University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Tomasz Dębiec

Weak convergence methods for equations of mathematical physics and biology

Doctoral dissertation

Supervisor:
dr hab. Agnieszka Świerczewska-Gwiazda, prof. UW
Institute of Applied Mathematics and Mechanics
University of Warsaw

April 2020

Author's declaration:

I hereby declare that this thesis is my own work.

20 April 2020 ....

Tomasz Dębiec

Supervisor's declaration:

The disertation is ready to be reviewed.

20 April 2020 ....

dr hab. Agnieszka Świerczewska-Gwiazda, prof. UW

Abstract

This dissertation is a collection of several results in the analysis of partial differential equations arising in mathematical physics and biology. The themes we explore are diverse in their quantitative mathematical properties and applications – yet, a common feature of our studies is the necessity of developing and implementing a variety of weak convergence and compactness methods, indispensable when dealing with nonlinear phenomena.

In the first part of the thesis, we investigate the relation between regularity and conserved quantities for some equations arising in fluid dynamics. The main results here are a sufficient regularity condition for local conservation of energy for weak solutions of the Euler-Korteweg system, and an analogous study for the compressible Euler and Navier-Stokes equations in the degenerate case of possible vacuum formation.

Next, we study the archetypal equation in the field of structured population dynamics, namely the growth-fragmentation equation, which is a linear integro-differential equation describing competition between the phenomena of aggregation and fragmentation. The main result of this part is proving that solutions emanating from initial data in the space of positive Radon measures converge, in an appropriate weighted sense, to a steady size distribution.

Finally, we consider a two-species model motivated by applications in tumour modelling. The two species are coupled by an elliptic equation tying their velocity potential to the total population pressure. This link is usually referred to as Brinkman's law. A further coupling is given by a relation between the pressure and a power of the total population density. We establish an existence and uniqueness result, and perform an incompressible limit as the stiffness of the pressure law tends to infinity. This establishes a rigorous connection with a free-boundary model of Hele-Shaw flavour.

Keywords: weak convergence, energy conservation, compressible Euler equations, vacuum, relative entropy, measure solutions, structured population models, tissue growth, incompressible limit.

AMS Subject Classification: 35Q92, 76N10, 92D25, 35Q31, 35Q30, 35L65, 35G50, 35B40, 35B45, 35K55, 35K57, 35K65.

Metody słabej zbieżności dla równań fizyki i biologii matematycznej

Streszczenie

Niniejsza rozprawa stanowi zbiór wyników dotyczących matematycznej analizy pewnych równań różniczkowych cząstkowych motywowanych zagadnieniami fizyki i biologii matematycznej. Tematy, które badamy, są różnorodne ze względu na własności jakościowe oraz zastosowania – jednakże wspólną ich cechą jest potrzeba starannego rozwijania całego wachlarza metod słabej zbieżności i zwartości, nieodzownych przy analizie zjawisk nieliniowych.

W pierwszej części rozprawy badamy związek pomiędzy regularnością a wielkościami zachowywanymi dla pewnych równań obecnych w mechanice płynów. Głównymi wynikami są tutaj warunki wystarczające do zapewnienia spełnienia lokalnej równości energetycznej przez słabe rozwiązania układu Eulera-Kortewega, oraz ściśliwego układu Eulera (a również Naviera-Stokesa) w zdegenerowanym przypadku występowania obszarów próżni.

Następnie badamy podstawowe równanie w dziedzinie dynamiki populacji ze strukturą, a mianowicie równanie wzrostu-podziału. Jest to liniowe równanie całkowo-różniczkowe opisujące współzawodnictwo pomiędzy procesami wzrostu komórkowego a fragmentacją. Głównym wynikiem tej części rozprawy jest wykazanie, że rozwiązanie pochodzące z danych początkowych w przestrzeni nieujemnych miar Radona zbiega, w odpowiedniej normie z wagą, do stanu stacjonarnego.

W ostatniej części rozprawy rozważamy dwugatunkowy model motywowany zastosowaniami w opisie wzrostu komórek nowotworowych. Równania zadające dynamikę obu gatunków są sprzężone poprzez prawo Brinkmana, tj. równanie eliptyczne wiążące ich prędkość z ciśnieniem, które jest z kolei proporcjonalne do potęgi całkowitej gęstości populacji. Uzyskane wyniki dotyczą istnienia oraz jednoznaczności słabych rozwiązań układu, oraz przejścia asymptotycznego z wykładnikiem zadającym związek pomiędzy ciśnieniem a całkowitą populacją. Ukazuje to powiązanie rozważanego modelu z geometrycznym modelem o swobodnym brzegu.

Słowa kluczowe: słaba zbieżność, zasada zachowania energii, ściśliwe równania Eulera, próżnia, relatywna entropia, rozwiązania miarowe, modele populacji ze strukturą, wzrost tkanek, nieściśliwa granica.

Acknowledgements

First and foremost, I am greatly indebted to my advisor, Agnieszka Świerczewska-Gwiazda, for imparting to me boundless amounts of guidance, support, patience, authority, and friendship. She has been a fantastic teacher of mathematical analysis, rigour, and professional competence. She remains an unceasing source of inspiration and wisdom. Thank you!

I would also like to express my most sincere gratitude towards Piotr Gwiazda, with whom I have shared countless hours of discussions, debates, and more or less professional musings and frivolities.

Many special thanks are due to Benoît Perthame, Athanasios Tzavaras and Emil Wiedemann, who were kind enough to extend their warm hospitality to me during my research stays at LJLL, KAUST, and Hannover and Ulm, respectively. Working and discussing with them has made me appreciate different work styles and increased my understanding of mathematics manifold.

During the past couple of years I have met, during conferences, workshops, schools, research visits, or by sheer chance, many fellow fledgling mathematicians, with whom I had shared many a discussion, idea, drink, experience, and laugh. I owe particular recognition to Rafał Martynek, Myrto Galanopoulou, Ibrohim Akramov, Jack Skipper, and mes amis bien-aimés from Paris: Gissell Estrada-Rodriguez, Sophie Hecht and Markus Schmidtchen.

None of this would have been even remotely possible without my parents and their unconditional support.

Last but not least, I could not be more thankful to my dearest girlfriend, Kaja, for her continual support, comfort, and motivating me to withstand my laziness and sluggishness.

Financial acknowledgement: I appreciate financial support of the National Science Center (through grants: 2012/05/E/ST1/02218, 2014/13/B/ST1/03094, 2017/27/B/ST1/01569, 2018/30/M/ST1/00423, 2018/31/N/ST1/02394, 2019/32/T/ST1/00435); National Agency for Academic Exchange (through the Iwanowska Programme), and University of Warsaw's Integrated Development Programme.

Contents

Chapter 1

Autoreferat

1.1 Wstęp i motywacja

Teoria równań różniczkowych cząstkowych zrodziła się z potrzeby odpowiedzi na pytania zadawane w fizyce, biologii, czy innych naukach przyrodniczych. Powszechnie uważa się, że wiele istotnych zjawisk fizycznych (np. ruch płynów, zachowanie materiałów sprężystych, czy wzrost komórkowy) jest dobrze opisanych układami równań różniczkowych cząstkowych (przeważnie nieliniowych). Wiele ważkich otwartych zagadnień – od podstawowych kwestii analitycznych, takich jak istnienie, jednoznaczność i stabilność rozwiązań, poprzez jakościowe własności rozwiązań, po walidację czysto matematycznego modelu w zastosowaniach – wciąż pozostaje bez rozwiązania. Praktyczna istotność tych zagadnień sprawia, że ta trudna dziedzina jest zarówno ekscytująca, jak i satysfakcjonująca, inspirując nieustające wysiłki badawcze kolejnych pokoleń matematyków.

Niniejsza rozprawa dotyczy następujących trzech wiodących zagadnień badawczych.

Związek pomiędzy regularnością rozwiązań a zasadą zachowania energii w mechanice płynów: Zadajemy tutaj następujące pytanie: *Jaka minimalna regularność słabych rozwiązań danego układu jest potrzebna, aby zagwarantować, że spełniony jest odpowiedni bilans energetyczny?* W szczególności badamy ściśliwy układ Eulera oraz układ Eulera-Kortewega. Ta część rozprawy składa się z następujących dwóch publikacji:

> T. Dębiec, P. Gwiazda, A. Świerczewska-Gwiazda, A. Tzavaras, Energy conservation for the Euler-Korteweg equations. *Calc. Var. Partial Differential Equations* 57(6): Art. 160, 2018;

> I. Akramov, T. Dębiec, J. Skipper, E. Wiedemann, Energy conservation for the compressible Euler and Navier-Stokes equations with vacuum. *Anal. PDE*, 13(3):789–811, 2020.

Zastosowania metody relatywnych entropii dla miarowych rozwiązań modeli populacji ze strukturą: Zajmujemy się analizą jednego z podstawowych równań w dziedzinie dynamiki populacyjnej, a mianowicie modelem wzrostu-podziału w dość ogólnej postaci. Głównym wynikiem jest zapewnienie, że rozwiązania pochodzące z danych początkowych w przestrzeni miar, zbiegają, w odpowiednim sensie, do profilu stacjonarnego. Tę część rozprawy stanowi następująca praca:

> T. Dębiec, M. Doumic, P. Gwiazda, E. Wiedemann, Relative entropy method for measure solutions of the growth-fragmentation equation. *SIAM J. Math. Anal.* 50(6):5811–5824, 2018.

Wielofazowe modele wzrostu żywych tkanek: W ostatniej części rozprawy badamy dwufazowy model z zastosowaniami w opisie rozwoju komórek nowotworowych, sprzężony poprzez prawo Brinkmana. Głównymi wynikami są twierdzenie o istnieniu i jednoznaczności słabych rozwiązań oraz ścisłe powiązanie rozważanego modelu z modelem geometrycznym, w którym guz opisany jest jako związany (nieściśliwy) płyn zanurzony w ściśliwym płynie (otaczająca zdrowa tkanka). Ta część rozprawy składa się z następującej publikacji:

> T. Dębiec, M. Schmidtchen, Incompressible limit for a two-species tumour model with coupling through Brinkman's law in one dimension. *Acta Appl. Math.*, w druku, 2020, https://doi.org/10.1007/s10440-020-00313-1.

W kolejnym rozdziale opiszemy bardziej szczegółowo powyższe zagadnienia oraz uzyskane wyniki.

1.2 Przegląd najważniejszych wyników

1.2.1 Zasada zachowania energii w mechanice płynów

Pierwsza część rozprawy dotyczy kwestii, czy słabe rozwiązania danego prawa zachowania lub bilansu spełniają dodatkowe *prawa stowarzyszone*, a w szczególności bilans energii. Ewolucyjne równania różniczkowe cząstkowe przeważnie posiadają pewne naturalne fizycznie wielkości (np. energia) które są, przynajmniej formalnie, zachowywane – tzn. zależą jedynie od danych początkowych i ich wartość nie zmienia się w czasie. Znane są jednak przypadki, kiedy takie formalnie wyprowadzone dodatkowe prawo zachowania przestaje być prawdziwe dla słabych rozwiązań o niskiej regularności. Istotnie, słabe rozwiązania, nie dość że niejednoznaczne, nierzadko wykazują niefizyczne własności, takie jak na przykład spontaniczne generowanie lub rozpraszanie energii.

Ścisła zależność między regularnością a zachowywanymi wielkościami jest związana np. z pojęciem renormalizacji w teorii DiPerny-Lionsa [14] i sławną *hipotezą Onsagera* [28] o zachowaniu energii dla nieściśliwego układu Eulera. Przewiduje ona, że istnieje próg regularności dla zachowania energii kinetycznej – wystarczająco gładkie słabe rozwiązania (mianowicie takie, które posiadają 1/3 pochodnej przestrzennej) zawsze będą zachowywały energię, podczas gdy poniżej tego progu istnieje możliwość rozpraszania energii (ang. *anomalous dissipation*). Ten kierunek badań cieszył się w ciągu ostatnich kilku dziesięcioleci ogromnym zainteresowaniem i doprowadził m.in. do rozwoju techniki wypukłego całkowania (ang. *convex integration*) w kontekście dynamiki płynów.

Obie części hipotezy stały się dynamicznymi programami badawczymi, których kulminacją był pełny dowód postulatu Onsagera [7, 24], a także liczne studia dotyczące innych układów równań pochodzących z dynamiki płynów, jak i ogólnych praw zachowania.

W artykule [12] (we współpracy z P. Gwiazdą, A. Świerczewską-Gwiazdą oraz A. Tzavarasem) rozważamy następujący układ równań Eulera-Kortewega:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) + \operatorname{div}_x(\rho u \otimes u) &= -\rho \nabla_x \left(h'(\rho) + \frac{\kappa'(\rho)}{2} |\nabla_x \rho|^2 - \operatorname{div}_x(\kappa(\rho) \nabla_x \rho) \right), \\ \frac{\partial \rho}{\partial t} + \operatorname{div}_x(\rho u) &= 0, \end{aligned} \tag{1.1}$$

w $(0, T) \times \mathbb{T}^d$ dla pewnego $T > 0$, gdzie

- $d > 1$ oznacza wymiar przestrzenny, a $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ płaski 1-wymiarowy torus;
- $\rho(t, x) \geq 0$ oznacza gęstość płynu w czasie $t \geq 0$ oraz w pozycji $x \in \mathbb{T}^d$;
- $u(t, x) \in \mathbb{R}^d$ jest lokalnym polem prędkości płynu;
- $h(\rho)$ jest gęstością energii, a $\kappa(\rho) > 0$ oznacza współczynnik kapilarności. Przyjmujemy następujące założenie:

  $$h, \kappa \in C^3(\mathcal{T}), \tag{1.2}$$

  gdzie, w zależności od konkretnej postaci funkcji h oraz κ, jako zbiór $\mathcal{T}$ możemy wybrać $[0, \infty)$ lub $(0, \infty)$.

Układ (1.1) jest znanym modelem opisującym mieszaniny ciekło-gazowe, zob. [16]. Można łatwo wykazać, że klasyczne rozwiązania spełniają następujący lokalny bilans energii

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\frac{1}{2} \rho |u|^2 + h(\rho) + \frac{1}{2} \kappa(\rho) |\nabla_x \rho|^2 \right) \\ &+ \operatorname{div}_x \left[\rho u \left(\frac{1}{2} |u|^2 + h'(\rho) + \frac{1}{2} \kappa'(\rho) |\nabla_x \rho|^2 - \operatorname{div}_x(\kappa(\rho) \nabla_x \rho) \right) + \kappa(\rho) \nabla_x \rho \operatorname{div}_x(\rho u) \right] = 0, \end{aligned} \tag{1.3}$$

w sensie dystrybucyjnym. Możliwe jest występowanie wysoce niejednoznacznych, niefizycznych słabych rozwiązań układu (1.1). Zostało to wykazane w pracy [15], gdzie, za pomocą metody wypukłego całkowania, autorzy konstruują globalne w czasie ograniczone rozwiązania dysypatywne. Istnienie zarówno rozwiązań zachowujących energię oraz rozwiązań niezachowawczych motywuje studia nad analogiem hipotezy Onsagera dla układu

Eulera-Kortewega. W pracy [12] wykazujemy następujący warunek konieczny (w odpowiednich przestrzeniach Biesowa), gwarantujący, że słabe rozwiązania będą spełniać prawo zachowania energii.

Zasada zachowania energii dla układu Eulera-Kortewega (Twierdzenie 1.1 w [12]):

Niech (ρ, u) będzie słabym rozwiązaniem układu (1.1) oraz

$$u \in (B_3^{\alpha,\infty} \cap L^\infty)((0,T) \times \mathbb{T}^d), \quad \rho, \nabla_x \rho, \Delta\rho \in (B_3^{\beta,\infty} \cap L^\infty)((0,T) \times \mathbb{T}^d),$$

gdzie $1 > \alpha \geq \beta > 0$ spełniają $\min(2\alpha + \beta, \alpha + 2\beta) > 1$.
Wówczas bilans energii (1.3) jest spełniony w sensie dystrybucji na $(0,T) \times \mathbb{T}^d$.

Przyjrzyjmy się bliżej założeniu (1.2). Nie jest ono szczególnie restryktywne w tym sensie, że nie wyklucza dwóch najczęściej badanych przypadków współczynnika kapilarności, a mianowicie stałej kapilarności $\kappa(\rho) \equiv \kappa$, oraz przypadku $\kappa(\rho) = \frac{\epsilon_0^2}{4\rho}$, gdzie ϵ_0 oznacza stałą Plancka, co prowadzi do tzw. układu hydrodynamiki kwantowej (ang. *quantum hydrodynamics*):

$$\frac{\partial}{\partial t}(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p(\rho) = \frac{\epsilon_0^2}{2}\rho \nabla_x \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right),$$
$$\frac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho u) = 0.$$

Z technicznego punktu widzenia założenie (1.2) jest wymagane do oszacowania pewnych wyrazów w oszacowaniach typu *commutator estimates*. We wcześniejszej pracy Feireisl i in. [17] wykazali podobny warunek wystarczający dla spełnienia zasady zachowania energii dla rozwiązań ściśliwego układu Eulera. Ich kluczowym założeniem jest aby ciśnienie, $p = p(\rho)$, było dwukrotnie różniczkowalne w sposób ciągły względem gęstości. Mogli wówczas zastosować metodę *commutator estimates*, która wcześniej była stosowana tylko do nieliniowości typu kwadratowego, jak w [7] dla nieściśliwego układu Eulera.

Okazuje się, że osłabienie wymagania, aby nieliniowości były klasy C^2 jest ciekawym i istotnie nietrywialnym zagadnieniem. W pracy [1] (we współpracy z I. Akramovem, J. Skipperem oraz E. Wiedemannem) rozważamy ściśliwy układ Eulera:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x p(\rho) &= 0, \\ \frac{\partial \rho}{\partial t} + \mathrm{div}_x(\rho u) &= 0. \end{aligned} \tag{1.4}$$

Lokalna forma równości energetycznej dla tego układu ma następującą postać:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho|u|^2 + P(\rho)\right) + \mathrm{div}_x\left[\left(\frac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\right)u\right] = 0, \tag{1.5}$$

w sensie dystrybucyjnym, gdzie tzw. potencjał ciśnienia, P, jest zdefiniowany poprzez

$$P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2}\, \mathrm{d}r.$$

Wspomniana wyżej praca [17] jest pierwszym odpowiednikiem wyniku Constantina i in. [7] dla dynamiki gazu ściśliwego. Autorzy rozważają słabe rozwiązania w klasie

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \overline{\rho} \text{ p.w. w } (0,T)\times\mathbb{T}^d,$$

dla pewnych stałych $\underline{\rho}$, $\overline{\rho}$ oraz $0 \le \alpha, \beta \le 1$, takich że $\min(2\alpha+\beta, \alpha+2\beta) > 1$. Wówczas, zakładając również, że $p \in C^2[\underline{\rho}, \overline{\rho}]$, wykazują, że prawo zachowania energii (1.5) jest spełnione w sensie dystrybucji na $(0,T)\times\mathbb{T}^d$, por. [17, Twierdzenie 4.1].

Zauważmy, że założenie o ciśnieniu jest oczywiście spełnione w przypadku politropowego równania stanu $p(\rho) = \kappa\rho^\gamma$, dla dowolnego wykładnika adiabaty $\gamma > 2$. Jednakże, w przedziale $1 < \gamma < 2$ należy wykluczyć stany próżniowe (tzn. $\underline{\rho} > 0$) – tymczasem zakres ten jest niewątpliwie znaczący fizycznie (istotnie, dla gazu monoatomowego mamy $\gamma = 5/3$, a dla gazu diatomowego $\gamma = 7/5$).

Potencjalna możliwość występowania obszarów próżni stanowi zdegenerowanie, które w wielu przypadkach znacznie utrudnia matematyczną analizę modeli ściśliwych. Na przykład ściśliwy układ Eulera przestaje być ściśle hiperboliczny w takich obszarach. W szczególności, gęstości bliskie zera sprawiają, że nie można zastosować wspomnianych wcześniej metod pozwalających na traktowanie wyrażeń nieliniowych jako nieliniowości kwadratowych w oszczowaniach typu *commutator estimates.*

W pracy [1] badamy różne warunki konieczne na regularność rozwiązań, które gwarantują prawdziwość równości energetycznej nawet w takich zdegenerowanych sytuacjach.

Po pierwsze, możemy założyć, że prędkość płynu należy do klasy tzw. *divergence-measure fields*, która jest dobrze znana w dziedzinach geometrycznej teorii miary oraz hiperbolicznych praw zachowania. Należy przyznać, że jest to dosyć silne założenie dla rozwiązań ściśliwego układu Eulera, jednakże jest ono motywowane natychmiastowym zastosowaniem w przypadku ściśliwego układu Naviera-Stokesa, zob. Wniosek 3.3 w [1].

Następujące twierdzenie formułuje warunek konieczny dla zachowania energii, zakładając dostępność dodatkowego oszacowania na iloraz funkcji gęstości oraz jej regularyzacji.

Zasada zachowania energii dla ściśliwego układu Eulera w obecności próżni (Twierdzenie 4.1 w [1]):

Niech (ρ, u) będzie słabym rozwiązaniem układu (1.4) oraz

$$u \in B_p^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_q^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \overline{\rho} \text{ p.w. w } (0,T)\times\mathbb{T}^d,$$

dla pewnych stałych $\underline{\rho}$, $\overline{\rho}$ oraz $0 \leq \alpha, \beta \leq 1$, takich że

$$\frac{1}{p}+\frac{2}{q} \leq 1, \qquad \frac{2}{p}+\frac{1}{q} \leq 1, \qquad p,q \geq 2, \qquad \alpha+\gamma\beta > 1 \qquad \text{oraz} \qquad 2\alpha+\beta > 1.$$

Zdefiniujmy zbiór $\mathcal{B}_{\epsilon^\beta} := \{x : 0 < \rho^\epsilon(x) < \epsilon^\beta \text{ oraz } \rho \neq 0\}$ i załóżmy, że

$$\left\| \frac{\rho^\epsilon - \rho}{\rho^\epsilon} \right\|_{L^q(\mathcal{B}_{\epsilon^\beta})} \leq C(\rho), \tag{1.6}$$

gdzie stała C nie zależy od ϵ. Załóżmy ponadto, że $p \in C^{1,(\gamma-1)}([\underline{\rho}, \overline{\rho}])$, oraz $p'(0) = 0$ jeśli $\underline{\rho} = 0$.
Wówczas energia jest lokalnie zachowywana, tzn. równość (1.5) jest spełniona w sensie dystrybucji na $(0,T) \times \mathbb{T}^d$.

Warunek (1.6) wynika z uważnej analizy *commutator estimates* związanych z obecnością ciśnienia w równaniu i potencjału ciśnienia w równości energetycznej. Jest to więc techniczny warunek, który może się wydawać raczej sztucznym. W artykule [1] analizujemy bardziej naturalne założenia, które gwarantowałyby spełnienie (1.6): zakładając hölderowską ciągłość rozwiązań (co odpowiada wzięciu $p = q = \infty$) wykazujemy, że energia jest zachowana niezależnie od tego, jak zachowuje się gęstość płynu w pobliżu obszarów próżni, por. [1, Wniosek 4.4]. Chcąc koniecznie pozostać w klasie Biesowa, musimy dodać założenia o sposobie zbliżania się do obszarów zerowej gęstości: na przykład, gęstość powinna zanikać dostatecznie szybko [1, Wniosek 4.6] lub dostatecznie powolni [1, Wniosek 4.10]. Otrzymane wyniki pozostają prawdziwe również w przypadku, gdy równanie jest postawione w obszarze ograniczonym (z odpowiednio regularnym brzegiem), przy założeniu o ciągłości składowej normalnej pola prędkości płynu, zob. [1, Wniosek 5.1].

W powiązanym tematycznie artykule [9], badam możliwości rozszerzenia powyższych wyników do przypadku ogólnych praw zachowania w postaci:

$$\operatorname{div}_X G(U(X)) = 0,$$

gdzie pole strumienia nie jest funkcją klasy C^2 (w przeciwieństwie do wcześniejszej pracy Gwiazdy i in. [20]), dla rozwiązań spełniających warunek typu Biesowa-VMO, który został zaproponowany po raz pierwszy przez Fjordholma i Wiedemanna [18] w kontekście nieściśliwego układu Eulera.

1.2.2 Asymptotyka długookresowa miarowych rozwiązań modeli populacji ze strukturą

Modele populacji ze strukturą są pokrewne równaniu transportu, a ich celem jest zrozumienie dynamiki danej populacji względem pewnej zmiennej (lub grupy zmiennych), określającej jej strukturę (np. wiek, rozmiar, czy cechy fenotypowe). Przykładami zjawisk opisywanych takimi równaniami są wzrost i podział komórkowy, polimeryzacja, nasycenie komórkowe, replikacja prionów, oraz wiele innych. Z praktycznego punktu widzenia istotne jest zrozumienie zachowania rozwiązań tego typu modeli w długim okresie czasu, zob. np. [32].

Zagadnienie asymptotyki długookresowej dla całkowalnych rozwiązań modeli populacji ze strukturą zostało podjęte już w latach 80. ubiegłego wieku, poczynając od najprostszego takiego równania – modelu McKendricka-von Foerstera. Różne metody znalazły zastosowanie w celu wykazania zbieżności (w przestrzeni L^1 z odpowiednią wagą) do profilu stacjonarnego.

Szczególnie owocną metodę zaproponowali Perthame i współpracownicy [25–27]. Ich uogólniona metoda relatywnych entropii (ang. *generalised relative entropy method*, GRE) umożliwia studiowanie asymptotyki długookresowej dużej klasy liniowych modeli, nawet przy braku luki spektralnej.

Niedawno Gwiazda i Wiedemann [19], przy pomocy metod współczesnego rachunku wariacyjnego oraz teorii miary, zastosowali metodę GRE do analizy własności asymptotycznych równania McKendricka-von Foerstera z danymi początkowymi w przestrzeni miar. Zaznaczmy, że takie studia są istotne z biologicznego punktu widzenia, ponieważ dopuszczają sytuację, gdy badana populacja jest początkowo skoncentrowana względem zmiennej strukturyzującej (i, w szczególności, nie jest absolutnie ciągła względem miary Lebesgue'a). Jest tak, na przykład, gdy populacja wywodzi się z pojedynczej komórki.

Ich strategia może być potencjalnie zastosowana do otrzymania podobnych wyników dla innych modeli populacji, jednak, w zależności od struktury entropijnej modelu, wymaga to rozwiązania znacznych trudności. W pracy [11] (we współpracy z M. Doumic, P. Gwiazdą oraz E. Wiedemannem) rozważamy następujące ogólne liniowe równanie wzrostu-podziału z danymi początkowymi w przestrzeni miar:

$$\begin{aligned} \frac{\partial n}{\partial t}(t,x) + \frac{\partial}{\partial x}(g(x)n(t,x)) + B(x)n(t,x) &= \int_x^\infty k(x,y)B(y)n(t,y)\,\mathrm{d}y, \\ g(0)n(t,0) &= 0, \\ n(0,x) &= n^0(x), \end{aligned} \tag{1.7}$$

gdzie

- $n(t,x)$ oznacza gęstość agentów o rozmiarze $x \geq 0$ w czasie $t > 0$;
- $g(x) \geq 0$ zadaje tempo wzrostu jednostek, a $B(x) \geq 0$ tempo ich podziału;
- $k(x,y)$ oznacza odsetek osobników rozmiaru x powstałych z podziału osobników rozmiaru y.

Powyższe równanie łączy w sobie bardzo istotnie w biologii zjawisko, a mianowicie współzawodnictwo pomiędzy podziałem a wzrostem. Oczywiście mają one przeciwny wpływ na całkowitą dynamikę, a w zależności od tego, który z czynników jest dominujący, zaobserwować można różną długookresową asymptotykę rozkładu populacji.

Podstawowym wymaganiem przy badaniu asymptotyki długookresowej metodą GRE jest istnienie i jednoznaczność wiodącego wektora własnego (λ, N, ϕ), tzn. rozwiązań następujących stacjonarnych zagadnień własnych:

$$\frac{\partial}{\partial x}(g(x)N(x)) + (B(x)+\lambda)N(x) = \int_x^\infty k(x,y)B(y)N(y)\,\mathrm{d}y,$$
$$g(0)N(0)=0, \quad N(x)>0, \ \text{dla} \ x>0, \quad \int_0^\infty N(x)\,\mathrm{d}x = 1,$$

$$-g(x)\frac{\partial}{\partial x}\phi(x) + (B(x)+\lambda)\phi(x) = B(x)\int_0^x k(y,x)\phi(y)\,\mathrm{d}y,$$
$$\phi(x)>0, \quad \int_0^\infty \phi(x)N(x)\,\mathrm{d}x = 1.$$

Wiemy, zob. np. [29], że gdy dane początkowe, n^0, należą do przestrzeni $L^1(\phi\,\mathrm{d}x)$[1], to rozwiązanie równania (1.7) spełnia następującą nierówność (ang. *relative entropy inequality*)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\infty \phi(x)N(x)H\left(\frac{n(t,x)e^{-\lambda t}}{N(x)}\right)\,\mathrm{d}x\right\} = -D^H(t) \le 0,$$

dla dowolnej nieujemnej funkcji wypukłej H, gdzie D^H oznacza tzw. *entropy dissipation*

$$D^H(t) = \int_0^\infty\int_0^\infty \phi(x)N(y)B(y)k(x,y)\left\{H\left(\frac{n(t,y)e^{-\lambda t}}{N(y)}\right) - H\left(\frac{n(t,x)e^{-\lambda t}}{N(x)}\right)\right.$$
$$\left. - H'\left(\frac{n(t,x)e^{-\lambda t}}{N(x)}\right)\left[\frac{n(t,y)e^{-\lambda t}}{N(y)} - \frac{n(t,x)e^{-\lambda t}}{N(x)}\right]\right\}\mathrm{d}x\,\mathrm{d}y.$$

Gdy dane początkowe są jedynie miarą, stosujemy aproksymację, aby uzyskać ciąg, n_ϵ, rozwiązań równania (1.7) pochodzących z odpowiednio regularnych danych – wówczas każdy element ciągu spełnia powyższą nierówność. W Stwierdzeniu 3.1 w [11] zapewniamy, że możemy przejść do granicy i uzyskać analogiczne nierówności dla rozwiązania miarowego (czy, bardziej ściśle, rozwiązania o wartościach w przestrzeni miar) generowanego przez ciąg n_ϵ.

Warto odnotować pewną szczególną trudność techniczną, jaką po drodze napotykamy – mianowicie wielkość D^H zawiera składnik będący iloczynem wyrazów, w których wartość zregularyzowanego rozwiązania liczona jest w dwóch różnych punktach "przestrzennych". Nie jest wówczas jasne, czy taki iloczyn zbiega do iloczynu odpowiednich słabych granic, ponieważ ewentualne oscylacje w czasie mogą prowadzić do braku zwartości. Mówiąc bardziej precyzyjnie, ciąg generujący miarę Younga sparametryzowaną w czasie i przestrzeni, niekoniecznie generuje tę samą miarę Younga punktowo względem czasu. Ta subtelna

[1] Przez $L^1(\phi\,\mathrm{d}x)$ oznaczamy przestrzeń tych funkcji mierzalnych $f:[0,\infty)\to\mathbb{R}$, dla których norma $\int_0^\infty |f(x)|\,\phi(x)\,\mathrm{d}x$ jest skończona.

różnica może zachodzić dokładnie w sytuacji, gdy ciąg oscyluje z dużą częstotliwością względem zmiennej czasowej.

Aby rozwiązać ten problem potrzebujemy zagwarantować ciągłość względem czasu w jakiejś słabej topologii (np. topologii indukowanej przez tzw. *bounded Lipschitz distance*). Dla powyższego równania wzrostu-podziału dodatkową informację o zwartości zapewnia wynik Carrillo i in. [5]. Dla bardziej ogólnych modeli niezbędne może być wykazanie odpowiednich oszacowań, prowadzących do zastosowania wersji twierdzenia Arzeli-Ascoliego.

Głównym wynikiem tej części rozprawy jest następujące twierdzenie dotyczące asymptotyki długookresowej.

Zbieżność rozwiązań miarowych do stanu stacjonarnego:

Niech $n^0 \in \mathcal{M}(\mathbb{R}_+; \phi)$[2] oraz niech n będzie rozwiązaniem równania wzrostu-podziału (1.7). Wówczas

$$\lim_{t\to\infty} \int_0^\infty \phi(x)\,\mathrm{d}|n(t,x) - m_0 N(x)\mathcal{L}^1| = 0,$$

gdzie $m_0 \coloneqq \int_0^\infty \phi(x)\,\mathrm{d}n^0(x)$, a $\mathcal{L}^1$ oznacza 1-wymiarową miarę Lebesgue'a.

1.2.3 Ściśliwe modele rozwoju tkanek

Modele opisujące żywe tkanki jako płyny stały się bardzo powszechne w biologii matematycznej i skupiają w ostatnich latach wiele uwagi, szczególnie ze względu na ich zastosowanie do modelowania rozwoju komórek nowotworowych. Opisują one dynamikę gęstości komórek, która jest napędzana przez mechaniczne ciśnienie oraz podział komórkowy. Najprostszy tego typu model bierze pod uwagę tylko ograniczenia związane z fizyczną przestrzenią – proliferacja jest hamowana przez "przeludnienie" dostępnego obszaru. Rozważa się wówczas następujące zagadnienie początkowe:

$$\begin{aligned}\frac{\partial n}{\partial t} - \mathrm{div}_x(nv) &= nG(p),\\ n(t=0,x) &= n^0(x) \geq 0,\\ v &= -\nabla_x p,\end{aligned}$$

gdzie

– $n(t,x) \geq 0$ jest gęstością populacji komórek w czasie $t \geq 0$ oraz w pozycji $x \in \mathbb{R}^d$;

– $p(t,x) \geq 0$ jest ciśnieniem wywołanym przez gęstość komórkową;

[2] Przez $\mathcal{M}(\mathbb{R}_+; \phi)$ oznaczamy przestrzeń tych skończonych miar Radona μ, dla których $\int_0^\infty \phi(x)\,\mathrm{d}|\mu|(x) < \infty$.

– $v(t,x)$ oznacza lokalne pole prędkości generowanej przez ciśnienie. Napędza ona ruch komórek i jest powiązana z gradientem ciśnienia poprzez prawo Darcy'ego $v = -\nabla_x p$;

– $G = G(p)$ odpowiada za wzrost i śmierć komórek. Zwykle przyjmuje się, że

$$G(0) > 0, \quad G' < 0, \quad G(p_H) = 0 \text{ dla pewnego } p_H > 0,$$

aby uwzględnić zahamowanie wzrostu, gdy ciśnienie staje się zbyt duże;

– często zakłada się, że ciśnienie spełnia następujące "ściśliwe" prawo stanu

$$p(t,x) \equiv \Pi(n) = \kappa n^\gamma, \;\; \gamma > 1.$$

Związek pomiędzy ciśnieniem i gęstością komórkową może być wykorzystany do odkrycia bogatej struktury analitycznej powyższego równania. Możemy je na przykład zapisać w następującej postaci

$$\frac{\partial n}{\partial t} - \Delta\left(\frac{\kappa\gamma}{\gamma+1} n^{\gamma+1}\right) = nG(p),$$

co, ignorując funkcję wzrostu, daje klasyczne równanie ośrodka porowatego. Ponadto, korzystając z zasady łańcuchowej, możemy wyprowadzić równanie na ciśnienie, p,

$$\frac{\partial p}{\partial t} - \gamma p \Delta p = |\nabla_x p|^2 + \gamma p G(p),$$

które można wykorzystać, aby zrozumieć, jakie są jakościowe różnice pomiędzy ciśnieniem, a gęstością komórkową (np. będą w ogólności miały różne wykładniki całkowalności).

W ostatnich latach badano również wiele pokrewnych, bardziej złożonych modeli z mniej uproszczonym opisem proliferacji komórek. W artykule [10] (we współpracy z M. Schmidtchenem) badamy następujący dwugatunkowy model w jednym wymiarze przestrzennym:

$$\begin{aligned}
&\frac{\partial n_k^{(1)}}{\partial t} - \frac{\partial}{\partial x}\left(n_k^{(1)} \frac{\partial W_k}{\partial x}\right) = n_k^{(1)} G^{(1)}(p_k), \\
&\frac{\partial n_k^{(2)}}{\partial t} - \frac{\partial}{\partial x}\left(n_k^{(2)} \frac{\partial W_k}{\partial x}\right) = n_k^{(2)} G^{(2)}(p_k), \\
&-\nu \frac{\partial^2}{\partial x^2} W_k + W_k = p_k,
\end{aligned} \tag{1.8}$$

zadany na $(0,T) \times \mathbb{R}$, gdzie $n^{(i)}$ reprezentuje gęstość normalnych (odp. anormalnych) komórek, dla $i = 1, 2$, a $k \in \mathbb{N}$ jest daną stałą modelującą "sztywność" ciśnienia, p_k. Mamy teraz dwojakie sprzężenie pomiędzy równaniami. Po pierwsze, tempo wzrostu obu populacji zależy od ciśnienia – które generowane jest przez *całkowitą* gęstość komórkową, $n_k = n_k^{(1)} + n_k^{(2)}$, tzn.

$$p_k := \frac{k}{k-1}\Big(n_k^{(1)} + n_k^{(2)}\Big)^{k-1} = \frac{k}{k-1} n_k^{k-1}.$$

Po drugie, potencjał prędkości, W_k, jest powiązany z ciśnieniem poprzez równanie eliptyczne, zwane zwyczajowo prawem Brinkmana. W przeciwieństwie do prawa Darcy'ego związek ten bierze pod uwagę zjawisko lepkości w poszczególnych gatunkach (istotnie, komórki mogą wchodzić ze sobą w kontakt nawet gdy nie są stłoczone).

Głównym celem wspomnianej wyżej pracy jest uzasadnienie przejścia granicznego ze współczynnikiem sztywności prawa stanu do nieskończoności, tj. $k \to \infty$. Nazwiemy to "nieściśliwą granicą", ponieważ ukazuje ona przejście asymptotyczne od mechanicznego modelu ściśliwego do modelu ze swobodnym brzegiem, który jest w pewnym sensie uogólnieniem klasycznego równania Hele-Shaw i stanowi inne powszechne w literaturze matematyczne podejście do opisu rozwoju komórek nowotworowych.

Odsyłamy czytelnika np. do prac [30, 31], oraz literatury w nich cytowanej, gdzie znajdzie obszerną analizę podobnego lepko-sprężystego modelu w przypadku jednogatunkowym. Jak wyżej, pole prędkości można wyznaczyć rozwiązując równanie eliptycznie związane z ciśnieniem, które w ich przypadku dane jest jako potęga jedynej gęstości komórkowej.

Wprowadzenie drugiego gatunku oraz sprzężenie równań drastycznie zmienia własności modelu. W szczególności metoda skompensowanej zwartości zastosowana w [30] nie może być zastosowana (a przynajmniej jej potencjalne zastosowanie jest dalece nieoczywiste), a zatem należy opracować inne metody, lepiej dostosowane do analizy modelu wielogatunkowego.

Zaznaczmy również, że nawet w przypadku nielepkim, $\nu = 0$, fakt, że mamy do czynienia z układem prowadzi do całej gamy dodatkowych trudności, zob. [3, 4, 21]. W tym przypadku ciśnienie zyskuje trochę na regularności – jest to jednak wystarczające jedynie do zapewnienia zwartości jego gradientu. Podkreślmy, że napotykamy podobne trudności również, gdy prawo stanu nie przyjmuje formy potęgowej, a ciśnienie posiada osobliwości, zob. np. [6, 8, 22].

Cechą wspólną dla każdego z powyższych przypadków jest to, że do ich analizy niezbędne jest dokładne przestudiowanie równania spełnionego przez ciśnienie populacji. Pozwala to na uzyskanie wyników o istnieniu rozwiązań oraz wyprowadzenie jednostajnych (względem parametru k) oszacowań. W pracy [10] również podążamy tą ścieżką. Proste zastosowanie zasady łańcuchowej prowadzi, w połączeniu z równaniem na całkowitą gęstość komórkową, $n_k = n_k^{(1)} + n_k^{(2)}$, do

$$\frac{\partial p_k}{\partial t} - \frac{\partial p_k}{\partial x}\frac{\partial W_k}{\partial x} = \frac{k-1}{\nu} p_k \Big[W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k) G^{(2)}(p_k)\Big], \tag{1.9}$$

gdzie wielkość $r_k := n_k^{(1)}/n_k$ spełnia

$$\frac{\partial r_k}{\partial t} - \frac{\partial r_k}{\partial x}\frac{\partial W_k}{\partial x} = r_k(1-r_k)\Big[G^{(1)}(p_k) - G^{(2)}(p_k)\Big].$$

Stawiając stosowne wymagania na dane początkowe, otrzymujemy następujące wyniki.

Istnienie rozwiązań (Twierdzenie 2.1 w [10]):

Dla dowolnego $k \geq 2$, układ (1.8) posiada jednoznaczne słabe rozwiązanie $(n_k^{(1)}, n_k^{(2)})$ z $n_k^{(i)} \in L^\infty(0,T;BV(\mathbb{R}))$, $i = 1,2$.

Segregacja (Lemat 3.3 w [10]):

Jeżeli obie populacje są oddzielone w czasie początkowym, tzn. $r_k(0,x)(1 - r_k(0,x)) = 0$, dla p.w. $x \in \mathbb{R}$, to pozostają oddzielone w każdej kolejnej chwili czasu, tzn. $r_k(t,x)(1-r_k(t,x)) = 0$ dla każdego $t \in [0,T]$ i p.w. $x \in \mathbb{R}$.

