = \{(p_1,\ldots,p_r) \in Y^r_{dis} \mid S[Y]/I(\{p_1,\ldots,p_r\}) \text{ has Hilbert function } h_{r,Y}\}.$$

We will use the following key observation.

Theorem 4.12 ([15, Thm. 1.4])**.** *In the above notation, Y^r_{gen} is an open subset of Y^r_{dis}. In particular, it has a natural scheme structure.*

The rest of this subsection is devoted to showing that $(p_1, \ldots, p_r) \mapsto [I(\{p_1, \ldots, p_r\})]$ defines a morphism $Y^r_{gen} \to \mathrm{Hilb}^{h_{r,Y}}_{S[Y]}$.

We start with constructing a morphism $Y^r_{dis} \to \mathcal{H}ilb_r(Y)$. Let $\mathscr{U}_{dis} \subseteq Y^r_{dis} \times Y$ be the (reduced) closed subscheme $\coprod_{i=1}^r Z_i$ where

$$Z_i = \{(p_1, \ldots, p_r), q \mid p_i = q\}.$$

Let $\pi_{dis} \colon \mathscr{U}_{dis} \to Y^r_{dis}$ be the projection. The family $\mathscr{U}_{dis}$ is flat over Y^r_{dis}, since each Z_i is mapped isomorphically to Y^r_{dis}. By construction, the fiber of π_{dis} over a closed point $(p_1, \ldots, p_r)$ of Y^r_{dis} is the (reduced) subscheme $\{p_1, \ldots, p_r\}$ of Y. Since Y is of finite type over $\mathbb{C}$, it follows that all fibers of π_{dis} are of length r. Therefore, the family $\mathscr{U}_{dis}$ defines a natural map $Y^r_{dis} \to \mathcal{H}ilb_r(Y)$ by the universal property of the Hilbert scheme.

We proceed to the construction of the map $Y^r_{gen} \to \mathrm{Hilb}^{h_{r,Y}}_{S[Y]}$. We shall denote by $\mathscr{U}$ the restriction of $\mathscr{U}_{dis}$ to $Y^r_{gen} \times Y$. Let $\pi \colon Y^r_{gen} \times Y \to Y^r_{gen}$ be the natural projection. Consider the exact sequence of sheaves of $\mathcal{O}_{Y^r_{gen} \times Y}$-modules

$$0 \to \bigoplus_{[D] \in \mathrm{Pic}(Y)} \mathcal{I}_{\mathscr{U}}(D) \to \bigoplus_{[D] \in \mathrm{Pic}(Y)} \mathcal{O}_{Y^r_{gen} \times Y}(D) \xrightarrow{\eta} \bigoplus_{[D] \in \mathrm{Pic}(Y)} \mathcal{O}_{\mathscr{U}}(D) \to 0. \tag{4.13}$$

Let $\mathcal{A} = \mathrm{im}(\pi_* \eta)$. We shall verify the following claims:

1. $\pi_*\big(\bigoplus_{[D] \in \mathrm{Pic}(Y)} \mathcal{O}_{Y^r_{gen} \times Y}(D)\big) \cong \mathcal{O}_{Y^r_{gen}} \otimes_{\mathbb{C}} S[Y]$;
2. $\mathcal{A}$ is a sheaf of $\mathcal{O}_{Y^r_{gen}} \otimes_{\mathbb{C}} S[Y]$-algebras;
3. $\mathcal{A}_{[D]}$ is a locally free sheaf of $\mathcal{O}_{Y^r_{gen}}$-modules of rank $h_{r,Y}([D])$ for every $[D] \in \mathrm{Pic}(Y)$.

The first claim follows from [47, Prop. III.9.3] since $\Gamma(Y, \mathcal{O}_Y(D)) \cong S[Y]_{[D]}$.

The second claim follows from the fact that in exact sequence (4.13) the $\mathcal{O}_{Y^r_{gen} \times Y}$-submodule $\bigoplus_{[D] \in \mathrm{Pic}(Y)} \mathcal{I}_{\mathscr{U}}(D)$ of the sheaf of $\mathcal{O}_{Y^r_{gen} \times Y}$-algebras $\bigoplus_{[D] \in \mathrm{Pic}(Y)} \mathcal{O}_{Y^r_{gen} \times Y}(D)$ is in fact a sheaf of ideals. Therefore, from left-exactness of the pushforward, it follows that the kernel of $\pi_*(\eta)$ is a sheaf of ideals of the sheaf $\mathcal{O}_{Y^r_{gen}} \otimes_{\mathbb{C}} S[Y]$.

Finally, we address the third claim. We will consider two cases:

(a) $h_{r,Y}([D]) = r$;

(b) $h_{r,Y}([D]) < r$.

Observe that by definition of $\mathscr{U}$, for every $y \in Y^r_{gen}$ we have

$$\dim_{\mathbb{C}} H^0\big((Y^r_{gen} \times Y)_y, (\mathcal{I}_{\mathscr{U}}(D))_y\big) = \dim_{\mathbb{C}} S[Y]_{[D]} - h_{r,Y}([D]).$$

Similarly, we have $\dim_{\mathbb{C}} H^0\big((Y^r_{gen} \times Y)_y, (\mathcal{O}_{\mathscr{U}}(D))_y\big) = r$ for every $y \in Y^r_{gen}$. Moreover, both $\mathcal{O}_{\mathscr{U}}(D)$ and $\mathcal{I}_{\mathscr{U}}(D)$ are flat over Y^r_{gen}. Therefore, by [47, Cor. III.12.9] the sheaves of $\mathcal{O}_{Y^r_{gen}}$-modules $\pi_*(\mathcal{I}_{\mathscr{U}}(D))$ and $\pi_*(\mathcal{O}_{\mathscr{U}}(D))$ are locally free of rank $\dim_{\mathbb{C}} S[Y]_{[D]} - h_{r,Y}([D])$ and r, respectively. In particular, this establishes claim 3. in the case (b) since then $\mathcal{A}_{[D]} \cong \mathcal{O}_{Y^r_{gen}} \otimes_{\mathbb{C}} S[Y]_{[D]}$.

Thus, it is enough to show that if $h_{r,Y}([D]) = r$ then $\pi_*(\eta)$ induces a surjection $\mathcal{O}_{Y^r_{gen}} \otimes_{\mathbb{C}} S[Y]_{[D]} \to \pi_*(\mathcal{O}_{\mathscr{U}}(D))$. This can be checked on stalks over closed points, and by Nakayama's lemma it is even enough to check this on fibers. Let $y \in Y^r_{gen}$ correspond to the subscheme

$Z \subseteq Y$ and let I_Z denote its $B(\Sigma_Y)$-saturated ideal. Using [47, Cor. III.12.9], it is enough to show that natural map

$$S[Y]_{[D]} \to \Gamma(Y, \mathcal{O}_Z(D))$$

is surjective for every $[D] \in \operatorname{Pic}(Y)$ such that $h_{r,Y}([D]) = r$. However, the kernel of this map is $(I_Z)_{[D]}$ and

$$r = \dim_{\mathbb{C}} \Gamma(Y, \mathcal{O}_Z(D)) = \dim_{\mathbb{C}} S[Y]_{[D]} - \dim_{\mathbb{C}}(I_Z)_{[D]}$$

by the choice of $[D]$. This finishes the proof of claims 1.-3.

Lemma 4.14. *In the above notation, there is a natural morphism $\psi_{r,Y} \colon Y^r_{gen} \to \operatorname{Hilb}^{h_{r,Y}}_{S[Y]}$ which on closed points maps $\{p_1, \dots, p_r\}$ to $[I(\{p_1, \dots, p_r\})]$.*

Proof. By properties 1.-3. above, $\mathcal{A}$ defines an admissible family over Y^r_{gen} for the Hilbert function $h_{r,Y}$. Thus, there is a morphism $Y^r_{gen} \to \operatorname{Hilb}^{h_{r,Y}}_{S[Y]}$. By construction, on closed points, it maps $\{p_1, \dots, p_r\}$, to $I(\{p_1, \dots, p_r\})$. □

4.2 Criterion based on a morphism of toric varieties

Let $f \colon X \to Y$ be a morphism between smooth projective toric varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let r be a positive integer. In this section we present a necessary condition for a closed point $[I] \in \operatorname{Hilb}^{h_{r,X}}_{S[X]}$ to be in the irreducible component $\operatorname{Slip}_{r,X}$. The most interesting case is when $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_{k+1}}$, $Y = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ for some positive integers $k, n_1, \dots, n_{k+1}$ and f is the natural projection.

Theorem 4.15. *Let $f \colon X \to Y$ be a morphism between smooth projective toric varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Let r be a positive integer and $[I] \in \operatorname{Hilb}^{h_{r,X}}_{S[X]}$ be a closed point. Let $\overline{f}^{\#} \colon S[Y] \to S[X]$ be a lift of f as in Definition 4.2. Then*

(i) $\overline{f}^{\#}$ induces a morphism $\pi \colon \operatorname{Hilb}^{h_{r,X}}_{S[X]} \to \operatorname{Hilb}^{h_{r,Y}}_{S[Y]}$ given on closed points by $[I] \mapsto [(\overline{f}^{\#})^{-1}(I)]$;

(ii) The morphism $\pi \colon \operatorname{Hilb}^{h_{r,X}}_{S[X]} \to \operatorname{Hilb}^{h_{r,Y}}_{S[Y]}$ from part (i) induces a surjection $\operatorname{Slip}_{r,X} \to \operatorname{Slip}_{r,Y}$.

Proof. The existence of a lift $\overline{f}^{\#}$ follows from Theorem 4.3.

(i) Using Lemma 4.10 we may identify (as $\operatorname{Pic}(Y)$-graded rings) the ring $S[Y]$ with the subring $\bigoplus_{[D] \in \operatorname{Pic} Y} S[X]_{f^*([D])}$ of $S[X]$. Under this identification,

$$(\overline{f}^{\#})^{-1}(I) = I|_{f^*(\operatorname{Pic}(Y))} := \bigoplus_{[D] \in \operatorname{Pic}(Y)} I_{f^*([D])}.$$

It follows that if $S[X]/I$ has Hilbert function $h_{r,X}$, then $S[Y]/(\overline{f}^{\#})^{-1}(I)$ has Hilbert function $h_{r,Y}$. Thus, we have a natural transformation of functors of points

$$\underline{\operatorname{Hilb}^{h_{r,X}}_{S[X]}} \to \underline{\operatorname{Hilb}^{h_{r,Y}}_{S[Y]}}$$

given on a $\mathbb{C}$-algebra R by

$$\underline{\operatorname{Hilb}^{h_{r,X}}_{S[X]}}(R) \ni I \mapsto I|_{f^*(\operatorname{Pic}(Y))} \in \underline{\operatorname{Hilb}^{h_{r,Y}}_{S[Y]}}(R).$$

Hence we have the corresponding morphism of schemes $\pi\colon \mathrm{Hilb}^{h_{r,X}}_{S[X]} \to \mathrm{Hilb}^{h_{r,Y}}_{S[Y]}$.

(ii) We first show that $\pi(\mathrm{Slip}_{r,X}) \subseteq \mathrm{Slip}_{r,Y}$ set-theoretically. If I is radical then $(\overline{f}^{\#})^{-1}(I)$ is also radical. Moreover, if I is saturated with respect to $B(\Sigma_X)$, then $(\overline{f}^{\#})^{-1}$ is saturated with respect to $B(\Sigma_Y)$ by Lemma 4.6. It follows that $\pi(\mathrm{Sip}_{r,X}) \subseteq \mathrm{Sip}_{r,Y}$ set-theoretically. Therefore, $\pi(\mathrm{Slip}_{r,X}) \subseteq \mathrm{Slip}_{r,Y}$ set-theoretically.

Now we show that in fact $\pi\colon \mathrm{Slip}_{r,X} \to \mathrm{Slip}_{r,Y}$ is surjective. Recall the definition of Y^r_{gen} from Subsection 4.1.5. Consider the product morphism $f^r\colon X^r \to Y^r$. By Chevalley's theorem [40, Thm. 10.20] the image of X^r_{gen} in Y^r is constructible. Moreover, f^r is projective and surjective and X^r_{gen} is dense in X^r by Theorem 4.1.5. It follows that $f^r(X^r_{gen})$ is dense in Y^r. Thus, there is an open subset $U \subseteq Y^r$ contained in $f^r(X^r_{gen})$ (see [47, Ex. II.3.18]). Let $V = U \cap Y^r_{gen}$ and $W = (f^r)^{-1}(V) \cap X^r_{gen}$. We have a diagram

$$\begin{tikzcd} \mathrm{Hilb}^{h_{r,X}}_{S[X]} \arrow[r, "\pi"] & \mathrm{Hilb}^{h_{r,Y}}_{S[Y]} \\ W \arrow[u, "(\psi_{r,X})|_W"] \arrow[r, "f^r|_W"] & V \arrow[u, "(\psi_{r,Y})|_V"'] \end{tikzcd}$$

where the maps $\psi_{r,X}$ and $\psi_{r,Y}$ are as in Lemma 4.14. We claim that this diagram is commutative. Let $(p_1, \ldots, p_r)$ be a point of W. Then $\psi_{r,Y} \circ f^r(p_1, \ldots, p_r) = [I(\{f(p_1), \ldots, f(p_r)\})]$. On the other hand $\pi \circ \psi_{r,X}(p_1, \ldots, p_r) = [(\overline{f}^{\#})^{-1}(I(\{p_1, \ldots, p_r\}))] = [I(\{f(p_1), \ldots, f(p_r)\})]$. Here the last equality follows from Lemma 4.11. We have shown that the diagram commutes.

By construction, $f^r(W) = V$, so it is dense in Y^r_{gen}. Since π is projective, it follows that

$$\mathrm{Slip}_{r,Y} = \overline{\psi_{r,Y} \circ f^r(W)} = \overline{\pi \circ \psi_{r,X}(W)} = \pi(\overline{\psi_{r,X}(W)}) = \pi(\mathrm{Slip}_{r,X})$$

set-theoretically.

□

We obtain the following corollary.

Corollary 4.16. *In the notation of Theorem 4.15, assume that* $[J] \in \mathrm{Slip}_{r,Y}$ *is such that there exists a unique closed point* $[I] \in \mathrm{Hilb}^{h_{r,X}}_{S[X]}$ *for which* $(\overline{f}^{\#})^{-1}(I) = J$. *Then* $[I]$ *is in* $\mathrm{Slip}_{r,X}$.

Remark 4.17. The usual Hilbert scheme $\mathcal{H}ilb_r(X)$ is usually not functorial in X. That is, let $f\colon X \to Y$ be a regular non-constant morphism of algebraic varieties. A general r-tuple of distinct points of X is mapped to an r-tuple of points of Y and this assignment induces a rational map of the smoothable components $\pi\colon \mathcal{H}ilb^{sm}_r(X) \dashrightarrow \mathcal{H}ilb^{sm}_r(Y)$. However, this map needs not to extend to a regular morphism $\mathcal{H}ilb_r(X) \to \mathcal{H}ilb_r(Y)$, or even $\mathcal{H}ilb^{sm}_r(X) \to \mathcal{H}ilb^{sm}_r(Y)$. To see some of the problems, for simplicity assume X and Y are smooth and projective, f is dominant, and $\dim Y \geq 2$.

1. If $\mathcal{H}ilb_r(X)$ has a component whose general point represents a subscheme of X with embedding dimension larger than $\dim Y$, then the image of such scheme has length less than r, thus it is hard or impossible to sensibly define $\mathcal{H}ilb_r(X) \to \mathcal{H}ilb_r(Y)$ on this component.

2. Even restricting to the smoothable components, the rational map π does not necessarily extend to a regular morphism. Already if $r = 2$, $Y = \mathbb{P}^2$ and f is the blowup of $\mathbb{P}^2$ at a single point (thus we are in the situation of Theorem 4.15), there are degree 2 finite subschemes contained in the exceptional divisor, which is contracted to a point. It is straightforward to verify that there is no continuous map that extends $\pi\colon \mathcal{H}ilb_2^{sm}(X) \dashrightarrow \mathcal{H}ilb_2^{sm}(\mathbb{P}^2)$ to the points representing such subschemes.

