University of Warsaw — Faculty of Mathematics, Informatics and Mechanics

Khovanov-Rozansky $\mathfrak{sl}_N$-homology for periodic links

PhD Dissertation

submitted by

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written under the supervision of

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Author's declaration:
I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

September 4, 2025 ....
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Supervisors' declaration:
The dissertation is ready to be reviewed.

September 4, 2025 ....
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....
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Abstract

The main goal of this dissertation is to construct equivariant $\mathfrak{sl}_N$ homology for periodic links. For this purpose, we use the approach to $\mathfrak{sl}_N$ homology via webs and foams. The rotation action of $\mathbb{Z}_m$ on webs and foams allows us to define equivariant Khovanov-Rozansky homology for periodic links.

Following this definition, we deal with Reshetikhin-Turaev polynomials for the newly constructed equivariant homology via the newly defined difference polynomials.

In the end, we provide a periodicity criterion originating from equivariant Khovanov-Rozansky $\mathfrak{sl}_N$ homology.

1. Introduction

Let $L \subset S^3$ be a link. For $m \geq 2$, we say that L is *m-periodic* if it is invariant under a semi-free $\mathbb{Z}_m$-action on S^3 and L is disjoint from the fixed point set. For a periodic link, we have a question: how is the symmetry of the link reflected in link invariants? As an example, we have the Murasugi formula [21] recalled in Theorem 3.6. Besides giving a useful periodicity criterion, it also establishes the relation between the Alexander polynomial of L and the Alexander polynomial of its quotient knot $\bar{L}$ with respect to $\mathbb{Z}_m$ rotation action.

Equivariant Khovanov homology for periodic links was defined in [24]. The group action on S^3 induces a well-defined group action on the Khovanov homology modules $\mathrm{Kh}(L;R)$. The $\mathfrak{sl}_N$-homology for links was introduced in [12, 13] by Khovanov and Rozansky as a generalization of Khovanov homology. The first method to construct $\mathfrak{sl}_N$-homology was matrix via factorization. Over the years, other methods were constructed, see [7, 26, 28]. In this thesis, the combinatorial definition approach sketched in Section 4 turns out to be well-suited for studying periodic links. Basically, in this approach, for any link diagram D we define a cochain complex $[\![D]\!]$ living in a suitably defined foam category. To get $\mathfrak{sl}_N$ homology, we pass to the category of $\mathbb{S}_N$-modules where $\mathbb{S}_N = \mathrm{Sym}(X_1, \ldots, X_N)$ denotes the ring of symmetric polynomials in $X_1, \ldots, X_N$ over $\mathbb{C}$. For this, we need the evaluation functor $\mathcal{F}$ which takes webs and sends them to $\mathbb{S}_N$-modules. The goal of this thesis is to generalize the result of [4, 24] in the case of $\mathfrak{sl}_N$-homology. We show that the action of the symmetry group $\mathbb{Z}_m$ of the periodic link induces a $\mathbb{Z}_m$ action on its $\mathbb{S}_N$-equivariant $\mathfrak{sl}_N$-homology. Precisely, we have the following theorems which help us to define $\mathbb{Z}_m$-equivariant $\mathfrak{sl}_N$-homology.

Theorem (see Proposition 5.22). Suppose D is a periodic link diagram. Then, there is an action of $\mathbb{Z}_m$ on $[\![D]\!]$ induced by rotating the resolution diagrams of D.

We note that Proposition 5.22 is stated and proved for *labelled* links diagrams, that is, for link diagrams that come with an assignment of an integer between 0 and N to every component. We return to labelling in Subsection 4.5. A classical link, unlabelled, can be viewed as a link whose all labels are 1.

By using the evaluation functor $\mathcal{F}$, we obtain a chain complex of $\mathbb{S}_N$-modules $\mathcal{F}([\![D]\!])$. By Proposition 5.26, $\mathcal{F}$ commutes with the $\mathbb{Z}_m$ action. The $\mathbb{Z}_m$ action on $[\![D]\!]$ gives a $\mathbb{S}_N[\mathbb{Z}_m]$-module structure on the modules of chain complex $\mathcal{F}([\![D]\!])$. We prove the following result.

Theorem (see Theorem 5.28). Suppose L is a $\mathbb{Z}_m$-periodic link and D and D' are $\mathbb{Z}_m$-equivalent $m-$periodic link diagrams of L then we have an induced quasi isomorphism between $\mathcal{F}([\![D]\!])$ and $\mathcal{F}([\![D']\!])$ in the $\mathrm{Kom}(\mathrm{Sym}_N[\mathbb{Z}_m])$ where $\mathrm{Kom}(\mathrm{Sym}_N[\mathbb{Z}_m])$ is the category of bounded chain complexes in Sym_N with $\mathbb{Z}_m$ action on chain complex.

Theorem 5.28 is stated and proved only for links whose labels are equal to 1, that is, for usual links. Next, we establish a skein spectral sequence for a change of an orbit of crossings in $\mathfrak{sl}_N$-homology. An analogous skein spectral sequence was considered in [24] for the Khovanov homology of a periodic link. The skein spectral sequence gives a relation between the so-called difference $\mathfrak{sl}_N$-polynomials after a change of an orbit of crossings. Refer to Section 8 for details.