Nieściśliwa granica oraz zasada komplementarności (Twierdzenie 2.2 w [10])

Możemy przejść do granicy $k \to \infty$ w równaniu na ciśnienie (1.9). Daje to tzw. zasadę komplementarności, tj.

$$0 = p_\infty\Big[W_\infty - p_\infty + \nu n_\infty^{(1)} G^{(1)}(p_\infty) + \nu n_\infty^{(2)} G^{(2)}(p_\infty)\Big], \tag{1.10}$$

w sensie dystrybucyjnym, gdzie $n_\infty^{(i)}$, $i = 1,2$, spełnia

$$\frac{\partial n_\infty^{(i)}}{\partial t} - \frac{\partial}{\partial x}\left(n_\infty^{(i)}\frac{\partial W_\infty}{\partial x}\right) = n_\infty^{(i)} G^{(i)}(p_\infty),$$

$$-\nu\frac{\partial^2 W_\infty}{\partial x^2} + W_\infty = p_\infty.$$

Ponadto mamy, prawie wszędzie,

$$p_\infty(n_\infty - 1) = 0.$$

Twierdzenie 2.2 dostarcza ścisłego związku pomiędzy opisem ewolucji dwu populacji komórek (1.8), a geometrycznym zagadnieniem ze swobodnym brzegiem typu Hele-Shaw. Obszary o dodatnim ciśnieniu odpowiadają w pełni wysyconym strefom, ponieważ mamy $p_\infty(n_\infty - 1) = 0$, a ciśnienie na takich obszarach zadane jest zasadą komplementarności (1.10). Ścisłe wyprowadzenie tego typu powiązania znane było dotychczas w przypadku jednogatunkowym (choć w dowolnym wymiarze przestrzennym), por. [30]. Podkreślmy, że możliwe jest występowanie nieciągłości skokowych w funkcji ciśnienia, co sprawia, że uzyskanie zwartości jest dalece nietrywialne.

Nasz dowód opiera się w głównej mierze na uzyskaniu jednostajnych oszacowań w przestrzeni BV jednocześnie dla poszczególnych gęstości komórkowych oraz dla całkowitej populacji. W połączeniu ze znanym kryterium zwartościowym (zob. [23, Lemma A]) wnioskujemy o silnej zwartości ciągu ciśnień, która jest wystarczająca do wykonania przejścia granicznego. Niestety, próby rozszerzenia tej strategii do przypadku wyżej wymiarowego, jak również zastosowania do układu wielogatunkowego metody skompensowanej zwartości z pracy [30], wydają się daremne.

Wspomnijmy jednak, że możliwe okazało się obejście powyższych problemów. Udało nam się to w pracy [13] (we współpracy z B. Perthame, M. Schmidtchenem oraz N. Vauchelet), gdzie otrzymujemy wynik analogiczny do cytowanego wyżej Twierdzenia 2.2, ale bez ograniczeń na wymiar przestrzeni. Stosujemy w tym celu połączenie kilku technik: tych stosowanych dla jednego gatunku w dowolnym wymiarze, dla układu w jednym wymiarze, oraz nielokalnego kryterium zwartościowego zaproponowanego przez Brescha i Jabina dla ściśliwego układu Naviera-Stokesa, zob. [2].

Chapter 2

Extended Abstract

2.1 Introduction and motivation

From its inception, PDE theory has been driven by questions arising in physics, biology, and other natural sciences. Many important physical phenomena (*e.g.*, motion of fluids, behaviour of elastic materials, cell growth) are believed to be reasonably well described by systems of, usually nonlinear, partial differential equations. Many substantial open problems are waiting to be solved, ranging from the most fundamental mathematical questions such as existence, uniqueness and stability of solutions, through qualitative properties of solutions, to validation of a purely mathematical PDE model via numerics and applications. The practical importance of these questions makes this challenging field both exciting and rewarding, inspiring relentless research efforts of generations of mathematicians.

This thesis concerns the following three leading research directions:

Relation between regularity and energy conservation in fluid mechanics: Here we investigate the following question: *How much regularity does a distributional solution to a given system of equations need to possess so that it satisfies an associated balance of energy?* In particular, we work with the compressible Euler equations and the Euler-Korteweg equations. This part of the thesis comprises of the following two publications:

> T. Dębiec, P. Gwiazda, A. Świerczewska-Gwiazda and A. Tzavaras, Energy conservation for the Euler-Korteweg equations. *Calc. Var. Partial Differential Equations* 57(6): Art. 160, 2018;
>
> I. Akramov, T. Dębiec, J. Skipper and E. Wiedemann, Energy conservation for the compressible Euler and Navier-Stokes equations with vacuum. *Anal. PDE*, 13(3):789–811, 2020.

Applications of the *generalised relative entropy* method for measure solutions

of structured population models: We investigate one of the key models in the field of population dynamics, namely the growth-fragmentation equation in a fairly general form. The main result is to guarantee that solutions emanating from data in the space of measures converge, in an appropriate sense, to a steady-state profile. This part of the thesis comprises of the following publication:

> T. Dębiec, M. Doumic, P. Gwiazda and E. Wiedemann, Relative entropy method for measure solutions of the growth-fragmentation equation. *SIAM J. Math. Anal.* 50(6):5811–5824, 2018.

Two-species models of tissue growth: Finally, we study a two-species model with applications in tumour modelling coupled through Brinkman's law. Here the main results are about existence and uniqueness of distributional solutions, as well as passing to the stiff pressure law limit to make a rigorous link with a geometric model, wherein the tumour is described as a constrained fluid immersed in an unconstrained compressible fluid (the surrounding tissue). This part of the thesis comprises of the following publication:

> T. Dębiec and M. Schmidtchen, Incompressible limit for a two-species tumour model with coupling through Brinkman's law in one dimension. *Acta Appl. Math.*, to appear, 2020, https://doi.org/10.1007/s10440-020-00313-1.

In the following section we describe in more detail these three areas of research and summarise the results obtained in the above papers.

2.2 Presentation of the main results

2.2.1 Energy conservation in fluid mechanics

The first part of this thesis is concerned with the issue of whether weak solutions to a given conservation or balance law obey additional *companion laws*, in particular an energy balance. Evolutionary partial differential equations usually come endowed with certain additional, physically meaningful, quantities – such as energy. These quantities can be shown to be, at least formally, *conserved*, *i.e.*, depend only on the initial conditions and not change in time. However, it is notoriously known that such a formally derived conservation law may break down for weak solutions of low regularity. Indeed, not only do weak solutions defy uniqueness, they often posses the unmannerly trait of exhibiting non-physical features, such as spontaneous creation or destruction of energy. The intimate relation between regularity and conserved quantities is related to the concept of renormalisation in the DiPerna-Lions [14] theory, and the famous *Onsager conjecture* [28] about energy conservation for the incompressible Euler equations. It states that there is a threshold regularity for conservation of kinetic energy – smooth enough weak solutions (namely those, which posses 1/3 of a spatial derivative) will always conserve energy, while below the threshold there is possibility of anomalous dissipation. This research direction enjoyed an enormous amount of interest in the past few decades, and famously motivated the development of convex integration techniques in the context of fluid dynamics. Both parts of the conjecture, the one guaranteeing energy conservation for smooth enough solutions, and the flexible one, became lively and potent research programmes, which culminated in the full proof of Onsager's prediction [7, 24], as well as numerous studies for other systems of equations coming from fluid dynamics, and general conservation laws.

In [12] (in collaboration with P. Gwiazda, A. Świerczewska-Gwiazda and A. Tzavaras) we consider the following Euler-Korteweg system of equations:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) + \operatorname{div}_x(\rho u \otimes u) &= -\rho \nabla_x \left(h'(\rho) + \frac{\kappa'(\rho)}{2} |\nabla_x \rho|^2 - \operatorname{div}_x(\kappa(\rho) \nabla_x \rho) \right), \\ \frac{\partial \rho}{\partial t} + \operatorname{div}_x(\rho u) &= 0, \end{aligned} \tag{2.1}$$

in $(0, T) \times \mathbb{T}^d$ for some $T > 0$, where

- $d > 1$ denotes the spatial dimension and $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is the flat 1-dimensional torus;
- $\rho(t, x) \geq 0$ is the fluid density at time $t \geq 0$ and position $x \in \mathbb{T}^d$;
- $u(t, x) \in \mathbb{R}^d$ is the local velocity field of the fluid;
- $h(\rho)$ is the energy density and $\kappa(\rho) > 0$ is the coefficient of capillarity. We place the following assumption on the functions h and κ:

$$h, \kappa \in C^3(\mathcal{T}), \tag{2.2}$$

  where, depending on the actual form of h and κ, the set $\mathcal{T}$ can be chosen to be $[0, \infty)$ or $(0, \infty)$.

System (2.1) is a well-known model for describing liquid-vapour flows, see, *e.g.*, [16] for a modern thermodynamical derivation. It can be easily shown that smooth solutions satisfy the following local energy balance

$$\begin{aligned} &\frac{\partial}{\partial t} \left(\frac{1}{2} \rho |u|^2 + h(\rho) + \frac{1}{2} \kappa(\rho) |\nabla_x \rho|^2 \right) \\ &+ \operatorname{div}_x \left[\rho u \left(\frac{1}{2} |u|^2 + h'(\rho) + \frac{1}{2} \kappa'(\rho) |\nabla_x \rho|^2 - \operatorname{div}_x(\kappa(\rho) \nabla_x \rho) \right) + \kappa(\rho) \nabla_x \rho \operatorname{div}_x(\rho u) \right] = 0, \end{aligned} \tag{2.3}$$

in the sense of distributions. The existence of "wild" solutions is possible for (2.1), as pointed out in the recent work of Donatelli et al. [15], where the method of convex integration is adapted to show non-uniqueness in the class of bounded dissipative global weak solutions. The possibility of both conservative and dissipative solutions raises the issue of studying the Onsager conjecture for the Euler-Korteweg system. In [12] we establish the following sufficient regularity condition (in appropriate Besov spaces) for weak solutions of (2.1) to conserve the total energy.

Energy conservation for the Euler-Korteweg equations (Theorem 1.1 in [12]):

Let (ρ, u) be a solution of (2.1) in the sense of distributions. Assume

$$u \in (B_3^{\alpha,\infty} \cap L^\infty)((0,T) \times \mathbb{T}^d), \quad \rho, \nabla_x \rho, \Delta\rho \in (B_3^{\beta,\infty} \cap L^\infty)((0,T) \times \mathbb{T}^d),$$

where $1 > \alpha \geq \beta > 0$ such that $\min(2\alpha + \beta, \alpha + 2\beta) > 1$.
Then the energy balance (2.3) holds in the sense of distributions on $(0,T) \times \mathbb{T}^d$.

Let us comment on assumption (2.2). It is not extremely restrictive in that it does not exclude the two most commonly studied cases of the capillarity coefficient, namely the constant capillarity case, $\kappa(\rho) \equiv \kappa$, and the choice $\kappa(\rho) = \frac{\epsilon_0^2}{4\rho}$, with ϵ_0 denoting the Planck constant, which leads to the quantum-hydrodynamics system:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x p(\rho) &= \frac{\epsilon_0^2}{2}\rho \nabla_x \left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \right), \\ \frac{\partial \rho}{\partial t} + \operatorname{div}_x(\rho u) &= 0. \end{aligned}$$

From a technical point of view, assumption (2.2) is required to bound certain terms in commutator estimates. In an earlier work, Feireisl et al. [17] proved an Onsager-type sufficient condition for weak solutions of the compressible Euler equations under the assumption that the pressure, $p = p(\rho)$, is twice continuously differentiable in the range of the density. They could then employ a commutator estimate method, which was earlier applied only to quadratic-type nonlinearities, as in [7] for the incompressible Euler system.

It turns out that removing the C^2 assumption on the nonlinearities of the system is an interesting and highly non-trivial problem. In [1] (in collaboration with I. Akramov, J. Skipper and E. Wiedemann) we consider the isentropic compressible Euler equations:

$$\begin{aligned} \frac{\partial}{\partial t}(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x p(\rho) &= 0, \\ \frac{\partial \rho}{\partial t} + \operatorname{div}_x(\rho u) &= 0. \end{aligned} \tag{2.4}$$

The local form of energy equality for this system takes the form

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho|u|^2 + P(\rho)\right) + \operatorname{div}_x\left[\left(\frac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\right)u\right] = 0, \tag{2.5}$$

in the sense of distributions, where P is the pressure potential defined by

$$P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2}\, \mathrm{d}r.$$

In [17] Feireisl et al. provide the first extension of the result of Constantin et al. [7] to compressible gas dynamics. They consider distributional solutions of (2.4) such that

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \overline{\rho} \text{ a.e. in } (0,T)\times\mathbb{T}^d,$$

for some constants $\underline{\rho}$, $\overline{\rho}$, and $0 \le \alpha, \beta \le 1$ such that $\min(2\alpha+\beta, \alpha+2\beta) > 1$. Then, assuming additionally that $p \in C^2[\underline{\rho}, \overline{\rho}]$, they show that the local energy conservation law (2.5) holds in the sense of distributions on $(0,T)\times\mathbb{T}^d$, *cf.* [17, Theorem 4.1].

Notice that the C^2 assumption on the pressure is clearly satisfied by the archetypal power law pressure $p(\rho) = \kappa\rho^\gamma$, for any $\gamma > 2$. However, in the range $1 < \gamma < 2$ vacuum states have to be excluded (*i.e.*, $\underline{\rho} > 0$). But this regime is certainly a relevant one (for instance, a monoatomic gas has $\gamma = 5/3$, while a diatomic gas has $\gamma = 7/5$).

The possible appearance of vacuum regions constitutes a degeneracy that in many aspects greatly complicates the mathematical analysis of compressible models. For instance, the compressible Euler equations cease to be strictly hyperbolic in such regions. In particular, densities close to zero invalidate the above mentioned methods designed to treat commutator errors due to nonlinearities as quadratic expressions in the commutator estimates.

In [1] we give a number of sufficient conditions to ensure energy conservation even after possible formation of vacuum. First, it is assumed that the velocity is a *divergence-measure field*, a well-known notion in geometric measure theory and hyperbolic conservation laws. Admittedly, it is a lot to ask of a weak solution to (2.4), however a justification comes from the compressible Navier-Stokes system, whose *a priori* estimates ensure this condition, see Corollary 3.3 in [1].

The following result provides a sufficient condition for energy conservation assuming an estimate for the quotient between the density and its mollification.

Energy conservation for the compressible Euler equations with vacuum (Theorem 4.1 in [1]):

Let ρ, u be a solution of (2.4) in the sense of distributions. Assume

$$u \in B_p^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_q^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \overline{\rho} \text{ a.e. in } (0,T)\times\mathbb{T}^d,$$

for some constants $\underline{\rho}$, $\overline{\rho}$ and $0 \le \alpha, \beta \le 1$ such that,

$$\frac{1}{p}+\frac{2}{q} \le 1, \qquad \frac{2}{p}+\frac{1}{q} \le 1, \qquad p, q \ge 2, \qquad \alpha+\gamma\beta > 1 \qquad \text{and} \qquad 2\alpha+\beta > 1.$$

Define $\mathcal{B}_{\epsilon^\beta} := \{x : 0 < \rho^\epsilon(x) < \epsilon^\beta \text{ and } \rho \ne 0\}$ and assume that

$$\left\| \frac{\rho^\epsilon - \rho}{\rho^\epsilon} \right\|_{L^q(\mathcal{B}_{\epsilon^\beta})} \le C(\rho), \tag{2.6}$$

where C does not depend on ϵ. Assume further that $p \in C^{1,(\gamma-1)}([\underline{\rho}, \overline{\rho}])$, and, in addition, $p'(0) = 0$ as soon as $\underline{\rho} = 0$.
Then the energy is locally conserved, *i.e.*, (2.5) holds in the sense of distributions on $(0, T) \times \mathbb{T}^d$.

Condition (2.6) arises from careful analysis of the commutators due to the pressure, and may seem rather artificial. Some more natural conditions that ensure (2.6) to hold are identified in [1]: Under the slightly stronger assumption of Hölder (instead of Besov) regularity, but with the expected exponents, we show energy conservation *no matter how the density behaves near vacuum* [1, Corollary 4.4]. If one does want to assume only Besov regularity, then one needs to make further assumptions on the density near vacuum: for example, density descending into vacuum sufficiently fast [1, Corollary 4.6] or sufficiently slowly [1, Corollary 4.10]. These results can also be extended to a smooth domain with boundary, under a continuity assumption on the normal component of the velocity [1, Corollary 5.1].

In a related study [9], I investigate a generalisation of the above work for general systems of conservation laws:

$$\operatorname{div}_X G(U(X)) = 0,$$

with non-C^2 fluxes, for weak solutions satisfying a Besov-VMO type condition, first introduced by Fjordholm and Wiedemann [18] in the context of the incompressible Euler equations.

2.2.2 Long-time behaviour of measure solutions to a structured population model

Structured population models are transport type equations developed to understand the dynamics of a population, taking into account its distribution along some "structuring" variables, such as age, size, or phenotypic traits. Examples include models of cell growth and division, polymerisation, cell saturation, prion proliferation, and many more. From the practical viewpoint it is important to understand the behaviour of solutions to such models over a long time, see, *e.g.*, [32].

The problem of long-time asymptotics for L^1 solutions of structured population models had already been addressed in the middle 80's, starting with arguably the simplest such model, the renewal equation. Various methods were considered, giving rise to results of convergence to a steady profile. An especially potent approach was developed by Perthame and collaborators [25–27]. Their *generalised relative entropy* (GRE) method provides a way to study long-time asymptotics of linear models even when no spectral gap is guaranteed. Recently Gwiazda and Wiedemann [19] combined the GRE method with variational and measure-theoretic tools, extending it to study the renewal equation with measure initial data. Let us stress that this is biologically relevant, as it allows for treating situations when the population is initially concentrated with respect to the structuring variable.

This is typically the case when departing from a population formed by a single cell. Although their general strategy can, in principle, be applied to obtain similar results for more general population models, there is a number of substantial difficulties which may have to be dealt with. This is seen in [11] (in collaboration with M. Doumic, P. Gwiazda and E. Wiedemann), were we treat a general linear growth-fragmentation equation with measure data:

$$\begin{aligned}\frac{\partial n}{\partial t}(t,x) + \frac{\partial}{\partial x}(g(x)n(t,x)) + B(x)n(t,x) &= \int_x^\infty k(x,y)B(y)n(t,y)\,\mathrm{d}y,\\ g(0)n(t,0) &= 0,\\ n(0,x) &= n^0(x),\end{aligned} \tag{2.7}$$

where

- $n(t,x)$ represents the concentration of individuals of size $x \geq 0$ at time $t > 0$;
- $g(x) \geq 0$ is the growth rate of the individuals, while $B(x) \geq 0$ is their division rate;
- $k(x,y)$ is the proportion of individuals of size x created out of the division of individuals of size y.

This equation incorporates a very important phenomenon in biology – competition between growth and fragmentation. Clearly they have opposite dynamics: growth drives the population towards a larger size, while fragmentation makes it smaller and smaller. Depending on which factor dominates, one can observe various long-time behaviours of the population distribution.

The fundamental tool in studying the long-time asymptotics with the GRE method is the existence and uniqueness of the first eigenelements (λ, N, ϕ), *i.e.*, solutions to the following primal and dual eigenproblems:

$$\begin{aligned}&\frac{\partial}{\partial x}(g(x)N(x)) + (B(x)+\lambda)N(x) = \int_x^\infty k(x,y)B(y)N(y)\,\mathrm{d}y,\\ &g(0)N(0) = 0, \quad N(x) > 0, \ \text{ for } \ x > 0, \quad \int_0^\infty N(x)\,\mathrm{d}x = 1,\end{aligned}$$

$$\begin{aligned}&-g(x)\frac{\partial}{\partial x}\phi(x) + (B(x)+\lambda)\phi(x) = B(x)\int_0^x k(y,x)\phi(y)\,\mathrm{d}y,\\ &\phi(x) > 0, \quad \int_0^\infty \phi(x)N(x)\,\mathrm{d}x = 1.\end{aligned}$$

It is known, see, *e.g.*, [29], that if $n^0 \in L^1(\phi\,\mathrm{d}x)$[1], then the solution to equation (2.7) satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\int_0^\infty \phi(x)N(x)H\left(\frac{n(t,x)e^{-\lambda t}}{N(x)}\right)\,\mathrm{d}x\right\} = -D^H(t) \leq 0,$$

[1] By $L^1(\phi\,\mathrm{d}x)$ we denote the space of those measurable functions $f : [0,\infty) \to \mathbb{R}$, for which the norm $\int_0^\infty |f(x)|\,\phi(x)\,\mathrm{d}x$ is finite.

for any non-negative convex function H, where D^H denotes the *entropy dissipation*

$$D^H(t) = \int_0^\infty \int_0^\infty \phi(x)N(y)B(y)k(x,y)\Bigg\{ H\left(\frac{n(t,y)e^{-\lambda t}}{N(y)}\right) - H\left(\frac{n(t,x)e^{-\lambda t}}{N(x)}\right) - H'\left(\frac{n(t,x)e^{-\lambda t}}{N(x)}\right)\left[\frac{n(t,y)e^{-\lambda t}}{N(y)} - \frac{n(t,x)e^{-\lambda t}}{N(x)}\right]\Bigg\}\,\mathrm{d}x\,\mathrm{d}y.$$

When the initial data is only a measure, we proceed by first approximating it to obtain a sequence, n_ϵ, of solutions to (2.7) with integrable data – then each element of the sequence satisfies the above relative entropy inequality. In [11, Proposition 3.1] we prove that it is possible to pass to the limit to obtain corresponding inequalities for the measure-valued solution generated by the sequence n_ϵ.

Let us remark about an additional technical difficulty found in this pursuit – namely, the entropy dissipation contains a product term wherein the regularised solution depends on two different values of the structuring variable. It is not immediate that this product converges to the corresponding product of the weak limits, since fast oscillations in time might lead to lack of compactness. More technically speaking, a sequence generating a Young measure parametrised in time and space need *not* generate the same Young measure pointwise in time, and this subtle difference occurs precisely due to possible oscillations in time. To deal with this problem one has to guarantee continuity with respect to time in some weak topology (*e.g.*, the one induced by the bounded Lipschitz distance). For the growth-fragmentation model the additional compactness is available by virtue of a result of Carrillo et al. [5], but for more general models one would have to develop with extra care the right estimates, leading eventually to the application of some version of the Arzela-Ascoli theorem.

The main result of this part of the thesis is the following long-time asymptotics result.

Convergence of measure solutions towards stationary state:

Let $n^0 \in \mathcal{M}(\mathbb{R}_+;\phi)^2$ and let n solve the growth-fragmentation equation (2.7). Then

$$\lim_{t\to\infty} \int_0^\infty \phi(x)\,\mathrm{d}|n(t,x) - m_0 N(x)\mathcal{L}^1| = 0,$$

where $m_0 \coloneqq \int_0^\infty \phi(x)\,\mathrm{d}n^0(x)$ and $\mathcal{L}^1$ denotes the 1-dimensional Lebesgue measure.

2.2.3 Compressible models for tumour growth

Fluid-based models for living tissue have become ubiquitous in mathematical biology and attracted a lot of interest and research output, notably due to their applications in tumour

[2] By $\mathcal{M}(\mathbb{R}_+;\phi)$ we denote the space of those positive Radon measures, μ, for which $\int \phi\,\mathrm{d}|\mu| < \infty$.

modelling. They describe dynamics of cell densities, governed by mechanical pressure and cell division. The simplest such model only takes into account the restrictions caused by availability of space – namely, contact inhibition prevents cell proliferation. One then considers the following initial-value problem

$$\begin{aligned}\frac{\partial n}{\partial t} - \mathrm{div}_x(nv) &= nG(p),\\ n(t=0,x) &= n^0(x) \geq 0,\\ v &= -\nabla_x p,\end{aligned}$$

where

- $n(t,x) \geq 0$ is the population density of cells at time $t \geq 0$ and position $x \in \mathbb{R}^d$;
- $p(t,x) \geq 0$ is the mechanical pressure induced by the cell density;
- $v(t,x)$ is the local velocity field generated by the pressure, which drives cell motion; it is related to the pressure gradient via Darcy's law $v = -\nabla_x p$;
- $G = G(p)$ accounts for the growth and death of cells, and is usually assumed to satisfy

  $$G(0) > 0, \quad G' < 0, \quad G(p_H) = 0 \text{ for some } p_H > 0,$$

  to account for an inhibited growth whenever the pressure gets too large;
- the pressure is assumed to satisfy the following compressible state law relation

  $$p(t,x) \equiv \Pi(n) = \kappa n^\gamma, \;\; \gamma > 1.$$

The relation between the pressure and population density can be exploited to unravel a rich analytical structure of the above equation. For instance, we can write

$$\frac{\partial n}{\partial t} - \Delta\left(\frac{\kappa\gamma}{\gamma+1}n^{\gamma+1}\right) = nG(p),$$

which, in the absence of growth rate, is the porous medium equation. Furthermore, using the chain rule, one can derive an equation for p

$$\frac{\partial p}{\partial t} - \gamma p \Delta p = |\nabla_x p|^2 + \gamma p G(p),$$

which can be studied to understand how the properties of the cell density and the pressure differ (*e.g.*, their integrability exponents will in general be different).

Many more complex variants of the above model have been studied in recent years by introducing less simplified descriptions of cell proliferation. In [10] we study (in collaboration with M. Schmidtchen) the following two-species model in one spatial dimension

$$\frac{\partial n_k^{(1)}}{\partial t} - \frac{\partial}{\partial x}\left(n_k^{(1)} \frac{\partial W_k}{\partial x}\right) = n_k^{(1)} G^{(1)}(p_k),$$

$$\frac{\partial n_k^{(2)}}{\partial t} - \frac{\partial}{\partial x}\left(n_k^{(2)} \frac{\partial W_k}{\partial x}\right) = n_k^{(2)} G^{(2)}(p_k), \tag{2.8}$$

$$-\nu \frac{\partial^2}{\partial x^2} W_k + W_k = p_k,$$

posed on $(0, T) \times \mathbb{R}$, where $n^{(i)}$ represents the normal (resp. abnormal) cells, for $i = 1, 2$, and $k \in \mathbb{N}$ is a given constant modelling the stiffness of the total population pressure, p_k. There is now a twofold coupling between the equations. First, the rate of growth of each population density depends on the pressure, generated by the *total* population density, *i.e.*,

$$p_k := \frac{k}{k-1}\left(n_k^{(1)} + n_k^{(2)}\right)^{k-1} \equiv \frac{k}{k-1} n_k^{k-1}.$$

Second, the potential, W_k, for the velocity is linked to the pressure by an elliptic equation. This connection is usually referred to as Brinkman's law. Compared to Darcy's law, this coupling accounts for viscosity effects in the individual species (the cells can touch even when they are not packed).

The main objective of [10] is to study the incompressible limit, $k \to \infty$, as the stiffness of the state law relating the pressure to the sum of the densities goes to infinity. In doing so, we make a connection between the above mechanical compressible system and a free-boundary problem wherein the tumour is described as a constrained geometric domain.

We refer to [30, 31], and references therein, for a treatise of the incompressible limit for a related one-species visco-elastic tumour model. As above, the velocity field satisfies an elliptic equation involving the pressure, itself given by a power of the single species. The behaviour of the coupled system of equations for the individual species turns out to be substantially different. In particular, the strategy of a kinetic reformulation used in [30] cannot be applied directly, at least not in a straightforward manner, and a different approach has to be sought.

In fact, the system nature of the problem causes additional difficulties even in the inviscid case $\nu = 0$, see [3, 4, 21]. The pressure does gain a bit in regularity – however, it is just about sufficient to obtain strong compactness of its gradient. We highlight that similar difficulties arise when the pressure is not given in form of a power law and contains singularities, *cf.* [6, 8, 22].

The common feature of all the above cases is their thorough study of the equation satisfied by the population pressure, which allows for proving the existence of solutions and obtaining estimates that are uniform in the stiffness parameter, k. We follow this path in [10] as well. An easy application of the chain rule in conjunction with an equation satisfied by the total population density, $n_k = n_k^{(1)} + n_k^{(2)}$, leads to

$$\frac{\partial p_k}{\partial t} - \frac{\partial p_k}{\partial x}\frac{\partial W_k}{\partial x} = \frac{k-1}{\nu} p_k \Big[W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k)G^{(2)}(p_k)\Big], \tag{2.9}$$

with the population fraction, $r_k := n_k^{(1)}/n_k$, given by

$$\frac{\partial r_k}{\partial t} - \frac{\partial r_k}{\partial x}\frac{\partial W_k}{\partial x} = r_k(1-r_k)\Big[G^{(1)}(p_k) - G^{(2)}(p_k)\Big].$$

Under appropriate assumptions on the initial data, we obtain the following results.

Existence of solutions (Theorem 2.1 in [10]):

For any $k \geq 2$, system (2.8) admits a unique distributional solution $(n_k^{(1)}, n_k^{(2)})$ with $n_k^{(i)} \in L^\infty(0,T; BV(\mathbb{R}))$, $i = 1, 2$.

Segregation (Lemma 3.3 in [10]):

If initially the species are segregated, *i.e.*, $r_k(0,x)(1 - r_k(0,x)) = 0$ for a.e. $x \in \mathbb{R}$, then they stay segregated at later times, *i.e.*, $r_k(t,x)(1 - r_k(t,x)) = 0$ for every $t \in [0,T]$ and a.e. $x \in \mathbb{R}$.

Incompressible limit and complementarity relation (Theorem 2.2 in [10])

We may pass to the limit $k \to \infty$ in the pressure equation (2.9). This yields the so-called complementarity relation

$$0 = p_\infty \Big[W_\infty - p_\infty + \nu n_\infty^{(1)} G^{(1)}(p_\infty) + \nu n_\infty^{(2)} G^{(2)}(p_\infty)\Big], \tag{2.10}$$

in the distributional sense, where $n_\infty^{(i)}$, $i = 1, 2$, satisfies

$$\frac{\partial n_\infty^{(i)}}{\partial t} - \frac{\partial}{\partial x}\left(n_\infty^{(i)} \frac{\partial W_\infty}{\partial x}\right) = n_\infty^{(i)} G^{(i)}(p_\infty),$$

$$-\nu \frac{\partial^2 W_\infty}{\partial x^2} + W_\infty = p_\infty.$$

Moreover, the following holds true almost everywhere

$$p_\infty(n_\infty - 1) = 0.$$

Theorem 2.2 provides a rigorous link between the mechanical model (2.8) for the evolution of the two populations, and a geometric free-boundary model reminiscent of the classical Hele-Shaw equation. The relation $p_\infty(n_\infty - 1) = 0$ means that positive pressure regions correspond to fully saturated areas. On such domains the pressure is given by the so-called *complementarity relation* (2.10). A rigorous derivation of the incompressible limit was already known for a visco-elastic tumour model in the one species case, in any dimension, *cf.* [30]. Let us emphasise the possibility of jump discontinuities in the pressure which renders the problem of obtaining compactness rather challenging.

Our proof mainly relies on establishing uniform BV-bounds for the two species as well as the total population. By virtue of the compactness criterion [23, Lemma A] we can infer strong compactness of the pressure which suffices to pass to the limit. Unfortunately, an extension of this strategy to higher dimensions appears futile, as does the extension of [30] to two species due to the contribution of the individual species and their role in the identification of weak-$\star$ limits in the kinetic reformulation.

Let us however remark that a remedy to these obstacles has been found recently (in collaboration with B. Perthame, M. Schmidtchen and N. Vauchelet) in the paper [13], which is not part of this thesis. We have found that a combination of techniques for the one-species case in any space dimension and the two-species case in one dimension is possible, which, in conjunction with a nonlocal compactness criterion, originally introduced and devised by Bresch and Jabin, *cf.* [2], yields the required compactness and allows the passage to the incompressible limit.

Chapter 3

Conservation of energy for the Euler-Korteweg equations [12]

Calc. Var. (2018) 57:160
https://doi.org/10.1007/s00526-018-1441-8

Calculus of Variations

Conservation of energy for the Euler–Korteweg equations

Tomasz Dębiec[1] · Piotr Gwiazda[2] · Agnieszka Świerczewska-Gwiazda[1] · Athanasios Tzavaras[3]

Received: 23 December 2017 / Accepted: 7 September 2018 / Published online: 29 September 2018

Abstract
In this article we study the principle of energy conservation for the Euler–Korteweg system. We formulate an Onsager-type sufficient regularity condition for weak solutions of the Euler–Korteweg system to conserve the total energy. The result applies to the system of Quantum Hydrodynamics.

Mathematics Subject Classification 76D45 · 35G50

1 Introduction

It is known since the works of Scheffer [28] and Shnirelmann [29] that weak solutions of the incompressible Euler equations exhibit behaviour very different to that of classical solutions. These "wild solutions", as they are called since the seminal works of DeLellis and Székelyhidi [10,11], are often highly unphysical—for instance there is a lack of uniqueness and the principle of conservation of energy can be violated.

Dissipative solutions of incompressible Euler have been extensively studied in relation to the seminal Onsager conjecture [27]. It states that there is a threshold regularity, namely $\frac{1}{3}$-Hölder continuity, above which kinetic energy must be conserved, and below which

Communicated by C. De Lellis.

Agnieszka Świerczewska-Gwiazda
aswiercz@mimuw.edu.pl

Tomasz Dębiec
t.debiec@mimuw.edu.pl

Piotr Gwiazda
pgwiazda@mimuw.edu.pl

Athanasios Tzavaras
athanasios.tzavaras@kaust.edu.sa

1 Institute of Applied Mathematics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

2 Institute of Mathematics, Polish Academy of Sciences, Śniadeckich 8, 00-656 Warsaw, Poland

3 Computer, Electrical, Mathematical Sciences and Engineering Division, King Abdullah University of Science and Technology (KAUST), Thuwal, Saudi Arabia

anomalous dissipation might occur. This conjecture has been recently fully resolved, with non-conservative solutions of class $\mathscr{C}([0,T];\mathscr{C}^{\frac{1}{3}-}(\mathbb{T}^3))$ constructed by Isett [23]. See also [5,24] for further developments on the subject.

The positive direction of Onsager's conjecture has been settled already in the 1990's by Constantin et al. [9] (after a partial result of Eyink [17]). The method of mollification and estimation of commutator errors was employed to prove that, if a weak solution u of the incompressible Euler system belongs to $L^3([0,T],B_3^{\alpha,\infty}(\mathbb{T}^3))\cap\mathscr{C}([0,T],L^2(\mathbb{T}^3))$ with $\alpha>\frac{1}{3}$, then the energy $\|u\|_{L^2(\mathbb{T}^3)}$ is conserved in time. The method of proof as well as the observation that Besov spaces provide a suitable environment for this kind of problem were later used by several authors in the context of other systems of fluid dynamics: like inhomogeneous incompressible Euler and compressible Euler [18], incompressible and compressible Navier–Stokes (resp. [14,15,26,31]), incompressible magnetohydrodynamics [8,25], and general systems of first order conservation laws [22]. Onsager's conjecture was recently studied for incompressible Euler equations in bounded domains, cf. [3]. An overview of these results can be found in [12].

In the present paper we adapt the strategy of Constantin et al. [9] and Feireisl et al. [18] to obtain an Onsager-type sufficient condition on the regularity of weak solutions to the Euler–Korteweg equations so that they conserve the total energy. We consider the isothermal Euler–Korteweg system in the from

$$\begin{aligned}\partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)&=-\rho\nabla\left(h'(\rho)+\frac{\kappa'(\rho)}{2}|\nabla\rho|^2-\operatorname{div}(\kappa(\rho)\nabla\rho)\right),\\ \partial_t\rho+\operatorname{div}(\rho u)&=0,\end{aligned}\tag{1.1}$$

in the domain $(0,T)\times\mathbb{T}^d$ for some fixed time $T>0$, where $\mathbb{T}^d$ is the d-dimensional torus. Here $\rho\geq 0$ is the scalar density of a fluid, u is its velocity, $h=h(\rho)$ is the energy density and $\kappa=\kappa(\rho)>0$ is the coefficient of capillarity. We place the assumption on the functions h and κ:

$$h,\kappa\in\mathscr{C}^3(\mathscr{T})\tag{1.2}$$

where, depending on the actual form of h and κ, the set $\mathscr{T}$ can be chosen to be $[0,\infty)$ or $(0,\infty)$. For instance when $\kappa(\rho)=\frac{1}{\rho}$, as for the QHD system below, then $\mathscr{T}=(0,\infty)$ and we have to be away from vacuum.

While the analysis of the above system dates back to the 19th century, when the mathematical theory of phase interfaces and capillary effects was introduced, it still attracts much attention. A modern derivation of the system can be found in [16]. Concerning smooth solutions: in [6,7] local-in-time well-posedness and stability of special solutions are analysed, respectively. A relative energy identity is developed in [21], exploiting the variational structure of the system, and is used to show that solutions of (1.1) converge to smooth solutions of the compressible Euler system (before shock formation) in the vanishing capillarity limit $\kappa\to 0$, see [20].

The situation with weak solutions is much less understood. Most results concern the Quantum Hydrodynamics system, obtained from (1.1) when $\kappa(\rho)=\frac{\varepsilon_0^2}{4\rho}$, with ε_0 denoting the Planck constant. This takes the form

$$\begin{aligned}\partial_t\rho+\operatorname{div}(\rho u)&=0,\\ \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla p(\rho)&=\frac{\varepsilon_0^2}{2}\rho\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right).\end{aligned}\tag{1.3}$$

The interesting connection between QHD and the Schrödinger equation is used in [19] to provide conservative weak solutions for the special case of zero pressure, $p(\rho) = 0$. Existence of weak solutions for a (relatively limited) class of pressure functions is provided in [1,2]. The existence of wild solutions is possible for (1.1), as pointed out in the recent work of Donatelli et al. [13], where the method of "convex integration" is adapted to show non-uniqueness in the class of dissipative global weak solutions.

The possibility of both conservative and dissipative solutions raises the issue of studying the Onsager conjecture for the Euler–Korteweg system (1.1). We use Besov spaces $B_p^{\alpha,\infty}(\Omega)$, with $1 \le p < \infty$, $0 < \alpha < 1$ (see Sect. 2.1 for the definition) and prove the following theorem:

Theorem 1.1 *Suppose that* (1.2) *holds. Let* (ρ, u) *be a solution of* (1.1) *in the sense of distributions. Assume*

$$u \in (B_3^{\alpha,\infty} \cap L^\infty)((0,T) \times \mathbb{T}^d), \quad \rho, \nabla\rho, \Delta\rho \in (B_3^{\beta,\infty} \cap L^\infty)((0,T) \times \mathbb{T}^d), \tag{1.4}$$

where $1 > \alpha \ge \beta > 0$ *such that* $\min(2\alpha + \beta, \alpha + 2\beta) > 1$.
Then the energy is locally conserved, i.e.

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} \left(\frac{1}{2}\rho|u|^2 + h(\rho) + \frac{1}{2}\kappa(\rho)|\nabla\rho|^2 \right) \partial_t\varphi \, dxdt \\
&\quad + \int_0^T \int_{\mathbb{T}^d} \left(\rho u \left(\frac{1}{2}|u|^2 + h'(\rho) + \frac{1}{2}\kappa'(\rho)|\nabla\rho|^2 - \operatorname{div}(\kappa(\rho)\nabla\rho) \right) \right. \\
&\quad \left. + \kappa(\rho)\nabla\rho \operatorname{div}(\rho u) \right) \cdot \nabla\varphi \, dxdt = 0
\end{aligned}$$

holds for every $\varphi \in \mathscr{C}_c^1((0,T) \times \mathbb{T}^d)$.

Remark 1.2 (1) Notice that if $\alpha \ge \beta$, then $3\alpha \ge \alpha + 2\beta > 1$ and so we must have $\alpha > \frac{1}{3}$.
(2) If in addition we assume the following conditions on u and ρ

$$\lim_{|\xi|,\tau\to 0} \frac{1}{\tau} \int_0^T \frac{1}{|\xi|} \int_{\mathbb{T}^d} |u(t+\tau, x+\xi) - u(t,x)|^3 dxdt = 0,$$
$$\lim_{|\xi|,\tau\to 0} \frac{1}{\tau} \int_0^T \frac{1}{|\xi|} \int_{\mathbb{T}^d} |\rho(t+\tau, x+\xi) - \rho(t,x)|^3 dxdt = 0,$$

then, as pointed out by Shvydkoy [30], see also Duchon and Robert [15], one can allow for the case $\alpha = \beta = \frac{1}{3}$. For details see e.g. Proposition 3 in [15].

The short proof of the main theorem is presented in the following section: it is preceded by an outline of Besov spaces and their basic relevant properties, some preliminary material on the structure of the Euler–Korteweg system, followed by he main part of the proof in Sect. 2.3.

2 Proof of the main theorem

2.1 Besov spaces

Let $\Omega = (0,T) \times \mathbb{T}^d$. The Besov space $B_p^{\alpha,\infty}(\Omega)$, with $1 \le p < \infty$, $0 < \alpha < 1$, is the space of functions $w \in L^p$ for which the norm

$$\|w\|_{B_p^{\alpha,\infty}(\Omega)} := \|w\|_{L^p(\Omega)} + \sup_{r>0} \left\{ r^{-\alpha} \sup_{|\xi|\le r} \|w(\cdot + \xi) - w\|_{L^p(\Omega\cap(\Omega-\xi))} \right\} \tag{2.1}$$

is finite, cf. [4]. In fact, we can replace the semi-norm in (2.1) with the following one

$$\sup_{\xi\in\Omega}\left\{|\xi|^{-\alpha}\|w(\cdot+\xi)-w\|_{L^p(\Omega\cap(\Omega-\xi))}\right\}. \tag{2.2}$$

Indeed, if ξ^* and r^* realize the suprema in (2.1) with $|\xi^*|<r^*$, then taking $|\xi^*|<r<r^*$ would contradict the supremality of r^*. Therefore neccesarily $|\xi^*|=r^*$, thus producing (2.2). We choose to think of the Besov norm in terms of (2.2), as it is more convienient for our purposes.

We observe that if $\alpha\geq\beta$, then there is an inclusion $B_p^{\alpha,\infty}(\Omega)\subset B_p^{\beta,\infty}(\Omega)$.

Let $\eta\in C_c^\infty(\mathbb{R}^{d+1})$ be a standard mollification kernel and we denote

$$\eta^\varepsilon(x)=\frac{1}{\varepsilon^{d+1}}\eta\left(\frac{x}{\varepsilon}\right),\quad w^\varepsilon=\eta^\varepsilon*w\quad\text{and}\quad f^\varepsilon(w)=f(w)*\eta^\varepsilon.$$

Note that the function w^ε is well-defined on $\Omega^\varepsilon=\{x\in\Omega:\operatorname{dist}(x,\partial\Omega)>\varepsilon\}$. The following inequalities will be extensively used in the proof of the main theorem.

