University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Tomasz Gałązka

Martingale methods in selected topics of harmonic analysis

Supervisor
dr hab. Adam Osękowski
Institute of Mathematics
University of Warsaw

March 2022

Author's declaration:
I hereby declare that this dissertation is my own work.

.... Date

.... Signature

Supervisor's declaration:
The dissertation is ready to be reviewed

.... Date

.... Signature

Summary

This thesis can be regarded as an illustration of the fruitful interplay between martingale theory and harmonic analysis. Specifically, we will be concerned with a number of martingale inequalities which are motivated by certain questions coming from the theory of Fourier multipliers and Fourier analysis on the unit disc. We will be particularly interested in obtaining the optimal values of the constants involved. Such extremal problems arise in various contexts in analysis, for example, when one tries to compute explicitly norms of some given operators, which, in turn, is often useful in the theory of PDEs and approximation.

It should be emphasized that the main contribution of the thesis is of probabilistic nature. In our considerations below, the main difficulty will lie in the proofs of sharp inequalities for martingales. The analytic component will serve as an interesting and intriguing motivation and application.

The dissertation consists of five chapters, which are devoted to the following topics.

Chapter 1 has a preliminary character and it contains some background on the objects which will appear later in the text. In particular, the reader can find there some basic information on the martingale theory and some foundations of the theory of Fourier multipliers (both in the Euclidean and non-Euclidean setting), as well as some simple material from harmonic analysis on the unit circle.

Chapter 2 is devoted to the alternative proof of the celebrated L^p-estimates for differentially subordinate martingales. This result was originally established by Burkholder in the eighties, our approach exploits a novel duality argument. In addition, we also present a number of basic applications: Littlewood-Paley-type estimates and L^p-inequalities for Riesz transforms on Lie groups and spheres. The material in this chapter was taken from the paper written jointly with R. Bañuelos and A. Osękowski.

Chapter 3 contains the significant extension of sharp strong- and weak-type estimates for martingale transforms and stochastic integrals. The novelty comes from the fact that we drop the assumption of the boundedness of the transforming sequence (integrand), which is typically imposed in the literature. Instead, we allow it to belong to an L^r space; consequently, the martingale transform (stochastic integral) acts boundedly as an operator from L^q to L^p, where $1/p = 1/q + 1/r$. The main result of Chapter 3 identifies explicitly the norm of this operator, along with its weak-type version. The contents is taken from the joint work with A. Osękowski.

Chapter 4 is concerned with the sharp weak-type estimate for the periodic Hilbert transform, a fundamental singular integral operator. To accomplish this, we construct a certain special superharmonic function on the plane, which yields an interesting estimate for martingales satisfying the orthogonality condition. This inequality leads directly to the estimate for the Hilbert transform, it is also shown to produce the corresponding result for Riesz transforms on compact Lie groups. The material from this chapter comes from the joint work with A. Osękowski.

Chapter 5 is the final part of the paper and concerns a slightly different type of problem which, however, will also be solved with the use of martingale methods. We leave the context of Fourier multipliers and move towards complex harmonic analysis on the unit circle. We will be interested in the quantitative version of H^1-BMO duality; more precisely, we will compute the best constant in Fefferman's inequality in the conformal setting. The proof will exploit several deep facts coming from the complex analysis of several variables.

This will allow us to construct a certain special plurisuperharmonic function, which, in turn, will furnish an appropriate sharp estimate for analytic martingales. The contents of the chapter is taken from the joint work with A. Osękowski.

2010 Mathematics Subject Classification. Primary: 32U05, 42B20, 42B30, 60G42, 60G44. Secondary: 46E30.

Keywords and phrases: Martingale; transform; differential subordination; Burkholder's method; Bellman function; Hilbert transform; weak-type inequality; Hardy space; bounded mean oscillation; plurisuperharmonic; analytic envelope; best constants.

Streszczenie

Celem niniejszej rozprawy jest zilustrowanie wybranych związków i zależności pomiędzy teorią martyngałów a analizą harmoniczną. Ściślej, będziemy badać pewne nierówności martyngałowe, motywacją dla których są naturalne pytania pojawiające się w teorii mnożników Fouriera i analizie Fourierowskiej na okręgu jednostkowym. Położymy szczególny nacisk na optymalność stałych występujących w badanych nierównościach, co ma istotne zastosowania w różnych dziedzinach analizy; przykładowo, wspomniana optymalność często prowadzi do wyznaczenia norm pewnych specjalnych operatorów, co z kolei ma znaczenie w teorii równań różniczkowych i aproksymacji.

Należy podkreślić, że główny wkład pracy ma probabilistyczny charakter. W rozważaniach poniżej, zasadnicza trudność będzie spoczywać na dowodach odpowiednich nierówności dla martyngałów, natomiast składnik analityczny będzie raczej pełnił rolę ciekawej motywacji (zastosowania).

Rozprawa składa się z pięciu rozdziałów, poświęconych następującym zagadnieniom.

Rozdział 1 ma charakter wprowadzenia i zawiera niezbędne definicje, które będą przydatne w dalszej części pracy. W szczególności, znajdują się tam podstawowe informacje na temat teorii martyngałów oraz teorii mnożników Fourierowskich (zarówno w kontekście euklidesowym, jak i nieeuklidesowym), jak również proste fakty z analizy harmonicznej na okręgu jednostkowym.

Rozdział 2 poświęcony jest alternatywnemu dowodowi oszacowania w L^p dla martyngałów spełniających warunek silnej dominacji. Pierwotnie wynik ten został uzyskany przez Burkholdera w latach osiemdziesiątych, w pracy prezentujemy pewne nowe, dualne podejście do tego problemu. Przedstawiamy również podstawowe zastosowania uzyskanego wyniku: oszacowanie typu Littlewooda-Paleya oraz nierówności w L^p dla transformat Riesza na grupach Liego i sferach. Rozdział jest oparty na wynikach ze wspólnego artykułu z R. Bañuelosem i A. Osękowskim.

Rozdział 3 zawiera znaczące uogólnienie optymalnego oszacowania silnego i słabego typu dla transformat martyngałowych i całek stochastycznych. Wzmocnienie polega na tym, że porzucamy założenie o ograniczoności ciągu transformującego (funkcji podcałkowej), które zazwyczaj widnieje w pokrewnych rezultatach w literaturze. Zamiast tego dopuszczamy, aby powyższy ciąg (funkcja) należał do przestrzeni L^r; w konsekwencji, transformata (całka stochastyczna) okazuje się być ograniczona jako operator działający z przestrzeni L^q do przestrzeni L^p, gdzie $1/p = 1/q + 1/r$. Głównym rezultatem tego rozdziału jest zidentyfikowanie dokładnej normy tego operatora, wraz z jego wersją dla słabych przestrzeni L^p. Wynik pochodzi ze wspólnej pracy z A. Osękowskim.

Rozdział 4 dotyczy optymalnej nierówności słabego typu dla okresowej transformaty Hilberta, fundamentalnego operatora singularnego. Konstruujemy pewną nadharmoniczną funkcję na płaszczyźnie, która pozwala uzyskać interesujące oszacowania dla martyngałów spełniających warunek prostopadłości. Oszacowania te prowadzą do nierówności dla transformaty Hilberta, a wynik rozszerzamy na transformaty Riesza na zwartych grupach Liego. Rozdział jest oparty na wynikach ze wspólnego artykułu z A. Osękowskim.

W ostatniej części pracy, Rozdziale 5, badamy nieco inny rodzaj problemu, wciąż używając metod martyngałowych. Opuszczamy kontekst mnożników Fouriera i skupiamy się na zespolonej analizie harmonicznej na okręgu jednostkowym. Zajmujemy się dualnością pomiędzy przestrzeniami H^1 oraz BMO, a dokładniej, identyfikujemy najlepszą stałą w nierówności Feffermana w przypadku konforemnym. Dowód opiera się na głębokich faktach z analizy zespolonej wielu zmiennych, które pozwalają skonstruować pewną specjalną funkcję plurinadharmoniczną - obiekt ten prowadzi do odpowiedniej nierówności dla martyngałów analitycznych. Wynik pochodzi ze wspólnej pracy z A. Osękowskim.

2010 Mathematics Subject Classification. Primary: 32U05, 42B20, 42B30, 60G42, 60G44. Secondary: 46E30.

Słowa kluczowe: Martyngał; transformata; silna dominacja; metoda Burkholdera; funkcja Bellmana; transformata Hilberta; nierówność słabego typu; przestrzenie Hardy'ego; funkcja o ograniczonych średnich oscylacjach; plurinadharmoniczność; obwiednia analityczna; optymalne stałe.

Acknowledgments

I would like to express my gratitude to my supervisor, professor Adam Osękowski, for devoting a great amount of time to help me along the way and all the interesting problems he shared with me. For his guidance and support in my research, endless patience and optimistic attitude.

The research presented in this dissertation was partially supported by the National Science Centre (Poland), grant no. 2018/30/Q/ST1/00072.

Contents

Chapter 1

Preliminaries

1.1. Martingales

Let us introduce the necessary probabilistic background for our further study. For the more detailed and systematic presentation of the theory of stochastic processes, we refer the reader to the monographs [15] and [17]. The discussion on martingale transforms arising in the context of manifolds is taken from the paper [1].

Discrete-time martingales. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, filtered by a nondecreasing family $(\mathcal{F}_n)_{n\geq 0}$ of sub-σ-algebras of $\mathcal{F}$. Let $f = (f_n)_{n\geq 0}$ be an adapted martingale; in our considerations below, we will study different contexts, in which f will take values in $\mathbb{R}$, $\mathbb{C}$, a separable Hilbert space $\mathbb{H}$ or some separable Banach space $\mathbb{B}$. Usually, we will assume with no loss of generality that the Hilbert space $\mathbb{H}$ is equal to ℓ^2 and we will denote the corresponding norm and scalar product by $|\cdot|$ and $\langle\cdot,\cdot\rangle$, respectively (the symbol $\langle\cdot,\cdot\rangle$ will also be used below to denote the standard scalar product in $\mathbb{R}^d$; the meaning should be clear from the context and should not lead to any confusion). The difference sequence $df = (df_n)_{n\geq 0}$ of a martingale f is defined by the identities $df_0 = f_0$ and $df_n = f_n - f_{n-1}$ for $n = 1, 2, \ldots$. Equivalently, this sequence is uniquely determined by the equations

$$f_n = \sum_{k=0}^{n} df_k, \qquad n = 0, 1, 2, \ldots.$$

The maximal function f^* of the martingale f is defined by $f^* = \sup_{n\geq 0} |f_n|$. We will also work with truncated versions of this object, given by the formula $f_N^* = \max_{0\leq n\leq N} |f_n|$ for $N = 0, 1, 2, \ldots$.

Suppose that $v = (v_n)_{n\geq 0}$ is a predictable sequence of random variables; here by predictability we mean that v_0 is $\mathcal{F}_0$-measurable and, for any $n = 1, 2, \ldots$, the random variable v_n is measurable with respect to $\mathcal{F}_{n-1}$. An adapted sequence $g = (g_n)_{n\geq 0}$ of random variables is said to be the (martingale) transform of f by v, if for any $n \geq 0$ we have the identity $dg_n = v_n df_n$. We will write $g = v \cdot f$ in such a case, for the sake of consistency with the context of stochastic integrals (see below). Observe that if f is a martingale and v is predictable, then the transform $v \cdot f$ is also a martingale.

In what follows, we will measure the size of martingales in strong and weak L^p spaces. Given a martingale f and a parameter $p \in (0, \infty]$, we define the L^p-norm of f by

$$\|f\|_{L^p} = \sup_{n\geq 0} \|f_n\|_{L^p}.$$

If $p \geq 1$, then the sequence $\|f_n\|_{L^p}$ is nondecreasing and hence $\|f\|_{L^p} = \lim_{n\to\infty} \|f_n\|_{L^p}$. As for the weak L^p-norms, the definition is analogous: given $p \in (0, \infty)$, we set

$$\|f\|_{L^{p,\infty}} = \sup_{n\geq 0} \|f_n\|_{L^{p,\infty}},$$

where $\|f_n\|_{L^{p,\infty}}$ is the weak L^p-norm of the random variable f_n. One usually considers the quasinorm $\|\xi\|_{L^{p,\infty}} = \sup_{\lambda>0} \left(\lambda^p \mathbb{P}(|\xi| \geq \lambda)\right)^{1/p}$, however, in the range $1 < p < \infty$ it will be more convenient for us to use a slightly different norming, given by

$$|||\xi|||_{L^{p,\infty}} = \sup \left\{ \mathbb{P}(A)^{1/p-1} \int_A |\xi| \mathrm{d}\mathbb{P} \,:\, A \in \mathcal{F},\, \mathbb{P}(A) > 0 \right\}.$$

It can be easily shown that this is indeed a norm. Furthermore, both norming are equivalent: there is a constant κ_p depending only on p such that $\|\cdot\|_{L^{p,\infty}} \leq |||\cdot|||_{L^{p,\infty}} \leq \kappa_p \|\cdot\|_{L^{p,\infty}}$ (here the assumption $1 < p < \infty$ plays the key role). See e.g. [23] for details.

One of the main themes of this dissertation is the study of certain sharp estimates involving a martingale and its transform by a predictable sequence bounded in absolute value by 1. This problem will be studied in depth in Chapters 3 and 4.

Continuous-time martingales. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, which is equipped with continuous-time filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions: that is, we assume that the filtration is right-continuous and $\mathcal{F}_0$ contains all the events of probability 0. Let $X = (X_t)_{t\geq 0}$ be an adapted local martingale; as previously, we may consider scalar- or vector-valued processes. In addition, we will impose standard assumptions on the regularity of the trajectories: we will restrict ourselves to càdlàg local martingales, i.e., those processes, whose paths are right-continuous and have limits from the left. For any $t \geq 0$, the symbol $\Delta X_t = X_t - X_{t-}$ will denote the jump of X at time t (with the standard convention $X_{0-} = 0$). The maximal function of X will be given by $X^* = \sup_{t\geq 0} |X_t|$, we will also use the notation $X_T^* = \sup_{0\leq t\leq T} |X_t|$ for the truncated version of this object. Let $[X, X] = ([X, X]_t)_{t\geq 0}$ be the square bracket (quadratic variation) of the local martingale X. In the real-valued setting, this object is given by the following limit in probability: for any $t \geq 0$,

$$[X, X]_t = \lim_{n\to\infty} \sum_{j=1}^{k_n} (X_{t_j^{(n)}} - X_{t_{j-1}^{(n)}})^2$$

where $(t_j^{(n)})_{j=0}^{k_n}$ is an arbitrary sequence of partitions $0 = t_0^{(n)} < t_1^{(n)} < t_2^{(n)} < \ldots < t_{k_n}^{(n)} = t$ of $[0, t]$ with a diameter tending to 0. We also set $[X, X]_\infty = \lim_{t\to\infty} [X, X]_t$. If X is an $\mathbb{H}$-valued process (recall that we have assumed $\mathbb{H} = \ell^2$), then we set $[X, X] = \sum_{m=0}^\infty [X^m, X^m]$, where X^m is the m-th coordinate of X and $[X^m, X^m]$ is the square bracket of the real-valued process X^m. By a standard polarization, the quadratic variation gives rise to the bilinear form defined on pairs of local martingales, given by $[X, Y] = ([X + Y, X + Y] - [X - Y, X - Y])/4$ or alternatively, by the limiting procedure as above:

$$[X, Y]_t = \mathbb{P} - \limsup_{n\to\infty} \sum_{j=1}^{k_n} (X_{t_j^{(n)}} - X_{t_{j-1}^{(n)}})(Y_{t_j^{(n)}} - Y_{t_{j-1}^{(n)}}). \tag{1.1.1}$$

Two local martingales are said to be orthogonal if their bracket is constant as a function of t: that is, $d[X, Y] = 0$. We will also need the associated total variation of the pair X, Y, which is given by

$$\int_0^t |\mathrm{d}[X, Y]_s| = \mathbb{P} - \limsup_{n\to\infty} \sum_{j=1}^{k_n} |X_{t_j^{(n)}} - X_{t_{j-1}^{(n)}}|\,|Y_{t_j^{(n)}} - Y_{t_{j-1}^{(n)}}|.$$

The following well-known fact from stochastic analysis will often be used. Namely, for any martingale X there is a unique continuous local martingale part X^c of X, which satisfies

$$[X,X]_t = |X_0|^2 + [X^c, X^c]_t + \sum_{0<s\leq t} |\Delta X_s|^2, \qquad t \geq 0. \tag{1.1.2}$$

Furthermore, we have the identity $[X^c, X^c] = [X,X]^c$, where the expression on the right is the pathwise continuous part of $[X,X]$.

Next, suppose that $H = (H_t)_{t\geq 0}$ is a real-valued predictable process (that is, H is measurable with respect to the predictable σ-algebra, when treated as a function on $\Omega \times [0,\infty)$). Then the symbol $H \cdot X$ will denote the stochastic integral of H with respect to X, i.e.,

$$(H \cdot X)_t = H_0 X_0 + \int_{0+}^t H_s \cdot \mathrm{d}X_s, \qquad t \geq 0.$$

Obviously, it is a continuous-time extension of martingale transform considered in the discrete case.

Finally, we define the L^p- and weak L^p-norms of X in a similar manner as previously. However, since the process is assumed to be a *local* martingale, the definitions need to refer to stopping times. Namely, given $0 < p < \infty$, we set

$$\|X\|_{L^p} = \sup_\tau \|X_\tau\|_{L^p}, \quad \|X\|_{L^{p,\infty}} = \sup_\tau \|X_\tau\|_{L^{p,\infty}}, \quad |||X|||_{L^{p,\infty}} = \sup_\tau |||X_\tau|||_{L^{p,\infty}},$$

where the suprema are taken over all finite stopping times τ relative to $(\mathcal{F}_t)_{t\geq 0}$.

Martingale transforms on manifolds. The results obtained in the thesis can be applied beyond the Euclidean setting. In our discussion below, we will use some elementary facts from differential geometry and the theory of Riemannian manifolds, which can be found, for example, in [29]. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space equipped with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Assume further that M is a Riemannian manifold of dimension n with Ricci curvature bounded from below. Let $\langle \cdot, \cdot \rangle$ be an inner product on the associated tangent space TM. A Brownian motion in M is an $(\mathcal{F}_t)_{t\geq 0}$-adapted process $W = (W_t)_{t\geq 0}$ taking values in M such that for all smooth functions $f : M \to \mathbb{R}$,

$$I_{df} = \left(f(W_t) - f(W_0) - \frac{1}{2} \int_{0+}^t \Delta_M f(W_s) \mathrm{d}s \right)_{t\geq 0} \tag{1.1.3}$$

is a real-valued continuous martingale. Here Δ_M stands for the Laplace-Beltrami operator on M. Now, let $\mathcal{K} = (\mathcal{K}_t)_{t\geq 0}$ be a continuous, adapted process with values in T^*M, the cotangent space of M. We say that $\mathcal{K}$ is *above* W, if for all $t \geq 0$ and $\omega \in \Omega$ we have $\mathcal{K}_t(\omega) \in T^*_{W_t(\omega)}M$. Then the Itô integral of $\mathcal{K}$, denoted by $I_{\mathcal{K}} = \left(\int_0^t \langle \mathcal{K}_s, \mathrm{d}W_s \rangle \right)_{t\geq 0}$, is determined by the following properties:

(i) if $\mathcal{K}_t = \mathrm{d}f(W_t)$ for some smooth function $f : M \to \mathbb{R}$, then $I_{\mathcal{K}}$ equals I_{df} defined in (1.1.3).

(ii) if $A = (A_t)_{t\geq 0}$ is a real-valued, continuous process, then $I_{A\mathcal{K}} = \left(\int_0^t A_s \mathrm{d}(I_{\mathcal{K}})_s \right)_{t\geq 0}$ is the classical Itô integral of A with respect to the continuous martingale $I_{\mathcal{K}}$.

It can be verified that if $\mathcal{K}$ is above W, then the process $I_{\mathcal{K}}$ is a continuous, real-valued martingale, and the corresponding square bracket is given by

$$[I_{\mathcal{K}}, I_{\mathcal{L}}]_t = \int_0^t \mathrm{Trace}(\mathcal{K}_s \otimes \mathcal{L}_s)\mathrm{d}s, \tag{1.1.4}$$

where $\otimes$ is the tensor product and $(\mathcal{K}_s \otimes \mathcal{L}_s)(\omega) = \mathcal{K}_s(\omega) \otimes \mathcal{L}_s(\omega) \in T^*_{W_s(\omega)} \otimes T^*_{W_s(\omega)}$. Next, we will define a certain transformation of the class of stochastic integrals. Assume $x \in M$ and let $End(T^*_x M)$ be the family of all linear maps from $T^*_x M$ to itself. Let $End(T^*M)$ be the collection of all $End(T^*_x M)$, $x \in M$. A bounded and continuous process A with values in $End(T^*M)$ is called *a martingale transformer with respect to* W, if for all $t \geq 0$ and $\omega \in \Omega$ we have $A_t(\omega) \in End(T^*_{W_t(\omega)} M)$ (in other words, A is above W). At last, suppose that $\mathcal{K}$ is a continuous, bounded process with values in T^*M which is above W, and let A be a martingale transformer with respect to W. Then $A * I_{\mathcal{K}}$, *the martingale transform of* $I_{\mathcal{K}}$ *by* A, is the real-valued martingale given by

$$A * I_{\mathcal{K}} = I_{A\mathcal{K}} = \left(\int_0^t \langle A_s \mathcal{K}_s, \mathrm{d}W_s \rangle \right)_{t \geq 0}.$$

We introduce the norm of martingale transformer A by

$$\|A\| = \sup |A_t(\omega) e|,$$

where the supremum is taken over all $\omega \in \Omega$, all $t \geq 0$ and all vectors $e \in T_{W_t(\omega)} M$ of length 1.

1.2. Fourier multipliers. Hilbert and Riesz transforms

Fourier multipliers. Most of the probabilistic statements established in this thesis will have some profound implications in the theory of Fourier multipliers. We start with some preliminary and very general information on this subject. For an arbitrary (appropriately integrable) function $f : \mathbb{R}^d \to \mathbb{C}$, its Fourier transform $\hat{f}$ is defined by the formula

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{i\langle \xi, x \rangle} \mathrm{d}x, \qquad \xi \in \mathbb{R}^d.$$

For any bounded measurable function $m : \mathbb{R}^d \to \mathbb{C}$, there is a bounded linear operator T_m on $L^2(\mathbb{R}^d)$, called the Fourier multiplier associated with the symbol m, which is defined by the identity $\widehat{T_m f} = m\hat{f}$. A straightforward use of Plancherel's theorem shows that the norm of T_m on $L^2(\mathbb{R}^d)$ is equal to $\|m\|_{L^\infty(\mathbb{R}^d)}$. A classical problem, which has been studied very intensively in the literature, is to analyze those symbols m, for which the associated multiplier extends to a bounded linear operator on $L^p(\mathbb{R}^d)$ (for all $p \in (1, \infty)$; for a fixed p from this interval; for all p belonging to some subinterval (a, b); etc.). One can also study the boundedness properties of T_m on other function spaces. A related important question is to compute explicitly the norm of a given Fourier multiplier, as an operator between two given spaces. The problems of this type are very challenging in general, and the range of techniques used in their investigation is very wide.

Periodic and non-periodic Hilbert transform. From the historical point of view, the first result in the above direction concerned the L^p boundedness of the Hilbert transform $\mathcal{H}^{\mathbb{R}}$ on the real line (sometimes referred to as the non-periodic Hilbert transform). This operator is defined as a Fourier multiplier with the symbol $m(\xi) = i\,\mathrm{sgn}\,\xi$, $\xi \in \mathbb{R}$. Alternatively, the transform can be expressed in terms of singular integrals

$$\mathcal{H}^{\mathbb{R}} f(x) = \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(x-y)}{y} \mathrm{d}y,$$

where "p.v." means that we consider the Cauchy principal integral:

$$\mathcal{H}^{\mathbb{R}} f(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} \mathrm{d}y.$$

There is a periodic companion to the above operator, the so-called Hilbert transform $\mathcal{H}^{\mathbb{T}}$ on the unit circle $\mathbb{T} = \{\zeta \in \mathbb{C} : |\zeta| = 1\} \simeq (-\pi, \pi]$, which is defined by

$$\mathcal{H}^{\mathbb{T}} f(e^{it}) = \text{p.v.} \int_{-\pi}^{\pi} f(s) \cot \frac{t-s}{2} \mathrm{d}\mu(s) \qquad \text{for } f \in L^1(\mathbb{T}).$$

Here and below, μ stands for the normalized Haar measure on $\mathbb{T}$. In 1927, M. Riesz [42] showed that $\mathcal{H}^{\mathbb{T}}$ is bounded as an operator on $L^p(\mathbb{T})$ if and only if $1 < p < \infty$. This immediately gives the same statement about the non-periodic Hilbert transform $\mathcal{H}^{\mathbb{R}}$: we have $\|\mathcal{H}^{\mathbb{T}}\|_{L^p(\mathbb{T}) \to L^p(\mathbb{T})} = \|\mathcal{H}^{\mathbb{R}}\|_{L^p(\mathbb{R}) \to L^p(\mathbb{R})}$ by a certain conformal mapping argument. The explicit value of the norm was identified by Pichorides [40] and Cole (unpublished; see the discussion in Gamelin [19]): we have

$$\|\mathcal{H}^{\mathbb{T}}\|_{L^p(\mathbb{T}) \to L^p(\mathbb{T})} = \cot\left(\frac{\pi}{2p^*}\right), \qquad 1 < p < \infty,$$

where $p^* = \max\{p, p/(p-1)\}$. For $p = 1$ the strong-type estimate fails, but, as a substitute, there is a related weak-type $(1,1)$ inequality. Namely, as proved by Kolmogorov in [27], we have $\|\mathcal{H}^{\mathbb{T}}\|_{L^1(\mathbb{T}) \to L^{1,\infty}(\mathbb{T})} < \infty$. The exact value of the norm was evaluated by Davis [13]:

$$\|\mathcal{H}^{\mathbb{T}}\|_{L^1(\mathbb{T}) \to L^{1,\infty}(\mathbb{T})} = \frac{1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots}{1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots} = 1.347\dots, \tag{1.2.1}$$

under the standard norming of $L^{p,\infty}$: $\|f\|_{L^{p,\infty}(\mathbb{T})} = \sup_{\lambda>0} \left(\lambda^p \mu(|f| \geq \lambda)\right)^{1/p}$. As previously, the identity above remains valid in the non-periodic setting. A related result for the norm of the Hilbert transform, as an operator from L^p to $L^{p,\infty}$ for $1 < p \leq 2$, was obtained by Janakiraman in [26] (see also [36]): we have the equality

$$\|\mathcal{H}^{\mathbb{T}}\|_{L^p(\mathbb{T}) \to L^{p,\infty}(\mathbb{T})} = \|\mathcal{H}^{\mathbb{R}}\|_{L^p(\mathbb{R}) \to L^{p,\infty}(\mathbb{R})} = \left(\frac{1}{\pi} \int_{\mathbb{R}} \frac{\left|\frac{2}{\pi} \log |t|\right|^p}{t^2+1} \mathrm{d}t\right)^{-1/p}.$$

The precise norm for $p > 2$ is still an open problem. However, under the equivalent norming

$$|||f|||_{L^{p,\infty}(\mathbb{T})} = \sup\left\{\frac{1}{\mu(E)^{1-1/p}} \int_E |f| \mathrm{d}\mu \,:\, E \subseteq \mathbb{T},\, \mu(E) > 0\right\} \tag{1.2.2}$$

(with an obvious modification for $L^{p,\infty}(\mathbb{R})$), Osękowski solved the problem in the full range $1 < p < \infty$.

Theorem 1.1. *Under the above norming,* $\|\mathcal{H}^{\mathbb{T}}\|_{L^p(\mathbb{T})\to L^{p,\infty}(\mathbb{T})} = \|\mathcal{H}^{\mathbb{R}}\|_{L^p(\mathbb{R})\to L^{p,\infty}(\mathbb{R})}$ *equals*

$$\begin{cases} \left[\dfrac{2^{p'+2}\Gamma(p'+1)}{\pi^{p'+1}}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{p'+1}}\right]^{1/p'} & \text{if } 1<p\le 2,\\ \left[\dfrac{2^{p'+2}\Gamma(p'+1)}{\pi^{p'}}\displaystyle\sum_{k=0}^{\infty}\frac{1}{(2k+1)^{p'}}\right]^{1/p'} & \text{if } 2<p<\infty, \end{cases}$$

where $p' = p/(p-1)$ *is the harmonic conjugate to* p.

Riesz transforms on $\mathbb{R}^d$. The Hilbert transform $\mathcal{H}^{\mathbb{R}}$ has a natural extension to higher dimensions. Given a positive integer d, we introduce the collection of the so-called directional Riesz transforms $R_1, R_2, \ldots, R_d$ on $\mathbb{R}^d$ as Fourier multipliers with the symbols $i\xi_j/|\xi|$, $j = 1, 2, \ldots, d$. Alternatively, these operators can be given by singular integrals

$$R_j f(x) = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{(d+1)/2}}\ \text{p.v.}\int_{\mathbb{R}^d}\frac{x_j-y_j}{|x-y|^{d+1}}f(y)\mathrm{d}y, \qquad j = 1, 2, \ldots, d.$$

One can also define R_j by the formula

$$R_j = \partial_j \circ (-\Delta_{\mathbb{R}^d})^{-1/2}, \tag{1.2.3}$$

which is very convenient when one tries to generalize Riesz transforms to the context of Riemannian manifolds. It turns out (see Calderón and Zygmund [12], Iwaniec and Martin [25]) that the norms of R_j as operators on $L^p(\mathbb{R}^d)$, $1 < p < \infty$, are the same as for the Hilbert transform on the circle:

$$\|R_j\|_{L^p(\mathbb{R}^d)\to L^p(\mathbb{R}^d)} = \cot\left(\frac{\pi}{2p^*}\right) \qquad j = 1, 2, \ldots, d.$$

Osękowski [37] proved that Theorem 1.1 remains valid if we replace $\mathcal{H}$ by R_j. Interestingly, it is not known whether the aforementioned estimates of Davis and Janakiraman hold true for Riesz transforms. Roughly speaking, the reason is that the Riesz transform is an average of directional Hilbert transforms, and the averaging procedure is not a contraction on $L^{p,\infty}$ under the standard norming (on contrary, it *is* a contraction when the norming $|||\cdot|||$ is used). It should also be emphasized that all the sharp results formulated above were proved with the use of martingale methods. In some papers, this probabilistic component is deeply hidden (e.g., martingales do not appear in Pichorides' work [40]), however, a closer look at the proof reveals the exploitation of appropriate stochastic structures.

The concept of Riesz transforms can be extended far beyond the Euclidean setting. We will see in the next section how to define these operators in the context of Lie groups and spheres.

1.3. Probabilistic representation of Riesz transforms

Now, following the classical paper by Gundy and Varopoulos [24] and Arcozzi [1], we will connect the previous two sections and briefly describe the probabilistic representation of (various versions of) Riesz transforms in the language of martingales. The rough, general idea can be expressed as follows: given a function f (on $\mathbb{R}^d$, Lie group, sphere, etc.) and the Riesz transform R, the pair (f, Rf) behaves in the same manner as the pair $(X, A\cdot X)$, where X is a certain martingale and $A\cdot X$ is its appropriate transformation (expressed in terms of stochastic integrals; it also involves a certain additional averaging and a limiting procedure). The bottom line is that any martingale estimate between X and $A\cdot X$ leads to the corresponding inequality for the Riesz transform.

Representation of Riesz transforms on $\mathbb{R}^d$. Suppose that $Z=(X,Y)$ is a Brownian motion in $\mathbb{R}^d\times\mathbb{R}$, starting from the origin. For any $y\geq 0$, we consider the stopping time $\tau(y)=\inf\{t\geq 0: Y_t=-y\}$. If f belongs to $C_0^\infty(\mathbb{R}^d)$, the class of smooth functions of compact support on $\mathbb{R}^d$, let $P[f]:\mathbb{R}^d\times[0,\infty)\to\mathbb{R}$ denote the Poisson extension of f to the upper half-space:

$$P[f](x,y):=\mathbb{E}f\left(x+X_{\tau(y)}\right).$$

Next, for any $(d+1)\times(d+1)$ matrix A we define the martingale transform $A{*}f$ by

$$A{*}f(x,y)=\int_{0+}^{\tau(y)}A\nabla P[f]((x,y)+Z_s)\cdot \mathrm{d}Z_s.$$

Note that $A*f(x,y)$ is a random variable for each x, y. Now, for any $f\in C_0^\infty(\mathbb{R}^d)$, any $y\geq 0$ and any matrix A as above, we define $\mathcal{T}_A^y f:\mathbb{R}^d\to\mathbb{R}$ by the bilinear form

$$\int_{\mathbb{R}^d}\mathcal{T}_A^y f(x)g(x)\,\mathrm{d}x=\int_{\mathbb{R}^d}\mathbb{E}\big[A{*}f(x,y)g(x+X_{\tau(y)})\big]\mathrm{d}x, \tag{1.3.1}$$

where g runs over $C_0^\infty(\mathbb{R}^d)$. Less formally, $\mathcal{T}_A^y f$ is given as the following conditional expectation with respect to the measure $\tilde{\mathbb{P}}=\mathbb{P}\otimes\mathrm{d}x$ ($\mathrm{d}x$ denotes Lebesgue's measure on $\mathbb{R}^d$): for any $w\in\mathbb{R}^d$,

$$\mathcal{T}_A^y f(w)=\tilde{\mathbb{E}}\left[A{*}f(x,y)|x+X_{\tau(y)}=w\right].$$

The interplay between the operators $\mathcal{T}_A^y$ and Riesz transforms is explained in the following theorem, consult [22, 24].

Theorem 1.2. *Let $A^j=[a^j_{\ell m}]$, $A^{j+}=[a^{j+}_{\ell m}]$, $j=1,\,2,\,\ldots,\,d$, be the $(d+1)\times(d+1)$ matrices given by*

$$a^j_{\ell m}=\begin{cases}1 & \text{if } \ell=d+1,\ m=j,\\ -1 & \text{if } \ell=j,\ m=d+1,\\ 0 & \text{otherwise,}\end{cases}\qquad a^{j+}_{\ell m}=\begin{cases}1 & \text{if } \ell=d+1,\ m=j,\\ 0 & \text{otherwise.}\end{cases}$$

Then $\mathcal{T}^y_{A^j}f\to R_jf$ and $\mathcal{T}^y_{A^{j+}}f\to\frac{1}{2}R_jf$ almost everywhere as $y\to\infty$.

Thus, we see that Riesz transforms can be regarded as an average of a certain martingale transformation.

Riesz transforms on Lie groups. Suppose that G is a d-dimensional compact, connected Lie group, endowed with a Riemannian bi-invariant metric, and denote by $\mathrm{d}x$ the usual Riemannian volume measure on G. Let $\mathfrak{g}$ be its Lie algebra and fix an orthonormal basis $\{X_1,\,X_2,\,\ldots,\,X_d\}$ of $\mathfrak{g}$. Consider the group $\tilde{G}=G\times\mathbb{R}$, with the product Riemannian metric and the corresponding Lie algebra $\mathfrak{g}\oplus\mathbb{R}$. Note that if $X_0=\partial/\partial y$ is the generator of the Lie algebra of $\mathbb{R}$, then $\{X_1,\,X_2,\,\ldots,\,X_d,\,X_0\}$ is an orthonormal basis of $\mathfrak{g}\oplus\mathbb{R}$. We define the Riesz transform on G in the direction X_j by $R_j=R_{X_j}=X_j\circ(-\Delta_G)^{-1/2}$, where Δ_G is the Laplace-Beltrami operator on G. This is a natural extension of the Riesz transforms defined on $\mathbb{R}^d$: see (1.2.3).