The graded Euler characteristic of the Khovanov homology is the Jones polynomial. In the presence of a $\mathbb{Z}_{p^\ell}$-action (with p prime) there is a refinement of the Jones polynomial, called the difference Jones polynomials see [23]. They essentially appear as the graded Euler characteristic associated with the eigenspaces of the action of $\mathbb{Z}_{p^\ell}$ on the Khovanov homology.

Similarly, the Euler characteristic of $\mathfrak{sl}_N$-homology gives a well-known polynomial, the Reshetikin-Turaev polynomial, also known as the $\mathfrak{sl}_N$-polynomial. For a periodic link, we define analogs of difference Jones polynomials in $\mathfrak{sl}_N$-homology. We call them *difference $\mathfrak{sl}_N$-polynomials.* We use the skein spectral sequence to study these polynomials for the link and its mirror. Moreover, we show that if a link where all labels are equal to 1, is p^l-periodic, then the Poincaré polynomial of its $\mathfrak{sl}_N$-homology admits a decomposition into a sum of polynomials with non-negative coefficients and satisfying specific congruence relations; see Theorem 8.17. The new periodicity criterion cannot distinguish 3 and 4 periodic links.

The thesis is an expanded version of the paper [5] joint with Maciej Borodzik and Wojciech Politarczyk.

2. Khovanov homology

In this chapter, we define Khovanov homology. To define it, we first introduce some basic concepts from knot theory and some concepts from homological algebra.

2.1. **Short introduction to knot theory.**

2.1.1. *Introduction.* This subsection is based on [31] and [1]. To understand the definition of the Khovanov homology, we need some basic definitions and facts about knots and links.

Definition 2.1. A *knot* is an embedding of a circle S^1 in the 3-dimensional Euclidean space or in the 3-dimensional sphere S^3.

If we embed more than one circle, we call the image a *link*. Generally, we are interested in regular projections of knots (links) onto a 2-dimensional Euclidean subspace, meaning that the projection is injective everywhere except at finitely many points, called the crossing points, where the knot projection crosses itself once. We will call the projection diagram where we have an over-strand and under-strand a knot (link) diagram.

Example 2.2. We have some well-known knot diagrams below

Right-handed trefoil   Left-handed trefoil   Figure eight knot

A link can be given an orientation. For these intersections of over-strand and under-strand, we have a specific name. We call these intersections positive crossing and negative crossing. Changing the orientation of one component of a link, might affect positivity of the crossings; however if we change the orientation of every component of the link, the positivity of all crossings is preserved. We will denote n_+ for the total number of positive crossings and n_- for the total number of negative crossings in a diagram.

positive crossing   negative crossing

For these two crossing we have 0 and 1 resolution of crossings. For crossing ⤫ we have 0 resolution ≍ and for 1 resolution we have)(. Furthermore, if we change under and over strand we swap the 0- and the 1-resolutions.

Definition 2.3. The writhe $\omega(D)$ of a diagram D of an oriented knot or an oriented link is the difference between the numbers of positive and negative crossings, i.e.,

$$\omega(D) = n_+ - n_-$$

Definition 2.4. The reverse rK of an oriented knot K is simply the same knot with the opposite orientation.

Change all crossing points from positive to negative and from negative to positive crossing. The final diagram will be called the mirror image $m(K)$ of a knot K.

Definition 2.5. The mirror image of a knot diagram is a diagram which is obtained by reflecting the knot diagram with respect to a line $\mathbb{R}$ in the plane.

We consider the following equivalence relation between knots. It applies also for links.

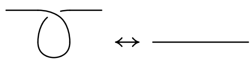
Definition 2.6. Two knots K_1 and K_2 are ambient isotopic if there is a smooth map $F : S^3 \times [0,1] \to S^3$ such that $F_x = F_{|S^3\times\{x\}}$ is a diffeomorphism for each $x \in [0,1]$, $F_{|S^3\times 0} = \mathrm{id}_{S^3}$, and $F_{|S^3\times 1}(K_1) = K_2$.

We want to understand if two knots are isotopic. The best way to understand this is by studying knot diagrams. We have an important theorem about equivalence in knot diagrams, but before this theorem, we need some definitions.

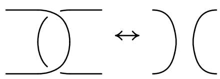
Definition 2.7. An isotopy of a knot projection is a continuous deformation of the knot diagram within the plane that preserves the number and type of crossings.

Definition 2.8. There are three local moves that are called Reidemeister moves for knot diagram equivalence.

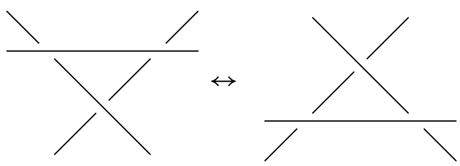
First Reidemeister move:



Second Reidemeister move:



Third Reidemeister move:



The following result was first proved by Reidemeister.