Lemma 2.1 *For any function $u\in B_p^{\alpha,\infty}(\Omega)$ we have*

$$\|u(\cdot+\xi)-u(\cdot)\|_{L^p(\Omega\cap(\Omega-\xi))}\leq|\xi|^\alpha\|u\|_{B_p^{\alpha,\infty}(\Omega)} \tag{2.3}$$

$$\|u^\varepsilon-u\|_{L^p(\Omega)}\leq\varepsilon^\alpha\|u\|_{B_p^{\alpha,\infty}(\Omega)} \tag{2.4}$$

$$\|\nabla u^\varepsilon\|_{L^p(\Omega)}\leq C\varepsilon^{\alpha-1}\|u\|_{B_p^{\alpha,\infty}(\Omega)} \tag{2.5}$$

where ∇ stands here for the space-time gradient.

Proof Inequality (2.3) follows directly from the definition of the norm in the space $B_p^{\alpha,\infty}(\Omega)$. To show (2.4) we write

$$|u^\varepsilon(x)-u(x)|\leq\int_{\operatorname{supp}\eta^\varepsilon}\eta^\varepsilon(y)|u(x-y)-u(x)|\,dy$$
$$\leq\left(\int_{\operatorname{supp}\eta^\varepsilon}\eta^\varepsilon(y)|u(x-y)-u(x)|^p\,dy\right)^{\frac{1}{p}}.$$

Therefore, by virtue of Fubini and (2.3)

$$\int_\Omega|u^\varepsilon(x)-u(x)|^p\,dx\leq\int_{\operatorname{supp}\eta^\varepsilon}\eta^\varepsilon(y)\int_\Omega|u(x-y)-u(x)|^p\,dx\,dy$$
$$\leq\int_{\operatorname{supp}\eta^\varepsilon}\eta^\varepsilon(y)|y|^{p\alpha}\|u\|^p_{B_p^{\alpha,\infty}(\Omega)}\,dy\leq\varepsilon^{p\alpha}\|u\|^p_{B_p^{\alpha,\infty}(\Omega)}.$$

For the last of the claimed inequalities we consider the convolution $\nabla u^\varepsilon=\nabla\eta^\varepsilon*u$ as a bounded linear operator $T:L^p(\Omega)\to L^p(\Omega)$. Then

$$\|Tu\|_{L^p}\leq C\varepsilon^{-1}\|u\|_{L^p}.$$

On the other hand, writing $\nabla u^\varepsilon=\eta^\varepsilon*\nabla u$, we can think of T as mapping $W^{1,p}(\Omega)$ into $L^p(\Omega)$. It then has unit norm.

Therefore, as the Besov space $B_p^{\alpha,\infty}$ is an interpolation space of exponent α for L^p and $W^{1,p}$ (cf. [4, Corollary 4.13]), T is bounded as an operator $B_p^{\alpha,\infty}(\Omega)\to L^p(\Omega)$ with

$$\|Tu\|_{L^p}\leq C\varepsilon^{-(1-\alpha)}\|u\|_{B_p^{\alpha,\infty}}.$$

□

Lemma 2.2 *Let $v \in B_p^{\alpha,\infty}(\Omega, \mathbb{R}^m)$. Suppose $f : \mathbb{R}^m \to \mathbb{R}$ is a C^1 function with $\frac{\partial f}{\partial v_i} \in L^\infty$ for each $i = 1, \dots, m$. Then*

$$\|\nabla f(v^\varepsilon)\|_{L^p} \le C\varepsilon^{\alpha-1}\|v\|_{B_p^{\alpha,\infty}}$$

Proof Since $\nabla f(v^\varepsilon) = \sum_{i=1}^m \frac{\partial f}{\partial v_i}(v^\varepsilon)\nabla v_i^\varepsilon$, we have

$$\begin{aligned}\|\nabla f(v^\varepsilon)\|_{L^p} &\le \sum_{i=1}^m \|\frac{\partial f}{\partial v_i}(v^\varepsilon)\|_{L^\infty}\|\nabla v_i^\varepsilon\|_{L^p} \le \max_{1\le i\le m}\|\frac{\partial f}{\partial v_i}\|_{L^\infty}\sum_{i=1}^m\|\nabla v_i^\varepsilon\|_{L^p}\\ &\le C\varepsilon^{\alpha-1}\sum_{i=1}^m\|v_i\|_{B_p^{\alpha,\infty}}\end{aligned}$$

where the last inequality follows from Lemma 2.1. □

2.2 Preliminaries

System (1.1) can be written in conservative form

$$\begin{aligned}\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) &= \operatorname{div}\mathbb{S},\\ \partial_t\rho + \operatorname{div}(\rho u) &= 0,\end{aligned} \tag{2.6}$$

where $\mathbb{S}$ is the Korteweg stress tensor

$$\mathbb{S} = \left(-p(\rho) - \frac{\rho\kappa'(\rho)+\kappa(\rho)}{2}|\nabla\rho|^2 + \operatorname{div}(\rho\kappa(\rho)\nabla\rho)\right)\mathbb{I} - \kappa(\rho)\nabla\rho\otimes\nabla\rho$$

with $\mathbb{I}$ denoting the d-dimensional identity matrix and the local pressure defined as

$$p(\rho) = \rho h'(\rho) - h(\rho).$$

It is routine to show that a strong solution (ρ, u) of the above system will satisfy the following local balance of total (kinetic and internal) energy

$$\begin{aligned}&\partial_t\left(\frac{1}{2}\rho|u|^2 + h(\rho) + \frac{1}{2}\kappa(\rho)|\nabla\rho|^2\right)\\ &\quad+\operatorname{div}\left(\rho u\left(\frac{1}{2}|u|^2 + h'(\rho) + \frac{1}{2}\kappa'(\rho)|\nabla\rho|^2 - \operatorname{div}(\kappa(\rho)\nabla\rho)\right) + \kappa(\rho)\nabla\rho\operatorname{div}(\rho u)\right) = 0.\end{aligned} \tag{2.7}$$

Theorem 1.1 gives sufficient conditions for regularity of weak solutions so that they obey the above energy equality in the sense of distributions. To prove the theorem we employ the strategy of [9], which was used in many works in the subject, including [18,22], where variants of the following lemma are an important ingredient.

Lemma 2.3 *Let $1 \le q < \infty$ and suppose $v \in L^{2q}((0,T)\times\mathbb{T}^d;\mathbb{R}^k)$ and $f \in \mathscr{C}^2(\mathbb{R}^k, \mathbb{R}^N)$. If*

$$\sup_{i,j}\|\frac{\partial^2 f}{\partial v_i\partial v_j}\|_{L^\infty} < \infty,$$

then there exists a constant $C > 0$ such that

$$\|f(v^\varepsilon) - f^\varepsilon(v)\|_{L^q} \le C\left(\|v^\varepsilon - v\|^2_{L^{2q}} + \sup_{(s,y)\in\operatorname{supp}\eta_\varepsilon} \|v(\cdot,\cdot) - v(\cdot - s, \cdot - y)\|^2_{L^{2q}}\right). \tag{2.8}$$

Proof We observe that by Taylor's theorem we have

$$\begin{aligned}&\left|f(v^\varepsilon(t,x)) - f(v(t,x))) - Df(v(t,x))(v^\varepsilon(t,x) - v(t,x))\right|\\ &\quad\le C|v^\varepsilon(t,x) - v(t,x)|^2\end{aligned} \tag{2.9}$$

where the constant C does not depend on the choice of x and t. Similarly

$$\begin{aligned}&|f(v(s,y)) - f(v(t,x)) - Df(v(t,x))(v(s,y) - v(t,x))|\\ &\quad\le C|v(s,y) - v(t,x)|^2.\end{aligned} \tag{2.10}$$

Mollification of the last inequality with respect to (s, y) yields, by virtue of Jensen's inequality

$$\begin{aligned}&\left|f^\varepsilon(v(t,x)) - f(v(t,x) - Df(v(t,x))(v^\varepsilon(t,x) - v(t,x))\right|\\ &\quad\le C|v(\cdot,\cdot) - v(t,x)|^2 *_{(s,y)} \eta^\varepsilon.\end{aligned} \tag{2.11}$$

Combining (2.9) and (2.11) and using the triangle inequality we deduce the estimate

$$\begin{aligned}&\left|f(v^\varepsilon(t,x)) - f^\varepsilon(v(t,x))\right| \le C\left(|v^\varepsilon(t,x) - v(t,x)|^2\right.\\ &\quad\left.+|v(\cdot,\cdot) - v(t,x)|^2 *_{(s,y)} \eta^\varepsilon\right).\end{aligned} \tag{2.12}$$

Finally, we observe that

$$\begin{aligned}&\int_{(0,T)\times\mathbb{T}^d} \left||v(\cdot,\cdot) - v(t,x)|^2 *_{(s,y)} \eta^\varepsilon\right|^q \, dxdt\\ &\quad\le \int_{\operatorname{supp}\eta^\varepsilon} \eta^\varepsilon(s,y) \int_{(0,T)\times\mathbb{T}^d} |v(t-s, x-y) - v(t,x)|^{2q} \, dxdt\, dyds\\ &\quad\le \sup_{(s,y)\in\operatorname{supp}\eta_\varepsilon} \|v(\cdot,\cdot) - v(\cdot - s, \cdot - y)\|^{2q}_{L^{2q}}.\end{aligned}$$

□

2.3 Energy equality

We begin the proof of the theorem by mollifying the momentum equation in both space and time with kernel and notation as in Sect. 2.1 to obtain

$$\partial_t(\rho u)^\varepsilon + \operatorname{div}(\rho u \otimes u)^\varepsilon = -\nabla p^\varepsilon(\rho) + \operatorname{div} S^\varepsilon(\rho, \nabla\rho, \Delta\rho), \tag{2.13}$$

where we define

$$S(\rho, q, r) = \left(\frac{1}{2}(\rho\kappa'(\rho) + \kappa(\rho))|q|^2 + \rho\kappa(\rho)r\right)\mathbb{I} - \kappa(\rho)q \otimes q, \tag{2.14}$$

and $S^\varepsilon(\rho, \nabla\rho, \Delta\rho) = S(\rho, \nabla\rho, \Delta\rho) * \eta^\varepsilon$. We note that

$$S(\rho, \nabla\rho, \Delta\rho) = \left(-\frac{1}{2}(\rho\kappa'(\rho) + \kappa(\rho))|\nabla\rho|^2 + \operatorname{div}(\rho\kappa(\rho)\nabla\rho)\right)\mathbb{I} - \kappa(\rho)\nabla\rho \otimes \nabla\rho,$$

cf. the definition of $\mathbb{S}$.
Equation (2.13) can be rewritten in terms of appropriate commutators to give

$$\begin{aligned}&\partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon) + \nabla p(\rho^\varepsilon) - \operatorname{div}(S(\rho^\varepsilon, \nabla\rho^\varepsilon, \Delta\rho^\varepsilon)) \\ &\quad = \partial_t(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon) + \operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon - (\rho u \otimes u)^\varepsilon) + \nabla\left(p(\rho^\varepsilon) - p^\varepsilon(\rho)\right) \\ &\qquad - \operatorname{div}\left(S(\rho^\varepsilon, \nabla\rho^\varepsilon, \Delta\rho^\varepsilon) - S^\varepsilon(\rho, \nabla\rho, \Delta\rho)\right).\end{aligned} \tag{2.15}$$

We observe the following identities

$$\operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon) = u^\varepsilon \operatorname{div}(\rho u)^\varepsilon + ((\rho u)^\varepsilon \cdot \nabla)u^\varepsilon$$

and

$$-\rho^\varepsilon\, \nabla\left(\frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2 - \operatorname{div}(\kappa(\rho^\varepsilon)\nabla\rho^\varepsilon)\right) = \operatorname{div} S(\rho^\varepsilon, \nabla\rho^\varepsilon, \Delta\rho^\varepsilon).$$

Thus the left-hand side of Eq. (2.15) can be written as

$$\begin{aligned}&(\partial_t\rho^\varepsilon)u^\varepsilon + \rho^\varepsilon\partial_t u^\varepsilon + u^\varepsilon \operatorname{div}(\rho u)^\varepsilon + ((\rho u)^\varepsilon \cdot \nabla)u^\varepsilon \\ &\quad + \rho^\varepsilon\, \nabla\left(h'(\rho^\varepsilon) - \frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2 - \kappa(\rho^\varepsilon)\Delta\rho^\varepsilon\right).\end{aligned}$$

Hence, upon multiplying with u^ε, Eq. (2.15) becomes

$$\begin{aligned}&\rho^\varepsilon\partial_t\left(\frac{1}{2}|u^\varepsilon|^2\right) + \left((\rho u)^\varepsilon \cdot \nabla\right)\frac{1}{2}|u^\varepsilon|^2 \\ &\quad + \rho^\varepsilon u^\varepsilon\, \nabla\left(h'(\rho^\varepsilon) - \frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2 - \kappa(\rho^\varepsilon)\Delta\rho^\varepsilon\right) \\ &\quad = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon + r_4^\varepsilon,\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}r_1^\varepsilon &= \partial_t(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon) \cdot u^\varepsilon, \\ r_2^\varepsilon &= \operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon - (\rho u \otimes u)^\varepsilon) \cdot u^\varepsilon, \\ r_3^\varepsilon &= \nabla\left(p(\rho^\varepsilon) - p^\varepsilon(\rho)\right) \cdot u^\varepsilon, \\ r_4^\varepsilon &= -\operatorname{div}\left(S(\rho^\varepsilon, \nabla\rho^\varepsilon, \Delta\rho^\varepsilon) - S^\varepsilon(\rho, \nabla\rho, \Delta\rho)\right) \cdot u^\varepsilon,\end{aligned}$$

and we have used the mollified continuity equation

$$\partial_t\rho^\varepsilon + \operatorname{div}(\rho u)^\varepsilon = 0. \tag{2.17}$$

Using (2.17) we can write the first two terms of (2.16) as

$$\begin{aligned}&\rho^\varepsilon\partial_t\left(\frac{1}{2}|u^\varepsilon|^2\right) + \left((\rho u)^\varepsilon \cdot \nabla\right)\frac{1}{2}|u^\varepsilon|^2 + \left(\partial_t\rho^\varepsilon + \operatorname{div}(\rho u)^\varepsilon\right)\frac{1}{2}|u^\varepsilon|^2 \\ &\quad = \partial_t\left(\frac{1}{2}\rho^\varepsilon|u^\varepsilon|^2\right) + \operatorname{div}\left((\rho u)^\varepsilon\, \frac{1}{2}|u^\varepsilon|^2\right).\end{aligned} \tag{2.18}$$

Combining Eqs. (2.16) and (2.18) we obtain

$$\begin{aligned}&\partial_t\left(\frac{1}{2}\rho^\varepsilon|u^\varepsilon|^2\right) + \operatorname{div}\left((\rho u)^\varepsilon\, \frac{1}{2}|u^\varepsilon|^2\right) \\ &\quad + \rho^\varepsilon u^\varepsilon\, \nabla\left(h'(\rho^\varepsilon) - \frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2 - \kappa(\rho^\varepsilon)\Delta\rho^\varepsilon\right) \\ &\quad = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon + r_4^\varepsilon.\end{aligned} \tag{2.19}$$

We now rewrite the mollified continuity Eq. (2.17) in the form

$$\partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon).$$

After multiplying this equation with

$$h'(\rho^\varepsilon) - \frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2 - \kappa(\rho^\varepsilon)\Delta\rho^\varepsilon$$

and rearranging, we obtain

$$\begin{aligned} &\partial_t\left(h(\rho^\varepsilon) + \frac{1}{2}\kappa(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2\right) - \operatorname{div}\left(\kappa(\rho^\varepsilon)\nabla\rho^\varepsilon\partial_t\rho^\varepsilon\right) \\ &\quad + \operatorname{div}(\rho^\varepsilon u^\varepsilon)\left(h'(\rho^\varepsilon) - \frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2 - \kappa(\rho^\varepsilon)\Delta\rho^\varepsilon\right) \\ &= r_5^\varepsilon + r_6^\varepsilon + r_7^\varepsilon, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} r_5^\varepsilon &= \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)\, h'(\rho^\varepsilon), \\ r_6^\varepsilon &= -\operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)\, \frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2, \\ r_7^\varepsilon &= -\operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)\kappa(\rho^\varepsilon)\Delta\rho^\varepsilon. \end{aligned}$$

Combining Eqs. (2.19) and (2.20) we obtain

$$\begin{aligned} &\partial_t\left(\frac{1}{2}\rho^\varepsilon|u^\varepsilon|^2 + h(\rho^\varepsilon) + \frac{1}{2}\kappa(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2\right) + \operatorname{div}\left((\rho u)^\varepsilon\,\frac{1}{2}|u^\varepsilon|^2\right) \\ &\quad + \operatorname{div}\left(\rho^\varepsilon u^\varepsilon\left(h'(\rho^\varepsilon) - \frac{1}{2}\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2 - \kappa(\rho^\varepsilon)\Delta\rho^\varepsilon\right) + \kappa(\rho^\varepsilon)\nabla\rho^\varepsilon\operatorname{div}(\rho^\varepsilon u^\varepsilon)\right) \\ &= r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon + r_4^\varepsilon + r_5^\varepsilon + r_6^\varepsilon + r_7^\varepsilon. \end{aligned} \tag{2.21}$$

It follows that to prove the theorem it is sufficient to show that each commutator error term converges to zero in the distributional sense on $(0, T) \times \mathbb{T}^d$ as $\varepsilon \to 0$.

2.4 Commutator estimates

Let $\varphi \in \mathscr{C}_c^1((0, T) \times \mathbb{T}^d)$ and take $\varepsilon > 0$ small enough so that $\operatorname{supp}\varphi \subset (\varepsilon, T - \varepsilon) \times \mathbb{T}^d$. We will show that for each $1 \le i \le 7$ we have

$$R_i^\varepsilon := \int_0^T \int_{\mathbb{T}^d} r_i^\varepsilon \varphi\, dx dt \xrightarrow{\varepsilon \to 0^+} 0.$$

The terms R_1^ε and R_2^ε are dealt with in the same way as in [18]. We recall these estimates for the reader's convenience. For R_1^ε we observe that

$$\begin{aligned} \rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon = (\rho^\varepsilon - \rho)(u^\varepsilon - u) \\ - \int_{-\varepsilon}^{\varepsilon}\int_{\mathbb{T}^d} \eta^\varepsilon(\tau, \xi)(\rho(t-\tau, x-\xi) \\ - \rho(t, x))(u(t-\tau, x-\xi) - u(t, x)) d\xi d\tau. \end{aligned} \tag{2.22}$$

The first part of R_1^ε therefore can be estimated by virtue of an integration by parts, Hölder inequality and estimates (2.4) and (2.5) as

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{T}^d} \varphi \partial_t \big((\rho^\varepsilon - \rho)(u^\varepsilon - u)\big) \cdot u^\varepsilon \, dxdt\right| \\
&\leq \int_0^T \int_{\mathbb{T}^d} |(\rho^\varepsilon - \rho)(u^\varepsilon - u)|(|\partial_t \varphi\, u^\varepsilon| + |\varphi \partial_t u^\varepsilon|)\, dxdt \\
&\leq \|\varphi\|_{\mathscr{C}^1} \|\rho^\varepsilon - \rho\|_{L^3} \|u^\varepsilon - u\|_{L^3} \|u^\varepsilon\|_{L^3} + \|\varphi\|_{\mathscr{C}^0} \|\rho^\varepsilon - \rho\|_{L^3} \|u^\varepsilon - u\|_{L^3} \|\partial_t u^\varepsilon\|_{L^3} \\
&\leq C\varepsilon^\beta \varepsilon^\alpha \|\rho\|_{B_3^{\beta,\infty}} \|u\|^2_{B_3^{\alpha,\infty}} + C\varepsilon^\beta \varepsilon^\alpha \varepsilon^{\alpha-1} \|\rho\|_{B_3^{\beta,\infty}} \|u\|^2_{B_3^{\alpha,\infty}}.
\end{aligned}$$

Note that since we assume Besov regularity of u also in time, in the above we can estimate the L^3-norm of $\partial_t u^\varepsilon$ according to (2.5).
For the second part of R_1^ε according to (2.22), we estimate (using integration by parts, Fubini (2.3) and (2.5))

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{T}^d} \varphi \partial_t \left(\int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{T}^d} \eta^\varepsilon(\tau, \xi)(\rho(t-\tau, x-\xi)\right.\right. \\
&\qquad \left.\left. -\rho(t,x))(u(t-\tau, x-\xi) - u(t,x)) d\xi d\tau\right) \cdot u^\varepsilon dxdt\right| \\
&\leq C\|\varphi\|_{\mathscr{C}^1} \varepsilon^\beta \varepsilon^\alpha \|\rho\|_{B_3^{\beta,\infty}} \|u\|^2_{B_3^{\alpha,\infty}} + C\|\varphi\|_{\mathscr{C}^0} \varepsilon^\beta \varepsilon^\alpha \varepsilon^{\alpha-1} \|\rho\|_{B_3^{\beta,\infty}} \|u\|^2_{B_3^{\alpha,\infty}}.
\end{aligned}$$

A similar estimation can be carried out for R_2^ε. We write

$$\begin{aligned}
(\rho u)^\varepsilon \otimes u^\varepsilon - (\rho u \otimes u)^\varepsilon &= ((\rho u)^\varepsilon - \rho u) \otimes (u^\varepsilon - u) \\
&- \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{T}^d} \eta^\varepsilon(\tau, \xi)(\rho u(t-\tau, x-\xi) - \rho u(t,x)) \\
&\otimes (u(t-\tau, x-\xi) - u(t,x)) d\xi d\tau.
\end{aligned}$$

To use the above decomposition to estimate R_2^ε we require that the product ρu belongs to the space $B_3^{\beta,\infty}((0,T) \times \mathbb{T}^d)$. To provide this regularity we needed to assume not only that ρ and u are in appropriate Besov spaces, but also belong to L^∞. Indeed, observe that since $\alpha \geq \beta$, we have $u \in B_3^{\beta,\infty}$ and

$$\begin{aligned}
\frac{\|(\rho u)(\cdot + \xi) - \rho u\|_{L^3}}{|\xi|^\beta} &\leq \frac{\|\rho(u(\cdot + \xi) - u)\|_{L^3}}{|\xi|^\beta} + \frac{\|(\rho(\cdot + \xi) - \rho) u(\cdot + \xi)\|_{L^3}}{|\xi|^\beta} \\
&\leq \|\rho\|_{L^\infty} \|u\|_{B_3^{\beta,\infty}} + \|u\|_{L^\infty} \|\rho\|_{B_3^{\beta,\infty}}
\end{aligned}$$

Thus the first part of R_2^ε can be estimated as

$$\begin{aligned}
&\left|\int_0^T \int_{\mathbb{T}^d} \operatorname{div}\big(((\rho u)^\varepsilon - \rho u) \otimes (u^\varepsilon - u)\big) \cdot \varphi u^\varepsilon \, dxdt\right| \\
&\leq \|\varphi\|_{C^1} \|(\rho u)^\varepsilon - \rho u\|_{L^3} \|u^\varepsilon - u\|_{L^3} \|u^\varepsilon\|_{L^3} \\
&\quad + \|\varphi\|_{C^0} \|(\rho u)^\varepsilon - \rho u\|_{L^3} \|u^\varepsilon - u\|_{L^3} \|\nabla u^\varepsilon\|_{L^3} \\
&\leq C\varepsilon^\beta \varepsilon^\alpha \|\rho\|_{B_3^{\beta,\infty}} \|u\|^2_{B_3^{\alpha,\infty}} + C\varepsilon^\beta \varepsilon^\alpha \varepsilon^{\alpha-1} \|\rho u\|_{B_3^{\beta,\infty}} \|u\|^2_{B_3^{\alpha,\infty}}.
\end{aligned}$$

Likewise, for the second part of R_2^ε we get

$$\begin{aligned}&\Big|\int_0^T\int_{\mathbb{T}^d}\operatorname{div}\Big\{\int_{-\varepsilon}^{\varepsilon}\int_{\mathbb{T}^d}\eta^\varepsilon(\tau,\xi)(\rho u(t-\tau,x-\xi)-\rho u(t,x))\\ &\quad\otimes(u(t-\tau,x-\xi)-u(t,x))d\xi d\tau\Big\}\cdot\varphi u^\varepsilon dxdt\Big|\\ &\quad\le C\|\varphi\|_{\mathscr{C}^0}\varepsilon^\beta\varepsilon^\alpha\varepsilon^{\alpha-1}\|\rho u\|_{B_3^{\beta,\infty}}\|u\|^2_{B_3^{\alpha,\infty}}+C\|\varphi\|_{\mathscr{C}^1}\varepsilon^\beta\varepsilon^\alpha\|\rho\|_{B_3^{\beta,\infty}}\|u\|^2_{B_3^{\alpha,\infty}}.\end{aligned}$$

These estimates show that R_1^ε and R_2^ε vanish as $\varepsilon\to 0$.
To estimate terms R_3^ε and R_4^ε we integrate by parts and apply Lemma 2.3 to get the following

$$\begin{aligned}|R_3^\varepsilon|&\le\|\varphi\|_{\mathscr{C}^1}\int_0^T\int_{\mathbb{T}^d}|p(\rho^\varepsilon)-p^\varepsilon(\rho)||u^\varepsilon|\,dxdt+\|\varphi\|_{\mathscr{C}^0}\int_0^T\int_{\mathbb{T}^d}|p(\rho^\varepsilon)\\ &\quad-p^\varepsilon(\rho)||\nabla u^\varepsilon|\,dxdt\\ &\le C\|p(\rho^\varepsilon)-p^\varepsilon(\rho)\|_{L^{3/2}}\left(\|u^\varepsilon\|_{L^3}+\|\nabla u^\varepsilon\|_{L^3}\right)\\ &\le C\left(\|\rho^\varepsilon-\rho\|^2_{L^3}+\sup_{y\in\operatorname{supp}\eta^\varepsilon}\|\rho(\cdot)-\rho(\cdot-y)\|^2_{L^3}\right)\left(1+\varepsilon^{\alpha-1}\right)\|u\|_{B_3^{\alpha,\infty}}\\ &\le C\left(\varepsilon^{2\beta}\|\rho\|^2_{B_3^{\beta,\infty}}+\sup_{|y|\le\varepsilon}|y|^{2\beta}\|\rho\|^2_{B_3^{\beta,\infty}}\right)\left(1+\varepsilon^{\alpha-1}\right)\|u\|_{B_3^{\alpha,\infty}}\\ &\le C(\varepsilon^{2\beta}+\varepsilon^{2\beta+\alpha-1})\|u\|_{B_3^{\alpha,\infty}}\|\rho\|^2_{B_3^{\beta,\infty}}\end{aligned}$$

and similarly

$$\begin{aligned}|R_4^\varepsilon|&\le C\|S(\rho^\varepsilon,\nabla\rho^\varepsilon,\Delta\rho^\varepsilon)-S^\varepsilon(\rho,\nabla\rho,\Delta\rho)\|_{L^{3/2}}(\|u^\varepsilon\|_{L^3}+\|\nabla u^\varepsilon\|_{L^3})\\ &\le C(\varepsilon^{2\beta}+\varepsilon^{2\beta+\alpha-1})\|u\|_{B_3^{\alpha,\infty}}(\|\rho\|^2_{B_3^{\beta,\infty}}+\|\nabla\rho\|^2_{B_3^{\beta,\infty}}+\|\Delta\rho\|^2_{B_3^{\beta,\infty}}).\end{aligned}$$

It now remains to estimate the last three commutator errors R_5^ε, R_6^ε and R_7^ε. To this end we employ Lemma 2.2 with function f being $h'(\rho)$, $\kappa'(\rho)|\nabla\rho|^2$, and $\kappa(\rho)\Delta\rho$, respectively. We observe that by assumptions (1.2) and (1.4) these functions belong to L^∞. Using again equality (2.22) we can estimate as follows.

$$\begin{aligned}|R_5^\varepsilon|&\le\int_0^T\int_{\mathbb{T}^d}|(\rho^\varepsilon-\rho)(u^\varepsilon-u)|(|h'(\rho^\varepsilon)\nabla\varphi|+|\varphi\nabla h'(\rho^\varepsilon)|)\,dxdt\\ &\le\|\varphi\|_{\mathscr{C}^1}\|\rho^\varepsilon-\rho\|_{L^3}\|u^\varepsilon-u\|_{L^3}\|h'(\rho^\varepsilon)\|_{L^3}\\ &\quad+\|\varphi\|_{\mathscr{C}^0}\|\rho^\varepsilon-\rho\|_{L^3}\|u^\varepsilon-u\|_{L^3}\|\nabla h'(\rho^\varepsilon)\|_{L^3}\\ &\le C\varepsilon^\beta\varepsilon^\alpha\|\rho\|_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}}+C\varepsilon^\beta\varepsilon^\alpha\varepsilon^{\beta-1}\|\rho\|^2_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}},\\ |R_6^\varepsilon|&\le C(\|\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2\|_{L^3}+\|\nabla(\kappa'(\rho^\varepsilon)|\nabla\rho^\varepsilon|^2)\|_{L^3})\|\rho^\varepsilon-\rho\|_{L^3}\|u^\varepsilon-u\|_{L^3}\\ &\le C\varepsilon^\beta\varepsilon^\alpha\|\rho\|_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}}+C\varepsilon^\beta\varepsilon^\alpha\varepsilon^{\beta-1}\|\rho\|^2_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}},\end{aligned}$$

and

$$\begin{aligned}|R_7^\varepsilon|&\le C(\|\kappa(\rho^\varepsilon)\Delta\rho^\varepsilon\|_{L^3}+\|\nabla(\kappa(\rho^\varepsilon)\Delta\rho^\varepsilon)\|_{L^3})\|\rho^\varepsilon-\rho\|_{L^3}\|u^\varepsilon-u\|_{L^3}\\ &\le C\varepsilon^\beta\varepsilon^\alpha\|\rho\|_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}}+C\varepsilon^\beta\varepsilon^\alpha\varepsilon^{\beta-1}\|\rho\|^2_{B_3^{\beta,\infty}}\|u\|_{B_3^{\alpha,\infty}}.\end{aligned}$$

For brevity the above calculations include only the first term coming from (2.22), with the second term easily seen to produce estimates of the same order.
Thus the proof of the theorem is complete.

Acknowledgements This work was partially supported by the Simons - Foundation Grant 346300 and the Polish Government MNiSW 2015-2019 matching fund; AET thanks the Institute of Mathematics of the Polish Academy of Sciences, Warsaw, for their hospitality during his stay as a Simons Visiting Professor, while P.G. and A.Ś-G thank KAUST for its hospitality during their stay. P.G. and A.Ś-G. received support from the National Science Centre (Poland), 2015/18/M/ST1/00075. T.D acknowledges the support of the National Science Centre (Poland), 2012/05/E/ST1/02218.

References

1. Antonelli, P., Marcati, P.: On the finite energy weak solutions to a system in quantum fluid dynamics. Commun. Math. Phys. **287**(2), 657–686 (2009)
2. Antonelli, P., Marcati, P.: The quantum hydrodynamics system in two space dimensions. Arch. Ration. Mech. Anal. **203**(2), 499–527 (2012)
3. Bardos, C., Titi, E.: Onsager's conjecture for the incompressible Euler equations in bounded domains. Arch. Ration. Mech. Anal. **228**, 197 (2017). https://doi.org/10.1007/s00205-017-1189-x
4. Bennett, C., Sharpley, R.: Interpolation of Operators. Pure and Applied Mathematics, vol. 129. Academic Press Inc., Boston (1988)
5. Buckmaster, T., De Lellis, C., Székelyhidi Jr., L., Vicol, V.: Onsager's conjecture for admissible weak solutions. Commun. Pure Appl. Math. (2018). https://doi.org/10.1002/cpa.21781
6. Benzoni-Gavage, S., Danchin, R., Descombes, S.: On the well-posedness for the Euler–Korteweg model in several space dimensions. Indiana Univ. Math. J. **56**, 1499–1579 (2007)
7. Benzoni-Gavage, S., Danchin, R., Descombes, S., Jamet, D.: Structure of Korteweg models and stability of diffuse interfaces. Interfaces Free Bound. **7**(4), 371–414 (2005)
8. Caflisch, R.E., Klapper, I., Steele, G.: Remarks on singularities, dimension and energy dissipation for ideal hydrodynamics and MHD. Commun. Math. Phys. **184**(2), 443–455 (1997)
9. Constantin, P., W, E., Titi, E.S.: Onsager's conjecture on the energy conservation for solutions of Euler's equation. Commun. Math. Phys. **165**(1), 207–209 (1994)
10. De Lellis, C., Székelyhidi Jr., L.: The Euler equations as a differential inclusion. Ann. Math. (2) **170**(3), 1417–1436 (2009)
11. De Lellis, C., Székelyhidi Jr., L.: On admissibility criteria for weak solutions of the Euler equations. Arch. Ration. Mech. Anal. **195**, 225–260 (2010)
12. Dębiec, T., Gwiazda, P., Świerczewska-Gwiazda, A.: A tribute to energy conservation for weak solutions. arXiv:1707.09794 (2017)
13. Donatelli, D., Feireisl, E., Marcati, P.: Well/ill posedness for the Euler–Korteweg–Poisson system and related problems. Commun. Partial Differ. Equ. **40**(7), 1314–1335 (2015)
14. Drivas, T.D., Eyink, G.L.: An Onsager singularity theorem for turbulent solutions of compressible Euler equations. Commun. Math. Phys. **359**, 733 (2017)
15. Duchon, J., Robert, R.: Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations. Nonlinearity **13**(1), 249–255 (2000)
16. Dunn, J.E., Serrin, J.: On the thermomechanics of interstitial working. Arch. Ration. Mech. Anal. **88**, 95–133 (1985)
17. Eyink, G.L.: Energy dissipation without viscosity in ideal hydrodynamics. I. Fourier analysis and local energy transfer. Physica D **78**(3–4), 222–240 (1994)
18. Feireisl, E., Gwiazda, P., Świerczewska-Gwiazda, A., Wiedemann, E.: Regularity and energy conservation for the compressible Euler equations. Arch. Ration. Mech. Anal. **223**(3), 1–21 (2017)
19. Gasser, I., Markowich, P.: Quantum hydrodynamics, Wigner transforms and the classical limit. Asymptot. Anal. **14**, 97–116 (1997)
20. Giesselmann, J., Tzavaras, A.: Stability properties of the Euler–Korteweg system with nonmonotone pressures. Appl. Anal. **96**(9), 1528–1546 (2017)
21. Giesselman, J., Lattanzio, C., Tzavaras, A.: Relative energy for the Korteweg theory and related hamiltonian flows in gas dynamics. Arch. Ration. Mech. Anal. **223**(3), 1427–1484 (2017)
22. Gwiazda, P., Michálek, M., Świerczewska-Gwiazda, A.: A note on weak solutions of conservation laws and energy/entropy conservation. Arch. Ration. Mech. Anal. **229**(3), 1223–1238 (2018)

23. Isett, P.: A proof of Onsager's conjecture. Ann. Math. (2018, to appear)
24. Isett, P.: On the endpoint regularity in Onsager's conjecture. arXiv:1706.0154 (2017)
25. Kang, E., Lee, J.: Remarks on the magnetic helicity and energy conservation for ideal magneto-hydrodynamics. Nonlinearity **20**(11), 2681–2689 (2007)
26. Leslie, T.M., Shvydkoy, R.: The energy balance relation for weak solutions of the density-dependent Navier–Stokes equations. J. Differ. Equ. **261**(6), 3719–3733 (2016)
27. Onsager, L.: Statistical hydrodynamics. Nuovo Cimento (9) **6**, 279–287 (1949). (Convegno Internazionale di Meccanica Statistica)
28. Scheffer, V.: An inviscid flow with compact support in space–time. J. Geom. Anal. **3**(4), 343–401 (1993)
29. Shnirelman, A.: Weak solutions with decreasing energy of incompressible Euler equations. Commun. Math. Phys. **210**(3), 541–603 (2000)
30. Shvydkoy, R.: On the energy of inviscid singular flows. J. Math. Anal. Appl. **349**, 583–595 (2009)
31. Yu, C.: Energy conservation for the weak solutions of the compressible Navier–Stokes equations. Arch. Ration. Mech. Anal. **225**(2), 1073–1087 (2017)

Chapter 4

Energy conservation for the compressible Euler and Navier-Stokes equations with vacuum [1]

ANALYSIS AND PDE
Vol. 13, No. 3, 2020
dx.doi.org/10.2140/apde.2020.13.789

ENERGY CONSERVATION FOR THE COMPRESSIBLE EULER AND NAVIER–STOKES EQUATIONS WITH VACUUM

IBROKHIMBEK AKRAMOV, TOMASZ DĘBIEC, JACK SKIPPER AND EMIL WIEDEMANN

We consider the compressible isentropic Euler equations on $[0, T] \times \mathbb{T}^d$ with a pressure law $p \in C^{1,\gamma-1}$, where $1 \leq \gamma < 2$. This includes all physically relevant cases, e.g., the monoatomic gas. We investigate under what conditions on its regularity a weak solution conserves the energy. Previous results have crucially assumed that $p \in C^2$ in the range of the density; however, for realistic pressure laws this means that we must exclude the vacuum case. Here we improve these results by giving a number of sufficient conditions for the conservation of energy, even for solutions that may exhibit vacuum: firstly, by assuming the velocity to be a divergence-measure field; secondly, imposing extra integrability on $1/\rho$ near a vacuum; thirdly, assuming ρ to be quasinearly subharmonic near a vacuum; and finally, by assuming that u and ρ are Hölder continuous. We then extend these results to show global energy conservation for the domain $[0, T] \times \Omega$ where Ω is bounded with a C^2 boundary. We show that we can extend these results to the compressible Navier–Stokes equations, even with degenerate viscosity.

1. Introduction

In recent years some substantial effort has been directed towards investigating the relation between energy (or, more generally, entropy) conservation and regularity of weak solutions to a given physical system of equations.

Onsager's conjecture states that a weak solution of the (three-dimensional) incompressible Euler system will conserve energy if it is Hölder regular with exponent greater than $\frac{1}{3}$. Otherwise it is possible for solutions to exist where anomalous dissipation of energy occurs. First results towards energy conservation for weak solutions are due to Eyink [1994] and Constantin, E, and Titi [Constantin et al. 1994]; see also [Duchon and Robert 2000]. The sharpest results in optimal Besov spaces are due to Cheskidov et al. [2008] and Fjordholm and Wiedemann [2018]. Further, Bardos and Titi [2018], Bardos, Titi, and Wiedemann [Bardos et al. 2018], and Drivas and Nguyen [2018] have extended these results to consider solutions on a bounded domain.

Investigating the possibility of analogous statements for other systems has become another lively direction of research. Sufficient regularity conditions for the energy to be conserved were studied for a number of models: inhomogeneous incompressible Euler [Chen and Yu 2019] and Navier–Stokes [Leslie and Shvydkoy 2016], compressible Euler [Feireisl et al. 2017], the full Euler system [Drivas and Eyink 2018], compressible Navier–Stokes [Yu 2017], and Euler–Korteweg [Dębiec et al. 2018]. A general class of first-order conservation laws was considered in [Gwiazda et al. 2018], and in [Bardos et al. 2019] on bounded domains.

MSC2010: primary 35Q31; secondary 35Q30, 35L65, 76N10.
Keywords: compressible Euler equations, compressible Navier–Stokes equations, vacuum, Onsager's conjecture, energy conservation.

Another direction of research was aimed towards the construction of $\left(\frac{1}{3}-\varepsilon\right)$-Hölder continuous solutions to the incompressible Euler system that do *not* conserve energy. With the application, and further refinements, of the method of convex integration this was achieved recently in [Isett 2018; Buckmaster et al. 2019]. Thus the famous conjecture of Lars Onsager for the incompressible Euler equations is fully resolved.

One of the major differences between incompressible and compressible fluid dynamics is the possible formation of *vacuum* in the latter case. This means that the density of the fluid may become zero in some region. More precisely, consider the isentropic compressible Euler system

$$\begin{aligned} \partial_t(\rho u)+\operatorname{div}(\rho u\otimes u)+\nabla p(\rho)&=0,\\ \partial_t\rho+\operatorname{div}(\rho u)&=0, \end{aligned} \tag{1-1}$$

where u denotes the velocity and ρ the density of the fluid. We will specify the constitutive pressure law $p=p(\rho)$ later. It is classically known that conservation laws like (1-1) may develop singularities (shocks) in finite time, which prohibits the use of a smooth notion of solution. Rather, one works with solutions in the sense of distributions, which may be very rough. Suppose now the density were initially bounded away from zero, $\rho^0\geq c>0$. If the solution were smooth, then from the continuity equation $\partial_t\rho+\operatorname{div}(\rho u)=0$ it would easily follow (see equation (7) in [DiPerna and Lions 1989]) that ρ remains bounded away from zero for all times. More precisely, this requires u to have bounded divergence. However, there seems to be no way to guarantee that the velocity component of a weak solution of (1-1) has bounded divergence, and thus it cannot be excluded that the solution spontaneously develops vacuum in finite time. In fact, to our knowledge it remains an outstanding open question whether this can actually occur for the compressible Euler or even Navier–Stokes equations.

The formation of vacuum constitutes a degeneracy that, in many situations, vastly complicates the mathematical analysis of compressible models. For instance, the compressible Euler equations cease to be strictly hyperbolic in vacuum regions. In the context of the current contribution, densities close to zero invalidate the methods and results from previous works like [Feireisl et al. 2017; Gwiazda et al. 2018; Bardos et al. 2019]: There, it is a crucial assumption that the nonlinearities depend on the dependent variables in a twice continuously differentiable fashion, in order to treat them like a quadratic expression in the commutator estimates. For the system (1-1), a typical and physically reasonable pressure law would be the polytropic one, i.e., $p(\rho)=\rho^\gamma$ with $\gamma>1$. The second derivative, however, is of order $\rho^{\gamma-2}$ and thus blows up at zero, at least if $\gamma<2$. But the regime $1<\gamma<2$ is precisely the relevant one (for instance, a monoatomic gas has $\gamma=\frac{5}{3}$).