We proceed to the martingale representation. Let X, Y be two independent Brownian motions in G and $\mathbb{R}$, respectively; then $Z = (X, Y)$ is a Brownian motion in the group $\tilde{G}$. Fix $\lambda > 0$ and suppose that Z_0, the initial distribution of $Z^\lambda = (Z_t)_{t\geq 0}$, is the product measure $\mathrm{d}x \times \delta_\lambda$, where δ_λ is the Dirac measure concentrated on $\{\lambda\}$. Denote $\tilde{G}^+ = G \times [0, \infty)$ and define the stopping time

$$\tau_0 = \inf\{t \geq 0 : Y_t \leq 0\},$$

the exit time of Z from $\tilde{G}^+$. Then $(Z_{\tau_0\wedge t})_{t\geq 0}$ is a Brownian motion in $\tilde{G}^+$, stopped at G. Let $A : \tilde{G}^+ \to End(T^*\tilde{G}^+)$ be a continuous section of the bundle $End(T^*\tilde{G}^+)$ and set $\tilde{A} = \big(A(Z_{\tau_0\wedge t})\big)_{t\geq 0}$. Then from the very definition, $\tilde{A}$ is a martingale transformer.
Now, for $f \in C_0^\infty(G)$, let F be its Poisson extension to $\tilde{G}^+$, i.e., the unique C^∞ function on $\tilde{G}$ satisfying

$$0 = \Delta_{\tilde{G}} F(x, y) = \Delta_G F(x, y) + \frac{\partial^2 F}{\partial y^2}(x, y), \qquad x \in G,\ y > 0,$$

such that $F(x, 0) = f(x)$ and F is bounded on $\tilde{G}^+$, see [20] and [45] for more on this topic. For A, f, F and λ as above, define

$$T_A^\lambda f(x) = \mathbb{E}\big[\tilde{A} * I_{dF} | Z_{\tau_0} = x\big],$$

the A-transform of f; the conditional expectation is taken with respect to the σ-algebra generated by Z_{τ_0}. Since Z_{τ_0} takes values in the boundary $G \times \{0\}$, $T_A^\lambda f$ can be treated as a function on the group G.

Now, consider the linear map $A^j : \mathfrak{g} \oplus \mathbb{R} \to \mathfrak{g} \oplus \mathbb{R}$ given by

$$A^j X_m = \begin{cases} X_j & \text{if } m = 0, \\ -X_0 & \text{if } m = j, \\ 0 & \text{otherwise.} \end{cases}$$

Since A^j is a smooth section of the bundle $End(T\tilde{G}^+)$, the natural identification between $\mathfrak{g} \oplus \mathbb{R}$ and its dual, lets us treat A^j as a martingale transform. Finally, the following fact by Arcozzi [1], provides a connection between aforementioned objects and allows to use probabilistic methods in studying Riesz transforms on Lie groups.

Theorem 1.3. *If $f \in C_0^\infty(G)$, then $\lim_{\lambda\to\infty} T_{A^j}^\lambda f = R_j f$ in $L^p(G)$ for $1 \leq p < \infty$. If we replace the entry $-X_0$ by 0, then we have the convergence to $-\frac{1}{2} R_j f$.*

Riesz transforms on spheres. We proceed to the concept of Riesz transforms on the Euclidean unit sphere $\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$, equipped with the standard Riemannian metric and normalized $SO(d)$ invariant measure. It turns out that there are several non-equivalent, yet meaningful definitions of Riesz transforms; roughly speaking, this is related to the fact that there are a few ways to treat $\mathbb{S}^{d-1}$ as the boundary of the d-dimensional manifold (see [2] for an overview of different types of Riesz transforms on $\mathbb{S}^{d-1}$).

Following [1] and [2], we will study two types of Riesz transforms. For $1 \leq l < m \leq d$, consider the differential operator $T_{lm} = x_l \partial_m - x_m \partial_l$. If $x_l + ix_m = re^{i\theta}$, then $T_{lm} = \frac{\partial}{\partial\theta}$ is the derivative with respect to the angular coordinate θ in the (x_l, x_m) plane, so the operators $(T_{lm})_{1\le l<m\le d}$ form a well-defined vector field on $\mathbb{S}^{d-1}$. The directional Riesz transform of cylinder type is defined by the identity

$$R^c_{lm} = T_{lm} \circ (-\Delta_{\mathbb{S}^{d-1}})^{-1/2}.$$

To define the second type of Riesz transforms on $\mathbb{S}^{d-1}$, let $\mathcal{H}_k$ be the space of spherical harmonics of degree k and let

$$\mathcal{E}_0 = \left\{ f : \mathbb{S}^{d-1} \to \mathbb{R} : f = \sum_{k=1}^{N} f_k,\ f_k \in \mathcal{H}_k,\ N = 1, 2, \dots \right\}$$

be the space of harmonic polynomials with null average on $\mathbb{S}^{d-1}$. For a fixed $f \in \mathcal{E}_0$, let J be the solution in $\mathbb{B}^d$ (the unit ball in $\mathbb{R}^d$) to the Neumann problem with boundary data f, normalized so that $J(0) = 0$. This is described by the equation

$$\left(\frac{\partial}{\partial v}\right)^{-1} f = J|_{\mathbb{S}^{d-1}},$$

where v is the outward pointing normal vector to $\mathbb{S}^{d-1}$. One can easily check that the operator $\left(\frac{\partial}{\partial v}\right)^{-1} : L^2_0(\mathbb{S}^{d-1}) \to L^2_0(\mathbb{S}^{d-1})$ acts on spherical harmonics f_k of degree $k \ge 1$ by $\left(\frac{\partial}{\partial v}\right)^{-1} f_k = f_k/k$. The directional Riesz transforms of ball type are given by

$$R^b_{lm} = T_{lm} \circ \left(\frac{\partial}{\partial v}\right)^{-1}.$$

The probabilistic representation of both R^c and R^b is very similar to what we have seen above in the context of Lie groups, but this time it will rest upon classical martingales (i.e., with values in $\mathbb{R}^d$). To this end, consider the standard Brownian motion $W = (W^1, W^2, \dots, W^d)$ in $\mathbb{R}^d$ starting from 0 and let $\tau = \inf\{t \ge 0 : W_t \notin \mathbb{B}^d\}$ be the first moment the process W exits the unit ball. Then W_τ is uniformly distributed on the unit sphere $\mathbb{S}^{d-1}$. Let A be a continuous function defined on $\mathbb{B}^d$ and taking values in the space of $d \times d$ matrices. Moreover, let $f : \mathbb{S}^{d-1} \to [0, 1]$ be a smooth function and let F be the Poisson extension of f to $\mathbb{B}^d$. We define the martingale transform $A * F$ by

$$A * F = \left(\int_0^{\tau\wedge t} A(W_s)\nabla_{\mathbb{B}^d} F(W_s) \cdot \mathrm{d}W_s \right)_{t\ge 0}.$$

For $x \in \mathbb{S}^{d-1}$ we introduce the following operator

$$T_A f(x) = \mathbb{E}\left[A * F | W_\tau = x\right].$$

The connection between the operator T_A and both cylindrical and ball directional Riesz transforms is described in the following theorem, see Arcozzi [1].

Theorem 1.4. *For given $1 \le l < m \le d$, a function $\varphi : [0, 1] \to \mathbb{R}$ and $x \in \overline{\mathbb{B}}^d$, let $A_{lm}(x)$ be the matrix with entries*

$$A^{ij}_{lm}(x) = \begin{cases} \varphi(|x|^2) & \text{if } i = l, j = m, \\ -\varphi(|x|^2) & \text{if } i = m, j = l, \\ 0 & \text{otherwise.} \end{cases}$$

i) if $\varphi \equiv 1$*, then* $T_{A_{lm}} = R^b_{lm}$.

ii) Suppose that $d \geq 3$ *and let* φ *be defined by the formula*

$$\varphi\left(e^{-2t/(d-2)}\right) = \frac{\int_0^t I_0(s)ds}{e^t - 1}, \qquad t \geq 0,$$

where $I_0(z) = \sum_{j=0}^{\infty}(z/2)^{2j}/(j!)^2, z \in \mathbb{C}$*, is the modified Bessel function of order* 0*. Then* $T_{A_{lm}} = R^c_{lm}$.

If we replace the entry $-\varphi(|x|^2)$ *by* 0*, then we have the convergence to* $\frac{1}{2}R^b_{lm}f$ *or* $\frac{1}{2}R^c_{lm}f$*, respectively.*

1.4. H^1 and BMO spaces

Let $f(\zeta) = \sum_{n\in\mathbb{Z}} \hat{f}(n)\zeta^n$ be a complex-valued integrable function on the unit circle $\mathbb{T}$, equipped with the normalized Lebesgue measure μ. Here for each $n \in \mathbb{Z}$, the symbol $\hat{f}(n)$ stands for the n-th Fourier coefficient of f, given by $\hat{f}(n) = \int_{\mathbb{T}} f(\zeta)\zeta^{-n}\mathrm{d}\mu(\zeta)$. Any such f can be extended to a harmonic function $P[f]$ on the closed unit disc $\overline{\mathbb{D}}$, with the use of the formula

$$P[f](z) = \int_{\mathbb{T}} f(\zeta)P(z,\zeta)\mathrm{d}\mu(\zeta).$$

Here $P : \overline{\mathbb{D}} \times \mathbb{T} \to \mathbb{R}$ is the Poisson kernel, defined by $P(z,\zeta) = (1-|z|^2)/|z-\zeta|^2$. The function f belongs to the Hardy space $H^1(\mathbb{T})$, a closed subspace of $L^1(\mathbb{T})$, if the coefficients $\hat{f}(n)$ vanish for all $n < 0$. In such a case the Poisson extension $P[f]$ of f is a holomorphic function inside the disc $\mathbb{D}$, with the Taylor series expansion

$$P[f](z) = \sum_{n\geq 0} \hat{f}(n)z^n, \qquad z \in \mathbb{D}.$$

More generally, for $1 \leq p < \infty$, the Hardy space $H^p(\mathbb{T})$ is defined as $H^1(\mathbb{T}) \cap L^p(\mathbb{T})$ and equipped with the norm $\|\cdot\|_{H^p(\mathbb{T})} = \|\cdot\|_{L^p(\mathbb{T},\mu)}$.

We turn our attention to the dual space $(H^1(\mathbb{T}))^*$. Recall that a function $f \in L^1(\mathbb{T})$ belongs to $BMO(\mathbb{T})$, the class of functions of bounded mean oscillation, if we have

$$\|f\|_{BMO} = \sup_I \frac{1}{\mu(I)}\int_I \left| f(\zeta) - \frac{1}{\mu(I)}\int_I f(\vartheta)\mathrm{d}\mu(\vartheta)\right| \mathrm{d}\mu(\zeta) < \infty,$$

where the supremum is taken over all intervals (arcs) I contained in $\mathbb{T}$. It is well-known that the spaces BMO allow for a number of different, but equivalent norms (or rather seminorms, to be precise). For example, given $1 \leq p < \infty$, one can define

$$\|f\|_{BMO_p} = \sup_I \left(\frac{1}{\mu(I)}\int_I \left| f(\zeta) - \frac{1}{\mu(I)}\int_I f(\vartheta)\mathrm{d}\mu(\vartheta)\right|^p \mathrm{d}\mu(\zeta)\right)^{1/p}$$

(with the supremum over the same class as above), and it can be shown that $\|f\|_{BMO_1} \leq \|f\|_{BMO_p} \leq c_p\|f\|_{BMO_1}$ for some constant c_p depending only on p. These norms, with a distinguished case $p = 2$, are used widely in the real harmonic analysis. In our considerations below, we will use a yet different norming, which seems to be more natural in complex analysis. Namely, from now on, we will work with the BMO seminorm

$$\begin{aligned} \|f\|_{BMO} &= \sup_{z\in\mathbb{D}} \left(\int_{\mathbb{T}} \left|f(\zeta) - P[f](z)\right|^2 P(z,\zeta)\mathrm{d}\mu(\zeta)\right)^{1/2} \\ &= \sup_{z\in\mathbb{D}} \left(P[|f|^2](z) - \left|P[f](z)\right|^2\right)^{1/2}. \end{aligned} \tag{1.4.1}$$

It can be shown that the dual space $(H^1(\mathbb{T}))^*$ is equal to the analytic BMO class $ABMO(\mathbb{T}) = H^1(\mathbb{T}) \cap BMO$ equipped with the seminorm $\|\cdot\|_{BMO}$. This follows from the classical result of Fefferman [18] and an appropriate conformal mapping argument. For the exposition of the key properties of the space $ABMO$ and its connections to other areas of complex analysis, we refer the reader to the survey article [21].

Chapter 2

A dual approach to Burkholder's L^p estimates

2.1. Introduction and statement of results

The purpose of this chapter is to provide a new proof of a certain important class of sharp martingale inequalities, established by Burkholder in the eighties. To present the result from an appropriate perspective, let us first discuss some motivation coming from very classical questions in harmonic analysis, which were studied intensively almost a century ago. Assume that $(h_n)_{n\geq 0}$ is the standard Haar system on $[0,1)$, i.e., the collection of functions on $[0,1)$ defined by

$$h_0 = \chi_{[0,1)}, \qquad h_1 = \chi_{[0,1/2)} - \chi_{[1/2,1)},$$
$$h_2 = \chi_{[0,1/4)} - \chi_{[1/4,1/2)}, \qquad h_3 = \chi_{[1/2,3/4)} - \chi_{[3/4,1)},$$
$$h_4 = \chi_{[0,1/8)} - \chi_{[1/8,1/4)}, \qquad h_5 = \chi_{[1/4,3/8)} - \chi_{[3/8,1/2)} \dots$$

and so on. As proved by Schauder [44], if $p \in [1,\infty)$, then the Haar system forms a basis of $L^p(0,1)$ (with the underlying Lebesgue measure). In addition, this basis is unconditional if $p > 1$: for any such p there exists a finite constant c_p with the following property. For any nonnegative integer n, any sequence a_0, a_1, a_2, ..., a_n of real numbers and any sequence ε_0, ε_1, ε_2, ..., ε_n of signs we have

$$\left\| \sum_{k=0}^{n} \varepsilon_k a_k h_k \right\|_{L^p} \leq c_p \left\| \sum_{k=0}^{n} a_k h_k \right\|_{L^p}. \tag{2.1.1}$$

This remarkable property, established by Marcinkiewicz [30], plays an important role in approximation theory and harmonic analysis. In addition, it has a natural and significant extension in probability theory. Suppose that $f = (f_n)_{n\geq 0}$ is a discrete-time martingale and $g = (g_n)_{n\geq 0}$ is its transform by a certain predictable sequence $v = (v_n)_{n\geq 0}$ bounded in absolute value by 1. A celebrated result of Burkholder [7] asserts the following.

Theorem 2.1. *For any $1 < p < \infty$ there is a finite constant C_p, depending on p only, such that for all f and g as above we have*

$$\|g\|_{L^p} \leq C_p \|f\|_{L^p}. \tag{2.1.2}$$

It is not difficult to prove that the optimal constants in (2.1.1) and (2.1.2) are the same. Indeed, the Haar system is a martingale difference sequence, with respect to its natural filtration, on the probability space $([0,1), \mathcal{B}(0,1), |\cdot|)$. Therefore the sequence $(a_n h_n)_{n\geq 0}$ also has this property, and hence setting v to be an arbitrary sequence of deterministic signs, we see that (2.1.2) implies the validity of (2.1.1) with $c_p = C_p$. As for the reverse implication, the idea is to embed a given martingale into an appropriate linear combination of Haar functions. More precisely, an arbitrary finite martingale difference sequence $df = (df_n)_{n=0}^N$ can be approximated in distribution by a sequence of the form $\left(\sum_{k=M_{n-1}}^{M_n} a_k h_k\right)_{n=0}^N$, for some coefficients a_0, a_1, a_2, ... and some increasing sequence $0 = M_{-1} < M_0 < M_1 < \ldots < M_N$ of integers (see Maurey [31] for the precise formulation). Therefore, (2.1.1) implies (2.1.2), with $C_p = c_p$.

There is a natural and intriguing question about the optimal value of the constant allowed in (2.1.1) and (2.1.2). This problem was solved by Burkholder [8]: it turns out that the best choice is $c_p = C_p = p^* - 1$, where $p^* = \max\{p, p/(p-1)\}$. This beautiful result is a starting point for numerous extensions and applications. For example, one can consider the less restrictive case in which the sequence a_0, a_1, a_2, ... (as well as the martingales f, g) take values in $\mathbb{C}$ or, more generally, in some given separable Hilbert space $\mathbb{H}$. Burkholder [8,10] proved that in this new context, the L^p estimates are still valid, with unchanged constant $p^* - 1$, which, of course, is still sharp. The situation becomes more complicated if we allow the coefficients and the martingales to take values in a separable Banach space $\mathbb{B}$. Denote the optimal constants in (2.1.1) and (2.1.2) by $c_p(\mathbb{B})$ and $C_p(\mathbb{B})$, respectively. Then the same argument as above shows that $c_p(\mathbb{B}) = C_p(\mathbb{B})$; however, there are spaces for which this value is infinite; for example, ℓ^1 and ℓ^∞ have this property. The "well-behaved" spaces $\mathbb{B}$, i.e., those for which $c_p(\mathbb{B}) < \infty$, are called UMD spaces (where the abbreviation comes from Unconditional for Martingale Differences). Roughly speaking, such spaces form an environment to which most of important results from the Hilbertian setting can be carried over. For example, as proved by Bourgain, Burkholder and McConnell (see [9]), the periodic Hilbert transform is bounded on $L^p(\mathbb{T}; \mathbb{B})$ for all $1 < p < \infty$ if and only if $\mathbb{B}$ is a UMD space.

There is another very interesting and important direction into which the inequality (2.1.2) can be extended: one can study the boundedness of martingale transforms in other function spaces. For example, the L^p estimate fails to hold for $p = 1$, but we have the corresponding weak-type estimate

$$\|g\|_{L^{1,\infty}} \leq C\|f\|_{L^1},$$

for some finite constant C (cf. [7]). One can ask about the optimal value of the constant C; actually, one can study this problem for weak-type (p,p) estimates in the full range $1 \leq p < \infty$. We have the following answer, provided by Burkholder [8] (for $1 \leq p \leq 2$) and Suh [47] (for $p > 2$).

Theorem 2.2. *Suppose that f is a Hilbert-space-valued martingale and g is its transform by a predictable sequence with values in $[-1, 1]$. Then for any $1 \leq p < \infty$ we have*

$$\|g\|_{L^{p,\infty}} \leq C_{p,\infty}\|f\|_{L^p},$$

where the optimal choice for the constant $C_{p,\infty}$ satisfies

$$C_{p,\infty}^p = \begin{cases} 2/\Gamma(p+1) & \text{if } 1 \leq p \leq 2, \\ p^{p-1}/2 & \text{if } p \geq 2. \end{cases} \tag{2.1.3}$$

We turn our attention to another extension of Theorem 2.1, which will be important for the results discussed in this dissertation. The idea is to allow a larger class of martingale pairs (f, g).

Definition 2.1. A martingale g is said to be differentially subordinate to f, if for any $n \geq 0$ we have the estimate $|dg_n| \leq |df_n|$ almost surely.

Note that this definition makes perfect sense also in the vector setting (i.e., for f, g taking values in some Banach space $\mathbb{B}$): one only needs to interpret $|df_n|$, $|dg_n|$ as the corresponding norms of the differences: $|df_n| = |df_n|_{\mathbb{B}}$, $|dg_n| = |dg_n|_{\mathbb{B}}$. Next, observe that if g is the transform of f by a certain predictable sequence with values in $[-1, 1]$, then g is automatically differentially subordinate to f.

Burkholder [8] proved the following extension of Theorem 2.1.

Theorem 2.3. *Suppose that f, g are Hilbert-space-valued martingales such that g is differentially subordinate to f. Then for any $1 < p < \infty$ we have*

$$\|g\|_{L^p} \leq (p^* - 1)\|f\|_{L^p} \tag{2.1.4}$$

and the constant $p^* - 1$ is the best possible: it is already optimal for real-valued processes, in the context of transforms.

It turns out that Theorem 2.2 also extends, with no change in the constants, to differentially subordinate martingales. Actually, the above statements can be pushed further, to cover the context of continuous-time local martingales. First, since stochastic integrals are continuous-time extensions of martingale transforms, a sharp version of Theorem 2.1 combined with standard approximation arguments yields the following fact.

Theorem 2.4. *Suppose that X is an arbitrary local martingale taking values in a separable Hilbert space and Y is the stochastic integral, with respect to X, of some predictable process H taking values in $[-1, 1]$. Then we have the sharp estimate*

$$\|Y\|_{L^p} \leq (p^* - 1)\|X\|_{L^p}, \qquad 1 < p < \infty.$$

To generalize Theorems 2.3 and 2.2 to the continuous-time context, we need an appropriate version of the differential subordination.

Definition 2.2. Suppose that X, Y are continuous-time local martingales. Then Y is differentially subordinate to X, if, almost surely, the process $([X, X]_t - [Y, Y]_t)_{t \geq 0}$ is nondecreasing and nonnegative as a function of t.

Note that if we treat the discrete-time martingales $f = (f_n)_{n \geq 0}$, $g = (g_n)_{n \geq 0}$ as continuous-time processes X, Y (via the identities $X_t = f_{\lfloor t \rfloor}$, $Y_t = g_{\lfloor t \rfloor}$ for all $t \geq 0$), then Y is differentially subordinate to X if and only if $\mathbb{P}(|dg_n| \leq |df_n|) = 1$ for all n. That is, the above extension is consistent with the original, discrete-time differential subordination introduced in Definition 2.1.

We have the following fact, established by Wang in [49] and Suh [47].

Theorem 2.5. *Suppose that X, Y are continuous-time local martingales taking values in a separable Hilbert space such that Y is differentially subordinate to X. Then for any $1 < p < \infty$ we have the estimate*

$$\|Y\|_{L^p} \leq (p^* - 1)\|X\|_{L^p} \tag{2.1.5}$$

and the constant $p^* - 1$ is the best possible. Furthermore, for $1 \leq p < \infty$ we have

$$\|Y\|_{L^{p,\infty}} \leq C_{p,\infty}\|X\|_{L^p},$$

where $C_{p,\infty}$ is given by (2.1.3). *The estimate is also sharp.*

The primary goal of this chapter is to provide a new proof of the estimate (2.1.5). The presentation below is based on the joint work [11] with Bañuelos and Osękowski.

2.2. On the approach: Bellman function method

Now we will describe the technique, invented in [10], which allowed Burkholder a successful treatment of Theorem 2.3 (and which, after some minor modifications, led to the continuous-time extension given in Theorem 2.5). The idea is to construct a certain special function, which enjoys appropriate size and concavity conditions. Specifically, suppose that $U : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ satisfies the following three requirements:

1° $U(x, y) \leq 0$ if $|y| \leq |x|$,

2° $U(x, y) \geq |y|^p - (p^* - 1)^p |x|^p$,

3° $(U(f_n, g_n))_{n\geq 0}$ is a supermartingale for any pair (f, g) of differentially subordinate martingales.

The existence of such a function immediately yields (2.1.4): indeed, for any nonnegative integer n we have the chain of inequalities

$$\mathbb{E}\big[|g_n|^p - (p^* - 1)^p |f_n|^p\big] \leq \mathbb{E}U(f_n, g_n) \leq \mathbb{E}U(f_0, g_0) \leq 0.$$

Here the first estimate follows from the majorization condition 2°, the second is due to the supermartingale property, while the final one is a consequence of the initial condition 1° and the observation that $|g_0| \leq |f_0|$, by the differential subordination. Thus, we have $\mathbb{E}|g_n|^p \leq (p^* - 1)^p \mathbb{E}|f_n|^p$ and it suffices to let $n \to \infty$ to obtain the desired bound.

Hence, the problem reduces to the search for an appropriate function. Burkholder [10] proved that

$$U(x, y) = p\left(1 - \frac{1}{p^*}\right)^{p-1} (|y| - (p^* - 1)|x|)(|x| + |y|)^{p-1} \tag{2.2.1}$$

has all the required properties. The discovery of this object requires a careful analysis of an appropriate second-order partial differential inequality: see [10, 48] for details, we will also encounter a related argumentation below. For a slightly different approach, which exploits a more complicated special function of three variables, consult the earlier paper [8] by Burkholder. We would also like to point out that the above argument applies, with no essential changes, to the continuous-time setting studied in Theorem 2.5.

There is a dual method of proving (2.1.4), developed by Nazarov, Treil and Volberg in [33, 34], which is also based on the construction of a certain special function. Let us start with the case $p = 2$, in which the description is particularly easy. Namely, consider the function $\mathbb{B}(x, z) = \frac{1}{2}(|x|^2 + |z|^2)$ given for $(x, z) \in \mathbb{H} \times \mathbb{H}$. This function satisfies the following analogues of the above properties 1°, 2° and 3°:

1°' $\mathbb{B}(x, z) \geq |xz|$,

2°' $\mathbb{B}(x, z) \leq \frac{1}{2}(|x|^2 + |z|^2)$,

3°' For any pair (f, h) of *arbitrary* $\mathbb{H}$-valued martingales and any $n \geq 1$ we have

$$\mathbb{E}\mathbb{B}(f_n, h_n) \geq \mathbb{E}\mathbb{B}(f_{n-1}, h_{n-1}) + \mathbb{E}|df_n||dh_n|. \tag{2.2.2}$$

Indeed, the first two conditions are trivial (we actually have equality in 2°'); to see that the third requirement also holds, simply observe that by the orthogonality of martingale differences,

$$\begin{aligned}
\mathbb{E}\mathbb{B}(f_n, h_n) &= \frac{1}{2}\mathbb{E}\big(|f_n|^2 + |h_n|^2\big) \\
&= \frac{1}{2}\mathbb{E}\big(|f_{n-1}|^2 + |h_{n-1}|^2\big) + \frac{1}{2}\mathbb{E}\big(|df_n|^2 + |dh_n|^2\big) \\
&\geq \frac{1}{2}\mathbb{E}\big(|f_{n-1}|^2 + |h_{n-1}|^2\big) + \mathbb{E}|df_n||dh_n| = \mathbb{E}\mathbb{B}(f_{n-1}, h_{n-1}) + \mathbb{E}|df_n||dh_n|.
\end{aligned}$$

Let us stress here that the martingales f and h appearing in 3°' are not related by any domination principle. These three conditions immediately give (2.1.4) (for $p = 2$). Indeed, fix an arbitrary pair (f, g) satisfying the differential subordination and let h be another martingale. By 1°', the inductive use of 3°' and finally 2°', we get that for $n = 0, 1, 2, \ldots,$

$$\mathbb{E}\sum_{k=0}^{n}|df_k||dh_k| \le \mathbb{E}\mathbb{B}(f_0, h_0) + \mathbb{E}\sum_{k=1}^{n}|df_k||dh_k| \le \mathbb{E}\mathbb{B}(f_n, h_n) \le \frac{1}{2}(\mathbb{E}|f_n|^2 + \mathbb{E}|h_n|^2).$$

Consequently, by the orthogonality of martingale differences and the differential subordination of g to f, we obtain (recall that $\cdot$ is the scalar product in $\mathbb{H}$)

$$\begin{aligned}\mathbb{E}\langle g_n, h_n\rangle &= \mathbb{E}\sum_{k=0}^{n}\langle dg_k, dh_k\rangle \\ &\le \mathbb{E}\sum_{k=0}^{n}|dg_k||dh_k| \le \mathbb{E}\sum_{k=0}^{n}|df_k||dh_k| \le \frac{1}{2}(\|f_n\|_{L^2}^2 + \|h_n\|_{L^2}^2).\end{aligned}$$

Now we make use of a standard homogenization argument: we fix a positive constant λ, apply the above inequality to the martingales f, g and h/λ, multiply throughout by λ and optimize over this parameter. As the result, we obtain $\mathbb{E}\langle g_n, h_n\rangle \le \|f_n\|_{L^2}\|h_n\|_{L^2}$, which implies the desired bound $\|g_n\|_{L^2} \le \|f_n\|_{L^2}$ by duality. There is a natural question whether this approach can be extended to other values of p. This problem was studied by Nazarov and Treil [33]. For $p > 2$, they introduced the following function: for $\zeta, \eta \in \mathbb{H}^2$,

$$\widetilde{\mathbb{B}}(\zeta, \eta) = |\zeta|^p + |\eta|^{p'} + \begin{cases} |\zeta|^2|\eta|^{2-p'} & \text{if } |\zeta|^p \le |\eta|^{p'}, \\ \dfrac{2}{p}|\zeta|^p + \left(\dfrac{2}{p'} - 1\right)|\eta|^{p'} & \text{if } |\zeta|^p \ge |\eta|^{p'}. \end{cases}$$

They showed that $\widetilde{\mathbb{B}}$ satisfies appropriate versions of 1°', 2°' and 3°', which yields (2.1.4), but with a suboptimal constant. Despite its non-optimality, this special function has found many applications in harmonic analysis and semigroup theory; see e.g. [14, 16]. A further improvement is due to Bañuelos and Osękowski [5], who identified the appropriate version of $\widetilde{\mathbb{B}}$ leading to the best constant $p^* - 1$ in the full range $1 < p < \infty$. However, this version involves *four* variables, which is a significant complication. Can this approach be simplified to produce a function on $\mathbb{H}^2$, as in the above proof for the case $p = 2$? The primary goal of this chapter is to answer this question in the affirmative and provide the explicit formula for the corresponding special functions.

Actually, our approach will allow us to study the above topic in the more general, continuous-time setting. We will prove the following statement.

Theorem 2.6. *Suppose that X, Y, Z are $\mathbb{H}$-valued local martingales such that Y is differentially subordinate to X. Then for any $1 < p < \infty$ we have the sharp inequality*

$$\|[Y, Z]_\infty\|_{L^1} \le (p^* - 1)\|X\|_{L^p}\|Z\|_{L^{p'}}. \tag{2.2.3}$$

The above estimate immediately yields (2.1.5). Indeed, by the properties of the square bracket, the difference $\langle Y, Z\rangle - [Y, Z]$ is a local martingale started at zero, so for any finite stopping time τ we have (up to localization)

$$\big|\mathbb{E}\langle Y_\tau, Z_\tau\rangle\big| = \big|\mathbb{E}[Y, Z]_\tau\big| \le \|[Y, Z^\tau]_\infty\|_{L^1} \le (p^* - 1)\|X\|_{L^p}\|Z^\tau\|_{L^{p'}},$$

where Z^τ is the local martingale Z stopped at time τ. This yields $\|Y_\tau\|_{L^p} \le (p^* - 1)\|X\|_{L^p}$ by duality, and taking the supremum over all τ yields the claim.

2.3. A dual approach

The purpose of this section is to show that the existence of a certain special function on $\mathbb{H}^2$, or rather $\mathbb{R}^d \times \mathbb{R}^d$, implies the validity of (2.2.3). Actually, all we need is an appropriate function defined on the first quadrant $(0,\infty)^2$. Namely, let $1 < p < \infty$ and $K > 0$ be fixed parameters. Suppose that $B : (0,\infty)^2 \to \mathbb{R}$ is a function of class C^2, which enjoys the following properties:

1° (Initial condition) We have $B(x,z) \geq xz$.
2° (Majorization) For any x, $z > 0$ we have

$$B(x,z) \leq \frac{Kx^p}{p} + \frac{z^{p'}}{p'}. \tag{2.3.1}$$

3° (Monotonicity) For any x, $z > 0$ we have

$$\frac{B_{xx}(x,z)}{|B_{xz}(x,z)|+1} \leq \frac{B_x(x,z)}{x} \quad \text{and} \quad \frac{B_{zz}(x,z)}{|B_{xz}(x,z)|+1} \leq \frac{B_z(x,z)}{z}. \tag{2.3.2}$$

4° (Concavity) For any x, $z > 0$ and any h, $k \in \mathbb{R}$,

$$B_{xx}(x,z)h^2 + 2B_{xz}(x,z)hk + B_{zz}(x,z)k^2 \geq 2|h||k|. \tag{2.3.3}$$

Several observations are in order. First, we see that 1° and 2° are perfect analogues of the conditions 1°', 2°' appearing in the previous section. Furthermore, the requirements 3° and 4° should be treated as *pointwise* conditions related to 3°'. Next, note that if B satisfies 4°, then we also have

$$B_{xx}(x,z)h^2 - 2|B_{xz}(x,z)|\,|h||k| + B_{zz}(x,z)k^2 \geq 2|h||k| \tag{2.3.4}$$

(simply plug $h := -\operatorname{sgn}(B_{xz}(x,z)hk)h$ into (2.3.3)). Consequently, we get

$$B_{xx}(x,z)h^2 - 2(|B_{xz}(x,z)|+1)hk + B_{zz}(x,z)k^2 \geq 0$$

for all h, $k \in \mathbb{R}$, which forces the corresponding discriminant to be nonpositive:

$$B_{xx}(x,z)B_{zz}(x,z) \geq (|B_{xz}(x,z)|+1)^2, \tag{2.3.5}$$

together with the inequality $B_{xx}(x,z) \geq 0$. Actually, it is easy to see that the implications can be reversed: the inequality (2.3.5), together with the inequality $B_{xx}(x,z) \geq 0$, implies the validity of (2.3.3).

We are ready to introduce the special function $\mathbb{B}$, the extension of B to higher dimensions. Namely, given $d \geq 2$, define $\mathbb{B} : (\mathbb{R}^d \times \mathbb{R}^d) \setminus \{(x,z) : |x||z| = 0\} \to \mathbb{R}$ by $\mathbb{B}(x,y) = B(|x|,|y|)$. As we shall see, this object will lead us to the proof of (2.2.3). In what follows, $\mathbb{B}_x$ and $\mathbb{B}_z$ will denote the vectors of partial derivatives of $\mathbb{B}$ with respect to the variables x_1, x_2, ..., x_d and z_1, z_2, ..., z_d, respectively. Furthermore, the symbol $D^2\mathbb{B}$ will stand for the Hessian matrix of $\mathbb{B}$.

Lemma 2.7. *If B satisfies 3° and 4°, then for any x, $z \in \mathbb{R}^d \setminus \{0\}$ and any h, $k \in \mathbb{R}^d$ satisfying $x+h \neq 0$ and $z+k \neq 0$, we have*

$$\mathbb{B}(x+h,z+k) \geq \mathbb{B}(x,z) + \langle \mathbb{B}_x(x,z), h\rangle + \langle \mathbb{B}_z(x,z), k\rangle + |h||k|. \tag{2.3.6}$$

Proof. By continuity, we may assume that x is not a multiple of h and similarly that z is not a multiple of k. That is, $x+th\neq 0$ and $z+tk\neq 0$ for all t. Consider the function $G(t)=\mathbb{B}(x+th,z+tk)$, given for $t\in\mathbb{R}$; then the assertion is equivalent to $G(1)\geq G(0)+G'(0)+|h||k|$. Observe that it is enough to show that $G''(t)\geq 2|h||k|$ for all t. Indeed, having proved this, we apply the mean value theorem to obtain $G(1)-G(0)-G'(0)=\frac{1}{2}G''(t_0)$ for some intermediate number $t_0\in(0,1)$, and the claim follows.

So, fix $t\in\mathbb{R}$. Setting $x'=(x+th)/|x+th|$, $z'=(z+tk)/|z+tk|$ and $w=(|x+th|,|z+tk|)$, we compute that

$$\begin{aligned} G''(t) &= \langle D^2\mathbb{B}(x+th,z+tk)(h,k),(h,k)\rangle \\ &= \Big\langle D^2B(w)\Big(\langle x',h\rangle,\langle z',k\rangle\Big),\Big(\langle x',h\rangle,\langle z',k\rangle\Big)\Big\rangle \\ &\quad + B_x(w)\cdot\frac{|h|^2-\langle x',h\rangle^2}{|x+th|}+B_z(w)\cdot\frac{|k|^2-\langle z',k\rangle^2}{|z+tk|} \\ &= J_1+J_2+J_3+J_4, \end{aligned} \tag{2.3.7}$$

where

$$\begin{aligned} J_1 &= \frac{|B_{xz}(w)|}{|B_{xz}(w)|+1}B_{xx}(w)\langle x',h\rangle^2+2B_{xz}(w)\langle x',h\rangle\langle z',k\rangle+\frac{|B_{xz}(w)|}{|B_{xz}(w)|+1}B_{zz}(w)\langle z',k\rangle^2, \\ J_2 &= \left[\frac{B_x(w)}{|x+th|}-\frac{B_{xx}(w)}{|B_{xz}(w)|+1}\right](|h|^2-\langle x',h\rangle^2), \\ J_3 &= \left[\frac{B_z(w)}{|z+tk|}-\frac{B_{zz}(w)}{|B_{xz}(w)|+1}\right](|k|^2-\langle z',k\rangle^2), \\ J_4 &= \frac{B_{xx}(w)}{|B_{xz}(w)|+1}|h|^2+\frac{B_{zz}(w)}{|B_{xz}(w)|+1}|k|^2. \end{aligned}$$

Let us analyze the terms J_1, J_2, J_3 and J_4. The first term is nonnegative. To see this, note that $2B_{xz}(w)\langle x',h\rangle\langle z',k\rangle\geq -2|B_{xz}(w)\langle x',h\rangle\langle z',k\rangle|$ and hence it is enough to show that

$$B_{xx}(w)\langle x',h\rangle^2-2(|B_{xz}(w)|+1)|\langle x',h\rangle|\,|\langle z',k\rangle|+B_{zz}(w)\langle z',k\rangle^2\geq 0.$$

But this estimate follows directly from (2.3.4). The terms J_2, J_3 are also nonnegative, which is an immediate consequence of 3°. Finally, observe that $J_4\geq 2|h||k|$: if we rewrite this in the equivalent form

$$B_{xx}(w)|h|^2-2|B_{xz}(w)||h||k|+B_{zz}(w)|k|^2\geq 2|h||k|,$$

we recognize (2.3.4) again. So, we have established the bound $G''(t)\geq 2|h||k|$ for all t. □

Remark 2.1. In particular, setting $t=0$ in (2.3.7), we obtain the estimate

$$\langle D^2\mathbb{B}(x,z)(h,k),(h,k)\rangle\geq\frac{B_{xx}(x,z)}{|B_{xz}(x,z)|+1}|h|^2+\frac{B_{zz}(x,z)}{|B_{xz}(x,z)|+1}|k|^2, \tag{2.3.8}$$

which will be useful later.