Theorem 2.9. *Two links are ambiently isotopic if and only if their diagrams are related by a finite number of Reidemeister moves and planar isotopies.*

A knot invariant is a property of a knot diagram that does not change under Reidemeister moves. For example, the writhe depends on the knot diagram, so it is not a knot invariant. A knot invariant only depends on the knot. Later, we will define the Jones polynomial and Khovanov homology. We will see that these are knot invariants.

2.2. **Jones Polynomial.** In this section, we will define the Jones polynomial. The Jones polynomial will be important for Khovanov homology. The definition of the Jones polynomial and its relation to Khovanov homology will be crucial to understanding concepts discussed in the following sections. We will start with the definition of the Kauffman bracket.

Definition 2.10. (see [1]) The Kauffman bracket is a function from the set of unoriented link diagrams in the plane to the ring of Laurent polynomials in variable q with integer coefficients. We denote by $\langle D \rangle \in \mathbb{Z}[q, q^{-1}]$ the Kauffman bracket of D. The Kauffman bracket is determined by the following three properties:

(1) $\langle \varnothing \rangle = 1$
(2) $\langle D \sqcup \bigcirc \rangle = (q^{-1} + q)\langle D \rangle$
(3) $\langle \times \rangle = \langle \asymp \rangle - q\langle)(\rangle$

where D is an unoriented diagram, $\varnothing$ is an empty diagram, and $\langle D \rangle$ is a Laurent polynomial.

The Kauffman bracket is invariant under RII and RIII moves. To make this definition invariant for the diagram D of an oriented link L under the Reidemeister 1 move, we have to multiply $\langle D \rangle$ by $(-1)^{n_-} q^{n_+ - 2n_-}$ where n_+ is the number of positive crossings and n_- is the number of negative crossings. The resulting polynomial is a knot invariant.

Definition 2.11. (see [1]) The unnormalized Jones polynomial of an oriented link L is defined as

$$\hat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle,$$

where D is a diagram of L.

In addition, we define the normalized Jones polynomial

$$J(L) = \hat{J}(D)(q + q^{-1})^{-1}.$$

We generally use the unnormalized version in this paper. We assign numbers to each crossing by $1, \dots, n$. By applying 0 or 1 resolution to each crossing we get 2^n diagrams that we can index with the sequence which has 0 and 1. We call such a diagram a *smoothing*. With these 2^n smoothings D_α where $\alpha \in \{0,1\}^n$, we have an n-dimensional cube. When we resolve all crossings, we get a union of circles. To compute the unnormalized Jones polynomial, we replace each union of k-circles with a term $(-1)^{r_\alpha} q^{n_+ - 2n_- + r_\alpha} (q + q^{-1})^{k_\alpha}$.

$$J(L) = \sum_{\alpha \in \{0,1\}^n} (-1)^{r_\alpha} q^{n_+ - 2n_- + r_\alpha} (q + q^{-1})^{k_\alpha}$$
$$r_\alpha = \text{Number of 1s in } \alpha$$
$$k_\alpha = \text{Number of circles in the } D_\alpha$$

We will define Khovanov homology, but for that, we need some homological algebra.

2.3. **Introduction to Homological Algebra.** In this section, we use [32] for the most definitions for some basic concepts of homological algebra that will be important for us

Definition 2.12. A chain complex $(C_\bullet, d_\bullet)$ is a sequence of modules $\cdots, C_{-2}, C_{-1}, C_0, C_1, C_2, \cdots$ connected by homomorphisms $d_n : C_n \to C_{n-1}$ where $d_{n-1} \circ d_n = 0$. We call $(C'_\bullet, d_\bullet)$ a subcomplex of $(C_\bullet, d_\bullet)$, if C'_i is a submodule of C_i and $d_n(C'_n) \subset C'_{n-1}$.

Definition 2.13. A cochain complex is a dual notion to a chain complex, it is a $(C_\bullet, d_\bullet)$ sequence of modules $\cdots, C_{-2}, C_{-1}, C_0, C_1, C_2, \cdots$ connected by homomorphism $d_n : C_n \to C_{n+1}$ where $d_{n+1} \circ d_n = 0$.

We define maps between chain complexes.

Definition 2.14. Assume we have $(C_\bullet, d_\bullet)$ and $(C'_\bullet, d'_\bullet)$ chain complexes. A chain map $F : C_\bullet \to C'_\bullet$ is a sequence of maps $\{F_n : C_n \to C'_n\}$ such that $F_{n-1} \circ d_n = d'_n \circ F_n$. In the diagram, we see that as below

$$\begin{array}{ccc} C_n & \xrightarrow{d_n} & C_{n-1} \\ {\scriptstyle F_n}\downarrow & & \downarrow{\scriptstyle F_{n-1}} \\ C'_n & \xrightarrow{d'_n} & C'_{n-1} \end{array}$$

Maps between cochain complexes can be defined similarly.

Definition 2.15. Assume we have a chain complex $(C_\bullet, d_\bullet)$, the homology of this sequence is $\ker(d_n)/\operatorname{im}(d_{n+1})$ and denoted by $H_n(C_\bullet)$.