The starting point of our current work is the result of Feireisl, Gwiazda, Świerczewska-Gwiazda, and Wiedemann [Feireisl et al. 2017] for the compressible Euler system, which we quote below. It gives sufficient conditions, in terms of Besov regularity of a weak solution, for energy conservation, but only as long as vacuum is excluded. In the presence of vacuum, the relevant commutator estimate involving the pressure completely breaks down, and it turns out that substantially new techniques are required to fix this. To our knowledge, the only other result on energy conservation for non-C^2 nonlinearities is the one on active scalar equations [Akramov and Wiedemann 2019], using however different techniques.

In the current article, we give a number of sufficient conditions to ensure energy conservation even after possible formation of vacuum.

First (Section 3), we consider the condition that the velocity be a so-called divergence-measure field; this notion is well known in geometric measure theory and hyperbolic conservation laws, but it may seem a bit unmotivated to consider in the present situation. However, justification comes from the compressible Navier–Stokes system, whose a priori estimates ensure this condition. We extensively discuss the ramifications of our result with respect to the Navier–Stokes equations in Section 3A, where we also compare it to recent work of Cheng Yu [2017].

In Section 4, we identify as a sufficient condition for energy conservation an estimate for the quotient between the density and its mollification; see (4-1). This, in itself, may seem rather artificial, and we go on to identify more natural conditions that will ensure (4-1) holds. Arguably, our strongest result is Corollary 4.4: under the slightly stronger assumption of Hölder (instead of Besov) regularity, but with the expected exponents, we can show energy conservation *no matter how the density behaves near vacuum*. It is surprising that this result is completely agnostic to the way that ρ approaches zero. It crucially relies on a new measure-theoretic observation (Lemma 4.3) that may be of independent interest.

If one does want to assume only Besov regularity, then one needs to make further assumptions on the density near vacuum; we show that energy is conserved provided the density descends into vacuum sufficiently fast (Corollary 4.7) or sufficiently slowly (Corollary 4.11).

Finally, in Section 5 we demonstrate how to extend our results, so far shown only under periodic boundary conditions, to the case of a bounded domain.

1A. ***The result of Feireisl et al.*** To formulate the local or global energy equality for (1-1) it is useful to define the so-called pressure potential by

$$P(\rho) = \rho \int_1^\rho \frac{p(r)}{r^2}\,\mathrm{d}r.$$

The following theorem was proven in [Feireisl et al. 2017, Theorem 4.1].

Theorem 1.1. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \bar{\rho} \ \textit{a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$, and $0 \le \alpha, \beta \le 1$ such that

$$\beta > \max\bigl\{1-2\alpha, \tfrac{1}{2}(1-\alpha)\bigr\}.$$

Assume further that $p \in C^2[\underline{\rho}, \bar{\rho}]$, and in addition

$$p'(0) = 0 \quad \textit{as soon as } \underline{\rho} = 0.$$

Then the energy is locally conserved; *i.e.*,

$$\partial_t\bigl(\tfrac{1}{2}\rho|u|^2 + P(\rho)\bigr) + \operatorname{div}\bigl[\bigl(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\bigr)u\bigr] = 0 \tag{1-2}$$

in the sense of distributions on $(0,T)\times\mathbb{T}^d$.

Our aim in the current paper is to improve the above theorem by relaxing the C^2 assumption on the pressure. This will allow one, for instance, to apply the theorem in the physically relevant case of the isentropic pressure law $p(\rho)=\kappa\rho^\gamma$ with the adiabatic coefficient $\gamma\in(1,2)$, without excluding vacuum.

2. Preliminaries

2A. ***Function spaces.*** For $\Omega:=(0,T)\times\mathbb{T}^d$ we recall the Besov space $B_p^{\alpha,\infty}(\Omega)$, which is the space of tempered distributions w for which the norm

$$\|w\|_{B_p^{\alpha,\infty}(\Omega)}:=\|w\|_{L^p(\Omega)}+\sup_{\xi\in\Omega}\frac{\|w(\,\cdot\,+\xi)-w\|_{L^p(\Omega\cap(\Omega-\xi))}}{|\xi|^\alpha} \tag{2-1}$$

is finite. The above norm provides a control over shifts of the distribution w, making Besov spaces a convenient environment for our analysis, as it relies on convolutions with a mollifying kernel.

Let $\eta\in C_c^\infty(\mathbb{R}^N)$ be a positive, radial function of integral 1 with

$$\eta(x)=\begin{cases}1 & \text{for } |x|\le\frac{1}{3},\\ 0 & \text{for } |x|\ge 1,\end{cases}$$

and for $N=1+d$ set

$$\eta^\varepsilon(x)=\frac{1}{\varepsilon^N}\eta\left(\frac{x}{\varepsilon}\right).$$

We define the notation $w^\varepsilon:=\eta^\varepsilon * w$. For any function w, w^ε is well-defined on $\Omega^\varepsilon=\{x\in\Omega:\mathrm{d}(x,\partial\Omega)>\varepsilon\}$.

It is then easy to check that the definition of the Besov spaces implies

$$\|w^\varepsilon-w\|_{L^p(\Omega^\varepsilon)}\le C\varepsilon^\alpha\|w\|_{B_p^{\alpha,\infty}(\Omega)}$$

and

$$\|\nabla w^\varepsilon\|_{L^p(\Omega^\varepsilon)}\le C\varepsilon^{\alpha-1}\|w\|_{B_p^{\alpha,\infty}(\Omega)}.$$

By $\mathscr{M}(\Omega)$ we denote the space of signed Radon measures equipped with the total variation norm

$$\|\mu\|_{TV}:=\int_\Omega \mathrm{d}|\mu|.$$

2B. ***Derivation of the local energy equality.*** The starting point in the proof of Theorem 1.1, as well as all our results, is to mollify the Euler equations, then derive the local energy equality for the regularized quantities, and finally estimate commutator errors generated by nonlinear terms. As this strategy is a common part in the proofs of our theorems, we devote this section to the said derivation, omitting the details of passing to the limit under the assumptions of Theorem 1.1.

We begin by mollifying the momentum equation in time and space to obtain

$$\partial_t(\rho u)^\varepsilon+\operatorname{div}(\rho u\otimes u)^\varepsilon+\nabla p^\varepsilon(\rho)=0, \tag{2-2}$$

or, in terms of commutators

$$\begin{aligned}&\partial_t(\rho^\varepsilon u^\varepsilon)+\operatorname{div}((\rho u)^\varepsilon\otimes u^\varepsilon)+\nabla p(\rho^\varepsilon)\\ &\qquad=\partial_t(\rho^\varepsilon u^\varepsilon-(\rho u)^\varepsilon)+\operatorname{div}((\rho u)^\varepsilon\otimes u^\varepsilon-(\rho u\otimes u)^\varepsilon)+\nabla(p(\rho^\varepsilon)-p^\varepsilon(\rho)).\end{aligned} \tag{2-3}$$

Making use of the identity

$$\operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon) = u^\varepsilon \operatorname{div}(\rho u)^\varepsilon + ((\rho u)^\varepsilon \cdot \nabla) u^\varepsilon,$$

we can see that multiplying (2-3) by u^ε yields

$$\rho^\varepsilon \partial_t\big(\tfrac{1}{2}|u^\varepsilon|^2\big) + ((\rho u)^\varepsilon \cdot \nabla)\tfrac{1}{2}|u^\varepsilon|^2 + \rho^\varepsilon u^\varepsilon\, \nabla(P'(\rho^\varepsilon)) = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon, \tag{2-4}$$

where

$$\begin{aligned} r_1^\varepsilon &= \partial_t(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon) \cdot u^\varepsilon, \\ r_2^\varepsilon &= \operatorname{div}((\rho u)^\varepsilon \otimes u^\varepsilon - (\rho u \otimes u)^\varepsilon) \cdot u^\varepsilon, \\ r_3^\varepsilon &= \nabla(p(\rho^\varepsilon) - p^\varepsilon(\rho)) \cdot u^\varepsilon. \end{aligned}$$

Using the mollified continuity equation

$$\partial_t \rho^\varepsilon + \operatorname{div}(\rho u)^\varepsilon = 0 \tag{2-5}$$

multiplied by $\frac{1}{2}|u^\varepsilon|^2$, we can rewrite (2-4) as

$$\partial_t\big(\tfrac{1}{2}\rho^\varepsilon|u^\varepsilon|^2\big) + \operatorname{div}\big((\rho u)^\varepsilon\, \tfrac{1}{2}|u^\varepsilon|^2\big) + \rho^\varepsilon u^\varepsilon\, \nabla(P'(\rho^\varepsilon)) = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon. \tag{2-6}$$

On the other hand writing (2-5) in the form

$$\partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)$$

and multiplying by $P'(\rho^\varepsilon)$ we get

$$\partial_t(P(\rho^\varepsilon)) + \operatorname{div}(\rho^\varepsilon u^\varepsilon) P'(\rho^\varepsilon) = \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)\, P'(\rho^\varepsilon). \tag{2-7}$$

Combining (2-6) and (2-7) we obtain

$$\partial_t\big(\tfrac{1}{2}\rho^\varepsilon|u^\varepsilon|^2 + P(\rho^\varepsilon)\big) + \operatorname{div}\big((\rho u)^\varepsilon\, \tfrac{1}{2}|u^\varepsilon|^2 + \rho^\varepsilon u^\varepsilon P'(\rho^\varepsilon)\big) = r_1^\varepsilon + r_2^\varepsilon + r_3^\varepsilon + s^\varepsilon, \tag{2-8}$$

where we set

$$s^\varepsilon := \operatorname{div}(\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon)\, P'(\rho^\varepsilon).$$

The proof of Theorem 4.1 in [Feireisl et al. 2017] shows that when ρ, u are Besov regular and p is of class C^2, the left-hand side of (2-8) converges to the left-hand side of (1-2) and each term on the right-hand side of (2-8) converges to zero, where each convergence is in the sense of distributions.

3. Energy conservation assuming the divergence of velocity is a bounded measure

Our first result establishes local energy conservation for weak solutions of (1-1) under the additional assumption that the velocity field u is a divergence-measure field.

Remark 3.1. See [Chen and Torres 2005] for details on the role of divergence-measure fields in the theory of hyperbolic conservation laws.

Theorem 3.2. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \bar{\rho} \text{ a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$, *and* $0 \le \alpha, \beta \le 1$ *such that*

$$\beta > \max\bigl\{1-2\alpha, \tfrac{1}{2}(1-\alpha)\bigr\}.$$

Assume further that

$$\operatorname{div} u \in \mathscr{M}((0,T)\times\mathbb{T}^d) \quad \text{and} \quad p \in C[\underline{\rho}, \bar{\rho}].$$

Then the energy is locally conserved; *i.e.*,

$$\partial_t\bigl(\tfrac{1}{2}\rho|u|^2 + P(\rho)\bigr) + \operatorname{div}\bigl[\bigl(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\bigr)u\bigr] = 0$$

in the sense of distributions on $(0,T)\times\mathbb{T}^d$.

Proof. Take a sequence $p^\delta \in C^2[\underline{\rho}, \bar{\rho}]$ that converges uniformly to $p \in C[\underline{\rho}, \bar{\rho}]$; that is, for each $\delta > 0$,

$$\|p - p^\delta\|_{L^\infty} \le \delta.$$

Then using p^δ in (2-2) we have

$$\partial_t(\rho u)^\varepsilon + \operatorname{div}(\rho u \otimes u)^\varepsilon + \nabla(p^\delta(\rho))^\varepsilon = \nabla[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]. \tag{3-1}$$

Now the left-hand side of the last equality satisfies all the conditions of Theorem 1.1, so for each fixed $\delta > 0$ we have, in the limit as $\varepsilon \to 0$,

$$\partial_t\bigl(\tfrac{1}{2}\rho|u|^2 + P^\delta(\rho)\bigr) + \operatorname{div}\bigl[\bigl(\tfrac{1}{2}\rho|u|^2 + p^\delta(\rho) + P^\delta(\rho)\bigr)u\bigr], \tag{3-2}$$

where

$$P^\delta(\rho) := \rho \int_1^\rho \frac{p^\delta(r)}{r^2}\,\mathrm{d}r.$$

We will now show that (3-2) converges as $\delta \to 0$ in the sense of distributions on $(0,T)\times\mathbb{T}^d$ to

$$\partial_t\bigl(\tfrac{1}{2}\rho|u|^2 + P(\rho)\bigr) + \operatorname{div}\bigl[\bigl(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\bigr)u\bigr].$$

Let $\varphi \in C_c^\infty((0,T)\times\mathbb{T}^d)$. From the choice of p^δ we have

$$\left|\int_0^T\int_{\mathbb{T}^d} \nabla\varphi\cdot(p^\delta(\rho) - p(\rho))u\,\mathrm{d}x\,\mathrm{d}t\right| \le C\|\varphi\|_{\mathscr{C}^1}\|p^\delta - p\|_{L^\infty}\|u\|_{L^3} \le C(\varphi, u)\delta.$$

For the terms containing $P^\delta(\rho)$ notice that

$$|P^\delta(\rho) - P(\rho)| \le \rho\int_1^\rho \frac{|p^\delta(r) - p(r)|}{r^2}\,\mathrm{d}r \le \|p^\delta - p\|_{L^\infty}\,\rho\left|\int_1^\rho \frac{1}{r^2}\,\mathrm{d}r\right| \le (1+\rho)\|p^\delta - p\|_{L^\infty}.$$

Hence we can estimate

$$\left|\int_0^T\int_{\mathbb{T}^d} \partial_t\varphi\,(P^\delta(\rho) - P(\rho))\,\mathrm{d}x\,\mathrm{d}t\right| \le C\|\varphi\|_{\mathscr{C}^1}(1 + \|\rho\|_{L^1})\delta \le C(\varphi)\delta,$$

and similarly for the divergence term. It follows that both terms of (3-2) containing P^δ converge as $\delta \to 0$ to the corresponding terms for P.

The final step of the proof is to consider the term coming into (2-8) from the right-hand side of (3-1). We need to show that

$$\nabla[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)] \cdot u^\varepsilon$$

converges to zero in the sense of distributions on $(0, T) \times \mathbb{T}^d$ as first ε and then δ tend to zero. Multiplying by $\varphi \in C_c^\infty((0, T) \times \mathbb{T}^d)$, integrating over time and space, and integrating by parts we obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^d} &\nabla[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\varphi u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T\int_{\mathbb{T}^d}[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\varphi \operatorname{div} u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_{\mathbb{T}^d}[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\nabla\varphi\cdot u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3-3}$$

For the second term on the right-hand side of the last equality we see that

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{T}^d}[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\nabla\varphi\cdot u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| &= \left|\int_0^T\int_{\mathbb{T}^d}[p^\delta(\rho) - p(\rho)]^\varepsilon\nabla\varphi\cdot u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| \\ &\le C\|\varphi\|_{C^1}\|(p^\delta - p)^\varepsilon\|_{L^\infty}\|u\|_{L^3} \\ &\le C\|\varphi\|_{C^1}\|p^\delta - p\|_{L^\infty}\|u\|_{L^3} \le C\delta.\end{aligned}$$

Finally, for the first term on the right-hand side of (3-3) we invoke the assumption that $\operatorname{div} u$ is a bounded Radon measure to see that

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{T}^d}\varphi[(p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)]\operatorname{div} u^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| &= \left|\int_0^T\int_{\mathbb{T}^d}\varphi[p^\delta(\rho) - p(\rho)]^\varepsilon(\operatorname{div} u)^\varepsilon \,\mathrm{d}x\,\mathrm{d}t\right| \\ &\le \|\varphi\|_{C^0}\|(p^\delta - p)^\varepsilon\|_{L^\infty}\|(\operatorname{div} u)^\varepsilon\|_{L^1} \\ &\le \|\varphi\|_{C^0}\|p^\delta - p\|_{L^\infty}\|\operatorname{div} u\|_{TV} \le C\delta\end{aligned}$$

and so we are done. □

3A. ***Application to the compressible Navier–Stokes equations.*** When studying the result of Theorem 3.2 we see that the condition $\operatorname{div} u \in \mathscr{M}((0, T) \times \mathbb{T}^d)$ is quite a strong assumption for solutions to the compressible Euler equations; however, it is given for the compressible Navier–Stokes equations where one obtains a priori from the diffusion term that $u \in L^2(0, T; H^1)$. Therefore a natural question to ask is what happens when we consider the solutions to the compressible Navier–Stokes equations with vacuum, and how these results relate to the current results in [Yu 2017].

The compressible Navier–Stokes equations are given by

$$\begin{aligned}\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) + \nabla p(\rho) &= \operatorname{div}\mathbb{S}(\nabla u),\\ \partial_t\rho + \operatorname{div}(\rho u) &= 0,\\ \mathbb{S}(\nabla u) := \mu\big(\nabla u + (\nabla u)^T - \tfrac{2}{3}\operatorname{div} u\mathbb{I}\big) &+ \nu\operatorname{div} u\mathbb{I},\end{aligned} \tag{3-4}$$

where we have the constants $\mu > 0$ and $\nu \geq 0$. Here we will use the main properties that $\mathbb{S}(\nabla u)$ is symmetric and positive definite. For degenerate viscosity, the momentum equation becomes, instead,

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) = \operatorname{div}(\rho \mathbb{S}(\nabla u)). \tag{3-5}$$

Corollary 3.3. *Let ρ, u be a solution of* (3-4) *or* (3-5) *in the sense of distributions. Assume*

$$u \in B_3^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad u \in L^2(0,T;H^1(\mathbb{T}^d)), \quad \rho, \rho u \in B_3^{\beta,\infty}((0,T)\times\mathbb{T}^d),$$
$$0 \leq \underline{\rho} \leq \rho \leq \bar{\rho} \text{ a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$, and $0 \leq \alpha, \beta \leq 1$ such that

$$\beta > \max\bigl\{1-2\alpha, \tfrac{1}{2}(1-\alpha)\bigr\}. \tag{3-6}$$

Assume further that $p \in C[\underline{\rho}, \bar{\rho}]$. Then the energy is locally conserved; *i.e.*,

$$\partial_t\bigl(\tfrac{1}{2}\rho|u|^2 + P(\rho)\bigr) + \mathbb{S}(\nabla u) : \nabla u + \operatorname{div}\bigl[\bigl(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho) + \mathbb{S}(\nabla u)\bigr)u\bigr] = 0 \tag{3-7}$$

for (3-4) *and*

$$\partial_t\bigl(\tfrac{1}{2}\rho|u|^2 + P(\rho)\bigr) + \rho\mathbb{S}(\nabla u) : \nabla u + \operatorname{div}\bigl[\bigl(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho) + \rho\mathbb{S}(\nabla u)\bigr)u\bigr] = 0 \tag{3-8}$$

for (3-5), *in the sense of distributions on* $(0,T)\times\mathbb{T}^d$.

Remark 3.4. The condition $\operatorname{div} u \in \mathscr{M}$ is trivially satisfied if we assume that $u \in L^2(0,T;H^1)$ and so does not appear in the statement of Corollary 3.3.

Remark 3.5. For $d \leq 3$ we can use Besov embedding theorems, see [Bahouri et al. 2011], to observe that $H^1 \hookrightarrow B_2^{1,\infty} \hookrightarrow B_3^{2/3,\infty}$ and so assuming that $u \in B_3^{\alpha_1,\infty}(0,T;B_3^{\alpha_2,\infty})$ and $\rho, \rho u \in B_3^{\beta_1,\infty}(0,T;B_3^{\beta_2,\infty})$ we have the same assumptions on the pairs (α_1,β_1) and (α_2,β_2) as (3-6) but can assume that $\alpha_2 \geq \frac{2}{3}$ and remove the assumption that $u \in L^2(0,T;H^1)$.

Remark 3.6. We have assumed that the density ρ is bounded above to simplify the proof, though this is not necessary. Indeed, we can assume that for some $C > 0$, $p^\delta(r) = p(r)$ for $r \geq C$ and so still obtain uniform convergence of p^δ to p for unbounded density.

Proof. We only have to consider the extra term $\operatorname{div}\mathbb{S}(\nabla u)$ in the derivation of the local energy equality that we performed previously. We see that

$$-\int_0^T\int_{\mathbb{T}^d} \operatorname{div}\mathbb{S}(\nabla u^\varepsilon)\cdot u^\varepsilon\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{T}^d} \mathbb{S}(\nabla u^\varepsilon) : \nabla u^\varepsilon\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d} (\mathbb{S}(\nabla u^\varepsilon)u^\varepsilon)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t$$

and so obtain (3-7). For (3-8) we perform the same calculation as above; however, with an extra ρ in the equation, the diffusion term is no longer linear and thus we pick up an extra commutator estimate

$$r_d^\varepsilon := \int_0^T\int_{\mathbb{T}^d} \operatorname{div}(\rho^\varepsilon\mathbb{S}(\nabla u^\varepsilon) - (\rho\mathbb{S}(\nabla u))^\varepsilon)\cdot\varphi u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t.$$

We can perform an integration by parts to obtain

$$|r_d^\varepsilon| \le \left| \int_0^T \int_{\mathbb{T}^d} [(\rho^\varepsilon \mathbb{S}(\nabla u^\varepsilon) - (\rho \mathbb{S}(\nabla u))^\varepsilon) u^\varepsilon] \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t \right| + \left| \int_0^T \int_{\mathbb{T}^d} (\rho^\varepsilon \mathbb{S}(\nabla u^\varepsilon) - (\rho \mathbb{S}(\nabla u))^\varepsilon) : \nabla u^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t \right|. \tag{3-9}$$

Note the pointwise identity where for any two functions f, g we have

$$f^\varepsilon g^\varepsilon - (fg)^\varepsilon = (f^\varepsilon - f)(g^\varepsilon - g) - \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{T}^d} \eta^\varepsilon(\tau, \xi)(f(t-\tau, x-\xi) - f(t,x))(g(t-\tau, x-\xi) - g(t,x)) \, \mathrm{d}\xi \, \mathrm{d}\tau. \tag{3-10}$$

Applying this allows us to split the two terms on the right-hand side of (3-9) into four more terms which we can estimate. We focus on the first of these terms only, as the other terms produce the same estimates, after applying Fubini's theorem, as seen in [Feireisl et al. 2017]. We see that

$$\begin{aligned} |r_d^\varepsilon| &\le \left| \int_0^T \int_{\mathbb{T}^d} [(\rho^\varepsilon - \rho)(\mathbb{S}(\nabla u^\varepsilon) - \mathbb{S}(\nabla u)) u^\varepsilon] \cdot \nabla \varphi \, \mathrm{d}x \, \mathrm{d}t \right| + \left| \int_0^T \int_{\mathbb{T}^d} (\rho^\varepsilon - \rho)(\mathbb{S}(\nabla u^\varepsilon) - \mathbb{S}(\nabla u)) : \nabla u^\varepsilon \varphi \, \mathrm{d}x \, \mathrm{d}t \right| \\ &\le \|\varphi\|_{C^1} \|\rho\|_{L^\infty} \|u\|_{L^2} \|\mathbb{S}(\nabla u^\varepsilon) - \mathbb{S}(\nabla u)\|_{L^2} + \|\varphi\|_{C^0} \|\rho\|_{L^\infty} \|\nabla u\|_{L^2} \|\mathbb{S}(\nabla u^\varepsilon) - \mathbb{S}(\nabla u)\|_{L^2}. \end{aligned}$$

Using the a priori estimate that $u \in L^2(0, T; H^1)$ we see that $\|\mathbb{S}(\nabla u^\varepsilon) - \mathbb{S}(\nabla u)\|_{L^2} \to 0$ as $\varepsilon \to 0$ and thus $r_d^\varepsilon \to 0$ as $\varepsilon \to 0$. □

The work of Cheng Yu [2017] also studies energy conservation for the compressible Navier–Stokes systems where a vacuum could occur. The result in [Yu 2017] treats the case where $p(\rho) = \rho^\gamma$ for $\gamma > 1$ and thus where $p \in C^{1,\gamma-1}$, with strong assumptions of spacial regularity where

$$\sqrt{\rho} \nabla u \in L^2(0, T; L^2(\Omega)) \quad \text{and} \quad \frac{\nabla \rho}{\sqrt{\rho}} \in L^\infty(0, T; L^2(\Omega)),$$

among other assumptions. However, [Yu 2017] only assumes integrability in time. The condition $\nabla \rho / \sqrt{\rho} \in L^\infty(0, T; L^2(\Omega))$ restricts the allowable vacuum cases and will only allow vacuum on measure-zero sets with a nice approach to this set. The result presented here complements the result in [Yu 2017] as we show that by assuming some differential regularity in time for both ρ and u then we can weaken the spacial regularity assumptions and only need continuity of the pressure p. Specifically, we can have vacuum on measurable subsets of the domain where the approach to this set can be quite generic.

4. Energy conservation assuming Hölder continuity of the pressure

For the next result we fix $1 < \gamma < 2$ and we will assume that the pressure p is of class $C^{1,(\gamma-1)}$, thus relaxing the regularity assumption of Theorem 1.1. The expense of this relaxation is that we require $\alpha + \gamma\beta > 1$ where before we only needed $\alpha + 2\beta > 1$.

Theorem 4.1. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u \in B_p^{\alpha,\infty}((0,T) \times \mathbb{T}^d), \quad \rho, \rho u \in B_q^{\beta,\infty}((0,T) \times \mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \bar{\rho} \ \text{a.e. in } (0,T) \times \mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$ *and* $0 \le \alpha, \beta \le 1$ *such that*

$$\frac{1}{p}+\frac{2}{q}\le 1, \quad \frac{2}{p}+\frac{1}{q}\le 1, \quad p,q\ge 2, \quad \alpha+\gamma\beta>1, \quad \text{and} \quad 2\alpha+\beta>1.$$

Define $\mathscr{B}_{\varepsilon^\beta} := \{x : 0 < \rho^\varepsilon(x) < \varepsilon^\beta \text{ and } \rho \neq 0\}$ *and assume that*

$$\left\| \frac{\rho^\varepsilon-\rho}{\rho^\varepsilon} \right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \le C(\rho), \tag{4-1}$$

where C *does not depend on* ε. *Assume further that* $p \in C^{1,(\gamma-1)}([\underline{\rho}, \bar{\rho}])$, *and, in addition*

$$p'(0)=0 \quad \text{as soon as } \underline{\rho}=0.$$

Then the energy is locally conserved; *i.e.*, (1-2) *holds in the sense of distributions on* $(0,T)\times\mathbb{T}^d$.

A large part of the proof of this theorem is identical to the proof of Theorem 1.1. In particular we regularize the balance equations to derive an energy balance for the smooth functions ρ^ε and u^ε. Then we need to show that the corresponding commutator errors vanish in the limit $\varepsilon \to 0$. This is done in the same way as in [Feireisl et al. 2017], the only difference being in the terms involving the pressure. In particular, we will have to estimate an appropriate norm of the difference $p(\rho)^\varepsilon - p(\rho^\varepsilon)$. This will be done by means of the following lemma, which is an adaptation to our present case of the argument in [Feireisl et al. 2017, p. 10]; see also [Gwiazda et al. 2018, Lemma 3.1].

Lemma 4.2. *Let* $\gamma \in (1,2)$ *and* $p \in C^{1,\gamma-1}([a,b])$. *If* $\rho \in B^{\beta,\infty}_{\gamma q}(\Omega; [a,b])$, *then*

$$\|p^\varepsilon(\rho)-p(\rho^\varepsilon)\|_{L^q} \le C\varepsilon^{\gamma\beta}\|\rho\|^{\gamma}_{B^{\beta,\infty}_{\gamma q}}.$$

Proof. First we note that by the fundamental theorem of calculus

$$\begin{aligned} p(s)-p(s_0) = \int_{s_0}^{s} p'(t)\,\mathrm{d}t &= \int_{s_0}^{s} p'(s_0)\,\mathrm{d}t + \int_{s_0}^{s} p'(t)-p'(s_0)\,\mathrm{d}t \\ &= p'(s_0)(s-s_0) + \int_{s_0}^{s} p'(t)-p'(s_0)\,\mathrm{d}t. \end{aligned}$$

Since $p' \in C^{0,\gamma-1}$, we have

$$\left| \int_{s_0}^{s} p'(t)-p'(s_0)\,\mathrm{d}t \right| \le \int_{s_0}^{s} |p'(t)-p'(s_0)|\,\mathrm{d}t \le C\int_{s_0}^{s}\mathrm{d}t \sup_{t\in[s_0,s]}|t-s_0|^{\gamma-1} \le C|s-s_0|^\gamma.$$

Thus,

$$|p(s)-p(s_0)-p'(s_0)(s-s_0)| \le C|s-s_0|^\gamma.$$

As the constant C is independent of s, s_0 we see that

$$|p(\rho^\varepsilon)-p(\rho)-p'(\rho)(\rho^\varepsilon-\rho)| \le C|\rho-\rho^\varepsilon|^\gamma, \tag{4-2}$$

and similarly,

$$|p(\rho(y))-p(\rho(x))-p'(\rho(x))(\rho(y)-\rho(x))| \le C|\rho(x)-\rho(y)|^\gamma. \tag{4-3}$$

Applying convolution against the function η^ε with respect to y in (4-3) and using Jensen's inequality we obtain

$$|p^\varepsilon(\rho)-p(\rho)-p'(\rho)(\rho^\varepsilon-\rho)| \le C|\rho-\rho(\,\cdot\,)|^\gamma *_y \eta^\varepsilon. \tag{4-4}$$

Combining (4-2) and (4-4) we get

$$|p^\varepsilon(\rho) - p(\rho^\varepsilon)| \le C|\rho - \rho^\varepsilon|^\gamma + C|\rho - \rho(\,\cdot\,)|^\gamma *_y \eta^\varepsilon. \tag{4-5}$$

Taking the L^q norm of both sides of (4-5) for the first term on the right-hand side we see that

$$C\| |\rho - \rho^\varepsilon|^\gamma \|_{L^q} = C\|\rho - \rho^\varepsilon\|^\gamma_{L^{\gamma q}}.$$

Finally, for the L^q norm of (4-5) for the second term on the right-hand side by Jensen's inequality and Fubini's theorem we have

$$\begin{aligned} C\| |\rho - \rho(\,\cdot\,)|^\gamma *_y \eta^\varepsilon\|_{L^q} &\le C\left(\iint |\rho(x) - \rho(x-y)|^{\gamma q}\,\mathrm{d}x\,\eta_\varepsilon(y)\,\mathrm{d}y\right)^{1/q} \\ &= C\left(\int \|\rho(\,\cdot\,) - \rho(\,\cdot\, - y)\|^{\gamma q}_{L^{\gamma q}}\eta_\varepsilon(y)\,\mathrm{d}y\right)^{1/q} \\ &\le C\sup_y |\eta_\varepsilon(y)|^{1/q}\left(\int_{\operatorname{supp}\eta_\varepsilon} \|\rho(\,\cdot\,) - \rho(\,\cdot\, - y)\|^{\gamma q}_{L^{\gamma q}}\,\mathrm{d}y\right)^{1/q} \\ &\le C \sup_{y\in\operatorname{supp}\eta_\varepsilon} \|\rho(\,\cdot\,) - \rho(\,\cdot\, - y)\|^{\gamma}_{L^{\gamma q}}. \end{aligned}$$

Finally, we use the definition of the Besov norm and (2-1) to write

$$\begin{aligned} \|p^\varepsilon(\rho) - p(\rho^\varepsilon)\|_{L^q} &\le C\big(\|\rho^\varepsilon - \rho\|^\gamma_{L^{\gamma q}} + \sup_{s\in\operatorname{supp}\eta^\varepsilon} \|\rho(\,\cdot\,) - \rho(\,\cdot\, - s)\|^\gamma_{L^{\gamma q}}\big) \\ &\le C\varepsilon^{\gamma\beta}\|\rho\|^\gamma_{B^{\beta,\infty}_{\gamma q}} + \sup_{s\in\operatorname{supp}\eta^\varepsilon} |s|^{\gamma\beta}\|\rho\|^\gamma_{B^{\beta,\infty}_{\gamma q}} \le C\varepsilon^{\gamma\beta}\|\rho\|^\gamma_{B^{\beta,\infty}_{\gamma q}}. \end{aligned}$$ □

Proof of Theorem 4.1. As remarked above the only novelty needed to establish the desired result is to estimate commutator errors due to nonlinearity of the pressure. Precisely, we need to show that the local versions of r_3^ε and s^ε, which we will denote by R^ε and S^ε, of (2-8) converge to zero as $\varepsilon \to 0$. For a test function $\varphi \in C_c^\infty((0,T)\times\mathbb{T}^d)$ we define

$$\begin{aligned} R^\varepsilon &:= \int_0^T\int_{\mathbb{T}^d} \nabla(p(\rho^\varepsilon) - p(\rho)^\varepsilon)\cdot\varphi u^\varepsilon\,\mathrm{d}x\,\mathrm{d}t, \\ S^\varepsilon &:= \int_0^T\int_{\mathbb{T}^d} \varphi\operatorname{div}[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\, P'(\rho^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{4-6}$$

Integrating (4-6) by parts and using Lemma 4.2 we obtain the estimate

$$\begin{aligned} |R^\varepsilon| &\le \|\varphi\|_{\mathscr{C}^1}\int_0^T\int_{\mathbb{T}^d} |p(\rho)^\varepsilon - p(\rho)^\varepsilon|(|\nabla u^\varepsilon| + |u^\varepsilon|)\,\mathrm{d}x\,\mathrm{d}t \\ &\le C\|\varphi\|_{\mathscr{C}^1}\|p(\rho^\varepsilon) - p(\rho)^\varepsilon\|_{L^{q/2}}(\|\nabla u^\varepsilon\|_{L^p} + \|u^\varepsilon\|_{L^p}) \\ &\le C(\varepsilon^{\gamma\beta+(\alpha-1)} + \varepsilon^{\gamma\beta+\alpha})\|\rho\|^\gamma_{B^{\beta,\infty}_{\gamma q/2}}\|u\|_{B^{\alpha,\infty}_p} \\ &\le C(\varepsilon^{\gamma\beta+(\alpha-1)} + \varepsilon^{\gamma\beta+\alpha})\|\rho\|^\gamma_{B^{\beta,\infty}_{q}}\|u\|_{B^{\alpha,\infty}_p}, \end{aligned}$$

where for the last inequality we used that $\frac{1}{2}\gamma q < q$, so we can embed $B^{\beta,\infty}_q$ into $B^{\beta,\infty}_{\gamma q/2}$.

We now investigate the term S^ε and see that we can integrate by parts to obtain

$$|S^\varepsilon| = \left| \int_0^T \int_{\mathbb{T}^d} \varphi \,\mathrm{div}[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\, P'(\rho^\varepsilon)\,\mathrm{d}x\,\mathrm{d}t \right| \\ \le \int_0^T \int_{\mathbb{T}^d} |\nabla\varphi \cdot [\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\, P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} |\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon] \cdot \nabla P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t. \tag{4-7}$$

We make note of the pointwise identity (3-10) but with f and g replaced by ρ and u respectively, that is,

$$\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon = (\rho^\varepsilon - \rho)(u^\varepsilon - u) \\ - \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{T}^d} \eta^\varepsilon(\tau,\xi)(\rho(t-\tau, x-\xi) - \rho(t,x))(u(t-\tau, x-\xi) - u(t,x))\,\mathrm{d}\xi\,\mathrm{d}\tau,$$

and using (3-10) allows us to split first term on the right-hand side of (4-7) into two terms. Here again we focus on the first of these terms only, as the other one produces the same estimates, after applying Fubini's theorem, as seen in [Feireisl et al. 2017]. We see that

$$\int_0^T \int_{\mathbb{T}^d} |\nabla\varphi \cdot (\rho^\varepsilon - \rho)(u^\varepsilon - u) P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t \le \|\varphi\|_{C^1} \varepsilon^\beta \|\rho\|_{B_q^{\beta,\infty}} \varepsilon^\alpha \|u\|_{B_p^{\alpha,\infty}} \|P'(\rho^\varepsilon)\|_{L^\infty}.$$

We will now focus on the second term on the right-hand side of (4-7), namely,

$$\int_0^T \int_{\mathbb{T}^d} |\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon] \cdot \nabla P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t,$$

and by letting $y = (t, x)$ we split $(0, T) \times \mathbb{T}^d$ into two disjoint domains $\mathscr{A} := \{y : \rho^\varepsilon(y) = 0\}$ and $\mathscr{A}^c$ and see that trivially on $\mathscr{A}$ we have $\rho(y) = 0$ a.e. For the integral over $\mathscr{A}$ we note that $\nabla P'(\rho^\varepsilon)$ is a distribution that may have a singular part but we see that $\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]$ is smooth and equals zero on $\mathscr{A}$ and so any singular part vanishes. Thus we are left with

$$\int_{\mathscr{A}^c} |\varphi[\rho^\varepsilon u^\varepsilon - (\rho u)^\varepsilon]\nabla P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t,$$

and using again the identity (3-10) we obtain

$$\int_{\mathscr{A}^c} |\varphi[(\rho^\varepsilon - \rho)(u^\varepsilon - u)]\nabla P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t.$$

For the integral over $\mathscr{A}^c$ we see that

$$\int_{\mathscr{A}^c} |\varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)\nabla P'(\rho^\varepsilon)|\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathscr{A}^c} |\varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)P''(\rho^\varepsilon) \cdot \nabla\rho^\varepsilon|\,\mathrm{d}x\,\mathrm{d}t$$

and we observe that by the definition of P we have $\rho^\varepsilon P''(\rho^\varepsilon) = p'(\rho^\varepsilon)$, and by assumption p' is bounded. Therefore we have the bound

$$\int_{\mathscr{A}^c} |\varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)P''(\rho^\varepsilon)\nabla\rho^\varepsilon|\,\mathrm{d}x\,\mathrm{d}t \le \int_{\mathscr{A}^c} \left|\varphi(\rho^\varepsilon - \rho)(u^\varepsilon - u)p'(\rho^\varepsilon)\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\right|\,\mathrm{d}x\,\mathrm{d}t.$$

We have assumed that $p'(0)=0$ and $p' \in C^{0,\gamma-1}$ and so take any ρ_1, ρ_2 such that $p'(\rho_2)=0$ and we obtain

$$|p'(\rho_1)| = |p'(\rho_1) - p'(\rho_2)| \le C|\rho_1-\rho_2|^{\gamma-1} \le C|\rho_1|^{\gamma-1}$$

using the definition of Hölder continuity. Thus letting $\rho_1 = \rho^\varepsilon(x)$ for each x we see that $|p'(\rho^\varepsilon)(x)| \le C|\rho^\varepsilon|^{\gamma-1}(x)$ and so we obtain

$$\int_{\mathscr{A}^c} \left| \varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)p'(\rho^\varepsilon)\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\right| \mathrm{d}x\,\mathrm{d}t \le C\int_{\mathscr{A}^c}\left|\varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)(\rho^\varepsilon)^{\gamma-1}\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\right|\mathrm{d}x\,\mathrm{d}t.$$

We will split the integral over $\mathscr{A}^c$ further into different disjoint domains, $\mathscr{B}_{\varepsilon^\beta} := \{y : 0 < \rho^\varepsilon(y) < \varepsilon^\beta\}$ and $\mathscr{C}_{\varepsilon^\beta} := \{y : \rho^\varepsilon(y) \ge \varepsilon^\beta\}$. For the integral over $\mathscr{B}_{\varepsilon^\beta}$ we see that

$$\begin{aligned}
&\left|\int_{\mathscr{B}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)(\rho^\varepsilon)^{\gamma-1}\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right| \\
&\qquad\le C\|\varphi\|_{C^0}\varepsilon^{\beta-1+\alpha}\|\rho\|_{B^{\beta,\infty}_q}\|u\|_{B^{\alpha,\infty}_p}\|(\rho^\varepsilon)^{\gamma-1}\|_{L^\infty(\mathscr{B}_{\varepsilon^\beta})}\left\|\frac{\rho^\varepsilon-\rho}{\rho^\varepsilon}\right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \\
&\qquad\le C\|\varphi\|_{C^0}\varepsilon^{\gamma\beta-1+\alpha}\|\rho\|_{B^{\beta,\infty}_q}\|u\|_{B^{\alpha,\infty}_p}\left\|\frac{\rho^\varepsilon-\rho}{\rho^\varepsilon}\right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})},
\end{aligned}$$

where for the last line, as $\rho^\varepsilon(y) \le \varepsilon^\beta$, we have $(\rho^\varepsilon(y))^{\gamma-1} \le \varepsilon^{\beta(\gamma-1)}$ as $\gamma-1>0$. We also have the assumption that $\|(\rho^\varepsilon-\rho)/\rho^\varepsilon\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \le C$ and so we have the bound $C\varepsilon^{\gamma\beta-1+\alpha}$ as wanted. We are left with the integral over $\mathscr{C}_{\varepsilon^\beta}$ and see that

$$\left|\int_{\mathscr{C}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)(\rho^\varepsilon)^{\gamma-1}\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right| \le C\|\varphi\|_{C^0}\varepsilon^{\beta-1+\alpha}\|\rho\|_{B^{\beta,\infty}_q}\|u\|_{B^{\alpha,\infty}_p}\left\|\frac{\rho^\varepsilon-\rho}{(\rho^\varepsilon)^{2-\gamma}}\right\|_{L^q(\mathscr{C}_{\varepsilon^\beta})}.$$

As $\rho^\varepsilon \ge \varepsilon^\beta$, we have $(\rho^\varepsilon)^{-1} \le \varepsilon^{-\beta}$, and so $(\rho^\varepsilon)^{\gamma-2} \le \varepsilon^{\beta(\gamma-2)}$, and we obtain

$$\left\|\frac{\rho^\varepsilon-\rho}{(\rho^\varepsilon)^{2-\gamma}}\right\|_{L^q(\mathscr{C}_{\varepsilon^\beta})} \le \|\rho^\varepsilon-\rho\|_{L^q(\mathscr{C}_{\varepsilon^\beta})}\varepsilon^{\beta(\gamma-2)} \le C\varepsilon^\beta\|\rho\|_{B^{\beta,\infty}_q}\varepsilon^{\beta(\gamma-2)} \le C\varepsilon^{\beta(\gamma-1)}.$$

We are thus done as we have obtained convergence to zero as long as $\gamma\beta+\alpha>1$.