Here is the main result of this section, which links the special functions to the L^p estimates for differentially subordinate (local) martingales.

Theorem 2.8. *If there is a function $B:(0,\infty)^2\to\mathbb{R}$ satisfying 1°-4°, then* (2.2.3) *holds, with p^*-1 replaced by $K^{1/p}$.*

Proof. By standard limiting arguments, it is enough to establish the desired estimate in the case when the processes take values in a finite- and at least two-dimensional subspace of $\mathbb{H}$. That is, we may set $\mathbb{H} = \mathbb{R}^d$ for some $d \geq 2$. Let $X = (X_t)_{t\geq 0}$, $Z = (Z_t)_{t\geq 0}$ be arbitrary local martingales with values in $\mathbb{R}^d$. We may restrict ourselves to the case $\|X\|_{L^p} < \infty$ and $\|Z\|_{L^{p'}} < \infty$, since otherwise there is nothing to prove. Furthermore, we may and do assume that X and Z are bounded away from zero. Indeed, fix $\varepsilon > 0$ and consider the $\mathbb{R}^{d+1}$-valued processes $X_t^\varepsilon = (X_t, \varepsilon)$, $Y_t^\varepsilon = (Y_t, 0)$ and $Z_t^\varepsilon = (Z_t, \varepsilon)$ for $t \geq 0$. Then X^ε and Z^ε are bounded away from zero, and having proved the estimate

$$\|[Y^\varepsilon, Z^\varepsilon]_\infty\|_{L^1} \leq (p^* - 1)\|X^\varepsilon\|_{L^p}\|Z^\varepsilon\|_{L^{p'}},$$

we obtain the desired bound (2.2.3) by letting $\varepsilon \to 0$ and exploiting Lebesgue's dominated convergence theorem (and the identity $[Y^\varepsilon, Z^\varepsilon]_\infty = [Y, Z]_\infty$).

For any integer N, consider the stopping time $T_N = \inf\{t \geq 0 : |[Y, Z]_t| \geq N\}$. By properties of stochastic integrals, we see that the processes $\left(\int_{0+}^t \mathbb{B}_x(X_{s-}, Z_{s-}) \cdot \mathrm{d}X_s\right)_{t\geq 0}$ and $\left(\int_{0+}^t \mathbb{B}_z(X_{s-}, Z_{s-}) \cdot \mathrm{d}Z_s\right)_{t\geq 0}$ are local martingales. Fix an arbitrary sequence $(\eta_n)_{n\geq 0}$ of stopping times which localizes these integrals and the processes X, Z. Fix n and set $\tau_n = \eta_n \wedge T_N$. The application of Itô's formula to the process $(\mathbb{B}(X_{\tau_n\wedge t}, Z_{\tau_n\wedge t}))_{t\geq 0}$ yields

$$\mathbb{B}(X_{\tau_n\wedge t}, Z_{\tau_n\wedge t}) = I_0 + I_1 + I_2/2 + I_3, \tag{2.3.9}$$

where

$$\begin{aligned}
I_0 &= \mathbb{B}(X_0, Z_0),\\
I_1 &= \int_{0+}^{\tau_n\wedge t} \mathbb{B}_x(X_{s-}, Z_{s-}) \cdot \mathrm{d}X_s + \int_{0+}^{\tau_n\wedge t} \mathbb{B}_z(X_{s-}, Z_{s-}) \cdot \mathrm{d}Z_s,\\
I_2 &= \int_{0+}^{\tau_n\wedge t} D^2\mathbb{B}(X_{s-}, Z_{s-})\mathrm{d}[X^c, Z^c]_s,\\
I_3 &= \sum_{0<s\leq\tau_n\wedge t} \Big[\mathbb{B}(X_s, Z_s) - \mathbb{B}(X_{s-}, Z_{s-}) - \mathbb{B}_x(X_{s-}, Z_{s-})\Delta X_s - \mathbb{B}_z(X_{s-}, Z_{s-})\Delta Z_s\Big].
\end{aligned}$$

Recall that $\Delta X_s = X_s - X_{s-}$ is the jump of X at time s. Let us analyze the behavior of the above terms. First, by 1° and the differential subordination of Y to X, we have

$$I_0 \geq |X_0||Z_0| \geq |Y_0||Z_0| = [Y, Z]_0.$$

Next, we have $\mathbb{E}I_1 = 0$, by the properties of stochastic integrals. To deal with I_2, fix $0 \leq s_0 < s_1 \leq t$. For any $\ell \geq 0$, let $(\eta_i^\ell)_{0\leq i\leq i_\ell}$ be a nondecreasing sequence of stopping times with $\eta_0^\ell = s_0$, $\eta_{i_\ell}^\ell = s_1$ such that $\lim_{\ell\to\infty} \max_{0\leq i\leq i_\ell - 1} |\eta_{i+1}^\ell - \eta_i^\ell| = 0$. Keeping ℓ fixed, we apply, for each $i = 0, 1, 2, \ldots, i_\ell$, the estimate (2.3.8) with $x = X_{s_0-}$, $z = Z_{s_0-}$ and $h = X^c_{\eta_{i+1}^\ell} - X^c_{\eta_i^\ell}$, $k = Z^c_{\eta_{i+1}^\ell} - Z^c_{\eta_i^\ell}$. We sum the obtained $i_\ell + 1$ inequalities and let $\ell \to \infty$. As the result, we obtain the estimate

$$\begin{aligned}
&\int_{s_0+}^{s_1} D^2\mathbb{B}(X_{s_0-}, Z_{s_0-})\mathrm{d}[X^c, Z^c]_s\\
&\geq \int_{s_0+}^{s_1} \frac{B_{xx}(X_{s_0-}, Z_{s_0-})}{|B_{xz}(X_{s_0-}, Z_{s_0-})| + 1}\mathrm{d}[X^c, X^c]_s + \int_{s_0+}^{s_1} \frac{B_{zz}(X_{s_0-}, Z_{s_0-})}{|B_{xz}(X_{s_0-}, Z_{s_0-})| + 1}\mathrm{d}[Z^c, Z^c]_s\\
&\geq \int_{s_0+}^{s_1} \frac{B_{xx}(X_{s_0-}, Z_{s_0-})}{|B_{xz}(X_{s_0-}, Z_{s_0-})| + 1}\mathrm{d}[Y^c, Y^c]_s + \int_{s_0+}^{s_1} \frac{B_{zz}(X_{s_0-}, Z_{s_0-})}{|B_{xz}(X_{s_0-}, Z_{s_0-})| + 1}\mathrm{d}[Z^c, Z^c]_s,
\end{aligned}$$

where in the last passage we have used the differential subordination of Y^c to X^c and the inequality $B_{xx}(x,z) \geq 0$. By the Kunita-Watanabe inequality and the estimate (2.3.5), the latter expression is not smaller than $2\int_{s_0+}^{s_1} \mathrm{d}[Y^c,Z^c]_s = 2[Y^c,Z^c]_{s_1} - 2[Y^c,Z^c]_{s_0}$ and hence, approximating I_2 by Riemann sums, we obtain $I_2 \geq 2[Y^c,Z^c]_{\tau_n\wedge t} - 2[Y^c,Z^c]_0 = 2[Y^c,Z^c]_{\tau_n\wedge t}$. Finally, the term I_3 is handled with the use of (2.3.6): we get

$$I_3 \geq \sum_{0<s\leq\tau_n\wedge t} |\Delta X_s||\Delta Z_s| \geq \sum_{0<s\leq\tau_n\wedge t} |\Delta Y_s||\Delta Z_s| \geq \sum_{0<s\leq\tau_n\wedge t} \langle \Delta Y_s, \Delta Z_s\rangle.$$

Plugging all these observations into (2.3.9) gives

$$\begin{aligned}\mathbb{B}(X_{\tau_n\wedge t}, Z_{\tau_n\wedge t}) &\geq |Y_0||Z_0| + I_1 + [Y^c,Z^c]_{\tau_n\wedge t} + \sum_{0<s\leq\tau_n\wedge t} \langle \Delta Y_s, \Delta Z_s\rangle \\ &\geq I_1 + [Y,Z]_{\tau_n\wedge t},\end{aligned} \tag{2.3.10}$$

where the last inequality is due to (1.1.2) (and polarization). The expressions above are integrable: by 1° and 2°, we have

$$0 \leq \mathbb{B}(X_{\tau_n\wedge t}, Z_{\tau_n\wedge t}) \leq \frac{K|X_{\tau_n\wedge t}|^p}{p} + \frac{|Z_{\tau_n\wedge t}|^{p'}}{p'},$$

and the right-hand side is integrable, since $\|X\|_{L^p} < \infty$ and $\|Z\|_{L^{p'}} < \infty$ (see the beginning of the proof). Furthermore, by the very definition of T_N and the differential subordination

$$|[Y,Z]_{\tau_n\wedge t}| \leq |[Y,Z]_{\tau_n\wedge t-}| + |\Delta Y_{\tau_n\wedge t}||\Delta Z_{\tau_n\wedge t}| \leq N + |\Delta X_{\tau_n\wedge t}||\Delta Z_{\tau_n\wedge t}|,$$

and the latter expression is integrable by the Young's inequality and the L^p / $L^{p'}$-boundedness of X and Z. Consequently, taking the expectation in (2.3.10), recalling that $\mathbb{E}I_1 = 0$, and applying the majorization condition 2°, we obtain

$$\mathbb{E}[Y,Z]_{\tau_n\wedge t} \leq \frac{K\mathbb{E}|X_{\tau_n\wedge t}|^p}{p} + \frac{\mathbb{E}|Z_{\tau_n\wedge t}|^{p'}}{p'} \leq \frac{K\|X\|_{L^p}^p}{p} + \frac{\|Z\|_{L^{p'}}^{p'}}{p'},$$

or, by a simple homogenization argument,

$$\mathbb{E}[Y,Z]_{\tau_n\wedge t} \leq K^{1/p}\|X\|_{L^p}\|Z\|_{L^{p'}}.$$

Letting $n\to\infty$, $t\to\infty$, $N\to\infty$ and using standard limit theorems, we get the desired estimate (2.2.3) (with p^*-1 replaced by $K^{1/p}$). □

As a by-product, we obtain the following interesting estimate for the total variation of X and Z.

Remark 2.2. The above reasoning can be easily adapted to yield the estimate

$$\mathbb{E}\int_0^\infty |\mathrm{d}[X,Z]_s| \leq K^{1/p}\|X\|_{L^p}\|Z\|_{L^{p'}}. \tag{2.3.11}$$

Indeed, when handling the terms I_2 and I_3, skip all the arguments which involve the martingale Y and the differential subordination.

2.4. Explicit special functions

In the light of the reasoning from the previous section, the inequality (2.2.3) will follow if we construct a special function B with $K = (p^*-1)^p$. The case $p=2$ has already been dealt with in the introduction. We consider the cases $1<p<2$ and $p>2$ separately.

2.4.1. The case $1 < p < 2$

We start with the introduction of a certain auxiliary function.

Lemma 2.9. *For any $s \geq 0$, there is a unique positive number $\varphi = \varphi(s)$ satisfying*

$$\varphi(s)(1+\varphi(s))^{p-2} = p^{p-2}s. \tag{2.4.1}$$

The resulting function $\varphi : [0,\infty) \to [0,\infty)$ is of class C^∞ and satisfies

$$\varphi'(s) = \frac{\varphi(s)(1+\varphi(s))}{s(1+(p-1)\varphi(s))}, \qquad s > 0. \tag{2.4.2}$$

In addition, we have $\varphi(s) \geq s^{1/(p-1)}$ for $s \leq (p-1)^{1-p}$ and $\varphi(s) < s^{1/(1-p)}$ for $s > (p-1)^{1-p}$.

Proof. The existence and uniqueness of $\varphi(s)$ follows at once from the fact that the function $\Phi(u) = u(1+u)^{p-2}$ satisfies $\Phi(0) = 0$ and $\Phi'(u) = (1+u)^{p-3}(1+(p-1)u) > 0$ for $u > 0$ and $\Phi(u) \to \infty$ as $u \to \infty$. The differentiability of φ is an immediate consequence of standard theorems on implicit functions. To show the identity (2.4.2), it suffices to differentiate both sides of (2.4.1) and rearrange terms. To prove the final part of the lemma, we invoke the monotonicity property of Φ described above. Namely, if $s \leq (p-1)^{1-p}$, then we have

$$\Phi(s^{1/(p-1)}) = s^{1/(p-1)}(1+s^{1/(p-1)})^{p-2} \leq s^{1/(p-1)}\big((p-1)s^{1/(p-1)} + s^{1/(p-1)}\big)^{p-2} = p^{p-2}s$$

and hence $\varphi(s) \geq s^{1/(p-1)}$. In the case $s > (p-1)^{1-p}$, we just reverse the estimates in the above reasoning. □

The central object, the special function $B : (0,\infty)^2 \to \mathbb{R}$, is defined by

$$B(x,z) = xz\left[\frac{p-1}{p}\varphi(x^{1-p}z) + \frac{2-p}{p(p-1)} + \frac{1}{p(p-1)\varphi(x^{1-p}z)}\right].$$

Theorem 2.10. *The function B satisfies the conditions 1°-4° listed in the previous section.*

Proof of 1°. This is easy: we have $as + bs^{-1} \geq 2\sqrt{ab}$, for any a, b, $s > 0$, so

$$\frac{p-1}{p}\varphi(x^{1-p}z) + \frac{2-p}{p(p-1)} + \frac{1}{p(p-1)\varphi(x^{1-p}z)} \geq \frac{2-p}{p(p-1)} + \frac{2}{p} = \frac{1}{p-1} \geq 1. \quad \square$$

Proof of 3° and 4°. As before, to keep the notation short, we will set $s = x^{1-p}z$. We compute directly that

$$\begin{aligned} &B_x(x,z) \\ &= z\left[\frac{p-1}{p}\varphi(s) + \frac{2-p}{p(p-1)} + \frac{1}{p(p-1)\varphi(s)}\right] + sz\left[-\frac{(p-1)^2}{p}\varphi'(s) + \frac{\varphi'(s)}{p\varphi^2(s)}\right]. \end{aligned}$$

The second expression on the right-hand side equals

$$\frac{sz\varphi'(s)}{p\varphi^2(s)}(1-(p-1)\varphi(s))(1+(p-1)\varphi(s)) = \frac{z(1+\varphi(s))(1-(p-1)\varphi(s))}{p\varphi(s)},$$

where the last equality is due to (2.4.2). Consequently,

$$B_x(x,z) = \frac{(2-p)z}{p-1} + \frac{z}{(p-1)\varphi(s)}.$$

Next, we see that

$$B_{xx}(x,z)=\frac{x^{-p}z^2\varphi'(s)}{\varphi^2(s)}$$

and, exploiting (2.4.2) again we have that,

$$\begin{aligned}
B_{xz}(x,z)&=-\frac{p-2}{p-1}+\frac{1}{(p-1)\varphi(s)}-\frac{s\varphi'(s)}{(p-1)\varphi^2(s)}\\
&=-\frac{p-2}{p-1}+\frac{1}{(p-1)\varphi(s)}-\frac{1+\varphi(s)}{(p-1)\varphi(s)(1+(p-1)\varphi(s))}\\
&=-\frac{p-2}{p-1}+\frac{p-2}{(p-1)(1+(p-1)\varphi(s))}\\
&=\frac{(2-p)\varphi(s)}{1+(p-1)\varphi(s)}.
\end{aligned}$$

Finally, note that

$$\begin{aligned}
&B_z(x,z)\\
&=x\left[\frac{p-1}{p}\varphi(s)+\frac{2-p}{p(p-1)}+\frac{1}{p(p-1)\varphi(s)}\right]+x\left[\frac{(p-1)s}{p}\varphi'(s)-\frac{s\varphi'(s)}{p(p-1)\varphi^2(s)}\right].
\end{aligned}$$

By (2.4.2), the second term on the right-hand side equals

$$\frac{xs\varphi'(s)}{p(p-1)\varphi^2(s)}((p-1)\varphi(s)+1)((p-1)\varphi(s)-1)=\frac{x(1+\varphi(s))((p-1)\varphi(s)-1)}{p(p-1)\varphi(s)}.$$

This, after some straightforward manipulations, yields $B_z(x,z)=x\varphi(s)$. In addition, we immediately obtain $B_{zz}(x,z)=x^{2-p}\varphi'(s)$.

We are now ready to check 3° and 4°. Note that

$$\begin{aligned}
\frac{B_{xx}(x,z)}{|B_{xz}(x,z)|+1}-\frac{B_x(x,z)}{x}&=\frac{z}{x}\left[\frac{s\varphi'(s)(1+(p-1)\varphi(s))}{\varphi^2(1+\varphi)}-\frac{1+(2-p)\varphi(s)}{(p-1)\varphi(s)}\right]\\
&=\frac{(p-2)z(1+\varphi(s))}{x(p-1)\varphi(s)}\leq 0
\end{aligned}$$

and

$$\frac{B_{zz}(x,z)}{|B_{xz}(x,z)|+1}-\frac{B_z(x,z)}{z}=\frac{x}{z}\left[\frac{s\varphi'(s)(1+(p-1)\varphi(s))}{1+\varphi}-\varphi(s)\right]=0.$$

Thus, 3° holds true. To check the concavity condition (2.3.3), note that $B_{xx}(x,z)\geq 0$ and hence it is enough to check the discriminant inequality (2.3.5). This estimate reads

$$\frac{s^2(\varphi'(s))^2}{\varphi^2(s)}\geq\left(\frac{1+\varphi(s)}{1+(p-1)\varphi(s)}\right)^2,$$

and follows from (2.4.2): actually, both sides are equal. □

Proof of 2°. It remains to handle the majorization property, with $K=(p^*-1)^p=(p-1)^{-p}$. The claim is equivalent to

$$\frac{p-1}{p}s\varphi(s)+\frac{2-p}{p(p-1)}s+\frac{s}{p(p-1)\varphi(s)}-\frac{(p-1)s^{p/(p-1)}}{p}\leq\frac{1}{p(p-1)^p},$$

where, as previously, $s = x^{1-p}z$. Denote the left-hand side of the above estimate by $L(s)$. Differentiating, we see that $L'(s) = B_z(1,s) - s^{1/(p-1)} = \varphi(s) - s^{1/(p-1)}$. Therefore, by the last part of Lemma 2.9, we conclude that L attains its maximum at the point $(p-1)^{1-p}$. It remains to check that $L((p-1)^{1-p}) = p^{-1}(p-1)^{-p}$. This follows easily from the identity $\varphi((p-1)^{1-p}) = s^{1/(p-1)}$, which can be verified directly in (2.4.1). $\square$

Remark 2.11. The function B can be extended to a continuous function on the closed quadrant $[0,\infty)^2$. Directly from the definition of φ, we infer that

$$\lim_{s\to 0}\frac{\varphi(s)}{s} = p^{p-2} \quad \text{and} \quad \lim_{s\to\infty}\frac{\varphi(s)}{s^{1/(p-1)}} = p^{(p-2)/(p-1)},$$

which implies

$$\begin{aligned}\lim_{(x,z)\to(0,z_0)} B(x,z) &= \lim_{(x,z)\to(0,z_0)} \frac{p-1}{p} xz\varphi(x^{1-p}z)\\ &= \frac{p-1}{p} z_0^{p/(p-1)} \lim_{(x,z)\to(0,z_0)} \frac{\varphi(x^{1-p}z)}{(x^{1-p}z)^{1/(p-1)}}\\ &= p^{-1/(p-1)}(p-1)z_0^{p/(p-1)}\end{aligned}$$

and

$$\lim_{(x,z)\to(x_0,0)} B(x,z) = \lim_{(x,z)\to(x_0,0)} \frac{x^p\cdot x^{1-p}z}{p(p-1)\varphi(x^{1-p}z)} = \frac{x_0^p}{p^{p-1}(p-1)}.$$

2.4.2. The case $p > 2$

We proceed in a similar manner, starting with an auxiliary function.

Lemma 2.12. *For any $s \geq 0$, there is a unique number $\varphi = \varphi(s) \in [p-2,\infty)$ satisfying*

$$p\left(1-\frac{1}{p}\right)^{p-1}(1+\varphi(s))^{p-2}(\varphi(s)-p+2) = s. \tag{2.4.3}$$

The resulting function $\varphi : [0,\infty) \to [p-2,\infty)$ is of class C^∞ and satisfies

$$\varphi'(s) = \frac{(1+\varphi(s))(\varphi(s)-p+2)}{(p-1)s(\varphi(s)-p+3)}, \qquad s > 0. \tag{2.4.4}$$

In addition, we have $\varphi(s) \geq s^{1/(p-1)}$ for $s \leq (p-1)^{p-1}$ and $\varphi(s) < s^{1/(1-p)}$ for $s > (p-1)^{p-1}$.

The special function B is given by

$$B(x,z) = \left(1-\frac{1}{p}\right)xz\left[\varphi(x^{1-p}z) + \frac{1}{\varphi(x^{1-p}z)-p+2}\right].$$

In the light of the previous section, the inequality (2.2.3) will follow once we show the following.

Theorem 2.13. *The function B satisfies the conditions 1°-4°.*

Proof of 1°. We have $s + s^{-1} \geq 2$ for any $s > 0$, so

$$\left(1-\frac{1}{p}\right)\left[\varphi(x^{1-p}z) + \frac{1}{\varphi(x^{1-p}z)-p+2}\right] \geq \left(1-\frac{1}{p}\right)(2+p-2) = p-1 \geq 1. \quad \square$$

Proof of 3° and 4°. The calculation are similar to those in the case $1 < p < 2$. A direct differentiation combined with (2.4.4) yields

$$\begin{aligned} B_x(x,z) &= \left(1-\frac{1}{p}\right) z\left[\varphi(s)+\frac{1}{\varphi(s)-p+2}\right] - \frac{(p-1)^2}{p} sz\left[\varphi'(s) - \frac{\varphi'(s)}{(\varphi(s)-p+2)^2}\right] \\ &= \left(1-\frac{1}{p}\right) z\left[\varphi(s)+\frac{1}{\varphi(s)-p+2}\right] - \frac{(p-1)z(1+\varphi(s))(\varphi(s)-p+1)}{p(\varphi(s)-p+2)} \\ &= \frac{(p-1)z}{\varphi(s)-p+2}. \end{aligned}$$

Therefore,

$$B_{xx}(x,z) = \frac{(p-1)^2 x^{-p} z^2 \varphi'(s)}{(\varphi(s)-p+2)^2}$$

and, by (2.4.4),

$$\begin{aligned} B_{xz}(x,z) &= \frac{p-1}{\varphi(s)-p+2} - \frac{(p-1)s\varphi'(s)}{(\varphi(s)-p+2} \\ &= \frac{p-1}{\varphi(s)-p+2} - \frac{1+\varphi(s)}{(\varphi(s)-p+2)(\varphi(s)-p+3)} = \frac{p-2}{\varphi(s)-p+3}. \end{aligned}$$

Moreover, arguing as above, we check that $B_z(x,z)$ equals

$$\left(1-\frac{1}{p}\right) x\left[\varphi(s)+\frac{1}{\varphi(s)-p+2}\right] + \left(1-\frac{1}{p}\right) sx\left[\varphi'(s) - \frac{\varphi'(s)}{(\varphi(s)-p+2)^2}\right] = x\varphi(s)$$

and hence $B_{zz}(x,z) = x^{2-p}\varphi'(s)$. Now we can establish 3°. We have

$$\frac{B_{xx}(x,z)}{|B_{xz}(x,z)|+1} - \frac{B_x(x,z)}{x} = \frac{(p-1)z}{x(\varphi(s)-p+2)}\left[\frac{(p-1)s\varphi'(s)(\varphi(s)-p+3)}{(\varphi(s)-p+2)(1+\varphi(s))} - 1\right] = 0$$

and

$$\frac{B_{zz}(x,z)}{|B_{xz}(x,z)|+1} - \frac{B_z(x,z)}{z} = \frac{(2-p)x(\varphi(s)+1)}{(p-1)z} \leq 0.$$

To check the concavity 4°, it is enough to verify the validity of (2.3.5) (since $B_{xx}(x,z) \geq 0$). However, we have

$$B_{xx}(x,z)B_{zz}(x,z) = \left(\frac{(p-1)s\varphi'(s)}{\varphi(s)-p+2}\right)^2,$$

which, by (2.4.4), is equal to $\left(\frac{1+\varphi(s)}{\varphi(s)-p+3}\right)^2 = (|B_{xz}(x,z)|+1)^2.$ □

Proof of 2°. We proceed as in the previous case. We let $K = (p^*-1)^p = (p-1)^p$ and note that the majorization is equivalent to

$$\left(1-\frac{1}{p}\right) s\left[\varphi(s)+\frac{1}{\varphi(s)-p+2}\right] - \frac{p-1}{p}s^{p/(p-1)} \leq \frac{(p-1)^p}{p}.$$

Denoting the left-hand side by $L(s)$, we compute that $L'(s) = B_z(1,s) - s^{1/(p-1)} = \varphi(s) - s^{1/(p-1)}$. By the last part of Lemma 2.12, L attains its maximal value at $s = (p-1)^{p-1}$. Since $\varphi((p-1)^{p-1}) = p-1$ (directly from (2.4.3)), we check that $L((p-1)^{p-1}) = (p-1)^p/p$, which establishes the desired majorization. □

Remark 2.14. As in the case $1 < p < 2$, one might wonder whether the function B can be extended continuously to the whole $[0,\infty)^2$. The answer is affirmative. By (2.4.3), we easily check that

$$\lim_{s\to 0}\frac{\varphi(s)-p+2}{s}=\frac{p^{p-2}}{(p-1)^{2p-3}}\quad\text{ and }\quad\lim_{s\to\infty}\frac{\varphi(s)}{s^{1/(p-1)}}=\frac{p^{p-2}}{(p-1)^{p-1}},$$

which, by similar calculations to those in Remark 2.11, gives

$$\lim_{(x,z)\to(0,z_0)}B(x,z)=\frac{p^{p-3}}{(p-1)^{p-2}}z_0^{p/(p-1)}$$

and

$$\lim_{(x,z)\to(x_0,0)}B(x,z)=\frac{(p-1)^{2p-2}}{p^{p-1}}x_0^p.$$

2.5. On the search of the Bellman function B

Now we will sketch some informal steps which lead to the discovery of the special functions B (and the optimal constant $K=(p^*-1)^p$); the reasoning will be based on a number of assumptions and guesses. A typical approach during the search for the Bellman function is to look at the concavity condition and assume its degeneracy. This usually gives rise to a corresponding second order partial differential equation. Next, one exploits structural properties of a general solution of the equation and from this one aims to come up with a reasonable candidate for the special function. We consider the cases $1 < p < 2$ and $p > 2$ separately.

2.5.1. The case $1 < p < 2$

It is convenient to split the argumentation into a few steps.

Step 1. Additional assumptions. We will impose a few extra assumptions on the function B. First, we guess that the partial derivative B_{xz} is *nonnegative* on the whole $(0,\infty)^2$. Our second assumption is that B can be extended to a function $\tilde{B}:\mathbb{R}^2\to\mathbb{R}$ satisfying $\tilde{B}(x,z)=\tilde{B}(\pm x,\pm z)$. Then, in particular, $\tilde{B}$ must satisfy

$$\tilde{B}_z(x,0)=0. \tag{2.5.1}$$

Step 2. The Monge-Ampère equation. In our case, the concavity is governed by the inequality (2.3.5). Setting $C(x,z)=\tilde{B}(x,z)+xz$ and recalling the assumption $B_{xz}>0$ for x, $z>0$, we see that the condition degenerates if and only if C satisfies the so-called Monge-Ampère equation

$$C_{xx}(x,z)C_{zz}(x,z)=(C_{xz}(x,z))^2.$$

From the general theory of such equations, we infer that the quadrant $(0,\infty)^2$ can be foliated, i.e., split into a union of pairwise disjoint line segments along which C is linear and the first-order partial derivatives of C are constant. In what follows, we will assume that these segments have negative slope; see Figure 2.1 below.

Next, by 2°, we have

$$C(x,z)\leq xz+\frac{Kx^p}{p}+\frac{z^{p'}}{p'}. \tag{2.5.2}$$

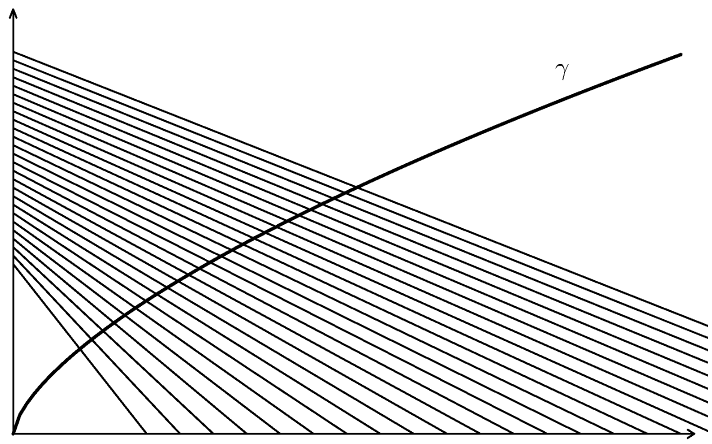

Figure 2.1. The foliation of $(0,\infty)^2$

Note that the right-hand side enjoys the following homogeneity property: if we fix $\lambda > 0$ and multiply x by $\lambda^{1/p}$ and z by $\lambda^{1/p'}$, then the whole expression is multiplied by λ. It seems plausible to assume that the same is true for the left-hand side: $C(\lambda^{1/p}x, \lambda^{1/p'}z) = \lambda C(x,z)$. Finally, if K is the optimal constant, then there should be a nonzero point (x_0, z_0) at which both sides of (2.5.2) are equal; by the aforementioned homogeneity, such point gives rise to the whole 'equality curve' $\gamma = \{(x,z) : z = s_0 x^{p-1}\}$, where $s_0 = z_0 x_0^{1-p}$. Note that by (2.5.2), the first-order partial derivatives of the functions C and $(x,z) \mapsto xz + Kx^p/p + z^{p'}/p'$ must match at each point from γ.

Step 3. The formula for C. Pick a point (x, s_0x^{p-1}) from the equality curve γ. Let I be the line segment of the foliation passing through this point. If $\alpha = \alpha(x)$ is the slope of this segment, we may write

$$C(x+d, s_0x^{p-1} + \alpha d) = C(x, s_0x^{p-1}) + C_x(x, s_0x^{p-1})d + C_z(x, s_0x^{p-1})\alpha d.$$

But $C(x, s_0x^{p-1}) = x^p(s_0 + K/p + s_0^{p'}/p')$, since $(x, s_0x^{p-1}) \in \gamma$. Moreover, from the last sentence of the previous step, we know that $C_x(x, s_0x^{p-1}) = (K + s_0)x^{p-1}$ and $C_z(x, s_0x^{p-1}) = x + (s_0x^{p-1})^{p'-1} = x(1 + s_0^{1/(p-1)})$. Furthermore, recall that C_z is constant along I and, by (2.5.1), $C_z(x, 0) = x$. Comparing the latter two expressions for C_z, we obtain that I intersects the x axis at the point $(x(1 + s_0^{1/(p-1)}), 0)$ and hence $\alpha = -s_0^{(p-2)/(p-1)}x^{p-2}$. Putting all the above facts together and calculating a little bit, we get the following explicit (or rather implicit) formula for C:

$$C(x+d, s_0x^{p-1} - s_0^{(p-2)/(p-1)}x^{p-2}d) = \left(s_0 + \frac{K}{p} + \frac{s_0^{p'}}{p'}\right)x^p + (K - s_0^{(p-2)/(p-1)})x^{p-1}d.$$

It remains to guess K and s_0. To this end, plug $d = s_0^{1/(p-1)}x$ to obtain

$$C\big(x(1 + s_0^{1/(p-1)}), 0\big) = \left(\frac{K}{p} + \frac{s_0^{p'}}{p'} + Ks_0^{1/(p-1)}\right)x^p$$

and hence, differentiating both sides with respect to x,

$$(1 + s_0^{1/(p-1)})C_x\big(x(1 + s_0^{1/(p-1)}), 0\big) = p\left(\frac{K}{p} + \frac{s_0^{p'}}{p'} + Ks_0^{1/(p-1)}\right)x^{p-1}.$$

But $C_x\big(x(1+s_0^{1/(p-1)}),0\big) = C_x(x, s_0x^{p-1}) = (K+s_0)x^{p-1}$ (again, by the constancy of partial derivatives along the segments of foliation). Combining the last two observations, we get the equation

$$p\left(\frac{K}{p}+\frac{s_0^{p'}}{p'}+Ks_0^{1/(p-1)}\right) = (1+s_0^{1/(p-1)})(K+s_0),$$

or, equivalently,

$$K = \frac{s_0^{(p-2)/(p-1)}+(2-p)s_0}{p-1}.$$

The right-hand side, considered as a function of s_0, attains its minimum $(p-1)^{-p}$ at the point $(p-1)^{1-p}$. Thus, it is natural to set $K=(p-1)^{-p}$ and $s_0=(p-1)^{1-p}$ as these extremal values. This leads to the function B studied in the previous section, but to check this, one has to carry out some lengthy calculations. We will give a brief sketch. The above analysis gives the formula for C, given implicitly as

$$C\left(x+d, \left(\frac{x}{p-1}\right)^{p-1} - \left(\frac{x}{p-1}\right)^{p-2}d\right) = p(p-1)^{-p}x^p + p(p-1)^{-p}(2-p)x^{p-1}d.$$

Set $X=x+d$ and $Z=\left(\frac{x}{p-1}\right)^{p-1}-\left(\frac{x}{p-1}\right)^{p-2}d$. One checks directly by (2.4.1) that

$$\varphi(X^{1-p}Z) = \frac{x+d-pd}{(x+d)(p-1)}.$$

Furthermore, calculating a little bit, we get

$$C(X,Z)-XZ = XZ\left[\frac{p-1}{p}\cdot\frac{x+d-pd}{(x+d)(p-1)}+\frac{2-p}{p(p-1)}+\frac{1}{p(p-1)}\frac{(x+d)(p-1)}{x+d-pd}\right],$$

and it remains to note that the right-hand side is $B(X,Z)$.

2.5.2. The case $p>2$

The reasoning is similar to that in the previous case, so we will be brief. We start with some additional assumptions on the partial derivatives of B. As previously, we work under the condition $B_{xz}\geq 0$ on $(0,\infty)^2$. Furthermore, we impose the vanishing requirement on one of the first-order derivatives: in contrast to the case $1<p<2$, now we assume that B_x is zero on the z axis: $B_x(0,z)=0$. Next, we consider the function $C(x,z)=\tilde{B}(x,z)+xz$ and note that the degeneration of (2.3.5) can be rewritten as the Monge-Ampère equation $C_{xx}(x,z)C_{zz}(x,z)=(C_{xz}(x,z))^2$. We assume that the foliation is of the same shape as in the case $1<p<2$; see Figure 2.1. Next we repeat, word by word, the analysis which leads to the equality curve γ and due to the assumption that $p>2$, this time it is a graph of a *convex* function.