In contrast, Theorem 4.15 shows that the multigraded Hilbert scheme $\mathrm{Hilb}_{S[X]}^{h_{r,X}}$ and the analogue of its smoothable component $\mathrm{Slip}_{r,X}$ behave nicely (functorially), at least under some special morphisms. In some sense, the induced map $\mathrm{Slip}_{r,X} \to \mathrm{Slip}_{r,Y}$ is a natural resolution of the rational map $\mathcal{H}ilb_r^{sm}(X) \dashrightarrow \mathcal{H}ilb_r^{sm}(Y)$.

4.3 Blowup of the closure of a torus orbit

In this section we study a special case of Theorem 4.15 - the blowup of a smooth projective toric variety at the closure of a torus orbit.

Let Y be a smooth projective n-dimensional toric variety associated with a fan $\Sigma_Y \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$. Let $f\colon X \to Y$ be the blowup of Y at the closure of the torus orbit $V(\tau) = \overline{O(\tau)}$ corresponding to a cone $\tau \in \Sigma_Y$ (see [28, Thm. 3.2.6]). In that case, as follows from [28, pages 132-133], the variety X is the toric variety associated with the fan $\Sigma_Y^*(\tau) \subseteq N \otimes_{\mathbb{Z}} \mathbb{R}$, whose construction we recall below. Moreover, the blowup $f\colon X \to Y$ corresponds to the identity map on the lattice N. Observe that the special case, when τ is n-dimensional (or, equivalently, when $V(\tau)$ is a torus invariant point of Y) is [28, Prop. 3.3.15].

Now we recall the construction of the fan $\Sigma_Y^*(\tau)$. Given a cone $\sigma \in \Sigma_Y$, we denote by $\sigma(1)$ the set of edges of σ. Let $\mathbf{u}_\tau = \sum_{\rho \in \tau(1)} \mathbf{u}_\rho$ be the sum of the ray generators of edges of τ. Let $\sigma \in \Sigma_Y$ be a cone containing τ and consider the set

$$(\Sigma_Y)_\sigma^*(\tau) = \{\mathrm{Cone}(A) \mid A \subseteq \{\mathbf{u}_\tau\} \cup \bigcup_{\rho \in \sigma(1)} \{\mathbf{u}_\rho\} \text{ such that } \tau(1) \not\subseteq A\}.$$

Then

$$\Sigma_Y^*(\tau) = \{\sigma \in \Sigma_Y \mid \tau \not\subseteq \sigma\} \cup \bigcup_{\tau \subseteq \sigma} (\Sigma_Y)_\sigma^*(\tau).$$

Let $\Sigma_Y(1) = \{\rho_1, \ldots, \rho_n, \rho_1', \ldots, \rho_t'\}$ and $\tau = \mathrm{Cone}(\rho_i \mid i = 1, \ldots, s)$ for some $1 \le s \le n$. Let $e_1, \ldots, e_n$ be the standard basis of M and let $\mathbf{e}_1^*, \ldots, \mathbf{e}_n^*$ be the dual basis. Since Y is smooth, we may assume that the ray generator $\mathbf{u}_{\rho_i}$ of ρ_i for $i = 1, \ldots, n$ is $\mathbf{e}_i^*$. We can express the ray generators of ρ_j' in terms of this basis

$$\mathbf{u}_{\rho_j'} = \sum_{i=1}^n a_{ij} \mathbf{e}_i^* \tag{4.18}$$

for some $a_{ij} \in \mathbb{Z}$. Then $\Sigma_X(1) = \Sigma_Y(1) \cup \{\rho_\tau\}$, where $\rho_\tau = \mathrm{Cone}(\mathbf{u}_\tau)$.

By [28, Thm. 4.1.3], the Picard group $\mathrm{Pic}(Y)$ is generated by the classes

$$[D_{\rho_1}], \ldots, [D_{\rho_n}], [D_{\rho_1'}], \ldots, [D_{\rho_t'}]$$

of prime torus invariant divisors modulo the relations

$$0 = [\operatorname{div}(\chi^{e_i})] = \sum_{j=1}^{n} \langle e_i, \mathbf{u}_{\rho_j} \rangle [D_{\rho_j}] + \sum_{j=1}^{t} \langle e_i, \mathbf{u}_{\rho'_j} \rangle [D_{\rho'_j}] \text{ for } i = 1, \ldots, n. \tag{4.19}$$

It follows from Equations (4.18) and (4.19) that

$$\operatorname{Pic}(Y) = \bigoplus_{i=1}^{t} \mathbb{Z}[D_{\rho'_i}]$$

where

$$[D_{\rho_i}] = \sum_{j=1}^{t} -a_{ij}[D_{\rho'_j}]$$

for $i = 1, \ldots, n$. We use this description to identify $\operatorname{Pic}(Y)$ with $\mathbb{Z}^t$.

Similarly, we obtain

$$\operatorname{Pic}(X) = \bigoplus_{i=1}^{t} \mathbb{Z}[D_{\rho'_i}] \oplus [D_{\rho_\tau}]$$

where

$$[D_{\rho_i}] = -\sum_{j=1}^{t} a_{ij}[D_{\rho'_j}] - [D_{\rho_\tau}]$$

for $i = 1, \ldots, s$ and

$$[D_{\rho_i}] = -\sum_{j=1}^{t} a_{ij}[D_{\rho'_j}]$$

for $i = s+1, \ldots, n$.

It follows that the Cox ring of Y is

$$S[Y] = \mathbb{C}[\beta_1, \ldots, \beta_n, \beta'_1, \ldots, \beta'_t]$$

with $\deg(\beta_i) = [D_{\rho_i}] = -\sum_{j=1}^{t} a_{ij}\mathbf{e}_i$ for $i = 1, \ldots, n$ and $\deg(\beta'_i) = \mathbf{e}_i$ where $\mathbf{e}_1, \ldots, \mathbf{e}_t$ is the standard basis of $\mathbb{Z}^t \cong \operatorname{Pic}(Y)$.

Similarly, the Cox ring of X is

$$S[X] = \mathbb{C}[\alpha_1, \ldots, \alpha_n, \alpha'_1, \ldots, \alpha'_t, \alpha'']$$

with

$$\begin{aligned}
\deg(\alpha_i) &= [D_{\rho_i}] = -\sum_{j=1}^{t} a_{ij}\mathbf{f}_j - \mathbf{f}_{t+1} \text{ for } i = 1, \ldots, s, \\
\deg(\alpha_i) &= [D_{\rho_i}] = -\sum_{j=1}^{t} a_{ij}\mathbf{f}_j \text{ for } i = s+1, \ldots, n, \\
\deg(\alpha'_i) &= [D_{\rho'_i}] = \mathbf{f}_i \text{ for } i = 1, \ldots, t \\
\deg(\alpha'') &= [D_{\rho_\tau}] = \mathbf{f}_{t+1}
\end{aligned}$$

where $\mathbf{f}_1, \dots, \mathbf{f}_{t+1}$ is the standard basis of $\mathbb{Z}^{t+1} \cong \mathrm{Pic}(X)$.

We will lift the map $f\colon X \to Y$ to a map of Cox rings $\overline{f}^{\#}\colon S[Y] \to S[X]$ as in Definition 4.2. We start with describing the pullback map $f^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$.

Lemma 4.20. *In the above notation, the pullback map $f^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ is given by $\mathbf{e}_i \mapsto \mathbf{f}_i$ for $i = 1, \dots, t$.*

Proof. For $i = 1, \dots, t$, let $\varphi_{D_{\rho_i'}}\colon N \to \mathbb{R}$ be the support function of the torus invariant Cartier divisor $D_{\rho_i'}$ on Y (see [28, Def. 4.2.11]). Then by [28, Prop. 6.2.7], $f^*([D_{\rho_i'}])$ is the class of the torus invariant Cartier divisor on X corresponding to the same support function. By definition

$$\varphi_{D_{\rho_i'}}(\mathbf{u}_\rho) = -\delta_{\rho\rho_i'} \text{ for } \rho \in \Sigma_Y(1).$$

In particular, $\varphi_{D_{\rho_i'}}(\mathbf{u}_\tau) = 0$ since $\varphi_{D_{\rho_i'}}$ is zero on each ray generator of the cone τ. Thus, $f^*(\mathbf{e}_i) = \mathbf{f}_i$ for $i = 1, \dots, t$. □

Now we describe a lift of $f\colon X \to Y$ to a map $\overline{f}^{\#}\colon S[Y] \to S[X]$.

Proposition 4.21. *In the above notation the $\mathbb{C}$-algebra homomorphism*

$$\overline{f}^{\#}\colon S[Y] \to S[X]$$

given by

$$\begin{aligned} &\beta_i \mapsto \alpha_i \cdot \alpha'' \text{ for } i = 1, \dots, s \\ &\beta_i \mapsto \alpha_i \text{ for } i = s+1, \dots, n \\ &\beta_i' \mapsto \alpha_i' \text{ for } i = 1, \dots, t \end{aligned}$$

is a lift of $f\colon X \to Y$ as in Definition 4.2. In particular, if r is a positive integer and $[I] \in \mathrm{Slip}_{r,X}$, then $[(\overline{f}^{\#})^{-1}(I)] \in \mathrm{Slip}_{r,Y}$.

Proof. By Lemma 4.20, the homomorphism $\overline{f}^{\#}$ is a map of graded rings with respect to the homomorphism $f^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ of grading groups, i.e.

$$\overline{f}^{\#}(S[Y]_{[D]}) \subseteq S[X]_{f^*([D])}$$

for every $[D] \in \mathrm{Pic}(Y)$.

It follows that $\overline{f}^{\#}$ defines an equivariant map $\overline{f}\colon \overline{X} = \operatorname{Spec} S[X] \to \operatorname{Spec} S[Y] = \overline{Y}$. Let $B(\Sigma_X) \subseteq S[X]$ and $B(\Sigma_Y) \subseteq S[Y]$ be the irrelevant ideals. We claim that $\overline{f}$ restricts to a map

$$\widehat{f}\colon \overline{X} \setminus V\big(B(\Sigma_X)\big) = \widehat{X} \to \widehat{Y} = \overline{Y} \setminus V\big(B(\Sigma_Y)\big).$$

Recall that $\Sigma_Y(n)$ is the set of maximal cones of the fan Σ_Y. For $\sigma \in \Sigma_Y(n)$ let $\beta^{\widehat{\sigma}}$ be the product of variables in $S[Y]$ corresponding to rays in $\Sigma_Y(1)$ which are not rays in $\sigma(1)$. Then,

$$B(\Sigma_Y) = (\beta^{\widehat{\sigma}})_{\sigma \in \Sigma_Y(n)}$$

by [28, page 207].

We will consider two types of maximal cones in $\Sigma_Y(n)$. Namely, $\Sigma_Y(n) = \Sigma'_Y(n) \cup \Sigma''_Y(n)$ where

$$\Sigma'_Y(n) = \{\sigma \in \Sigma_Y(n) \mid \tau \nsubseteq \sigma\}$$

and

$$\Sigma''_Y(n) = \{\sigma \in \Sigma_Y(n) \mid \tau \subseteq \sigma\}.$$

For a cone $\sigma \in \Sigma''_Y(n)$ and $i \in \{1,\dots,s\}$ we define $\sigma_i = \operatorname{Cone}\Big(\big(\bigcup_{\rho\in\sigma(1),\rho\neq\rho_i}\{\mathbf{u}_\rho\}\big) \cup \{\mathbf{u}_\tau\}\Big)$. Then $\Sigma_X(n) = \Sigma'_Y(n) \cup \bigcup_{\sigma\in\Sigma''_Y(n)} \bigcup_{i=1}^{s}\{\sigma_i\}$. For a cone $\sigma \in \Sigma_X(n)$, let $\alpha^{\widehat{\sigma}}$ be the product of variables of $S[X]$ corresponding to rays from $\Sigma_X(1) \setminus \sigma(1)$. Then

$$B(\Sigma_X) = (\alpha^{\widehat{\sigma}})_{\sigma\in\Sigma_X(n)}.$$

The map $\overline{f}\colon \mathbb{A}^{n+t+1} \to \mathbb{A}^{n+t}$ is given by

$$p := (a_1,\dots,a_n,a'_1,\dots,a'_t,a'') \mapsto (a_1a'',\dots,a_sa'',a_{s+1},\dots,a_n,a'_1,\dots,a'_t).$$

Assume that $\overline{f}(p) \in V\big(B(\Sigma_Y)\big)$. We shall show that $p \in V\big(B(\Sigma_X)\big)$.

Let $\sigma \in \Sigma'_Y(n)$. Then

$$0 = \beta^{\widehat{\sigma}}\big(\overline{f}(p)\big) = (a'')^k \cdot \big(\beta^{\widehat{\sigma}}(a_1,\dots,a_n,a'_1,\dots,a'_t)\big) = (a'')^{k-1} \cdot \big(\alpha^{\widehat{\sigma}}(p)\big)$$

where $s \geq k \geq 1$ is the number of rays in $\tau(1) \setminus \sigma(1)$. It follows that $\alpha^{\widehat{\sigma}}(p) = 0$.

Let $\sigma \in \Sigma''_Y(n)$ and $i \in \{1,\dots,s\}$. Then

$$\alpha^{\widehat{\sigma_i}}(p) = a_i \cdot \Big(\beta^{\widehat{\sigma}}\big(\overline{f}(p)\big)\Big) = 0.$$

We have shown that $p \in V\big(B(\Sigma_X)\big)$.

By Lemma 4.4, in order to verify that $\overline{f}^{\#}$ is a lift of f, it suffices to show that

$$\prod_{i=1}^{s}(\alpha_i\alpha'')^{\langle m,\mathbf{u}_{\rho_i}\rangle} \cdot \prod_{i=s+1}^{n} \alpha_i^{\langle m,\mathbf{u}_{\rho_i}\rangle} \cdot \prod_{i=1}^{t} {\alpha'}_i^{\langle m,\mathbf{u}_{\rho'_i}\rangle} = \prod_{i=1}^{s} \alpha_i^{\langle m,\mathbf{u}_{\rho_i}\rangle} \cdot \prod_{i=s+1}^{n} \alpha_i^{\langle m,\mathbf{u}_{\rho_i}\rangle} \cdot \prod_{i=1}^{t} {\alpha'}_i^{\langle m,\mathbf{u}_{\rho'_i}\rangle} \cdot {\alpha''}^{\langle m,\mathbf{u}_\tau\rangle}$$

for every $m \in M$. This holds since $\mathbf{u}_\tau = \sum_{i=1}^{s}\mathbf{u}_{\rho_i}$.

The last part of the proposition follows from Theorem 4.15. □

4.4 Toric bundle

In this section we study another special case of Theorem 4.15 - where X is a decomposable toric vector bundle.

Let Y be a smooth projective toric variety defined by a fan $\Sigma_Y \subseteq N_{\mathbb{R}}$. Let n be a positive integer and consider torus invariant divisors $D_i = \sum_{\rho\in\Sigma_Y(1)} a_{i\rho}D_\rho$ for $i = 0,\dots,n$, where D_ρ's are prime torus invariant divisors of Y corresponding to rays of Σ_Y and $a_{i\rho}$'s are integers. Let $\mathscr{E} = \mathcal{O}_Y(D_0) \oplus \dots \oplus \mathcal{O}_Y(D_n)$ and let $X = \mathbb{P}(\mathscr{E})$. Then X is a smooth projective toric variety (see [28, Prop. 7.3.3.]). We construct a lift $\overline{f}^{\#}\colon S[Y] \to S[X]$ of the natural projection $f\colon X \to Y$. A special case of interest is when $Y = \mathbb{P}^a \times \mathbb{P}^b$, $X = \mathbb{P}^a \times \mathbb{P}^b \times \mathbb{P}^c$ and f is the projection.