We have thus shown that, under the assumptions of the theorem, we have $R^\varepsilon, S^\varepsilon \to 0$. The result follows. □

We have written Theorem 4.1 in the most general from but observe that the condition

$$\left\|\frac{\rho^\varepsilon-\rho}{\rho^\varepsilon}\right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \le C$$

feels rather artificial and is not in the $p \in C^2$ result from [Feireisl et al. 2017]. We will now focus on finding conditions on ρ for different L^q norms that will control this term.

Our first result will show that when we assume that $q=1$ and so u, ρ are Hölder continuous, not just Besov functions, we can control this term directly as expected and do not have to ask for any special extra conditions.

Lemma 4.3. *Let $w \in L^1(\Omega)$ be nonnegative, where $\Omega \subset (0,T)\times\mathbb{T}^d$ satisfies $|\Omega| \neq 0$ and $w^\varepsilon|_\Omega > 0$. Then $\|(w^\varepsilon - w)/w^\varepsilon\|_{L^1(\Omega)} \le C$, where C does not depend on ε but may depend on w and Ω.*

Proof. It suffices to show that $\|w/w^\varepsilon\|_{L^1(\Omega)} \le C$. Indeed, since $|\Omega| \le C$,

$$\left\|\frac{w^\varepsilon - w}{w^\varepsilon}\right\|_{L^1(\Omega)} \le \|1\|_{L^1(\Omega)} + \left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)} = C + \left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)}.$$

Fix $\varepsilon > 0$, $N = d+1$, and let $\{Q_j\}_{j=1}^n$ be a partition of $(0,T)\times\mathbb{T}^d$ into disjoint cubes with side length ε/C_N, where C_N is a constant depending only on the dimension, and select the cubes such that $|\Omega \cap Q_j| \neq 0$. Decomposing w as $w = \sum_{j=1}^n w\chi_{Q_j}$, we see that

$$\left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)} = \left\|\frac{\sum_{j=1}^n w\chi_{Q_j}}{\sum_{k=1}^n (w\chi_{Q_k})^\varepsilon}\right\|_{L^1(\Omega)} \le \sum_{j=1}^n \left\|\frac{w\chi_{Q_j}}{(w\chi_{Q_j})^\varepsilon}\right\|_{L^1(\Omega)}.$$

We now want to bound $(w\chi_{Q_j})^\varepsilon$ from below. Recalling from Section 2 that $\eta = 1$ for $|x| < \frac{1}{3}$, we have, for $x \in Q_j$, that

$$\begin{aligned}(w\chi_{Q_j})^\varepsilon(x) &\ge \frac{1}{\varepsilon^N}\int_{\{|(x-y)/\varepsilon|\le 1/3\}} \eta\left(\frac{x-y}{\varepsilon}\right)(w\chi_{Q_j})(y)\,\mathrm{d}y = \frac{1}{\varepsilon^N}\int_{B_{\varepsilon/3}(x)} (w\chi_{Q_j})(y)\,\mathrm{d}y \\ &= \frac{\omega_N}{|B_\varepsilon|}\int_{B_{\varepsilon/3}(x)} (w\chi_{Q_j})(y)\,\mathrm{d}y \ge \frac{\omega_N}{|B_\varepsilon|}\int_{Q_j} w(y)\,\mathrm{d}y,\end{aligned}$$

where we obtain the last inequality provided C_N is large enough so that $B_{\varepsilon/3}(x) \supset Q_j$ for all $x \in Q_j$. Thus we obtain

$$\left\|\frac{w}{w^\varepsilon}\right\|_{L^1(\Omega)} \le \sum_{j=1}^n \left\|\frac{w\chi_{Q_j}}{(w\chi_{Q_j})^\varepsilon}\right\|_{L^1(\Omega)} \le \sum_{j=1}^n \frac{|B_\varepsilon|}{\omega_N}\frac{\int_{Q_j} w\,\mathrm{d}x}{\int_{Q_j} w\,\mathrm{d}x} \le C\sum_{j=1}^n |Q_j| \le C, \tag{4-8}$$

where we have used a dimensional constant to relate the measure of the balls to the associated cubes. □

As a consequence we obtain the following corollary, where by assuming Hölder continuity of u and ρ we obtain a natural extension of Theorem 1.1 to the case where $p \in C^{1,\gamma-1}$.

Corollary 4.4. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u \in C^\alpha((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in C^\beta((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \bar{\rho} \ \text{ a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$ and $0 \le \alpha, \beta \le 1$ such that

$$\alpha + \gamma\beta > 1 \quad \text{and} \quad 2\alpha + \beta > 1.$$

Assume further that $p \in C^{1,(\gamma-1)}([\underline{\rho}, \bar{\rho}])$, and, in addition

$$p'(0) = 0 \quad \text{as soon as } \underline{\rho} = 0.$$

Then the energy is locally conserved; *i.e.*, (1-2) *holds in the sense of distributions on* $(0,T)\times\mathbb{T}^d$.

Proof. For the integral over $\mathscr{B}_{\varepsilon^\beta}$, in the proof of Theorem 4.1, we see that

$$\begin{aligned}\left|\int_{\mathscr{B}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)(\rho^\varepsilon)^{\gamma-1}\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right| & \\ &\le C\|\varphi\|_{C^0}\varepsilon^{\beta-1+\alpha}\|\rho\|_{\mathscr{C}^\beta}\|u\|_{\mathscr{C}^\alpha}\|(\rho^\varepsilon)^{\gamma-1}\|_{L^\infty(\mathscr{B}_{\varepsilon^\beta})}\left\|\frac{\rho^\varepsilon-\rho}{\rho^\varepsilon}\right\|_{L^1(\mathscr{B}_{\varepsilon^\beta})}\\ &\le C\|\varphi\|_{C^0}\varepsilon^{\gamma\beta-1+\alpha}\|\rho\|_{\mathscr{C}^\beta}\|u\|_{\mathscr{C}^\alpha}.\end{aligned}$$

For the other bounds, as we are on a domain with finite measure, we can bound the Besov norms by the Hölder norms. □

Remark 4.5. Notice that the conditions $u\in C^\alpha((0,T)\times\mathbb{T}^d)$ and $\rho\in C^\beta((0,T)\times\mathbb{T}^d)$ imply that $\rho u\in C^{\min(\alpha,\beta)}((0,T)\times\mathbb{T}^d)$. Therefore, if one has $\alpha\ge\beta$, then the requirement that ρu be in $C^\beta((0,T)\times\mathbb{T}^d)$ can be dropped. See also Remark 3.2(2) in [Feireisl et al. 2017].

When we still want to consider Besov spaces for ρ and u we have to consider extra conditions on ρ in order to control the term $\|(\rho^\varepsilon-\rho)/\rho^\varepsilon\|_{L^q(\mathscr{B}_{\varepsilon^\beta})}$. Our first method will be to ask for an integrability condition on $1/\rho$.

Lemma 4.6. *Assume that $1/w\in L^p$ and $w\in L^q$. Then*

$$\left\|\frac{w^\varepsilon-w}{w^\varepsilon}\right\|_{L^r}\le C\quad \textit{for}\ \frac{1}{p}+\frac{1}{q}\le\frac{1}{r},$$

and in fact if $r<\infty$,

$$\lim_{\varepsilon\to0}\left\|\frac{w^\varepsilon-w}{w^\varepsilon}\right\|_{L^r}=0\quad \textit{for}\ \frac{1}{p}+\frac{1}{q}\le\frac{1}{r}.$$

Proof. Using Hölder's inequality and then Jensen's inequality, as the integral of the mollifier is 1 and $1/x$ is a convex function, we get that $\|1/w^\varepsilon\|\le\|(1/w)^\varepsilon\|\le\|1/w\|$ and so

$$\left\|\frac{w^\varepsilon-w}{w^\varepsilon}\right\|_{L^r}\le\|w^\varepsilon-w\|_{L^q}\left\|\frac{1}{w^\varepsilon}\right\|_{L^p}\le\|w^\varepsilon-w\|_{L^q}\left\|\frac{1}{w}\right\|_{L^p}\le C.$$

As long as $q<\infty$ we see that this, in fact, converges to zero. □

We now obtain the following corollary adding this condition into Theorem 4.1. We note that when $p=q=3$, we obtain the best result with the weakest integrability assumption in the Besov norms.

Corollary 4.7. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u\in B_p^{\alpha,\infty}((0,T)\times\mathbb{T}^d),\quad \rho,\rho u\in B_q^{\beta,\infty}((0,T)\times\mathbb{T}^d),\quad 0\le\underline{\rho}\le\rho\le\bar\rho\ \textit{ a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar\rho$ and $0\le\alpha,\beta\le1$ such that

$$\frac{1}{p}+\frac{2}{q}\le1,\quad \frac{2}{p}+\frac{1}{q}\le1,\quad p,q\ge2,\quad \alpha+\gamma\beta>1,\quad \textit{and}\quad 2\alpha+\beta>1.$$

Define $\mathscr{E}:=\{x:\rho\ne0\}$ and assume that

$$\frac{1}{\rho}\in L^q(\mathscr{E}).$$

Assume further that $p \in C^{1,(\gamma-1)}([\underline{\rho}, \bar{\rho}])$, *and, in addition*

$$p'(0) = 0 \quad \textit{as soon as } \underline{\rho} = 0.$$

Then the energy is locally conserved; *i.e.*, (1-2) *holds in the sense of distributions on* $(0, T) \times \mathbb{T}^d$.

Proof. For the integral over $\mathscr{B}_{\varepsilon^\beta}$, in the proof of Theorem 4.1, we see that as $\rho \in L^\infty$ and $\varepsilon^\beta \geq \rho^\varepsilon$ then

$$\begin{aligned}\left|\int_0^T\int_{\mathscr{B}_{\varepsilon^\beta}} \varphi(\rho^\varepsilon-\rho)(u^\varepsilon-u)(\rho^\varepsilon)^{\gamma-1}\frac{\nabla\rho^\varepsilon}{\rho^\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right| \\ &\leq C\|\varphi\|_{C^0}\varepsilon^{\beta-1+\alpha}\|\rho\|_{B_q^{\beta,\infty}}\|u\|_{B_p^{\alpha,\infty}}\|(\rho^\varepsilon)^{\gamma-1}\|_{L^\infty(\mathscr{B}_{\varepsilon^\beta})}\left\|\frac{\rho^\varepsilon-\rho}{\rho^\varepsilon}\right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \\ &\leq C\|\varphi\|_{C^0}\varepsilon^{\gamma\beta-1+\alpha}\|\rho\|_{B_q^{\beta,\infty}}\|u\|_{B_p^{\alpha,\infty}}\left\|\frac{1}{\rho}\right\|_{L^q(\mathscr{E})},\end{aligned}$$

and so we are done, using Lemma 4.6 for the final step. □

Remark 4.8. Even though we have written $1/\rho \in L^q(\mathscr{E})$, we can fix some $\delta > 0$ and only need this condition on some $\mathscr{B}_\delta$, as for $\varepsilon^1 > \varepsilon^2$ we have $\mathscr{B}_{\varepsilon^2} \subset \mathscr{B}_{\varepsilon^1}$, and so when $\varepsilon^\beta < \delta$ we have $\mathscr{B}_{\varepsilon^\beta} \subset \mathscr{B}_\delta$.

One can see that the condition $1/\rho \in L^q(\mathscr{B}_\delta)$ is quite a strong assumption and requires a quick approach of the function to the null set. Above we used conventional bounds to obtain a general integral result but do not consider the local structure of the function. We notice that a pointwise estimate $\rho \leq C\rho^\varepsilon$ would allow us to control the L^q norm of $(\rho^\varepsilon - \rho)/\rho^\varepsilon$ and, though convexity of ρ would do, we will now show a nice link between this and quasinearly subharmonic functions which are much more general functions than subharmonic, quasisubharmonic and nearly subharmonic functions [Pavlović and Riihentaus 2011]. The main motivation for the study of this notion in this paper is that it happens, as will be shown below, to be equivalent to the L^∞-boundedness of our problem term $(\rho^\varepsilon - \rho)/\rho^\varepsilon$.

Definition 4.9. Let $X \subset \mathbb{R}^d$ be a set and $u : X \to [0, +\infty)$ be Borel measurable. Then u is quasinearly subharmonic on X, that is $u \in \mathrm{QNS}(X)$, if there is a constant $\varepsilon_0 = \varepsilon_0(u)$, $0 < \varepsilon_0 < 1$, such that for each open set $O \subset X$, $O \neq X$, for each $x \in O$ and each r, $0 < r \leq \varepsilon_0\delta^O(x)$, one has $u \in L^1(B_r(x))$ and

$$u(x) \leq \frac{C}{|B_r(x)|}\int_{B_r(x)} u(y)\,\mathrm{d}y \quad \text{for some constant } C \geq 1, \tag{4-9}$$

where C is independent of r, $|B_r(x)| = \omega_d r^d$ is the volume of the ball and

$$\delta^O(x) = \mathrm{dist}(x, O^c) \quad \text{for the complement } O^c \text{ of } O \text{ in } X.$$

Lemma 4.10. *Let* $u : X \to [0, +\infty)$ *be a Borel measurable function. Then u is quasinearly subharmonic if and only if for every* $O \Subset X$ *there exist* M, ε_0 *such that for any* $0 < \varepsilon < \varepsilon_0$

$$u(x) \leq M u^\varepsilon(x) \quad \textit{for any } x \in O.$$

Proof. Let $u : X \to [0, +\infty)$ be a quasinearly subharmonic function. Then for any $\varepsilon < \mathrm{dist}(O, \partial X)$, u^ε is a well-defined smooth function on O. Suppose that $O \Subset X$ is a precompact set. Then $\delta_0 = \mathrm{dist}(O, \partial X)$

is a positive number and for $\varepsilon < \delta_0$

$$O \subset \{x : \mathrm{dist}(x, \partial X) > \varepsilon\}$$

and u^ε is well-defined on O. We prove that there exist M and ε_0 such that

$$u(x) \le M u^\varepsilon(x) \quad \text{for any } x \in O,\ 0 < \varepsilon < \varepsilon_0.$$

Indeed, we have

$$u^\varepsilon(x) = \frac{1}{\varepsilon^d} \int_X \eta\Big(\frac{x-y}{\varepsilon}\Big) u(y)\,\mathrm{d}y.$$

Note that $y \in X$ for $x \in O$ and $|x-y| < \varepsilon$. Since $u \ge 0$ and recalling that from the definition of η we know that $\eta = 1$ for $|x| < \frac{1}{3}$, we have

$$\begin{aligned} u^\varepsilon(x) &\ge \frac{1}{\varepsilon^d} \int_{\{|(x-y)/\varepsilon| \le 1/3\}} \eta\Big(\frac{x-y}{\varepsilon}\Big) u(y)\,\mathrm{d}y \\ &= \frac{1}{\varepsilon^d} \int_{\{|(x-y)/\varepsilon| \le 1/3\}} u(y)\,\mathrm{d}y = \frac{\omega_d}{3^d |B_{\varepsilon/3}(x)|} \int_{B_{\varepsilon/3}(x)} u(y)\,\mathrm{d}y \ge \frac{\omega_d u(x)}{3^d C} \end{aligned}$$

for sufficiently small ε. Therefore, we obtain

$$u(x) \le \frac{3^d C u^\varepsilon(x)}{\omega_d} \quad \text{for sufficiently small } \varepsilon \le \varepsilon_0 \delta^O(x).$$

On the other hand, if $u(x) \le M u^\varepsilon(x)$, then we have

$$u(x) \le \frac{M}{\varepsilon^d} \int_X \eta\Big(\frac{x-y}{\varepsilon}\Big) u(y)\,\mathrm{d}y = \frac{M\omega_d}{\omega_d \varepsilon^d} \int_{|x-y|\le\varepsilon} \eta\Big(\frac{x-y}{\varepsilon}\Big) u(y)\,\mathrm{d}y \le \frac{M\omega_d}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\,\mathrm{d}y. \tag{4-10}$$

Hence we deduce

$$u(x) \le \frac{C}{|B_\varepsilon(x)|} \int_{B_\varepsilon(x)} u(y)\,\mathrm{d}y. \qquad \square$$

From this pointwise control showing that $\rho(x) \le M\rho^\varepsilon(x)$ we obtain another corollary to our main result.

Corollary 4.11. *Let ρ, u be a solution of* (1-1) *in the sense of distributions. Assume*

$$u \in B_p^{\alpha,\infty}((0,T)\times\mathbb{T}^d), \quad \rho, \rho u \in B_q^{\beta,\infty}((0,T)\times\mathbb{T}^d), \quad 0 \le \underline{\rho} \le \rho \le \bar{\rho} \ \text{ a.e. in } (0,T)\times\mathbb{T}^d$$

for some constants $\underline{\rho}$, $\bar{\rho}$ and $0 \le \alpha, \beta \le 1$ such that

$$\frac{1}{p} + \frac{2}{q} \le 1, \quad \frac{2}{p} + \frac{1}{q} \le 1, \quad p, q \ge 2, \quad \alpha + \gamma\beta > 1, \quad \text{and} \quad 2\alpha + \beta > 1.$$

Assume that $\rho \in \mathrm{QNS}(\mathscr{B}_\delta)$ for some $\delta > 0$ and $p \in \mathscr{C}^{1,(\gamma-1)}([\underline{\rho}, \bar{\rho}])$ with

$$p'(0) = 0 \quad \text{as soon as } \underline{\rho} = 0.$$

Then the energy is locally conserved; *i.e.*, (1-2) *holds in the sense of distributions on* $(0,T)\times\mathbb{T}^d$.

Proof. For the integral over $\mathscr{B}_{\varepsilon^\beta}$, in the proof of Theorem 4.1, we see that

$$\left\| \frac{\rho^\varepsilon - \rho}{\rho^\varepsilon} \right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \leq \left\| \frac{\rho^\varepsilon + C\rho^\varepsilon}{\rho^\varepsilon} \right\|_{L^q(\mathscr{B}_{\varepsilon^\beta})} \leq C \quad \text{for } \varepsilon^\beta < \delta$$

and so we are done. □

Remark 4.12. (1) The condition $\rho \in \mathrm{QNS}(B_\delta)$ deals with ρ, $\rho_\varepsilon = 0$ without splitting into cases and so using this condition the proof is simplified.

(2) We note that this condition is weaker than local convexity of ρ on $\mathscr{B}_\delta$, which would also give the same result.

(3) In view of Lemma 4.10, it is essentially a matter of taste if one prefers to formulate Corollary 4.11 in terms of quasinearly subharmonicity or directly under the assumption $\rho \leq C\rho^\varepsilon$.

4A. ***Counterexample for the L^p case.*** We indicate in this subsection why Lemma 4.3 is no longer true when the L^1-norm is replaced with the L^p-norm for $p > 1$. This shows that the Hölder assumption of Corollary 4.4 cannot easily be relaxed.

We can see $\rho^\varepsilon(x)$ is like a weighted average of ρ over the ball $B_\varepsilon(x)$ and so heuristically we can see

$$\frac{\rho - \rho^\varepsilon}{\rho^\varepsilon} \simeq \frac{\rho(x) - (1/|B_\varepsilon|)\int_{B_\varepsilon(x)} \rho(y)\,\mathrm{d}y}{(1/|B_\varepsilon|)\int_{B_\varepsilon(x)} \rho(y)\,\mathrm{d}y}$$

(which is rigorous for $\eta_\varepsilon = (1/|B_\varepsilon|)\chi_{B_\varepsilon(0)}(x)$), and assuming the right-hand side is bounded and rearranging gives the condition (4-9). We see that a condition of the form

$$\left\| \frac{\rho(\,\cdot\,) - (1/|B_\varepsilon|)\int_{B_\varepsilon(\,\cdot\,)} \rho(y)\,\mathrm{d}y}{(1/|B_\varepsilon|)\int_{B_\varepsilon(\,\cdot\,)} \rho(y)\,\mathrm{d}y} \right\|_{L^p} < C,$$

in a sense a "relatively weighted L^p mean oscillation condition", could potentially be the weakest condition to control (4-1).

We notice that for the L^1 norm we obtain perfect cancellation in the fraction when calculating (4-8), as a mollifier acts like a local weighted average. However, when we perform the calculation in (4-8), but in L^p, then instead we obtain

$$\sum_{j=1}^{n} \frac{|B_\varepsilon|}{\omega_N} \frac{\|w\chi_{Q_j}\|_{L^p}}{\int_{Q_j} w\,\mathrm{d}x} = \sum_{j=1}^{n} \frac{|B_\varepsilon|}{\omega_N} \frac{\left(\int_{Q_j} w^p\,\mathrm{d}x\right)^{1/p}}{\int_{Q_j} w\,\mathrm{d}x}$$

and if we assume that $w = 1$ then we get $\sum_{j=1}^{n}(|B_\varepsilon|/\omega_N)|Q_j|^{1/p-1}$. As $1/p - 1 < 0$, for certain functions this term could blow up.

In fact if one chooses a function made of separated spikes where the supports get smaller and smaller then we can show this blow-up. We will formulate a simple counterexample so that it is in one dimension, discontinuous and nonnegative, though more regular counterexamples can be constructed in higher dimensions that are, for instance, even smooth and strictly positive.

Firstly, note that if we show that $\|f/f^\varepsilon\|_{L^p}$ blows up as $\varepsilon \to 0$ then $\|f/f^\varepsilon - f^\varepsilon/f^\varepsilon\|_{L^p}$ will also blow up. We can take $x \in \mathbb{T}$ and define our counterexample

$$f(x) := \sum_{i=1}^{\infty} \chi_{[1/i,\,1/i+1/2^i]}(x).$$

It is easy to see that $f \in B^{\alpha,\infty}_p(\mathbb{T})$ for $p > 1$ and any $0 < \alpha < 1 - 1/p$ by regularizing and using Lemma 2.49 from [Bahouri et al. 2011]. Thus we have the sum of separated spikes so they are further than $1/i^2$ apart yet have supports of size $1/2^i$. Let $\varepsilon = 1/(2i^2)$ and see that as f is nonnegative we can bound the sum below by just the i-th spike and see that as mollification only acts locally, the value on the denominator is only dependent on the i-th spike; thus we obtain

$$\left\|\frac{f}{f^\varepsilon}\right\|_{L^p(\mathbb{T})} \geq \left\|\frac{1}{f^\varepsilon}\right\|_{L^p(1/i,\,1/i+1/2^i)} = \|((\chi_{[1/i,\,1/i+1/2^i]})^{1/(2i^2)})^{-1}\|_{L^p(1/i,\,1/i+1/2^i)}. \tag{4-11}$$

We can then the bound mollification of $\chi_{[1/i,\,1/i+1/2^i]}$ in a similar method to (4-10) but in one dimension and so we can bound (4-11) below by

$$\left\|\frac{f}{f^\varepsilon}\right\|_{L^p(\mathbb{T})} \geq C\frac{2^i}{2i^2}\|1\|_{L^p(1/i,\,1/i+1/2^i)} = C\frac{2^i}{2i^2}2^{-i/p} = C\frac{2^{i(1-1/p)}}{2i^2}.$$

As f is the sum of infinitely many spikes there will exist an appropriate spike for any ε_i and thus we can send $i \to \infty$ and, as $1 - 1/p > 0$, we have $C2^{i(1-1/p)}/(2i^2) \to \infty$, which implies $\|f/f^\varepsilon\|_{L^p(\mathbb{T})} \to \infty$.

5. Energy conservation on domains with boundary

We have derived the local energy conservation equations on $(0, T) \times \mathbb{T}^d$ and so for an $\varphi \in C_c^\infty((0, T) \times \mathbb{T}^d)$ we have

$$\int_0^T\!\!\int_{\mathbb{T}^d} \partial_t\varphi \cdot \left(\tfrac{1}{2}\rho|u|^2 + P(\rho)\right) + \nabla\varphi \cdot \left[\left(\tfrac{1}{2}\rho|u|^2 + p(\rho) + P(\rho)\right)u\right] \mathrm{d}t\,\mathrm{d}x = 0. \tag{5-1}$$

The local energy equation is derived by taking momentum balance equations and testing with $(\varphi u^\varepsilon)^\varepsilon$ and using that mollification is symmetric to regularize the equation. For the continuity equation we just use φ^ε to test the equation and again move the mollification onto the equation. Once this is done, all the calculations are done locally on $\operatorname{supp}(\varphi)$.

When studying the isentropic Euler equations on a bounded domain with Lipschitz boundary Ω we have

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p(\rho) &= 0 \quad \text{in } [0, T] \times \Omega, \\ \partial_t\rho + \operatorname{div}(\rho u) &= 0 \quad \text{in } [0, T] \times \Omega, \\ u \cdot n &= 0 \quad \text{on } [0, T] \times \partial\Omega, \end{aligned} \tag{5-2}$$

where n denotes the outward normal vector field for $\partial\Omega$. For any $\varphi \in C_c^\infty((0, T) \times \Omega)$ we can find an $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ we have φ^ε, $(\varphi u^\varepsilon)^\varepsilon \in C_c^\infty((0, T) \times \Omega)$, and so we can apply the same method as above to obtain a local energy equation on $(0, T) \times \Omega$ of the form (5-1). Here we are assuming the same conditions on u, ρ and p as in the previous theorems and in the corollaries in Sections 3 and 4, yet making the appropriate changes so that u and ρ are defined on the domain $(0,T)\times\Omega$ rather than $(0,T)\times\mathbb{T}^d$.

The following theorem and its proof follow ideas from [Bardos et al. 2018]:

Theorem 5.1. *Let ρ, u be a solution of* (5-2) *in the sense of distributions. Assume that ρ, u, and p satisfy the conditions necessary to derive the local energy equality* (5-1). *Assume further that $\rho\in L^\infty((0,T)\times\partial\Omega)$, $\partial\Omega$ is C^2, and $u\cdot n$ is continuous at the boundary. Then we have energy conservation on Ω; that is, for $\Theta(t)\in C_c^\infty(0,T)$*

$$\int_0^T\int_\Omega \partial_t\Theta(t)\cdot\big(\tfrac12\rho|u|^2+P(\rho)\big)\,\mathrm{d}t\,\mathrm{d}x=0 \tag{5-3}$$

and further if u, ρ are weakly continuous in time then

$$\int_\Omega \tfrac12\rho|u|^2(t_1,x)+P(\rho)(t_1,x)\,\mathrm{d}x=\int_\Omega \tfrac12\rho|u|^2(t_2,x)+P(\rho)(t_2,x)\,\mathrm{d}x \tag{5-4}$$

for any $t_1,t_2\in[0,T]$.

Proof. For any $\varphi\in C_c^\infty((0,T)\times\Omega)$ we can find an $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$ we have $\varphi^\varepsilon,(\varphi u^\varepsilon)^\varepsilon\in C_c^\infty((0,T)\times\Omega)$ and so assuming sufficient regularity of ρ, u and p we obtain

$$\int_0^T\int_\Omega \partial_t\varphi\cdot\big(\tfrac12\rho|u|^2+P(\rho)\big)+\nabla\varphi\cdot\big[\big(\tfrac12\rho|u|^2+p(\rho)+P(\rho)\big)u\big]\,\mathrm{d}t\,\mathrm{d}x=0. \tag{5-5}$$

Let $\chi:\mathbb{R}^+\to\mathbb{R}$ be a nonnegative, smooth function such that

$$\chi(s):=\begin{cases}0 & \text{if } s<1,\\ 1 & \text{if } s>2,\end{cases}$$

and define for $x\in\overline{\Omega}$ the function $d_{\partial\Omega}(x)$ as the euclidean distance from x to the closest point on the boundary. We can then define for any $\delta>0$ the composition $\chi(d_{\partial\Omega}(x)/\delta)$ and see that as $\delta\to0$ so does $\chi(d_{\partial\Omega}(x)/\delta)\to\mathbb{1}_\Omega$. Further, let $\Theta(t)\in C_c^\infty(0,T)$.

We can for any $\delta>0$ let $\varphi(x,t)=\chi(d_{\partial\Omega}(x)/\delta)\Theta(t)$ in (5-5) and we obtain

$$\int_\Omega\chi\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)\int_0^T\partial_t\Theta(t)\cdot\big(\tfrac12\rho|u|^2+P(\rho)\big)\,\mathrm{d}x\,\mathrm{d}t \\ +\int_0^T\Theta(t)\int_\Omega\nabla\chi\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)\cdot\big[\big(\tfrac12\rho|u|^2+p(\rho)+P(\rho)\big)u\big]\,\mathrm{d}t\,\mathrm{d}x=0,$$

and by the chain rule we see that

$$\nabla\chi\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)=\frac1\delta\chi'\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)\nabla d_{\partial\Omega}(x),$$

and so

$$0=\int_\Omega\chi\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)\int_0^T\partial_t\Theta(t)\cdot\big(\tfrac12\rho|u|^2+P(\rho)\big)\,\mathrm{d}x\,\mathrm{d}t \\ +\int_0^T\Theta(t)\int_\Omega\frac1\delta\chi'\Big(\frac{d_{\partial\Omega}(x)}{\delta}\Big)\nabla d_{\partial\Omega}(x)\cdot\big[\big(\tfrac12\rho|u|^2+p(\rho)+P(\rho)\big)u\big]\,\mathrm{d}t\,\mathrm{d}x. \tag{5-6}$$

As $\chi(d_{\partial\Omega}(x)/\delta) \to \mathbb{1}_\Omega$ strongly, the first integral on the right-hand side of (5-6) will converge to

$$\int_0^T\int_\Omega \partial_t\Theta(t)\cdot\big(\tfrac{1}{2}\rho|u|^2+P(\rho)\big)\,\mathrm{d}x\,\mathrm{d}t$$

as we wanted. All that is left is to show that the other term on the right-hand side of (5-6) vanishes in the limit.

As $\partial\Omega$ is C^2 we can use [Gilbarg and Trudinger 1977], specifically Lemma 14.16, to see that there exists an $a>0$ such that $d_{\partial\Omega}(x)\in C^2(\Gamma_a)$, where $\Gamma_a := \{x\in\overline{\Omega} : d_{\partial\Omega}(x) < a\}$. Further, in a similar argument to [Bardos et al. 2014, Section 7], when $x\in\Omega$ is sufficiently close to $\partial\Omega$, there exists a unique point $\hat{x}\in\partial\Omega$ such that $x=\hat{x}+n(\hat{x})\,d_{\partial\Omega}(x)$, where $n(\hat{x})$ is the unit outward normal to the boundary at x. We see that we can bound the modulus for the second term on the right-hand side of (5-6) by

$$\left\|\chi'\left(\frac{d_{\partial\Omega}}{\delta}\right)\right\|_{L^\infty}\int_0^T \Theta(t)\frac{1}{\delta}\int_{\Gamma_{2\delta}}|\nabla d_{\partial\Omega}(x)\cdot u|\,\big|\big(\tfrac{1}{2}\rho|u|^2+p(\rho)+P(\rho)\big)\big|\,\mathrm{d}t\,\mathrm{d}x \le C\int_0^T \Theta(t)\frac{1}{\delta}\int_{\Gamma_{2\delta}}|\nabla d_{\partial\Omega}(x)\cdot u|\,\mathrm{d}t\,\mathrm{d}x \tag{5-7}$$

as we know that $\|\chi'(d_{\partial\Omega}/\delta)\|_{L^\infty}\le C$ and by our assumptions $\big\|\big(\tfrac{1}{2}\rho|u|^2+p(\rho)+P(\rho)\big)\big\|_{L^\infty}\le C$ as well. For $2\delta<a$ we know that $d_{\partial\Omega}\in C^2$ and furthermore as $\nabla d_{\partial\Omega}\in C^1$, in the region $\Gamma_{2\delta}$, $|\nabla d_{\partial\Omega}(x)\cdot u|\to C|n(\hat{x})\cdot u(\hat{x})|$ as long as $u(x)\to u(\hat{x})$ as $x\to\hat{x}$, and for this the assumption that $u\cdot n$ is continuous at the boundary will suffice. Thus as $\partial\Omega$ is at least Lipschitz so $|\Gamma_{2\delta}|\le C\delta|\partial\Omega|$ and so we can apply the Lebesgue differentiation theorem to (5-7) and see that as $\delta\to 0$,

$$C\int_0^T \Theta(t)\frac{1}{\delta}\int_{\Gamma_{2\delta}}|\nabla d_{\partial\Omega}(x)\cdot u|\,\mathrm{d}t\,\mathrm{d}x \to C\int_0^T\Theta(t)\int_{\partial\Omega}|n(\hat{x})\cdot u(\hat{x})|\,\mathrm{d}t\,\mathrm{d}\hat{x}=0$$

as $n(\hat{x})\cdot u(\hat{x})=0$ and so we have shown (5-3).

We now want to show (5-4) with the extra assumptions of weak continuity in time of both u and ρ. To do this we define the sequence of functions $\Theta_\nu : [0,T]\to\mathbb{R}$ which are nonnegative and smooth, where for any point $t_1, t_2\in[0,T]$ with $t_1<t_2$ we have

$$\Theta_\nu(\tau) := \begin{cases} 0 & \text{if } \tau<t_1+\nu \text{ or } \tau>t_2-\nu,\\ 1 & \text{if } \tau>t_1+2\nu \text{ or } \tau<t_2-2\nu,\end{cases}$$

and see similarly that as $\nu\to 0$ we have $\Theta_\nu(t)\to\mathbb{1}_{[t_1,t_2]}$. We see that $\Theta_\nu\in C_c^\infty(0,T)$ for every $\nu>0$ and so substituting this function into (5-6) we obtain

$$\int_0^T\int_\Omega \partial_t\Theta_\nu(t)\cdot\big(\tfrac{1}{2}\rho|u|^2+P(\rho)\big)\,\mathrm{d}t\,\mathrm{d}x = 0$$

for every ν. From our choice of Θ_ν we see that

$$\int_0^T\int_{\mathbb{T}^d}\partial_t\Theta_\nu(t)\cdot\big(\tfrac{1}{2}\rho|u|^2+P(\rho)\big)\,\mathrm{d}t\,\mathrm{d}x = \int_{t_1}^{t_1+2\nu}\partial_t\Theta_\nu(t)\cdot\int_\Omega\big(\tfrac{1}{2}\rho|u|^2+P(\rho)\big)\,\mathrm{d}t\,\mathrm{d}x + \int_{t_2-2\nu}^{t_2}\partial_t\Theta_\nu(t)\cdot\int_\Omega\big(\tfrac{1}{2}\rho|u|^2+P(\rho)\big)\,\mathrm{d}t\,\mathrm{d}x.$$

We know that $\int_{t_1}^{t_1+2\nu} \partial_t \Theta_\nu(t)\,\mathrm{d}t = 1$ and $\int_{t_2-2\nu}^{t_2} \partial_t \Theta_\nu(t)\,\mathrm{d}t = -1$ by the fundamental theorem of calculus and as $\nu \to 0$ these terms approximate the identity at t_1 and t_2, and thus these terms converge to

$$\int_\Omega \tfrac{1}{2}\rho|u|^2(t_1,x) + P(\rho)(t_1,x)\,\mathrm{d}x \quad \text{and} \quad -\int_\Omega \tfrac{1}{2}\rho|u|^2(t_2,x) + P(\rho)(t_2,x)\,\mathrm{d}x$$

respectively, assuming weak continuity of ρ and u in time. Thus we are done. □

Acknowledgements

We would like to thank the universities of Hannover and Warsaw for creating great research environments. Dębiec acknowledges support from the National Science Centre (Poland), 2012/05/E/ST1/02218. The research funding for Skipper is graciously given by German research foundation (DFG) grant no. WI 4840/1-1.

References

[Akramov and Wiedemann 2019] I. Akramov and E. Wiedemann, "Renormalization of active scalar equations", *Nonlinear Anal.* **179** (2019), 254–269. MR Zbl

[Bahouri et al. 2011] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Math. Wissenschaften **343**, Springer, 2011. MR Zbl

[Bardos and Titi 2018] C. Bardos and E. S. Titi, "Onsager's conjecture for the incompressible Euler equations in bounded domains", *Arch. Ration. Mech. Anal.* **228**:1 (2018), 197–207. MR Zbl

[Bardos et al. 2014] C. Bardos, L. Székelyhidi, Jr., and E. Wiedemann, "Non-uniqueness for the Euler equations: the effect of the boundary", *Uspekhi Mat. Nauk* **69**:2(416) (2014), 3–22. In Russian; translated in *Russian Math. Surv.* **69**:2 (2014), 189–207. MR Zbl

[Bardos et al. 2018] C. Bardos, E. Titi, and E. Wiedemann, "Onsager's conjecture with physical boundaries and an application to the vanishing viscosity limit", preprint, 2018. arXiv

[Bardos et al. 2019] C. Bardos, P. Gwiazda, A. Świerczewska-Gwiazda, E. S. Titi, and E. Wiedemann, "On the extension of Onsager's conjecture for general conservation laws", *J. Nonlinear Sci.* **29**:2 (2019), 501–510. MR Zbl

[Buckmaster et al. 2019] T. Buckmaster, C. de Lellis, L. Székelyhidi, Jr., and V. Vicol, "Onsager's conjecture for admissible weak solutions", *Comm. Pure Appl. Math.* **72**:2 (2019), 229–274. MR Zbl

[Chen and Torres 2005] G.-Q. Chen and M. Torres, "Divergence-measure fields, sets of finite perimeter, and conservation laws", *Arch. Ration. Mech. Anal.* **175**:2 (2005), 245–267. MR Zbl

[Chen and Yu 2019] R. M. Chen and C. Yu, "Onsager's energy conservation for inhomogeneous Euler equations", *J. Math. Pure Appl.* **131** (2019), 1–16. MR Zbl

[Cheskidov et al. 2008] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy, "Energy conservation and Onsager's conjecture for the Euler equations", *Nonlinearity* **21**:6 (2008), 1233–1252. MR Zbl

[Constantin et al. 1994] P. Constantin, W. E, and E. S. Titi, "Onsager's conjecture on the energy conservation for solutions of Euler's equation", *Comm. Math. Phys.* **165**:1 (1994), 207–209. MR Zbl

[DiPerna and Lions 1989] R. J. DiPerna and P.-L. Lions, "Ordinary differential equations, transport theory and Sobolev spaces", *Invent. Math.* **98**:3 (1989), 511–547. MR Zbl

[Dębiec et al. 2018] T. Dębiec, P. Gwiazda, A. Świerczewska-Gwiazda, and A. Tzavaras, "Conservation of energy for the Euler–Korteweg equations", *Calc. Var. Partial Differential Equations* **57**:6 (2018), art. id. 160. MR Zbl

[Drivas and Eyink 2018] T. D. Drivas and G. L. Eyink, "An Onsager singularity theorem for turbulent solutions of compressible Euler equations", *Comm. Math. Phys.* **359**:2 (2018), 733–763. MR Zbl

[Drivas and Nguyen 2018] T. D. Drivas and H. Q. Nguyen, "Onsager's conjecture and anomalous dissipation on domains with boundary", *SIAM J. Math. Anal.* **50**:5 (2018), 4785–4811. MR Zbl

[Duchon and Robert 2000] J. Duchon and R. Robert, "Inertial energy dissipation for weak solutions of incompressible Euler and Navier–Stokes equations", *Nonlinearity* **13**:1 (2000), 249–255. MR Zbl

[Eyink 1994] G. L. Eyink, "Energy dissipation without viscosity in ideal hydrodynamics, I: Fourier analysis and local energy transfer", *Phys. D* **78**:3-4 (1994), 222–240. MR Zbl

[Feireisl et al. 2017] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann, "Regularity and energy conservation for the compressible Euler equations", *Arch. Ration. Mech. Anal.* **223**:3 (2017), 1375–1395. MR Zbl

[Fjordholm and Wiedemann 2018] U. S. Fjordholm and E. Wiedemann, "Statistical solutions and Onsager's conjecture", *Phys. D* **376/377** (2018), 259–265. MR Zbl

[Gilbarg and Trudinger 1977] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Grundlehren der Math. Wissenschaften **224**, Springer, 1977. MR Zbl

[Gwiazda et al. 2018] P. Gwiazda, M. Michálek, and A. Świerczewska-Gwiazda, "A note on weak solutions of conservation laws and energy/entropy conservation", *Arch. Ration. Mech. Anal.* **229**:3 (2018), 1223–1238. MR Zbl

[Isett 2018] P. Isett, "A proof of Onsager's conjecture", *Ann. of Math.* (2) **188**:3 (2018), 871–963. MR Zbl

[Leslie and Shvydkoy 2016] T. M. Leslie and R. Shvydkoy, "The energy balance relation for weak solutions of the density-dependent Navier–Stokes equations", *J. Differential Equations* **261**:6 (2016), 3719–3733. MR Zbl

[Pavlović and Riihentaus 2011] M. Pavlović and J. Riihentaus, "Quasi-nearly subharmonic functions in locally uniformly homogeneous spaces", *Positivity* **15**:1 (2011), 1–10. MR Zbl

[Yu 2017] C. Yu, "Energy conservation for the weak solutions of the compressible Navier–Stokes equations", *Arch. Ration. Mech. Anal.* **225**:3 (2017), 1073–1087. MR Zbl

Received 16 Aug 2018. Revised 18 Dec 2018. Accepted 25 Mar 2019.

IBROKHIMBEK AKRAMOV: ibrokhimbek.akramov@uni-ulm.de
Institute of Applied Analysis, Ulm University, Ulm, Germany

TOMASZ DĘBIEC: t.debiec@mimuw.edu.pl
Institute of Applied Mathematics and Mechanics, University of Warsaw, Warsaw, Poland

JACK SKIPPER: jack.skipper@uni-ulm.de
Institute of Applied Analysis, Ulm University, Ulm, Germany

EMIL WIEDEMANN: emil.wiedemann@uni-ulm.de
Institute of Applied Analysis, Ulm University, Ulm, Germany

mathematical sciences publishers

Chapter 5

Relative entropy method for measure solutions of the growth-fragmentation equation [11]

SIAM J. MATH. ANAL.
Vol. 50, No. 6, pp. 5811–5824

RELATIVE ENTROPY METHOD FOR MEASURE SOLUTIONS OF THE GROWTH-FRAGMENTATION EQUATION*

TOMASZ DĘBIEC†, MARIE DOUMIC‡, PIOTR GWIAZDA§, AND EMIL WIEDEMANN¶

Abstract. The aim of this study is to generalize recent results of the two last authors on entropy methods for measure solutions of the renewal equation to other classes of structured population problems. Specifically, we develop a generalized relative entropy inequality for the growth-fragmentation equation and prove asymptotic convergence to a steady-state solution, even when the initial datum is only a nonnegative measure.