Similarly, we define cohomology.

Definition 2.16. Assume we have a cochain complex

$$\cdots \to C^{n-1} \xrightarrow{d^{n-1}} C^n \to \cdots$$

The cohomology of this sequence is $\ker(d^n)/\operatorname{im}(d^{n-1})$ and denoted by $H^i(C^\bullet)$.

Proposition 2.17. *A chain map $F : C_\bullet \to C'_\bullet$ induces a homomorphism between the homology groups of these two complexes.*

Between two chain homotopy maps, we have equivalence also.

Definition 2.18. Suppose we have chain maps f and g between $(C_\bullet, d_\bullet)$ and $(C'_\bullet, d'_\bullet)$. A chain homotopy ϕ between f and g is a sequence of morphisms $\phi_n : C_n \to C'_{n+1}$ such that $f_n - g_n = d'_{n+1} \circ \phi_n + \phi_{n-1} \circ d_n$. We call f and g chain-homotopic chain maps and denote this relation $f \simeq g$.

We can define equivalence between two chain complexes.

Definition 2.19. We say chain complexes A and B are homotopy equivalent if and only if we have chain maps $f : (A_\bullet, d_\bullet) \to (B_\bullet, d'_\bullet)$ and $g : (B_\bullet, d'_\bullet) \to (A_\bullet, d_\bullet)$ such that $f \circ g \simeq id_{B_\bullet}$ and $g \circ f \simeq id_{A_\bullet}$.

Chain maps induce homomorphisms between the homology groups of chain complexes. Do we have any relation between the induced maps f_* and g_* where chain maps are chain-homotopic? The next proposition shows us this relation.

Proposition 2.20. *If we have f and g chain-homotopic chain maps, their induced maps f_* and g_* are the same on homology groups (i.e., $f_* = g_*$).*

Definition 2.21. Suppose $M_1, M_2, \cdots, M_n$ are modules over the fixed ring R, and $P_1, P_2, \cdots, P_n$ are module homomorphisms. We say that

$$M_1 \xrightarrow{P_1} M_2 \xrightarrow{P_2} M_3 \cdots \xrightarrow{P_{n-1}} M_n$$

is an exact sequence if $\operatorname{im}(P_{n-1}) = \ker(P_n)$.

Definition 2.22. Suppose A, B, C are modules over the fixed ring R. We say that

$$0 \to A \xrightarrow{\text{i}} B \xrightarrow{\text{p}} C \to 0$$

is a short exact sequence if i is a monomorphism, p is an epimorphism, and $\operatorname{im}(i) = \ker(p)$.

Furthermore, we define a short exact sequence in the category of chain complexes.

Definition 2.23. Suppose A, B, C are chain complexes, and i and p are chain maps. We say that the sequence

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

is a short exact sequence if the induced sequence of maps

$$0 \to A_n \xrightarrow{i_n} B_n \xrightarrow{p_n} C_n \to 0$$

is a short exact sequence of modules.

Similarly, we define a long exact sequence for modules, and from the short exact sequence, we get a long exact sequence of homology groups.

Theorem 2.24. *Suppose A, B, C are chain complexes, and we have a short exact sequence of complexes given by:*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

then we obtain a long homology sequence of homology groups

$$\cdots H_n(A) \xrightarrow{i_*} H_n(B) \xrightarrow{j_*} H_n(C) \xrightarrow{\delta} H_{n-1}(A) \xrightarrow{i_*} H_{n-1}(B) \xrightarrow{j_*} H_{n-1}(C) \xrightarrow{\delta} \cdots$$

Proof. See [11, Theorem 2.16]. □

We have the same theory for cochain complexes

Theorem 2.25. *Suppose A,B and C are cochain complexes, and we have a short exact sequence of complexes given by*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$$

then we obtain a long cohomology sequence of cohomology groups

$$\cdots H^n(A) \xrightarrow{i_*} H^n(B) \xrightarrow{j_*} H^n(C) \xrightarrow{\delta} H^{n+1}(A) \xrightarrow{i_*} H^{n+1}(B) \xrightarrow{j_*} H^{n+1}(C) \xrightarrow{\delta} \cdots$$

Definition 2.26. (see 1.5.1 [32]) Assume we have E and F be graded cochain complexes and $E \xrightarrow{f} F$ a chain map that preserves gradings. The mapping cone is a chain complex given in a degree k by

$$\text{Cone}(f)_k = E_k \bigoplus F_{k-1}$$

with differential

$$\partial_{\text{Cone}(f)} = \begin{pmatrix} -\partial_E & 0 \\ f & \partial_F \end{pmatrix} : \text{Cone}(f)_k \to \text{Cone}(f)_{k+1}.$$

We have the following lemma.