Some substantial differences occur when we turn to the identification of the explicit formula for C. As before, we fix a point $(x, s_0x^{p-1})\in\gamma$, denote by α the slope of the corresponding leaf of foliation and write

$$C(x+d, s_0x^{p-1}+\alpha d) = C(x,s_0x^{p-1}) + C_x(x,s_0x^{p-1})d + C_z(x,s_0x^{p-1})\alpha d. \tag{2.5.3}$$

Now, we have $C(x, s_0x^{p-1}) = x^p(s_0 + K/p + s_0^{p'}/p')$ (because $(x, s_0x^{p-1}) \in \gamma$) and $C_x(x, s_0x^{p-1}) = (K + s_0)x^{p-1}$ and $C_z(x, s_0x^{p-1}) = x + (s_0x^{p-1})^{p'-1} = x(1 + s_0^{1/(p-1)})$. Now, the assumption $B_x(0, z) = 0$ implies $C_x(0, z) = z$. Since C_x is constant along the leaves of foliation, if we compare this to $C_x(x, s_0x^{p-1})$, we get that the line segment I intersects the z axis at the point $(0, (K + s_0)x^{p-1})$. Therefore, we have $\alpha = -Kx^{p-2}$. Plugging all this information into (2.5.3) gives

$$C(x + d, s_0x^{p-1} - Kx^{p-2}d) = \left(\frac{K}{p} + \frac{s_0^{p'}}{p'} + s_0\right) x^p - (Ks_0^{1/(p-1)} - s_0)x^{p-1}d.$$

To guess K and s_0, we compute the derivative C_z along the z axis. By the above formula for C, we have $C(0, (K + s_0)x^{p-1}) = \left(\frac{K}{p} + \frac{s_0^{p'}}{p'} + Ks_0^{1/(p-1)}\right) x^p$ and hence

$$(K + s_0)C_z(0, (K + s_0)x^{p-1}) = \frac{p}{p-1}\left(\frac{K}{p} + \frac{s_0^{p'}}{p'} + Ks_0^{1/(p-1)}\right) x.$$

On the other hand, C_z is constant along I, so $C_z(0, (K + s_0)x^{p-1}) = C_z(x, s_0x^{p-1}) = x(1 + s_0^{1/(p-1)})$. Combining this with the previous equation yields an identity which is equivalent to

$$K = \frac{s_0(p-1)}{s_0^{1/(p-1)} - p + 2}.$$

The right-hand side, considered as a function of $s_0 \in (p-2, \infty)$, attains its minimal value $(p-1)^p$ at the point $(p-1)^{p-1}$. Setting $K = (p-1)^p$, $s_0 = (p-1)^{p-1}$, one can check that the function $(x, z) \mapsto C(x, z) - xz$ is the special function we used in the Section 2.3. Indeed, substituting the above values of K and s_0 into the formula for C, we get

$$C\Big(x + d, ((p-1)x)^{p-1} - (p-1)^2((p-1)x)^{p-2}d\Big) = p'((p-1)x)^p - p(p-2)((p-1)x)^{p-1}d.$$

If we set $X = x + d$ and $Z = ((p-1)x)^{p-1} - (p-1)^2((p-1)x)^{p-2}d$, we check directly from (2.4.3) that $\varphi(X^{1-p}Z) = (px - x - d)/(x + d)$ and

$$C(X, Z) - XZ = \left(1 - \frac{1}{p}\right) XZ \left[\frac{px - x - d}{x + d} + \frac{1}{\frac{px-x-d}{x+d} - p + 2}\right].$$

The right-hand side is precisely $B(X, Z)$ and this shows that the function we discovered coincides with that used in Section 2.3.

2.6. Applications

We take the opportunity and present two well-known applications of the probabilistic results obtained above.

2.6.1. Littlewood-Paley estimates

Let $W = (W_t)_{t\geq 0}$ be a standard Brownian motion in $\mathbb{R}^d$ and let $H = (H_t)_{t\geq 0}$, $K = (K_t)_{t\geq 0}$ be two predictable processes also taking values in $\mathbb{R}^d$. Then the stochastic integrals

$$X_t = \int_0^t H_s \cdot \mathrm{d}W_s, \quad Z_t = \int_0^t K_s \cdot \mathrm{d}W_s, \qquad t \geq 0,$$

are martingales and hence the estimate (2.3.11) yields

$$\mathbb{E}\int_0^\infty |H_s||K_s|\mathrm{d}s \le K^{1/p}\|X\|_{L^p}\|Z\|_{L^{p'}} \le \frac{K\|X\|_{L^p}^p}{p} + \frac{\|Z\|_{L^{p'}}^{p'}}{p'}. \tag{2.6.1}$$

Now, consider arbitrary sufficiently regular functions $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$. Let u_f, u_g stand for the corresponding heat extensions to the upper half-space: $u_f(t,x) = P_t f(x)$ and $u_g(t,x) = P_t g(x)$, where $(P_t)_{t\ge 0}$ is the semigroup with the kernel

$$p_t(x,y) = (2\pi t)^{-d/2}\exp(-|x-y|^2/2t), \qquad x,\, y \in \mathbb{R}^d,\, t > 0.$$

Then u_f, u_g satisfy the heat equation in the interior of the halfspace and hence for any fixed $T > 0$ and $x \in \mathbb{R}^d$, the processes $X^{T,x} = (u_f(T-t, x+W_t))_{0\le t\le T}$, $Z^{T,x} = (u_g(T-t, x+W_t))_{0\le t\le T}$ are martingales. In addition, Itô's formula yields the representations

$$X_t^{T,x} = u_f(T,x) + \int_0^t \nabla_x u_f(T-s, x+W_s)\cdot \mathrm{d}W_s,$$

$$Z_t^{T,x} = u_g(T,x) + \int_0^t \nabla_x u_g(T-s, x+W_s)\cdot \mathrm{d}W_s.$$

Thus, an application of (2.6.1) gives

$$\begin{aligned}\mathbb{E}\int_0^T &|\nabla_x u_f(T-s, x+W_s)||\nabla_x u_g(T-s,x+W_s)|\mathrm{d}s \\ &\le \frac{K\mathbb{E}|f(x+W_T)|^p}{p} + \frac{\mathbb{E}|g(x+W_T)|^{p'}}{p'}.\end{aligned}$$

Integrating both sides over $x \in \mathbb{R}^d$ (with respect to the Lebesgue measure) and using Fubini's theorem, we obtain

$$\int_0^T\int_{\mathbb{R}^d} |\nabla_x u_f(T-s,x)||\nabla_x u_g(T-s,x)|\mathrm{d}x\mathrm{d}s \le \frac{K\|f\|_{L^p(\mathbb{R}^d)}^p}{p} + \frac{\|g\|_{L^{p'}(\mathbb{R}^d)}^{p'}}{p'}.$$

Hence, changing the variables $s := T - s$ on the left, letting $T \to \infty$ and applying a homogenization argument, we get the Littlewood-Paley-type inequality

$$\int_0^\infty\int_{\mathbb{R}^d} |\nabla_x u_f(t,x)||\nabla_x u_g(t,x)|\mathrm{d}x\mathrm{d}t \le K^{1/p}\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{p'}(\mathbb{R}^d)}.$$

By a simple approximation argument, this extends to general $f \in L^p(\mathbb{R}^d)$ and $g \in L^{p'}(\mathbb{R}^d)$, without any additional regularity assumptions. A similar reasoning, which exploits the Poisson semigroup instead of $(P_t)_{t\ge0}$ and the stopped Brownian motion in $[0,\infty)\times\mathbb{R}^d$ instead of $((T-t, x+W_t))_{0\le t\le T}$, yields the corresponding estimate

$$\int_0^\infty\int_{\mathbb{R}^d} 2t|\nabla v_f(t,x)||\nabla v_g(t,x)|\mathrm{d}x\mathrm{d}t \le K^{1/p}\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{p'}(\mathbb{R}^d)},$$

where v_f, v_g denote the Poisson extensions of f and g to the upper halfspace.

Similar inequalities hold for other semigroups including those arising from nonlocal operators. For example, consider the semigroup $(P_t)_{t\ge0}$ arising from the process of a Lévy measure as ν under the assumption in [3]. Then

$$\begin{aligned}\int_{\mathbb{R}^d}\int_0^\infty\int_{\mathbb{R}^d} &|P_tf(x+y) - P_tf(x)|\,|P_tg(x+y) - P_tg(x)|\;\nu(\mathrm{d}y)\mathrm{d}t\mathrm{d}x \\ &\le K^{1/p}\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{p'}(\mathbb{R}^d)}.\end{aligned}$$

For details on how the martingales arise in this case, see [3, pp 470-471].

2.6.2. L^p estimates for Riesz transforms

Now we will show how the estimate (2.1.5) leads to tight estimates for the *vectors* of Riesz transforms. For the sake of clarity, we will present the details only in the Euclidean setting; in the remaining cases (i.e., in the context of Lie groups and spheres) the reasoning is analogous. We will use the notation of Section 1.3. Suppose that d is a fixed dimension and let $\mathcal{R}f = (R_1 f, R_2 f, \ldots, R_d f)$ be a vector of Riesz transforms, for $f \in C_0^\infty(\mathbb{R}^d)$. Recall that $P[f]$ is the Poisson extension of f to $\mathbb{R}^d \times [0, \infty)$. For a fixed (x, y), consider two processes $\zeta = \zeta^{x,y}$ and $\eta = \eta^{x,y}$, given by

$$\zeta_t = P[f]((x,y) + Z_{\tau(y)\wedge t}) = P[f](x,y) + \int_{0+}^{\tau(y)\wedge t} \nabla P[f]((x,y)+Z_s)\cdot \mathrm{d}Z_s$$

and $\eta = (\eta^1, \eta^2, \ldots, \eta^d)$, where

$$\eta_t^j = (A^{j+} * f)_t(x,y) = \int_{0+}^{\tau(y)\wedge t} \partial_j P[f]((x,y)+Z_s)\mathrm{d}Y_s.$$

By properties of stochastic integrals, ζ and η are martingales taking values in $\mathbb{R}^d$. Furthermore, η is differentially subordinate to ζ, since

$$[\zeta,\zeta]_t = |\zeta_0|^2 + \int_{0+}^{\tau(y)\wedge t} |\nabla P[f]((x,y)+Z_s)|^2 \mathrm{d}s$$

and

$$[\eta,\eta]_t = \sum_{j=1}^d [\eta^j,\eta^j]_t = \int_{0+}^{\tau(y)\wedge t} |\nabla P[f]((x,y)+Z_s)|^2 \mathrm{d}s.$$

Consequently, Theorem 2.5 yields $\|\eta^{x,y}\|_{L^p} \le (p^*-1)\|\zeta^{x,y}\|_{L^p}$, where the norm on the right-hand side is simply equal to $\|\zeta^{x,y}_{\tau(y)}\|_{L^p} = \|f(x+X_{\tau(y)})\|_{L^p}$. So, for any function $g = (g_1,\, g_2,\, \ldots,\, g_d) \in C_0^\infty(\mathbb{R}^d;\mathbb{R}^d)$ we have, by the Young's inequality and Fubini's theorem,

$$\begin{aligned}
\sum_{j=1}^d \int_{\mathbb{R}^d} \mathcal{T}^y_{A^{j+}} f(x) g_j(x)\mathrm{d}x &= \sum_{j=1}^d \int_{\mathbb{R}^d} \mathbb{E}\left[A^{j+} * f(x,y) g_j(x+X_{\tau(y)})\right]\mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \frac{\|\eta^{x,y}\|^p_{L^p}}{p} + \frac{\left\|g(x+X_{\tau(y)})\right\|^{p'}_{L^{p'}}}{p'}\mathrm{d}x \\
&\le \frac{(p^*-1)^p}{p}\|f\|^p_{L^p(\mathbb{R}^d)} + \frac{\|g\|^{p'}_{L^{p'}(\mathbb{R}^d)}}{p'}.
\end{aligned}$$

This yields $\sum_{j=1}^d \int_{\mathbb{R}^d} \mathcal{T}^y_{A^{j+}} f(x) g_j(x)\mathrm{d}x \le (p^*-1)\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{p'}(\mathbb{R}^d)}$ by a simple homogenization argument. But g was arbitrary, so we obtain

$$\left\| \left(\sum_{j=1}^d (\mathcal{T}^y_{A^{j+}} f)^2 \right)^{1/2} \right\|_{L^p(\mathbb{R}^d)} \le (p^*-1)\|f\|_{L^p(\mathbb{R}^d)}.$$

Letting $y \to \infty$ and applying the second part of Theorem 1.2 gives the desired L^p-bound for the vectorial Riesz transform acting on smooth, compactly supported functions. The standard density argument allows us to extend the estimate to the full L^p.

Chapter 3

Sharp estimates for martingale transforms with unbounded transforming sequences

3.1. Introduction and statement of results

The L^p-estimates for martingale transforms discussed in the previous chapter concerned the case in which the transforming sequence v was bounded by 1. There is a very interesting question about strong- and weak–type estimates under the assumption that the transforming sequence belongs to L^r for some given $r < \infty$. More specifically, the above question is to study, for given parameters p, q and r, the optimal constants $C_{p,q,r}$ and $c_{p,q,r}$ in the inequalities

$$\|g\|_{L^p} \leq C_{p,q,r}\|f\|_{L^q}\|v^*\|_{L^r} \tag{3.1.1}$$

and

$$|||g|||_{L^{p,\infty}} \leq c_{p,q,r}\|f\|_{L^q}\|v^*\|_{L^r}. \tag{3.1.2}$$

A simple argument shows that the exponents must satisfy the condition $\frac{1}{p} \geq \frac{1}{q} + \frac{1}{r}$, otherwise the constants are infinite. In what follows, we will assume that this condition is satisfied with equality sign.

Here is the main result of this chapter, obtained in a joint work with A. Osękowski. In the formulation below, ϕ is a certain special concave function on $[0, \infty)$, described precisely in Theorem 3.7.

Theorem 3.1. *Let f be a discrete-time martingale taking values in a separable Hilbert space $\mathbb{H}$. Assume further that g is the transform of f by a predictable sequence v and let $1 < p,\ q,\ r < \infty$ be parameters satisfying $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$. Then the estimates* (3.1.1) *and* (3.1.2) *hold with*

$$C_{p,q,r} = \begin{cases} p-1 & \text{if } p > 2, \\ (q-1)^{-1} & \text{if } p < q < 2, \\ 1 & \text{otherwise} \end{cases}$$

and

$$c_{p,q,r} = \begin{cases} (p^{p-1}/2)^{1/p} & \text{if } p > 2, \\ \left(\frac{q}{r'}\right)^{1/r'} \left(\frac{r'(2-r')}{2(q-r')}\phi(0)^{r'}\right)^{1/p'} & \text{if } p < q < 2, \\ 1 & \text{otherwise.} \end{cases}$$

The constants are the best possible even if $\mathbb{H} = \mathbb{R}$.

There is a natural question whether the above theorem can be extended to the context of differential subordination. In particular, one needs to provide an appropriate domination principle which would generalize the operation of transforming by L^r-valued sequences. We will prove the following statement.

Theorem 3.2. *Suppose that f, g are $\mathbb{H}$-valued martingales and v is a predictable sequence such that g is differentially subordinate to $v \cdot f$. Then for any parameters $1 < p, q, r < \infty$ satisfying $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ the estimates* (3.1.1) *and* (3.1.2) *hold true.*

Note that by a simple limiting argument, the above theorem yields the sharp weak– and strong–type inequalities of Burkholder and Suh, discussed in the previous chapter.

In fact, we will establish the more general continuous-time version of Theorem 3.2.

Theorem 3.3. *Suppose that X, Y are $\mathbb{H}$-valued martingales and let H be a left-continuous process such that Y is differentially subordinate to $H \cdot X$. Then for any parameters $1 < p, q, r < \infty$ satisfying $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ we have the sharp estimates*

$$\|Y\|_{L^p} \leq C_{p,q,r} \|X\|_{L^q} \|H^*\|_{L^r} \tag{3.1.3}$$

and

$$|||Y|||_{L^{p,\infty}} \leq c_{p,q,r} \|X\|_{L^q} \|H^*\|_{L^r}. \tag{3.1.4}$$

We see that in the above statement we consider integrands H with left-continuous trajectories. This is slightly more restrictive than the predictability condition which is typically imposed in the context of stochastic integrals. The reason is that the analysis of the behavior of H^* will be based on Itô's formula, for which this enhanced regularity seems to be necessary.

The next section is devoted to two special estimates which serve as 'building blocks' in the proofs of (3.1.3) and (3.1.4). The strong–type estimate (3.1.3) is established in Subsection 3.3.1, we also prove the sharpness of (3.1.1) there. The last two sections contain the proof of the weak–type estimate (3.1.4) and address the sharpness of (3.1.2).

3.2. Two auxiliary inequalities

Introduce the domain $D = \{(x, y) \in \mathbb{H} \times \mathbb{H} : |x| + |y| < 1\}$ and let u_1, $u_\infty : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be two special functions, given by

$$u_1(x, y) = \begin{cases} |y|^2 - |x|^2 & \text{if } (x, y) \in D, \\ 1 - 2|x| & \text{if } (x, y) \notin D \end{cases}$$

and

$$u_\infty(x, y) = \begin{cases} 0 & \text{if } (x, y) \in D, \\ (|y| - 1)^2 - |x|^2 & \text{if } (x, y) \notin D. \end{cases}$$

The function u_1 was invented by Burkholder in [8] and it played the key role in the proof of the weak–type (1,1) estimate for martingale transforms. The function u_∞ first appeared in [4] and it can be regarded as an appropriate dual to u_1. See the monograph [35] for the detailed discussion and much more on the subject.

Later on, we will need the following property of these functions. Namely, if $(x, y) \in D$ and h, $k \in \mathbb{H}$ satisfy $|k| \leq |h|$, then

$$u_1(x + h, y + k) \leq u_1(x, y) + \langle u_{1x}(x, y), h \rangle + \langle u_{1y}(x, y), k \rangle \tag{3.2.1}$$

and similarly,

$$u_\infty(x + h, y + k) \leq u_\infty(x, y) + \langle u_{\infty x}(x, y), h \rangle + \langle u_{\infty y}(x, y), k \rangle. \tag{3.2.2}$$

Here u_{1x}, u_{1y}, $u_{\infty x}$ and $u_{\infty y}$ stand for the appropriate partial derivatives of u_1 and u_∞. Note that (3.2.2) is equivalent to saying that $u_\infty(x+h, y+k) \leq 0$.

Before we proceed, let us record a useful fact, proved by Wang (see Lemma 1 in [49]). Recall that X^c stands for the unique continuous local martingale part of X.

Lemma 3.4. *If Y is differentially subordinate to X, then Y^c is differentially subordinate to X^c and with probability 1 we have $|Y_0| \leq |X_0|$ and $|\Delta Y_t| \leq |\Delta X_t|$ for all $t \geq 0$. In addition, if X, Y are orthogonal, then Y has continuous paths and X^c, Y are orthogonal.*

We are ready for the main result of this section. In what follows, H^*_+ is the càdlàg maximal function of H, defined by $H^*_{t+} = \inf_{s>t} H^*_s$.

Theorem 3.5. *Let $t \geq 0$. Suppose that X, Y are martingales and H is a left-continuous process such that Y is differentially subordinate to $H \cdot X$.*

(i) If H_0 is bounded away from zero, then

$$\mathbb{E}u_1(X_t, Y_t/H^*_{t+}) \leq 0. \tag{3.2.3}$$

(ii) If $H^*_{t+}X_t$ and Y_t are square integrable, then

$$\mathbb{E}u_\infty(H^*_{t+}X_t, Y_t) \leq 0. \tag{3.2.4}$$

Proof of (3.2.3). Introduce the stopping time $\tau = \inf\{s \geq 0 : (X_s, Y_s/H^*_{s+}) \notin D\}$, with the usual convention $\inf \emptyset = +\infty$. Let us start with the obvious identity

$$\mathbb{E}u_1(X_t, Y_t/H^*_{t+}) = \mathbb{E}u_1(X_t, Y_t/H^*_{t+})\chi_{\{\tau \leq t\}} + \mathbb{E}u_1(X_t, Y_t/H^*_{t+})\chi_{\{\tau > t\}}.$$

Note that $u_1(x, y) \leq 1 - 2|x|$ for all x, $y \in \mathbb{H}$. Therefore, using the supermartingale property of $(1 - 2|X_s|)_{s \geq 0}$, we may write

$$\begin{aligned}\mathbb{E}u_1(X_t, Y_t/H^*_{t+})\chi_{\{\tau \leq t\}} &\leq \mathbb{E}(1 - 2|X_t|)\chi_{\{\tau \leq t\}} \\ &\leq \mathbb{E}(1 - 2|X_\tau|)\chi_{\{\tau \leq t\}} = \mathbb{E}u_1(X_\tau, Y_\tau/H^*_{\tau+})\chi_{\{\tau \leq t\}},\end{aligned}$$

which combined with the preceding identity gives

$$\mathbb{E}u_1(X_t, Y_t/H^*_{t+}) \leq \mathbb{E}u_1(X_{\tau\wedge t}, Y_{\tau\wedge t}/H^*_{\tau\wedge t+}).$$

Hence it is enough to prove that the right-hand side is nonpositive. To this end, denote $Z_s = (X_s, Y_s/H^*_{s+})$ and apply Itô's formula to obtain

$$u_1(X_{\tau\wedge t}, Y_{\tau\wedge t}/H^*_{\tau\wedge t+}) = I_0 + I_1 + I_2 + I_3/2 + I_4, \tag{3.2.5}$$

where

$$\begin{aligned}
I_0 &= u_1(X_0, Y_0/H^*_{0+}),\\
I_1 &= \int_{0+}^{\tau\wedge t} u_{1x}(Z_{s-}) \cdot \mathrm{d}X_s + \int_{0+}^{\tau\wedge t} \frac{u_{1y}(Z_{s-})}{H^*_s} \cdot \mathrm{d}Y_s,\\
I_2 &= -\int_{0+}^{\tau\wedge t} u_{1y}(Z_{s-}) \cdot \frac{Y_{s-}}{(H^*_s)^2}\mathrm{d}H^*_{s+} + \sum_{0<s\leq\tau\wedge t} u_{1y}(Z_{s-}) \cdot \frac{Y_{s-}}{(H^*_s)^2}\Delta H^*_{s+},\\
I_3 &= \int_{0+}^{\tau\wedge t} u_{1xx}(Z_{s-}) \cdot \mathrm{d}[X,X]^c_s + \int_{0+}^{\tau\wedge t} u_{1yy}(Z_{s-})(H^*_s)^{-2} \cdot \mathrm{d}[Y,Y]^c_s,\\
I_4 &= \sum_{0<s\leq\tau\wedge t} \Big[u_1(Z_s) - u_1(Z_{s-}) - \langle u_{1x}(Z_{s-}), \Delta X_s\rangle - \langle u_{1y}(Z_{s-}), (\Delta Y_s)/H^*_s\rangle\Big].
\end{aligned}$$

Let us make several helpful observations here. The quantity I_1 and the integral in I_2 is just the sum of all first-order terms, while the expression I_3 is the sum of all second-order terms (note that for $(x,y) \in D$ we have $u_{1xy}(x,y) = 0$, so the mixed integral does not appear in I_3). The second half of I_2 and the whole I_4 correspond to the jump part.

Let us study the behavior of the terms I_0 through I_4. By the differential subordination of Y to $H \cdot X$, we have $|Y_0| \leq |H_0||X_0| \leq H^*_{0+}|X_0|$ and hence $I_0 \leq 0$ (indeed, we have $u_1(x,y) \leq 0$ if $|y| \leq |x|$). The stochastic integrals in I_1 are martingales (as processes indexed by t), since by the definition of τ, Z_- is bounded on $(0,\tau]$. The term I_2 is nonpositive: indeed, we have $u_{1y}(Z_{s-}) \cdot Y_{s-} = 2|Y_{s-}|^2 \geq 0$, the process $\left(H^*_{s+} - \sum_{0<u\leq s} \Delta H^*_{u+}\right)_{s\geq 0}$ is nondecreasing, and

$$I_2 = -\int_{0+}^{\tau\wedge t} u_{1y}(Z_{s-}) \cdot \frac{Y_{s-}}{(H^*_s)^2} \mathrm{d}\left(H^*_{s+} - \sum_{0<u\leq s} \Delta H^*_{u+}\right).$$

Next, we compute that

$$I_3 = -2\int_{0+}^{\tau\wedge t} \mathrm{d}[X,X]^c_s + 2\int_{0+}^{\tau\wedge t} (H^*_s)^{-2} \mathrm{d}[Y,Y]^c_s \leq 0,$$

by the differential subordination and Lemma 3.4 above. Finally, each summand appearing in I_4 is nonpositive, by virtue of (3.2.1) applied to $(x,y) = Z_{s-}$ and $(h,k) = (\Delta X_s, (\Delta Y_s)/H^*_s)$ (the estimate $|k| \leq |h|$ follows by the differential subordination of Y to $H \cdot X$: see Lemma 3.4).

Putting all the above facts together, we get the desired assertion. □

Proof of (3.2.4). Let $\tau = \inf\{s \geq 0 : (H^*_{s+}X_s, Y_s) \notin D\}$. The first step is to show that

$$\mathbb{E}u_\infty(H^*_{\tau\wedge t+}X_{\tau\wedge t}, Y_{\tau\wedge t}) \leq 0. \tag{3.2.6}$$

To this end, we write the trivial identity

$$\begin{aligned}&\mathbb{E}u_\infty(H^*_{\tau\wedge t+}X_{\tau\wedge t}, Y_{\tau\wedge t})\\ &\qquad = \mathbb{E}u_\infty(H^*_{\tau\wedge t+}X_{\tau\wedge t}, Y_{\tau\wedge t})\chi_{\{\tau>t\}} + \mathbb{E}u_\infty(H^*_{\tau\wedge t+}X_{\tau\wedge t}, Y_{\tau\wedge t})\chi_{\{\tau\leq t\}}.\end{aligned}$$

The first summand on the right is equal to zero: by the definitions of u_∞ and the stopping time τ, the random variable under the expectation vanishes. To handle the second summand, we apply, on the set $\{\tau \leq t\}$, the inequality (3.2.2) with $x = H^*_\tau X_{\tau-}$, $y = Y_{\tau-}$, $h = H^*_\tau \Delta X_\tau$ and $k = \Delta Y_\tau$. Note that $|k| \leq |h|$, by Lemma 3.4 and hence we get $u_\infty(H^*_\tau X_\tau, Y_\tau) \leq 0$. Thus, we also have $u_\infty(H^*_{\tau+}X_\tau, Y_\tau) \leq 0$, since $u_\infty(x,y)$ decreases as $|x|$ increases. Integrating, we get $\mathbb{E}u_\infty(H^*_{\tau+}X_\tau, Y_\tau)\chi_{\{\tau\leq t\}} \leq 0$, which proves (3.2.6).

The next step is to establish the inequality

$$\mathbb{E}u_\infty(H^*_{t+}X_t, Y_t) \leq \mathbb{E}u_\infty(H^*_{\tau\wedge t+}X_{\tau\wedge t}, Y_{\tau\wedge t}), \tag{3.2.7}$$

or equivalently,

$$\mathbb{E}u_\infty(H^*_{t+}X_t, Y_t)\chi_{\{\tau\leq t\}} \leq \mathbb{E}u_\infty(H^*_{\tau+}X_\tau, Y_\tau)\chi_{\{\tau\leq t\}}. \tag{3.2.8}$$

To show this bound, note that $u_\infty(x,y) \leq (|y|-1)^2 - |x|^2$ for all $(x,y) \in \mathbb{H}$, and hence

$$\mathbb{E}u_\infty(H^*_{t+}X_t, Y_t)\chi_{\{\tau\leq t\}} \leq \mathbb{E}\big((|Y_t|-1)^2 - |H^*_{t+}X_t|^2\big)\chi_{\{\tau\leq t\}}.$$

Arguing as above, by Doob's optional sampling theorem and the supermartingale property of the process $(1-2|Y_s|)_{s\geq 0}$, the estimate (3.2.8) will follow if we manage to prove that

$$\mathbb{E}\big(|Y_t|^2 - |H^*_{t+}X_t|^2\big)\chi_{\{\tau\leq t\}} \leq \mathbb{E}\big(|Y_\tau|^2 - |H^*_{\tau+}X_\tau|^2\big)\chi_{\{\tau\leq t\}}. \tag{3.2.9}$$

This is done by Itô's formula. The calculations are essentially the same as in the proof of (3.2.3) and hence we omit the details. □

3.3. Strong–type estimates

3.3.1. Proof of (3.1.3)

With no loss of generality, we may assume that X is bounded in L^q and $H^* \in L^r$, since otherwise there is nothing to prove. Furthermore, we may assume that H_0 is bounded away from zero, replacing it with $|H_0| + \varepsilon$ and letting $\varepsilon \downarrow 0$ at the very end of the proof. These assumptions imply that for each t the random variables $H^*_{t+} X_t$ and Y_t belong to L^p. Indeed, we have $\|H^*_{t+} X_t\|_{L^p} \leq \|X\|_{L^q} \|H^*\|_{L^r}$ by Young's inequality, while Y is handled with the use of Burkholder-Gundy inequality and the differential subordination:

$$\begin{aligned}\|Y_t\|_{L^p} \lesssim_p \|[Y,Y]_t^{1/2}\|_{L^p} &\leq \left\| \left(\int_0^t (H^*_s)^2 \mathrm{d}[X,X]_s \right)^{1/2} \right\|_{L^p} \\ &\leq \|H^*_{t+}[X,X]_t^{1/2}\|_{L^p} \leq \|[X,X]_t^{1/2}\|_{L^q} \|H^*_{t+}\|_{L^r} \lesssim_q \|X\|_{L^q} \|H^*\|_{L^r}.\end{aligned} \tag{3.3.1}$$

Now we consider separately three cases.

The case $p \geq 2$. If $p = 2$, the claim follows from (3.3.1): all the intermediate inequalities hold with the constant 1. Hence we may restrict ourselves to p strictly bigger than 2. Consider the functions U, $V : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ given by

$$V(x,y) = |y|^p - (p-1)^p |x|^p$$

and

$$U(x,y) = p^{2-p}(p-1)^{p-1}(|y| - (p-1)|x|)(|x|+|y|)^{p-1}.$$

Burkholder [10] showed that we have the majorization

$$U \geq V \qquad \text{on } \mathbb{H} \times \mathbb{H}. \tag{3.3.2}$$

The function U has the following remarkable representation in the language of u_∞:

$$U(x,y) = \alpha_p \int_0^\infty \lambda^{p-1} u_\infty(x/\lambda, y/\lambda) \mathrm{d}\lambda,$$

where $\alpha_p = p^{3-p}(p-1)^p(p-2)/2$ (see [4]). Therefore, by (3.2.4) and Fubini's theorem,

$$\mathbb{E} U(H^*_{t+} X_t, Y_t) \leq 0, \qquad t \geq 0. \tag{3.3.3}$$

To see that Fubini's theorem is applicable, note that

$$|u_\infty(x,y)| \leq \begin{cases} 0 & \text{if } |x|+|y| \leq 1, \\ |x|^2 + |y|^2 & \text{if } |x|+|y| > 1, \end{cases}$$

which implies

$$\int_0^\infty \lambda^{p-1} |u_\infty(x/\lambda, y/\lambda)| \mathrm{d}\lambda \lesssim_p (|x|^2 + |y|^2)(|x|+|y|)^{p-2} \lesssim_p |x|^p + |y|^p. \tag{3.3.4}$$

Since $H^*_{t+} X_t$ and Y_t belong to L^p, we have the necessary integrability and (3.3.3) follows. Thus, by (3.3.2), we conclude that $\mathbb{E} V(H^*_{t+} X_t, Y_t) \leq 0$, or

$$\|Y_t\|_{L^p} \leq (p-1) \|H^*_{t+} X_t\|_{L^p} \leq (p-1) \|X\|_{L^q} \|H^*\|_{L^r}.$$

Since t was arbitrary, the inequality is established.

The case $p < q \le 2$. We may actually assume that $q < 2$, the case $q = 2$ follows from a limiting argument. The reasoning goes along similar to those above, but we need some additional effort. Let U_q, $V_q : \mathbb{H} \times \mathbb{H} \to \mathbb{R}$ be defined by

$$V_q(x, y) = |y|^q - (q-1)^{-q}|x|^q, \qquad U_q(x, y) = \frac{q^{2-q}}{q-1}((q-1)|y| - |x|)(|x| + |y|)^{q-1}.$$

As shown by Burkholder in [10], we have

$$U_q \ge V_q \qquad \text{on } \mathbb{H} \times \mathbb{H}. \tag{3.3.5}$$

Furthermore, the function U_q admits the representation (cf. [4])

$$U_q(x, y) = \alpha_q \int_0^\infty \lambda^{q-1} u_1(x/\lambda, y/\lambda) \mathrm{d}\lambda,$$

where $\alpha_q = q^{3-q}(2-q)/2$. Now it is natural to try to use (3.2.3) and Fubini's theorem to obtain $\mathbb{E}U_q(X_t, Y_t/H^*_{t+}) \le 0$ for all $t \ge 0$. The function U_q enjoys an appropriate boundedness: we have

$$|u_1(x, y)| \le \begin{cases} |x|^2 + |y|^2 & \text{if } |x| + |y| \le 1, \\ |x| + |y| & \text{if } |x| + |y| > 1 \end{cases}$$

and hence

$$\int_0^\infty \lambda^{q-1} |u_1(x/\lambda, y/\lambda)| \mathrm{d}\lambda \lesssim_q |x|^q + |y|^q. \tag{3.3.6}$$

So, to use Fubini's theorem, we need to establish the L^q-boundedness of the process Y/H^*_+. This, in contrast to the previous situation, does not seem to follow from Burkholder-Gundy inequality. To overcome this difficulty, we apply localization. Given an arbitrary positive integer M, consider the stopping time

$$\sigma_M = \inf\{s \ge 0 : |X_s| + |Y_s/H^*_s| \ge M\}.$$

By the differential subordination of Y to $H \cdot X$, we have

$$|\Delta(Y_{\sigma_M}/H^*_{\sigma_M})| = |\Delta Y_{\sigma_M}|/H^*_{\sigma_M} \le |\Delta X_{\sigma_M}|,$$

which implies that $|Y_{\sigma_M \wedge t}/H^*_{\sigma_M \wedge t}| \le M + |\Delta X_{\sigma_M \wedge t}|$, in particular, $Y_{\sigma_M \wedge t}/H^*_{\sigma_M \wedge t}$, and hence also $Y_{\sigma_M \wedge t}/H^*_{\sigma_M \wedge t+}$, belong to L^q. The stopped martingale Y^{σ_M} is differentially subordinate to $H^{\sigma_M} \cdot X^{\sigma_M}$, so (3.2.3) and Fubini's theorem give

$$\mathbb{E}U_q(X_{\sigma_M \wedge t}, Y_{\sigma_M \wedge t}/H^*_{\sigma_M \wedge t+}) \le 0, \qquad t \ge 0. \tag{3.3.7}$$

Combining this estimate with (3.3.5), we get $\mathbb{E}V_q(X_{\sigma_M \wedge t}, Y_{\sigma_M \wedge t}/H^*_{\sigma_M \wedge t+}) \le 0$ and hence

$$\begin{aligned} \|Y_{\sigma_M \wedge t}\|_{L^p} &\le \|Y_{\sigma_M \wedge t}/H^*_{\sigma_M \wedge t+}\|_{L^q} \|H^*_{\sigma_M \wedge t+}\|_{L^r} \\ &\le (q-1)^{-1} \|X_{\sigma_M \wedge t}\|_{L^q} \|H^*_{\sigma_M \wedge t+}\|_{L^r} \le (q-1)^{-1} \|X\|_{L^q} \|H^*\|_{L^r}. \end{aligned}$$

Letting $M \to \infty$ and $t \to \infty$, we get the claim, by Fatou's lemma.

The case $p < 2 < q$. For this choice of p and q, the assertion will follow by applying (3.1.3) twice, in the range already covered by the above considerations. Specifically, take

$s = 2p/(2-p)$, $\alpha = r(2-p)/(2p)$ and write the stochastic integral $H \cdot X$ in the alternative form

$$\int H_t \mathrm{d}X_t = \int |H_t|^\alpha \frac{H_t}{|H_t|^\alpha} \mathrm{d}X_t,$$

i.e., as the stochastic integral of the process $|H|^\alpha$ with respect to the martingale $H|H|^{-\alpha} \cdot X$. So, Y is differentially subordinate to $|H|^\alpha \cdot (H|H|^{-\alpha} \cdot X)$, and hence (3.1.3), applied with $1/p = 1/2 + 1/s$ (then $C_{p,2,s} = 1$, as we have shown above), gives

$$\|Y\|_{L^p} \leq \left\|H|H|^{-\alpha} \cdot X\right\|_{L^2} \|(H^*)^\alpha\|_{L^s} = \left\|H|H|^{-\alpha} \cdot X\right\|_{L^2} \|H^*\|_{L^r}^{r/s}.$$

The term $\left\|H|H|^{-\alpha} \cdot X\right\|_{L^2}$ is again handled by (3.1.3). Namely, we have $1/2 = 1/q + (q-2)/(2q)$ and $C_{2,q,(2q)/(q-2)} = 1$, so

$$\left\|H|H|^{-\alpha} \cdot X\right\|_{L^2} \leq \|X\|_{L^q} \|(H^*)^{1-\alpha}\|_{L^{2q/(q-2)}} = \|X\|_{L^q} \|H^*\|_{L^r}^{r(q-2)/(2q)}.$$

Putting all the above facts together, we get the desired estimate.