Key words. measure solutions, growth-fragmentation equation, structured population, relative entropy, generalized Young measure

AMS subject classifications. 35B40, 35Q92, 92D25

DOI. 10.1137/18M117981X

1. Introduction.

Structured population models were developed for the purpose of understanding the evolution of a population over time—and in particular to adequately describe the dynamics of a population by its distribution along some "structuring" variables representing e.g., age, size, or cell maturity. These models, often taking the form of an evolutionary partial differential equation, have been extensively studied for many years. The first age structure was considered in the early 20th century by Sharpe and Lotka [36], who already made predictions on the question of asymptotic behavior of the population; see also [25, 26]. In the second half of the 20th century size-structured models appeared first in [3, 37]. These studies gave rise to other physiologically structured models (age size, saturation, cell maturity, etc.).

The object of this paper is the growth-fragmentation model, which is found fitting in many different contexts: cell division, polymerization, neurosciences, prion proliferation, or even telecommunication. In its general linear form this model takes the form of the following equation:

$$
\begin{aligned}
\partial_t n(t,x) + \partial_x(g(x)n(t,x)) + B(x)n(t,x) &= \int_x^\infty k(x,y)B(y)n(t,y)\,\mathrm{d}y, \\
g(0)n(t,0) &= 0, \\
n(0,x) &= n^0(x).
\end{aligned}
\tag{1.1}
$$

*Received by the editors April 10, 2018; accepted for publication (in revised form) August 22, 2018; published electronically November 15, 2018.
http://www.siam.org/journals/sima/50-6/M117981.html
Funding: This work was partially supported by the Simons Foundation grant 346300 and the Polish Government MNiSW 2015-2019 matching fund. The research of the first author was supported by National Science Center (Poland) 2014/13/B/ST1/03094. The research of the second author was supported by the Wolfgang Pauli Institute (Vienna) and the ERC Starting grant SKIPPER^{AD} (306321). The research of the third author was supported by the National Science Center (Poland) 2015/18/M/ST1/00075.
†Institute of Applied Mathematics and Mechanics, University of Warsaw, 02-097 Warszawa, Poland (t.debiec@mimuw.edu.pl).
‡Sorbonne Universités, INRIA, UPMC Univ. Paris 06, Lab. J. L. Lions UMR CNRS 7598, Paris F-75005, France, and Wolfgang Pauli Institute, c/o University of Vienna, 1090 Wien, Austria (marie.doumic@inria.fr).
§Institute of Mathematics, Polish Academy of Sciences, 00-656 Warszawa, Poland (pgwiazda@mimuw.edu.pl).
¶Institute of Applied Analysis, Universität Ulm, 89081 Ulm, Germany (wiedemann@uni-ulm.de).

Here $n(t, x)$ represents the concentration of individuals of size $x \geq 0$ at time $t > 0$, $g(x) \geq 0$ is their growth rate, $B(x) \geq 0$ is their division rate, and $k(x, y)$ is the proportion of individuals of size x created out of the division of individuals of size y. This equation incorporates a very important phenomenon in biology—a competition between growth and fragmentation. Clearly they have opposite dynamics: Growth drives the population towards a larger size, while fragmentation makes it smaller and smaller. Depending on which factor dominates, one can observe various long-time behavior of the population distribution.

Many authors have studied the long-time asymptotics (along with well-posedness) of variants of the growth-fragmentation equation; see e.g. [8, 17, 29, 32, 35]. The studies which establish convergence, in a proper sense, of a (renormalized) solution towards a steady profile were until recently limited only to initial data in weighted L^1 spaces. The methods used for such studies include the semigroup theory [32], the Mellin transform [16], and very recently a probabilistic approach using a Feynman–Kac representation of the semigroup [6]. These methods could also provide an exponential rate of convergence, linked to the existence of a spectral gap.

A different approach was developed by Perthame et al. in a series of papers [30, 31, 35]. Their generalized relative entropy (GRE) method provides a way to study long-time asymptotics of linear models even when no spectral gap is guaranteed but fails to provide a rate of convergence, unless an entropy-entropy dissipation inequality is obtained [8]. Recently Gwiazda and Wiedemann [24] extended the GRE method to the case of the renewal equation with initial data in the space of nonnegative Radon measures. Their result is motivated by the increasing interest in measure solutions to models of mathematical biology; see, e.g., [9, 22] for some recent results concerning well-posedness and stability theory in the space of nonnegative Radon measures. The clear advantage of considering measure data is that it is biologically justified—it allows for treating the situation when a population is initially concentrated with respect to the structuring variable (and is, in particular, not absolutely continuous with respect to the Lebesgue measure). This is typically the case when departing from a population formed by a unique cell. We refer also to the recent result of Gabriel [20], who uses the Doeblin method to analyze the long-time behavior of measure solutions to the renewal equation.

Let us remark that the method of analysis employed in the current paper is inspired by the classical relative entropy method introduced by Dafermos in [11]. In recent years this method was extended to yield results on measure-valued–strong uniqueness for equations of fluid dynamics [7, 19, 23] and general conservation laws [10, 13, 21]. See also [12] and references therein.

Comparison with the renewal equation. In [24], the generalized relative entropy method was combined with variational and measure-theoretic tools (such as recession values, weak lower semicontinuity, Reshetnyak's continuity theorem) to study the long-time asymptotics for the renewal equation, arguably the simplest model of structured population dynamics, with measure data. The purpose of the current contribution is to achieve similar results in the case of a general growth-fragmentation equation.

Although our general strategy, relying on generalized relative entropy and approximation arguments, is comparable to that of [24], we wish to point out some substantial differences, which require several novel features in order to handle the growth-fragmentation equation. There are two main issues that had to be overcome compared to [24]:

1. The approximation argument for the renewal equation (see Theorems 4.1 and 5.1 in [24]) relied on the availability of a weak limit, pointwise in time, of the approximations. This, in turn, was justified from a result of Gwiazda et al. [22], which, however, does not include the growth-fragmentation equation. Even worse, due to the nonconvex nature of the entropy dissipation (3.1), it would seem difficult to apply weak lower semicontinuity arguments as in [24]. These problems can be overcome by the use of so-called generalized Young measures, which enjoy (by design) useful compactness properties. As explained in section 2.3, such measure-valued solutions form a larger class of solutions than the measure solutions considered in [24]. Hence, as a by-product of our analysis, we can show the long-time convergence to equilibrium even for a possibly larger class of solutions. We also believe that the framework of generalized Young measures will be most suitable to tackle other and more general structured population models.
2. The appearance of the term $H'(u_\varepsilon(t,x))u_\varepsilon(t,y)$ in the entropy dissipation (see (3.9) below) mixes dependences on the variables x and y. It is then not at all clear whether this product converges to the corresponding product of the weak limits, as fast oscillations in time might lead to lack of compactness. More technically speaking, a sequence generating a Young measure parametrized in time and space need *not* generate the same Young measure pointwise in time, and this subtle difference occurs precisely due to possible oscillations in time. Fortunately, for the growth-fragmentation model, additional compactness is available by virtue of a result of Carrillo et al. [9]

The current paper is structured as follows. In section 2 we recall some basic results on Radon measures, recession functions, and Young measures as well as introduce the assumptions of our model. In section 3 we state and prove the GRE inequality, which is then used to prove a long-time asymptotics result in section 4.

2. Description of the model.

2.1. Preliminaries. In what follows we denote by $\mathbb{R}_+$ the set $[0,\infty)$. By $\mathcal{M}(\mathbb{R}_+)$ we denote the space of signed Radon measures on $\mathbb{R}_+$. By Lebesgue's decomposition theorem, for each $\mu \in \mathcal{M}(\mathbb{R}_+)$ we can write

$$\mu = \mu^a + \mu^s,$$

where μ^a is absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$ and μ^s is singular. The space $\mathcal{M}(\mathbb{R}_+)$ is endowed with the total variation norm

$$\|\mu\|_{\mathrm{TV}} \coloneqq \int_{\mathbb{R}_+} \mathrm{d}|\mu|.$$

By the Riesz Representation Theorem we can identify this space with the dual space to the space $\mathcal{C}_0(\mathbb{R}_+)$ of continuous functions on $\mathbb{R}_+$ which vanish at infinity. The duality pairing is given by

$$\langle \nu, f\rangle \coloneqq \int_0^\infty f(\xi)\; d\mu(\xi).$$

By $\mathcal{M}^+(\mathbb{R}_+)$ we denote the set of positive Radon measures of bounded total variation. We further define the φ-weighted total variation by

$$\|\mu\|_{\mathrm{TV}_\varphi} \coloneqq \int_{\mathbb{R}_+} \varphi \mathrm{d}|\mu|$$

and correspondingly the space $\mathcal{M}^+(\mathbb{R}_+;\varphi)$ of positive Radon measures whose φ-weighted total variation is finite. Of course we require that the function φ be nonnegative. In fact, for our purposes φ will be strictly positive and bounded on each compact subset of $(0,\infty)$.

We say that a sequence $\nu_n \in \mathcal{M}(\mathbb{R}_+)$ converges *weakly*^* to some measure $\nu \in \mathcal{M}(\mathbb{R}_+)$ if

$$\langle \nu_n, f\rangle \longrightarrow \langle \nu, f\rangle$$

for each $f \in \mathcal{C}_0(\mathbb{R}_+)$.

By a *Young measure* on $\mathbb{R}_+\times\mathbb{R}_+$ we mean a parameterized family $\nu_{t,x}$ of probability measures on $\mathbb{R}_+$. More precisely, it is a weak*-measurable function $(t,x)\mapsto \nu_{t,x}$, i.e., such that the mapping

$$(t,x)\mapsto \langle \nu_{t,x}, f\rangle$$

is measurable for each $f\in C_0(\mathbb{R}_+)$. Young measures are often used to describe limits of weakly converging approximating sequences to a given problem. They serve as a way of describing weak limits of nonlinear functions of the approximate solution. Indeed, it is a classical result that a uniformly bounded measurable sequence u_n generates a Young measure by which one represents the limit of $f(u_n)$, where f is some nonlinear function; see [38] for sequences in L^∞ and [2] for measurable sequences.

This framework was used by DiPerna in his celebrated paper [14], where he introduced the concept of an admissible measure-valued solution to scalar conservation laws. However, in more general contexts (e.g., for hyperbolic systems, where there is usually only one entropy-entropy-flux pair) one needs to be able to describe limits of sequences which exhibit oscillatory behavior as well as concentrate mass. Such a framework is provided by *generalized Young measures*, first introduced in the context of incompressible Euler equations in [15] and later developed by many authors. We follow the exposition of Alibert and Bouchitté [1] and Kristensen and Rindler [28].

Suppose $f:\mathbb{R}\to\mathbb{R}_+$ is an even continuous function with at most linear growth, i.e.,

$$|f(x)| \le C(1+|x|),$$

for some constant C. We define, whenever it exists, the *recession value* of f as

$$f^\infty = \lim_{s\to\infty}\frac{f(s)}{s} = \lim_{s\to\infty}\frac{f(-s)}{s}.$$

Generally, for functions f of n variables, the recession value is a *function* on the $(n-1)$-dimensional unit sphere. Here, however, since we assume f to depend only on one variable and to be even, the recession function is completely determined by its value at 1 and is therefore a constant.

Definition 2.1. *The set $\mathcal{F}(\mathbb{R})$ of continuous functions $f:\mathbb{R}\to\mathbb{R}_+$ for which f^∞ exists is called the class of* admissible integrands.

By a *generalized Young measure* on $\Omega=\mathbb{R}_+\times\mathbb{R}_+$ we mean a parameterized family $(\nu_{t,x}, m)$, where for $(t,x)\in\Omega$, $\nu_{t,x}$ is a family of probability measures on $\mathbb{R}$ and m is a nonnegative Radon measure on Ω. In the following, we may omit the indices for $\nu_{t,x}$ and denote it simply (ν,m).

The following result gives a way of representing weak* limits of sequences bounded in L^1 via a generalized Young measure. It was first proved in [1, Theorem 2.5]. We state an adaptation to our simpler case.

PROPOSITION 2.2. *Let* (u_n) *be a bounded sequence in* $L^1_{loc}(\Omega;\mu,\mathbb{R})$, *where* μ *is a measure on* Ω. *There exists a subsequence* (u_{n_k}), *a nonnegative Radon measure* m *on* Ω, *and a parametrized family of probabilities* (ν_ζ) *such that for any even function* $f \in \mathcal{F}(\mathbb{R})$ *we have*

$$f(u_{n_k}(\zeta))\mu \overset{*}{\rightharpoonup} \langle \nu_\zeta, f\rangle\mu + f^\infty m. \tag{2.1}$$

Proof. We apply Theorem 2.5. and Remark 2.6 in [1], simplified by the fact that f is even and that we only test against functions f independent of ζ. Note that the weak* convergence then has to be understood in the sense of compactly supported test functions $\varphi \in \mathcal{C}_0(\Omega,\mathbb{R})$. ∎

The above proposition can in fact be generalized to say that every bounded sequence of generalized Young measures possesses a weak* convergent subsequence; cf. [28, Corollary 2.]

PROPOSITION 2.3. *Let* (ν^n, m^n) *be a sequence of generalized Young measures on* Ω *such that*

- *the map* $\zeta \mapsto \langle \nu^n_\zeta, |\cdot|\rangle$ *is uniformly bounded in* L^1;
- *the sequence* $(m^n(\bar\Omega))$ *is uniformly bounded.*

Then there is a generalized Young measure (ν, m) *on* Ω *such that* (ν^n, m^n) *converges weakly* to* (ν, m).

2.2. The model. We consider the growth-fragmentation equation under a general form:

$$\begin{aligned} \partial_t n(t,x) + \partial_x(g(x)n(t,x)) + B(x)n(t,x) &= \int_x^\infty k(x,y)B(y)n(t,y)\,\mathrm{d}y, \\ g(0)n(t,0) &= 0, \\ n(0,x) &= n^0(x). \end{aligned} \tag{2.2}$$

We assume $n^0 \in \mathcal{M}^+(\mathbb{R}_+;\varphi)$, where φ is the solution of (2.4) as explained below.

The fundamental tool in studying the long-time asymptotics with the GRE method is the existence and uniqueness of the first eigenelements (λ, N, φ), i.e., solutions to the following primal and dual eigenproblems:

$$\begin{gathered} \frac{\partial}{\partial x}(g(x)N(x)) + (B(x)+\lambda)N(x) = \int_x^\infty k(x,y)B(y)N(y)\,\mathrm{d}y \\ g(0)N(0) = 0, \quad N(x) > 0, \text{ for } x > 0, \quad \int_0^\infty N(x)\mathrm{d}x = 1, \end{gathered} \tag{2.3}$$

$$\begin{gathered} -g(x)\frac{\partial}{\partial x}(\varphi(x)) + (B(x)+\lambda)\varphi(x) = B(x)\int_0^x k(y,x)\varphi(y)\,\mathrm{d}y \\ \varphi(x) > 0, \quad \int_0^\infty \varphi(x)N(x)\mathrm{d}x = 1. \end{gathered} \tag{2.4}$$

We make the following assumptions on the parameters of the model:

$$B \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}_+^*), \quad g \in W^{1,\infty}(\mathbb{R}_+, \mathbb{R}_+^*) \quad \forall\, x \geq 0,\ g \geq g_0 > 0, \quad xB(x) \to_{x\to\infty} \infty, \tag{2.5}$$

$$k \in \mathcal{C}_b(\mathbb{R}_+ \times \mathbb{R}_+), \qquad \int_0^y k(x,y)\mathrm{d}x = 2, \qquad \int_0^y xk(x,y)\mathrm{d}x = y, \tag{2.6}$$

$$k(x, y < x) = 0, \qquad k(x, y > x) > 0. \tag{2.7}$$

These guarantee in particular existence and uniqueness of a solution $n \in \mathcal{C}(\mathbb{R}_+; L^1_\varphi(\mathbb{R}_+))$ for L^1 initial data (see, e.g., [33]), existence of a unique measure solution for data in $\mathcal{M}^+(\mathbb{R}_+;\varphi)$ (cf. [9]), as well as existence and uniqueness of a dominant eigentriplet $(\lambda > 0, N(x), \varphi(x))$; cf. [17]. In particular the functions N and φ are continuous, N is bounded, and φ has at most polynomial growth. In what follows N and φ will always denote the solutions to problems (2.3) and (2.4), respectively. Let us remark that in the L^1 setting we have the following conservation law:

$$\int_0^\infty n_\varepsilon(t,x)e^{-\lambda t}\varphi(x)\mathrm{d}x = \int_0^\infty n^0(x)\varphi(x)\mathrm{d}x. \tag{2.8}$$

2.3. Measure and measure-valued solutions. Let us observe that there are two basic ways to treat the above model in the measure setting. The first one is to consider a *measure solution*, i.e., a narrowly continuous map $t \mapsto \mu_t \in \mathcal{M}^+(\mathbb{R}_+)$, which satisfies (2.2) in the weak sense, i.e., for each $\psi \in \mathcal{C}_c^1(\mathbb{R}_+ \times \mathbb{R}_+)$

$$\begin{aligned} -\int_0^\infty \int_0^\infty (\partial_t\psi(t,x) + \partial_x\psi(t,x)g(x))d\mu_t(x)\mathrm{d}t + \int_0^\infty \int_0^\infty \psi(t,x)B(x)d\mu_t(x)\mathrm{d}t \\ = \int_0^\infty \int_0^\infty \psi(t,x)\int_x^\infty k(x,y)B(y)d\mu_t(y)\mathrm{d}x\mathrm{d}t + \int_0^\infty \psi(0,x)dn^0(x). \end{aligned} \tag{2.9}$$

Thus, a measure solution is a family of time-parameterized nonnegative Radon measures on the structure-physical domain $\mathbb{R}_+$.

The second way is to work with generalized Young measures and corresponding measure-valued solutions. To prove the generalized relative entropy inequality, which relies on considering a family of nonlinear renormalizations of the equation, we choose to work in this second framework.

A *measure-valued solution* is a generalized Young measure (ν, m), where the oscillation measure is a family of parameterized probabilities over the state domain $\mathbb{R}_+$ such that (2.2) is satisfied by its barycenters $\langle \nu_{t,x}, \xi\rangle$; i.e. the equation

$$\begin{aligned} \partial_t\left(\langle\nu_{t,x},\xi\rangle + m\right) + \partial_x\left(g(x)(\langle\nu_{t,x},\xi\rangle + m)\right) + B(x)(\langle\nu_{t,x},\xi\rangle + m) \\ = \int_x^\infty k(x,y)B(y)\langle\nu_{t,x},\xi\rangle\mathrm{d}y + \int_x^\infty k(x,y)B(y)dm(y) \end{aligned} \tag{2.10}$$

holds in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}_+^*$. Here and in the following, we write $\langle\nu_{t,x},\xi\rangle := \int_{\mathbb{R}} \xi d\nu_{t,x}(\xi)$ for the barycenter of the measure $\nu_{t,x}$.

It is proven in [22] that (2.2) has a unique measure solution. To this solution one can associate a measure-valued solution—for example, given a measure solution $t \mapsto \mu_t$ one can define a measure-valued solution by

$$\left\langle \delta_{\left\{\frac{d\mu_t^a}{d\mathcal{L}^1}\right\}}, \mathrm{id} \right\rangle = \mu_t^a, \qquad m = \mu_t^s,$$

where $\frac{d\mu_1}{d\mu_2}$ denotes the Radon–Nikodym derivative of μ_1 with respect to μ_2.

However, clearly, the measure-valued solutions are not unique—since the equation is linear, there is freedom in choosing the Young measure as long as the barycenter satisfies (2.10). For example, a different measure-valued solution can be defined by

$$\langle \frac{1}{2}\delta_{\left\{2\frac{d\mu_t^a}{d\mathcal{L}^1}\right\}} + \frac{1}{2}\delta_{\{0\}}, \mathrm{id}\rangle = \mu_t^a.$$

Uniqueness of measure-valued solution can be ensured by requiring that the generalized Young measure satisfies not only the equation but also a family of nonlinear renormalizations. This was the case in the work of DiPerna [14]; see also [12].

To establish the GRE inequality which will then be used to prove an asymptotic convergence result, we consider the measure-valued solution generated by a sequence of regularized solutions. This allows us to use the classical GRE method established in [34]. Careful passage to the limit will then show that analogous inequalities hold for the measure-valued solution.

3. GRE inequality. In this section we formulate and prove the generalized relative entropy inequality, our main tool in the study of long-time asymptotics for (2.2). We take advantage of the well-known GRE inequalities in the L^1 setting. To do so we consider the growth-fragmentation equation (2.2) for a sequence of regularized data and prove that we can pass to the limit, thus obtaining the desired inequalities in the measure setting.

Let $n_\varepsilon^0 \in L^1_\varphi(\mathbb{R}_+)$ be a sequence of regularizations of n^0 converging weakly* to n^0 in the space of measures and such that $\|n_\varepsilon^0\|_{\mathrm{TV}_\varphi} \to \|n^0\|_{\mathrm{TV}_\varphi}$. Let n_ε denote the corresponding unique solution to (2.2) with n_ε^0 as an initial condition. Then for any differentiable strictly convex admissible integrand H we define the usual relative entropy

$$\mathcal{H}_\varepsilon(t) := \int_0^\infty \varphi(x)N(x)H\left(\frac{n_\varepsilon(t,x)e^{-\lambda t}}{N(x)}\right)\mathrm{d}x$$

and entropy dissipation

(3.1)
$$D_\varepsilon^H(t) = \int_0^\infty \int_0^\infty \varphi(x)N(y)B(y)k(x,y)\Bigg\{H\left(\frac{n_\varepsilon(t,y)e^{-\lambda t}}{N(y)}\right) - H\left(\frac{n_\varepsilon(t,x)e^{-\lambda t}}{N(x)}\right) - H'\left(\frac{n_\varepsilon(t,x)e^{-\lambda t}}{N(x)}\right)\left[\frac{n_\varepsilon(t,y)e^{-\lambda t}}{N(y)} - \frac{n_\varepsilon(t,x)e^{-\lambda t}}{N(x)}\right]\Bigg\}\,\mathrm{d}x\mathrm{d}y.$$

Then, as shown, e.g., in [30], one can show that

$$\frac{d}{dt}\left\{\int_0^\infty \varphi(x)N(x)H\left(\frac{n_\varepsilon(t,x)e^{-\lambda t}}{N(x)}\right)\,\mathrm{d}x\right\} = -D_\varepsilon^H(t) \tag{3.2}$$

with the right-hand side being nonpositive due to convexity of H. Hence, the relative entropy is nonincreasing. It follows that $\mathcal{H}_\varepsilon(t) \le \mathcal{H}_\varepsilon(0)$ and, since $H \ge 0$,

$$\int_0^\infty D_\varepsilon^H(t)\,\mathrm{d}t \le \mathcal{H}_\varepsilon(0). \tag{3.3}$$

In the next proposition we prove corresponding inequalities for the measure-valued solution generated by the sequence n_ε. This result is an analogue of Theorem 5.1 in [24], although its proof requires new ingredients as explained in the introduction.

PROPOSITION 3.1. *With notation as above, there exists a subsequence (not relabeled), generating a generalized Young measure* (ν, m) *with* $m = m_t \otimes \mathrm{d}t$ *for a family of positive Radon measures* m_t*, such that*

$$\lim_{\varepsilon\to 0}\int_0^\infty \chi(t)\mathcal{H}_\varepsilon(t)\ \mathrm{d}t = \int_0^\infty \chi(t)\left(\int_0^\infty \varphi(x)N(x)\langle \nu_{t,x}(\alpha), H(\alpha)\rangle \mathrm{d}x + \int_0^\infty \varphi(x)N(x)H^\infty dm_t(x)\right)\mathrm{d}t \tag{3.4}$$

for any $\chi \in C_c([0,\infty))$ *and*

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^\infty D_\varepsilon^H(t)\ \mathrm{d}t = &\int_0^\infty\int_0^\infty\int_0^\infty \varphi(x)N(y)B(y)k(x,y)\langle \nu_{t,y}(\xi)\otimes\nu_{t,x}(\alpha), H(\xi)-H(\alpha)-H'(\alpha)(\xi-\alpha)\rangle \mathrm{d}x\mathrm{d}y\mathrm{d}t\\ &+\int_0^\infty\int_0^\infty\int_0^\infty \varphi(x)N(y)B(y)k(x,y)\langle \nu_{t,x}(\alpha), H^\infty - H'(\alpha)\rangle dm_t(y)\mathrm{d}x\mathrm{d}t \ge 0.\end{aligned} \tag{3.5}$$

We denote the limits on the left-hand sides of the above equations by $\int_0^\infty \chi(t)\mathcal{H}(t)\ \mathrm{d}t$ *and* $\int_0^\infty D^H(t)\ \mathrm{d}t$*, respectively, thus defining the measure-valued relative entropy and entropy dissipation for almost every* t*. We further set*

$$\mathcal{H}(0) := \int_0^\infty \varphi(x)N(x)H\left(\frac{(n^0)^a(x)}{N(x)}\right)\mathrm{d}x + \int_0^\infty \varphi(x)H^\infty\left(\frac{(n^0)^s}{|(n^0)^s|}(x)\right)d|(n^0)^s(x)|. \tag{3.6}$$

We then have

$$\frac{d}{dt}\mathcal{H}(t) \le 0 \quad \textit{in the sense of distributions} \tag{3.7}$$

and

$$\int_0^\infty D^H(t)dt \le \mathcal{H}(0). \tag{3.8}$$

Proof. The function $t \mapsto \int_0^\infty n_\varepsilon(t,x)e^{-\lambda t}\varphi(x)\mathrm{d}x$ is constant, and the function N is strictly positive on $(0,\infty)$. Therefore, the sequence $u_\varepsilon(t,x) := \frac{n_\varepsilon(t,x)e^{-\lambda t}}{N(x)}$ is uniformly bounded in $L^\infty(\mathbb{R}_+; L^1_{\varphi,\mathrm{loc}}(\mathbb{R}_+))$. Hence, we can apply Proposition 2.2 to obtain a generalized Young measure (ν, m) on $\mathbb{R}_+\times\mathbb{R}_+$.

Since $u_\varepsilon \in L^\infty(\mathbb{R}_+; L^1_{\varphi,\mathrm{loc}}(\mathbb{R}_+))$, we have $m \in L^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{R}_+;\varphi))$. By a standard disintegration argument (see, for instance, [18, Theorem 1.5.1]) we can write the slicing measure for m, $m(\mathrm{d}t,\mathrm{d}x) = m_t(\mathrm{d}x)\otimes\mathrm{d}t$, where the map $t\mapsto m_t \in \mathcal{M}^+(\mathbb{R}_+;\varphi)$ is measurable and bounded.

By Proposition 2.2 we have the weak* convergence

$$H(u_\varepsilon(t,x))(\mathrm{d}t\otimes\varphi(x)\mathrm{d}x) \overset{*}{\rightharpoonup} \langle \nu_{t,x}, H\rangle(\mathrm{d}t\otimes\varphi(x)\mathrm{d}x) + H^\infty m.$$

Testing with $(t,x) \mapsto \chi(t)N(x)$, where $\chi \in \mathcal{C}_c(\mathbb{R}_+)$, we obtain (3.4). Further, the convergence $\int_0^\infty \chi(t)\mathcal{H}_\varepsilon(t)\mathrm{d}t \to \int_0^\infty \chi(t)\mathcal{H}(t)\mathrm{d}t$ implies (3.7) since for $\mathcal{H}_\varepsilon$ we have the corresponding inequality (3.2).

We now investigate the limit as $\varepsilon \to 0$ of $\int_0^\infty D_\varepsilon^H(t)\mathrm{d}t$. Denoting $\Phi(x,y) \coloneqq k(x,y)N(y)B(y)$ we have

(3.9)
$$D_\varepsilon^H(t) = \int_0^\infty \int_0^\infty \Phi(x,y)\varphi(x)[H(u_\varepsilon(t,y)) - H(u_\varepsilon(t,x)) - H'(u_\varepsilon(t,x))u_\varepsilon(t,y) + H'(u_\varepsilon(t,x))u_\varepsilon(t,x)]\mathrm{d}x\mathrm{d}y.$$

We consider each of the four terms of the sum separately on the restricted domain $[0,T]\times[\eta,K]^2$ for fixed $T>0$ and $K>\eta>0$. Let $D^H_{\varepsilon,\eta,K}$ denote the entropy dissipation with the integrals of (3.9) each taken over the subsets $[\eta,K]$ of $\mathbb{R}_+$.

We now apply Proposition 2.2 to the sequence u_ε, the measure $\mathrm{d}t\otimes\varphi(x)\mathrm{d}x$ on the set $[0,T]\times[\eta,K]$. The first two and the last integrands of $D^H_{\varepsilon,\eta,K}(t)$ depend on t and only either on x or on y. Therefore, we can pass to the limit as $\varepsilon\to 0$ by Proposition 2.2 using a convenient test function. More precisely, testing with $(t,x)\mapsto \int_\eta^K \Phi(x,y)\mathrm{d}y$, we obtain the convergence

$$\begin{aligned}
-\int_0^T\int_\eta^K\int_\eta^K &\Phi(x,y)\varphi(x)H(u_\varepsilon(t,x))\mathrm{d}y\mathrm{d}x\mathrm{d}t\\
&\longrightarrow -\int_0^T\int_\eta^K\int_\eta^K \Phi(x,y)\varphi(x)\langle\nu_{t,x},H\rangle\mathrm{d}y\mathrm{d}x\mathrm{d}t\\
&\qquad -\int_0^T\int_\eta^K\int_\eta^K \Phi(x,y)\varphi(x)H^\infty dm_t(x)\mathrm{d}y\mathrm{d}t,
\end{aligned}$$

and, noticing that the recession value of $\alpha\mapsto\alpha H'(\alpha)$ is H^∞,

$$\begin{aligned}
\int_0^T\int_\eta^K\int_\eta^K &\Phi(x,y)\varphi(x)H'(u_\varepsilon(t,x))u_\varepsilon(t,x)\mathrm{d}y\mathrm{d}x\mathrm{d}t\\
&\longrightarrow \int_0^T\int_\eta^K\int_\eta^K \Phi(x,y)\varphi(x)\langle\nu_{t,x},\alpha H'(\alpha)\rangle\mathrm{d}y\mathrm{d}x\mathrm{d}t\\
&\qquad +\int_0^T\int_\eta^K\int_\eta^K \Phi(x,y)\varphi(x)H^\infty dm_t(x)\mathrm{d}y\mathrm{d}t.
\end{aligned}$$

Likewise, using $(t,y)\mapsto \frac{1}{\varphi(y)}\int_\eta^K \Phi(x,y)\varphi(x)\mathrm{d}x$, we obtain

$$\begin{aligned}
\int_0^T\int_\eta^K\int_\eta^K &\Phi(x,y)\varphi(x)H(u_\varepsilon(t,y))\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\to \int_0^T\int_\eta^K\int_\eta^K \Phi(x,y)\varphi(x)\langle\nu_{t,y},H\rangle\mathrm{d}x\mathrm{d}y\mathrm{d}t\\
&\qquad +\int_0^T\int_\eta^K\int_\eta^K \Phi(x,y)\varphi(x)H^\infty dm_t(y)\mathrm{d}x\mathrm{d}t.
\end{aligned}$$

There remains the term of $D^H_{\varepsilon,\eta,K}$, in which the dependence on u_ε combines x and y. To deal with this term we separate variables by testing against functions of the form $f_1(x)f_2(y)$. We then consider

$$
\begin{aligned}
-\int_0^T \iint_{[\eta,K]^2} & f_1(x)f_2(y)H'(u_\varepsilon(t,x))u_\varepsilon(t,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= -\int_0^T \left(\int_\eta^K f_1(x)H'(u_\varepsilon(t,x))\mathrm{d}x\right)\left(\int_\eta^K f_2(y)u_\varepsilon(t,y)\mathrm{d}y\right)\mathrm{d}t.
\end{aligned}
$$

The integrands are now split, one containing the x dependence and one the y dependence. However, extra care is required here to pass to the limit. As the Young measures depend on both time and space, it is possible for the oscillations to appear in both directions. We therefore require appropriate time regularity of at least one of the sequences to guarantee the desired behavior of the limit of the product.

Such requirement is met by noticing that since $u_\varepsilon \in \mathcal{C}([0,T];L^1_\varphi(\mathbb{R}_+))$ uniformly, we have u_ε uniformly in $W^{1,\infty}([0,T];(\mathcal{M}^+(\mathbb{R}_+;\varphi),\|\cdot\|_{(W^{1,\infty})^*}))$; cf. [9, Lemma 4.1]. Assuming $f_2 \in W^{1,\infty}(\mathbb{R}_+)$ we therefore have

$$
\left(t \mapsto \int_\eta^K f_2(y)u_\varepsilon(t,y)\mathrm{d}y\right) \in W^{1,\infty}([0,T]).
$$

This in turn implies strong convergence of $\int_\eta^K f_2(y)u_\varepsilon(t,y)\mathrm{d}y$ in $\mathcal{C}([0,T])$ by virtue of the Arzelà–Ascoli theorem. Therefore, we have (noting that $(H')^\infty \equiv 0$ by sublinear growth of H)

$$
\begin{aligned}
-\int_0^T \iint_{[\eta,K]^2} & f_1(x)f_2(y)H'(u_\varepsilon(t,x))u_\varepsilon(t,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&= -\int_0^T \left(\int_\eta^K f_1(x)H'(u_\varepsilon(t,x))\mathrm{d}x\right)\left(\int_\eta^K f_2(y)u_\varepsilon(t,y)\mathrm{d}y\right)\mathrm{d}t \\
&\longrightarrow -\int_0^T \left(\int_\eta^K f_1(x)\langle\nu_{t,x},H'\rangle\mathrm{d}x\right)\left(\int_\eta^K f_2(y)\langle\nu_{t,y},\mathrm{id}\rangle\mathrm{d}y\right)\mathrm{d}t \\
&\qquad -\int_0^T \left(\int_\eta^K f_1(x)\langle\nu_{t,x},H'\rangle\mathrm{d}x\right)\left(\int_\eta^K f_2(y)dm_t(y)\right)\mathrm{d}t \\
&= -\int_0^T \iint_{[\eta,K]^2} f_1(x)f_2(y)\langle\nu_{t,x},H'(\alpha)\rangle\langle\nu_{t,y},\xi\rangle\mathrm{d}x\mathrm{d}y \\
&\qquad -\int_0^T \iint_{[\eta,K]^2} f_1(x)f_2(y)\langle\nu_{t,x},H'(\alpha)\rangle dm_t(y)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

By density of the linear space spanned by separable functions in the space of bounded continuous functions of (x,y) we obtain

$$
\begin{aligned}
-\int_0^T \iint_{[\eta,K]^2} & \Phi(x,y)\varphi(x)H'(u_\varepsilon(t,x))u_\varepsilon(t,y)\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\longrightarrow \int_0^T \iint_{[\eta,K]^2} \Phi(x,y)\varphi(x)\langle\nu_{t,x},H'(\alpha)\rangle\langle\nu_{t,y},\xi\rangle\mathrm{d}x\mathrm{d}y\mathrm{d}t \\
&\qquad -\int_0^T \iint_{[\eta,K]^2} \Phi(x,y)\varphi(x)\langle\nu_{t,x},H'(\alpha)\rangle dm_t(y)\mathrm{d}x\mathrm{d}t.
\end{aligned}
$$

Gathering all the terms we thus obtain the convergence as $\varepsilon \to 0$

$$\int_0^T D^H_{\varepsilon,\eta,K}(t)\mathrm{d}t \longrightarrow \int_0^T D^H_{\eta,K}(t)\mathrm{d}t$$

with

$$\begin{aligned} D^H_{\eta,K}(t) \coloneqq & \iint_{[\eta,K]^2} \Phi(x,y)\varphi(x)\langle \nu_{t,y}(\xi)\otimes\nu_{t,x}(\alpha), H(\xi)-H(\alpha)-H'(\alpha)(\xi-\alpha)\rangle \mathrm{d}x\mathrm{d}y \\ & + \iint_{[\eta,K]^2} \Phi(x,y)\varphi(x)\langle \nu_{t,x}(\alpha), H^\infty - H'(\alpha)\rangle dm_t(y)\mathrm{d}x. \end{aligned}$$

Observe that since Φ is nonnegative and H is convex, the integrand of $D^H_{\varepsilon,\eta,K}$ is nonnegative. Hence, so is the integrand of the limit. Therefore, by monotone convergence, we can pass to the limit $\eta \to 0$, $K \to \infty$, and $T \to \infty$ to obtain

$$\begin{aligned} 0 \le \lim_{\varepsilon\to 0} \int_0^\infty D^H_\varepsilon(t)\ \mathrm{d}t = & \\ \int_0^\infty\int_0^\infty\int_0^\infty \varphi(x)N(y)B(y)k(x,y)\langle \nu_{t,y}(\xi)\otimes\nu_{t,x}(\alpha), H(\xi)-H(\alpha)-H'(\alpha)(\xi-\alpha)\rangle \mathrm{d}x\mathrm{d}y\mathrm{d}t & \\ + \int_0^\infty\int_0^\infty\int_0^\infty \varphi(x)N(y)B(y)k(x,y)\langle \nu_{t,x}(\alpha), H^\infty - H'(\alpha)\rangle dm_t(y)\mathrm{d}x\mathrm{d}t. & \end{aligned}$$

Finally we note that by the Reshetnyak continuity theorem (cf. [24, 27]) we have the convergence $\mathcal{H}_\varepsilon(0) \to \mathcal{H}(0)$. Together with (3.3) this implies (3.8). ∎

4. Long-time asymptotics. In this section we use the result of the previous section to prove that a measure-valued solution of (2.2) converges to the steady-state solution. More precisely we prove

THEOREM 4.1. *Let $n^0 \in \mathcal{M}(\mathbb{R}_+;\varphi)$, and let n solve the growth-fragmentation equation* (2.2). *Then*

$$\lim_{t\to\infty} \int_0^\infty \varphi(x)\mathrm{d}|n(t,x) - m_0 N(x)\mathcal{L}^1| = 0, \tag{4.1}$$

where $m_0 \coloneqq \int_0^\infty \varphi(x)\mathrm{d}n^0(x)$ and $\mathcal{L}^1$ denotes the one-dimensional Lebesgue measure.

Proof. From inequality (3.8) we see that D^H belongs to $L^1(\mathbb{R}_+)$. Therefore, there exists a sequence of times $t_n \to \infty$ such that

$$\lim_{n\to\infty} D^H(t_n) = 0.$$

Consider the corresponding sequence of generalized Young measures $(\nu_{t_n,x}, m_{t_n})$. Thanks to the inequality $\mathcal{H}(t) \le \mathcal{H}(0)$ this sequence is uniformly bounded in the sense that

$$\sup_n \left\{ \int_0^\infty \varphi(x)N(x)\langle \nu_{t,x}(\alpha), |\alpha|\rangle \mathrm{d}x + \int_0^\infty \varphi(x)N(x)dm_{t_n}(x) \right\} < \infty. \tag{4.2}$$

Therefore, by the compactness property of Proposition 2.3 there is a subsequence, not relabeled, and a generalized Young measure $(\bar\nu_x, \bar m)$ such that

$$(\nu_{t_n,x}, m_{t_n}) \overset{*}{\rightharpoonup} (\bar\nu_x, \bar m)$$

in the sense of measures. We now show that the corresponding "entropy dissipation" (4.3)

$$D^H_\infty := \int_0^\infty \int_0^\infty \Phi(x,y)\varphi(x)\langle \bar\nu_y(\xi)\otimes\bar\nu_x(\alpha), H(\xi)-H(\alpha)-H'(\alpha)(\xi-\alpha)\rangle \mathrm{d}x\mathrm{d}y + \int_0^\infty \int_0^\infty \Phi(x,y)\varphi(x)\langle \bar\nu_x(\alpha), H^\infty - H'(\alpha)\rangle d\bar m(y)\mathrm{d}x$$

is zero. To this end we argue that

$$D^H_\infty = \lim_{n\to\infty} D^H(t_n).$$

Indeed this follows by the same arguments as in the proof of Proposition 3.1. In fact now the "mixed" term poses no additional difficulty, as there is no time integral. It therefore follows that

$$D^H_\infty = 0. \tag{4.4}$$

As H is convex, both integrands in (4.3) are nonnegative. Therefore, (4.4) implies that both the integrals of D^H_∞ are zero. In particular

$$\int_0^\infty \int_0^\infty H(\xi)-H(\alpha)-H'(\alpha)(\xi-\alpha) d\bar\nu_x(\alpha)d\bar\nu_y(\xi) = 0,$$

and since the integrand vanishes if and only if $\xi=\alpha$, this implies that the Young measure $\bar\nu$ is a Dirac measure concentrated at a constant. Then the vanishing of the second integral of D^H_∞ implies that $\bar m = 0$. Moreover, the constant can be identified as

$$m_0 := \int_0^\infty \varphi(x)dn^0(x) \tag{4.5}$$

by virtue of the conservation in time of

$$\int_0^\infty \varphi(x)e^{-\lambda t}\langle \nu_{t,x},\cdot\rangle \mathrm{d}x + \int_0^\infty \varphi(x)e^{-\lambda t}dm_t(x).$$

By virtue of Proposition 2.2 with $H = |\cdot - m_0|$ it then follows that

$$\lim_{n\to\infty}\int_0^\infty \varphi(x)d|n(t_n,x)e^{-\lambda t_n} - m_0N(x)\mathcal{L}^1| = 0,$$

which is the desired result, at least for our particular sequence of times.