We start with describing the Cox ring of X. Let $f_1,\dots,f_m$ be a basis of the lattice M

and let $\mathbf{f}_1^*, \ldots, \mathbf{f}_m^*$ be its dual basis. Consider $\mathbb{Z}^n$ with standard $\mathbb{Z}$-basis $\mathbf{e}_1^*, ..., \mathbf{e}_n^*$ and let $\mathbf{e}_0^* = -\mathbf{e}_1^* - ... - \mathbf{e}_n^*$. Given a ray generator $\mathbf{u}_\rho \in N_\mathbb{R}$ with $\rho \in \Sigma_Y(1)$ we define

$$\mathbf{v}_\rho = \mathbf{u}_\rho + (a_{1\rho} - a_{0\rho})\mathbf{e}_1^* + ... + (a_{n\rho} - a_{0\rho})\mathbf{e}_n^* \in N_\mathbb{R} \times \mathbb{R}^n.$$

The cones of the fan $\Sigma_X \subseteq N_\mathbb{R} \times \mathbb{R}^n$ of X are of the form

$$\sigma_i = \mathrm{Cone}(\mathbf{v}_\rho \mid \rho \in \sigma(1)) + \mathrm{Cone}(\mathbf{e}_0^*, ..., \widehat{\mathbf{e}_i^*}, ..., \mathbf{e}_n^*)$$

together with their faces, where $\sigma \in \Sigma_Y$, $i \in \{0, ..., n\}$ and $\widehat{\mathbf{e}_i^*}$ means that $\mathbf{e}_i^*$ is omitted ([28, Prop. 7.3.3.]). Thus, the ray generators of the fan of X are

$$\{\mathbf{v}_\rho \mid \rho \in \Sigma_Y(1)\} \cup \{\mathbf{e}_0^*, ..., \mathbf{e}_n^*\}.$$

In particular, by [28, Thm. 4.1.3] the Picard group of X is generated by the classes of torus invariant divisors F_ρ for $\rho \in \Sigma_Y(1)$ corresponding to $\mathbf{v}_\rho$ and the classes of $E_0, ..., E_n$ corresponding to $\mathbf{e}_0^*, ..., \mathbf{e}_n^*$. Moreover, these generators are subject to the relations

$$0 = [\mathrm{div}(\chi^{f_i})] = \sum_{\rho \in \Sigma_Y(1)} \langle f_i, \mathbf{u}_\rho \rangle [F_\rho] \text{ for } i = 1, ..., m$$

and

$$0 = [\mathrm{div}(\chi^{e_i})] = [E_i] - [E_0] + \sum_{\rho \in \Sigma_Y(1)} (a_{i\rho} - a_{0\rho})[F_\rho] \text{ for } i = 1, ..., n.$$

Therefore, we have an isomorphism $\mathrm{Pic}(X) \cong \mathrm{Pic}(Y) \times \mathbb{Z}$ given by

$$\begin{aligned} &[F_\rho] \mapsto ([D_\rho], 0) && \text{for } \rho \in \Sigma_Y(1) \\ &[E_i] \mapsto (-\sum_{\rho \in \Sigma_Y(1)} (a_{i\rho} - a_{0\rho})[D_\rho], 1) && \text{for } i = 0, ..., n. \end{aligned}$$

In particular, $[E_0] \in \mathrm{Pic}(X)$ corresponds to $(0, 1) \in \mathrm{Pic}(Y) \times \mathbb{Z}$. From these considerations, it follows that the Cox rings of Y and X are

$$S[Y] = \mathbb{C}[\{\beta_\rho \mid \rho \in \Sigma_Y(1)\}] \text{ and } S[X] = \mathbb{C}[\{\alpha_\rho \mid \rho \in \Sigma_Y(1)\}, \alpha'_0, ..., \alpha'_n]$$

with

$$\begin{aligned} &\deg(\beta_\rho) = [D_\rho] \in \mathrm{Pic}(Y) \text{ for } \rho \in \Sigma_Y(1), \\ &\deg(\alpha_\rho) = [F_\rho] = ([D_\rho], 0) \in \mathrm{Pic}(Y) \times \mathbb{Z} \text{ for } \rho \in \Sigma_Y(1) \text{ and} \\ &\deg(\alpha'_i) = [E_i] \text{ for } i = 0, \ldots, n. \end{aligned}$$

Let $\phi\colon N \times \mathbb{Z}^n \to N$ be the natural surjection of lattices. If τ is a face of $\sigma_i \in \Sigma_X$ for some $\sigma \in \Sigma_Y$ and $i \in \{0, \ldots, n\}$, then $\phi_\mathbb{R}(\tau)$ is a face of σ. Therefore, the map ϕ is compatible with the fans of X and Y. Thus, it induces a toric morphism $f\colon X \to Y$ [28, Thm. 3.3.4]. We want to lift this morphism to a homomorphism of Cox rings as in Definition 4.2. First we describe the pullback map $f^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$.

Lemma 4.22. *In the above notation, the pullback map $f^*\colon \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ maps $[D_\rho]$ to $[F_\rho]$*

for every $\rho \in \Sigma_Y(1)$.

Proof. Let $\varphi_{D_\rho}\colon N_{\mathbb{R}} \to \mathbb{R}$ be the support function corresponding to D_ρ, i.e. it is linear on each cone of Σ_Y and $\varphi_{D_\rho}(\mathbf{u}_{\rho'}) = -\delta_{\rho\rho'}$ (see [28, Thm. 4.2.12]). Then by [28, Thm. 6.2.7] $f^*([D_\rho])$ is the class of the torus invariant Cartier divisor with support function

$$\psi\colon N_{\mathbb{R}} \times \mathbb{R}^n \xrightarrow{\phi_{\mathbb{R}}} N_{\mathbb{R}} \xrightarrow{\varphi_{D_\rho}} \mathbb{R}.$$

We have $\psi(\mathbf{v}_{\rho'}) = \varphi_{D_\rho}(\mathbf{u}_{\rho'}) = -\delta_{\rho\rho'}$ and $\psi(\mathbf{e}_i^*) = 0$. Thus, ψ is the support function of F_ρ. $\square$

Now we describe a lift of f to a map $\overline{f}^\#\colon S[Y] \to S[X]$.

Proposition 4.23. *In the above notation, the* $\mathbb{C}$*-algebra homomorphism* $\overline{f}^\#\colon S[Y] \to S[X]$ *given by* $\beta_\rho \mapsto \alpha_\rho$ *for* $\rho \in \Sigma_Y(1)$ *is a lift of* f *as in Definition 4.2. In particular, if* $[I] \in \mathrm{Slip}_{r,X}$ *is a closed point, then* $[(\overline{f}^\#)^{-1}(I)] \in \mathrm{Slip}_{r,Y}$.

Proof. By Lemma 4.22, the homomorphism $\overline{f}^\#$ is a homomorphism of graded rings, i.e.

$$\overline{f}^\#(S[Y]_{[D]}) \subseteq S[X]_{f^*([D])}$$

for every $[D] \in \mathrm{Pic}(Y)$. Therefore, it induces an equivariant map $\overline{f}\colon \overline{X} \to \overline{Y}$. We claim that it restricts to a morphism $\widehat{f}\colon \widehat{X} \to \widehat{Y}$. The map $\overline{f}$ is defined by

$$(a_1, \ldots, a_s, a_0', \ldots, a_n') \mapsto (a_1, \ldots, a_s),$$

where the first $s = \#\Sigma_Y(1)$ coordinates of the affine space $\overline{X}$ correspond to α_ρ's. By $\Sigma_Y(m)$ we denote the set of m-dimensional cones of Σ_Y. Given $\sigma \in \Sigma_Y$, let $\beta^{\widehat{\sigma}} = \prod_{\rho\in\Sigma_Y(1)\setminus\sigma(1)} \beta_\rho$. Recall that $B(\Sigma_Y) = (\beta^{\widehat{\sigma}} \mid \sigma \in \Sigma_Y(m))$. Similarly, we have $B(\Sigma_X) = (\alpha^{\widehat{\sigma}_i} \mid \sigma \in \Sigma_Y(m), i \in \{0, ..., n\})$. Assume that $(a_1, \ldots, a_s) \in V(B(\Sigma_Y))$. Let $\sigma \in \Sigma_Y(m)$ and $i \in \{0, ..., n\}$. Then $\beta^{\widehat{\sigma}}(a_1, \ldots, a_s) = 0$ implies that

$$0 = a_i' \cdot \beta^{\widehat{\sigma}}(a_1, \ldots, a_s) = \alpha^{\widehat{\sigma}_i}(a_1, \ldots, a_s, a_0', \ldots, a_n').$$

Thus, $(a_1, \ldots, a_s, a_0', \ldots, a_n') \in V(B(\Sigma_X))$.

We have shown, that $\overline{f}\colon \overline{X} \to \overline{Y}$ restricts to a map $\widehat{f}\colon \widehat{X} \to \widehat{Y}$. Therefore, we also have an induced morphism $f'\colon X \to Y$. We claim that $f = f'$. By Lemma 4.4, it is enough to show that

$$\prod_{\rho\in\Sigma_Y(1)} \alpha_\rho^{\langle m,\mathbf{u}_\rho\rangle} = \prod_{\rho\in\Sigma_Y(1)} \alpha_\rho^{\langle\psi(m),\mathbf{v}_\rho\rangle} \cdot \prod_{i=0}^{n} \alpha_i'^{\langle\psi(m),\mathbf{e}_i^*\rangle}$$

for every $m \in M$, where $\psi\colon M \to M \times \mathbb{Z}^n$ is dual to $N \times \mathbb{Z}^n \to N$ (i.e., it is the natural inclusion). The claimed equality follows from the definition of $\mathbf{v}_\rho$.

The last part of the proposition is implied by Theorem 4.15. $\square$

4.5 Product of projective spaces

Let X be the product of projective spaces $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ for some positive integers $k \geq 2, n_1, \ldots, n_k$. Proposition 4.23 gives a necessary condition for $[I] \in \mathrm{Hilb}^{h_{r,X}}_{S[X]}$ to be in $\mathrm{Slip}_{r,X}$. In this section, we will present another condition that must be fulfilled for $[I]$ to be in the irreducible component $\mathrm{Slip}_{r,X}$.

The Cox ring of X is of the form

$$S[X] = \mathbb{C}[\alpha_{1,0}, \ldots, \alpha_{1,n_1}, \alpha_{2,0}, \ldots, \alpha_{2,n_2}, \ldots, \alpha_{k,0}, \ldots, \alpha_{k,n_k}].$$

It has a grading in $\mathbb{Z}^k$. In the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_k$ of $\mathbb{Z}^k$ we have $\deg(\alpha_{i,j}) = \mathbf{e}_i$ for $i \in \{1, \ldots, k\}$ and $j \in \{0, \ldots, n_i\}$. The Cox ring of the i-th factor of $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ is the polynomial ring $\mathbb{C}[\alpha_{i,0}, \ldots, \alpha_{i,n_i}]$ with the standard $\mathbb{Z}$-grading, i.e. $\deg(\alpha_{i,j}) = 1$ for $j \in \{0, \ldots, n_i\}$. The irrelevant ideal $B(\Sigma_i)$ of $\mathbb{P}^{n_i}$ is $B(\Sigma_i) = (\alpha_{i,0}, \ldots, \alpha_{i,n_i})$. The irrelevant ideal of X is $B(\Sigma_X) = B(\Sigma_1) \cdot \ldots \cdot B(\Sigma_k) = \big(S[X]_{(1,1,\ldots,1)}\big)$.

We will use the following lemma about Hilbert functions of quotient algebras of homogeneous ideals in $S[X]$.

Lemma 4.24. *Let $I \neq (1)$ be a $\mathbb{Z}^k$-graded ideal in $S[X]$. Assume that I is saturated with respect to the irrelevant ideal $B(\Sigma_X)$. Then for all $i \in \{1, \ldots, k\}$ there is a homogeneous element $\gamma_i \in S[X]_{\mathbf{e}_i}$ such that γ_i is a non-zero divisor on $S[X]/I$. Therefore:*

(i) For all $\mathbf{u} \in \mathbb{Z}^k$ and for all $i \in \{1, 2, \ldots, k\}$ we have $H_{S[X]/I}(\mathbf{u}) \leq H_{S[X]/I}(\mathbf{u} + \mathbf{e}_i)$.

(ii) Let $\mathbf{u} \in \mathbb{Z}^k_{\geq 0}$ and assume that there is $i \in \{1, 2, \ldots, k\}$ such that

$$H_{S[X]/I}(\mathbf{u}) = H_{S[X]/I}(\mathbf{u} + \mathbf{e}_i).$$

Then $H_{S[X]/I}(\mathbf{u} + \mathbf{e}_i) = H_{S[X]/I}(\mathbf{u} + 2\mathbf{e}_i)$.

Proof. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the associated primes of $S[X]/I$. These are $\mathbb{Z}^k$-graded ideals. We claim that for each i, the $\mathbb{C}$-vector subspace $\bigcup_{j=1}^s (\mathfrak{p}_j)_{\mathbf{e}_i} \subseteq S[X]_{\mathbf{e}_i}$ is a proper subspace. Indeed, otherwise, there is $j \in \{1, \ldots, s\}$ such that $(\mathfrak{p}_j)_{\mathbf{e}_i} = S[X]_{\mathbf{e}_i}$. Therefore, $B(\Sigma_X) \subseteq \mathfrak{p}_j$. We obtain $(I\colon B(\Sigma_X)) \neq I$ which contradicts the assumption that I is saturated with respect to $B(\Sigma_X)$. Having established the claim, we proceed to the proof of the lemma.

(i) By the above claim, for every $i \in \{1, \ldots, k\}$ there is a homogeneous non-zero divisor γ_i on $S[X]/I$ of degree $\mathbf{e}_i$. It follows that the map $(S[X]/I)_{\mathbf{u}} \to (S[X]/I)_{\mathbf{u}+\mathbf{e}_i}$ given by multiplication by γ_i is injective.

(ii) Let $[\Theta] \in (S[X]/I)_{\mathbf{u}+2\mathbf{e}_i}$. Then there are $\Theta_0, \ldots, \Theta_{n_i} \in S[X]_{\mathbf{u}+\mathbf{e}_i}$ such that

$$[\Theta] = [\alpha_{i,0}\Theta_0 + \ldots + \alpha_{i,n_i}\Theta_{n_i}].$$

Using the notation of the proof of part (i), multiplication by γ_i gives an isomorphism $(S[X]/I)_{\mathbf{u}} \to (S[X]/I)_{\mathbf{u}+\mathbf{e}_i}$. Therefore, there are $\Gamma_0, \ldots, \Gamma_{n_i} \in S[X]_{\mathbf{u}}$ such that $[\Theta_j] = [\gamma_i \Gamma_j]$ for $j \in \{0, \ldots, n_i\}$. It follows that $[\Theta] = \gamma_i([\alpha_{i,0}\Gamma_0 + \ldots + \alpha_{i,n_i}\Gamma_{n_i}])$. Thus, the injective map $(S[X]/I)_{\mathbf{u}+\mathbf{e}_i} \to (S[X]/I)_{\mathbf{u}+2\mathbf{e}_i}$ given by multiplication by γ_i is in fact bijective.

□

We present a necessary condition for $[I] \in \mathrm{Hilb}^{h_{r,X}}_{S[X]}$ to be in the irreducible component $\mathrm{Slip}_{r,X}$.

Theorem 4.25. *Let $X = \mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_k}$ for some positive integers $k \geq 2, n_1, \ldots, n_k$. For $i \in \{1, \ldots, k\}$ let $B(\Sigma_i) \subseteq S[X]$ be the extension of the irrelevant ideal of $\mathbb{P}^{n_i}$ under the natural inclusion $S[\mathbb{P}^{n_i}] \to S[X]$. If $[I] \in \mathrm{Slip}_{r,X}$ for some positive integer r, then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{S[X]}\Big(I + B(\Sigma_i)^2, S[X]/\big(I + B(\Sigma_i)^2\big)\Big)_0 \geq r(n_1 + \ldots + n_k)$$

for $i \in \{1, \ldots, k\}$.