Lemma 2.27. *We have a short exact sequence which includes* Cone(f)

$$0 \to F[1] \xrightarrow{i} \text{Cone}(f) \xrightarrow{p} E \to 0$$

where $F[1]_n = F_{n-1}$, $i(a) = (0, a)$ for $a \in F$ and $p(e', a') = -e'$, so we get a long exact sequence by Theorem 2.25

$$\cdots \to H^d(E) \xrightarrow{H(f)} H^d(F) \xrightarrow{i_*} H^d(\text{Cone}(f)) \xrightarrow{p_*} H^{d+1}(E) \to \cdots$$

Definition 2.28. Let C be an Abelian category. A homologically graded spectral sequence is a family of objects with differentials $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ which satisfy the rule $d^r \circ d^r = 0$ where $p, q, r \in \mathbb{Z}$. Moreover, for $E^{r+1}_{p,q}$ and $E^r_{p,q}$ for any r we have

$$E^{r+1}_{p,q} \equiv H(E^r_{p,q}) = \ker(d^r_{p,q})/\operatorname{Im}(d^r_{p-r,q+r-1})$$

For a fixed r, the family $E^r_{p,q}$ is called the page of the spectral sequence. Here we can think spectral sequences as a book. When we turn next page it means we increase r by 1 and take homology of the old page.

Definition 2.29. Let H_n be a collection of objects in category C.

- We say spectral sequence weakly converges to H_* if there is a filtration

$$\ldots \subseteq F_{p-1}H_n \subseteq F_pH_n \subseteq F_{p+1}H_n \subseteq \ldots \subseteq H_n$$

  and isomorphism

$$\beta_{pq} : E^\infty_{pq} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$$

- We say spectral sequence approaches to H_* if it weakly converges to H_* and

$$H_n = \bigcup F_pH_n \text{ and } \bigcap F_pH_n = 0$$

- We say sequence converges to H_* if it approaches to H_* and

$$H_n = \lim_{\leftarrow}(H_n/F_pH_n)$$

Convergence is denoted by $E^r_{pq} \implies H_{p+q}$

Definition 2.30. A first quadrant spectral sequence is a type of spectral sequence where all the information or data contained in its pages is confined or concentrated within the region of the (p,q)-plane where

$$p < 0 \text{ or } q < 0 \implies E_r^{p,q} = 0.$$

Proposition 2.31. *If the r-th page is confined to the first quadrant, then the (r+1)st page will also be so. Therefore, if the first one is, then all subsequent pages will be as well.*

Proposition 2.32. *For every first quadrant spectral sequence, convergence occurs at position (p,q) starting from the r-th term where r is greater than the maximum of p and $q+1$.*

$$E^{p,q}_{\max(p,q+1)+1} = E^{p,q}_\infty$$

Proposition 2.33. *If a first quadrant spectral sequence converges*

$$E_r^{p,q} \implies H^{p+q}$$

then each H^n has a filtration of length $n+1$

$$0 = F^{n+1}H^n \subset F^nH^n \subset \dots F^1H^n \subset F^0H^n = H^n$$

We also have

- $F^nH^n \simeq E^{n,0}_\infty$
- $H^n/F^1H^n \simeq E^{0,n}_\infty$

2.4. **Introduction to the Khovanov homology.** In this paper, our main goal is to define $\mathfrak{sl}_N$ homology via web and foams. For $n = 2$, $\mathfrak{sl}_N$ homology is called Khovanov homology. In this subsection, we will define Khovanov homology in a basic way that will help us to understand $\mathfrak{sl}_N$ homology. We need the Khovanov bracket definition to define Khovanov homology. The definition is similar to the Kaufmann bracket definition. In this section we generally use papers [1] and [31].

Definition 2.34. We say that the vector space V is a graded vector space, if V can be decomposed into the direct sum of the form $V = \oplus_{n\in\mathbb{N}}V_n$ where V_n is a vector space for any n. Elements of V_n are called homogeneous element of degree n.

Definition 2.35 (see [1, Definition 3.1])**.** The q dimension for this new vector space is

$$\mathrm{qdim}(V) := \sum_m q^m \dim(V_m)$$

Example 2.36. Suppose we have field $\mathbb{F}$, and we have graded vector space $\mathbb{F}_{-1} \oplus \mathbb{F}_1$, where the subscript denotes the grading of generators. Then $\mathrm{qdim}(\mathbb{F}_{-1} \oplus \mathbb{F}_1) = q + q^{-1}$.

In this section we use vector space $V = \langle v_+, v_- \rangle$ where $\deg v_+ = 1$ and $\deg v_+ = -1$. The $\mathrm{qdim}(V) = q + q^{-1}$.

Definition 2.37 (see [1])**.** Khovanov bracket of a diagram D of a link L, denoted $[\![D]\!]$, is a cochain complex of graded $\mathbb{Z}$-vector spaces. It is characterized by the following properties:

(1) $[\![\varnothing]\!] = 0 \to \mathbb{Z} \to 0$
(2) $[\![\bigcirc \sqcup D]\!] = V \otimes [\![D]\!]$
(3) $[\![\times]\!] = \mathrm{Cone}\left(0 \to [\![\asymp]\!] \xrightarrow{d} [\![)(]\!]\{1\} \to 0\right)$

Here, the $\{1\}$ operator is the degree shift operation $V\{l\}_m = V_{m-l}$.