3.3.2. Sharpness for martingale transforms

Observe that $C_{p,q,r} \geq 1$ for all p, q, r satisfying $1/p = 1/q + 1/r$: this is easily seen by considering the constant sequences $f = g = v \equiv 1$. Therefore, the estimate (3.1.1) is sharp for $p \leq 2 \leq q$ and from now on we may assume that $p > 2$ or $p < q < 2$. Actually, by the lemma below, we may restrict ourselves to the first possibility.

Lemma 3.6. *Let $C^{tr}_{p,q,r}$ denote the optimal constant in* (3.1.1)*, restricted to real-valued martingales. Then we have $C^{tr}_{p,q,r} = C^{tr}_{q',p',r}$ for all p, q and r satisfying $1/p = 1/q + 1/r$.*

Proof. Let $p' = p/(p-1)$ be the Hölder conjugate to p. Assume that $\varphi = (\varphi_n)_{n\geq 0}$ is an arbitrary $L^{p'}$-bounded, real-valued martingale with $\|\varphi\|_{L^{p'}} \leq 1$ and let $\psi = (\psi_n)_{n\geq 0}$ be the transform of φ by v. Since the martingale differences are orthogonal, we may write

$$\mathbb{E} g_n \varphi_n = \mathbb{E} \sum_{k=0}^n dg_k d\varphi_k = \mathbb{E} \sum_{k=0}^n df_k d\psi_k = \mathbb{E} f_n \psi_n.$$

However, we have $1/q' = 1/p' + 1/r$, so

$$\mathbb{E} f_n \psi_n \leq \|f_n\|_{L^q} \|\psi_n\|_{L^{q'}} \leq C^{tr}_{q',p',r} \|f_n\|_{L^q} \|\varphi\|_{L^{p'}} \|v^*\|_{L^r} \leq C^{tr}_{q',p',r} \|f\|_{L^q} \|v^*\|_{L^r}.$$

Combining this with the previous identity and using the fact that φ was chosen arbitrarily, we conclude that $\|g_n\|_{L^p} \leq C^{tr}_{q',p',r} \|f\|_{L^q} \|v^*\|_{L^r}$ and hence, taking the supremum over n, we obtain that $C^{tr}_{p,q,r} \leq C^{tr}_{q',p',r}$. Switching from (p,q) to (q',p'), we get the reverse bound. The proof is complete. □

Thus, from now on, we assume that $p > 2$ and proceed to the construction of the appropriate extremal examples. The analysis splits naturally into several steps.

Step 1. The filtered probability space. Assume that the probability space is the interval $(0,1]$ with its Borel subsets and the Lebesgue measure. Let $a > q$ and $\delta > 0$ be fixed parameters, and set $Q = 1 - a\delta$. We start with defining a certain decreasing sequence $(p_n)_{n\geq 0}$ with values in $(0,1]$. Namely, for any $n \geq 0$ we put

$$p_{2n} = Q^n \quad \text{and} \quad p_{2n+1} = \frac{Q^n + Q^{n+1}}{2} = \frac{p_{2n} + p_{2n+2}}{2}.$$

This sequence gives rise to the filtration $(\mathcal{F}_n)_{n\geq 0}$ such that for a fixed n, the σ-field $\mathcal{F}_n$ is generated by the intervals $(0,p_n]$, $(0,p_{n-1}]$, $(0,p_{n-2}]$, ..., $(0,p_0]$. That is, the atoms of $\mathcal{F}_n$ are precisely $(0,p_n]$, $(p_n,p_{n-1}]$, $(p_{n-1},p_{n-2}]$, ..., $(p_1,p_0]$.

Step 2. The variable f. Introduce the function (random variable) $f:(0,1]\to\mathbb{R}$ by

$$f=\sum_{n=0}^{\infty}(1+\delta)^n\Big(\chi_{(p_{2n+1},p_{2n}]}-\chi_{(p_{2n+2},p_{2n+1}]}\Big).$$

Note that f is measurable with respect to $\sigma(\mathcal{F}_n : n\geq 0)$. It is easy to check that f is integrable, it actually belongs to L^q, at least for sufficiently small δ. Indeed, we compute directly that

$$\mathbb{E}|f|^q=\sum_{n=0}^{\infty}(1+\delta)^{nq}(p_{2n}-p_{2n+2})=a\delta\sum_{n=0}^{\infty}\Big[(1+\delta)^q(1-a\delta)\Big]^n<\infty,$$

where the last inequality follows from the estimate $a>q$ (which guarantees that the ratio of the geometric series is less than 1). Furthermore, note that if a is chosen close to q, then

$$\lim_{\delta\to 0}\mathbb{E}|f|^q=\lim_{\delta\to 0}\frac{a\delta}{1-(1+\delta)^q(1-a\delta)}=\frac{a}{a-q}$$

and hence

$$\lim_{a\downarrow q}\lim_{\delta\downarrow 0}\|f\|_{L^q}=\infty.$$

Step 3. On the martingale $(f_n)_{n\geq 0}$ generated by f. For any nonnegative integer n, we let $f_n=\mathbb{E}(f|\mathcal{F}_n)$. By the very definition of f and $(\mathcal{F}_n)_{n\geq 0}$, we check that

$$f_{2n}=\begin{cases}0 & \text{on } (0,p_{2n}],\\ f & \text{on } (p_{2n},1].\end{cases}$$

Indeed, on $(0,p_{2n}]$ we have

$$f_{2n}=\frac{1}{|(0,p_{2n}]|}\int_0^{p_{2n}}f\,\mathrm{d}x=0,$$

by symmetry: for each k, the point p_{2k+1} is the middle of (p_{2k+2},p_{2k}). Similarly, we get

$$f_{2n+1}=\begin{cases}-\dfrac{1-Q}{1+Q}(1+\delta)^n & \text{on } (0,p_{2n+1}],\\ f & \text{on } (p_{2n+1},1].\end{cases}$$

To check the first formula, note that $\int_0^{p_{2n+2}}f=0$ (as we have seen above), so on $(0,p_{2n+1}]$,

$$\begin{aligned}f_{2n+1}=\frac{1}{|(0,p_{2n+1}]|}\int_0^{p_{2n+1}}f\,\mathrm{d}x&=\frac{2}{Q^n+Q^{n+1}}\int_{p_{2n+2}}^{p_{2n+1}}f\,\mathrm{d}x\\&=-\frac{2}{Q^n+Q^{n+1}}(1+\delta)^n(p_{2n+1}-p_{2n+2})\\&=-\frac{1-Q}{1+Q}(1+\delta)^n.\end{aligned}$$

Passing to the difference sequence df, we obtain that $df_0 = f_0 = 0$ and

$$df_{2n+1} = \begin{cases} -\dfrac{1-Q}{1+Q}(1+\delta)^n & \text{on } (0, p_{2n+1}], \\ (1+\delta)^n & \text{on } (p_{2n+1}, p_{2n}], \\ 0 & \text{on } (p_{2n}, 1], \end{cases}$$

$$df_{2n+2} = \begin{cases} \dfrac{1-Q}{1+Q}(1+\delta)^n & \text{on } (0, p_{2n+2}], \\ -(1+\delta)^n \cdot \dfrac{2Q}{1+Q} & \text{on } (p_{2n+2}, p_{2n+1}], \\ 0 & \text{on } (p_{2n+1}, 1] \end{cases}$$

for $n = 0, 1, 2, \ldots$.

Step 4. The predictable sequence v and its properties. We introduce $v = (v_n)_{n\geq 0}$ by $v_0 \equiv 1$ and, for $n \geq 0$,

$$v_{2n+1} = -(1+\delta)^{nq/r}\chi_{(0,p_{2n}]}, \qquad v_{2n+2} = (1+\delta)^{nq/r}\chi_{(0,p_{2n+1}]}.$$

Obviously, v is predictable: we have $(0, p_n] \in \mathcal{F}_n$ for each n. Furthermore, on the set $(p_{n+1}, p_n]$ we have $|v_0| \leq |v_1| \leq |v_2| \leq \ldots \leq |v_{n+1}|$ and $v_{n+2} = v_{n+3} = \ldots = 0$. Consequently,

$$v^* = \sum_{n=0}^{\infty}(1+\delta)^{nq/r}\chi_{(p_{2n+2},p_{2n}]} = |f|^{q/r},$$

so $\|v^*\|_{L^r} = \|f\|_{L^q}^{q/r}$ and hence in particular $\|f\|_{L^q}\|v^*\|_{L^r} = \|f\|_{L^q}^{q/p}$.

Step 5. On the transform. Let g be the transform of f by v. We will compute the explicit formula for g on each interval of the form $(p_{n+1}, p_n]$. We start with an even n. Directly from the above construction, we see that $df_{2n+2} = 0$ on $(p_{2n+1}, p_{2n}]$ and hence

$$\begin{aligned} g_{2n+2} = g_{2n+1} &= v_0 df_0 + v_1 df_1 + \ldots + v_{2n} df_{2n} + v_{2n+1} df_{2n+1} \\ &= \frac{2(1-Q)}{1+Q}\left[1 + (1+\delta)^{1+q/r} + \ldots + (1+\delta)^{(n-1)(1+q/r)}\right] - (1+\delta)^{n(1+q/r)} \\ &= \frac{2(1-Q)}{1+Q} \cdot \frac{(1+\delta)^{n(1+q/r)} - 1}{(1+\delta)^{1+q/r} - 1} - (1+\delta)^{n(1+q/r)} \\ &= (1+\delta)^{n(1+q/r)}\left[\frac{2}{1+Q} \cdot \frac{1-Q}{(1+\delta)^{1+q/r}-1} - 1\right] - \frac{2}{1+Q} \cdot \frac{1-Q}{(1+\delta)^{1+q/r}-1}. \end{aligned}$$

On $(p_{2n+2}, p_{2n+1}]$ the calculations are similar, but slightly more complicated: we get

$$\begin{aligned} g_{2n+2} &= v_0 df_0 + v_1 df_1 + \ldots + v_{2n+1} df_{2n+1} + v_{2n+2} df_{2n+2} \\ &= \frac{2(1-Q)}{1+Q}\left[1 + (1+\delta)^{1+q/r} + \ldots + (1+\delta)^{(n-1)(1+q/r)}\right] \\ &\quad + \frac{1-Q}{1+Q}(1+\delta)^{n(1+q/r)} - (1+\delta)^{n(1+q/r)} \cdot \frac{2Q}{1+Q} \\ &= (1+\delta)^{n(1+q/r)}\left[\frac{2}{1+Q} \cdot \frac{1-Q}{(1+\delta)^{1+q/r}-1} - 1\right] - \frac{2}{1+Q} \cdot \frac{1-Q}{(1+\delta)^{1+q/r}-1} \\ &\quad + \frac{2(1-Q)}{1+Q}(1+\delta)^{n(1+q/r)}. \end{aligned}$$

Finally, note that $dg_{n+1} = dg_{n+2} = \ldots = 0$ on $(p_n, p_{n-1}]$. Therefore, we have that g, the pointwise limit of $(g_n)_{n\geq 0}$, can be rewritten in the form $g = g^{(1)} + g^{(2)} + g^{(3)}$, where

$$g^{(1)} = \sum_{n=0}^{\infty}(1+\delta)^{n(1+q/r)}\left[\frac{2}{1+Q}\cdot\frac{1-Q}{(1+\delta)^{1+q/r}-1}-1\right]\chi_{(p_{2n+2},p_{2n}]},$$

$$g^{(2)} = -\sum_{n=0}^{\infty}\frac{2}{1+Q}\cdot\frac{1-Q}{(1+\delta)^{1+q/r}-1}\chi_{(p_{2n+2},p_{2n}]},$$

$$g^{(3)} = \sum_{n=0}^{\infty}\frac{2(1-Q)}{1+Q}(1+\delta)^{n(1+q/r)}\chi_{(p_{2n+2},p_{2n+1}]}.$$

Step 6. The analysis of $g^{(j)}$. Observe that $|g^{(1)}| = |f|^{q/p}\cdot\left[\frac{2}{1+Q}\cdot\frac{1-Q}{(1+\delta)^{1+q/r}-1}-1\right]$ and the expression in the square brackets enjoys the following behavior:

$$\lim_{\delta\to 0}\left[\frac{2}{1+Q}\cdot\frac{1-Q}{(1+\delta)^{1+q/r}-1}-1\right] = \lim_{\delta\to 0}\left(\frac{2a\delta}{(2-a\delta)((1+\delta)^{1+q/r}-1)}-1\right) = \frac{ra}{r+q}-1.$$

The limit $ra/(r+q)-1$ can be made arbitrarily close to $rq/(r+q)-1 = p-1$, if a is chosen sufficiently close to q. Consequently,

$$\lim_{a\downarrow q}\lim_{\delta\downarrow 0}\frac{\|g^{(1)}\|_{L^p}}{\|f\|_{L^q}^{q/p}} = \lim_{a\downarrow q}\lim_{\delta\downarrow 0}\frac{\|g^{(1)}\|_{L^p}}{\|f\|_{L^q}\|v^*\|_{L^r}} = p-1.$$

Now we will show that the contribution of the variables $g^{(2)}$ and $g^{(3)}$ is negligible. Note that

$$|g^{(2)}| = \frac{2(1-Q)}{(1+Q)((1+\delta)^{1+q/r}-1)}$$

is deterministic and converges to $ra/(r+q)$ as $\delta\to 0$. Combining this with the analysis at the end of Step 2, we see that

$$\lim_{a\downarrow q}\lim_{\delta\downarrow 0}\frac{\|g^{(2)}\|_{L^p}}{\|f\|_{L^q}^{q/p}} = \lim_{a\downarrow q}\lim_{\delta\downarrow 0}\frac{\|g^{(2)}\|_{L^p}}{\|f\|_{L^q}\|v^*\|_{L^r}} = 0.$$

Finally, note that $|g^{(3)}| \leq \frac{2(1-Q)}{1+Q}|f|^{q/p}$, and hence

$$\lim_{a\downarrow q}\lim_{\delta\downarrow 0}\frac{\|g^{(3)}\|_{L^p}}{\|f\|_{L^q}^{q/p}} = \lim_{a\downarrow q}\lim_{\delta\downarrow 0}\frac{\|g^{(3)}\|_{L^p}}{\|f\|_{L^q}\|v^*\|_{L^r}} = 0.$$

Step 7. Completion of the proof. Let us put the above facts together. We fix $\varepsilon > 0$ and take $a > q$ such that

$$\left|\frac{ra}{r+q}-1-(p-1)\right| < \varepsilon.$$

Then for sufficiently small δ we have

$$\frac{\|g^{(1)}\|_{L^p}}{\|f\|_{L^q}\|v^*\|_{L^r}} > p-1-\varepsilon, \qquad \frac{\|g^{(2)}\|_{L^p}}{\|f\|_{L^q}\|v^*\|_{L^r}} < \varepsilon, \qquad \frac{\|g^{(3)}\|_{L^p}}{\|f\|_{L^q}\|v^*\|_{L^r}} < \varepsilon$$

and hence

$$\frac{\|g\|_{L^p}}{\|f\|_{L^q}\|v^*\|_{L^r}} > p-1-3\varepsilon.$$

Since ε was arbitrary, the sharpness follows.

3.4. Weak-type estimates, $p \geq 2$

3.4.1. Proof of (3.1.4)

As in the case of strong–type estimates, we may and do assume that $\|X\|_{L^q} < \infty$ and $\|H^*\|_{L^r} < \infty$; we may also assume that the norms are strictly positive since otherwise, the claim is obvious. Then $H^*_+ X$ and Y belong to L^p, as we checked in the preceding section. Consider the functions U, V on $\mathbb{H} \times \mathbb{H}$, given by

$$U(x, y) = \beta_p \int_0^{1-p^{-1}} \lambda^{p-1} u_\infty(x/\lambda, y/\lambda) \mathrm{d}\lambda$$

and

$$V(x, y) = p\,(|y| - 1 + 1/p)_+ - \frac{p^{p-1}}{2}|x|^p,$$

where $\beta_p = p^p(p-1)^{2-p}(p-2)/4$. It was proved in [6] that

$$U \geq V \qquad \text{on } \mathbb{H} \times \mathbb{H}. \tag{3.4.1}$$

Applying (3.2.4) and Fubini's theorem, we get $\mathbb{E}V(H^*_{t+}X_t, Y_t) \leq \mathbb{E}U(H^*_{t+}X_t, Y_t) \leq 0$ for $t \geq 0$. Fubini's theorem is applicable, since

$$\int_0^{1-p^{-1}} \lambda^{p-1} |u_\infty(x/\lambda, y/\lambda)| \mathrm{d}\lambda \leq \int_0^\infty \lambda^{p-1} |u_\infty(x/\lambda, y/\lambda)| \mathrm{d}\lambda \lesssim_p |x|^p + |y|^p,$$

as we already verified in (3.3.4). Therefore, we obtain

$$\mathbb{E}(p|Y_t| - p + 1)_+ \leq \frac{p^{p-1}}{2} \mathbb{E}|H^*_{t+}X_t|^p \leq \frac{p^{p-1}}{2} \|X\|^p_{L^q} \|H^*\|^p_{L^r}.$$

Fix an arbitrary event A of positive probability. Then

$$\mathbb{E}(p|Y_t| - p + 1)1_A \leq \mathbb{E}(p|Y_t| - p + 1)_+ \leq \frac{p^{p-1}}{2} \|X\|^p_{L^q} \|H^*\|^p_{L^r},$$

or equivalently,

$$\int_A |Y_t| \mathrm{d}\mathbb{P} \leq \frac{p^{p-2}}{2} \|X\|^p_{L^q} \|H^*\|^p_{L^r} + \frac{p-1}{p} \mathbb{P}(A).$$

The differential subordination of Y to $H \cdot X$ is preserved if we multiply X and Y by a fixed positive constant λ. Applying the above estimate to the modified triple λX, λY and H, we obtain

$$\lambda \int_A |Y_t| \mathrm{d}\mathbb{P} \leq \lambda^p \frac{p^{p-2}}{2} \|X\|^p_{L^q} \|H^*\|^p_{L^r} + \frac{p-1}{p} \mathbb{P}(A).$$

Dividing both sides by λ and optimizing over λ (specifically, the best choice is $\lambda = (2\mathbb{P}(A)/p^{p-1})^{1/p} \|X\|^{-1}_{L^q} \|H^*\|^{-1}_{L^r}$), we get

$$\int_A |Y_t| \mathrm{d}\mathbb{P} \leq \left(\frac{p^{p-1}}{2}\right)^{1/p} \|X\|_{L^q} \|H^*\|_{L^r} \cdot \mathbb{P}(A)^{1-1/p}.$$

This yields $|||Y|||_{L^{p,\infty}} \leq (p^{p-1}/2)^{1/p} \|X\|_{L^q} \|H^*\|_{L^r}$, since A and t were arbitrary.

3.4.2. Sharpness for martingale transforms

The calculations are quite similar to those appearing in the previous section. We take $\delta > 0$, fix a positive integer N and set $Q = 1 - (p-1)q\delta/p$. Then we define the sequence $(p_n)_{n\geq 0}$ as before and consider the probability space $((0,1], \mathcal{B}(0,1), |\cdot|)$. We consider the σ-algebras $\mathcal{F}_0$, $\mathcal{F}_1$, ..., $\mathcal{F}_{2N}$ as previously, and $\mathcal{F}_{2N+1} = \mathcal{F}_{2N+2} = \ldots$ is the σ-field with atoms $(0, p_{2N}/2]$, $(p_{2N}/2, p_{2N}]$, $(p_{2N}, p_{2N-1}]$, $(p_{2N-1}, p_{2N-2}]$, ..., $(p_1, p_0]$, that is, $\mathcal{F}_{2N+1} = \mathcal{F}_{2N+2} = \ldots = \sigma(\mathcal{F}_{2N}, (0, p_{2N}/2])$. Consider the function f given by the finite sum

$$f = \sum_{n=0}^{N-1} (1+\delta)^n \Big(\chi_{(p_{2n+1}, p_{2n}]} - \chi_{(p_{2n+2}, p_{2n+1}]}\Big) + (1+\delta)^N \Big(\chi_{(0, p_{2N}/2]} - \chi_{(p_{2N}/2, p_{2N}]}\Big).$$

This function is measurable with respect to $\mathcal{F}_{2N+1}$ and satisfies

$$\begin{aligned} \mathbb{E}|f|^q &= \sum_{n=0}^{N-1} (1+\delta)^{qn} (p_{2n} - p_{2n+2}) + (1+\delta)^{qN} p_{2N} \\ &\leq (Q(1+\delta)^q)^N \cdot \frac{(1+\delta)^q Q - Q}{(1+\delta)^q Q - 1}. \end{aligned} \tag{3.4.2}$$

It is easy to see that the formulas for df_n, $n = 0, 1, 2, \ldots, 2N$, are the same as in the previous section. This follows from the fact that f has not been changed on $(p_{2N}, 1]$ and it still has a vanishing integral on $(0, p_{2N}]$. To complete the description of the difference sequence, note that $df_{2N+1} = f\chi_{(0,p_{2N}]}$ and $df_{2N+2} = df_{2N+3} = \ldots = 0$.

The transforming sequence $v = (v_n)_{n\geq 0}$ is given by $v_0 \equiv 1$; for $n = 0, 1, 2, \ldots, N-1$ we put $v_{2n+1} = -(1+\delta)^{nq/r}\chi_{(0,p_{2n}]}$ and $v_{2n+2} = (1+\delta)^{nq/r}\chi_{(0,p_{2n+1}]}$; finally, for $n > 2N$ we set $v_n = (1+\delta)^{Nq/r}\chi_{(0,p_{2N}]}$. So, in comparison to the formulas from the previous section, we see that v_0, v_1, ..., v_{2N} are the same. Consequently, we may repeat the analysis and obtain that $v^* = |f|^{q/r}$; furthermore, on $(0, p_{2N}/2]$ we have

$$\begin{aligned} g_{2N+1} &= v_0 df_0 + v_1 df_1 + \ldots + v_{2N+1} df_{2N+1} \\ &= \frac{2(1-Q)}{1+Q} \left[1 + (1+\delta)^{1+q/r} + \ldots + (1+\delta)^{(N-1)(1+q/r)}\right] + (1+\delta)^{N(1+q/r)} \\ &= \frac{2(1-Q)}{1+Q} \cdot \frac{(1+\delta)^{N(1+q/r)} - 1}{(1+\delta)^{1+q/r} - 1} + (1+\delta)^{N(1+q/r)}. \end{aligned}$$

Denoting the latter expression by λ, we see that

$$\frac{|||g|||_{L^{p,\infty}}}{\|f\|_{L^q}\|v^*\|_{L^r}} \geq \frac{\|g\|_{L^{p,\infty}}}{\|f\|_{L^q}^{1+q/r}} \geq \frac{\lambda|(0, p_{2N}/2]|^{1/p}}{\|f\|_{L^q}^{q/p}} \geq \lambda \left(\frac{(1+\delta)^q Q - 1}{2Q(1+\delta)^{qN}((1+\delta)^q - 1)}\right)^{1/p},$$

where the last inequality is due to (3.4.2). Now we need to perform an appropriate limiting procedure. Letting $N \to \infty$, the latter expression converges to

$$\left(\frac{2(1-Q)}{(1+Q)((1+\delta)^{1+q/r} - 1)} + 1\right) \left(\frac{(1+\delta)^q Q - 1}{2Q((1+\delta)^q - 1)}\right)^{1/p}.$$

Now if we let $\delta \to 0$, the above quantity tends to $p \cdot (2p)^{-1/p} = (p^{p-1}/2)^{1/p}$. This yields the desired lower bound for the weak–type constant.

3.5. Weak-type estimates, $p < 2$

3.5.1. Proof of (3.1.4)

If $q \geq 2$, then the estimate follows at once from the strong–type bound: we have

$$|||Y|||_{L^{p,\infty}} \leq \|Y\|_{L^p} \leq \|X\|_{L^q}\|H^*\|_{L^r}.$$

The main difficulty lies in proving the weak–type inequality for $1 < p < q < 2$; one easily checks that $1 < r' < q$ in such a case. Fix X, Y and H as in the statement; we may assume that $\|X\|_{L^q} < \infty$, $\|H^*\|_{L^r} < \infty$ and $|H_0|$ is bounded away from zero. Then $\|Y\|_{L^p} < \infty$, by the strong–type estimate which we have established in Section 3.3.

We will make use of Burkholder's method: this time the definitions of the appropriate special functions are much more involved. To avoid notational confusion, in our considerations below we will use the letter α for the number $r' = r/(r-1)$. Consider the differential equation

$$\alpha(2-\alpha)\phi'(x) + \alpha = q(q-1)x^{q-2}\phi(x)^{2-\alpha}. \tag{3.5.1}$$

We have the following fact, which appears as Theorem 2.1 in [38].

Theorem 3.7. *There exists a unique nondecreasing, concave solution $\phi : [0,\infty) \to [0,\infty)$ of (3.5.1) satisfying $\phi(0) > 0$ and $\phi'(t) \to 0$, $\phi(t) \to \infty$, as $t \to \infty$.*

From now on, ϕ stands for the solution described in the above theorem. Let $\Phi : [\phi(0),\infty) \to [0,\infty)$ be the inverse to $t \mapsto t + \phi(t)$. We have $\phi(\Phi(t)) + \Phi(t) = t$, which in particular yields

$$\phi(\Phi(t)) \leq t \quad \text{and} \quad \phi'(\Phi(t))\Phi'(t) \leq 1 \tag{3.5.2}$$

for $t > 0$. For the notational convenience, let us distinguish the constant

$$L_{\alpha,q} = \frac{(2-\alpha)\phi(0)^\alpha}{2}$$

and consider the auxiliary kernel

$$w(\lambda) = \frac{\alpha(2-\alpha)}{2}\phi(\Phi(\lambda))^{\alpha-3}\phi'(\Phi(\lambda))\Phi'(\lambda)\lambda^2, \qquad \lambda > 0.$$

We are ready for the definitions of the functions V, $U : \mathbb{H}\times\mathbb{H} \to \mathbb{R}$ which will lead us to the weak–type estimate. Set

$$V(x,y) = \big(|y|^\alpha - L_{\alpha,q}\big)_+ - |x|^q$$

and

$$U(x,y) = \int_{\phi(0)}^\infty w(\lambda)u_1(x/\lambda, y/\lambda)\mathrm{d}\lambda. \tag{3.5.3}$$

One can derive the explicit formula for U, but it will not be needed in our considerations. The only property which matters to us is the majorization of V by U (see Lemma 3.5 in [39]). Furthermore, by (3.5.2) we have $w(\lambda) \lesssim_{\alpha,q} \lambda^{q-1}$ and hence, computing as in (3.3.6),

$$\int_{\phi(0)}^\infty w(\lambda)|u_1(x/\lambda, y/\lambda)|\mathrm{d}\lambda \lesssim_{\alpha,q} |x|^q + |y|^q.$$

Thus by (3.2.3), Fubini's theorem and the majorization $U \geq V$,

$$\mathbb{E}V(X_t, Y_t/H^*_{t+}) \leq \mathbb{E}U(X_t, Y_t/H^*_{t+}) \leq 0, \qquad t \geq 0. \tag{3.5.4}$$

Now we argue as in the case $p > 2$. For an arbitrary event A of positive probability, we may write

$$\mathbb{E}\big(|Y_t/H^*_{t+}|^\alpha - L_{\alpha,q}\big)1_A \leq \mathbb{E}(|Y_t/H^*_{t+}|^\alpha - L_{\alpha,q})_+ \leq \mathbb{E}|X_t|^q,$$

where the last passage is equivalent to (3.5.4). Therefore, we get

$$\int_A |Y_t/H^*_{t+}|^\alpha \mathrm{d}\mathbb{P} \leq \|X\|^q_{L^q} + L_{\alpha,q}\mathbb{P}(A).$$

The differential subordination of Y to $H \cdot X$ is not affected if we multiply X and Y by a fixed positive constant λ. Therefore, the above inequality gives

$$\int_A |Y_t/H^*_{t+}|^\alpha \mathrm{d}\mathbb{P} \leq \lambda^{q-\alpha}\|X\|^q_{L^q} + \lambda^{-\alpha}L_{\alpha,q}\mathbb{P}(A),$$

and the optimization over λ yields

$$\int_A |Y_t/H^*_{t+}|^\alpha \mathrm{d}\mathbb{P} \leq \frac{q}{\alpha}\left(\frac{\alpha}{q-\alpha}L_{\alpha,q}\right)^{1-\alpha/q} \|X\|^\alpha_{L^q}\mathbb{P}(A)^{1-\alpha/q}.$$

Consequently, recalling that α is the Hölder conjugate to r, we may write

$$\begin{aligned}\int_A |Y_t|\mathrm{d}\mathbb{P} &\leq \left(\int_A |Y_t/H^*_{t+}|^\alpha \mathrm{d}\mathbb{P}\right)^{1/\alpha} \|H^*\|_{L^r}\\ &\leq \left(\frac{q}{r'}\right)^{1/r'}\left(\frac{r'}{q-r'}L_{r',q}\right)^{1-1/p} \|X\|_{L^q}\|H^*\|_{L^r}\mathbb{P}(A)^{1-1/p}.\end{aligned}$$

This is precisely the desired weak–type bound, since A and t were chosen arbitrarily.

3.5.2. Sharpness for martingale transforms

As previously, we may restrict ourselves to the case $1 < p < q < 2$: for $q \geq 2$, the constant is 1, which is achieved for $f = g = v \equiv 1$.

Fix $\varepsilon > 0$. Our starting point is the strong–type estimate

$$\|\varphi\|_{L^{r'}} \leq K_{r',q}\|f\|_{L^q},$$

where f is an arbitrary L^q-bounded martingale and φ is its transform by the deterministic sequence $w_n = (-1)^n$, $n = 0, 1, 2, \ldots$. The optimal value of the constant $K_{r',q}$ was identified in [38]: it is equal to $c_{p,q,r}$ and the almost-extremal examples have the following structure: see Figure 3.1 below to gain some intuition. Fix a small parameter $\delta > 0$. The pair (f, φ) starts from $(\phi(0)/2, \phi(0)/2)$ and at the first move it goes to $(0, \phi(0))$ or to $(\phi(0), 0)$. Then the evolution is governed by the following rules:

· if (f, φ) lies on one of the curves $y = \phi(x)$ or $y = -\phi(x)$, it stops ultimately;

· if we have $(f, \varphi) = (x, 0)$ for some $x > 0$, then the pair jumps, along the line of slope 1, to $(x + \delta, \delta)$ or onto the curve $y = -\phi(x)$;

· if we have $(f, \varphi) = (x + \delta, \delta)$ for some $x > 0$, then the pair jumps, along the line of slope -1, to $(x + 2\delta, 0)$ or onto the curve $y = \phi(x)$.

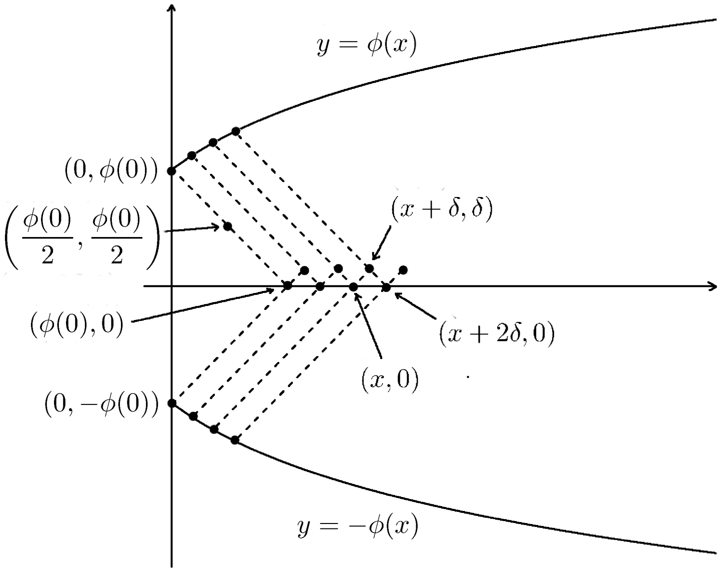

Figure 3.1. The structure of the extremal examples. The dots • indicate the possible locations of the pair (f, φ).

Let us gather some basic information about f and φ, which will be needed later. First, the martingales are unbounded, but they are both bounded in L^q. Furthermore, it can be extracted from [38] that

$$\lim_{\delta \downarrow 0} \frac{\|\varphi\|_{L^{r'}}}{\|f\|_{L^q}} = c_{p,q,r}.$$

Next, we make some observations concerning the behavior of the differences df_0, df_1, ... We easily see that with probability 1, first several differences are positive; then there is a negative term; and then the remaining differences are zero. Let us be more specific. We have $df_0 = \phi(0)/2 > 0$ and then there are two possible scenarios:

(a) $df_1 = -\phi(0)/2$ and $df_2 = df_3 = \ldots = 0$; then $(df)^* = \phi(0)/2$ and $\varphi^* = \varphi = \phi(0)$;

(b) $df_1 = \phi(0)/2$. Then there is an integer $m \geq 2$ such that $df_2 = df_3 = \ldots = df_{m-1} = \delta > 0$, $df_m < 0$ and $df_{m+1} = df_{m+2} = \ldots = 0$. In this case, we have $(df)^* = |df_m|$ and $\varphi^* = |\varphi| \geq (df)^*$.

We define the transforming sequence v by $v_0 = \phi(0)^{r'-1}$, $v_1 = -\phi(0)^{r'-1}$ and $v_n = (-1)^n |\varphi_{n-1}|^{r'-1}$ for $n \geq 2$. Obviously, this sequence is predictable and we have $v^* = (\varphi^*)^{r'-1} = |\varphi|^{r'-1}$. To understand the behavior of g, note that in the scenario (a),

$$g = \phi(0)^{r'-1} \cdot \phi(0)/2 - \phi(0)^{r'-1} \cdot (-\phi(0)/2) = \phi(0)^{r'} = |\varphi|^{r'}.$$

On the other hand, in the scenario (b) we have $v_0 df_0 + v_1 df_1 = 0$ and

$$g = v_2 df_2 + v_3 df_3 + \ldots + v_m df_m.$$

But the sequence $(v_n)_{n\geq 0}$ is alternating and $(|v_n|)_{n\geq 0}$ is nondecreasing, while $df_2 = df_3 = \ldots = df_{m-1} = \delta$ and $df_m < 0$. Consequently, $|g| \geq |v_m||df_m| = v^*(df)^* > (1-\varepsilon)v^*\varphi^* = (1-\varepsilon)|\varphi|^{r'}$, if δ is sufficiently small. Putting all these facts together, we obtain the inequality

$$|||g|||_{L^{p,\infty}} \geq \int_\Omega |g| \mathrm{d}\mathbb{P} \geq (1-\varepsilon)\mathbb{E}|\varphi|^{r'} = (1-\varepsilon)\|\varphi\|_{L^{r'}}\|v^*\|_{L^r} \geq (1-\varepsilon)(c_{p,q,r}-\varepsilon)\|f\|_{L^q}\|v^*\|_{L^r},$$

provided δ is sufficiently small. This is precisely the desired claim, since ε can be chosen arbitrarily small.

Chapter 4

Sharp $L^p \to L^{q,\infty}$ estimates for Hilbert and Riesz transforms on compact Lie groups

4.1. Introduction and statement of results

The main objective of this chapter is to study the boundedness of the Hilbert and Riesz transforms as operators from L^p to $L^{q,\infty}$ for $p \neq q$. Obviously, this problem makes sense only if $q < p$, since for $q > p$ the norm is infinite. Note that $\|\mathcal{H}^{\mathbb{R}}\|_{L^p(\mathbb{R}) \to L^{q,\infty}(\mathbb{R})} = \infty$, by a simple dilation argument; therefore, we will assume that the base space has finite measure. That is, we will restrict ourselves to the context of Hilbert transforms on the circle and, more generally, Riesz transforms on compact Lie groups and spheres.

To formulate our main result, we need to introduce an auxiliary object. Given $0 < p < \infty$, let $\omega_p : [0,1] \to [0,\infty)$ be the L^p-modulus of continuity of the function $u \mapsto \frac{1}{\pi}\ln|\tan\frac{\pi u}{4}|$, $u \in (-2,2)$:

$$\omega_p(t) = \left(\frac{1}{4}\int_{-2}^{2}\left|\frac{1}{\pi}\ln\left|\tan\frac{\pi(s+t)}{4}\right| - \frac{1}{\pi}\ln\left|\tan\frac{\pi(s-t)}{4}\right|\right|^p \mathrm{d}s\right)^{1/p}. \tag{4.1.1}$$

We will prove the following fact. We have decided to state it in the classical, periodic setting only; the more general formulation in the presence of compact Lie groups is postponed until Section 4.4.