Finally, we can argue that the last convergence holds for the entire time limit $t\to\infty$, invoking the monotonicity of the relative entropy $\mathcal{H}$. Indeed, the choice $H=|\cdot - m_0|$ in (3.6) yields the monotonicity in time of

$$\int_0^\infty \varphi(x)d|n(t,x)e^{-\lambda t} - m_0N(x)\mathcal{L}^1|,$$

and the result follows. ∎

Conclusion. In this paper, we have proved the long-time convergence of measure-valued solutions to the growth-fragmentation equation. This result extends previously obtained results for L^1_φ solutions [31]. As for the renewal equation [24], it is based on extending the generalized relative entropy inequality to measure-valued solutions, thanks to recession functions. Generalized Young measures provide an adequate framework to represent the measure-valued solutions and their entropy functionals.

Under slightly stronger assumptions on the fragmentation kernel k, e.g., the ones assumed in [8], it has been proved that an entropy-entropy dissipation inequality could be obtained. Under such assumptions, we could obtain in a simple way a stronger result of exponential convergence; see the proof of Theorem 4.1 in [24]. However, the above-seen method allows us to extend the convergence to spaces where no spectral gap exists [5].

A specially interesting case of application of this method would be critical cases where the dominant eigenvalue is not unique but is given by a countable set of eigenvalues. It has been proved that for L^2 initial conditions, the solution then converges to its projection on the space spanned by the dominant eigensolutions [4]. In the case of measure-valued initial condition, due to the fact that the equation has no regularization effect, the asymptotic limit is expected to be the periodically oscillating measure: projection of the initial condition on the space of measures spanned by the dominant eigensolutions. This is a subject for future work.

Acknowledgments. T. D. would like to thank the Institute for Applied Mathematics of Leibniz University of Hannover for its warm hospitality during his stay, when part of this work was completed.

REFERENCES

[1] J. J. Alibert and G. Bouchitté, *Non-uniform integrability and generalized Young measures*, J. Convex Anal., 4 (1997), pp. 129–147, http://eudml.org/doc/225088.

[2] J. M. Ball, *A version of the fundamental theorem for Young measures*, in PDEs and Continuum Models of Phase Transitions (Nice, 1988), Lecture Notes in Phys. 344, Springer, Berlin, 1989, pp. 207–215, https://doi.org/10.1007/BFb0024945.

[3] G. I. Bell and E. C. Anderson, *Cell growth and division:* I. *A mathematical model with applications to cell volume distributions in mammalian suspension cultures*, Biophy. J., 7 (1967), pp. 329–351, https://doi.org/10.1016/S0006-3495(67)86592-5, http://www.sciencedirect.com/science/article/pii/S0006349567865925.

[4] E. Bernard, M. Doumic, and P. Gabriel, *Cyclic asymptotic behaviour of a population reproducing by fission into two equal parts*, Kinet. Relat. Models, to appear.

[5] E. Bernard and P. Gabriel, *Asymptotic Behavior of the Growth-Fragmentation Equation with Bounded Fragmentation Rate*, preprint, https://hal.archives-ouvertes.fr/hal-01313817, 2016.

[6] J. Bertoin and A. R. Watson, *A probabilistic approach to spectral analysis of growth-fragmentation equations*, J. Funct. Anal., 274 (2018), pp. 2163–2204.

[7] Y. Brenier, C. De Lellis, and L. Székelyhidi, Jr., *Weak-strong uniqueness for measure-valued solutions*, Comm. Math. Phys., 305 (2011), pp. 351–361, https://doi.org/10.1007/s00220-011-1267-0.

[8] M. J. Cáceres, J. A. Cañizo, and S. Mischler, *Rate of convergence to the remarkable state for fragmentation and growth-fragmentation equations*, J. Math. Pures Appl., 96 (2011), pp. 334–362.

[9] J. Carrillo, R. Colombo, P. Gwiazda, and A. Ulikowska, *Structured populations, cell growth and measure valued balance laws*, Journal Differential Equations, 252 (2012), pp. 3245–3277, https://doi.org/http://dx.doi.org/10.1016/j.jde.2011.11.003, http://www.sciencedirect.com/science/article/pii/S0022039611004621.

[10] C. Christoforou and A. E. Tzavaras, *Relative entropy for hyperbolic-parabolic systems and application to the constitutive theory of thermoviscoelasticity*, Arch. Ration. Mech. Anal., 229 (2018), pp. 1–52.

[11] C. Dafermos, *The second law of thermodynamics and stability*, Arch. Ration. Mech. Anal., 70 (1979), pp. 167–179.

[12] T. Dębiec, P. Gwiazda, K. Łyczek, and A. Świerczewska-Gwiazda, *Relative entropy method for measure-valued solutions in natural sciences*, Topol. Methods Nonlin. Anal., 52 (2018), pp. 311–335.

[13] S. Demoulini, D. M. A. Stuart, and A. E. Tzavaras, *Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics*, Arch. Ration. Mech. Anal., 205 (2012), pp. 927–961, https://doi.org/10.1007/s00205-012-0523-6.

[14] R. J. DiPerna, *Measure-valued solutions to conservation laws*, Arch. Ration. Mech. Anal., 88 (1985), pp. 223–270, https://doi.org/10.1007/BF00752112.
[15] R. J. DiPerna and A. J. Majda, *Oscillations and concentrations in weak solutions of the incompressible fluid equations*, Comm. Math. Phys., 108 (1987), pp. 667–689, http://projecteuclid.org/euclid.cmp/1104116630.
[16] M. Doumic and M. Escobedo, *Time asymptotics for a critical case in fragmentation and growth-fragmentation equations*, Kinet. Relat. Models, 9 (2016), 251–297, https://doi.org/10.3934/krm.2016.9.251.
[17] M. Doumic and P. Gabriel, *Eigenelements of a general aggregation-fragmentation model*, Math. Models Methods Appl. Sci., 20 (2009), p. 757, arXiv:0907.5467.
[18] L. C. Evans, *Weak Convergence Methods for Nonlinear Partial Differential Equations*, no. 74, American Mathematical Society, Providence, RI, 1990.
[19] E. Feireisl, P. Gwiazda, A. Świerczewska Gwiazda, and E. Wiedemann, *Dissipative measure-valued solutions to the compressible Navier-Stokes system*, Calc. Var. Partial Differential Equations, 55 (2016), pp. Art. 141, 20.
[20] P. Gabriel, *Measure solutions to the conservative renewal equation*, ESAIM Proc. Surveys, 62 (2018), pp. 68–78.
[21] P. Gwiazda, O. Kreml, and A. Świerczewska-Gwiazda, *Dissipative Measure Valued Solutions for General Conser- Vation Laws*, ArXiv preprint, 2018.
[22] P. Gwiazda, T. Lorenz, and A. Marciniak-Czochra, *A nonlinear structured population model: Lipschitz continuity of measure valued solutions with respect to model ingredients*, J. Differential Equations, 248 (2010), pp. 2703–2735.
[23] P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann, *Weak-strong uniqueness for measure-valued solutions of some compressible fluid models*, Nonlinearity, 28 (2015), pp. 3873–3890, http://stacks.iop.org/0951-7715/28/i=11/a=3873.
[24] P. Gwiazda and E. Wiedemann, *Generalized entropy method for the renewal equation with measure data*, Commun. Math. Sci., 15 (2017), pp. 577–586.
[25] W. O. Kermack and A. G. McKendrick, *A contribution to the mathematical theory of epidemics*, Proc. R. Soc. Lond. Ser. A, 115 (1927), pp. 700–721.
[26] W. O. Kermack and A. G. McKendrick, *Contribution to the mathematical theory of epidemics.* II. *The problem of endemicity*, Proc. R. Soc. Lond. Ser. A, 138 (1932), pp. 55–83.
[27] J. Kristensen and F. Rindler, *Relaxation of signed integral functionals in BV*, Calc. Var. Partial Differential Equations, 37 (2010), pp. 29–62.
[28] J. Kristensen and F. Rindler, *Characterization of generalized gradient young measures generated by sequences in $W^{1,1}$ and BV*, Arch. Ration. Mech. Anal., 197 (2012), pp. 539–598.
[29] P. Michel, *Existence of a solution to the cell division eigenproblem*, Math. Models Methods Appl. Sci., 16 (2006), pp. 1125–1153.
[30] P. Michel, S. Mischler, and B. Perthame, *General entropy equations for structured population models and scattering*, C. R. Math. Acad. Sci. Paris, 338 (2004), pp. 697–702.
[31] P. Michel, S. Mischler, and B. Perthame, *General relative entropy inequality: An illustration on growth models*, J. Math. Pures Appl. (9), 84 (2005), pp. 1235–1260.
[32] S. Mischler and J. Scher, *Spectral analysis of semigroups and growth-fragmentation equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 33 (2016), pp. 849–898, https://doi.org/doi:10.1016/j.anihpc.2015.01.007.
[33] B. Perthame, *Kinetic Formulation of Conservation Laws*, Oxford Lecture Series in Mathematics and Its Applications 21, Oxford University Press, Oxford, 2002.
[34] B. Perthame, *Transport Equations in Biology*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2007.
[35] B. Perthame and L. Ryzhik, *Exponential decay for the fragmentation or cell-division equation*, J. Differential Equations, 210 (2005), pp. 155–177.
[36] F. R. Sharpe and A. J. Lotka, *A problem in age-distribution*, Philos. Mag., 21 (1911), pp. 435–438.
[37] J. Sinko and W. Streifer, *A new model for age-size structure of a population*, Ecology, 48 (1967), pp. 910–918.
[38] L. Tartar, *Compensated compactness and applications to partial differential equations*, in Nonlinear Analysis and Mechanics: Heriot-Watt Symposium, Vol. IV, Res. Notes in Math. 39, Pitman, Boston, 1979, pp. 136–212.

Chapter 6

Incompressible limit for a two-species tumour model with coupling through Brinkman's law in one dimension [10]

Acta Appl Math
https://doi.org/10.1007/s10440-020-00313-1

Incompressible Limit for a Two-Species Tumour Model with Coupling Through Brinkman's Law in One Dimension

Tomasz Dębiec[1] · **Markus Schmidtchen**[2]

Received: 21 October 2019 / Accepted: 24 January 2020

Abstract We present a two-species model with applications in tumour modelling. The main novelty is the coupling of both species through the so-called Brinkman law which is typically used in the context of visco-elastic media, where the velocity field is linked to the total population pressure via an elliptic equation. The same model for only one species has been studied by Perthame and Vauchelet in the past. The first part of this paper is dedicated to establishing existence of solutions to the problem, while the second part deals with the incompressible limit as the stiffness of the pressure law tends to infinity. Here we present a novel approach in one spatial dimension that differs from the kinetic reformulation used in the aforementioned study and, instead, relies on uniform BV-estimates.

Keywords Systems of PDEs · Mathematical biology · Tumour growth

1 Introduction

In recent years there has been an increasing interest in multi-phase models applied to tumour growth. Traditionally, tumour growth was modelled using a single equation describing the evolution of the abnormal cell density. This paper is dedicated to studying the two-species model

$$\begin{cases} \dfrac{\partial n_k^{(i)}}{\partial t} - \nabla \cdot \left(n_k^{(i)} \nabla W_k \right) = n_k^{(i)} G^{(i)}(p_k), \\ \qquad -\nu \Delta W_k + W_k = p_k, \end{cases}$$

T. Dębiec
t.debiec@mimuw.edu.pl

M. Schmidtchen
schmidtchen@ljll.math.upmc.fr

1 Institute of Applied Mathematics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

2 Sorbonne Université, Laboratoire Jacques-Louis Lions, Paris 75005, France

where $n^{(i)}$ represents the normal (resp. abnormal) cells, for $i = 1, 2$, and $k \in \mathbb{N}$ is a given constant modelling the stiffness of the total population pressure, p_k, which is generated by both species, *i.e.*,

$$p_k := \frac{k}{k-1}\left(n_k^{(1)} + n_k^{(2)}\right)^{k-1}.$$

In addition, $\nu > 0$ is a fixed positive constant that is understood as a measure of viscosity. The elliptic equation linking the macroscopic velocity potential, W_k, with the pressure p_k is typically referred to as *Brinkman's law*, for instance *cf.* [1]. The growth of the two densities is assumed to be modulated by two functions $G^{(i)}$, for $i = 1, 2$, that are assumed to be decreasing in their variable, p_k, similar to [7, 20].

Throughout, we shall use the shorthand notation $n_k := n_k^{(1)} + n_k^{(2)}$, in order to denote the total population. Upon adding up the two equations for the individual species, we obtain an equation for the total population density, n_k, *i.e.*,

$$\frac{\partial n_k}{\partial t} - \nabla \cdot (n_k \nabla W_k) = n_k \left(r_k G^{(1)}(p_k) + (1 - r_k) G^{(2)}(p_k)\right), \tag{1}$$

where r_k is the population fraction $r_k := n_k^{(1)}/n_k$. Related models have been extensively studied in the past. We refer to [17, 19], and references therein, for a treatise of the incompressible limit for a single-species visco-elastic tumour model. As above, the velocity potential is given by an elliptic equation involving the pressure that, in their case, is just given by a power of the sole species. Introducing the coupling of the two equations for the individual species drastically changes the behaviour and the same tool employed in [19] cannot be applied, at least not in a straightforward manner, and a different strategy has to be found. Even in the case $\nu = 0$ corresponding to the inviscid case, the system nature of the problem gives rise to a whole range of difficulties, *cf.* [6, 8, 13]. At first glance, the pressure gains in regularity, however, it gains just enough regularity to obtain compactness of its gradient, requiring a minute derivation of suitable estimates. Let us stress that the same type of difficulties are also encountered when the pressure is not given as a power law, *cf.* [11, 12, 14]. A key tool in obtaining existence results and stable (with respect to the parameter k) estimates is to devise and manipulate the equation satisfied by the (joint) population pressure, *cf.* [6, 8, 11–14, 16, 18, 19]. In this work we shall follow this path. An easy application of the chain rule in conjunction with Eq. (1) leads to

$$\frac{\partial p_k}{\partial t} - \nabla p_k \cdot \nabla W_k = \frac{k-1}{\nu} p_k \left[W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1 - r_k) G^{(2)}(p_k)\right],$$

where the population fraction r_k satisfies

$$\frac{\partial r_k}{\partial t} - \nabla r_k \cdot \nabla W_k = r_k(1 - r_k)\left[G^{(1)}(p_k) - G^{(2)}(p_k)\right].$$

The change to these new variables was first introduced in [2–4] in the context of a two-species system where the two species avoid overcrowding. In a way, their works paved the way for more modern approaches to tumour models linked through Darcy's law, *cf.* [5, 6, 8, 13, 18].

The rest of this paper is organised as follows. In the subsequent section we set up precisely the problem and state our assumptions. In Sect. 3 we establish existence of solutions to the main system under consideration, Eqs. (2a), (2b), and discuss their regularity necessary for our purposes. Section 4 is dedicated to establishing a range of a priori estimates necessary in the analysis of the incompressible limit. Section 5 is devoted to establishing the strong compactness of the pressure, which is key in passing to the stiff limit. Finally, with all information at hand, we pass to the incompressible limit in the pressure equation and derive the so-called *complementarity relation* in Sect. 6. We round off the analytical results in Sect. 7 by presenting some numerical simulations for different parameter choices.

2 Preliminaries and Statement of the Main Results

We study the system

$$\begin{cases} \dfrac{\partial n_k^{(1)}}{\partial t} - \dfrac{\partial}{\partial x}\left(n_k^{(1)} \dfrac{\partial W_k}{\partial x}\right) = n_k^{(1)} G^{(1)}(p_k), \\ \dfrac{\partial n_k^{(2)}}{\partial t} - \dfrac{\partial}{\partial x}\left(n_k^{(2)} \dfrac{\partial W_k}{\partial x}\right) = n_k^{(2)} G^{(2)}(p_k), \end{cases} \tag{2a}$$

posed on the whole domain $\mathbb{R}$. It is coupled through the Brinkman law

$$-\nu \frac{\partial^2}{\partial x^2} W_k + W_k = p_k. \tag{2b}$$

The system is equipped with non-negative initial data

$$n_{0,k}^{(i)} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}), \tag{3a}$$

for any integer $k \geq 2$. Moreover, we assume that there exists a constant, $C > 0$, such that

$$\int_{\mathbb{R}} \left| \frac{\partial n_{0,k}^{(i)}}{\partial x} \right| \mathrm{d}x \leq C, \tag{3b}$$

for $i = 1, 2$, and every $k \geq 2$. As before, the pressure is given in form of a power of the joint population, *i.e.*,

$$p_k := \frac{k}{k-1}\left(n_k^{(1)} + n_k^{(2)}\right)^{k-1} = \frac{k}{k-1} n_k^{k-1}.$$

Recall that the pressure satisfies

$$\frac{\partial p_k}{\partial t} - \frac{\partial p_k}{\partial x}\frac{\partial W_k}{\partial x} = \frac{k-1}{\nu} p_k \left[W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k) G^{(2)}(p_k)\right], \tag{4}$$

with the population fraction, $r_k := n_k^{(1)}/n_k$, given by

$$\frac{\partial r_k}{\partial t} - \frac{\partial r_k}{\partial x}\frac{\partial W_k}{\partial x} = r_k(1-r_k)\left[G^{(1)}(p_k) - G^{(2)}(p_k)\right]. \tag{5}$$

Throughout the paper we assume the following regularity and properties of the growth functions $G^{(i)}$, $i = 1, 2$,

$$G^{(i)} \in C^1(\mathbb{R}), \quad G_p^{(i)} \leq -\alpha < 0, \quad \text{as well as } G^{(i)}(p_M) = 0, \tag{6}$$

for some $p_M > 0$, where $G_p^{(i)}$ denotes the derivative of the function $G^{(i)}$. The pressure p_M is often called the *homeostatic pressure*.

Recall that a solution W_k to Brinkman's equation $-\nu\partial_x^2 W_k + W_k = p_k$ can be written as $W_k = K \star p_k$, where K is the fundamental solution to the equation $-\nu\partial_x^2 K + K = \delta_0$, *i.e.*,

$$K(x) = \frac{1}{4\pi}\int_0^\infty \exp\left[-(\pi|x|^2/4s\nu + s/4\pi)\right]s^{-1/2}\,\mathrm{d}s = \frac{1}{2\sqrt{\nu}}\exp\left(-\nu^{-1/2}|x|\right). \quad (7)$$

Then $K \geq 0$, $\int K(x)\,\mathrm{d}x = 1$ and $K, \partial_x K \in L^q(\mathbb{R})$ for $1 \leq q \leq \infty$. By the elliptic regularity theory we have $W_k(t,\cdot) \in W^{2,q}(\mathbb{R})$, for any $t \in [0,T]$, $1 \leq q \leq \infty$.

Below we formulate the main results of this work.

Theorem 2.1 (Existence of Solutions) *For any initial data satisfying* (3a), (3b), *system* (2a), (2b) *admits a unique solution* $n_k^{(1)}, n_k^{(2)} \in L^\infty(0,T; BV(\mathbb{R}) \cap L^\infty(\mathbb{R}))$.

We highlight the fact that solutions are essentially bounded since these bounds are not a consequence of the BV-bounds. Rather, they are obtained independently. This may prove useful for an extension to higher dimensions in future works.

Theorem 2.2 (Incompressible Limit and Complementarity Relation) *We may pass to the limit* $k \to \infty$ *in the pressure equation*, *Eq.* (4). *This yields the so-called complementarity relation*

$$0 = p_\infty\left(W_\infty - p_\infty + \nu n_\infty^{(1)} G^{(1)}(p_\infty) + \nu n_\infty^{(2)} G^{(2)}(p_\infty)\right),$$

in the distributional sense, *where* $n_\infty^{(i)}$, $i = 1, 2$, *satisfies*

$$\begin{cases} \dfrac{\partial n_\infty^{(i)}}{\partial t} - \dfrac{\partial}{\partial x}\left(n_\infty^{(i)}\dfrac{\partial W_\infty}{\partial x}\right) = n_\infty^{(i)} G^{(i)}(p_\infty), \\ \qquad -\nu\dfrac{\partial^2 W_\infty}{\partial x^2} + W_\infty = p_\infty. \end{cases}$$

Moreover, *the following holds true*

$$p_\infty(n_\infty - 1) = 0.$$

The subsequent sections are concerned with the proof of the two main theorems.

3 Existence of Solutions and Regularity

This section is dedicated to proving the existence of solutions to the (p,r)-system. The proof is based on an application of Banach's fixed point theorem. Let $k \geq 2$ be fixed throughout this section. Further, assume for now, that the initial data $u_0^{(i)}$ are Lipschitz continuous. For given functions $p, r \in L^\infty(0,T; L^\infty(\mathbb{R}))$ we construct solutions $u^{(i)}$ to the linearised system, $i = 1, 2$,

$$\frac{\partial u^{(i)}}{\partial t} - \frac{\partial u^{(i)}}{\partial x}\frac{\partial W}{\partial x} = \frac{k-1}{\nu} f^{(i)}(p, r), \quad (8)$$

where

$$f^{(1)}(p,r) = K \star p - p + \nu r G^{(1)}(p) + \nu(1-r)G^{(2)}(p),$$

and

$$f^{(2)}(p,r) = r(1-r)\left[G^{(1)}(p) - G^{(2)}(p)\right].$$

For the fixed p from above, we may construct the backward flow

$$\begin{cases} \dfrac{\mathrm{d}X_{(x,t)}}{\mathrm{d}s}(s) = -\dfrac{\partial W}{\partial x}(X_{(x,t)}, s), \\ X_{(x,t)}(t) = x. \end{cases}$$

We readily observe that

$$u^{(i)}(t,x) = u_0^{(i)}(X_{(x,t)}(s=0)) + \int_0^t f^{(i)}(p(\tau,x), r(\tau,x))\,\mathrm{d}\tau,$$

$i = 1, 2$, solve the linearised system (8). Now, considering another element $(\tilde{p}, \tilde{r})$ in $L^\infty(0,T; L^\infty(\mathbb{R}))$, we observe that

$$\begin{aligned} \left|u^{(i)}(t,x) - \tilde{u}^{(i)}(t,x)\right| &= \left|u_0^{(i)}(X_{(x,t)}(s=0)) - u_0^{(i)}(\tilde{X}_{(x,t)}(s=0))\right| \\ &\le \mathrm{Lip}(u_0^{(i)}) \left|X_{(x,t)}(s=0) - \tilde{X}_{(x,t)}(s=0)\right| \\ &\le \mathrm{Lip}(u_0^{(i)}) \int_t^0 \left|\frac{\partial W}{\partial x}(X_{(x,t)}(s), s) - \frac{\partial \tilde{W}}{\partial x}(\tilde{X}_{(x,t)}(s), s)\right| \mathrm{d}s \\ &\le \mathrm{Lip}(u_0^{(i)}) \int_t^0 \left|\frac{\partial K}{\partial x} \star (p - \tilde{p})\right| \mathrm{d}s \\ &\le \mathrm{Lip}(u_0^{(i)}) \int_t^0 \left\|\frac{\partial K}{\partial x}\right\|_{L^1} \|p - \tilde{p}\|_{L^\infty}\,\mathrm{d}s \\ &\le \mathrm{Lip}(u_0^{(i)}) T \left\|\frac{\partial K}{\partial x}\right\|_{L^1} \|p - \tilde{p}\|_{L^\infty}. \end{aligned}$$

Thus, upon passing to the supremum, we obtain the following stability estimate for two solutions

$$\left\|u^{(i)} - \tilde{u}^{(i)}\right\|_{L^\infty} \le CT\|p - \tilde{p}\|_{L^\infty}.$$

In particular, for $T_1 > 0$ small enough the estimate gives rise to a contraction in the Banach space $L^\infty(0, T_1; L^\infty(\mathbb{R}))$, which is sufficient to infer the existence of a unique fixed point, by an application of Banach's fixed point theorem. Since the supremum norm of the solution does not blow up, a finite number of iterations of the above argument leads to existence of a unique solution for all times $T > 0$.

For the subsequent analysis, let us call this fixed point $(u_*^{(1)}, u_*^{(2)})$. It remains to prove the expected BV-regularity of solutions. This is an easy consequence of the "transport nature" of the system, *i.e.*,

$$\frac{\partial}{\partial t}\frac{\partial u_*^{(i)}}{\partial x} = \frac{\partial u_*^{(i)}}{\partial x}\frac{\partial W}{\partial x} + \frac{k-1}{\nu}\left[f_p^{(i)}(u_*^{(1)}, u_*^{(2)})\frac{\partial u_*^{(1)}}{\partial x} + f_r^{(i)}(u_*^{(1)}, u_*^{(2)})\frac{\partial u_*^{(2)}}{\partial x}\right].$$

Multiplying by $\mathrm{sign}(\partial_x u_*^{(i)})$ and adding the two equations, for $i = 1, 2$, we obtain, after integrating

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left|\frac{\partial u_*^{(1)}}{\partial x}\right| + \left|\frac{\partial u_*^{(2)}}{\partial x}\right| \mathrm{d}x \le C\int_{\mathbb{R}} \left|\frac{\partial u_*^{(1)}}{\partial x}\right| + \left|\frac{\partial u_*^{(2)}}{\partial x}\right| \mathrm{d}x,$$

where the constant $C > 0$ depends only on the Lipschitz constants of the functions $f^{(i)}$ and the L^∞-bounds on the fixed point. In particular, from Gronwall's inequality we deduce a control on the BV-seminorm and, more importantly, the existence of solutions even in cases where $u_0^{(i)}$ is not Lipschitz continuous but only of bounded variation.

Using the fact that

$$n^{(1)} = \left(\frac{k-1}{k}u_*^{(1)}\right)^{\frac{1}{k-1}} u_*^{(2)}, \quad \text{and} \quad n^{(2)} = \left(\frac{k-1}{k}u_*^{(1)}\right)^{\frac{1}{k-1}} \left(1 - u_*^{(2)}\right),$$

the existence and uniqueness result transfers to the original system for $n_k^{(i)}$, $i = 1, 2$.

Remark 3.1 (Extension to Higher Dimensions) Let us remark here that the same strategy can be easily extended to higher dimensions since the transport nature is the same in any dimension. In fact, the only "problematic" point in our strategy is the contraction argument which depends on $\|\partial_x K\|_{L^1}$. However, this norm is finite in any dimension, and therefore our existence result holds in any dimension.

4 A Priori Estimates

In this section we derive some bounds for the main quantities of interests, uniformly in k. These will be vital when passing to the limit with $k \to \infty$.

Lemma 4.1 (A priori estimates I) *The following hold uniformly in k for any* $T > 0$.

(i) $n_k \in L^\infty(0, T; L^1(\mathbb{R}))$,
(ii) $p_k \in L^\infty(0, T; L^\infty(\mathbb{R}))$,
(iii) $n_k \in L^\infty(0, T; L^\infty(\mathbb{R}))$,
(iv) $p_k \in L^\infty(0, T; L^1(\mathbb{R}))$, *and*
(v) $n_k^{(i)} \in L^\infty(0, T; L^\infty(\mathbb{R}))$, *for* $i = 1, 2$.

Proof Clearly when $n_k(t = 0) \ge 0$, then n_k stays non-negative at all times. Integrating Eq. (1) in space and time we deduce that $n_k \in L^\infty(0, T; L^1(\mathbb{R}))$ uniformly in k. By the maximum principle we have the bound $0 \le p_k \le p_M$. Then using $n_k \simeq p^{\frac{1}{k-1}}$ we deduce $n_k \in L^\infty(0, T; L^\infty(\mathbb{R}))$ uniformly. Writing $p_k \le n_k \|n_k\|_\infty^{k-2}$ we see that $p_k \in L^\infty(0, T; L^1(\mathbb{R}))$. Finally, we use that $n_k^{(1)} = r_k n_k$ and $0 \le r_k \le 1$ to deduce the last bounds. □

Using the above Lemma and the boundedness of W_k, we have the following result.

Lemma 4.2 (Integrability and Segregation) *If both species are segregated initially*, i.e.,

$$\int_{\mathbb{R}} r_k^0(1-r_k^0)\,\mathrm{d}x = 0,$$

then there holds

$$\int_{\mathbb{R}} r_k(t,x)\ (1-r_k(t,x))\,\mathrm{d}x = 0,$$

for all times $0 \leq t \leq T$. *In particular,* $r_k^0(1-r_k^0) \in L^1(\mathbb{R})$ *implies* $r_k(1-r_k) \in L^\infty(0,T;L^1(\mathbb{R}))$.

Proof Here and henceforth we shall employ the notation

$$\left\|G^{(i)}\right\|_\infty := \sup_{0\leq p\leq P_M} \left|G^{(i)}(p)\right|.$$

The supremum is taken only up to p_M, because in principle the functions $G^{(i)}$ can decrease arbitrarily. The uniform bound obtained in the previous proof shows however, that only the range $0 \leq p_k \leq p_M$ is relevant.

Using the equation for the population fraction and boundedness of the growth functions $G^{(i)}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} r_k\,(1-r_k)\,\mathrm{d}x &= \int_{\mathbb{R}}(1-2r_k)\left(\frac{\partial r_k}{\partial x}\frac{\partial W_k}{\partial x} + r_k(1-r_k)\left[G^{(1)}(p_k)-G^{(2)}(p_k)\right]\right)\mathrm{d}x\\ &\leq \max_{i=1,2}\left\|G^{(i)}\right\|_\infty \int_{\mathbb{R}} r_k(1-r_k)\,\mathrm{d}x + \int_{\mathbb{R}}\frac{\partial}{\partial x}\left(r_k(1-r_k)\right)\frac{\partial W_k}{\partial x}\,\mathrm{d}x\\ &\leq \max_{i=1,2}\left\|G^{(i)}\right\|_\infty \int_{\mathbb{R}} r_k(1-r_k)\,\mathrm{d}x - \int_{\mathbb{R}} r_k(1-r_k)\frac{\partial^2 W_k}{\partial x^2}\,\mathrm{d}x.\end{aligned}$$

Using Brinkman's law (2b), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} r_k\,(1-r_k)\,\mathrm{d}x &\leq \max_{i=1,2}\left\|G^{(i)}\right\|_\infty \int_{\mathbb{R}} r_k(1-r_k)\,\mathrm{d}x + \int_{\mathbb{R}} r_k(1-r_k)\frac{p_k-W_k}{\nu}\,\mathrm{d}x\\ &\leq C\int_{\mathbb{R}} r_k(1-r_k)\,\mathrm{d}x,\end{aligned}$$

having used the a priori bounds on the pressure, p_k. □

The following lemma establishes an L^1-bound on the right-hand side of the pressure equation.

Lemma 4.3 (A priori estimates II) *The following estimate holds for any* $T > 0$

$$k\int_0^T\int_{\mathbb{R}} p_k|W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k)G^{(2)}(p_k)|\,\mathrm{d}x\,\mathrm{d}t \leq C(T),$$

for a constant $C(T) > 0$, *independent of* k. *Furthermore, the following bounds hold uniformly in* k:

(i) $\dfrac{\partial W_k}{\partial t} \in L^1(0, T; L^q(\mathbb{R}))$, *for* $1 \le q \le \infty$,
(ii) $\dfrac{\partial}{\partial t}\dfrac{\partial W_k}{\partial x} \in L^1(0, T; L^q(\mathbb{R}))$, *for* $1 < q < \infty$.

Proof Let us introduce the following notation

$$Q_k := W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1 - r_k)G^{(2)}(p_k),$$

and follow the strategy of [19]. Using

$$\frac{\partial W_k}{\partial t} = K \star \left[\frac{\partial p_k}{\partial x}\frac{\partial W_k}{\partial x} + \frac{k-1}{\nu} p_k Q_k\right], \tag{9}$$

we derive the equation

$$\begin{aligned}\frac{\partial Q_k}{\partial t} &- \frac{\partial Q_k}{\partial x}\frac{\partial W_k}{\partial x} + \frac{k-1}{\nu} p_k Q_k \left[1 - \nu r_k G_p^{(1)}(p_k) - \nu\left(1 - r_k\right) G_p^{(2)}(p_k)\right] \\ &= -\left|\frac{\partial W_k}{\partial x}\right|^2 + K \star \left[\frac{\partial p_k}{\partial x}\frac{\partial W_k}{\partial x} + \frac{k-1}{\nu} p_k Q_k\right] + \nu\left(G^{(1)}(p_k) - G^{(2)}(p_k)\right)^2 r_k(1 - r_k),\end{aligned}$$

and consequently,

$$\begin{aligned}\frac{\partial\left|Q_k\right|}{\partial t} &- \frac{\partial\left|Q_k\right|}{\partial x}\frac{\partial W_k}{\partial x} + \frac{k-1}{\nu} p_k \left|Q_k\right| \left[1 - \nu r_k G_p^{(1)}(p_k) - \nu(1 - r_k) G_p^{(2)}(p_k)\right] \\ &\le -\left|\frac{\partial W_k}{\partial x}\right|^2 + \left|K \star \left[\frac{\partial p_k}{\partial x}\frac{\partial W_k}{\partial x} + \frac{k-1}{\nu} p_k \left|Q_k\right|\right]\right| \\ &\quad + \nu\left(G^{(1)}(p_k) - G^{(2)}(p_k)\right)^2 r_k\left(1 - r_k\right).\end{aligned}$$

Integrating in space-time and using the assumption that $|G_p^{(i)}| \ge \alpha > 0$, we obtain

$$\begin{aligned}\alpha(k-1)\int_0^T\int_{\mathbb{R}} p_k\left|Q_k\right| \mathrm{d}x\,\mathrm{d}t \le &\int_{\mathbb{R}} \left|Q_k(x,0)\right| - \left|Q_k(x,T)\right| \mathrm{d}x + \int_0^T\int_{\mathbb{R}}\left|\frac{\partial W_k}{\partial x}\right|^2 \mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T\int_{\mathbb{R}} \nu^{-1}\left|Q_k\right|(p_k - W_k) + \left|K \star \left[\frac{\partial p_k}{\partial x}\frac{\partial W_k}{\partial x}\right]\right| \mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T\int_{\mathbb{R}} \nu\left(G^{(1)}(p_k) - G^{(2)}(p_k)\right)^2 r_k(1 - r_k)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The first three terms on the right-hand side are controlled uniformly, as is the very last term. For the remaining two terms we write

$$\nu^{-1}\int_0^T\int_{\mathbb{R}} |Q_k|(p_k - W_k)\,\mathrm{d}x\,\mathrm{d}t \le \nu^{-1}\int_0^T\int_{\mathbb{R}} |Q_k| p_k\,\mathrm{d}x\,\mathrm{d}t,$$

which, for k large enough, is controlled by the left-hand side of the last inequality, and

$$\begin{aligned}K \star \left[\frac{\partial p_k}{\partial x}\frac{\partial W_k}{\partial x}\right] &= \frac{\partial K}{\partial x} \star \left[p_k \frac{\partial W_k}{\partial x}\right] - K \star \left[p_k \frac{\partial^2 W_k}{\partial x^2}\right] \\ &= \frac{\partial K}{\partial x} \star \left[p_k \frac{\partial K}{\partial x} \star p_k\right] - K \star \left[p_k \frac{\partial^2 W_k}{\partial x^2}\right].\end{aligned}$$

Using Lemma 4.1, we see that the right-hand side is uniformly bounded in $L^\infty(0,T;L^q(\mathbb{R}))$, $1 \le q \le \infty$. It follows that

$$\alpha(k-1)\int_0^T \int_{\mathbb{R}} p_k |Q_k| \,\mathrm{d}x\,\mathrm{d}t \le C(T),$$

as desired.

Now, using Eq. (9) and the above computations, it is clear that $\partial_t W_k$ is uniformly bounded in $L^\infty(0,T;L^q(\mathbb{R}))$, for $1 \le q \le \infty$. Finally we write

$$\frac{\partial}{\partial t}\frac{\partial W_k}{\partial x} = \frac{\partial^2 K}{\partial x^2} \star \left(p_k \frac{\partial W_k}{\partial x}\right) - \frac{\partial K}{\partial x} \star \left(p_k \frac{\partial^2 W_k}{\partial x^2}\right) + \frac{k-1}{\nu}\frac{\partial K}{\partial x} \star (p_k Q_k),$$

and use the definition of K, *cf.* Eq. (7), to conclude the proof. □

Remark 4.4 All the results of this section remain valid in any spatial dimension $d \ge 1$, see for example [19] for the a priori estimates, and the L^1-bound on the quantity $kp_k Q_k$.

5 Strong Compactness of the Pressure

This section is solely dedicated to the derivation of suitable estimates in order to obtain strong compactness of the pressure, p_k. A key step in this pursuit is the following BV-estimate on the individual species as well as the total population.

Lemma 5.1 (Regularity of $n_k^{(i)}$ and n_k) *For $i = 1, 2$, we have the following*

$$\left|\frac{\partial n_k^{(i)}}{\partial x}\right|, \left|\frac{\partial n_k}{\partial x}\right| \in L^\infty(0,T;L^1(\mathbb{R})),$$

uniformly in $k \ge 2$.

Proof For $i = 1, 2$, we consider

$$\frac{\partial n_k^{(i)}}{\partial t} = \frac{\partial}{\partial x}\left(n_k^{(i)} \frac{\partial W_k}{\partial x}\right) + n_k^{(i)} G^{(i)}(p_k).$$

Upon differentiating in space, we obtain

$$\frac{\partial}{\partial t}\frac{\partial n_k^{(i)}}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial n_k^{(i)}}{\partial x}\frac{\partial W}{\partial x}\right) + \frac{\partial}{\partial x}\left(n_k^{(i)}\frac{\partial^2 W}{\partial x^2}\right) + \frac{\partial n_k^{(i)}}{\partial x} G^{(i)}(p_k) + n_k^{(i)} G_p^{(i)}(p_k)\frac{\partial p_k}{\partial x}, \quad (10)$$

for $i = 1, 2$. Upon adding up both equations we get

$$\frac{\partial}{\partial t}\frac{\partial n_k}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial n_k}{\partial x}\frac{\partial W_k}{\partial x}\right) + \frac{\partial}{\partial x}\left(n_k\frac{\partial^2 W_k}{\partial x^2}\right) + \sum_{i=1,2}\frac{\partial n_k^{(i)}}{\partial x} G^{(i)}(p_k) + n_k^{(i)} G_p^{(i)}(p_k)\frac{\partial p_k}{\partial x}. \quad (11)$$

Multiplying the equation for the individual species by $\sigma^{(i)} := \operatorname{sign}(\partial_x n_k^{(i)})$ and the equation for the total population by $\sigma := \operatorname{sign}(\partial_x n_k)$, we get, upon adding the two equations in Eq. (10) and the one in Eq. (11), and integrating in space

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left|\frac{\partial n_k^{(1)}}{\partial x}\right| + \left|\frac{\partial n_k^{(2)}}{\partial x}\right| + \left|\frac{\partial n_k}{\partial x}\right| \mathrm{d}x \\
&\quad = \int_{\mathbb{R}} \frac{\partial}{\partial x}\left(\left|\frac{\partial n_k^{(1)}}{\partial x}\right| \frac{\partial W_k}{\partial x}\right) + \frac{\partial}{\partial x}\left(n_k^{(i)} \frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(1)} + \left|\frac{\partial n_k^{(1)}}{\partial x}\right| G^{(1)}(p_k) \\
&\qquad + \sigma^{(1)} n_k^{(1)} G_p^{(1)}(p_k)\frac{\partial p_k}{\partial x} \\
&\qquad + \frac{\partial}{\partial x}\left(\left|\frac{\partial n_k^{(2)}}{\partial x}\right| \frac{\partial W_k}{\partial x}\right) + \frac{\partial}{\partial x}\left(n_k^{(2)} \frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(2)} + \left|\frac{\partial n_k^{(2)}}{\partial x}\right| G^{(2)}(p_k) \\
&\qquad + \sigma^{(2)} n_k^{(2)} G_p^{(2)}(p_k)\frac{\partial p_k}{\partial x} \\
&\qquad + \frac{\partial}{\partial x}\left(\left|\frac{\partial n_k}{\partial x}\right| \frac{\partial W_k}{\partial x}\right) + \frac{\partial}{\partial x}\left(n_k \frac{\partial^2 W_k}{\partial x^2}\right)\sigma + \sum_{i=1,2} \sigma \frac{\partial n_k^{(i)}}{\partial x} G^{(i)}(p_k) \\
&\qquad + n_k^{(i)} G_p^{(i)}(p_k)\left|\frac{\partial p_k}{\partial x}\right| \mathrm{d}x.
\end{aligned}$$

First we notice that the exact derivatives vanish and the estimate simplifies to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left|\frac{\partial n_k^{(1)}}{\partial x}\right| + \left|\frac{\partial n_k^{(2)}}{\partial x}\right| + \left|\frac{\partial n_k}{\partial x}\right| \mathrm{d}x \\
&\quad \le \int_{\mathbb{R}} \frac{\partial}{\partial x}\left(n_k^{(1)} \frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(1)} + \left|\frac{\partial n_k^{(1)}}{\partial x}\right| G^{(1)}(p_k) + n_k^{(1)} \left|G_p^{(1)}(p_k)\right| \left|\frac{\partial p_k}{\partial x}\right| \\
&\qquad + \frac{\partial}{\partial x}\left(n_k^{(2)} \frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(2)} + \left|\frac{\partial n_k^{(2)}}{\partial x}\right| G^{(2)}(p_k) + n_k^{(2)} \left|G_p^{(2)}(p_k)\right| \left|\frac{\partial p_k}{\partial x}\right| \\
&\qquad + \frac{\partial}{\partial x}\left(n_k \frac{\partial^2 W_k}{\partial x^2}\right)\sigma + \sum_{i=1,2} \left|\frac{\partial n_k^{(i)}}{\partial x}\right| \left\|G^{(i)}\right\|_\infty + n_k^{(i)} G_p^{(i)}(p_k)\left|\frac{\partial p_k}{\partial x}\right| \mathrm{d}x.
\end{aligned}$$

Next, we notice that all the terms involving the pressure gradient cancel due to opposite signs, whence

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left|\frac{\partial n_k^{(1)}}{\partial x}\right| + \left|\frac{\partial n_k^{(2)}}{\partial x}\right| + \left|\frac{\partial n_k}{\partial x}\right| \mathrm{d}x \\
&\quad \le \int_{\mathbb{R}} \frac{\partial}{\partial x}\left(n_k^{(1)} \frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(1)} + \left|\frac{\partial n_k^{(1)}}{\partial x}\right| G^{(1)}(p_k)
\end{aligned}$$

$$+\frac{\partial}{\partial x}\left(n_k^{(2)}\frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(2)}+\left|\frac{\partial n_k^{(2)}}{\partial x}\right|G^{(2)}(p_k)$$
$$+\frac{\partial}{\partial x}\left(n_k\frac{\partial^2 W_k}{\partial x^2}\right)\sigma+\sum_{i=1,2}\left|\frac{\partial n_k^{(i)}}{\partial x}\right|\left\|G^{(i)}\right\|_\infty \mathrm{d}x.$$

Thus we are left with

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left|\frac{\partial n_k^{(1)}}{\partial x}\right|+\left|\frac{\partial n_k^{(2)}}{\partial x}\right|+\left|\frac{\partial n_k}{\partial x}\right|\mathrm{d}x\\
\le C\int_{\mathbb{R}}\left|\frac{\partial n_k^{(1)}}{\partial x}\right|+\left|\frac{\partial n_k^{(2)}}{\partial x}\right|+\left|\frac{\partial n_k}{\partial x}\right|\mathrm{d}x\\
+\int_{\mathbb{R}}\frac{\partial}{\partial x}\left(n_k^{(1)}\frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(1)}+\frac{\partial}{\partial x}\left(n_k^{(2)}\frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(2)}+\frac{\partial}{\partial x}\left(n_k\frac{\partial^2 W_k}{\partial x^2}\right)\sigma\,\mathrm{d}x.\end{aligned}\tag{12}$$

Using the fact that

$$-\nu\frac{\partial^2 W_k}{\partial x^2}+W_k=p_k,$$

the integrand of the last line of Eq. (12) may be simplified to

$$\begin{aligned}&\frac{\partial}{\partial x}\left(n_k^{(1)}\frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(1)}+\frac{\partial}{\partial x}\left(n_k^{(2)}\frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(2)}+\frac{\partial}{\partial x}\left(n_k\frac{\partial^2 W_k}{\partial x^2}\right)\sigma\\
&=\nu^{-1}\left|\frac{\partial n_k^{(1)}}{\partial x}\right|(W_k-p_k)+\nu^{-1}n_k^{(1)}\frac{\partial}{\partial x}(W_k-p_k)\sigma^{(1)}\\
&\quad+\nu^{-1}\left|\frac{\partial n_k^{(2)}}{\partial x}\right|(W_k-p_k)+\nu^{-1}n_k^{(2)}\frac{\partial}{\partial x}(W_k-p_k)\sigma^{(2)}\\
&\quad+\nu^{-1}\left|\frac{\partial n_k}{\partial x}\right|(W_k-p_k)+\nu^{-1}n_k\frac{\partial}{\partial x}(W_k-p_k)\sigma.\end{aligned}\tag{13}$$

Using the fact that $|\sigma^{(i)}|, |\sigma|\le 1$ and exploiting the bounds

$$n_k^{(i)}, n_k\in L^\infty(0,T;L^1(\mathbb{R})),\quad\text{and}\quad\frac{\partial W_k}{\partial x}\in L^\infty(0,T;L^\infty(\mathbb{R})),$$

we may bound the terms of Eq. (13), and the last line of Eq. (12) becomes

$$\begin{aligned}\int_{\mathbb{R}}\frac{\partial}{\partial x}\left(n_k^{(1)}\frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(1)}+\frac{\partial}{\partial x}\left(n_k^{(2)}\frac{\partial^2 W_k}{\partial x^2}\right)\sigma^{(2)}+\frac{\partial}{\partial x}\left(n_k\frac{\partial^2 W_k}{\partial x^2}\right)\sigma\,\mathrm{d}x\\
=C\nu^{-1}\int_R\left|\frac{\partial n_k^{(1)}}{\partial x}\right|+\left|\frac{\partial n_k^{(2)}}{\partial x}\right|+\left|\frac{\partial n_k}{\partial x}\right|\mathrm{d}x\\
+C+\nu^{-1}\int_{\mathbb{R}}n_k^{(1)}\left|\frac{\partial p_k}{\partial x}\right|+n_k^{(2)}\left|\frac{\partial p_k}{\partial x}\right|-n_k\left|\frac{\partial p_k}{\partial x}\right|\mathrm{d}x.\end{aligned}\tag{14}$$

The last integral in Eq. (14) vanishes due to the fact that $n_k = n_k^{(1)} + n_k^{(2)}$. Thus, substituting Eq. (14) into Eq. (12), an application of Gronwall's lemma yields the BV-estimate in space. □

Corollary 5.2 *From the proof of the preceding lemma we deduce*

$$\int_0^T \int_{\mathbb{R}} n_k \left|\frac{\partial p_k}{\partial x}\right| \mathrm{d}x\, \mathrm{d}t \le C,$$

where $C > 0$ *is independent of* k.