The first axiom is about empty diagram, bracket sends empty diagram to cochain complex with 0 and $\mathbb{Z}$. The second axiom says that if we have diagram D which can be written as a disjoint sum of a circle and a diagram D', then to calculate $[\![D]\!]$ we need to calculate only $[\![D']\!]$. The third axiom gives a recipe how to find the Khovanov bracket of a general link diagram. If we have a link diagram D, the third axiom allows us to write

$$C^{i,*}(D) = C^{i,*}(D_0) + C^{i-1,*}(D_1)\{1\}$$

where D_0 and D_1 are the diagrams which we get them by resolving a fixed crossing by 0 and 1 respectively on the diagram D. In other words,the third axiom says that for a link diagram D, $C^{i,*}(D)$ is the mapping cone of $C^{i,*}(D_0)$ and $C^{i-1,*}(D_1)$ with the map d between $C^{i,*}(D_0)$ and $C^{i-1,*}(D_1)$, the map d will be defined in 2.42.

Now we define the modules that we use in the definition of Khovanov homology, see [31, Chapter 1.3]. We begin with the definition of the space V_α.

$$V_\alpha = V^{\otimes k_\alpha}\{r_\alpha + n_+ - 2n_-\},$$

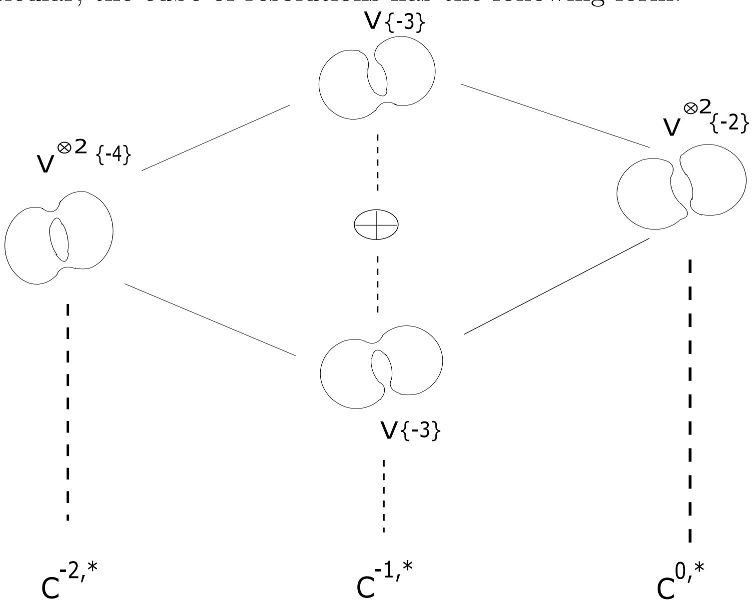
where $\alpha \in \{0,1\}^n$, and:

- k_α = the number of circles in the diagram D_α,
- r_α = the number of 1's in α,
- n_+ = number of positive crossings in L,
- n_- = number of negative crossings in L.

We define our module now.

$$C^{i,*}(D) = \bigoplus_{\substack{\alpha \in \{0,1\}^n \\ i = r_\alpha - n_-}} V_\alpha$$

Example 2.38 ([31, Figure 4])**.**) For the negative Hopf link . It is easy to see that $n_+ = 0$ and $n_- = 2$. In particular, the cube of resolutions has the following form:



2.5. **Definition of boundary map for Khovanov homology.** Have defined the modules underlying the Khovanov chain complex, we need to describe the boundary map. Consider a cube where nodes are diagrams which we get by different resolutions. There are edges between nodes. We define a map for the edge between two nodes which we get from different specific resolutions. We define the map d_ϵ where ϵ is the edge of our cube that lies between two resolutions that differ at one crossing. This edge can be labeled by sequences in $\{0,1,*\}$ where the height of the ϵ is denoted by $|\epsilon|$ and is defined by the number of '1' in the domain of the d_ϵ. We turn edges into arrows by the rule $* = 0$ gives the tail and $* = 1$ gives the head. For instance, the edge between resolutions 001 and 011 is $0 * 1$ and the map between them is d_{0*1}. Prior to defining d_ϵ, we need to describe some elementary maps, from which d_ϵ is constructed. It might be helpful to remind here that V is vector space which is generated by v_+ and v_- where $\deg(v_+) = 1$ and $\deg(v_-) = -1$.

First, we define a map m that corresponds to merging two circles to one circle. Namely:

Definition 2.39. The *multiplication map* $m: V \otimes V \to V$ is defined as:

$$v_+ \otimes v_+ \mapsto v_+$$
$$v_+ \otimes v_- \mapsto v_-$$
$$v_- \otimes v_+ \mapsto v_-$$
$$v_- \otimes v_- \mapsto 0.$$

We extend it linearly to $V \otimes V$.

In addition to that, we define a map corresponding to splitting one circle into two circles:

Definition 2.40. The *comultiplication map* $\Delta : V \to V \otimes V$ is defined as:

$$v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+$$
$$v_- \mapsto v_- \otimes v_-$$

and it can be extended linearly on V.