Proof. Assume that $[I] \in \mathrm{Hilb}^{h_{r,X}}_{S[X]}$ where $I \subseteq S[X]$ is an ideal saturated with respect to $B(\Sigma_X)$. Fix an integer $i \in \{1, \ldots, k\}$ and let

$$\mathcal{A}_i = \{\mathbf{u} = (u_1, \ldots, u_k) \in \mathbb{Z}^k \mid u_j \geq 0 \text{ for } j \in \{1, \ldots, k\} \text{ and } u_i \in \{0, 1\}\}.$$

Let J be the ideal of $S[X]$ generated by

$$\bigoplus_{\mathbf{u} \in \mathcal{A}_i} I_{\mathbf{u}}.$$

We claim that $(J : B(\Sigma_X)^\infty) = I$. We first show how to conclude the proof using the claim. Let g be the Hilbert function of $S[X]/(I + B(\Sigma_i)^2)$ and consider the natural map

$$\chi\colon \mathrm{Hilb}^{h_{r,X}}_{S[X]} \to \mathrm{Hilb}^{g}_{S[X]}$$

given on closed points by $\chi([I]) = [I + B(\Sigma_i)^2]$. It follows from the claim that χ is injective on points corresponding to ideals that are saturated with respect to $B(\Sigma_X)$. Indeed, the inverse map is $[I'] \mapsto [\big(\bigoplus_{\mathbf{u} \in \mathcal{A}_i} I'_{\mathbf{u}}\big) \colon B(\Sigma_X)^\infty]$. Since a general closed point of $\mathrm{Slip}_{r,X}$ corresponds to an ideal of $S[X]$ that is saturated with respect to $B(\Sigma_X)$, it follows that the image of $\mathrm{Slip}_{r,X}$ under χ is of dimension $\dim \mathrm{Slip}_{r,X} = r(n_1 + \ldots + n_k)$. Therefore, if $[I] \in \mathrm{Slip}_{r,X}$ then the tangent space to $\mathrm{Hilb}^{g}_{S[X]}$ at $\chi([I])$ is of dimension at least $r(n_1 + \ldots + n_k)$. Application of Theorem 2.74 finishes the proof of the theorem.

We are left with proving the claim. Let $K = (J : B(\Sigma_X)^\infty)$. Since $J \subseteq K \subseteq I$, it follows that for $\mathbf{u} \in \mathcal{A}_i$ we have $K_{\mathbf{u}} = J_{\mathbf{u}} = I_{\mathbf{u}}$. Let $\mathbf{u} = (u_1, \ldots, u_k) \in \mathcal{A}_i$ with $u_j \geq r$ for $j \neq i$ and $u_i = 0$. Since

$$H_{S[X]/K}(\mathbf{u}) = H_{S[X]/K}(\mathbf{u} + \mathbf{e}_i) = r,$$

it follows from Lemma 4.24(ii) that

$$H_{S[X]/K}(\mathbf{u}) = r$$

for all $\mathbf{u} = (u_1, \ldots, u_k) \in \mathbb{Z}^k$ with $u_j \geq r$ for $j \in \{1, 2, \ldots, k\}$. Therefore, $H_{S[X]/K}(\mathbf{u}) \leq r$ for all $\mathbf{u} \in \mathbb{Z}^k$ by Lemma 4.24(i). Since $K \subseteq I$ and $S[X]/I$ has Hilbert function $h_{r,X}$, it follows that the Hilbert function of $S[X]/K$ is also $h_{r,X}$. Thus, $K = I$ as claimed. □

4.6 Small examples of reducible multigraded Hilbert schemes

In this section we show that $\mathrm{Hilb}^{h_{2,X}}_{S[X]}$ need not be irreducible for a smooth projective toric surface X. We present two examples: Hirzebruch surface $\mathscr{H}_1 = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big)$ and $\mathbb{P}^1 \times \mathbb{P}^1$. These examples illustrate that the necessary condition described in Theorem 4.15 is in general not sufficient even for small values of r and $\dim X$. In fact, as we shall see, this condition is trivially satisfied in these two cases.

Both special versions of Theorem 4.15 studied in Sections 4.3 and 4.4 apply to multigraded Hilbert scheme $\mathrm{Hilb}^{h_{2,\mathscr{H}_1}}_{S[\mathscr{H}_1]}$ since $\mathscr{H}_1$ is also the blowup of $\mathbb{P}^2$ at a torus invariant point. However, we present also the example of $\mathrm{Hilb}^{h_{2,\mathbb{P}^1 \times \mathbb{P}^1}}_{S[\mathbb{P}^1 \times \mathbb{P}^1]}$ to demonstrate that Theorem 4.25 gives some insight into $\mathrm{Slip}_{2,\mathbb{P}^1 \times \mathbb{P}^1}$ even though Theorem 4.15 is of no use in this case.

Proposition 4.26. *Let $S[\mathbb{P}^1 \times \mathbb{P}^1] = \mathbb{C}[\alpha_0, \alpha_1, \beta_0, \beta_1]$ be the Cox ring of $\mathbb{P}^1 \times \mathbb{P}^1$ with* $\deg(\alpha_0) = \deg(\alpha_1) = (1,0)$ *and* $\deg(\beta_0) = \deg(\beta_1) = (0,1)$. *Then* $\mathrm{Hilb}^{h_{2,\mathbb{P}^1\times\mathbb{P}^1}}_{S[\mathbb{P}^1\times\mathbb{P}^1]}$ *is not irreducible. In fact,* $[(\alpha_0\alpha_1, \alpha_0\beta_0, \alpha_1\beta_0, \beta_0\beta_1)] \in \mathrm{Hilb}^{h_{2,\mathbb{P}^1\times\mathbb{P}^1}}_{S[\mathbb{P}^1\times\mathbb{P}^1]} \setminus \mathrm{Slip}_{2,\mathbb{P}^1\times\mathbb{P}^1}$.

Proof. Let $I = (\alpha_0\alpha_1, \alpha_0\beta_0, \alpha_1\beta_0, \beta_0\beta_1)$. Then $[I] \in \mathrm{Hilb}^{h_{2,\mathbb{P}^1\times\mathbb{P}^1}}_{S[\mathbb{P}^1\times\mathbb{P}^1]}$. We claim that $[I] \notin \mathrm{Slip}_{2,\mathbb{P}^1\times\mathbb{P}^1}$. Let $\mathfrak{a} = (\alpha_0, \alpha_1)$ and $J = I + \mathfrak{a}^2$. Then $\mathrm{Hom}_{S[\mathbb{P}^1\times\mathbb{P}^1]}(J, S[\mathbb{P}^1\times\mathbb{P}^1]/J)_0 = 2$. Thus, $[I] \notin \mathrm{Slip}_{2,\mathbb{P}^1\times\mathbb{P}^1}$ by Theorem 4.25. $\square$

The case of the Hirzebruch surface $\mathscr{H}_1$ will be more involved since we lack a criterion analogous to Theorem 4.25.

Proposition 4.27. *Let $\mathscr{H}_1 = \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1)\big)$ be the Hirzebruch surface. The multigraded Hilbert scheme* $\mathrm{Hilb}^{h_{2,\mathscr{H}_1}}_{S[\mathscr{H}_1]}$ *is not irreducible.*

Proof. We start with calculating the Cox ring of $\mathscr{H}_1$. Let $S[\mathbb{P}^2] = \mathbb{C}[\beta_0, \beta_1, \beta_2]$ with $\deg(\beta_0) = \deg(\beta_1) = \deg(\beta_2) = 1$. The Hirzebruch surface $\mathscr{H}_1$ can be constructed as the blowup of $\mathbb{P}^2$ at the torus invariant point $[0:1:0]$.

Then the Cox ring $S[\mathscr{H}_1]$ of $\mathscr{H}_1$ is $\mathbb{C}[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ with $\deg(\alpha_0) = \deg(\alpha_2) = (1,-1)$, $\deg(\alpha_1) = (1,0)$ and $\deg(\alpha_3) = (0,1)$ (see Section 4.3). Moreover, by Proposition 4.21 the graded homomorphism of graded rings $S[\mathbb{P}^2] \to S[\mathscr{H}_1]$ given by $\beta_0 \mapsto \alpha_0\alpha_3$, $\beta_1 \mapsto \alpha_1$, $\beta_2 \mapsto \alpha_2\alpha_3$ is a lift of the natural map $\mathscr{H}_1 \to \mathbb{P}^2$. We identify $S[\mathbb{P}^2]$ with its image in $S[\mathscr{H}_1]$.

Let W be the locus of those points $[I]$ of $\mathrm{Hilb}^{h_{2,\mathbb{P}^2}}_{S[\mathbb{P}^2]}$ for which the unique linear generator of I is of the form $a\alpha_0\alpha_3 + b\alpha_2\alpha_3$ for some $a, b \in \mathbb{C}$ (or geometrically, the locus of points defining subschemes contained in a line passing through the center of the blowup). We claim that W is irreducible and 3-dimensional. Indeed, it is a $\mathrm{Hilb}^{h_{2,\mathbb{P}^1}}_{S[\mathbb{P}^1]}$-bundle over the projective line $\mathbb{P}(\mathrm{lin}\{\alpha_0\alpha_3, \alpha_2\alpha_3\})$. Since $\mathrm{Hilb}^{h_{2,\mathbb{P}^1}}_{S[\mathbb{P}^1]} \cong \mathbb{P}^2$ (see Proposition 3.35), our claim follows.

Let $\pi\colon \mathrm{Hilb}^{h_{2,\mathscr{H}_1}}_{S[\mathscr{H}_1]} \to \mathrm{Hilb}^{h_{2,\mathbb{P}^2}}_{S[\mathbb{P}^2]}$ be the natural map from Theorem 4.15. We claim that the set-theoretic inverse image V of W is of dimension at least 4. Let $[I] \in W$ be a closed point. We may assume that

$$I = (\alpha_0\alpha_3, \theta_2), \text{ where } \theta_2 = A\alpha_2^2\alpha_3^2 + B\alpha_1\alpha_2\alpha_3 + C\alpha_1^2$$

for some $A, B, C \in \mathbb{C}$, not all zero. It is enough to show that the fiber over $[I]$ is of positive dimension. Let $[a:b] \in \mathbb{P}^1$. We claim that

$$[J] = [(\alpha_0\alpha_3, \theta_2, \alpha_0(a\alpha_0 + b\alpha_2), \alpha_0\alpha_1)]$$

is a point of that fiber. We need to check two things:

1. $J \cap S[\mathbb{P}^2] = I$;

2. $S[\mathscr{H}_1]/J$ has Hilbert function $h_{2,\mathscr{H}_1}$.

We start with 1. Clearly, $I \subseteq J \cap S[\mathbb{P}^2]$. Suppose that $f \in (\alpha_0(a\alpha_0 + b\alpha_2), \alpha_0\alpha_1)_{(d,0)}$ for some positive integer d. We shall show that $f \in (\alpha_0\alpha_3)$. Observe that $\deg(\alpha_0(a\alpha_0 + b\alpha_2)) = (2,-2)$ and $\deg(\alpha_0\alpha_1) = (2,-1)$. Since $\deg(f) = (d,0)$, we get that α_3 divides f. This shows that 1. is fulfilled.

Now we show that 2. holds. We have $J \subseteq K = (\alpha_0, \theta_2)$. Moreover, if $J_{(a,b)} \neq K_{(a,b)}$ then $b = -a < 0$ and $\dim_{\mathbb{C}}(S[\mathscr{H}_1]/K)_{(a,-a)} = \dim_{\mathbb{C}}(S[\mathscr{H}_1]/J)_{(a,-a)} - 1$. Thus, it is enough to show that

$$\dim_{\mathbb{C}}(S[\mathscr{H}_1]/K)_{(a,b)} = \begin{cases} h_{2,\mathscr{H}_1}(a,b) - 1 = 1 & \text{if } b = -a < 0; \\ h_{2,\mathscr{H}_1}(a,b) & \text{otherwise.} \end{cases}$$

We can rewrite this as follows:

$$\dim_{\mathbb{C}}(S[\mathscr{H}_1]/K)_{(a,b)} = \begin{cases} 0 & \text{if } a+b<0 \text{ or } a<0; \\ 1 & \text{if } a+b=0 \text{ and } a \geq 0; \\ 1 & \text{if } a=0 \text{ and } b \geq 0; \\ 2 & \text{if } a+b>0 \text{ and } a>0. \end{cases} \tag{4.28}$$

Let $R = \mathbb{C}[\alpha_1, \alpha_2, \alpha_3] \subseteq S[\mathscr{H}_1]$. Then

$$\dim_{\mathbb{C}} R_{(a,b)} = \begin{cases} 0 & \text{if } a+b<0 \text{ or } a<0; \\ a+b+1-\max\{0,b\} & \text{otherwise.} \end{cases} \tag{4.29}$$

Indeed, $\deg(\alpha_1) = (1,0)$, $\deg(\alpha_2) = (1,-1)$ and $\deg(\alpha_3) = (0,1)$ so the case $a+b<0$ or $a<0$ is clear. On the other hand, if $a+b \geq 0$ and $a \geq 0$ then $R_{(a,b)}$ is spanned by

$$\{\alpha_1^{a+b-c}\alpha_2^{c-b}\alpha_3^{c} \mid \max\{0,b\} \leq c \leq a+b\}.$$

We have

$$\dim_{\mathbb{C}}(S[\mathscr{H}_1]/K)_{(a,b)} = \dim_{\mathbb{C}}(R/(\theta_2))_{(a,b)} = \dim_{\mathbb{C}} R_{(a,b)} - \dim_{\mathbb{C}} R_{(a-2,b)}. \tag{4.30}$$

Equations (4.29) and (4.30) imply Equation (4.28) and thus, finish the proof that V is of dimension at least 4.

Suppose that $\mathrm{Hilb}^{h_{2,\mathscr{H}_1}}_{S[\mathscr{H}_1]}$ is irreducible. Then it is of dimension $\dim \mathrm{Slip}_{2,\mathscr{H}_1} = 4$. It follows that $V = \mathrm{Hilb}^{h_{2,\mathscr{H}_1}}_{S[\mathscr{H}_1]}$ set-theoretically. This contradicts Theorem 4.15 since $W \neq \mathrm{Hilb}^{h_{2,\mathbb{P}^2}}_{S[\mathbb{P}^2]} = \mathrm{Slip}_{2,\mathbb{P}^2}$. $\square$

We conclude this chapter with a remark.

Remark 4.31. In this section we considered two examples of toric morphisms $f\colon X \to Y$ between smooth projective toric varieties such that $f_*\mathcal{O}_X \cong \mathcal{O}_Y$. Hence Theorem 4.15 applies to these cases. However, we have $Y = \mathbb{P}^1$ or $\mathbb{P}^2$ (depending on example) and $r = 2$. Therefore, $\mathrm{Hilb}^{h_{2,Y}}_{S[Y]} = \mathrm{Slip}_{2,Y}$ (see Proposition 3.37). Thus, the necessary condition from Theorem 4.15 is trivially satisfied. Nevertheless, $\mathrm{Slip}_{2,X} \neq \mathrm{Hilb}^{h_{2,X}}_{S[X]}$ in both cases.

Chapter 5

Applications of border apolarity to secant varieties

In this chapter we present some applications of the border apolarity lemma (Proposition 2.91) for studying secant varieties. Section 5.1 deals with existence of homogeneous wild polynomials, i.e. polynomials whose border rank is strictly smaller than the smoothable rank. In Subsection 5.1.1 we show that there are no wild degree d polynomials in three variables of border rank at most $d+2$. In Subsection 5.1.2 we prove that there is no wild quartic in four variables of border rank at most 6. In Subsection 5.1.3 we give an example of a wild quintic in four variables of border rank 7. In Subsection 5.1.4 we show that the known example of a wild cubic of border rank 5 in five variables (see [12, Thm. 1.3]) is a unique such example up to a change of variables. Results from Section 5.1 depend on the criteria developed in Chapter 3. Subsections 5.1.1, 5.1.2 and 5.1.4 are based on [66].

Sections 5.2, 5.3 and 5.4 are based on [39]. This paper uses Proposition 2.91 in a simple form. Namely, we do not use any criteria for distinguishing $\mathrm{Slip}_{r,n}$ from $\mathrm{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$. Therefore, these results are not directly related to the rest of the thesis. Consequently, we present them in special versions, where the proofs are simpler. In Section 5.2 we calculate cactus and border cactus rank of a homogeneous subspace of a divided power ring that is divisible by a large power of a linear form. This is then used in Section 5.3 to describe the points in $\kappa_{14}\big(\nu_d(\mathbb{P}^6)\big) \setminus \sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$ for $d \geq 7$. Results from Section 5.3 have their analogues for $\kappa_{8,3}\big(\nu_d(\mathbb{P}^4)\big) \setminus \sigma_{8,3}\big(\nu_d(\mathbb{P}^4)\big)$. In Section 5.4 we state the main theorem in that direction.