Theorem 4.1. *For any $1 < q < p < \infty$ we have the sharp estimate*

$$\left\|\left|\mathcal{H}^{\mathbb{T}} f\right|\right\|_{L^{q,\infty}(\mathbb{T})} \leq C_{p,q}\|f\|_{L^p(\mathbb{T})}, \tag{4.1.2}$$

where

$$C_{p,q} = \begin{cases} \left[\dfrac{2^{p'+2}\Gamma(p'+1)}{\pi^{p'+1}}\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{p'+1}}\right]^{1/p'} & \text{if } 1 < p \leq 2, \\ \displaystyle\sup_{0<t\leq 1}\left(t^{-1/q'}\omega_{p'}(t)\right) & \text{if } p > 2. \end{cases}$$

So, we see that if $1 < p \leq 2$, the weak norm does not change if we vary q: we have the identity $C_{p,q} = \|\mathcal{H}^{\mathbb{T}}\|_{L^p(\mathbb{T}) \to L^{p,\infty}(\mathbb{T})}$. On the contrary, for $p > 2$ there is a nontrivial dependence on q. There is a natural question whether, for $p > 2$, the constant can be expressed in a more explicit form, but we believe that this is not possible.

Our approach will rest on the construction of a certain special superharmonic function on the strip $\{z \in \mathbb{C} : |\operatorname{Re} z| \leq 1\}$, which will satisfy an appropriate majorization condition: see Section 4.2. This function, in turn, will allow us to establish a general probabilistic estimate involving orthogonal martingales satisfying the differential subordination: see Section 4.3 for details. In the final part of the chapter we will pass from the probabilistic to the analytic realm. The martingale inequality will yield (4.1.2) and its extension to the directional Riesz transforms on Lie groups and spheres, with the use of stochastic representation of these operators established by Arcozzi [1]. The sharpness of (4.1.2) will be obtained by the construction of the extremal examples; we will also apply a certain transference argument to deduce the optimality of the constant for the d-dimensional torus.

4.2. A special superharmonic function

In our argumentation below, we will often use the identification $\mathbb{C} \simeq \mathbb{R}^2$ and switch from $z = x + iy$ to (x, y) and back; this should not lead to any confusion. Throughout, $a > 0$ and $1 < p \leq 2$ are fixed parameters. Consider the planar domain $D = D_a = ([-1, 1] \times \mathbb{R}) \setminus \{(0, y) : |y| \geq a\}$ and let $H = H_a$ be the map given by

$$H(z) = i\left(\frac{e^{\pi a - i\pi z} - 1}{e^{\pi a} - e^{-i\pi z}}\right)^{1/2}, \qquad z \in \mathbb{C}. \tag{4.2.1}$$

Here we use the following branch of the square root on the complex plane: $(re^{i\varphi})^{1/2} = r^{1/2}e^{i\varphi/2}$, where $r \geq 0$ and $\varphi \in (-\pi, \pi]$. It is easy to check that H is a conformal mapping which sends the interior of D onto the open upper half-plane $\mathbb{R}^2_+ := \mathbb{R} \times (0, \infty)$. Next, let $\mathcal{U} = \mathcal{U}^{p,a} : \mathbb{R}^2_+ \to \mathbb{R}$ be given by the Poisson integral

$$\mathcal{U}(\alpha, \beta) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\beta}{(t-\alpha)^2 + \beta^2}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2}\right|\right|^p - C\chi_{\{|\ln|t|| \leq \pi a/2\}}\right)\mathrm{d}t,$$

where

$$C = C_{p,a} = (4\sinh(\pi a/2))^{-1}\int_{\mathbb{R}}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2}\right|\right|^p - a^p\right)\mathrm{d}t.$$

Obviously, the function $\mathcal{U}$ is harmonic and satisfies the boundary behavior

$$\lim_{\beta \downarrow 0}\mathcal{U}(\alpha, \beta) = \left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}\alpha^2 - 1}{e^{\pi a} - \alpha^2}\right|\right|^p - C\chi_{\{|\ln|\alpha|| \leq \pi a/2\}} \tag{4.2.2}$$

for $\alpha \in \mathbb{R} \setminus \{\pm e^{\pm \pi a/2}, 0\}$. Finally, let U be a function defined on the interior of D by the formula

$$U(x, y) = \mathcal{U}(H(x, y)).$$

Then U is harmonic, being the composition of a harmonic function with a conformal mapping. Furthermore, by (4.2.2), we have

$$\lim_{(x,y) \to (\pm 1, u)} U(x, y) = |u|^p - C$$

and

$$\lim_{(x,y) \to (0, u)} U(x, y) = |u|^p \qquad \text{for } |u| \geq a.$$

In other words, U is the continuous solution to the Dirichlet problem

$$\begin{cases} \Delta U = 0 & \text{inside } D, \\ U(x, y) = |y|^p - C|x| & \text{for } (x, y) \in \partial D. \end{cases}$$

In particular, the function U satisfies the symmetry condition

$$U(x, y) = U(|x|, |y|) \qquad \text{for } (x, y) \in D \tag{4.2.3}$$

(this can be also proved directly, by performing appropriate substitutions in the integral defining $\mathcal{U}$).

The function U is of fundamental importance to our considerations. The remaining part of this section is devoted to the study of the properties of U which will be needed later. We start with a technical lemma.

Lemma 4.2. *We have*

$$\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{t^4+3t^2\beta^2}{(t^2+\beta^2)^2}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)dt=\frac{2pa^{p-1}(e^{\pi a}-e^{-\pi a})}{\pi}.$$

Proof. We will use twice the following simple property of the Poisson integral: if $f:\mathbb{R}\to\mathbb{R}$ is a locally integrable function satisfying $\lim_{x\to\pm\infty}f(x)=M$, then

$$\lim_{\beta\to\infty}\frac{1}{\pi}\int_{\mathbb{R}}\frac{\beta f(t)}{t^2+\beta^2}\mathrm{d}t=M. \tag{4.2.4}$$

This implies

$$\begin{aligned}\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{3t^2}{t^2+\beta^2}&\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\mathrm{d}t\\ &=\lim_{t\to\infty}3t^2\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right).\end{aligned} \tag{4.2.5}$$

Furthermore, integrating by parts and applying (4.2.4) again, we obtain

$$\begin{aligned}-\lim_{\beta\to\infty}\frac{2\beta}{\pi}\int_{\mathbb{R}}\frac{t^4}{(t^2+\beta^2)^2}&\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\mathrm{d}t\\ &=-\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{1}{t^2+\beta^2}\left\{t^3\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\right\}'\mathrm{d}t\\ &=-\lim_{t\to\infty}\left\{t^3\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\right\}',\end{aligned}$$

which added to (4.2.5) gives

$$\begin{aligned}\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{t^4+3t^2\beta^2}{(t^2+\beta^2)^2}&\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\mathrm{d}t\\ &=-\lim_{t\to\infty}t^3\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)'=\frac{2pa^{p-1}(e^{\pi a}-e^{-\pi a})}{\pi}.\end{aligned}$$

□

In the lemma below, we establish an appropriate "smooth-fit" property at the point $(0,a)$.

Lemma 4.3. *We have* $\lim_{y\uparrow a}U_y(0,y)=pa^{p-1}$.

Proof. The equality follows from the definition of C. Observe that $U(0,y)=\mathcal{U}(H(0,y))$ and

$$H(0,y)=i\left(\frac{e^{\pi a+\pi y}-1}{e^{\pi a}-e^{\pi y}}\right)^{1/2}$$

is purely imaginary. Consequently,

$$U_y(0,y)=\mathcal{U}_\beta\left(0,\left(\frac{e^{\pi a+\pi y}-1}{e^{\pi a}-e^{\pi y}}\right)^{1/2}\right)\cdot H_y(0,y).$$

For brevity, let us denote $\left(\frac{e^{\pi a+\pi y}-1}{e^{\pi a}-e^{\pi y}}\right)^{1/2}$ by β. Clearly, when y increases to a, then β tends to infinity. Furthermore, we compute directly that

$$H_y(0,y)\sim\frac{\pi}{2}\frac{(e^{\pi a+\pi y}-1)^{1/2}}{(e^{\pi a}-e^{\pi y})^{3/2}}\cdot e^{\pi y}\sim\frac{\pi e^{\pi a}}{2(e^{2\pi a}-1)}\beta^3,$$

where the symbol $\sim$ above means that the ratio of the expressions on both sides of it tends to 1 as $y\uparrow a$. Consequently, we see that

$$\lim_{y\uparrow a}U_y(0,y)=\frac{\pi e^{\pi a}}{2(e^{2\pi a}-1)}\lim_{\beta\to\infty}U_\beta(0,\beta)\beta^3.$$

By the definition of $\mathcal{U}$, we compute that $\mathcal{U}(0,\beta)$ equals

$$\frac{1}{\pi}\int_{\mathbb{R}}\frac{\beta}{t^2+\beta^2}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p\mathrm{d}t-\frac{2C}{\pi}\left[\arctan(e^{\pi a/2}/\beta)-\arctan(e^{-\pi a/2}/\beta)\right].$$

A direct differentiation with respect to β yields

$$\left(\arctan(e^{\pi a/2}/\beta)\right)'=-\frac{e^{\pi a/2}}{e^{\pi a}+\beta^2}=-\frac{e^{\pi a/2}}{\beta^2}+O(\beta^{-4})$$

and

$$\left(\arctan(e^{-\pi a/2}/\beta)\right)'=-\frac{e^{-\pi a/2}}{e^{-\pi a}+\beta^2}=-\frac{e^{-\pi a/2}}{\beta^2}+O(\beta^{-4}),$$

which implies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\beta}&\left\{\frac{2C}{\pi}\left[\arctan(e^{\pi a/2}/\beta)-\arctan(e^{-\pi a/2}/\beta)\right]\right\}\\&=\frac{2C(-e^{\pi a/2}+e^{-\pi a/2})}{\pi\beta^2}+O(\beta^{-4}).\end{aligned}$$

Furthermore,

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}\beta}\left\{\frac{1}{\pi}\int_{\mathbb{R}}\frac{\beta}{t^2+\beta^2}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p\mathrm{d}t\right\}\\&=\frac{\mathrm{d}}{\mathrm{d}\beta}\left\{\frac{1}{\pi}\int_{\mathbb{R}}\frac{\beta}{t^2+\beta^2}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\mathrm{d}t\right\}\\&=\frac{1}{\pi}\int_{\mathbb{R}}\frac{t^2-\beta^2}{(t^2+\beta^2)^2}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\mathrm{d}t=I_1+I_2,\end{aligned}$$

where

$$I_1=-\frac{1}{\beta^2}\cdot\frac{1}{\pi}\int_{\mathbb{R}}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\mathrm{d}t=\frac{2C(-e^{\pi a/2}+e^{-\pi a/2})}{\pi\beta^2}$$

and

$$I_2=\frac{1}{\beta^2}\cdot\frac{1}{\pi}\int_{\mathbb{R}}\frac{t^4+3t^2\beta^2}{(t^2+\beta^2)^2}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\mathrm{d}t.$$

Putting the above facts together, we obtain

$$\lim_{\beta\to\infty}\mathcal{U}_\beta(0,\beta)\beta^3=\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{t^4+3t^2\beta^2}{(t^2+\beta^2)^2}\left(\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p-a^p\right)\mathrm{d}t.$$

It remains to use the previous lemma to get the claim. □

Lemma 4.4. *We have $U_y(x,y) \le py^{p-1}$ for $x \in [-1,1]$ and $y \ge 0$.*

Proof. Fix an arbitrary point (x,y) belonging to the half-strip $(0,1)\times\mathbb{R}$. The function U is continuous on $[0,1]\times\mathbb{R}$, so we have

$$U(x,y) = \int_{\{0,1\}\times\mathbb{R}} U(u,v)\mathrm{d}\mu_{x,y}(u,v),$$

where $\mu_{x,y}$ is the harmonic measure on $\{0,1\}\times\mathbb{R}$ with respect to the point (x,y). Since $[0,1]\times\mathbb{R}$ is invariant with respect to vertical translations, we also have

$$U(x,y+h) = \int_{\{0,1\}\times\mathbb{R}} U(u,v+h)\mathrm{d}\mu_{x,y}(u,v)$$

and hence, by Lebesgue's dominated convergence theorem,

$$U_y(x,y) = \lim_{h\to 0}\frac{U(x,y+h)-U(x,y)}{h} = \int_{\{0,1\}\times\mathbb{R}} U_y(u,v)\mathrm{d}\mu_{x,y}(u,v).$$

Since $y\mapsto U_y(0,y)$ and $y\mapsto U_y(1,y)$ are continuous, we conclude that U_y extends to a continuous function on $[0,1]\times\mathbb{R}$, and hence also to a continuous function on $[-1,1]\times\mathbb{R}$. Now, consider the upper half-strip $S^+ = ((-1,1)\times(0,\infty))\setminus\{(0,y): y\ge a\}$. The crucial observation is that on the boundary of S^+, U_y coincides with the function $W(x,y)=py^{p-1}$, which is superharmonic in the interior of S^+. Indeed, the equalities $U_y(\pm 1,y)=py^{p-1}$ and $U_y(0,y)=py^{p-1}$ for $y\ge a$ are obvious, while $U_y(x,0)=0$ follows from the symmetry condition (4.2.3). Since U_y is continuous on S^+, we obtain $U_y\le W$ on S^+, which completes the proof. □

Lemma 4.5. *We have $U_{yy}\ge 0$ in the interior of D and $\lim_{y\downarrow a} U_x(0+,y) = U_x(0,a)=0$.*

Proof. Fix arbitrary (x,y), $(x,y+\delta)\in D$, where $\delta\in(0,a)$ is a small positive number. Consider the auxiliary domain $\mathcal{D}_{a,\delta} = ((-1,1)\times\mathbb{R})\setminus\{(0,v): v\ge a-\delta \text{ or } v\le -a\}$. Since $\mathcal{D}_{a,\delta}$ and its translation $i\delta+\mathcal{D}_{a,\delta}$ are contained in the interior of D, we may write

$$U_y(x,y) = \int_{\partial\mathcal{D}_{a,\delta}} U_y(u,v)\mathrm{d}\mu_{x,y}^{\mathcal{D}_{a,\delta}}(u,v)$$

and

$$U_y(x,y+\delta) = \int_{\partial\mathcal{D}_{a,\delta}} U_y(u,v+\delta)\mathrm{d}\mu_{x,y}^{\mathcal{D}_{a,\delta}}(u,v),$$

so

$$U_y(x,y+\delta)-U_y(x,y) = \int_{\partial\mathcal{D}_{a,\delta}} \big(U_y(u,v+\delta)-U_y(u,v)\big)\mathrm{d}\mu_{x,y}^{\mathcal{D}_{a,\delta}}(u,v).$$

But the integrand is nonnegative. Indeed, if $u=\pm1$, or $u=0$ and $|v+\delta|,|v|\ge a$, then $U_y(u,v+\delta)-U_y(u,v) = p|v+\delta|^{p-1}\operatorname{sgn}(v+\delta) - p|v|^{p-1}\operatorname{sgn}(v)\ge 0$. If $u=0$ and $v+\delta\ge a>v$, then $v\ge 0$ (here we use the assumption $\delta<a$) and by the previous lemma,

$$U_y(u,v+\delta)-U_y(u,v) \ge p|v+\delta|^{p-1}\operatorname{sgn}(v+\delta) - p|v|^{p-1}\operatorname{sgn}(v)\ge 0. \tag{4.2.6}$$

Finally, if $v+\delta>-a\ge v$, then by the symmetry of U we have $U_y(u,v+\delta)-U_y(u,v) = U_y(u,-v)-U_y(u,-v-\delta)\ge 0$, by (4.2.6). Consequently, we have shown that for each $x\in[-1,1]$, the function $U_y(x,\cdot)$ is nondecreasing, which yields the first part of the claim.

To handle the second part, note that $U_{xx} = -U_{yy} \leq 0$ in the interior of D. Furthermore, by the symmetry condition (4.2.3), we have $U_x(0, y) = 0$ for $|y| < a$. These two facts imply $U(x, y) \leq U(0, y)$ for all $x \in [-1, 1]$ and $y \in (-a, a)$, which, by the continuity of U, is also true for $y = a$. This gives $U_x(0+, a) \leq 0$; to see that both sides are equal, note that if $U(\cdot, a)$ had a concave cusp at $x = 0$, then we would have $\lim_{y\uparrow a} U_y(0, y) = \infty$, by elementary facts about harmonic functions. This proves that $U_x(0, a) = 0$.

Next, we will show that the function $y \mapsto U_x(0+, y)$ is nonincreasing on $[a, \infty)$. Pick $y' > y > a$. By the previous lemma we may write, for any $x \in (0, 1)$,

$$\begin{aligned}
&\frac{U(x, y') - U(0, y')}{x} \\
&\quad = \frac{U(x, y') - U(x, y) + U(x, y) - U(0, y) + U(0, y) - U(0, y')}{x} \\
&\quad \leq \frac{\int_y^{y'} ps^{p-1}\mathrm{d}s + U(x, y) - U(0, y) + y^p - (y')^p}{x} \\
&\quad = \frac{U(x, y) - U(0, y)}{x}.
\end{aligned}$$

Hence, letting $x \downarrow 0$ gives the desired monotonicity $U_x(0+, y') \leq U_x(0, y)$; in particular, this shows that the limit $\lim_{y\downarrow a} U_x(0+, y)$ exists and is at most zero. However, if we had $\lim_{y\downarrow a} U_x(0+, y) = M < 0$, then we would have $U_x(0+, y) \leq M$ for all $y > a$ and the estimate $U_{xx} \leq 0$ would imply that

$$U(x, y) \leq U(0, y) + U_x(0+, y)x \leq y^p + Mx$$

for all $y > a$ and $x \in (0, 1)$. Letting $y \downarrow a$, we would obtain $U_x(0+, a) \leq M$, a contradiction. □

Remark 4.1. The second half of the above lemma can be computed directly. Let us briefly outline the proof. We start from the observation that if $x \downarrow 0$, then

$$H(x, a) \sim \left(\frac{(e^{2\pi a} - 1)^2}{2e^{2\pi a}(1 - \cos \pi x)}\right)^{1/4} \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \sim (\beta, \beta),$$

for $\beta = \left(\frac{\sinh a}{\pi x}\right)^{1/2} \to \infty$. A direct differentiation shows that both the real and the imaginary parts of $H_x(x, a)$ are equal and behave like β^3, up to a universal multiplicative constant. Consequently, it is enough to show that

$$\lim_{\beta\to\infty} (\mathcal{U}_x(\beta, \beta) + \mathcal{U}_y(\beta, \beta))\beta^3 = 0. \tag{4.2.7}$$

For simplicity, denote $g(t) = \left|\frac{1}{\pi} \ln \left|\frac{e^{\pi a}t^2 - 1}{e^{\pi a} - t^2}\right|\right|^p - a^p$ and note that $K = \lim_{t\to\infty} t^2 g(t)$ is finite. Some tedious, but rather straightforward computations reveal that

$$\begin{aligned}
\mathcal{U}_x(\beta, \beta) + \mathcal{U}_y(\beta, \beta) &= \frac{1}{\pi} \int_{\mathbb{R}} \frac{(\beta - t)^2 - \beta^2 + 2\beta(t - \beta)}{((\beta - t)^2 + \beta^2)^2} g(t)\mathrm{d}t \\
&\quad - \frac{C}{\pi}\left[- \frac{e^{\pi a/2}}{\beta^2 + (\beta - e^{\pi a/2})^2} + \frac{e^{-\pi a/2}}{\beta^2 + (\beta - e^{-\pi a/2})^2}\right. \\
&\qquad \left. + \frac{e^{-\pi a/2}}{\beta^2 + (\beta + e^{-\pi a/2})^2} - \frac{e^{\pi a/2}}{\beta^2 + (\beta + e^{\pi a/2})^2}\right] \\
&= \frac{1}{\pi} \int_{\mathbb{R}} \frac{t^2 - 2\beta^2}{((\beta - t)^2 + \beta^2)^2} g(t)\mathrm{d}t + \frac{C}{\pi}\left[2\beta^{-2} \sinh \frac{\pi a}{2} + O(\beta^{-4})\right]
\end{aligned}$$

and hence, by the very definition of C, the limit in (4.2.7) equals

$$\lim_{\beta\to\infty}\frac{\beta^3}{\pi}\int_{\mathbb{R}}\left(\frac{t^2-2\beta^2}{((\beta-t)^2+\beta^2)^2}+\frac{1}{2\beta^2}\right)g(t)\mathrm{d}t$$
$$=\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{t^4-4\beta t^3+10\beta^2t^2}{((\beta-t)^2+\beta^2)^2}g(t)\mathrm{d}t-\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{8t\beta^3}{((\beta-t)^2+\beta^2)^2}g(t)\mathrm{d}t=I_1-I_2.$$

As for the I_1, the calculations are simple and similar to Lemma 4.2. The substitution $t=\beta s$ yields

$$\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{t^4-4\beta t^3+10\beta^2t^2}{((\beta-t)^2+\beta^2)^2}g(t)\mathrm{d}t=\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{t^2-4\beta t+10\beta^2}{((\beta-t)^2+\beta^2)^2}\left[t^2g(t)\right]\mathrm{d}t$$
$$=\frac{1}{\pi}\int_{\mathbb{R}}\frac{s^2-4s+10}{((1-s)^2+1)^2}\left[(s\beta)^2g(s\beta)\right]\mathrm{d}s.$$

It is not difficult to see that we can pull the limit inside the integral, obtaining $I_1=K\frac{1}{\pi}\int_{\mathbb{R}}\frac{s^2-4s+10}{((1-s)^2+1)^2}\mathrm{d}s=4K$. Next, we rewrite the expression I_2 in the form

$$I_2=\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}}\frac{8\beta^3}{t((\beta-t)^2+\beta^2)^2}\left[t^2g(t)\right]\mathrm{d}t.$$

If we bound the integral away from the singularity point 0, then we perform calculations similar to those above, obtaining

$$\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]}\frac{8\beta^3}{t((\beta-t)^2+\beta^2)^2}\left[t^2g(t)\right]\mathrm{d}t$$
$$=\lim_{\beta\to\infty}\frac{1}{\pi}\int_{\mathbb{R}\setminus[-\varepsilon/\beta,\varepsilon/\beta]}\frac{8}{s((1-s)^2+1)^2}\left[(s\beta)^2g(s\beta)\right]\mathrm{d}s$$
$$=\text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}}\frac{8K}{s((1-s)^2+1)^2}\mathrm{d}s=4K.$$

Near the singularity point, it is enough to notice that $t^2g(t)=O(t^4)$ (as $t\to 0$), so

$$\lim_{\beta\to\infty}\frac{\beta}{\pi}\int_{-\varepsilon}^{\varepsilon}\frac{8\beta^3}{t((\beta-t)^2+\beta^2)^2}\left[t^2g(t)\right]\mathrm{d}t=0.$$

Hence, subtracting I_1 from I_2, we obtain that the limit in (4.2.7) equals $4K-4K=0$.

The following statement is the main result of the section.

Theorem 4.6. *The function U is a superharmonic majorant of the function $V:D\to\mathbb{R}$ given by $V(x,y)=|y|^p-C|x|$.*

Proof. To show the superharmonicity, fix an arbitrary ball $K\subset[-1,1]\times\mathbb{R}$ of center (x,y) and radius r. Let $W=(W^{(1)},W^{(2)})$ be a two-dimensional Brownian motion started at (x,y) and stopped upon reaching the boundary of K. The function U is of class C^2 on $(0,1)\times\mathbb{R}$, of class C^1 on $[0,1]\times\mathbb{R}$ and satisfies (4.2.3), so Itô's formula gives

$$U(W_t)=U(|W_t^{(1)}|,W_t^{(2)})=U(|W_0^{(1)}|,W_0^{(2)})+I_1+\frac{1}{2}I_2, \tag{4.2.8}$$

where

$$I_1=\int_0^t U_x(|W_s^{(1)}|,W_s^{(2)})\mathrm{d}|W^{(1)}|_s+\int_0^t U_y(|W_s^{(1)}|,W_s^{(2)})\mathrm{d}W_s^{(2)},$$

and
$$I_2 = \int_0^t \Delta U(|W_s^{(1)}|, W_s^{(2)})\mathrm{d}s.$$

Note that $\mathrm{d}|W^{(1)}|_s = \operatorname{sgn}(W_s^{(1)})\mathrm{d}W_s^{(1)} + \mathrm{d}\ell_s$, where ℓ is the local time of $W^{(1)}$ at zero. Since the local time is a monotone process which increases on the set $\{t : W_t^{(1)} = 0\}$ and $U_x(0+, y) \le 0$ for all y, we have

$$\begin{aligned}\int_0^t U_x(|W_s^{(1)}|, W_s^{(2)})\mathrm{d}|W^{(1)}|_s &= \int_0^t U_x(|W_s^{(1)}|, W_s^{(2)}) \operatorname{sgn}(W_s^{(1)})\mathrm{d}W_s^{(1)} + \int_0^t U_x(0, W_s^{(2)})\mathrm{d}\ell_s \\ &\le \int_0^t U_x(|W_s^{(1)}|, W_s^{(2)}) \operatorname{sgn}(W_s^{(1)})\mathrm{d}W_s^{(1)}.\end{aligned}$$

But the latter integral, as well as the second integral in I_1, has zero expectation: this follows at once from the properties of stochastic integrals. Finally, I_2 vanishes, since U is harmonic inside $[0, 1] \times \mathbb{R}$. Thus, taking the expectation in (4.2.8), we obtain $\mathbb{E}U(W_t) \le U(|W_0^{(1)}|, W_0^{(2)}) = U(x, y)$. Letting $t \to \infty$ yields the superharmonicity, since the random variable W_∞ is uniformly distributed at the boundary of K.

Concerning the majorization $U(x, y) \ge V(x, y)$, let us first show it for $x \in \{0, 1\}$ and $y \ge 0$. We have $U(1, y) = V(1, y)$ for all y, and $U(0, y) = V(0, y)$ for $|y| \ge a$. The estimate $U(0, y) \ge V(0, y)$, for $y \in [0, a]$, follows at once from the equality $U_y(0, a) = V_y(0, a)$ (see Lemma 4.3) and the estimate $U_y(0, y) \le py^{p-1}$ proved in Lemma 4.4. Now we extend the majorization to $x \in \{0, 1\}$ and $y \in \mathbb{R}$, using the symmetry of U and V. Since U is harmonic on $[0, 1] \times \mathbb{R}$ and V is subharmonic on this strip, we deduce the estimate $U \ge V$ on $[0, 1] \times \mathbb{R}$; finally, using the symmetry with respect to the variable x, we obtain the majorization on the full range. □

4.3. Martingale inequalities

Now we will exploit the function U constructed in the previous section to obtain an appropriate stochastic version of (4.1.2). For the sake of convenience, we have decided to split the contents into two separate parts.

4.3.1. Inequalities in the classical context

Let us introduce necessary tools; assume that X, Y are two adapted real-valued cádlág martingales. Here is our main probabilistic result, which can be regarded as the dual to (4.1.2) in the range $1 < p \le 2$. Recall the function ω_p defined in (4.1.1).

Theorem 4.7. *Assume that X, Y are orthogonal martingales such that Y is differentially subordinate to X, $\|X\|_\infty \le 1$ and $Y_0 = 0$. Then for any $1 < p \le 2$ we have the estimate*

$$\|Y\|_{L^p} \le \omega_p\big(\|X\|_{L^1}\big). \tag{4.3.1}$$

The inequality is sharp in the following sense: for any $T \in [0, 1]$ there is a pair X, Y as above with $\|X\|_{L^1} = T$ and $\|Y\|_{L^p} = \omega_p(T)$.

Proof. Fix a parameter $a > 0$; its value will be specified in a moment. The reasoning rests on Itô's formula, applied to the composition of U with the two-dimensional martingale (X, Y). However, since U is not of class C^2, we need an additional mollification argument to guarantee the regularity. Let $g : \mathbb{R}^2 \to [0,\infty)$ be a C^∞ radial function, supported on the unit ball and satisfying $\int_{\mathbb{R}^2} g = 1$. For any $\delta > 0$ define $U^\delta = U^{a,\delta} : [-1,1] \times \mathbb{R} \to \mathbb{R}$ by the convolution

$$U^\delta(x,y) = \int_{[-1,1]^2} U((1-\delta)x + \delta u, (1-\delta)y + \delta v) g(u,v)\mathrm{d}u\mathrm{d}v.$$

This function is superharmonic and inherits the convexity with respect to the variable y. Furthermore, directly by the definition of U^δ, we have

$$U^\delta(x,0) \leq U((1-\delta)x,0) \leq U(0,0). \tag{4.3.2}$$

Indeed, the first inequality holds since U is superharmonic and g is radial, while the second passage follows from (4.2.3) and the concavity of $U(\cdot,0)$. Finally, by the majorization property established in Theorem 4.6, we easily check that

$$U^\delta(x,y) \geq \big|(1-\delta)|y| - \delta\big|^p - C(1-\delta)|x| - C\delta. \tag{4.3.3}$$

Now fix a stopping time (it will be specified in a moment) and apply Itô's formula to $U^\delta(X_{\tau\wedge t}, Y_{\tau\wedge t})$. As the result, we obtain

$$U^\delta(X_{\tau\wedge t}, Y_{\tau\wedge t}) = I_0 + I_1 + \frac{1}{2}I_2 + I_3, \tag{4.3.4}$$

where

$$\begin{aligned}
I_0 &= U^\delta(X_0, Y_0),\\
I_1 &= \int_{0+}^{\tau\wedge t} U_x^\delta(X_{s-}, Y_s)\mathrm{d}X_s^c + \int_{0+}^{\tau\wedge t} U_y^\delta(X_{s-}, Y_s)\mathrm{d}Y_s,\\
I_2 &= \int_{0+}^{\tau\wedge t} U_{xx}^\delta(X_{s-}, Y_s)\mathrm{d}[X,X]_s^c + \int_{0+}^{\tau\wedge t} U_{yy}^\delta(X_{s-}, Y_s)\mathrm{d}[Y,Y]_s,\\
I_3 &= \sum_{0<s\leq\tau\wedge t} \Big[U^\delta(X_s, Y_s) - U^\delta(X_{s-}, Y_s) - U_x^\delta(X_{s-}, Y_s)\Delta X_s\Big].
\end{aligned}$$

Here in I_2 the summand $2\int_{0+}^{\tau\wedge t} U_{xy}^\delta(X_{s-}, Y_s)\mathrm{d}[X^c, Y]_s$ is not present, since X^c and Y are orthogonal, by Lemma 3.4. This lemma implies also that the martingale Y has continuous paths, so we write Y_s instead of Y_{s-} under the integrals, and we do not have the jump term $U_y^\delta(X_{s-}, Y_s)\Delta Y_s$ in I_3.

Let us study the properties of I_0-I_3. First, by the assumption $Y_0 = 0$ and the estimate (4.3.2), we have $I_0 = U^\delta(X_0, 0) \leq U(0,0)$. By the general theory of stochastic integrals, both processes

$$\left(\int_{0+}^t U_x^\delta(X_{s-}, Y_s)\mathrm{d}X_s^c\right)_{t\geq 0}, \qquad \left(\int_{0+}^t U_y^\delta(X_{s-}, Y_s)\mathrm{d}Y_s\right)_{t\geq 0}$$

are local martingales. Let $(\tau_n)_{n\geq 0}$ be some localizing sequence for them and put $\tau = \tau_n$ for some n. Then the two integrals in I_1 have zero expectation. Next, the differential subordination of Y to X implies $\mathrm{d}[Y,Y] \leq \mathrm{d}[X,X]^c$ (again, see Lemma 3.4), so the estimate $U_{yy}^\delta \geq 0$ gives

$$I_2 \leq \int_{0+}^{\tau\wedge t} \Delta U^\delta(X_{s-}, Y_s)\mathrm{d}[X,X]_s^c \leq 0,$$

since U^δ is superharmonic. Finally, each summand in I_3 is nonpositive: this follows at once from the mean-value theorem and the fact that $U^\delta_{xx} \leq -U^\delta_{yy} \leq 0$. Therefore, taking the expectation of both sides of (4.3.4) and using all the above observations, we obtain $\mathbb{E}U^\delta(X_{\tau\wedge t}, Y_{\tau\wedge t}) \leq U(0,0)$. Combining this with (4.3.3), we get

$$\mathbb{E}|(1-\delta)|Y_{\tau\wedge t}| - \delta|^p \leq U(0,0) + C(1-\delta)\mathbb{E}|X_{\tau\wedge t}| + C\delta.$$

Now we let $\tau = \tau_n \to \infty$ and $t \to \infty$. Since X is bounded, both X and Y converge pointwise and in L^2 to some terminal variables, say, X_∞ and Y_∞. Therefore, we obtain

$$\mathbb{E}|(1-\delta)|Y_\infty| - \delta|^p \leq U(0,0) + C(1-\delta)\mathbb{E}|X_\infty| + C\delta$$

and letting $\delta \to 0$ gives

$$\mathbb{E}|Y_\infty|^p \leq U(0,0) + C\mathbb{E}|X_\infty|.$$

It is high time to specify a: we plug $a = \frac{2}{\pi}\ln\left(\tan\left(\frac{\pi}{4}\left(\mathbb{E}|X_\infty|+1\right)\right)\right)$, obtaining

$$\begin{aligned} U(0,0) &= \mathcal{U}(0,1) \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t^2+1}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p \mathrm{d}t - \frac{2C}{\pi}\left[\arctan e^{\pi a/2} - \arctan e^{-\pi a/2}\right] \\ &= \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t^2+1}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p \mathrm{d}t - C\mathbb{E}|X_\infty|. \end{aligned}$$

Hence,

$$\mathbb{E}|Y_\infty|^p \leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t^2+1}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{e^{\pi a}-t^2}\right|\right|^p \mathrm{d}t = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}\tan^2 s-1}{e^{\pi a}-\tan^2 s}\right|\right|^p \mathrm{d}s.$$

But by the definition of a, we have $e^{\pi a/2} = \tan\left(\frac{\pi}{4}\left(\mathbb{E}|X_\infty|+1\right)\right)$, so

$$\begin{aligned} \frac{e^{\pi a}\tan^2 s - 1}{e^{\pi a}-\tan^2 s} &= \frac{e^{\pi a/2}\tan s - 1}{e^{\pi a/2}-\tan s}\cdot\frac{e^{\pi a/2}\tan s + 1}{e^{\pi a/2}+\tan s} \\ &= \tan\left(\frac{\pi}{4}(\mathbb{E}|X_\infty|+1)+s\right)\cdot\tan\left(\frac{\pi}{4}(\mathbb{E}|X_\infty|+1)-s\right). \end{aligned}$$

The latter expression, considered as a function of s, is π-periodic. Therefore, plugging it above and substituting $s := s + \frac{\pi}{4}$ in the integral, we get

$$\mathbb{E}|Y_\infty|^p \leq \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\left|\Psi\left(s+\frac{\pi}{4}\mathbb{E}|X_\infty|\right) - \Psi\left(s-\frac{\pi}{4}\mathbb{E}|X_\infty|\right)\right|^p \mathrm{d}s,$$

where $\Psi(u) = \frac{1}{\pi}\ln|\tan u|$. By a simple change of variables in the latter expression, we obtain $\mathbb{E}|Y_\infty|^p \leq \left(\omega_p\left(\mathbb{E}|X_\infty|\right)\right)^p$. This is the desired estimate (4.3.1), since $\|Y\|_{L^p} = (\mathbb{E}|Y_\infty|^p)^{1/p}$, by the L^2-boundedness of Y.

The sharpness will follow from the results of the next section. See Remark 4.2 below.

□

4.3.2. Inequalities for martingales on manifolds

Now we will extend the above results to the context of manifolds. We will use the notation introduced and discussed in Section 1.1 above. Here is an appropriate version of Theorem 4.7.