We define the map d_ϵ.

Definition 2.41. We define d_ϵ as the identity on the tensor factors associated to circles which stay the same after smoothing. If two circles merge into one circle, d_ϵ is the map m on tensor factors associated to these two circles, see Definition 2.39. Another case is when we divide one circle into two circles, d_ϵ is the linear map Δ on this circle, see Definition 2.42.

We are ready to define the Khovanov differentials $d^i : C^{i,*}(D) \to C^{i+1,*}(D)$

Definition 2.42. For $v \in V_\alpha \subset C^{i,*}(D)$

$$d^i(v) = \sum_{\substack{\epsilon \\ \text{tail}(\epsilon)=\alpha}} \text{sign}(\epsilon) d_\epsilon(v)$$

where $\text{sign}(\epsilon) = (-1)^{\text{number of 1's to the left of the change place}}$, see Chapter 1 of [31].

For example, suppose we have ϵ the edge between 010 and 011, then $\text{sign}(\epsilon)$ is -1 because there is just one 1 before the change from 0 to 1 in the edge.

It can be shown that m and Δ preserve the quantum grading, and since d_ϵ is the sum of them, we say that d_ϵ preserve the q-grading.

With this definition, we have a lemma below:

Lemma 2.43 (see [31])**.** $d^r \circ d^{r-1} = 0$.

The above lemma shows that d^i is indeed a boundary map.

We defined the chain complex, so we define Khovanov homology on this chain complex.

Definition 2.44 (see [31])**.** $\text{Kh}^{*,*}(D) = H(C^{*,*}(D), d)$ where Kh stands for Khovanov homology.

The graded Euler characteristic of $C^{i,*}(L)$ for a link diagram L is

$$\sum_i (-1)^i \,\text{qdim}(C^{i,*}(D))$$

This is equal to the unnormalized Jones polynomial of the knot diagram $\langle D \rangle$ of a link L. See [31].

In order to say that this definition gives a well-defined link invariant, we need to show that if we have two different diagrams D_1 and D_2 of the same link L, we have $H(D_1) \simeq H(D_2)$. In particular, we need to check if homology will be the same after we apply Reidemeister move to link diagram. (See [1, Theorem 2])

Theorem 2.45. *Assume we have two diagrams D_1 and D_2 which are connected to each other with a finite sequence of Reidemeister move, then $H(D_1) \simeq H(D_2)$.*

Theorem 2.45 has in fact three parts, each corresponding to a different Reidemeister move.

There exist pairs of links where they have the same Jones polynomials but have different Khovanov homologies. This shows us that Khovanov homology is a stronger invariant.

Example 2.46 ([31, Example 3.2])**.** Two knots 5_1 and 10_{132} are the knots with the same Jones polynomial but different Khovanov homology. For the unnormalized Jones polynomial, we have $\hat{J}(10_{132}) = \hat{J}(5_1) = q^{-3} + q^{-5} + q^{-7} + q^{-15}$ whereas we have different Khovanov homology.

$$\begin{aligned} \mathrm{Kh}_{\mathbb{Q}}(5_1) &= \mathbb{Q}_{(0,-3)} + \mathbb{Q}_{(0,-5)} + \mathbb{Q}_{(-2,-7)} + \mathbb{Q}_{(-3,-11)} + \mathbb{Q}_{(-4,-11)} + \mathbb{Q}_{(-5,-15)} \\ \mathrm{Kh}_{\mathbb{Q}}(10_{132}) &= \mathbb{Q}_{(0,-1)} + \mathbb{Q}_{(0,-3)} + (\mathbb{Q} \oplus \mathbb{Q})_{(-2,-5)} + \mathbb{Q}_{(-3,-5)} + \mathbb{Q}_{(-3,-9)} + \mathbb{Q}_{(-4,-7)} + \mathbb{Q}_{(-4,-9)} + \\ &\quad \mathbb{Q}_{(-5,-11)} + \mathbb{Q}_{(-6,-11)} + \mathbb{Q}_{(-7,-15)}, \end{aligned}$$

where $\mathbb{Q}_{i,j}$ means at the i and j th degree we have a copy of $\mathbb{Q}$.

Definition 2.47. (see [1]) From the Khovanov homology $\mathrm{Kh}^{i,j}(L)$, where i is the homological grading and j is the quantum grading, one defines the *Khovanov polynomial* as:

$$\mathrm{Kh}(L; t, q) = \sum_{i,j} \dim \mathrm{Kh}^{i,j}(L) \cdot q^j t^i,$$

which serves as a categorification of the Jones polynomial.

Remark 2.48. The unnormalized Jones polynomial is equal to the Khovanov polynomial where we have $t = -1$. In other words, We have the equation $\mathrm{Kh}(K, -1, q) = J(K)$.