5.1 Wild polynomials

In this section we assume that the base field $\Bbbk$ is the field of complex numbers $\mathbb{C}$ since we cite results from papers in which this is assumed. Let n be a positive integer and $S = \mathbb{C}[\alpha_0, \dots, \alpha_n]$ be the polynomial ring with standard $\mathbb{Z}$-grading. We consider the dual polynomial ring $S^* = \mathbb{C}[x_0, \dots, x_n]$ with the structure of an S-module on S^* given by partial differentiation. We denote this action by $\lrcorner$. Given a homogeneous polynomial $F \in S^*$ we denote by $\mathrm{Ann}(F)$ the ideal $\{\theta \in S \mid \theta \lrcorner F = 0\}$. Recall the definitions of the border rank, the smoothable rank and the cactus rank from Subsection 2.4.2.

We will use the following consequence of the border apolarity lemma (Proposition 2.91).

Corollary 5.1. *Let d be a positive integer and $F \in S^*_d$. Assume that $\operatorname{br}(F) \le r < \operatorname{cr}(F)$ for some integer r. Then there exists an ideal $I \subseteq \operatorname{Ann}(F)$ such that $[I] \in \operatorname{Slip}_{r,n}$ and $\overline{I}_d \ne I_d$.*

Proof. By Proposition 2.91, there is an ideal $[I] \in \operatorname{Slip}_{r,n}$ such that $I \subseteq \operatorname{Ann}(F)$. If $\overline{I}_d = I_d$, then $\overline{I} \subseteq \operatorname{Ann}(F)$ by [11, Prop. 3.4]. Thus, $\operatorname{cr}(F) \le r$ follows from Proposition 2.90. This is a contradiction. □

We always have $\operatorname{br}(F) \le \operatorname{sr}(F)$ (see [55, Lem. 5.17]). Recall that we say that F is wild, if the inequality is strict. Wild polynomials are more difficult to control using standard, existing methods. Therefore, new methods need to be developed in order to study them effectively. For example, see [3, Prop. 11] and its applications, [19, Rmk. 1.5] and [39].

We will study wild homogeneous polynomials F such that $\operatorname{br}(F) \le \deg(F) + 2$. If $\operatorname{br}(F) \le \deg(F) + 1$ then F is not wild. This is established in [11, Prop. 2.5] based on a result in [3]. Therefore, we will assume that $\operatorname{br}(F) = \deg(F) + 2$.

We shall use the following observation.

Lemma 5.2. *Let $r \ge 2$ be an integer and $[I] \in \operatorname{Hilb}^{h_{r,n}}_{S[\mathbb{P}^n]}$ be a closed point. If $\overline{I}_{r-2} \ne I_{r-2}$ then $S/\overline{I}$ has Hilbert function $h_{r,1}$.*

Proof. Let g be the Hilbert function of $S/\overline{I}$. The Hilbert polynomial of $S/\overline{I}$ is r. Therefore, by Lemma 2.9(ii) we have $g(r-2) \le r-1$. Lemma 2.9(iii) implies that $g(0) = 1 < g(1) < g(2) < \ldots < g(r-2) \le r-1$. It follows that $g(a) = a+1$ for every $a \in \{0, \ldots, r-2\}$. Using Lemma 2.9 again we obtain $g(a) = r$ for $a \ge r-1$. Thus, $g = h_{r,1}$. □

5.1.1 Polynomials in three variables of small border rank

In this subsection we prove that there is no wild homogeneous polynomial F in three variables of border rank at most $\deg(F) + 2$.

We start with the following observation.

Lemma 5.3. *Let d be a positive integer and let $e = \lceil \frac{d+1}{2} \rceil$. Let $H_{d-1} \in \mathbb{C}[x_1, x_2]_{d-1}$ and $H_d \in \mathbb{C}[x_1, x_2]_d$ be homogeneous polynomials. Then there exists an element $\alpha_0\xi_{e-1} + \xi_e \in \operatorname{Ann}(x_0H_{d-1} + H_d)$ with $\xi_e \ne 0$, where $\xi_{e-1} \in \mathbb{C}[\alpha_1, \alpha_2]_{e-1}$ and $\xi_e \in \mathbb{C}[\alpha_1, \alpha_2]_e$.*

Proof. Let $T^* = \mathbb{C}[x_1, x_2]$ and $T = \mathbb{C}[\alpha_1, \alpha_2]$. We will consider the restriction of the action of $S = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2]$ on $S^* = \mathbb{C}[x_0, x_1, x_2]$ to an action of T on T^*. If $H \in T^* \subseteq S^*$, we write $\operatorname{Ann}_T(H)$ if we compute the annihilator ideal with respect to the T action.

Let $F = x_0H_{d-1} + H_d$. We have $(\alpha_0\xi_{e-1} + \xi_e) \lrcorner F = x_0(\xi_e \lrcorner H_{d-1}) + (\xi_{e-1} \lrcorner H_{d-1} + \xi_e \lrcorner H_d)$. Therefore, we need to choose $\xi_e \in \operatorname{Ann}_T(H_{d-1})$. If there exists a non-zero $\xi_e \in \operatorname{Ann}_T(H_{d-1}) \cap \operatorname{Ann}_T(H_d)$ we can set $\xi_{e-1} = 0$ and we are done.

Otherwise, let h be the Hilbert function of $T/\operatorname{Ann}_T(H_{d-1})$. The $\mathbb{C}$-vector space

$$\operatorname{lin}\{\xi_e \lrcorner H_d \mid \xi_e \in \operatorname{Ann}_T(H_{d-1})_e\}$$

has dimension $e + 1 - h(e)$. On the other hand the vector space

$$\operatorname{lin}\{\xi_{e-1} \lrcorner H_{d-1} \mid \xi_{e-1} \in T_{e-1}\}$$

is of dimension $h(e-1)$. It is enough to show that these two vector subspaces of T^*_{d-e} have a non-zero intersection. It suffices to establish that

$$e+1-h(e)+h(e-1) \geq d-e+2. \tag{5.4}$$

By the definition of e we have $d+1 \leq 2e$. We claim that $h(e-1)-h(e) \geq 0$. If $\operatorname{Ann}_T(H_{d-1})_{e-1} \neq 0$ then it follows from Lemma 2.12 that $h(e-1)-h(e) \geq 0$. On the other hand, if $\operatorname{Ann}_T(H_{d-1})_{e-1} = 0$ and $h(e) > h(e-1)$, then $\operatorname{Ann}_T(H_{d-1})_e = 0$. Since $T/\operatorname{Ann}_T(H_{d-1})$ is Gorenstein, we get $h(d-1-e) = h(e) = e+1$. This gives a contradiction with $d+1 \leq 2e$. These remarks imply Equation (5.4). $\square$

Now we present the main result of this subsection.

Proposition 5.5. *Let $S = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2]$ be a polynomial ring with dual ring $S^* = \mathbb{C}[x_0, x_1, x_2]$. Let $F \in S^*_d$ be a non-zero polynomial for some $d \in \mathbb{Z}_{>0}$. If the border rank of F is at most $d+2$, then* $\operatorname{cr}(F) = \operatorname{sr}(F) = \operatorname{br}(F)$.

Proof. By [11, Prop. 2.5] we may assume that $\operatorname{br}(F) = d+2$. Furthermore, $\operatorname{sr}(F) = \operatorname{cr}(F)$ since $\mathcal{H}ilb_r(\mathbb{P}^2)$ is irreducible for every positive integer r. If $\operatorname{cr}(F) \leq d+2$, it follows from

$$d+2 = \operatorname{br}(F) \leq \operatorname{sr}(F) = \operatorname{cr}(F) \leq d+2$$

that $\operatorname{cr}(F) = \operatorname{sr}(F) = \operatorname{br}(F)$. Assume that $\operatorname{cr}(F) > d+2$. From Corollary 5.1 we obtain that there is an ideal $I \subseteq \operatorname{Ann}(F)$ such that $[I] \in \operatorname{Slip}_{d+2,2}$ and $I_d \neq \overline{I}_d$. From Lemma 5.2 we get that $S/\overline{I}$ has Hilbert function $h_{d+2,1}$. We may assume that $\overline{I} = \big(\alpha_0, F_{d+2}(\alpha_1, \alpha_2)\big)$. It follows from Theorem 3.65 that $(\alpha_0^2) \cdot (\alpha_0, \alpha_1, \alpha_2)^{d-2} \subseteq I \subseteq \operatorname{Ann}(F)$. Thus, $\alpha_0^2 \in \operatorname{Ann}(F)$ and consequently, F is of the form $F = x_0 H_{d-1} + H_d$ with $H_{d-1} \in \mathbb{C}[x_1, x_2]_{d-1}$ and $H_d \in \mathbb{C}[x_1, x_2]_d$.

Let $e = \lceil \frac{d+1}{2} \rceil$. By Lemma 5.3, there is an element $\eta_e = \alpha_0 \xi_{e-1} + \xi_e \in \operatorname{Ann}(F)$ with $\xi_{e-1} \in \mathbb{C}[\alpha_1, \alpha_2]_{e-1}$ and non-zero $\xi_e \in \mathbb{C}[\alpha_1, \alpha_2]_e$. Let $J = (\alpha_0^2, \eta_e)$. Then $J \subseteq \operatorname{Ann}(F)$ and S/J has Hilbert polynomial $2e \leq d+2$. Moreover, J is saturated. Indeed, let $>$ be the lex order with $\alpha_2 > \alpha_1 > \alpha_0$ and let J' be the initial ideal of J with respect to the order $>$. Then $J' = (\alpha_0^2, M)$ where $M \in \{\alpha_1^e, \alpha_1^{e-1}\alpha_2, \ldots, \alpha_2^e\}$. In particular, J' is saturated. Thus, so is J by Lemma 2.7. It follows from Proposition 2.90 that $\operatorname{cr}(F) \leq d+2$. $\square$

Remark 5.6. In [3, page 37] in the paragraph above Remark 13, there is an example that suggests that there could exist a wild polynomial in $\sigma_8(\nu_6(\mathbb{P}^2))$. Proposition 5.5 shows that there is no such polynomial.

In the context of Proposition 5.5, there is the following natural question.

Problem 5.7. Does there exist a homogeneous polynomial $F \in \mathbb{C}[x_0, x_1, x_2]$ such that $\operatorname{br}(F) \neq \operatorname{sr}(F)$? If it does, what is the smallest possible degree of such a polynomial?

It follows from Proposition 5.5 that if there exists a wild polynomial $F \in \mathbb{C}[x_0, x_1, x_2]_d$, then $d \geq 6$ since otherwise $\sigma_{d+2}(\nu_d(\mathbb{P}^2)) = \mathbb{P}^{\binom{2+d}{d}-1}$ by the Alexander–Hirschowitz theorem [7, Thm. 1.2].

5.1.2 Quartics in four variables of small border rank

There are no wild cubics in four variables [12, Thm. 1.3]. In this subsection we prove that there are no wild homogeneous quartics in four variables of border rank at most 6.

Proposition 5.8. *Let $S = \mathbb{C}[\alpha_0, \dots, \alpha_3]$ be a polynomial ring with dual ring $S^* = \mathbb{C}[x_0, \dots, x_3]$. Let $F \in S^*_4$ be non-zero. If the border rank of F is at most 6, then $\operatorname{cr}(F) = \operatorname{sr}(F) = \operatorname{br}(F)$.*

Proof. By [11, Prop. 2.5] we may assume that $\operatorname{br}(F) = 6$. Since $\mathcal{H}ilb_6(\mathbb{P}^3)$ is irreducible (see [20, Thm. 1.1]), it is enough to show that $\operatorname{cr}(F) \leq 6$. Assume that it does not hold.

By Corollary 5.1 there is an ideal $[I] \in \operatorname{Slip}_{6,3}$ such that $I \subseteq \operatorname{Ann}(F)$ and $I_4 \neq \overline{I}_4$. It follows from Lemma 5.2 that $S/\overline{I}$ has Hilbert function $h_{6,1}$. We may assume that $\overline{I}_1 = (\alpha_0, \alpha_1)_1$. Then, by Theorem 3.65 we get $(\alpha_0, \alpha_1)^2 \subseteq \operatorname{Ann}(F)$. Thus, we may restrict our attention to the case that $F = x_0C_1 + x_1C_2 + D$ where $C_1, C_2 \in \mathbb{C}[x_2, x_3]_3$ and $D \in \mathbb{C}[x_2, x_3]_4$.

There is a polynomial $\theta \in \mathbb{C}[\alpha_2, \alpha_3]_3$ such that $\theta \lrcorner C_1 = \theta \lrcorner C_2 = 0$. By a linear change of variables in $\mathbb{C}[\alpha_2, \alpha_3]$ we may assume that θ is one of the following:

1. $\theta = \alpha_2^3$;
2. $\theta = \alpha_2^2\alpha_3$;
3. $\theta = \alpha_2\alpha_3(\alpha_2 - \alpha_3)$.

We study this case by case. We will further simplify F by a linear change of variables and in each case we will find a homogeneous ideal $J \subseteq \operatorname{Ann}(F)$ whose initial ideal with respect to the lex order with $\alpha_2 > \alpha_3 > \alpha_1 > \alpha_0$ will be saturated and the Hilbert polynomial of the corresponding quotient algebra will be 6. Thus, $\operatorname{cr}(F) \leq 6$ by Lemma 2.7 and Proposition 2.90. In each case the given set of generators of J will be a Gröbner basis. We may assume that $\operatorname{Ann}(F)_1 = 0$ by Proposition 5.5 and [12, §3.1].

We start with case 1. Up to a linear change of variables in S^* we have one of the cases:

1.A $F = x_0(x_2^2x_3 + ax_3^3) + x_1x_2x_3^2 + Q$ with $a \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$;

1.B $F = x_0(x_2^2x_3 + ax_2x_3^2) + x_1x_3^3 + Q$ with $a \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$;

1.C $F = x_0x_2x_3^2 + x_1x_3^3 + Q$ with $Q \in \mathbb{C}[x_2, x_3]_4$.

Let $\alpha_2^3 \lrcorner Q = Ax_2 + Bx_3$. Corresponding to the above cases, the following ideals contained in $\operatorname{Ann}(F)$ show that the cactus rank of F is at most 6:

1.A $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_2 - \alpha_1\alpha_3, \alpha_1\alpha_2^2, \alpha_2^3 - \frac{A}{2}\alpha_0\alpha_2\alpha_3 - \frac{B}{2}\alpha_1\alpha_2\alpha_3)$;

1.B $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_1\alpha_2, \alpha_0\alpha_2^2 - \frac{1}{3}\alpha_1\alpha_3^2, \alpha_2^3 - \frac{A}{2}\alpha_0\alpha_2\alpha_3 + \frac{aA-B}{6}\alpha_1\alpha_3^2)$;

1.C $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_1\alpha_2, \alpha_0\alpha_2^2, \alpha_2^3 - \frac{A}{2}\alpha_0\alpha_3^2 - \frac{B}{6}\alpha_1\alpha_3^2)$.

Now we consider case 2, namely we assume that $\alpha_2^2\alpha_3 \lrcorner C_1 = \alpha_2^2\alpha_3 \lrcorner C_2 = 0$. Then, up to a linear change of variables in S^* and excluding possibilities already considered in case 1, we have one of the cases:

2.A $F = x_0(x_2^3 + ax_3^3) + x_1(x_2x_3^2 + bx_3^3) + Q$ with $a, b \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$;

2.B $F = x_0(x_2^3 + ax_2x_3^2) + x_1x_3^3 + Q$ with $a \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$.

Let $\alpha_2^2\alpha_3 \lrcorner Q = Ax_2 + Bx_3$. Corresponding to the above cases, the following ideals contained in $\operatorname{Ann}(F)$ show that the cactus rank of G is at most 6:

2.A If $a \neq 0$ take $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, a\alpha_1\alpha_2 - \frac{1}{3}\alpha_0\alpha_3, \alpha_0\alpha_2\alpha_3, \alpha_2^2\alpha_3 - \frac{A}{6}\alpha_0\alpha_2^2 - \frac{B}{2}\alpha_1\alpha_2\alpha_3)$.
If $a = 0$ take $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_3, \alpha_1\alpha_2^2, \alpha_2^2\alpha_3 - \frac{A}{6}\alpha_0\alpha_2^2 - \frac{B}{2}\alpha_1\alpha_2\alpha_3)$;

2.B Take $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_1\alpha_2, \alpha_0\alpha_2\alpha_3 - \frac{a}{3}\alpha_1\alpha_3^2, \alpha_2^2\alpha_3 - \frac{A}{6}\alpha_0\alpha_2^2 - \frac{B}{6}\alpha_1\alpha_3^2)$.