Theorem 4.8. *Let $\mathcal{K}$ be a bounded, continuous, T^*M-valued process above B. Assume that A is a martingale transformer satisfying the conditions $\|A\| \leq 1$ and $\langle A_t(\omega)\xi, \xi\rangle = 0$ for all $t \geq 0$, $\omega \in \Omega$ and $\xi \in T^*_{B_t(\omega)}M$. Then we have the estimate*

$$\|A * I_{\mathcal{K}}\|_{L^p} \leq \omega_p(\|I_{\mathcal{K}}\|_{L^1}), \qquad 1 < p \leq 2. \tag{4.3.5}$$

Proof. This is a simple application of Theorem 4.7, we only need to verify the differential subordination and orthogonality for the martingales $A * I_{\mathcal{K}}$ and $\|A\|I_{\mathcal{K}}$. Pick $t \geq 0$, $\omega \in \Omega$ and let $x = B_t(\omega) \in M$. Suppose that e_1, e_2, ..., e_n is an orthonormal basis for T_xM, the tangent space to M at x. Then we have

$$\begin{aligned}\text{Trace}\big(A\mathcal{K}_t(\omega) \otimes A\mathcal{K}_t(\omega)\big) &= \sum_{k=1}^n \big(A\mathcal{K}_t(\omega) \otimes A\mathcal{K}_t(\omega)\big)(e_k, e_k) \\ &= \sum_{k=1}^n \big| < A\mathcal{K}_t(\omega), e_k > \big|^2 = |A\mathcal{K}_t(\omega)|^2,\end{aligned}$$

where $< \cdot, \cdot >: T_x^*M \times T_xM \to \mathbb{R}$ is the duality bracket. Consequently, the identity (1.1.4) gives that for any $0 \leq s < t$,

$$\begin{aligned}[A * I_{\mathcal{K}}, A * I_{\mathcal{K}}]_t - [A * I_{\mathcal{K}}, A * I_{\mathcal{K}}]_s &= \int_{s+}^t \text{Trace}(A\mathcal{K}_u \otimes A\mathcal{K}_u)\mathrm{d}u \\ &= \int_{s+}^t |A\mathcal{K}_u|^2 \mathrm{d}u \\ &\leq \|A\|^2 \int_{s+}^t |\mathcal{K}_u|^2 \mathrm{d}u \\ &= [\|A\|I_{\mathcal{K}}, \|A\|I_{\mathcal{K}}]_t - [\|A\|I_{\mathcal{K}}, \|A\|I_{\mathcal{K}}]_s.\end{aligned}$$

which is the desired differential subordination. The proof of the orthogonality is analogous: one shows that $\text{Trace}(A\mathcal{K}_t(\omega) \otimes \mathcal{K}_t(\omega)) = 0$ for all t and ω, which yields $\mathrm{d}[A * I_{\mathcal{K}}, \|A\|I_{\mathcal{K}}] = 0$, directly from (1.1.4). □

4.4. Inequalities for Riesz transforms

4.4.1. Riesz transforms on Lie groups

The next step of our analysis is to apply the above martingale inequalities to obtain weak-type bounds for directional Riesz transforms in the context of compact Lie groups. In particular, if the group is equal to $\mathbb{T}$, then we get the sharp weak-type estimate (4.1.2) for the periodic Hilbert transform; the specification to the group $G = \mathbb{T}^d$ will yield the estimates for the directional Riesz transforms on the torus.

Theorem 4.9. *For any $1 < q < p < \infty$ and any j we have*

$$\|R_j\|_{L^p(G) \to L^{q,\infty}(G)} \leq C_{p,q}. \tag{4.4.1}$$

Proof. If $1 < p \leq 2$, then $\|R_j\|_{L^p(G) \to L^{q,\infty}(G)} \leq \|R_j\|_{L^p(G) \to L^{p,\infty}(G)} \leq C_{p,q}$, where the last estimate was established in [6]. Hence, it is enough to show the claim for $p > 2$. Fix a function $f \in C^\infty(G)$ bounded by 1 and consider the martingale transformer A^j; obviously, we have $\|A^j\| = 1$. By the inequality (4.3.5), we get

$$\|A^j * I_{dF}\|_{L^{p'}(G)} \leq \omega_{p'}(\|I_{dF}\|_{L^1(G)}),$$

which by Jensen's inequality and Theorem 1.3 implies

$$\|R_j f\|_{L^{p'}(G)} \leq \omega_{p'}(\|f\|_{L^1(G)}). \tag{4.4.2}$$

By a standard approximation, this result continues to hold if we skip the regularity and assume only that f is a function bounded by 1.

To deduce the assertion, we need an appropriate duality argument. Consider the decomposition of $L^2(G) = \bigoplus_{k=1}^\infty \mathcal{H}_k$ into eigenspaces for Δ_G, provided by Peter-Weyl theorem [45]. Thus, $\mathcal{H}_k \subset C_0^\infty(G)$ and $\Delta_G f = -\mu_k f$ for $f \in \mathcal{H}_k$, where $0 < \mu_1 < \mu_2 < \ldots$ is the sequence of eigenvalues of $-\Delta_G$. Fix $f = \sum_{k=1}^N f_k$, with $f_k \in \mathcal{H}_k$, $k = 1, 2, \ldots, N$, and put $g = \chi_E R_j f / |R_j f|$ ($g = \chi_E$ if the denominator is zero) for any subset E in G. Let $g = \sum_{k=1}^\infty g_k$ be the decomposition of g, with $g_k \in \mathcal{H}_k$ for each k. The metric on G is bi-invariant, so Δ_G commutes with all X_j and hence $\int_G R_j f_k g_m \mathrm{d}x = 0$ for $k \neq m$. Therefore, integrating by parts, we get

$$\begin{aligned}
\int_E |R_j f(x)| \mathrm{d}x &= \int_G R_j f(x)\, g(x)\, \mathrm{d}x \\
&= \sum_{k=1}^N \int_G R_j f_k(x)\, g_k(x)\, \mathrm{d}x \\
&= \sum_{k=1}^N \frac{1}{\sqrt{\mu_k}} \int_G X_j f_k(x)\, g_k(x)\, \mathrm{d}x \\
&= -\sum_{k=1}^N \frac{1}{\sqrt{\mu_k}} \int_G f_k(x)\, X_j g_k(x)\, \mathrm{d}x \\
&= -\sum_{k=1}^N \int_G f_k(x)\, R_j g_k(x)\, \mathrm{d}x = -\int_G f(x) R_j g(x)\, \mathrm{d}x.
\end{aligned}$$

Now we apply Hölder's inequality and (4.4.2) (with g), to obtain

$$\int_E |R_j f(x)| \mathrm{d}x \leq \|f\|_{L^p(G)} \|R_j g\|_{L^{p'}(G)} \leq \|f\|_{L^p(G)} \omega_{p'}(\|g\|_{L^1(G)}).$$

Since $\|g\|_{L^1(G)} = |E| = \int_G \chi_E \mathrm{d}x$, the volume measure of E, we get

$$\frac{1}{|E|^{1/q'}} \int_E |R_j f(x)| \mathrm{d}x \leq \frac{\omega_{p'}(|E|)}{|E|^{1/q'}} \|f\|_{L^p(G)} \leq C_{p,q} \|f\|_{L^p(G)}.$$

The proof is complete. □

In particular, if we set $G = \mathbb{T}$, then there is a unique Riesz transform: the periodic Hilbert transform, and hence the above theorem yields (4.1.2). Similarly, one can apply the above result to the Lie group $\mathbb{T}^d$, the d-dimensional torus, and obtain the estimate $\|R_j\|_{L^p(\mathbb{T}^d) \to L^{q,\infty}(\mathbb{T}^d)} \leq C_{p,q}$ for all $j \in \{1, 2, \ldots, d\}$.

4.4.2. Sharpness on the circle and the torus

Proof of $\|\mathcal{H}^{\mathbb{T}}\|_{L^p(\mathbb{T})\to L^{q,\infty}(\mathbb{T})} \geq C_{p,q}$, the case $p \leq 2$. Consider the conformal map $F : \mathbb{D} \to [-1,1]\times\mathbb{R}$, given by

$$F(z) = \frac{2i}{\pi}\log\left[\frac{iz-1}{z-i}\right] + 1.$$

Then F maps the unit circle onto the boundary $\{-1,1\}\times\mathbb{R}$. We easily check the following explicit formulas on $\mathbb{T}$:

$$\varphi(e^{it}) := \operatorname{Re} F(e^{it}) = -\chi_{\{|t|\leq \pi/2\}} + \chi_{\{|t|>\pi/2\}}$$

and

$$\mathcal{H}^{\mathbb{T}}\varphi(e^{it}) = \operatorname{Im} F(e^{it}) = \frac{2}{\pi}\ln\left|\frac{1+\sin t}{\cos t}\right|.$$

Set $f = -|\mathcal{H}^{\mathbb{T}}\varphi|^{p'-2}\mathcal{H}^{\mathbb{T}}\varphi$. Since φ takes values in $\{-1,1\}$, we have

$$\left|\left|\left|\mathcal{H}^{\mathbb{T}} f\right|\right|\right|_{L^{q,\infty}(\mathbb{T})} \geq \frac{1}{\mu(\mathbb{T})}\int_{\mathbb{T}} \mathcal{H}^{\mathbb{T}} f\varphi \mathrm{d}\mu = -\int_{\mathbb{T}} f\mathcal{H}^{\mathbb{T}}\varphi\mathrm{d}\mu = \int_{\mathbb{T}} |\mathcal{H}^{\mathbb{T}}\varphi|^{p'}\mathrm{d}\mu.$$

However, we compute that

$$\begin{aligned}
\|\mathcal{H}^{\mathbb{T}}\varphi\|^{p'}_{L^{p'}(\mathbb{T})} &= \int_{-\pi}^{\pi}\left|\frac{2}{\pi}\ln\left|\frac{1+\sin t}{\cos t}\right|\right|^{p'}\frac{\mathrm{d}t}{2\pi} = \frac{1}{\pi}\int_{\mathbb{R}}\frac{\left|\frac{2}{\pi}\log|t|\right|^{p'}}{t^2+1}\mathrm{d}t \\
&= \frac{2^{p'+1}}{\pi^{p'+1}}\int_0^{\infty}\frac{|\log t|^{p'}}{t^2+1}\mathrm{d}t = \frac{2^{p'+1}}{\pi^{p'+1}}\int_{-\infty}^{\infty}\frac{|s|^{p'}e^s}{e^{2s}+1}\mathrm{d}s \\
&= \frac{2^{p'+2}}{\pi^{p'+1}}\int_0^{\infty} s^{p'}e^{-s}\sum_{k=0}^{\infty}(-e^{-2s})^k\mathrm{d}s = \frac{2^{p'+2}}{\pi^{p'+1}}\Gamma(p'+1)\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{p'+1}} \\
&= C_{p,q}^{p'}.
\end{aligned}$$

Combining this with the preceding estimate, we obtain

$$\left|\left|\left|\mathcal{H}^{\mathbb{T}} f\right|\right|\right|_{L^{q,\infty}(\mathbb{T})} \geq \|\mathcal{H}^{\mathbb{T}}\varphi\|_{L^{p'}(\mathbb{T})}\cdot\|\mathcal{H}^{\mathbb{T}}\varphi\|^{p'-1}_{L^{p'}(\mathbb{T})} = C_{p,q}\|f\|_{L^p(\mathbb{T})}.$$

Hence the constant $C_{p,q}$ in (4.1.2) cannot be improved. □

Proof of $\|\mathcal{H}^{\mathbb{T}}\|_{L^p(\mathbb{T})\to L^{q,\infty}(\mathbb{T})} \geq C_{p,q}$, the case $p > 2$. Fix an arbitrary parameter T belonging to $(0,1]$ and set $a = \frac{2}{\pi}\ln\left(\tan\frac{\pi(T+1)}{4}\right) > 0$. Recall the set D introduced at the beginning of Section 4.2. Let G be a conformal mapping which sends the unit disc $\mathbb{D}$ onto the set D and satisfies $G(0) = 0$. Finally, put $\varphi = \operatorname{Re} G|_{\mathbb{T}}$, $E = \{\varphi \neq 0\}$ and $f = -|\mathcal{H}^{\mathbb{T}}\varphi|^{p'-2}\mathcal{H}^{\mathbb{T}}\varphi$. Note that $\varphi \in \{0,\pm 1\}$, which gives

$$\int_E |\mathcal{H}^{\mathbb{T}} f|\mathrm{d}\mu \geq \int_{\mathbb{T}} \mathcal{H}^{\mathbb{T}} f\varphi\mathrm{d}\mu = -\int_{\mathbb{T}} f\mathcal{H}^{\mathbb{T}}\varphi\mathrm{d}\mu = \int_{\mathbb{T}}|\mathcal{H}^{\mathbb{T}}\varphi|^{p'}\mathrm{d}\mu.$$

To evaluate the latter integral, we apply appropriate conformal changes of variables. First, note that

$$\int_{\mathbb{T}}|\mathcal{H}^{\mathbb{T}}\varphi|^{p'}\mathrm{d}\mu = \int_{\partial D}|v|^{p'}\mathrm{d}\mu_{(0,0)}(u,v).$$

Next, recall the mapping H defined in (4.2.1). It sends D onto the upper halfplane $\mathbb{R}^2_+$ and 0 to i. Since $\frac{dt}{\pi(1+t^2)}$ is the harmonic measure on $\partial\mathbb{R}^2_+$ with respect to i, the latter integral equals

$$\int_{\partial\mathbb{R}^2_+} |\operatorname{Im}(H^{-1}(0,t))|^{p'} \frac{\mathrm{d}t}{\pi(1+t^2)} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{1+t^2}\left|\frac{1}{\pi}\ln\left|\frac{e^{\pi a}t^2-1}{t^2-e^{\pi a}}\right|\right|^{p'} \mathrm{d}t = \big(\omega_{p'}(T)\big)^{p'}.$$

(To see the last equality, repeat the calculations from the proof of Theorem 4.7.) A similar reasoning reveals that

$$\mu(E) = \int_{\mathbb{T}} |\varphi|\mathrm{d}\mu = \frac{4}{\pi}\arctan e^{\pi a/2} - 1 = T.$$

Putting all the above facts together, we obtain

$$\begin{aligned}\big\|\big|\mathcal{H}^{\mathbb{T}}f\big|\big\|_{L^{q,\infty}(\mathbb{T})} \geq \frac{1}{\mu(E)^{1-1/q}}\int_E |\mathcal{H}^{\mathbb{T}}f|\mathrm{d}\mu &\geq \frac{\|\mathcal{H}^{\mathbb{T}}\varphi\|_{L^{p'}(\mathbb{T})}}{\mu(E)^{1-1/q}}\cdot \|\mathcal{H}^{\mathbb{T}}\varphi\|^{p'-1}_{L^{p'}(\mathbb{T})} \\ &\geq T^{-1/q'}\omega_{p'}(T)\|f\|_{L^p(\mathbb{T})}.\end{aligned}$$

Taking the supremum over all T, we see that the constant in (4.1.2) is indeed the best possible. □

Remark 4.2. The calculations above show that the constants in (4.3.1) and (4.3.5) are also optimal. Indeed, for any $T \in (0,1]$, we have constructed above a function $\varphi : \mathbb{T} \to [-1,1]$ for which $\|\mathcal{H}^{\mathbb{T}}\varphi\|_{L^{p'}(\mathbb{T})} = \omega_{p'}(T)$ and $\|\varphi\|_{L^1(\mathbb{T})} = T$: this shows the sharpness, since the probabilistic inequalities are stronger. The boundary case $T = 0$, not covered by the above reasoning, is trivial (equality is attained for functions/martingales equal to zero).

Sharpness on the torus. If $G = \mathbb{T}^d$ is the d-dimensional torus, endowed with the standard, Riemannian product metric, then the constant $C_{p,q}$ is also optimal in (4.4.1), as we show now. Our starting observation is that for any $j = 1, 2, \ldots, d$, the Riesz transform R_j is the Fourier multiplier with the symbol $i\ell_j/|\ell|$, $\ell \in \mathbb{Z}^d \setminus \{0\}$: that is, we have

$$R_j(e^{i\ell\cdot\theta}) = \frac{i\ell_j}{|\ell|}e^{i\ell\cdot\theta}$$

for all ℓ as above. We will apply a transference argument. Fix j and consider the operator K mapping a function $f : \mathbb{T} \simeq (-\pi,\pi] \to \mathbb{R}$ to a function $Kf : \mathbb{T}^d \simeq (-\pi,\pi]^d \to \mathbb{R}$, given by $Kf(\theta) = f(\theta_j)$. We easily check that

$$K \circ \mathcal{H}^{\mathbb{T}} = R_j \circ K. \tag{4.4.3}$$

Next, pick an arbitrary set $E \in \mathbb{T}$ with $\mu(E) > 0$ and an arbitrary function $f \in L^p(\mathbb{T},\mu)$ of norm one. Then we also have $\|Kf\|_{L^p(\mathbb{T}^d,\mu_{\mathbb{T}^d})} = 1$, by a straightforward application of Fubini's theorem. Suppose further that φ is a function supported on E, taking values ±1 there. Then $K\varphi$ is supported on $K(E) = \{\theta \in \mathbb{T}^d : \theta_j \in E\}$ and also takes values ±1 there. Consequently, by (4.4.3), we obtain

$$\begin{aligned}\frac{\int_{K(E)}|R_j(Kf)|\mathrm{d}\mu_{\mathbb{T}^d}}{\mu_{\mathbb{T}^d}(K(E))^{1/q'}} &\geq \frac{\int_{\mathbb{T}^d} R_j(Kf)K\varphi\,\mathrm{d}\mu_{\mathbb{T}^d}}{\|K\varphi\|_{L^{q'}(\mathbb{T}^d,\mu_{\mathbb{T}^d})}} \\ &= \frac{\int_{\mathbb{T}^d} K(\mathcal{H}^{\mathbb{T}}f)K\varphi\mathrm{d}\mu_{\mathbb{T}^d}}{\|K\varphi\|_{L^{q'}(\mathbb{T}^d,\mu_{\mathbb{T}^d})}} = \frac{\int_{\mathbb{T}}\mathcal{H}^{\mathbb{T}}f\varphi\mathrm{d}\mu}{\|\varphi\|_{L^{q'}(\mathbb{T},\mu)}}.\end{aligned}$$

Therefore, if we put $\varphi = \mathrm{sgn}(\mathcal{H}^{\mathbb{T}} f)\chi_E$ (with the convention $\mathrm{sgn}(0) = 1$), then we obtain the estimate

$$\frac{\int_{K(E)} |R_j(Kf)| \mathrm{d}\mu_{\mathbb{T}^d}}{\mu_{\mathbb{T}^d}(K(E))^{1/q'}} \geq \frac{\int_E |\mathcal{H}^{\mathbb{T}} f| \mathrm{d}\mu}{\mu(E)^{1/q'}}. \tag{4.4.4}$$

This gives the desired estimate $\|R_j\|_{L^p(\mathbb{T}^d)\to L^{q,\infty}(\mathbb{T}^d)} \geq \|\mathcal{H}^{\mathbb{T}}\|_{L^p(\mathbb{T})\to L^{q,\infty}(\mathbb{T})}$. □

4.4.3. Inequalities for Riesz transforms on spheres

Now we proceed to the weak-type bounds for Riesz transforms on the Euclidean unit sphere. We start with the analogue of the inequality (4.3.5).

Theorem 4.10. *Suppose that $1 < p \leq 2$. Then for any $f : \mathbb{S}^{d-1} \to [-1, 1]$ and for $R \in \{R^c_{lm}, R^b_{lm}\}$, we have*

$$\|Rf\|_{L^p(\mathbb{S}^{d-1})} \leq \omega_p \left(\|f\|_{L^1(\mathbb{S}^{d-1})}\right). \tag{4.4.5}$$

Proof. Let W be the standard Brownian motion in $\mathbb{R}^d$ and let τ denote its exit time from the unit ball. The conditional Jensen inequality yields

$$\int_{\mathbb{S}^{d-1}} |Rf(x)|^p \mathrm{d}x = \mathbb{E}\, |T_{A_{lm}} f(W_\tau)|^p \leq \mathbb{E}\, |A_{lm} * F|^p .$$

Consider two martingales:

$$\eta_t = A_{lm} * F = \left(\int_0^{\tau\wedge t} A_{lm}(W_s) \nabla_{\mathbb{B}^d} F(W_s) \cdot \mathrm{d}W_s\right)_{t\geq 0}$$
$$\zeta_t = F(W_{\tau\wedge t}) = \left(\int_0^{\tau\wedge t} \nabla_{\mathbb{B}^d} F(W_s) \cdot \mathrm{d}W_s\right)_{t\geq 0}.$$

Then the martingale η is differentially subordinate to ζ, since

$$[\zeta, \zeta]_t - [\eta, \eta]_t = \sum_{k\notin\{l,m\}} \int_0^{\tau\wedge t} \left|\frac{\partial F}{\partial x_k}(W_s)\right|^2 \mathrm{d}s$$

is nonnegative and nondecreasing as a function of t. Moreover, the martingales are orthogonal, since $\langle A_{lm}x, x\rangle = 0$ for all $x \in \mathbb{R}^d$. Therefore, by the martingale inequality (4.3.1), we obtain

$$\|Rf\|_{L^p(\mathbb{S}^{d-1})} \leq \|A_{lm} * F\|_{L^p} \leq \omega_p \left(\|F(W_\tau)\|_{L^1}\right).$$

This finishes the proof, since W_τ is uniformly distributed on the sphere. □

Observe that from Green's formula and the properties of the Laplace-Beltrami operator (cf. [45, 46]), the adjoint of both cylindrical and ball directional Riesz transform R is equal to $-R$. This allows us to repeat, word by word, the reasoning of the proof of the weak-type estimate for the Hilbert transform, obtaining

Theorem 4.11. *For $1 < q < p < \infty$ and $R \in \{R^c_{lm}, R^b_{lm}\}$, we have*

$$|||Rf|||_{L^{q,\infty}\left(\mathbb{S}^{d-1}\right)} \leq C_{p,q} \|f\|_{L^p\left(\mathbb{S}^{d-1}\right)}. \tag{4.4.6}$$

There is a natural question whether the estimates (4.4.5) and (4.4.6) are sharp. We have been unable to answer it; no transference arguments seem to work here.

Chapter 5

Sharp analytic version of Fefferman's inequality

5.1. Introduction and statement of results

The purpose of this chapter is to study a quantitative version of the H^1-BMO duality in the holomorphic context. More specifically, we will be interested in the estimate related to the inclusion $H^1(\mathbb{T})\cap BMO \subseteq (H^1(\mathbb{T}))^*$. This inclusion implies that there is a finite constant $C>0$ such that for all $f\in H^1(\mathbb{T})$ and $g\in ABMO(\mathbb{T})$ satisfying $\int_{\mathbb{T}} g\mathrm{d}\mu=0$, we have

$$\left|\int_{\mathbb{T}} \overline{f(\zeta)}g(\zeta)\mathrm{d}\mu(\zeta)\right| \leq C\|f\|_{H^1(\mathbb{T})}\|g\|_{BMO(\mathbb{T})}. \tag{5.1.1}$$

Our main result identifies the optimal constant in this inequality.

Theorem 5.1. *The least constant allowed in* (5.1.1) *is equal to* $C=\sqrt{e^2+1}=2.896387\ldots$. *The constant is already the best possible in the weaker estimate*

$$\mathrm{Re}\left(\int_{\mathbb{T}} \overline{f(\zeta)}g(\zeta)d\mu(\zeta)\right) \leq \sqrt{e^2+1}\|f\|_{H^1(\mathbb{T})}\|g\|_{BMO(\mathbb{T})}. \tag{5.1.2}$$

Following the main theme of this dissertation, we will study the appropriate martingale analogue of the above result. Suppose that W is a planar Brownian motion started at zero and stopped upon reaching the boundary of the unit disc $\mathbb{D}$. For a square-integrable function g on $\mathbb{T}$, introduce the associated sharp maximal function on $\mathbb{T}$ by

$$g^{\#}(\zeta)=\mathbb{E}\left[\sup_{t\geq 0}\left(P[|g|^2](W_t)-|P[g](W_t)|^2\right)^{1/2}\,\middle|\, W_\infty=\zeta\right].$$

We will establish the following fact.

Theorem 5.2. *For any* $f\in H^1(\mathbb{T})$ *and any* $g\in H^2(\mathbb{T})$, *we have the sharp estimate*

$$\int_{\mathbb{D}} \left|\nabla P[f](z)\cdot\nabla P[g](z)\right|\cdot\frac{1}{\pi}\ln\frac{1}{|z|}dz \leq \sqrt{e^2+1}\left\|fg^{\#}\right\|_{L^1(\mathbb{T})}, \tag{5.1.3}$$

where $\nabla=\partial_z$ *is the complex derivative with respect to the variable* z.

It should be emphasized that (5.1.3) also applies to functions g with an unbounded $g^{\#}$, i.e., for functions outside the class $ABMO$. Comparing the above estimate to (5.1.1), we see that the left-hand side is increased and the right-hand side is decreased. Indeed, $\frac{1}{\pi}\ln\frac{1}{|z|}$ is the Green function for the disc, so we have

$$\begin{aligned}\int_{\mathbb{D}} \left|\nabla P[f](z)\cdot\nabla P[g](z)\right|\cdot\frac{1}{\pi}\ln\frac{1}{|z|}\mathrm{d}z &\geq \left|\int_{\mathbb{D}} \overline{\nabla P[f](z)}\cdot\nabla P[g](z)\cdot\frac{1}{\pi}\ln\frac{1}{|z|}\mathrm{d}z\right| \\ &= \left|\int_{\mathbb{T}} \overline{f(\zeta)}g(\zeta)\mathrm{d}\mu(\zeta)\right|.\end{aligned}$$

Furthermore, we have $\|g^\#\|_{L^\infty(\mathbb{T})} \leq \|g\|_{BMO(\mathbb{T})}$ and hence $\left\|fg^\#\right\|_{L^1(\mathbb{T})} \leq \|f\|_{H^1(\mathbb{T})}\|g\|_{BMO(\mathbb{T})}$. Thus, the above result generalizes Theorem 5.1 in two directions.

The chapter is organized as follows. The next section is devoted to the lower bound for the optimal constant in (5.1.2). Using the theory of analytic envelopes [41], we introduce a certain abstract plurisuperharmonic function associated with (5.1.2) and exploit its properties to show that the optimal constant in the estimate is at least $\sqrt{e^2+1}$. The analysis presented there leads us to a related, explicit special function, which is the main tool in the proof of (5.1.3), presented in Section 5.3.

5.2. On the lower bound for the constant

Throughout this section, we assume that C is a fixed positive constant such that for any $f \in H^1(\mathbb{T})$ and any $g \in ABMO(\mathbb{T})$ with $\|g\|_{BMO(\mathbb{T})} \leq 1$ and $\int_{\mathbb{T}} g\mathrm{d}\mu = 0$, we have

$$\operatorname{Re}\int_{\mathbb{T}} \bar{f}g\mathrm{d}\mu \leq C\|f\|_{H^1(\mathbb{T})}. \tag{5.2.1}$$

Our goal is to show that $C \geq \sqrt{e^2+1}$. One could try to provide appropriate examples, but these seem to have quite a complicated structure. Hence, we have decided to use a different approach and apply the theory of the so-called disc envelopes, a topic of complex analysis developed intensively during the last thirty years (see e.g. [28, 32, 41]). As a by-product, we will obtain some additional insight into certain special functions which will be used in the proof of (5.1.3).

We start with the necessary background and notation. Suppose that $\mathcal{X}$ is a Banach space and $\mathcal{D}$ is a domain in $\mathcal{X}$. A lower semicontinuous function G on $\mathcal{D}$ is called plurisuperharmonic, if for any $x \in \mathcal{D}$ and $w \in \mathcal{X}$ there is $r > 0$ such that

$$G(x) \geq \frac{1}{2\pi}\int_0^{2\pi} G(x + e^{i\theta}tw)\mathrm{d}\theta$$

for all $t \in (0, r)$. An analytic disc in $\mathcal{D}$ is a holomorphic mapping ρ of $\overline{\mathbb{D}}$, the closure of $\mathbb{D}$, into the domain $\mathcal{D}$. The collection of all analytic discs in $\mathcal{D}$ is denoted by $\mathcal{A}(\mathcal{D})$. For a given $x \in \mathcal{D}$, the symbol $\mathcal{A}_x(\mathcal{D})$ denotes the subclass of $\mathcal{A}(\mathcal{D})$ which consists of all ρ satisfying $\rho(0) = x$. A classical theorem of Poletsky [41] asserts that for a given lower semicontinuous function $H : \mathcal{D} \to \mathbb{R}$, the associated disc envelope

$$B(x) = \sup_{\rho\in\mathcal{A}_x(\mathcal{D})} \frac{1}{2\pi}\int_0^{2\pi} H(\rho(e^{i\theta}))\mathrm{d}\theta, \qquad x \in \mathcal{D},$$

is plurisuperharmonic.

In our considerations below, we will apply the above result with $\mathcal{X} = \mathbb{C}^3$ and $\mathcal{D} = \{(z_1, z_2, z_3) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C} : |z_2|^2 \leq \operatorname{Re} z_3 \leq |z_2|^2 + 1\}$. We will also use the following notation: for $\rho = (u, v, w) \in \mathcal{A}(\mathcal{D})$, we will write $(f, g, h) = \rho|_{\mathbb{T}}$. The above special choice of $\mathcal{D}$ is linked with the BMO property, as we explain in a simple lemma below.

Lemma 5.3. *If $\rho = (u, v, w) \in \mathcal{A}(\mathcal{D})$, then $g = v|_{\mathbb{T}}$ belongs to the class $ABMO(\mathbb{T})$ and $\|g\|_{BMO(\mathbb{T})} \leq 1$.*

Proof. Obviously, $g \in H^1(\mathbb{T})$, so it is enough to establish the estimate for the BMO norm. By the very definition of $\mathcal{D}$, we see that $|v|^2 \leq \operatorname{Re} w$ on $\overline{\mathbb{D}}$. Hence in particular, setting $h = w|_{\mathbb{T}}$, we get that for any $z \in \mathbb{D}$,

$$P[|g|^2](z) \leq P[\operatorname{Re} h](z) = \operatorname{Re} w(z) \leq |v(z)|^2 + 1 = \left|P[g](z)\right|^2 + 1. \quad \square$$

Next, consider the continuous function $H : \mathcal{D} \to \mathbb{R}$ given by $H(z_1, z_2, z_3) = \mathrm{Re}(\bar{z}_1 z_2) - C|z_1|$. By the aforementioned result of Poletsky, the associated disc envelope

$$\begin{aligned} &B(z_1, z_2, z_3) \\ &= \sup\left\{\frac{1}{2\pi}\int_0^{2\pi} H(\rho(e^{i\theta}))\mathrm{d}\theta \,:\, \rho \in \mathcal{A}_{(z_1,z_2,z_3)}(\mathcal{D})\right\} \\ &= \sup\left\{\frac{1}{2\pi}\int_0^{2\pi}\Big[\mathrm{Re}(\bar{f}(e^{i\theta})g(e^{i\theta})) - C|f(e^{i\theta})|\Big]\mathrm{d}\theta \,:\, \rho \in \mathcal{A}_{(z_1,z_2,z_3)}(\mathcal{D}),\ (f,g,h) = \rho|_{\mathbb{T}}\right\} \end{aligned}$$

is plurisuperharmonic on $\mathcal{D}$. In the sequence of lemmas below, we will study the key properties of this object. We start with the following homogeneity-type condition.

Lemma 5.4. *There exists a function $\varphi : [-1, 0] \to [0, C]$ such that*

$$B(z_1, z_2, z_3) = \mathrm{Re}(\bar{z}_1 z_2) - |z_1|\varphi(|z_2|^2 - \mathrm{Re}\, z_3). \tag{5.2.2}$$

Proof. First, note that B depends on z_3 through $\mathrm{Re}\, z_3$, that is, we have $B(z_1, z_2, z_3) = B(z_1, z_2, z_3')$ if $\mathrm{Re}\, z_3 = \mathrm{Re}\, z_3'$. This follows at once from the definition of B and the fact that neither $\mathcal{A}_{(z_1,z_2,z_3)}(\mathcal{D})$ nor $H(z_1, z_2, z_3)$ depends on $\mathrm{Im}\, z_3$. Next, we show the identity (5.2.2) for $z_2 = 0$. To this end, fix an arbitrary $\lambda \in \mathbb{C} \setminus \{0\}$ and note that if $(u, v, w) \in \mathcal{A}_{(z_1,0,z_3)}(\mathcal{D})$, then $(\lambda u, \lambda v/|\lambda|, w) \in \mathcal{A}_{(\lambda z_1,0,z_3)}(\mathcal{D})$, so

$$\begin{aligned} B(\lambda z_1, 0, z_3) &\geq \frac{1}{2\pi}\int_0^{2\pi}\Big[\mathrm{Re}(\overline{\lambda f}(e^{i\theta}) \cdot \lambda g(e^{i\theta})/|\lambda|) - C|\lambda||f(e^{i\theta})|\Big]\mathrm{d}\theta \\ &= \frac{|\lambda|}{2\pi}\int_0^{2\pi}\Big[\mathrm{Re}(\bar{f}(e^{i\theta})g(e^{i\theta})) - C|f(e^{i\theta})|\Big]\mathrm{d}\theta. \end{aligned}$$

Taking the supremum over all $(u, v, w) \in \mathcal{A}_{(z_1,0,z_3)}(\mathcal{D})$ on the right, we obtain

$$B(\lambda z_1, 0, z_3) \geq |\lambda| B(z_1, 0, z_3) \qquad \text{for all } z_1,\ z_3.$$

But we actually have equality here, which can be seen by applying the estimate to slightly different parameters. Indeed, we have $B(\lambda^{-1}(\lambda z_1), 0, z_3) \geq |\lambda^{-1}| B(\lambda z_1, 0, z_3)$, or $B(\lambda z_1, 0, z_3) \leq |\lambda| B(z_1, 0, z_3)$. Therefore, we have

$$B(z_1, 0, z_3) = |\lambda^{-1}| B(\lambda z_1, 0, z_3) \qquad \text{for all } z_1,\ z_3. \tag{5.2.3}$$

Now, if $z_1 \neq 0$, we put $\lambda = z_1^{-1}$, obtaining $B(z_1, 0, z_3) = |z_1| B(1, 0, z_3) = |z_1| B(1, 0, \mathrm{Re}\, z_3)$, so the claim holds with $\varphi(s) = -B(1, 0, -s)$. If $z_1 = 0$, then we let $\lambda \to \infty$ in (5.2.3) and get the equality $B(z_1, 0, z_3) = 0$; hence the claim is true also in this case.

For $z_2 \neq 0$, we observe the following translation condition: if $(u, v, w) \in \mathcal{A}_{(z_1,z_2,z_3)}(\mathcal{D})$, then $(u, v - \eta, w - 2v\bar{\eta} + |\eta|^2) \in \mathcal{A}_{(z_1,z_2-\eta,z_3-2z_2\bar{\eta}+|\eta|^2)}(\mathcal{D})$ for any $\eta \in \mathbb{C}$. Indeed, $(u, v - \eta, w - 2v\bar{\eta} + |\eta|^2)$ is holomorphic, equal to $(z_1, z_2 - \eta, z_3 - 2z_2\bar{\eta} + |\eta|^2)$ at zero and

$$|v - \eta|^2 - \mathrm{Re}\left(w - 2v\bar{\eta} + |\eta|^2\right) = |v|^2 - \mathrm{Re}\, w \in [0, 1]$$

on $\mathbb{D}$. Consequently, by the definition of B,

$$\begin{aligned} &B(z_1, z_2 - \eta, z_3 - 2z_2\bar{\eta} + |\eta|^2) \\ &\qquad \geq \frac{1}{2\pi}\int_0^{2\pi}\Big[\mathrm{Re}(\bar{f}(e^{i\theta})(g(e^{i\theta}) - \eta)) - C\big|f(e^{i\theta})\big|\Big]\mathrm{d}\theta \\ &\qquad = -\mathrm{Re}(\bar{z}_1\eta) + \frac{1}{2\pi}\int_0^{2\pi}\Big[\mathrm{Re}(\bar{f}(e^{i\theta})g(e^{i\theta})) - C\big|f(e^{i\theta})\big|\Big]\mathrm{d}\theta, \end{aligned}$$

which gives $B(z_1, z_2 - \eta, z_3 - 2z_2\bar{\eta} + |\eta|^2) \geq -\operatorname{Re}(\bar{z}_1\eta) + B(z_1, z_2, z_3)$, by taking the supremum over all $(u, v, w) \in \mathcal{A}_{(z_1,z_2,z_3)}(\mathcal{D})$. Again, as previously, the estimate can be reversed, by applying it to $z_1 := z_1$, $z_2 := z_2 - \eta$, $z_3 := z_3 - 2z_2\bar{\eta} + |\eta|^2$ and $\eta := -\eta$. Therefore, taking $\eta = z_2$, we obtain

$$B(z_1, z_2, z_3) = \operatorname{Re}(\bar{z}_1 z_2) + B(z_1, 0, z_3 - |z_2|^2) = \operatorname{Re}(\bar{z}_1 z_2) - |z_1|\varphi(|z_2|^2 - \operatorname{Re} z_3),$$

which completes the proof of (5.2.2).