Proof Let us revisit the equation for $\partial_t n_k$, *i.e.*,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} \left|\frac{\partial n_k}{\partial x}\right| \mathrm{d}x \le C + C\int_{\mathbb{R}} \left|\frac{\partial n_k^{(1)}}{\partial x}\right| + \left|\frac{\partial n_k^{(2)}}{\partial x}\right| + \left|\frac{\partial n_k}{\partial x}\right| \mathrm{d}x$$
$$+ \int_{\mathbb{R}} \left(n_k^{(1)} G_p^{(1)}(p_k) + n_k^{(2)} G_p^{(2)}(p_k) - \nu^{-1} n_k\right) \left|\frac{\partial p_k}{\partial x}\right| \mathrm{d}x.$$

Now we use the bounds $G_p^{(i)} \le -\alpha < 0$, for $i = 1, 2$, and integrate in time to see that

$$(\nu^{-1} + \alpha)\int_0^T \int_{\mathbb{R}} n_k \left|\frac{\partial p_k}{\partial x}\right| \mathrm{d}x\, \mathrm{d}t \le 2\left\|\frac{\partial n_k}{\partial x}\right\|_{L^\infty(0,T;L^1(\mathbb{R}))} + CTR,$$

where

$$R := \left\|\frac{\partial n_k}{\partial x}\right\|_{L^\infty(0,T;L^1(\mathbb{R}))} + \left\|\frac{\partial n_k^{(1)}}{\partial x}\right\|_{L^\infty(0,T;L^1(\mathbb{R}))} + \left\|\frac{\partial n_k^{(2)}}{\partial x}\right\|_{L^\infty(0,T;L^1(\mathbb{R}))} + 1.$$

Thus we infer that $n_k \partial_x p_k$ is uniformly bounded in $L^1(0, T; L^1(\mathbb{R}))$. □

Lemma 5.3 (Strong Compactness of the Pressure) *There exists a function*

$$p_\infty \in L^\infty\left(0, T; L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})\right),$$

such that, up to a subsequence, there holds

$$p_k \longrightarrow p_\infty,$$

in $L^p_{\mathrm{loc}}(0, T; L^q(\mathbb{R}))$, *as* $k \to \infty$ *for any* $2 \le p, q < \infty$. *In addition, the convergence also holds in the almost everywhere sense.*

Proof Let us write the quantity $n_k\, |\partial_x p_k|$ as a spatial derivative of a non-decreasing function of the pressure. We compute as follows

$$n_k \left|\frac{\partial p_k}{\partial x}\right| = \left(\frac{k-1}{k}\right)^{\frac{k}{k-1}} \operatorname{sign}\left(\frac{\partial p_k}{\partial x}\right) \frac{\partial}{\partial x}\left(p^{\frac{k}{k-1}}\right) = \left(\frac{k-1}{k}\right)^{\frac{k}{k-1}} \left|\frac{\partial}{\partial x}\left(p^{\frac{k}{k-1}}\right)\right|.$$

Let $\phi_k(z) := z^{\frac{k}{k-1}}$. Then

$$\int_0^T \int_{\mathbb{R}} n_k \left|\frac{\partial p_k}{\partial x}\right| \mathrm{d}x\, \mathrm{d}t \ge \frac{1}{4}\int_0^T \int_{\mathbb{R}} \left|\frac{\partial \phi_k}{\partial x}(p_k)\right| \mathrm{d}x\, \mathrm{d}t,$$

i.e., $\partial_x \phi_k(p_k) \in L^1(0,T;L^1(\mathbb{R}))$, uniformly in k. Moreover, we have the same L^1-bound for the time derivative of $\phi_k(p_k)$. Indeed

$$\begin{aligned}\frac{\partial \phi_k}{\partial t}(p_k) = \phi_k'(p_k)\frac{\partial p_k}{\partial t}\\ &= \phi_k'(p_k)\left(\frac{\partial p_k}{\partial x}\frac{\partial W_k}{\partial x} + \frac{k-1}{\nu} p_k Q_k\right)\\ &= \frac{\partial \phi_k}{\partial x}(p_k)\frac{\partial W_k}{\partial x} + \frac{k}{k-1}\frac{k-1}{\nu} p_k^{\frac{1}{k-1}} p_k Q_k,\end{aligned}$$

and therefore

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\left|\frac{\partial \phi_k}{\partial t}(p_k)\right| \mathrm{d}x\,\mathrm{d}t &\le \left\|\frac{\partial \phi_k}{\partial t}(p_k)\right\|_{L^1(0,T;L^1(\mathbb{R}))}\left\|\frac{\partial W}{\partial x}\right\|_{L^\infty(0,T;L^\infty(\mathbb{R}))}\\ &\quad + 2p_M^{\frac{1}{k-1}}\frac{k-1}{\nu}\int_0^T\int_{\mathbb{R}} p_k\,|Q_k|\,\mathrm{d}x\,\mathrm{d}t\\ &\le C,\end{aligned}$$

where we have used Lemma 4.3.

We conclude that the sequence $(\phi_k(p_k))_k$ converges strongly in $L^2((0,T)\times\mathbb{R})$. On the other hand, from the uniform bounds on p_k we infer that $p_k \rightharpoonup p_\infty$, weakly in $L^2((0,T)\times\mathbb{R})$, up to the subsequence. We can therefore apply Lemma 8.1 to conclude that

$$\phi_k(p_k) \to p_\infty,$$

strongly in $L^2_{\text{loc}}((0,T)\times\mathbb{R})$. We claim that this in fact implies strong convergence of the sequence of pressures $(p_k)_k$ itself. Indeed, using the triangle inequality yields

$$\|p_k - p_\infty\|_{L^2(0,T;L^2(\mathbb{R}))} \le \left\|p_k - \phi_k(p_k)\right\|_{L^2(0,T;L^2(\mathbb{R}))} + \left\|\phi_k(p_k) - p_\infty\right\|_{L^2(0,T;L^2(\mathbb{R}))},$$

and

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\left|p_k - p_k^{\frac{k}{k-1}}\right|^2 \mathrm{d}x\,\mathrm{d}t &= \int_0^T\int_{\mathbb{R}} |p_k|\left|\sqrt{p_k} - p_k^{\frac{1}{k-1}+\frac{1}{2}}\right|^2 \mathrm{d}x\,\mathrm{d}t\\ &\le \sup_{0\le z\le p_M}\left|\sqrt{z} - \sqrt{z}z^{\frac{1}{k-1}}\right|^2 \|p_k\|_{L^1(0,T;L^1(\mathbb{R}))},\end{aligned}$$

with the right-hand side of the last line converging to zero. We conclude that

$$p_k \to p_\infty,$$

strongly in $L^2_{\text{loc}}((0,T)\times\mathbb{R})$. In combination with the L^∞-bounds, we deduce that this convergence holds strongly in $L^p_{\text{loc}}(0,T;L^q(\mathbb{R}))$, for any $2\le p,q<\infty$, using the dominated convergence theorem. Moreover, the convergence is also true almost everywhere. □

6 Incompressible Limit and Complementarity Relation

We have garnered all information necessary to pass to the incompressible limit in the pressure equation (4) and prove Theorem 2.2.

Proof of Theorem 2.2 Having established strong convergence of the sequence $(p_k)_k$, and weak convergence of $(n_k)_k$ due to the a priori estimates, we can pass to the limit in the relation

$$n_k p_k = \left(\frac{k-1}{k}\right)^{\frac{1}{k-1}} p_k^{\frac{k}{k-1}}, \tag{15}$$

to deduce the relation $(1-n_\infty)p_\infty = 0$, almost everywhere. For a test function $\varphi \in C_c^1((0,T)\times\mathbb{R})$, let us recall the weak formulation of the equation for the pressure

$$\int_0^T \int_{\mathbb{R}} \frac{\partial \varphi}{\partial t} p_k - \frac{\partial \varphi}{\partial x} p_k \frac{\partial W_k}{\partial x} - \varphi p_k \frac{\partial^2 W_k}{\partial x^2} \,\mathrm{d}x\,\mathrm{d}t$$
$$= -\int_0^T \int_{\mathbb{R}} \frac{k-1}{\nu} \varphi p_k \left[W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k) G^{(2)}(p_k)\right] \mathrm{d}x\,\mathrm{d}t.$$

Due to the uniform bounds on the right-hand side, *cf.* Lemma 4.3, we may divide by $k-1$ to obtain

$$0 = \lim_{k\to\infty} \int_0^T \int_{\mathbb{R}} \varphi p_k \left[W_k - p_k + \nu r_k G^{(1)}(p_k) + \nu(1-r_k) G^{(2)}(p_k)\right] \mathrm{d}x\,\mathrm{d}t.$$

Note that, writing $n_k^{(1)} = n_k r_k$ and expressing n_k in terms of p_k, in a fashion similar to Eq. (15), we may readily pass to the limit in all of these terms due to the strong convergence of the pressure and the a priori bounds of Lemma 4.1. We thus obtain

$$0 = p_\infty \left(W_\infty - p_\infty + \nu n_\infty^{(1)} G^{(1)}(p_\infty) + \nu n_\infty^{(2)} G^{(2)}(p_\infty)\right),$$

in the weak sense, where $n_\infty^{(i)}$ satisfies

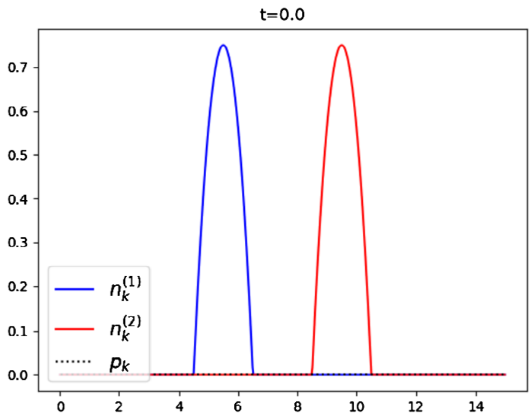
$$\frac{\partial n_\infty^{(i)}}{\partial t} - \frac{\partial}{\partial x}\left(n_\infty^{(i)} \frac{\partial W_\infty}{\partial x}\right) = n_\infty^{(i)} G^{(i)}(p_\infty),$$

for $i = 1, 2$. Indeed these equations follow by passing to the limit in the weak formulation of (2a), (2b)

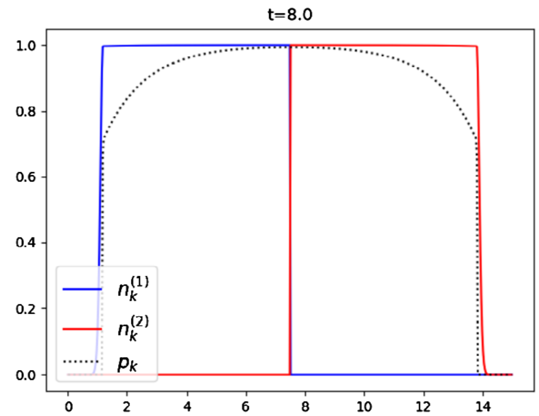
$$\int_0^T \int_{\mathbb{R}} \frac{\partial \varphi}{\partial t} n_k^{(i)} - \frac{\partial \varphi}{\partial x} n_k^{(i)} \frac{\partial W_k}{\partial x} \,\mathrm{d}x\,\mathrm{d}t = -\int_0^T \int_{\mathbb{R}} \varphi n_k^{(i)} G^{(i)}(p_k) \,\mathrm{d}x\,\mathrm{d}t,$$

where $\varphi \in C_c^1((0,T)\times\mathbb{R})$. □

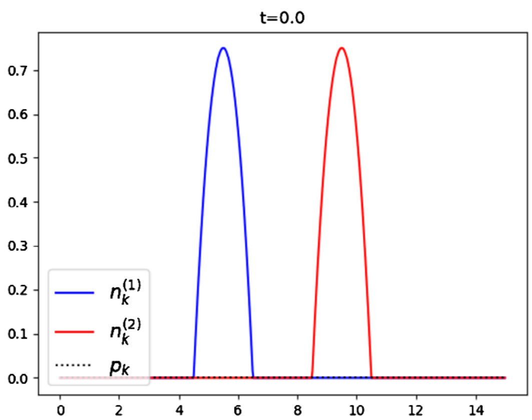
Remark 6.1 In fact, using the strategy of the previous section, *i.e.*, the BV-bounds in space, in conjunction with a control on the time derivative obtained from bounding the right-hand side of the equation for the individual species, one can deduce also strong convergence of the sequence $(n_k)_k$. As a consequence, the limit functions n_∞, $n_\infty^{(i)}$ are of bounded variation in time and space.



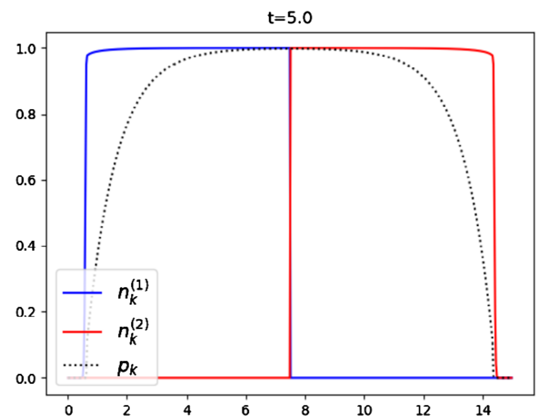
(a) Initial data, $\nu = 1$.



(b) Final time, $T = 8$.



(c) Initial data, $\nu = 0.01$.



(d) Final time, $T = 5$.

Fig. 1 We run the simulation for the same initial data for two different values of ν, *i.e.*, $\nu = 1$ in the upper row and $\nu = 0.01$ in the bottom row. In both cases, we chose $k = 100$ since we are interested in the limiting behaviour. The individual species are represented by solid lines in red and blue, the pressure is superimposed as a black dotted line. In the upper row the pressure drops to zero immediately, whereas in the bottom row we can see an almost smooth transition

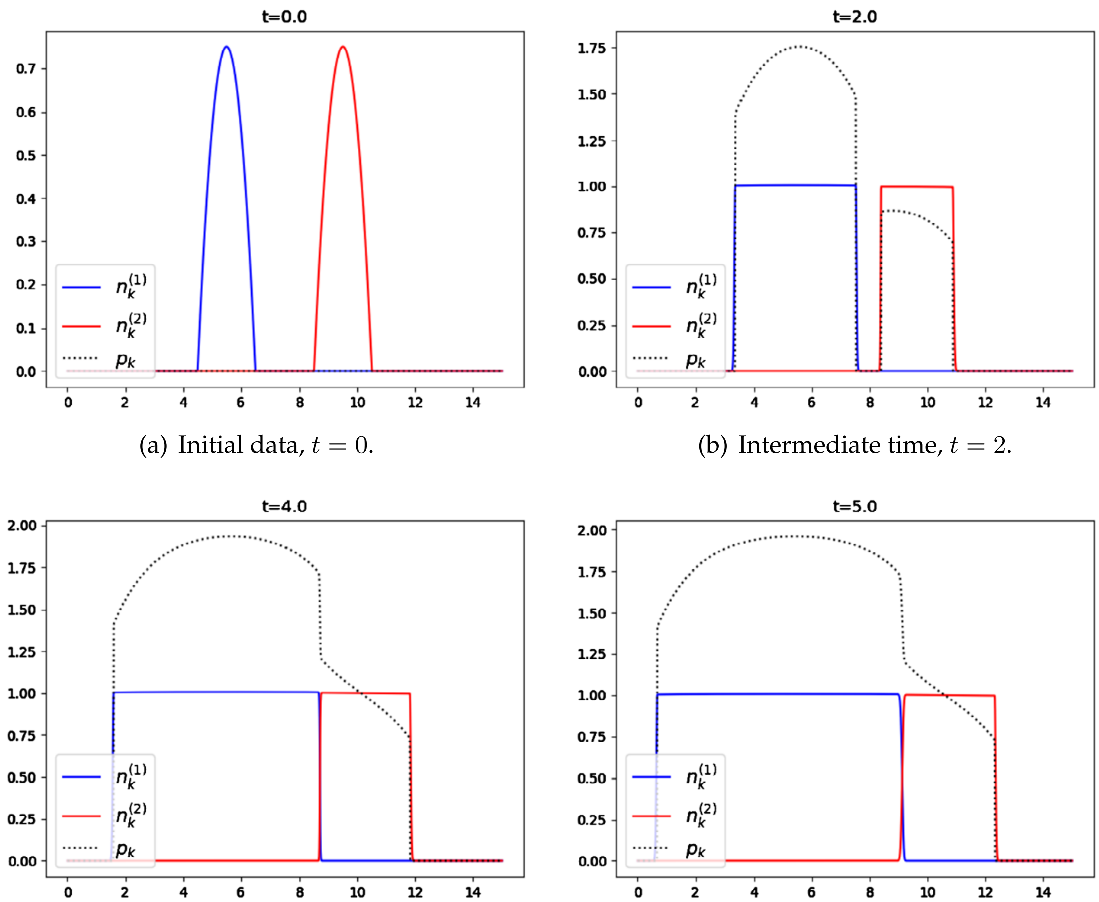
7 Numerical Investigations

In this section, we revisit the results from the preceding sections and showcase certain properties of the system. The numerical simulations are performed using the positivity-preserving upwind finite volume scheme proposed for a system of two interacting species in [9, 10] where the reaction terms are computed on each finite volume cell as simple ODEs. The implementation hinges on the fact that the elliptic Brinkman law (2b) can be solved using the integral representation (7).

Figure 1 displays the role of the viscosity parameter, ν. The same initial data

$$n_{k,0}^{(1)}(x) = m(x - 4.5)(6.5 - x), \quad \text{and} \quad n_{k,0}^{(2)}(x) = m(x - 8.5)(10.5 - x),$$

are used in both cases and $m > 0$ is chosen to normalise the initial mass to 1. In both cases we used $k = 100$, as we are interested in the incompressible regime. In addition, we chose $G^{(i)}(p) = 1 - p$, for $i = 1, 2$, corresponding to a homeostatic pressure of $p_M = 1$, *cf.* Eq. (6). In both cases we observe the propagation of segregation in agreement with Lemma 4.2. Moreover, we observe a drastic drop in the pressure in Fig. 1(b). This was already observed



(a) Initial data, $t = 0$.

(b) Intermediate time, $t = 2$.

(c) Intermediate time, $t = 4$.

(d) Final time, $T = 5$.

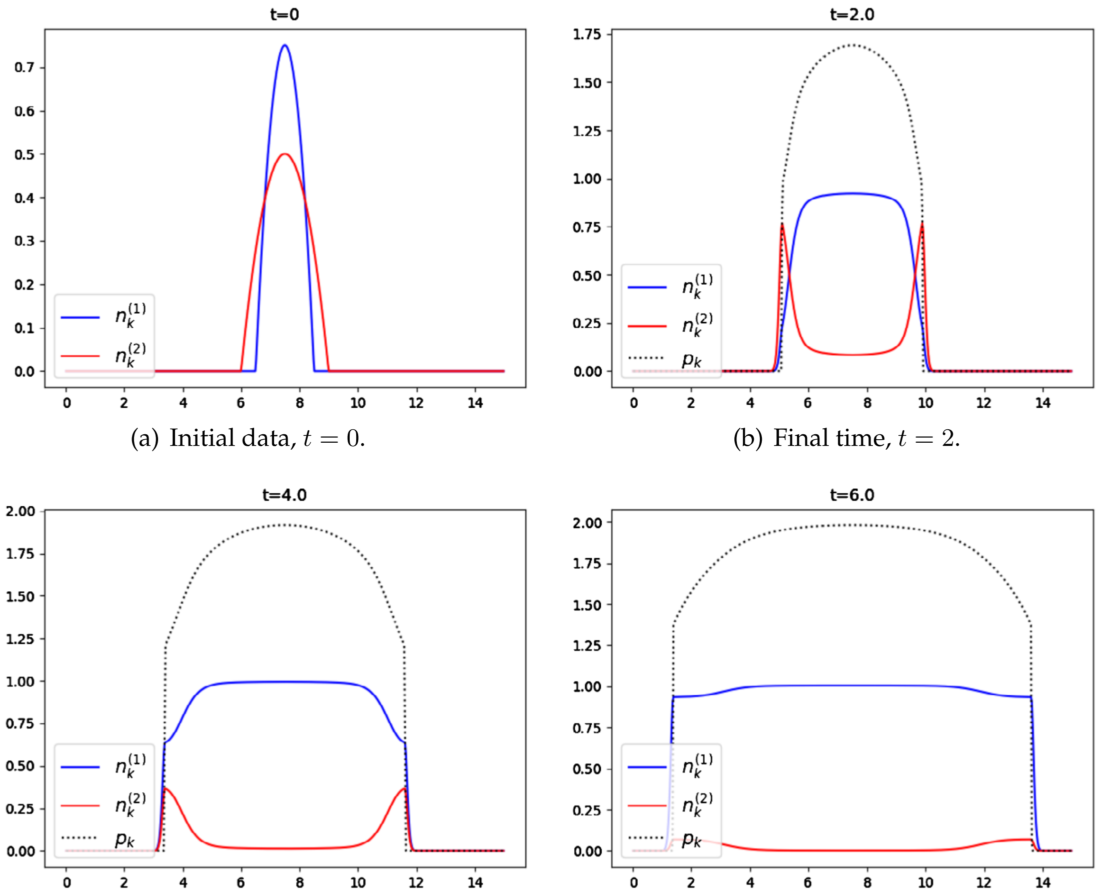
Fig. 2 We run the simulation for the same initial data for two different growth functions, $G^{(i)}(p)$. In both cases, we chose $k = 100$ and the individual species are represented by solid lines in red and blue, the pressure is superimposed as a black dotted line, as before. The pressure drops not only at the boundary of its support. We also observe jumps in internal layers

in the one species case, *cf.* [19], where the fact was exploited that the limiting pressure has an integral representation formula. In stark contrast, Fig. 1(d) shows an almost smooth transition of the pressure indicating a much higher regularity. This is in perfect alignment with the findings of [6], as the case $\nu = 0$ yields, at least formally, the system studied in the latter. As a matter of fact, the pressure gradient was shown to be square-integrable in the Darcy case, *i.e.*, $\nu = 0$. We conclude, by remarking that the front propagation is much faster in the regime of small ν, another fact that was already observed in the single-species case.

In Fig. 2 we present the effect of different growth terms of the tumour cells and healthy tissue. To be more precise, we choose the same initial condition as above but use

$$G^{(1)}(p) = 2 - p, \quad \text{and} \quad G^{(2)}(p) = 1 - p,$$

as growth terms for the two species. We see that the first species, $n_k^{(1)}$, proliferates much faster compared to the second one. More interestingly, we see that the pressure not only has a jump at the boundaries of the support of the total population, but also at the internal layer.



(a) Initial data, $t = 0$. (b) Final time, $t = 2$.

(c) Initial data, $t = 4$ (d) Final time, $T = 6$.

Fig. 3 The simulation shows the invasion of abnormal cells surrounded by healthy tissue. As time evolves, the tumour spreads and the density of normal cells is diminished and nearly vanishes, *cf.* Figure 3(d). As before, $\nu = 1$ and $k = 100$

Figure 3 shows the evolution of system (2a), (2b) for initial data representing a regime where healthy tissue has already been intruded by cancerous cells, *i.e.*,

$$n_{k,0}^{(1)}(x) = m(x - 6.5)(8.5 - x), \quad \text{and} \quad n_{k,0}^{(2)}(x) = m(x - 6)(9 - x),$$

where, again, $m > 0$ normalises the mass. In addition, we choose the same unequal growth functions, $G^{(i)}$, as before, thus promoting the tumour growth compared to the normal tissue.

8 Conclusions

The goal of the paper was twofold. We extended an established tumour growth model to an interaction system of two cell populations, *i.e.*, normal and abnormal cells. The interaction is given through the Brinkman flow, an elliptic equation that yields the velocity potentials for each cell population. In the first part of this paper we proved the existence of solutions to the interaction system, *cf.* Theorem 2.1. Building upon this result, we passed to the "incompressible" limit in the pressure equation, Eq. (4), and obtained the limiting equation, also referred to as *complementarity relation*, *cf.* Theorem 2.2. This way we were able to derive a geometric model from the cell-density model we presented.

Note that both the existence result and the incompressible limit rely on strategies different from the ones adapted for related models (either in the parabolic two-species case when Brinkman's law is replaced by Darcy's law ($\nu = 0$) or the one-species model with Brinkman flow). The results are complemented with a numerical investigation showcasing the segregation result, the discontinuities in the pressure and the two individual population densities which is why we do not expect better regularity than bounded variation.

In summary, this paper extends known results in the literature to two species. As far as the existence of solutions is concerned, no additional difficulties are expected in the multi-dimensional case. However, when it comes to the stiff limit not only our method fails but also the kinetic reformulation that was employed in the one-species case, *cf.* [19], would need a serious make-over that is, at this stage, far from clear — even in one dimension. New singularities appear at internal layers when the two species meet and it seems that different tools are required, such as the extension of the kinetic reformulation to systems, which, to our knowledge, does not exist. The exploration of such a technique is left for future works.

In addition, the rigorous inviscid limit, $\nu \to 0$, remains an open question that is left for future work.

Acknowledgements The authors are grateful to Benoît Perthame and Nicolas Vauchelet for suggesting this problem and delightful discussions. This work was completed while T.D. was a visitor at Laboratoire Jacques-Louis Lions, whose kind hospitality he appreciates. T.D. recognises the support of the Polish National Agency for Academic Exchange (NAWA) and National Science Center (Poland), grant no 2018/30/M/ST1/00423. M.S. fondly acknowledges the support of the Fondation Sciences Mathématiques de Paris (FSMP) for the postdoctoral fellowship.

Appendix

For the readers' convenience we shall recall here the compactness method invoked in [15] in the context of the fast reaction limit in a cross-diffusion system with growth and death processes. Roughly speaking, it allows to identify the limit of the composition of a uniformly compact nonlinear function and a weakly convergent sequence.

Lemma 8.1 ("Lemma A") *Let $\Omega \subset \mathbb{R}$ be a compact domain and set $Q_T := (0,T) \times \Omega$. Furthermore, let $\{u_n\} \subset L^\infty(Q_T)$ and $\{f_n\} \subset C(\mathbb{R})$ be sequences with the properties*

(i) *$u_n \rightharpoonup u$, weakly in $L^2(Q_T)$,*
(ii) *f_n is nondecreasing,*
(iii) *$f_n \to f$, uniformly on compact subsets of $\mathbb{R}$, and*
(iv) *$f_n(u_n) \to \chi$, strongly in $L^2(Q_T)$.*

Then

$$\chi = f(u).$$

For the sake of exposition, we recall here that the assumptions of the above lemma are indeed met in our case.

Remark 8.2 (The assumptions are met) The first assumption is the easiest to check as it follows directly from the uniform L^∞-bounds on the pressure. Similarly, it is readily verified that each element of the sequence of functions, in our context given by $\phi_k(x) = x^{k/k-1}$, is indeed nondecreasing. Moreover, the uniform convergence towards the identity is straightforward. Thus the only requirement that needs a more minute argument is (iv) which we present in the first part of the proof of Lemma 5.3.

References

1. Allaire, G.: Homogenization of the Navier-Stokes equations and derivation of Brinkman's law. In: Mathématiques appliquées aux sciences de l'ingénieur, Santiago, 1989, pp. 7–20. Cépaduès, Toulouse (1991)
2. Bertsch, M., Gurtin, M.E., Hilhorst, D.: On a degenerate diffusion equation of the form $c(z)_t = \varphi(z_x)_x$ with application to population dynamics. J. Differ. Equ. **67**(1), 56–89 (1987)
3. Bertsch, M., Gurtin, M.E., Hilhorst, D.: On interacting populations that disperse to avoid crowding: the case of equal dispersal velocities. Nonlinear Anal., Theory Methods Appl. **11**(4), 493–499 (1987)
4. Bertsch, M., Gurtin, M.E., Hilhorst, D., Peletier, L.A.: On interacting populations that disperse to avoid crowding: preservation of segregation. J. Math. Biol. **23**(1), 1–13 (1985)
5. Bertsch, M., Hilhorst, D.A., Izuhara, H., Mimura, M.: A nonlinear parabolic-hyperbolic system for contact inhibition of cell-growth. Differ. Equ. Appl. **4**(1), 137–157 (2012)
6. Bubba, F., Perthame, B., Pouchol, C., Schmidtchen, M.: Hele-Shaw limit for a system of two reaction-(cross-)diffusion equations for living tissues. Arch. Ration. Mech. Anal. (2019)
7. Byrne, H., Drasdo, D.: Individual-based and continuum models of growing cell populations: a comparison. J. Math. Biol. **58**(4–5), 657–687 (2009)
8. Carrillo, J.A., Fagioli, S., Santambrogio, F., Schmidtchen, M.: Splitting schemes and segregation in reaction cross-diffusion systems. SIAM J. Math. Anal. **50**(5), 5695–5718 (2018)
9. Carrillo, J.A., Filbet, F., Schmidtchen, M.: Convergence of a Finite Volume Scheme for a System of Interacting Species with Cross-Diffusion. ArXiv e-prints (2018)
10. Carrillo, J.A., Huang, Y., Schmidtchen, M.: Zoology of a nonlocal cross-diffusion model for two species. SIAM J. Appl. Math. **78**(2), 1078–1104 (2018)
11. Chertock, A., Degond, P., Hecht, S., Vincent, J.-P.: Incompressible limit of a continuum model of tissue growth with segregation for two cell populations. ArXiv e-prints (2018)
12. Degond, P., Hecht, S., Vauchelet, N.: Incompressible limit of a continuum model of tissue growth for two cell populations. Netw. Heterog. Media **15**(1), 57–85 (2020)
13. Gwiazda, P., Perthame, B., Świerczewska-Gwiazda, A.: A two-species hyperbolic–parabolic model of tissue growth. Commun. Partial Differ. Equ. **44**(12), 1605–1618 (2019)
14. Hecht, S., Vauchelet, N.: Incompressible limit of a mechanical model for tissue growth with non-overlapping constraint. Commun. Math. Sci. **15**(7), 1913 (2017)
15. Hilhorst, D., van der Hout, R., Peletier, L.A.: Nonlinear diffusion in the presence of fast reaction. Nonlinear Anal., Theory Methods Appl. **41**(5–6), 803–823 (2000)
16. Mellet, A., Perthame, B., Quirós, F.: A Hele-Shaw problem for tumor growth. J. Funct. Anal. **273**(10), 3061–3093 (2017)
17. Perthame, B., Quirós, F., Tang, M., Vauchelet, N.: Derivation of a Hele-Shaw type system from a cell model with active motion. Interfaces Free Bound. **16**, 489–508 (2014)
18. Perthame, B., Quirós, F., Vázquez, J.L.: The Hele-Shaw asymptotics for mechanical models of tumor growth. Arch. Ration. Mech. Anal. **212**(1), 93–127 (2014)
19. Perthame, B., Vauchelet, N.: Incompressible limit of a mechanical model of tumour growth with viscosity. Philos. Trans. R. Soc. A **373**, 20140283, 16 (2050). 2015
20. Ranft, J., Basan, M., Elgeti, J., Joanny, J.-F., Prost, J., Jülicher, F.: Fluidization of tissues by cell division and apoptosis. Proc. Natl. Acad. Sci. **107**(49), 20863–20868 (2010)

References

[1] I. Akramov, T. Dębiec, J. Skipper, and E. Wiedemann. Energy conservation for the compressible Euler and Navier-Stokes equations with vacuum. *Anal. PDE*, 13(3):789–811, 2020.

[2] D. Bresch and P.-E. Jabin. Global existence of weak solutions for compressible Navier-Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. *Ann. of Math. (2)*, 188(2):577–684, 2018.

[3] F. Bubba, B. Perthame, C. Pouchol, and M. Schmidtchen. Hele–shaw limit for a system of two reaction-(cross-)diffusion equations for living tissues. *Arch. Ration. Mech. Anal.*, 236(2):735–766, 2020.

[4] J. A. Carrillo, S. Fagioli, F. Santambrogio, and M. Schmidtchen. Splitting schemes and segregation in reaction cross-diffusion systems. *SIAM J. Math. Anal.*, 50(5):5695–5718, 2018.

[5] J. A. Carrillo, R.M. Colombo, P. Gwiazda, and A. Ulikowska. Structured populations, cell growth and measure valued balance laws. *J. Differ. Equ.*, 252(4):3245–3277, 2012.

[6] A. Chertock, P. Degond, S. Hecht, and J.-P. Vincent. Incompressible limit of a continuum model of tissue growth with segregation for two cell populations. *Math. Biosci. Eng.*, 16(5):5804–5835, 2019.

[7] P. Constantin, W. E, and E. S. Titi. Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Commun. Math. Phys.*, 165(1):207–209, 1994.

[8] P. Degond, S. Hecht, and N. Vauchelet. Incompressible limit of a continuum model of tissue growth for two cell populations. *Netw. Heterog. Media*, 15(1):57–85, 2020.

[9] T. Dębiec. On entropy conservation for general systems of conservation laws. *ArXiv preprint, https://arxiv.org/abs/1910.05793, submitted*, Oct. 2019.

[10] T. Dębiec and M. Schmidtchen. Incompressible limit for a two-species tumour model with coupling through Brinkman's law in one dimension. *Acta Appl. Math.*, to appear, 2020, https://doi.org/10.1007/s10440-020-00313-1.

[11] T. Dębiec, M. Doumic, P. Gwiazda, and E. Wiedemann. Relative Entropy Method for Measure Solutions of the Growth-Fragmentation Equation. *SIAM J. Math. Anal.*, 50(6):5811–5824, 2018.

[12] T. Dębiec, P. Gwiazda, A. Świerczewska-Gwiazda, and A. Tzavaras. Conservation of energy for the Euler-Korteweg equations. *Calc. Var. Partial Differ. Equ*, 57(6):Art. 160, 2018.

[13] T. Dębiec, B. Perthame, M. Schmidtchen, and N. Vauchelet. Incompressible limit for a two-species model with coupling through Brinkman's law in any dimension. https://hal.archives-ouvertes.fr/hal-02461406, January 2020.

[14] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[15] D. Donatelli, E. Feireisl, and P. Marcati. Well/ill posedness for the Euler-Korteweg-Poisson system and related problems. *Commun. Partial. Differ. Equ.*, 40(7):1314–1335, 2015.

[16] J. E. Dunn and J. Serrin. On the thermomechanics of interstitial working. *Arch. Ration. Mech. Anal.*, 88(2):95–133, 1985.

[17] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda, and E. Wiedemann. Regularity and energy conservation for the compressible Euler equations. *Arch. Ration. Mech. Anal.*, 223(3):1375–1395, 2017.

[18] U. S. Fjordholm and E. Wiedemann. Statistical solutions and Onsager's conjecture. *Phys. D*, 376/377:259–265, 2018.

[19] P. Gwiazda and E. Wiedemann. Generalized entropy method for the renewal equation with measure data. *Commun. Math. Sci.*, 15(2):577–586, 2016.

[20] P. Gwiazda, M. Michálek, and A. Świerczewska-Gwiazda. A note on weak solutions of conservation laws and energy/entropy conservation. *Arch. Ration. Mech. Anal.*, 229(3):1223–1238, 2018.

[21] P. Gwiazda, B. Perthame, and A. Świerczewska-Gwiazda. A two-species hyperbolic–parabolic model of tissue growth. *Commun. Partial. Differ. Equ.*, 44(12):1605–1618, 2019.

[22] S. Hecht and N. Vauchelet. Incompressible limit of a mechanical model for tissue growth with non-overlapping constraint. *Commun. Math. Sci.*, 15(7):1913, 2017.

[23] D. Hilhorst, R. van der Hout, and L. A. Peletier. Nonlinear diffusion in the presence of fast reaction. *Nonlinear Anal.*, 41(5-6, Ser. A: Theory Methods):803–823, 2000.

[24] P. Isett. A proof of Onsager's conjecture. *Ann. of Math. (2)*, 188(3):871–963, 2018.

[25] P. Michel, S. Mischler, and B. Perthame. General entropy equations for structured population models and scattering. *C. R. Math. Acad. Sci. Paris*, 338(9):697–702, 2004.

[26] P. Michel, S. Mischler, and B. Perthame. General relative entropy inequality: an illustration on growth models. *J. Math. Pures Appl. (9)*, 84(9):1235–1260, 2005.

[27] S. Mischler, B. Perthame, and L. Ryzhik. Stability in a nonlinear population maturation model. *Math. Models Methods Appl. Sci.*, 12(12):1751–1772, 2002.

[28] L. Onsager. Statistical hydrodynamics. *Nuovo Cimento (9)*, 6(Supplemento, 2 (Convegno Internazionale di Meccanica Statistica)):279–287, 1949.

[29] B. Perthame. *Transport equations in biology.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2007.

[30] B. Perthame and N. Vauchelet. Incompressible limit of a mechanical model of tumour growth with viscosity. *Philos. Trans. Roy. Soc. A*, 373(2050):20140283, 16, 2015.

[31] B. Perthame, F. Quirós, M. Tang, and N. Vauchelet. Derivation of a hele-shaw type system from a cell model with active motion. *Interfaces Free Boundaries*, 16:489–508, 2014.

[32] G. F. Webb. *Theory of nonlinear age-dependent population dynamics*. Marcel Dekker, Inc., 1985.