Finally we consider case 3, that is we assume that $\alpha_2\alpha_3(\alpha_2 - \alpha_3)\lrcorner C_1 = \alpha_2\alpha_3(\alpha_2 - \alpha_3)\lrcorner C_2 = 0$. Then, up to a linear change of variables in S^* and excluding possibilities considered in case 1, we have one of the cases:

3.A $F = x_0(x_2^3 + ax_3^3) + x_1(x_2^2x_3 + x_2x_3^2 + bx_3^3) + Q$ with $a, b \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$;

3.B $F = x_0(x_2^3 + ax_2^2x_3 + ax_2x_3^2) + x_1x_3^3 + Q$ with $a \in \mathbb{C}$ and $Q \in \mathbb{C}[x_2, x_3]_4$.

Let $(\alpha_2^2\alpha_3 - \alpha_2\alpha_3^2)\lrcorner Q = Ax_2 + Bx_3$. Corresponding to the above cases, the following ideals show that the cactus rank of G is at most 6:

3.A If $a \neq 0$ take $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, a\alpha_1\alpha_2 + \frac{a}{3}\alpha_0\alpha_2 - a\alpha_1\alpha_3 + (b - \frac{1}{3})\alpha_0\alpha_3, \alpha_0\alpha_2\alpha_3, \alpha_2^2\alpha_3 - \alpha_2\alpha_3^2 - \frac{A}{6}\alpha_0\alpha_2^2 - \frac{B}{2}\alpha_1\alpha_2^2)$.
If $a = 0$ take $J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_0\alpha_3, \alpha_1\alpha_2^2 + \frac{1}{3}\alpha_0\alpha_2^2 - \alpha_1\alpha_2\alpha_3, \alpha_2^2\alpha_3 - \alpha_2\alpha_3^2 - \frac{A}{6}\alpha_0\alpha_2^2 - \frac{B}{2}\alpha_1\alpha_2^2)$;

3.B If $a = 0$ then we are in case 2.B. Therefore, assume that $a \neq 0$. Take

$$J = (\alpha_0^2, \alpha_0\alpha_1, \alpha_1^2, \alpha_1\alpha_2, \alpha_0\alpha_2\alpha_3 - \alpha_0\alpha_3^2 - \frac{a}{3}\alpha_1\alpha_3^2, \alpha_2^2\alpha_3 - \alpha_2\alpha_3^2 - \frac{A}{6}\alpha_0\alpha_2^2 + \frac{aA - 3B}{18}\alpha_1\alpha_3^2).$$

□

5.1.3 Wild quintic in four variables of border rank 7

In Proposition 5.8 we showed that there are no wild quartics in four variables of border rank 6. In this subsection, we give an example of a wild quintic in four variables of border rank 7.

Proposition 5.9. *Let $S = \mathbb{C}[\alpha_0, \alpha_1, \alpha_2, \alpha_3]$ be polynomial ring with graded dual ring $S^* = \mathbb{C}[x_0, x_1, x_2, x_3]$. Let $F = x_0x_2^4 + x_0x_2^3x_3 + x_1x_2^2x_3^2 + x_1x_3^4$. Then $\operatorname{br}(F) = 7$ and $\operatorname{cr}(F) > 7$. Thus, F is wild.*

Proof. The Hilbert scheme $\mathcal{H}ilb_7(\mathbb{P}^3)$ is irreducible by [20, Thm. 1.1]. Therefore, $\operatorname{sr}(F) = \operatorname{cr}(F)$ so it is enough to show that $\operatorname{br}(F) = 7 < \operatorname{cr}(F)$. Furthermore, by [11, Prop. 2.5] it suffices to show that $\operatorname{br}(F) \leq 7 < \operatorname{cr}(F)$.

The following claims were obtained by computation in Macaulay2 [42].

We have $(\alpha_0, \alpha_1)^2 \subseteq \operatorname{Ann}(F)$. Let $J = (\operatorname{Ann}(F)_{\leq 3}) + (\alpha_2^7)$. Then $[J] \in \operatorname{Hilb}_S^{h_{7,3}}$ and it follows from Theorem 3.65 that there is an ideal $[J'] \in \operatorname{Slip}_{7,3}$ such that $(J')_{\geq 5} = J_{\geq 5}$. In particular, $J'_5 = J_5 \subseteq \operatorname{Ann}(F)_5$ so $J' \subseteq \operatorname{Ann}(F)$. It follows from Proposition 2.91 that $\operatorname{br}(F) \leq 7$.

Now we show that $\operatorname{cr}(F) > 7$. Otherwise, by Proposition 2.90 there exists a homogeneous, saturated ideal $K \subseteq \operatorname{Ann}(F)$ such that S/K has Hilbert polynomial 7. Since $H_{S/\operatorname{Ann}(F)}(a) = h_{7,3}(a)$ for $a \leq 3$, we have $K_{\leq 3} = \operatorname{Ann}(F)_{\leq 3}$. In particular, $(\alpha_0, \alpha_1) = \overline{(\operatorname{Ann}(F)_{\leq 3})} \subseteq K$. This is a contradiction since $K_1 \subseteq \operatorname{Ann}(F)_1 = 0$. □

5.1.4 Cubics in five variables of minimal border rank

In this subsection we let $S = \mathbb{C}[\alpha_0, \dots, \alpha_4]$ be a polynomial ring and $S^* = \mathbb{C}[x_0, \dots, x_4]$ be the dual ring. Let $C \in S^*$ be a homogeneous polynomial of degree 3. We say that C is concise if $\operatorname{Ann}(C)_1 = 0$. It is known that there exists a wild concise cubic in S^* of border rank 5 (see [12, Thm. 1.3]) and that a concise cubic in S^* of border rank 5 is wild if and only if its Hessian is zero (see [52, Thm. 4.9]).

Using Theorem 3.65 we obtain in a simple way that up to a linear change of variables, the cubic given in [12, Thm. 1.3] is the unique wild cubic in S^* of border rank 5.

Proposition 5.10. *Let $S = \mathbb{C}[\alpha_0, \dots, \alpha_4]$ be a polynomial ring with graded dual ring $S^* = \mathbb{C}[x_0, \dots, x_4]$. Up to a linear change of variables, the cubic*

$$x_0x_3^2 - x_1(x_3 + x_4)^2 + x_2x_4^2. \tag{5.11}$$

is the unique wild cubic in S^* of border rank 5.

Proof. Let $C \in S$ be a wild cubic of border rank 5. By [12, Thm. 1.3] we may assume that C is concise. By Proposition 2.91 there is an ideal $I \subseteq \operatorname{Ann}(C)$ such that $[I] \in \operatorname{Slip}_{5,4}$. If the Hilbert function of $S/\overline{I}$ is not $h_{5,1}$, then $I_3 = \overline{I}_3$ by Lemma 5.2. Thus, $\operatorname{cr}(C) \leq 5$ by Corollary 5.1. Consequently, C is not wild since $\operatorname{cr}(C) = \operatorname{sr}(C)$ (see [20, Thm. 1.1]). Therefore, we may assume that $\overline{I} = \big(\alpha_0, \alpha_1, \alpha_2, F(\alpha_3, \alpha_4)\big)$ for some $F \in \mathbb{C}[\alpha_3, \alpha_4]_5$. Since $[I] \in \operatorname{Slip}_{5,4}$ it follows from Theorem 3.65 that $(\alpha_0, \alpha_1, \alpha_2)^2 \cdot (\alpha_0, \alpha_1, \dots, \alpha_4) \subseteq \operatorname{Ann}(C)$ and thus, $(\alpha_0, \alpha_1, \alpha_2)^2 \subseteq \operatorname{Ann}(C)$. Hence

$$C = x_0Q_0 + x_1Q_1 + x_2Q_2 + C' \text{ where } Q_0, Q_1, Q_2 \in \mathbb{C}[x_3, x_4]_2 \text{ and } C' \in \mathbb{C}[x_3, x_4]_3.$$

Moreover, $Q_0, Q_1, Q_2 \in \mathbb{C}[x_3, x_4]_2$ are linearly independent since C is concise. Therefore, after a linear change of variables we may reduce C to the form given in Equation (5.11). □

Remark 5.12. The annihilator ideal $\operatorname{Ann}(C)$ of a concise cubic C has a minimal generator of degree 3 (see [13, Thm. 5.4] for a vast generalization). Therefore, the form given in Equation (5.11) should be compared to the form given in [16, Thm. 4.5].

5.2 (Border) cactus rank of a homogeneous subspace divisible by a large power of a linear form

In this section we compute the cactus rank and the border cactus rank of a homogeneous subspace of a divided power ring divisible by a large power of a linear form. This result is based on [39, Thm. 4.2 and 4.3]. However, here we strengthen the assumptions to omit technical difficulties.

Let $\Bbbk$ be an algebraically closed field, let n be a positive integer and consider polynomial rings $S = \Bbbk[\alpha_1, \dots, \alpha_n] \subseteq \Bbbk[\alpha_0, \dots, \alpha_n] = T$ with graded dual rings $S^* = \Bbbk_{dp}[x_1, \dots, x_n] \subseteq \Bbbk_{dp}[x_0, \dots, x_n] = T^*$. Let d_1 be a positive integer and $W \subseteq S^*_{\leq d_1}$ be a linear subspace. For a non-negative integer d_2 we define

$$W^{hom,d_2} = \{f^{hom,d_2} \mid f \in W\} \subseteq T^*_{d_1+d_2}.$$

where

$$f^{hom,d_2} = \sum_{i=0}^{\deg(f)} F_i x_0^{[d_2+d_1-i]} \in T^*_{d_1+d_2}$$

for $f = F_{\deg f} + \ldots + F_0 \in W$ with $F_i \in S_i^*$. We shall show that

$$\operatorname{cr}(W^{hom,d_2}) = \operatorname{bcr}(W^{hom,d_2}) = \dim_{\Bbbk} S/\operatorname{Ann}(W)$$

if $d_2 \geq d_1$.

We start with a lemma which is based on [4, Lem. 2].

Lemma 5.13 ([39, Lem. 3.6 and 3.8]). *In the above notation we have:*

(i) $\operatorname{Ann}(W)^{hom} \subseteq \operatorname{Ann}(W^{hom,d_2})$*, and*

(ii) $(\operatorname{Ann}(W)^{hom})_{\leq d_2} = \operatorname{Ann}(W^{hom,d_2})_{\leq d_2}$.

Proof. The proof of the lemma is based on the following calculation. Let

$$\Gamma = \alpha_0^d \Theta_0 + \alpha_0^{d-1}\Theta_1 + \ldots + \Theta_d,$$

where $\Theta_i \in S_i$ and let $f \in W$. We can rewrite $\Gamma \lrcorner f^{hom,d_2}$ as follows

$$\begin{aligned}
\Gamma \lrcorner f^{hom,d_2} &= \sum_{e=0}^{d_1} \sum_{j=0}^{\min(d_1-e,d)} (\alpha_0^{d-j}\Theta_j) \lrcorner (x_0^{[d_1+d_2-(e+j)]} F_{e+j}) \\
&= \sum_{e=0}^{d_1} \sum_{j=0}^{\min(d_1-e,d)} (\alpha_0^{d-j} \lrcorner x_0^{[d_1+d_2-(e+j)]})(\Theta_j \lrcorner F_{e+j}) \\
&= \sum_{e=0}^{\min(d_1,d_1+d_2-d)} \sum_{j=0}^{\min(d_1-e,d)} x_0^{[d_1+d_2-d-e]}(\Theta_j \lrcorner F_{e+j}) \\
&= \sum_{e=0}^{\min(d_1,d_1+d_2-d)} x_0^{[d_1+d_2-d-e]} \sum_{j=0}^{\min(d_1-e,d)} \Theta_j \lrcorner F_{e+j}.
\end{aligned} \tag{5.14}$$

(i) Let $\theta = \Theta_0 + \cdots + \Theta_d \in \operatorname{Ann}(W)$, where Θ_i is homogeneous of degree i. We show that $\theta^{hom} = \alpha_0^d\Theta_0 + \alpha_0^{d-1}\Theta_1 + \ldots + \Theta_d \in \operatorname{Ann}(W^{hom,d_2})$. We have

$$\operatorname{Ann}(W^{hom,d_2}) = \bigcap_{f \in W} \operatorname{Ann}(f^{hom,d_2}). \tag{5.15}$$

Thus, it suffices to show that $\theta^{hom} \in \operatorname{Ann}(f^{hom,d_2})$ for every $f \in W$. Pick $f \in W$. Then $\theta \in \operatorname{Ann}(f)$. We set $\Gamma = \theta^{hom}$ in Equation (5.14). For every $e = 0, \ldots, \min(d_1, d_1 + d_2 - d)$ the sum $\sum_{j=0}^{\min(d_1-e,d)} \Theta_j \lrcorner F_{e+j}$ is zero since $\theta \lrcorner f = 0$. Hence $\theta^{hom} \lrcorner f^{hom,d_2} = 0$, as claimed.

(ii) We have $\operatorname{Ann}(W)^{hom} \subseteq \operatorname{Ann}(W^{hom,d_2})$ by part (i). We claim that

$$(\Theta|_{\alpha_0=1}) \in \operatorname{Ann}(f) \text{ for every } f \in W \text{ and } \Theta \in \operatorname{Ann}(f^{hom,d_2})_{\leq d_2}. \tag{5.16}$$

Before proving the claim, we show how to conclude the proof of part (ii). Let $\Theta \in \operatorname{Ann}(W^{hom,d_2})_{\leq d_2}$. Then it follows from Equation (5.15) that $\Theta \in \operatorname{Ann}(f^{hom,d_2})$ for every

$f \in W$. Thus, by Equation (5.16) we get

$$\Theta|_{\alpha_0=1} \in \bigcap_{f\in W} \operatorname{Ann}(f) = \operatorname{Ann}(W).$$

As a result $\Theta \in (\operatorname{Ann}(W)^{hom})_{\leq d_2}$.

We are left with proving the claimed Equation (5.16). Pick $f \in W$. Assume that $d \leq d_2$ and let $\Theta = \alpha_0^d\Theta_0 + \alpha_0^{d-1}\Theta_1 + \ldots + \Theta_d$, where $\Theta_i \in S_i$, be such that $\Theta \lrcorner f^{hom,d_2} = 0$. We claim that $(\Theta|_{\alpha_0=1}) \lrcorner f = 0$. By Equation (5.14) (with $\Gamma = \Theta$) we have

$$0 = \sum_{e=0}^{d_1} x_0^{[d_1+d_2-d-e]} \sum_{j=0}^{\min(d_1-e,d)} \Theta_j \lrcorner F_{e+j}.$$

Since the exponents at x_0 are pairwise different, we get

$$\sum_{j=0}^{\min(d_1-e,d)} \Theta_j \lrcorner F_{e+j} = 0 \text{ for every } d_1 \geq e \geq 0.$$

This implies that $(\Theta|_{\alpha_0=1}) \lrcorner f = 0$.

□

We will use the following result, which bounds the degree from which $T/\operatorname{Ann}(W)^{hom}$ agrees with its Hilbert polynomial.

Lemma 5.17 ([39, Cor. 3.3])**.** *Let $W \subseteq S^*_{\leq d_1}$ be a linear subspace. Then*

$$H(T/\operatorname{Ann}(W)^{hom}, e) = \dim_{\Bbbk} S/\operatorname{Ann}(W)$$

for $e \geq d_1$.

Now we present the main result of this section. Note that the version for polynomials instead of arbitrary subspaces can be strengthened, see [39, Thm. 4.3]. Recall the notion of a standard Hilbert function from Definition 2.92.

Theorem 5.18 ([39, Thm. 4.2])**.** *Let $W \subseteq S^*_{\leq d_1}$ be a linear subspace and $r = \dim_{\Bbbk} S/\operatorname{Ann}(W)$. Let d_2 be a non-negative integer. We have the following:*

(i) The cactus rank of W^{hom,d_2} is at most r.