It remains to handle the range of φ. Since (5.2.1) holds, we have $B(z_1, 0, z_3) \leq 0$ for all $z_1 \in \mathbb{C}$ and $z_3 \in \mathbb{C}$ with $\operatorname{Re} z_3 \in [0, 1]$. This implies $\varphi \geq 0$, directly by (5.2.2). Furthermore, since the constant triple $(1, 0, z_3)$ belongs to $\mathcal{A}_{(1,0,z_3)}(\mathcal{D})$, the very definition of B implies $-\varphi(-\operatorname{Re} z_3) = B(1, 0, z_3) \geq -C$ and hence $\sup \varphi \leq C$. □

The next step is to translate the plurisuperharmonicity of B into a differential inequality for φ. However, since B (and hence also φ) need not be a priori sufficiently smooth, we will perform an additional mollification argument. Suppose that $\gamma : \mathbb{R} \to [0, \infty)$ is a C^∞ function, supported on the interval $[-1, 1]$ and satisfying $\int_{\mathbb{R}} \gamma = 1$. Given $\delta \in (0, 1/2)$, define $\phi : [-1, -2\delta] \to \mathbb{R}$ by the convolution

$$\phi(s) = \int_{[-1,1]} \varphi(s + \delta + \delta u)\gamma(u)\mathrm{d}u.$$

Clearly, ϕ takes values in the interval $[0, C]$, since so does φ. In addition, it is straightforward to check that the modified function

$$\tilde{B}(z_1, z_2, z_3) = \operatorname{Re}(\bar{z}_1 z_2) - |z_1|\phi(|z_2|^2 - \operatorname{Re} z_3),$$

given on $\mathcal{D}_\delta = \{(z_1, z_2, z_3) : |z_2|^2 + 2\delta \leq \operatorname{Re} z_3 \leq |z_2|^2 + 1\}$, inherits the plurisuperharmonicity from B. We will interpret this condition in the language of ϕ.

Lemma 5.5. *We have*

$$\left(\phi(s) - \frac{(\phi'(s))^2}{\phi''(s)}\right)\phi'(s) \geq 1. \tag{5.2.4}$$

Proof. The function $\tilde{B}$ is of class C^2 on $\mathcal{D} \setminus \{(0, z_2, z_3)\}$ and plurisuperharmonic, so

$$\sum_{j,k=1}^{3} \tilde{B}_{\bar{z}_j z_k}(z_1, z_2, z_3)\bar{w}_j w_k \leq 0$$

for all $(z_1, z_2, z_3) \in \mathcal{D}$, $z_1 \neq 0$, and all $(w_1, w_2, w_3) \in \mathbb{C}^3$. This is equivalent to

$$\begin{aligned} \operatorname{Re}(\bar{w}_1 w_2) - \frac{\phi(s)}{4|z_1|}|w_1|^2 - \frac{\phi'(s)}{|z_1|}\operatorname{Re}\left(\bar{z}_1 w_1\left(z_2\bar{w}_2 - \frac{\bar{w}_3}{2}\right)\right) \\ - |z_1|\left(\phi''(s)\left|z_2\bar{w}_2 - \frac{\bar{w}_3}{2}\right|^2 + \phi'(s)|w_2|^2\right) \leq 0, \end{aligned} \tag{5.2.5}$$

where $s = |z_2|^2 - \operatorname{Re} z_3$. To obtain the claim, we will now specify the values of the parameters z_j and w_k. Assume that $|z_1| = 1$ and that the argument of z_1 is chosen so that $\operatorname{Re}(\bar{z}_1 w_1(z_2\bar{w}_2 - \frac{\bar{w}_3}{2})) = -|\bar{z}_1 w_1(z_2\bar{w}_2 - \frac{\bar{w}_3}{2})| = -|w_1||\bar{z}_2 w_2 - \frac{w_3}{2}|$. Second, suppose that the argument of w_2 satisfies $\operatorname{Re}(\bar{w}_1 w_2) = |w_1||w_2|$. Then the above estimate yields

$$|w_1||w_2| - \frac{|w_1|^2}{4}\phi(s) + |w_1|\left|\bar{z}_2 w_2 - \frac{w_3}{2}\right|\phi'(s) - |w_2|^2\phi'(s) - \left|\bar{z}_2 w_2 - \frac{w_3}{2}\right|^2\phi''(s) \leq 0.$$

Note that we must have $\phi'(s) > 0$; otherwise, the above inequality would be violated with $w_3 = 2\bar{z}_2 w_2$ and sufficiently large $|w_2|$. Consequently, the second derivative $\phi''(s)$ is also positive (in particular, non-zero), since for $\phi''(s) \leq 0$ the above estimate does not hold for small $|w_1|$, $|w_2|$ and large $|w_3|$. This enables us to rewrite the above bound in the form

$$|w_1||w_2| - \frac{|w_1|^2}{4}\left(\phi(s) - \frac{(\phi'(s))^2}{\phi''(s)}\right) - |w_2|^2\phi'(s) - \left(\frac{|w_1|\phi'(s)}{2\phi''(s)} - \left|\bar{z}_2 w_2 - \frac{w_3}{2}\right|\right)^2 \phi''(s) \leq 0.$$

With an appropriate choice of the parameter w_3, the last term on the left vanishes. It is easy to see that the sum of the remaining three terms is nonpositive (for all possible values of $|w_1|$ and $|w_2|$) if and only if the assertion holds. □

Remark 5.1. The above reasoning shows that a stronger version of (5.2.5) holds, in which the term $\mathrm{Re}(\bar{w}_1 w_2)$ is replaced with $|w_1 w_2|$.

We are ready for the proof of the main result of this section.

Theorem 5.6. *We have* $C \geq \sqrt{e^2+1}$.

Proof. Since $\phi' > 0$ and $\phi'' > 0$, the estimate (5.2.4) is equivalent to

$$\left(\frac{\phi}{\phi'} - \frac{1}{2(\phi')^2}\right)' \leq 0. \tag{5.2.6}$$

Furthermore, again by (5.2.4) (and the estimate $\phi' > 0$ we established earlier), we have $\phi\phi' > 1$, which implies $\left(\frac{\phi}{\phi'} - \frac{1}{2(\phi')^2}\right) \geq \frac{1}{2(\phi')^2}$. Consequently, we get that the limit $\alpha = \lim_{s\downarrow -1}\left(\frac{\phi(s)}{\phi'(s)} - \frac{1}{2(\phi'(s))^2}\right)$ exists and is strictly positive. Next, the trivial estimate

$$\frac{\phi^2(s)}{2} \geq \frac{\phi(s)}{\phi'(s)} - \frac{1}{2(\phi'(s))^2},$$

combined with the monotonicity of ϕ, implies that for any $t \in (-1, -2\delta)$,

$$\frac{\phi^2(t)}{2} \geq \lim_{s\downarrow -1}\frac{\phi^2(s)}{2} \geq \alpha. \tag{5.2.7}$$

Coming back to (5.2.6), we see that for t as above we have $\frac{\phi(t)}{\phi'(t)} - \frac{1}{2(\phi'(t))^2} \leq \alpha$, or

$$2\alpha(\phi'(t))^2 - 2\phi(t)\phi'(t) + 1 \geq 0.$$

Let us solve this quadratic inequality (with respect to $\phi'(t)$). The discriminant is equal to $4(\phi^2(t) - 2\alpha)$, which is nonnegative, by (5.2.7). Therefore, we obtain that either

$$\phi'(t) \leq \frac{\phi(t) - \sqrt{\phi^2(t) - 2\alpha}}{2\alpha} \quad \text{or} \quad \phi'(t) \geq \frac{\phi(t) + \sqrt{\phi^2(t) - 2\alpha}}{2\alpha}. \tag{5.2.8}$$

The first possibility cannot hold: we would have

$$\phi(t)\phi'(t) \leq \phi(t) \cdot \frac{\phi(t) - \sqrt{\phi^2(t) - 2\alpha}}{2\alpha} = \frac{\phi(t)}{\phi(t) + \sqrt{\phi^2(t) - 2\alpha}} \leq 1,$$

a contradiction. Therefore, we have proved that the second inequality in (5.2.8) holds. This is equivalent to saying that $(F(\phi))' \geq 1$, where

$$F(r) = \int \frac{2\alpha \mathrm{d}r}{r + \sqrt{r^2 - 2\alpha}} = \frac{r(r - \sqrt{r^2 - 2\alpha})}{2} + \alpha \log(r + \sqrt{r^2 - 2\alpha}). \tag{5.2.9}$$

By the very definition, F is increasing and hence we may write

$$F(C) \geq F(\phi(-2\delta)) \geq \lim_{s \downarrow -1} F(\phi(s)) + 1 - 2\delta \geq F(\sqrt{2\alpha}) + 1 - 2\delta,$$

which is equivalent to

$$\frac{C^2 - 2\alpha - C\sqrt{C^2 - 2\alpha}}{2} + \alpha \log\left(\frac{C}{\sqrt{2\alpha}} + \sqrt{\frac{C^2}{2\alpha} - 1}\right) \geq 1 - 2\delta.$$

The parameter δ was chosen arbitrarily and we may now send it to zero. Dividing both sides by $C^2/2$ and substituting $s = 2\alpha/C^2$ (which belongs to $(0,1)$: we have $\sqrt{2\alpha} < \phi(-1/2) \leq C$), we obtain

$$\frac{2}{C^2} \leq 1 - s - \sqrt{1-s} + s \log\left(\frac{1}{\sqrt{s}} + \sqrt{\frac{1}{s} - 1}\right).$$

Let us compute the maximum of the right-hand side over $s \in (0,1)$: a direct differentiation shows that the biggest value is attained for $s_0 = \left(\frac{2e}{e^2+1}\right)^2$. Plugging this above, we get $2/C^2 \leq 2/(e^2+1)$, which is the desired lower bound. □

Assuming equalities in appropriate places in the proof of the above theorem, we come up with a special function which will be of key importance in the next section. Specifically, suppose that $C = \sqrt{e^2+1}$ and set $\alpha = C^2 s_0/2 = 2e^2/(e^2+1)$. There exists a continuous function $\psi : [-1,0] \to [0,C]$ satisfying the differential equation $(F(\psi(s)))' = 1$ for $s \in (-1,0)$ (the function F is given by (5.2.9)) and the initial condition $\psi(-1) = \sqrt{2\alpha}$. Indeed, F is a function on $[\sqrt{2\alpha}, \infty)$ which increases from $F(\sqrt{2\alpha}) = \alpha + \frac{1}{2}\alpha \log(2\alpha)$ to $F(\infty) = \infty$, and therefore ψ given explicitly by

$$\psi(s) = F^{-1}\left(s + 1 + \alpha + \frac{1}{2}\alpha \log(2\alpha)\right) \tag{5.2.10}$$

has all the required properties. It is easy to see that this definition allows to extend ψ to some neighborhood of zero (actually, the formula makes perfect sense for $s \in [-1, \infty)$); on contrary, one cannot go below -1. Note that $\psi(0) = \sqrt{e^2+1}$.

We conclude the above analysis by observing that

$$\psi'(s) = \frac{\psi(s) + \sqrt{\psi^2(s) - 2\alpha}}{2\alpha},$$

so $\psi'(s) > 0$, $\psi''(s) > 0$ and $(\psi(s) - (\psi'(s))^2/\psi''(s))\,\psi'(s) = 1$. If we revert the reasoning from the proof of Lemma 5.5, we see that (5.2.5) holds, with ϕ replaced by ψ (actually, a stronger estimate mentioned in Remark 5.1 is valid). This in particular implies that the function $b(z_1, z_2, z_3) = \mathrm{Re}(\bar{z}_1 z_2) - |z_1|\psi(|z_2|^2 - \mathrm{Re}\, z_3)$, defined on $\mathcal{D}$, is plurisuperharmonic. This allows for a quick proof of (5.1.2). Although we will prove the stronger estimate (5.1.3) later, we take the opportunity to discuss here the former inequality, for which the argument is purely analytic.

Proof of (5.1.2). Pick any $f \in H^1(\mathbb{T})$ and $g \in ABMO(\mathbb{T})$ with $\int_{\mathbb{T}} g \mathrm{d}\mu = 0$ satisfying $\|g\|_{BMO(\mathbb{T})} \leq 1$. Let $h \in H^1(\mathbb{T})$ be given by $h = a_0 + 2\sum_{n>0} a_n \zeta^n$, where $\sum_{n\in\mathbb{Z}} a_n\zeta^n$ is the Fourier expansion of $|g|^2$. Observe that $\operatorname{Re} h = |g|^2$: this follows at once from the fact that $a_n = \overline{a_{-n}}$ for all n (since $|g|^2$ is real-valued). Hence, the BMO condition implies that the holomorphic function $(P[f], P[g], P[h])$ takes values in $\mathcal{D}$. Therefore, the composition $b(P[f], P[g], P[h])$ is well-defined and gives a superharmonic function on $\mathbb{D}$, so

$$\int_{\mathbb{T}} b(f(\zeta), g(\zeta), h(\zeta)) \mathrm{d}\mu(\zeta) \leq b(P[f](0), P[g](0), P[h](0)). \tag{5.2.11}$$

Now, since $\operatorname{Re} h = |g|^2$, the left-hand side is equal to

$$\begin{aligned}\int_{\mathbb{T}} &\left\{ \operatorname{Re}\left(\overline{f(\zeta)} g(\zeta)\right) - |f(\zeta)| \cdot \psi(0) \right\} \mathrm{d}\mu(\zeta) \\ &= \operatorname{Re}\left(\int_{\mathbb{T}} \overline{f(\zeta)} g(\zeta) \mathrm{d}\mu(\zeta) \right) - \sqrt{e^2+1} \|f\|_{H^1(\mathbb{T})}.\end{aligned}$$

It remains to note that $P[g](0) = \int_{\mathbb{T}} g \mathrm{d}\mu = 0$, which implies that the right hand side of (5.2.11) is equal to $b(P[f](0), P[g](0), P[h](0)) = -|P[f](0)|\psi(-\operatorname{Re} P[h](0)) \leq 0$. □

The estimate (5.1.3) will require more effort, in particular we will need some machinery from the stochastic analysis. This will be done in the next section.

5.3. Proof of (5.1.3)

We start with the introduction of some basic notions for complex martingales. To avoid confusion, we have decided not to present them in Chapter 1 and postponed the definitions until now. For a pair $X = (X_t)_{t\geq 0}$, $Y = (Y_t)_{t\geq 0}$ of continuous-path martingales taking values in $\mathbb{C}$, the total variation is defined, as previously, by

$$\int_0^t |\mathrm{d}[X,Y]_s| = \limsup_{n\to\infty} \sum_{k=1}^{2^n} \left| (X_{tk\cdot 2^{-n}} - X_{t(k-1)\cdot 2^{-n}})(Y_{tk\cdot 2^{-n}} - Y_{t(k-1)\cdot 2^{-n}}) \right|,$$

and we let $\int_0^\infty |\mathrm{d}[X,Y]_t| = \sup_{t\geq 0} \int_0^t |\mathrm{d}[X,Y]_s|$. The key difference, in comparison to the previous chapters, lies in the definition of the square bracket. Namely, for complex-valued martingales it is customary to put

$$\begin{aligned}[][X,Y] &= \Big[\operatorname{Re} X + i \operatorname{Im} X, \operatorname{Re} Y + i \operatorname{Im} Y \Big] \\ &= \Big([\operatorname{Re} X, \operatorname{Re} Y] - [\operatorname{Im} X, \operatorname{Im} Y] \Big) + i \Big([\operatorname{Re} X, \operatorname{Im} Y] + [\operatorname{Im} X, \operatorname{Re} Y] \Big),\end{aligned}$$

where the square brackets in the last line are the usual quadratic variations of real-valued martingales. Now, a martingale X is called analytic (or conformal), if we have $d[X,X] = 0$, that is, the square bracket $[X,X]$ is constant. This can be extended to the vector setting: a martingale $(X^1, X^2, \ldots, X^n)$ with values in $\mathbb{C}^n$ is called analytic, if for any z_1, z_2, $\ldots$, $z_n \in \mathbb{C}$, the linear combination $z_1X^1 + z_2X^2 + \ldots + z_nX^n$ is a conformal martingale in $\mathbb{C}$. It is easy to check that this is the case if and only if $d[X^j, X^k] = 0$ for any j, $k \in \{1, 2, \ldots, n\}$. For example, a planar Brownian motion is a conformal martingale; more generally, analytic martingales arise naturally as compositions of holomorphic functions (of one or several variables) with a planar Brownian motion. See Chapter 5 in [43] for more on the subject.

We are ready for the proof of our main estimate.

Proof of (5.1.3). For the sake of clarity, we split the reasoning into four steps.

Step 1. Notation. Let $W = (W_t)_{t\geq 0}$ be a planar Brownian motion, started at zero and stopped upon reaching the boundary of the unit circle. Pick $f \in H^1(\mathbb{T})$ and $g \in H^2(\mathbb{T})$. In the proof of (5.1.3), we may assume that $\int_{\mathbb{T}} f \mathrm{d}\mu \neq 0$, by adding a small $\varepsilon > 0$ to f if necessary, and letting $\varepsilon \to 0$ at the very end. Let $h \in H^1(\mathbb{T})$ be given by $h = a_0 + 2\sum_{n>0} a_n\zeta^n$, where $\sum_{n\in\mathbb{Z}} a_n\zeta^n$ is the Fourier expansion of $|g|^2$. As we proved at the end of the previous section, we have $\operatorname{Re} h = |g|^2$.

Consider the three-dimensional process (X, Y, Z), defined by

$$X_t = P[f](W_t), \quad Y_t = P[g](W_t), \quad Z_t = P[h](W_t), \qquad t \geq 0.$$

It is easy to see that it is an analytic continuous-path martingale: any linear combination of X, Y, Z can be written in the form $G(W)$ for some analytic function G on the unit disc. The triple (X, Y, Z) takes values in the set $\{(z_1, z_2, z_3) : \operatorname{Re} z_3 \geq |z_2|^2\}$, since

$$\operatorname{Re} Z_t = \operatorname{Re} P[h](W_t) = P[|g|^2](W_t) \geq |P[g](W_t)|^2 = |Y_t|^2.$$

By the well-known properties of analytic martingales (see [43], p. 191), with probability 1 the process X does not visit the origin in finite time. So if we define, for a given $\varepsilon \in (0, 1)$, the stopping time $\tau(\varepsilon) = \inf\{t : |X_t| \notin [\varepsilon, \varepsilon^{-1}] \text{ or } |Y_t| \geq \varepsilon^{-1}\}$, then $\tau(\varepsilon) \to \infty$ as $\varepsilon \to 0$. We introduce an additional nondecreasing process $U_t = \left(\max_{0\leq s\leq t}(\operatorname{Re} Z_s - |Y_s|^2)\right)^{1/2}$, $t \geq 0$, which will be responsible for the control over the sharp maximal function of g. Note that we may assume that the process U is strictly positive. Indeed, we have $U_t \geq U_0 = \operatorname{Re} Z_0 - |Y_0|^2 = P[|g|^2](0) - |P[g](0)|^2 > 0$, unless g is constant a.e. on $\mathbb{T}$ (in which case the estimate is trivial).

Step 2. A Bellman function and the application of Itô's formula. Let $\psi : [-1, \infty) \to [0, \infty)$ be the special function given in (5.2.10). Take an arbitrary $\kappa > 1$ and put

$$b(u, z_1, z_2, z_3) = u|z_1|\psi\left(\frac{|z_2|^2 - \operatorname{Re} z_3}{\kappa u^2}\right)$$

for all $u > 0$ and all (z_1, z_2, z_3) such that $\operatorname{Re} z_3 - |z_2|^2 \leq u^2$. This function is of class C^2, unless $z_1 = 0$; the role of the parameter κ in the above definition is to bound the argument of ψ away from -1 (in which a singularity of ψ arises). Hence we may apply Itô's formula to the composition of b with the stopped quadruple $\xi = (U^{\tau(\varepsilon)}, X^{\tau(\varepsilon)}, Y^{\tau(\varepsilon)}, Z^{\tau(\varepsilon)})$. We obtain

$$b(\xi_t) = I_0 + I_1 + I_2, \tag{5.3.1}$$

where

$$I_0 = b(\xi_0), \quad I_1 = \int_0^t \nabla b(\xi_s) \cdot \mathrm{d}\xi_s, \quad I_2 = \int_0^t D^2 b(\xi_s)\mathrm{d}[\xi, \xi]_s.$$

Here we have used the shortened notation: I_1, I_2 are simply the sums of all first- and second-order terms, respectively. That is,

$$I_1 = \int_0^t b_u(\xi_s)\mathrm{d}U_s^{\tau(\varepsilon)} + \int_0^t b_{z_1}(\xi_s)\mathrm{d}X_s^{\tau(\varepsilon)} + \int_0^t b_{\bar{z}_1}(\xi_s)\mathrm{d}\bar{X}_s^{\tau(\varepsilon)} + \int_0^t b_{z_2}(\xi_s)\mathrm{d}Y_s^{\tau(\varepsilon)} + \ldots$$

and

$$I_2 = \int_0^t b_{\bar{z}_1 z_1}(\xi_s)\mathrm{d}[\bar{X}, X]_s^{\tau(\varepsilon)} + \int_0^t b_{\bar{z}_1 z_2}(\xi_s)\mathrm{d}[\bar{X}, Y]_s^{\tau(\varepsilon)} + \int_0^t b_{\bar{z}_1 z_3}(\xi_s)\mathrm{d}[\bar{X}, Z]_s^{\tau(\varepsilon)} + \ldots.$$

Note that in the definition of I_2, there are no derivatives with respect to u, which is due to the fact that U is a nondecreasing (and hence finite-variation) process. Furthermore, only the mixed ('conjugate-nonconjugate') derivatives appear in I_2: the remaining integrals are zero, because (X, Y, Z) is analytic - the appropriate square brackets are constant.

Step 3. The analysis of I_0, I_1 and I_2. First, note that we have $I_0 \geq 0$, since ψ takes values in $[0, \infty)$. The integral

$$\int_0^t b_u(\xi_s)\mathrm{d}U_s^{\tau(\varepsilon)}$$

in I_1 is also nonnegative. To see this, note that $U_s^{\tau(\varepsilon)}$ increases only on the set

$$\Big\{s : \max_{0\leq r\leq s}(\operatorname{Re} Z_r - |Y_r|^2) = \operatorname{Re} Z_s - |Y_s|^2\Big\},$$

and on this set we have $b_u(\xi_s) = |X_s^{\tau(\varepsilon)}|\big(\psi(-\kappa^{-1}) + 2\psi'(-\kappa^{-1})\big) \geq 0$. The remaining stochastic integrals appearing in I_1 are L^2-bounded martingales of expectation zero: by the definition of $\tau(\varepsilon)$, the stopped processes $X^{\tau(\varepsilon)}$, $Y^{\tau(\varepsilon)}$ and $Z^{\tau(\varepsilon)}$ are bounded, and $X^{\tau(\varepsilon)}$ is bounded away from zero. The main difficulty lies in the understanding of the term I_2. We will make use of the estimate mentioned in Remark 5.1. It implies that for any (z_1, z_2, z_3) with $z_1 \neq 0$ and $\operatorname{Re} z_3 - |z_2|^2 \leq u^2$, and all w_1, w_2, $w_3 \in \mathbb{C}$, we have

$$\sum_{j,k=1}^{3} b_{\bar{z}_j z_k}(u, z_1, z_2, z_3)\overline{w}_j w_k \geq |w_1 w_2|.$$

Fix $0 \leq s_0 < s_1 \leq t$. For any $\ell \geq 0$, let $(\eta_i^\ell)_{0\leq i\leq i_\ell}$ be a nondecreasing sequence of stopping times with $\eta_0^\ell = s_0$, $\eta_{i_\ell}^\ell = s_1$ such that $\lim_{\ell\to\infty}\max_{0\leq i\leq i_\ell-1}|\eta_{i+1}^\ell - \eta_i^\ell| = 0$. Keeping ℓ fixed, we apply, for each $i = 0, 1, 2, \ldots, i_\ell - 1$, the above estimate with $u = U_{s_0}^{\tau(\varepsilon)}$, $z_1 = X_{s_0}^{\tau(\varepsilon)}$, $z_2 = Y_{s_0}^{\tau(\varepsilon)}$, $z_3 = Z_{s_0}^{\tau(\varepsilon)}$ and $w_1 = X_{\eta_{i+1}^\ell}^{\tau(\varepsilon)} - X_{\eta_i^\ell}^{\tau(\varepsilon)}$, $w_2 = Y_{\eta_{i+1}^\ell}^{\tau(\varepsilon)} - Y_{\eta_i^\ell}^{\tau(\varepsilon)}$, $w_3 = Z_{\eta_{i+1}^\ell}^{\tau(\varepsilon)} - Z_{\eta_i^\ell}^{\tau(\varepsilon)}$. We sum the obtained i_ℓ inequalities and let $\ell \to \infty$, arriving at

$$D^2 b(\xi_{s_0})([\xi, \xi]_{s_1} - [\xi, \xi]_{s_0}) \geq \int_{s_0}^{s_1} |\mathrm{d}[X^{\tau(\varepsilon)}, Y^{\tau(\varepsilon)}]_s|.$$

By the Itô's formula again, we have

$$X_t = X_0 + \int_0^t \nabla P[f](W_s)\mathrm{d}W_s, \qquad Y_t = Y_0 + \int_0^t \nabla P[g](W_s)\mathrm{d}W_s,$$

and hence

$$\int_{s_0}^{s_1} |\mathrm{d}[X^{\tau(\varepsilon)}, Y^{\tau(\varepsilon)}]_s| = \int_{\tau(\varepsilon)\wedge s_0}^{\tau(\varepsilon)\wedge s_1} \big|\nabla P[f](W_s) \cdot \nabla P[g](W_s)\big|\mathrm{d}s.$$

If we approximate the integral I_2 by Riemann sums and use the above bound, we get

$$I_2 \geq \int_0^{\tau(\varepsilon)\wedge t} \big|\nabla P[f](W_s) \cdot \nabla P[g](W_s)\big|\mathrm{d}s.$$

Step 4. The completion of the proof. Plugging all these observations into (5.3.1) and taking expectation of both sides, we obtain

$$\mathbb{E}b(\xi_t) \geq \mathbb{E}\int_0^{\tau(\varepsilon)\wedge t} \big|\nabla P[f](W_s) \cdot \nabla P[g](W_s)\big|\mathrm{d}s.$$

By the definition of b, we have

$$\mathbb{E}b(\xi_t) = \mathbb{E}\left[U_t^{\tau(\varepsilon)}|X_t^{\tau(\varepsilon)}|\psi\left(\frac{|Y_t^{\tau(\varepsilon)}|^2 - \operatorname{Re} Z_t^{\tau(\varepsilon)}}{\kappa(U_t^{\tau(\varepsilon)})^2}\right)\right].$$

But $\psi \leq \sqrt{e^2+1}$ on $[-1,0]$ and $|X_t^{\tau(\varepsilon)}| \leq \mathbb{E}\big(|X_\infty|\big|\mathcal{F}_{\tau(\varepsilon)\wedge t}\big)$, since X is L^1-bounded martingale; furthermore, we have $U_t^{\tau(\varepsilon)} \leq U_\infty = \sup_{s\geq 0}\big(P[|g|^2](W_s) - |P[g](W_s)|^2\big)^{1/2}$. Therefore, we get

$$\begin{aligned}\mathbb{E}\int_0^{\tau(\varepsilon)\wedge t}&\big|\nabla P[f](W_s)\cdot\nabla P[g](W_s)\big|\mathrm{d}s\\ &\leq \sqrt{e^2+1}\,\mathbb{E}\big(|X_\infty|U_\infty\big)\\ &= \sqrt{e^2+1}\,\mathbb{E}\left[\big|P[f](W_\infty)\big|\sup_{s\geq 0}\big(P[|g|^2](W_s) - |P[g](W_s)|^2\big)^{1/2}\right]\\ &= \sqrt{e^2+1}\,\mathbb{E}\Big[\big|P[f](W_\infty)\big|g^\#(W_\infty)\Big].\end{aligned}$$

Letting $\varepsilon \to 0$ and $t \to \infty$ we obtain, by Lebesgue's monotone convergence theorem,

$$\mathbb{E}\int_0^\infty \big|\nabla P[f](W_s)\cdot\nabla P[g](W_s)\big|\mathrm{d}s \leq \sqrt{e^2+1}\,\mathbb{E}\Big[\big|P[f](W_\infty)\big|g^\#(W_\infty)\Big].$$

Since W_∞ is uniformly distributed on the unit circle, the expectation on the right-hand side equals $\|fg^\#\|_{L^1(\mathbb{T})}$. Furthermore, exploiting the formula for the Green function on the disc, we see that the expression on the left is precisely

$$\int_{\mathbb{D}} |\nabla P[f](z)\cdot\nabla P[g](z)|\cdot\frac{1}{\pi}\ln\frac{1}{|z|}\mathrm{d}z.$$

This completes the proof. □

Bibliography

[1] N. Arcozzi, *Riesz transforms on compact Lie groups, spheres and Gauss space*, Ark. Mat. **36** (1998), 201–231.

[2] N. Arcozzi and X. Li, *Riesz transforms on spheres*, Math. Res. Lett. **4** (1997), 401–412.

[3] R. Bañuelos, K. Bogdan and T. Luks, *Hardy-Stein identities and square functions for semigroups*, J. London Math. Soc. **94** (2016), 462–478.

[4] A. Baernstein II and S. J. Montgomery-Smith, *Some conjectures about integral means of* ∂f *and* $\bar{\partial} f$, Complex analysis and differential equations (Uppsala, 1997), Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist., vol. 64, Uppsala Univ., Uppsala, 1999, 92–109.

[5] R. Bañuelos, A. Osękowski, *On the Bellman function of Nazarov, Treil and Volberg*, Math. Z. *278* (2014), 385–399.

[6] R. Bañuelos, A. Osękowski, *Sharp martingale inequalities and applications to Riesz transforms on manifolds, Lie groups and Gauss space*, J. Funct. Anal. **269** (2015), 1652–1713.

[7] D. L. Burkholder, *Martingale transforms*, Ann. Math. Statist. **37** (1966), 1494–1504.

[8] D. L. Burkholder, *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Probab. **12** (1984), 647–702.

[9] D. L. Burkholder, *Martingales and Fourier analysis in Banach spaces*, Probability and Analysis (Varenna, 1985) Lecture Notes in Math. **1206**, Springer, Berlin (1986), pp. 61–108.

[10] D. L. Burkholder, *Explorations in martingale theory and its applications*, École d'Ete de Probabilités de Saint-Flour XIX—1989, 1–66, Lecture Notes in Math., 1464, Springer, Berlin, 1991.

[11] R. Bañuelos, T. Gałązka, A. Osękowski, *A dual approach to Burkholder's Lp estimates*, to appear in Bulletin of the London Mathematical Society.

[12] A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[13] B. Davis, *On the weak type* $(1, 1)$ *inequality for conjugate functions*, Proc. Amer. Math. Soc. **44** (1974), 307–311.

[14] A. Carbonaro and O. Dragičević, *Functional calculus for generators of symmetric contraction semigroups*, Duke Math. J. **166** (2017), 937–974.

[15] C. Dellacherie and P.-A. Meyer, *Probabilities and potential B: Theory of martingales*, North Holland, Amsterdam, 1982.

[16] O. Dragičević and A. Volberg, *Bellman functions and dimensionless estimates of Littlewood-Paley type*, J. Op. Theory **56** (2006), 167–198.

[17] R. J. Elliott, Stochastic calculus and applications. Applications of Mathematics (New York), 18. Springer-Verlag, New York, 1982.

[18] C. Fefferman, *Characterization of bounded mean oscillation*, Bull. Amer. Math. Soc. **77** (1971), 587–588.

[19] T. W. Gamelin, *Uniform Algebras and Jensen Measures*, London Math. Soc. Lecture Notes Series, Vol. 32, Cambridge Univ. Press, Cambridge/New York, 1978.

[20] R. F. Gundy, *Some Topics in Probability and Analysis*, Am. Math. Soc. **70**, Providence, Rhode Island, 1986.

[21] D. Girela, Analytic functions of bounded mean oscillation. Complex function spaces (Mekrijärvi, 1999), 61–170, Univ. Joensuu Dept. Math. Rep. Ser., 4, Univ. Joensuu, Joensuu, 2001.

[22] R. K. Getoor and M. J. Sharpe, *Conformal martingales*, Invent. Math. **16** (1972), 271–308.

[23] L. Grafakos, Classical Fourier analysis. Second edition. Graduate Texts in Mathematics, 249. Springer, New York, 2008.

[24] R.F. Gundy and N. Th. Varopoulos, *Les transformations de Riesz et les integrales stochastiques*, C. R. Acad. Sci. Paris Ser. A-B 289 (1979), A13-A16.

[25] T. Iwaniec and G. Martin, *The Beurling-Ahlfors transform in* $\mathbb{R}^n$ *and related singular integrals*, J. Reine Angew. Math. 473, 25–57.

[26] P. Janakiraman, *Best weak-type* (p,p) *constants,* $1 \le p \le 2$ *for orthogonal harmonic functions and martingales*, Illinois J. Math. 48 No. 3 (2004), 909–921.

[27] A. N. Kolmogorov, *Sur les fonctions harmoniques conjugées et les séries de Fourier*, Fund. Math. 7 (1925), 24–29.

[28] F. Lárusson and E. A. Poletsky, *Plurisubharmonic subextensions as envelopes of disc functionals*, Michigan Math. J. **62** (2013), no. 3, 551–565.

[29] J. M. Lee, Manifolds and differential geometry. Graduate Studies in Mathematics, 107. American Mathematical Society, Providence, RI, 2009.

[30] J. Marcinkiewicz, Quelques théoremes sur les séries orthogonales, Ann. Soc. Polon. Math. **16** (1937), 84–96.

[31] B. Maurey, Syst'eme de Haar, Seminaire Maurey-Schwartz (1974-1975), Ecole Polytechnique, Paris.

[32] B. S. Magnússon, *Extremal* ω*-plurisubharmonic functions as envelopes of disc functionals*, Ark. Mat. **49** (2011), no. 2, 383–399.

[33] F. L. Nazarov and S. R. Treil, *The hunt for a Bellman function: applications to estimates for singular integral operators and to other classical problems of harmonic analysis,* St. Petersburg Math. J. **8** (1997), 721–824.

[34] F. L. Nazarov, S. R. Treil and A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers*, J. Amer. Math. Soc., **12** (1999), 909–928.

[35] A. Osękowski, *Survey article: Bellman function method and sharp inequalities for martingales*, Rocky Mountain J. Math. **43** (2013), 1759–1823.

[36] A. Osękowski, *Sharp weak type inequalities for differentially subordinated martingales*, Bernoulli **15**, Vol. 3 (2009), 871–897.

[37] A. Osękowski, *Sharp inequalities for Riesz transforms*, Studia Math. **222** (2014), 1-18.

[38] A. Osękowski, *Sharp moment inequalities for differentially subordinated martingales*, Studia Math. **201** (2010), 103–131.

[39] A. Osękowski, *Sharp localized inequalities for Fourier multipliers*, Canadian J. Math. **66** (2014), 1358–1381.

[40] S. K. Pichorides, *On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov*, Studia Math. 44 (1972), 165–179.

[41] E. A. Poletsky, Plurisubharmonic functions as solutions of variational problems. Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989), 163–171, Proc. Sympos. Pure Math., 52, Part 1, Amer. Math. Soc., Providence, RI, 1991.

[42] M. Riesz, *Sur les fonctions conjugées*, Math. Z. 27 (1927), 218–244.

[43] D. Revuz and M. Yor, Continuous martingales and Brownian motion, *volume 293 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences].* Springer-Verlag, Berlin, third edition, 1999.

[44] J. Schauder, *Eine Eigenschaft des Haarschen Orthogonalsystems*, Math. Z. **28** (1928), 317–320.

[45] E. M. Stein, *Topics in Harmonic Analysis Related to the Littlewood-Paley Theory*, Princeton University Press, Princeton, 1970.

[46] E. M. Stein, Singular integrals and Differentiability Properties of Functions, Princeton University Press, Princeton, 1970.

[47] Y. Suh, *A sharp weak type* (p,p) *inequality* $(p > 2)$ *for martingale transforms and other subordinate martingales*, Trans. Amer. Math. Soc. 357 (2005), 1545–1564.

[48] V. Vasyunin and A. Volberg, *Burkholder's function via Monge-Ampere equation*, Illinois J. Math. **54** (2010), no. 4, 1393–1428.

[49] G. Wang, *Differential subordination and strong differential subordination for continuous time martingales and related sharp inequalities*, Ann. Probab. **23** (1995), 522–551.