(ii) If $d_2 \geq d_1$, then there is no homogeneous ideal $J \subseteq \operatorname{Ann}(W^{hom,d_2})$ such that T/J has an $(r-1, n+1)$-standard Hilbert function. In particular, the border cactus rank $\operatorname{bcr}(W^{hom,d_2})$ of W^{hom,d_2} equals r.

(iii) If $d_2 \geq d_1 + 1$, and $J \subseteq \operatorname{Ann}(W^{hom,d_2})$ is a homogeneous ideal such that T/J has an $(r, n+1)$-standard Hilbert function, then $\overline{J} = \operatorname{Ann}(W)^{hom}$.

Proof. (i) We have $\operatorname{Ann}(W)^{hom} \subseteq \operatorname{Ann}(W^{hom,d_2})$ by Lemma 5.13(i). The Hilbert polynomial of $T/\operatorname{Ann}(W)^{hom}$ is $\dim_{\Bbbk} S/\operatorname{Ann}(W) = r$. Moreover, the ideal $\operatorname{Ann}(W)^{hom}$ is saturated. Hence the claim follows from Proposition 2.90.

(ii) We have $H(T/\operatorname{Ann}(W)^{hom}, d_1) = r$ by Lemma 5.17. Therefore, by Lemma 5.13(ii) we have

$$H(T/\operatorname{Ann}(W^{hom,d_2}), d_1) = r.$$

Thus, there exists no ideal $J \subseteq \operatorname{Ann}(W^{hom,d_2})$ such that T/J has an $(r-1, n+1)$-standard Hilbert function. By Proposition 2.93 we get $\operatorname{bcr}(W^{hom,d_2}) \geq r$, which together with part (i) implies that $\operatorname{bcr}(W^{hom,d_2}) = r$.

(iii) Assume that $J \subseteq \operatorname{Ann}(W^{hom,d_2})$ is such that T/J has an $(r, n+1)$-standard Hilbert function. By Lemmas 5.13(ii) and 5.17

$$H(T/\operatorname{Ann}(W^{hom,d_2}), d_2) = H(T/\operatorname{Ann}(W)^{hom}, d_2) = r.$$

In particular, $J_{d_2} = (\operatorname{Ann}(W)^{hom})_{d_2}$. Since $\operatorname{Ann}(W)^{hom}$ is generated in degrees smaller or equal $d_1 + 1 \leq d_2$, it follows that $J_d \supseteq (\operatorname{Ann}(W)^{hom})_d$ for every $d \geq d_2$.

Ideals J and $\operatorname{Ann}(W)^{hom}$ have the same Hilbert polynomial. Hence we have

$$\overline{J} = \overline{(\operatorname{Ann}(W)^{hom})} = \operatorname{Ann}(W)^{hom}. \qquad \square$$

5.3 Distinguishing secant from cactus varieties

In this section we work over the field of complex numbers. We show that for $d \geq 7$, the cactus variety $\kappa_{14}\big(\nu_d(\mathbb{P}^6)\big)$ has two irreducible components: $\eta_{14}\big(\nu_d(\mathbb{P}^6)\big)$ and the secant variety $\sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$. Moreover, we describe the points of $\eta_{14}\big(\nu_d(\mathbb{P}^6)\big)$ and present an algorithm that verifies whether a point in $\kappa_{14}\big(\nu_d(\mathbb{P}^6)\big)$ is in $\sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$. These results are special cases of [39, Thm. 1.4 and Thm. 1.6] which address the case of $\kappa_{14}\big(\nu_d(\mathbb{P}^n)\big)$ for $n \geq 6$ and $d \geq 5$ ([39, Thm. 1.4]) or $d \geq 6$ ([39, Thm. 1.6]). Our presentation follows [39] with minor changes and some simplifications due to additional assumptions.

For $X = \mathbb{A}^n$ or $\mathbb{P}^n$, we denote by $\mathcal{H}ilb_r^{Gor}(X)$ the open subset of the Hilbert scheme $\mathcal{H}ilb_r(X)$ consisting of points corresponding to Gorenstein subschemes. Let $\mathcal{H}ilb_r^{Gor,sm}(X)$ denote its smoothable component. It is a key observation, that the cactus variety $\kappa_r\big(\nu_d(\mathbb{P}^n)\big)$ is actually determined by the Gorenstein locus of the Hilbert scheme. More precisely, we have

$$\kappa_r(\nu_d(\mathbb{P}^n)) = \overline{\bigcup \{\langle \nu_d(R)\rangle \mid [R] \in \mathcal{H}ilb_r^{Gor}(\mathbb{P}^n)\}} \tag{5.19}$$

by [11, Prop. 2.2]. Therefore, if $\mathcal{H}ilb_r^{Gor}(\mathbb{P}^n)$ is irreducible then $\kappa_r\big(\nu_d(\mathbb{P}^n)\big) = \sigma_r\big(\nu_d(\mathbb{P}^n)\big)$. Note that a description of the cactus variety, similar to the one given by Equation (5.19), works over an arbitrary field (see [17, Cor. 6.20]).

If either $r \leq 13$ or $r = 14$ and $n < 6$, the scheme $\mathcal{H}ilb_r^{Gor}(\mathbb{A}^n)$ is irreducible by [21, Thm. A]. Therefore, in that case, $\kappa_r\big(\nu_d(\mathbb{P}^n)\big) = \sigma_r\big(\nu_d(\mathbb{P}^n)\big)$. Thus, the cactus variety $\kappa_{14}\big(\nu_d(\mathbb{P}^6)\big)$ that we study in this section is the simplest example of a reducible cactus variety. We assume that $d \geq 7$. However, the presented results hold for $d \geq 6$, and some of them, even for $d = 5$, with more technical proofs. See [39] for this as well as the case $n > 6$.

We start with defining for $d \geq 3$ an irreducible, closed subset $\eta_{14}\big(\nu_d(\mathbb{P}^6)\big)$. Consider the following rational map φ, which assigns to a scheme R its projective linear span $\langle v_d(R)\rangle$

$$\varphi : \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6) \dashrightarrow \operatorname{Gr}(14, \operatorname{Sym}^d \mathbb{C}^7).$$

Let $U \subseteq \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6)$ be a dense open subset on which φ is regular. Consider the projectivized universal bundle $\mathbb{P}\mathcal{S}$ over $\mathrm{Gr}(14, \mathrm{Sym}^d \mathbb{C}^7)$, given as a set by

$$\mathbb{P}\mathcal{S} = \{([P],[p]) \in \mathrm{Gr}(14, \mathrm{Sym}^d \mathbb{C}^7) \times \mathbb{P}(\mathrm{Sym}^d \mathbb{C}^7) \mid p \in P\},$$

together with the inclusion $i : \mathbb{P}\mathcal{S} \hookrightarrow \mathrm{Gr}(14, \mathrm{Sym}^d \mathbb{C}^7) \times \mathbb{P}(\mathrm{Sym}^d \mathbb{C}^7)$. We pull the commutative diagram

$$\begin{tikzcd} \mathbb{P}\mathcal{S} \arrow[rr, hook, "i"] \arrow[dr, "\pi"'] & & \mathrm{Gr}(14, \mathrm{Sym}^d \mathbb{C}^7) \times \mathbb{P}(\mathrm{Sym}^d \mathbb{C}^7) \arrow[dl, "\mathrm{pr}_1"] \\ & \mathrm{Gr}(14, \mathrm{Sym}^d \mathbb{C}^7) & \end{tikzcd}$$

back along φ to U, getting the commutative diagram

$$\begin{tikzcd} \varphi^*(\mathbb{P}\mathcal{S}) \arrow[rr, hook, "\varphi^* i"] \arrow[dr, "\varphi^*\pi"'] & & U \times \mathbb{P}(\mathrm{Sym}^d \mathbb{C}^7) \arrow[dl, "\mathrm{pr}_1"] \\ & U. & \end{tikzcd}$$

Let Y be the closure of $\varphi^*(\mathbb{P}\mathcal{S})$ inside $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6) \times \mathbb{P}(\mathrm{Sym}^d \mathbb{C}^7)$. The scheme Y has two irreducible components, Y_1 and Y_2, corresponding to two irreducible components of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6)$, the schemes $\mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{P}^6)$ and $\mathcal{H}_{1661}$, respectively. For the description of irreducible components of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6)$ see [21].

Then

$$\sigma_{14}\big(\nu_d(\mathbb{P}^6)\big) = \mathrm{pr}_2(Y_1), \text{ and we define}$$
$$\eta_{14}\big(\nu_d(\mathbb{P}^6)\big) = \mathrm{pr}_2(Y_2).$$

By construction, if $\kappa_{14}\big(\nu_d(\mathbb{P}^6)\big)$ is reducible, then $\eta_{14}\big(\nu_d(\mathbb{P}^6)\big)$ and $\sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$ are its irreducible components.

In the next lemma, we bound the dimension of $\eta_{14}\big(\nu_d(\mathbb{P}^6)\big)$.

Lemma 5.20 ([39, Prop. 5.5])**.** *The irreducible set $\eta_{14}\big(\nu_d(\mathbb{P}^6)\big)$ has dimension at most* 89.

Proof. We have the following commutative diagram

$$\begin{tikzcd} \mathbb{P}(\mathrm{Sym}^d \mathbb{C}^7) \supseteq \sigma \cup \eta & Y_1 \cup Y_2 \arrow[l] \arrow[r, dashed] \arrow[d, "\chi"] & \mathbb{P}\mathcal{S} \arrow[d] \\ \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6) \arrow[r, equal] & \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{P}^6) \cup \mathcal{H}_{1661} \arrow[r, dashed] & \mathrm{Gr}(14, \mathrm{Sym}^d \mathbb{C}^7) \end{tikzcd}$$

where σ and η denote $\sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$ and $\eta_{14}\big(\nu_d(\mathbb{P}^6)\big)$ respectively, and $\chi : Y_1 \cup Y_2 \to \mathcal{H}ilb_{14}^{Gor}(\mathbb{P}^6)$ is the projection. Then $\dim \eta_{14}(\nu_d(\mathbb{P}^6)) \leq \dim(Y_2) = m + 13$, where $m = \dim \mathcal{H}_{1661}$ and 13 is the dimension of the general fiber of the map $\chi|_{Y_2} : Y_2 \to \mathcal{H}_{1661}$. It follows from [58, Thm. 1.1], that $m = 76$ and therefore, $\dim \eta_{14}\big(\nu_d(\mathbb{P}^6)\big) \leq 89$. $\square$

One of the reasons why the case of $\mathbb{P}^6$ is simpler than the case of $\mathbb{P}^n$ for $n > 6$ is the following lemma which follows from [58].

Lemma 5.21. *Let $S = \mathbb{C}[\alpha_1, \dots, \alpha_6]$ be a polynomial ring and let $S^* = \mathbb{C}[x_1, \dots, x_6]$ be its graded dual ring. For $f = F_3 + F_2 + F_1 + F_0 \in S^*_{\leq 3}$ such that $F_i \in S^*_i$ consider the conditions:*

(i) $\operatorname{Apolar}(f)$ *has (local) Hilbert function* $(1, 6, 6, 1)$*;*

(ii) $\operatorname{Apolar}(F_3)$ *has Hilbert function* $(1, 6, 6, 1)$*;*

(iii) $[\operatorname{Spec} \operatorname{Apolar}(f)] \in \mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^6) \setminus \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^6)$*;*

(iv) $[\operatorname{Spec} \operatorname{Apolar}(F_3)] \in \mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^6) \setminus \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^6)$*.*

Then conditions (i) and (ii) are equivalent. Conditions (iii) and (iv) are equivalent.

Proof. The equivalence of (i) and (ii) follows from [58, Rmk. 2.2] and the other equivalence is a consequence of [58, Thm. 1.1]. □

In Lemma 5.22 we identify some points in $\kappa_{14}\big(\nu_d(\mathbb{P}^6)\big) \setminus \sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$ for $d \geq 7$. Later, we will show that the closure of the locus of these points is the irreducible component $\eta_{14}\big(\nu_d(\mathbb{P}^6)\big)$.

Lemma 5.22 ([39, Prop. 5.6])**.** *Let* $T = \mathbb{C}[\alpha_0, \dots, \alpha_6]$ *be a polynomial ring with graded dual ring* $T^* = \mathbb{C}[x_0, \dots, x_6]$*. Let* $(y_0, y_1, \dots, y_6)$ *be a* $\mathbb{C}$*-basis of* T_1^**. Assume that* $G = y_0^{d-3}P$ *for some natural number* $d \geq 7$ *and* $P \in T_3^*$*. Define* $f := P|_{y_0=1} = F_3 + F_2 + F_1 + F_0 \in R^* := \mathbb{C}[y_1, \dots, y_6]$*. If* f *satisfies the following conditions:*

(a) $\operatorname{Apolar}(f)$ *has (local) Hilbert function* $(1, 6, 6, 1)$*,*

(b) $[\operatorname{Spec} \operatorname{Apolar}(f)] \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^6)$*,*

then $[G] \in \kappa_{14}\big(\nu_d(\mathbb{P}^6)\big) \setminus \sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$*.*

Proof. Let $F'_i = (d-i)!F_i$ for $i = 0, 1, 2, 3$ and let $f' = F'_3 + F'_2 + F'_1 + F'_0$. Then

$$G = \sum_{i=0}^{3} y_0^{[d-i]} F'_i. \tag{5.23}$$

By Lemma 5.21, conditions (a) and (b) hold with f' instead of f. By condition (a) we have $\dim_{\mathbb{C}}(R/\operatorname{Ann}(f')) = 14$. Therefore, from Theorem 5.18(i) and Equation (5.23) we get $\operatorname{cr}(G) \leq 14$.

From Proposition 2.91, if $[G] \in \sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$ then there exists $J \subseteq \operatorname{Ann}(G)$ with $[J] \in \operatorname{Slip}_{14,\mathbb{P}T_1^*} \subseteq \operatorname{Hilb}_T^{h_{14,6}}$. Thus, $[\operatorname{Proj}(T/\overline{J})] \in \mathcal{H}ilb_{14}^{sm}(\mathbb{P}^6)$. From Theorem 5.18(iii) it follows that $\overline{J} = \operatorname{Ann}(f')^{hom}$, so

$$[\operatorname{Spec}(R/\operatorname{Ann}(f'))] \in \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^6).$$

This contradicts condition (b). □

The following lemma is an analogue of [39, Lem. 5.3].

Lemma 5.24. *Let* $S = \mathbb{C}[\alpha_1, \dots, \alpha_6]$ *be a polynomial ring and* $S^* = \mathbb{C}[x_1, \dots, x_6]$ *be the graded dual ring. Define subsets*

$$\widehat{A} = \{f \in S^*_{\leq 3} \mid S/\operatorname{Ann}(f) \text{ has local Hilbert function } (1, 6, 6, 1)\}$$

and

$$\widehat{B} = \{f \in \widehat{A} \mid [\operatorname{Spec} S/\operatorname{Ann}(f)] \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^6)\}.$$

Then $\widehat{A}$ *is irreducible and* 84 *dimensional. Furthermore,* $\widehat{B}$ *is dense in* $\widehat{A}$*.*

Proof. It follows from Lemma 5.21 that for $f = F_3 + F_2 + F_1 + F_0 \in S^*_{\leq 3}$ we have $f \in \widehat{A}$ (respectively, $f \in \widehat{B}$) if and only if $F_3 \in \widehat{A}$ (respectively, $F_3 \in \widehat{B}$). Moreover, $\widehat{A}$ is open in $S^*_{\leq 3}$ so $\dim \widehat{A} = \dim S^*_{\leq 3} = 84$.

We have a well defined morphism

$$\pi\colon \widehat{A} \to \mathcal{H}ilb^{Gor}_{14}(\mathbb{A}^6) \subseteq \mathcal{H}ilb_{14}(\mathbb{A}^6)$$

that maps f to $[\operatorname{Spec} S/\operatorname{Ann}(f)]$ (see [39, Thm. 7.1] which is based on [58, Prop. 2.12]). By definition, $\widehat{B} = \pi^{-1}\big(\mathcal{H}_{1661} \setminus \mathcal{H}ilb^{Gor,sm}_{14}(\mathbb{A}^6)\big)$, where $\mathcal{H}_{1661}$ is the irreducible component of $\mathcal{H}ilb^{Gor}_{14}(\mathbb{A}^6)$ other than the smoothable component. It follows that $\widehat{B}$ is open, and hence dense in $\widehat{A}$. $\square$

Now we present the main result of this section.

Theorem 5.25 ([39, Thm. 1.1])**.** *Let $d \geq 7$ be an integer and $T^* = \mathbb{C}[x_0, x_1, \ldots, x_6]$. Then the cactus variety $\kappa_{14}\big(\nu_d(\mathbb{P}T^*_1)\big)$ has two irreducible components: the secant variety $\sigma_{14}\big(\nu_d(\mathbb{P}T^*_1)\big)$ and the other one, denoted by $\eta_{14}\big(\nu_d(\mathbb{P}T^*_1)\big)$. Consider the map $\psi : \mathbb{P}T^*_1 \times \mathbb{P}T^*_3 \to \mathbb{P}T^*_d$ given by $([y_0], [P]) \mapsto [y_0^{d-3}P]$. Its image is $\eta_{14}\big(\nu_d(\mathbb{P}T^*_1)\big)$.*

Proof. Let

$$U_0 = \{[a_0x_0 + \ldots + a_6x_6] \in \mathbb{P}T^*_1 \mid a_i \in \mathbb{C} \text{ and } a_0 \neq 0\}.$$

Let $S^* = \mathbb{C}[x_1, \ldots, x_6]$. Given $[y_0] \in U_0$ with $y_0 = a_0x_0 + \ldots + a_6x_6$ and $P \in T^*_d$, we can consider $P|_{y_0=1}$ as an element of S^*. Note that it is independent of the choice of representative y_0 of the class $[y_0]$. Indeed, it is obtained by substituting $x_0 = 1 - \sum_{i=1}^{6} \frac{a_i}{a_0} x_i$. Recall the definition of the set $\widehat{B}$ from Lemma 5.24. We will use the following subset of $\mathbb{P}T^*_1 \times \mathbb{P}T^*_3$:

$$D = \{[y_0], [P] \in \mathbb{P}T^*_1 \times \mathbb{P}T^*_3 \mid [y_0] \in U_0 \text{ and } P|_{y_0=1} \in \widehat{B}\}.$$

Observe that the condition $P|_{y_0=1} \in \widehat{B}$ is independent of the choice of representatives y_0 and P of $[y_0]$ and $[P]$. By construction and Lemma 5.24, D is irreducible and of dimension $6 + \dim \widehat{B} - 1 = 89$. We have $\dim(\mathbb{P}T^*_1 \times \mathbb{P}T^*_3) = 6 + 83 = 89$. Thus, $\overline{D} = \mathbb{P}T^*_1 \times \mathbb{P}T^*_3$. The morphism ψ is closed. Hence

$$\psi(\mathbb{P}T^*_1 \times \mathbb{P}T^*_3) = \overline{\psi(D)}.$$

By Lemma 5.22 the set-theoretic image $\psi(D)$ is contained in $\eta_{14}\big(\nu_d(\mathbb{P}T^*_1)\big)$. Therefore, $\psi(\mathbb{P}T^*_1 \times \mathbb{P}T^*_3) \subseteq \eta_{14}\big(\nu_d(\mathbb{P}T^*_1)\big)$. Observe that ψ has finite fibers. It follows that $\psi(\mathbb{P}T^*_1 \times \mathbb{P}T^*_3)$ is an 89-dimensional, irreducible closed subset of $\eta_{14}\big(\nu_d(\mathbb{P}T^*_1)\big)$. The latter variety is irreducible and of dimension at most 89 (see Lemma 5.20). Thus, $\psi(\mathbb{P}T^*_1 \times \mathbb{P}T^*_3) = \eta_{14}\big(\nu_d(\mathbb{P}T^*_1)\big)$. $\square$

Having described the irreducible component of $\kappa_{14}\big(\nu_d(\mathbb{P}^6)\big)$ other than $\sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$ we can algorithmically check whether a given point $[G] \in \kappa_{14}\big(\nu_d(\mathbb{P}^6)\big)$ belongs to $\sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$.

We start with the following lemma which characterizes the points of $\kappa_{14}\big(\nu_d(\mathbb{P}^6)\big) \setminus \sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$.

Lemma 5.26. *Let $T = \mathbb{C}[\alpha_0, \ldots, \alpha_6]$ be a polynomial ring and $T^* = \mathbb{C}[x_0, \ldots, x_6]$ be the graded dual ring. Let $d \geq 7$. The point $[G] \in \kappa_{14}\big(\nu_d(\mathbb{P}^6)\big)$ does not belong to $\sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$ if and only if there exist $y_0 \in T^*_1$ and $P \in T^*_3$ such that $G = y_0^{d-3}P$ and for any completion of y_0 to a basis $(y_0, \ldots, y_6)$ of T^*_1 (with dual basis equal to $(\beta_0, \ldots, \beta_6)$) we have:*

(a) $\operatorname{Apolar}(P|_{y_0=1})$ *has Hilbert function* $(1, 6, 6, 1)$,

(b) $[\operatorname{Spec}\operatorname{Apolar}(P|_{y_0=1})] \notin \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^6)$.

Proof. If $y_0 \in T_1^*$ and $P \in T_3^*$ are such that $G = y_0^{d-3}P$ and there exists a completion of y_0 to a basis $(y_0, \dots, y_6)$ of T_1^* for which conditions (a),(b) hold, we get $[G] \notin \sigma_{14}(\nu_d(\mathbb{P}^6))$ by Lemma 5.22.

Assume that $[G] \notin \sigma_{14}(\nu_d(\mathbb{P}^6))$. Then by Theorem 5.25 there exists a linear form $y_0 \in T_1^*$ such that $y_0^{d-3}|G$. We claim that G is not divisible by y_0^{d-2}. Indeed, otherwise, from Theorem 5.18(ii) we get $\operatorname{bcr}(G) \le 8$, so $[G] \in \kappa_8(\nu_d(\mathbb{P}^6)) = \sigma_8(\nu_d(\mathbb{P}^6)) \subseteq \sigma_{14}(\nu_d(\mathbb{P}^6))$. We showed that $G = y_0^{d-3}P$ for some $P \in T_3^*$. Extend y_0 to a basis $y_0, y_1, \dots, y_6$. Let $f = P|_{y_0=1} \in \mathbb{C}[y_1, \dots, y_6]$. Suppose $f = F_3 + F_2 + F_1 + F_0$.

Now we prove that conditions (a),(b) hold. Let $f' = F_3' + F_2' + F_1' + F_0' \in \mathbb{C}[y_1, \dots, y_6]$ where $F_i' = (d-i)!F_i$. By Lemma 5.21, it is enough to show that conditions (a) and (b) hold for f' instead of $f = P|_{y_0=1}$. We have

$$G = \sum_{i=0}^{3} y_0^{[d-i]} F_i'.$$

By Lemma 5.13 (i)

$$\operatorname{Ann}(f')^{hom} \subseteq \operatorname{Ann}(G).$$

If $\dim_{\mathbb{C}}\left(\operatorname{Apolar}(f')\right) \le 13$, then $\operatorname{cr}(G) \le 13$ by Proposition 2.90, since $\operatorname{Ann}(f')^{hom}$ is saturated and $T/\operatorname{Ann}(f')^{hom}$ has Hilbert polynomial $\dim_{\mathbb{C}}\left(\operatorname{Apolar}(f')\right)$. Then, $[G] \in \kappa_{13}(\nu_d(\mathbb{P}^6)) = \sigma_{13}(\nu_d(\mathbb{P}^6)) \subseteq \sigma_{14}(\nu_d(\mathbb{P}^6))$, a contradiction.

From Theorem 5.18(ii) we obtain $\dim_{\mathbb{C}}\left(\operatorname{Apolar}(f')\right) \le 14$, and thus, $\dim_{\mathbb{C}}\left(\operatorname{Apolar}(f')\right) = 14$. We claim that $[\operatorname{Proj} T/\operatorname{Ann}(f')^{hom}] \notin \mathcal{H}ilb_{14}^{sm}(\mathbb{P}^6)$. Indeed, otherwise, there is a point $[J] \in \operatorname{Slip}_{14,6}$ such that

$$\overline{J} = \operatorname{Ann}(f')^{hom} \subseteq \operatorname{Ann}(G).$$

By Proposition 2.91, this contradicts the assumption that $[G] \notin \sigma_{14}(\nu_d(\mathbb{P}^6))$. We showed that $[\operatorname{Proj} T/\operatorname{Ann}(f')^{hom}] \notin \mathcal{H}ilb_{14}^{sm}(\mathbb{P}^6)$ and therefore, condition (b) holds. Hence the algebra $\operatorname{Apolar}(f')$ has Hilbert function $(1,6,6,1)$ by [58, Thm. 1.1]. Thus, condition (a) also holds. $\square$

We present the aforementioned algorithm.

Theorem 5.27 ([39, Thm. 1.6])**.** *Let $T = \mathbb{C}[\alpha_0, \dots, \alpha_6]$ be a polynomial ring with graded dual ring $T^* = \mathbb{C}[x_0, \dots, x_6]$. Given an integer $d \ge 7$ and $[G] \in \kappa_{14}(\nu_d(\mathbb{P}T_1^*)) \subseteq \mathbb{P}T_d^*$ the following algorithm checks if $[G] \in \sigma_{14}(\nu_d(\mathbb{P}T_1^*))$.*

Step 1 *Compute the ideal $\mathfrak{a} = \sqrt{((\operatorname{Ann} G)_{\le d-3})} \subseteq T$.*

Step 2 *If $\mathfrak{a}_1$ is not 6-dimensional, then $[G] \in \sigma_{14}(\nu_d(\mathbb{P}T_1^*))$ and the algorithm terminates. Otherwise, compute $\{K \in T_1^* \mid \mathfrak{a}_1 \lrcorner K = 0\}$. Let y_0 be a generator of this one dimensional $\mathbb{C}$-vector space.*

Step 3 *Let e be the maximal integer such that y_0^e divides G. If $e \neq d-3$, then $[G] \in \sigma_{14}(\nu_d(\mathbb{P}T_1^*))$ and the algorithm terminates. Otherwise, let $G = y_0^{d-3}P$. Pick a basis $y_0, y_1, \dots, y_6$ of T_1^* and compute $f = P|_{y_0=1} \in R^* := \mathbb{C}[y_1, \dots, y_6]$.*

Step 4 *Let $I = \operatorname{Ann}(f) \subseteq R$. If the (local) Hilbert function of R/I is not $(1,6,6,1)$, then $[G] \in \sigma_{14}(\nu_d(\mathbb{P}T_1^*))$, and the algorithm terminates.*

Step 5 *Compute $r = \dim_{\mathbb{C}} \operatorname{Hom}_R(I, R/I)$. Then $[G] \in \sigma_{14}(\nu_d(\mathbb{P}T_1^*))$ if and only if $r > 76$.*

Proof. Assume that $[G] \notin \sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$. Then there exist a basis $(y_0, \dots, y_6)$ of T_1^* and $P \in T_3^*$ as in Lemma 5.26. Let $f = P|_{y_0=1}$ and define $f' = F_3' + F_2' + F_1' + F_0' \in \mathbb{C}[y_1, \dots, y_6]$ where $F_i' = (d-i)!F_i$. Then $G = y_0^{[d-3]}F_3' + y_0^{[d-2]}F_2' + y_0^{[d-1]}F_1' + y_0^{[d]}F_0'$. By Lemma 5.13(ii), we have $\mathrm{Ann}(G)_{\leq d-3} = (\mathrm{Ann}(f')^{hom})_{\leq d-3}$. Moreover,

$$\big((\mathrm{Ann}(f')^{hom})_{\leq d-3}\big) = \mathrm{Ann}(f')^{hom}$$

since $d-3 \geq 4 > \deg(f')$. Therefore, we have

$$\mathfrak{a} = \sqrt{(\mathrm{Ann}(G)_{\leq d-3})} = \sqrt{\mathrm{Ann}(f')^{hom}} = (\beta_1, \dots, \beta_6),$$

where $\beta_1, \dots, \beta_6 \in T_1$ are dual to $y_1, \dots, y_6 \in T_1^*$. This shows that if the $\mathbb{C}$-linear space $\big(\sqrt{(\mathrm{Ann}(G)_{\leq d-3})}\big)_1$ is not 6-dimensional then $[G] \in \sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$. Therefore, in that case, algorithm stops correctly at Step 2.

Assume that the algorithm did not stop at Step 2. Then if G is of the form as in Lemma 5.26, then y_0 divides G exactly $(d-3)$-times. Otherwise $[G] \in \sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$ and the algorithm stops correctly at Step 3.

Assume that the algorithm did not stop at Step 3. Then the algorithm does not stop at Step 4 if and only if condition (a) of Lemma 5.26 is fulfilled.

Assume that the algorithm did not stop at Step 4. Then P satisfies condition (a) from Lemma 5.26. Hence $[G]$ is in $\sigma_{14}\big(\nu_d(\mathbb{P}^6)\big)$ if and only if P does not satisfy condition (b). The irreducible component $\mathcal{H}ilb_{1661}$ of $\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^6)$ is 76-dimensional and $\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^6)$ is smooth at points in $\mathcal{H}ilb_{1661} \setminus \mathcal{H}ilb_{14}^{Gor,sm}(\mathbb{A}^6)$ (see [58, Thm. 1.1 and Claim 3]). Thus, P does not satisfy condition (b) from Lemma 5.26 if and only if

$$\dim_{\mathbb{C}} \mathrm{Hom}_R(I, R/I) > 76,$$

since the left term is the dimension of the tangent space $\mathbf{T}_{[\mathrm{Spec}\, R/I]}\mathcal{H}ilb_{14}^{Gor}(\mathbb{A}^6)$ by [48, Prop. 2.3]. □

An implementation in Macaulay2 [42] of the algorithm from Theorem 5.27 is presented in [39, §A].

5.4 Distinguishing Grassmann secant from Grassmann cactus varieties

Let $d \geq 5$ be an integer. In this section, we state a theorem from [39] describing the irreducible component $\eta_{8,3}\big(\nu_d(\mathbb{P}^4)\big)$ of the cactus variety $\kappa_{8,3}\big(\nu_d(\mathbb{P}^4)\big)$ other than the secant variety $\sigma_{8,3}\big(\nu_d(\mathbb{P}^4)\big)$. This is analogous to Theorem 5.25.

Theorem 5.28 ([39, Thm. 1.2])**.** *Let $d \geq 5$ be an integer and $T^* = \mathbb{C}[x_0, x_1, \dots, x_4]$. Then the Grassmann cactus variety $\kappa_{8,3}\big(\nu_d(\mathbb{P}T_1^*)\big)$ has two irreducible components: the Grassmann secant variety $\sigma_{8,3}\big(\nu_d(\mathbb{P}T_1^*)\big)$ and the other one, denoted by $\eta_{8,3}\big(\nu_d(\mathbb{P}T_1^*)\big)$. Consider the map $\psi : \mathbb{P}T_1 \times \mathrm{Gr}(3, T_2^*) \to \mathrm{Gr}(3, T_d^*)$ given by $([y_0],[U]) \mapsto [y_0^{d-2}U]$. Its image is $\eta_{8,3}\big(\nu_d(\mathbb{P}T_1^*)\big)$.*

Theorem 5.28 can be generalized for $n \geq 4$ (see [39, Thm. 1.5]). However, we present the simpler version due to its similarity to Theorem 5.25.

By [20, Thm. 1.1], for $r \leq 8$, the Hilbert scheme $\mathcal{H}ilb_r(\mathbb{P}^n)$ is reducible if and only if $n \geq 4$ and $r = 8$. Furthermore, for $n \geq 4$ and $r = 8$, a general point of the non-smoothable irreducible component of $\mathcal{H}ilb_r(\mathbb{P}^n)$ corresponds to a subscheme whose coordinate ring has local Hilbert function $(1, 4, 3)$. Therefore, Theorem 5.28 describes the other irreducible component of the Grassmann cactus variety in a minimal case when such a component exists.

As in Section 5.3, we can characterize the points of $\eta_{8,3}\big(\nu_d(\mathbb{P}^n)\big) \setminus \sigma_{8,3}\big(\nu_d(\mathbb{P}^n)\big)$ (see [39, Lem. 6.9]) and obtain an algorithm analogous to the one from Theorem 5.27 (see [39, Thm 6.8]).

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