After defining the Khovanov bracket and the Jones polynomial, we define Reshetikhin–Turaev (RT) polynomial, which is a generalization of the Jones polynomial. The RT polynomial is computed via a diagrammatic approach, which assigns a polynomial invariant to a link diagram. The construction begins by applying a set ofresolution rules to each crossing in a link diagram, reducing it to a linear combination of simpler diagrams without crossings. These resolutions are replacing a crossing with planar configurations according to specific local patterns, as illustrated in Figure 1. After all crossings are resolved, the resulting diagrams may contain loops, caps,cups

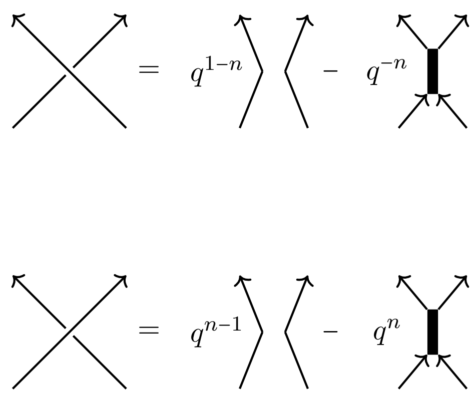


FIGURE 1. Resolution of positive and negative crossing points.

and strands labelled by positive integers. The rules to evaluate the diagram are defined in the Figure 2.

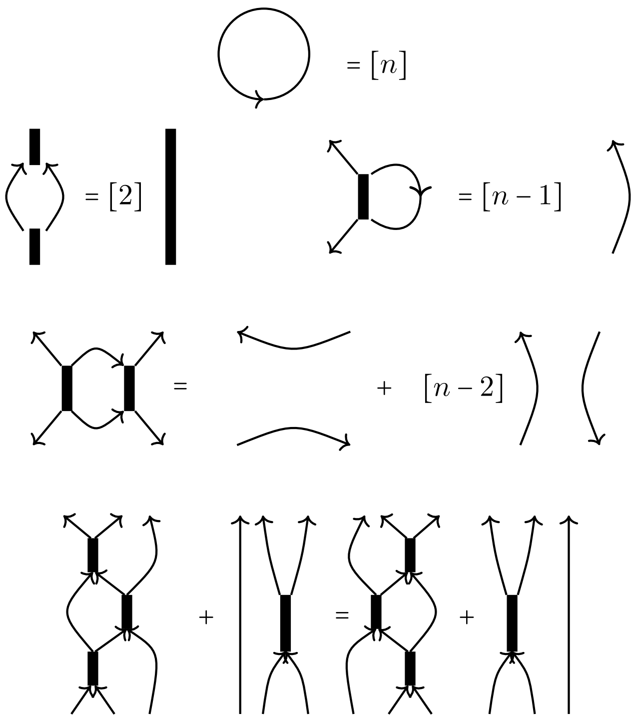


FIGURE 2. Evalution rules of loops, caps and cups

We have

$$[k] := \frac{q^k - q^{-k}}{q - q^{-1}}$$

With this resolving and evaluating, we obtain polynomial associated to the original link diagram. This invariant is known as the Reshetikhin–Turaev polynomial. The details of the planar graph calculus shown in Figure 2 are explained in [20]. We only make use of the part of their calculus involving edges labeled 1 and 2. In our diagrams, these labels are omitted, and edges labeled 2 are indicated by thick lines.

Definition 2.49. (Reshetikhin-Turaev polynomials) The polynomial invariant $P_n(L)$ of an oriented link L can be computed by selecting a planar diagram D of L and applying a resolution to each crossing according to the specified rules in Figure 1 and take a sum of all $P_n(\Gamma)$ for all resolutions Γ.

$$Pn(L) = P_n(D) := \sum_{\text{resolutions } \Gamma} q^{\alpha(\Gamma)} P_n(\Gamma),$$

where $\alpha(\Gamma)$ comes from the rules in Figure 1. From the equations in Figure 2, we can get that the definition is independent of the choice of the diagram D of a link L.

3. Periodic links

Definition 3.1. Consider a link L in S^3 and a semi-free $\mathbb{Z}_m$ action on S^3, that is to say, there is a non-empty fixed point set, such that the action is free on its complement. All the $\mathbb{Z}_m$ actions on S^3 are classified by the solution to the Smith conjecture. In particular, if the fixed point set of the action is one dimensional, it is an unknotted circle. We say L is m-periodic for the semi-free $\mathbb{Z}_m$ rotation action of order m on S^3, if the set of fixed points f of action is disjoint with L and L is invariant under the $\mathbb{Z}_m$ action.

Similarly, we define an action for a link diagram.

Definition 3.2. We say that the link diagram $D \subset \mathbb{R}^2$ of an m-periodic link L is m-periodic if it is invariant under the rotation action of $\mathbb{R}^2$ of order m, and D is disjoint from the set of fixed points of the action (that is, the coordinate center of $\mathbb{R}^2$). In other words, an m-periodic link diagram is a diagram that is carried to itself by a rotation of $(360/m)^\circ$ about the origin.

Every m-periodic link admits an m-periodic link diagram.

Example 3.3. The trefoil knot is a 3-periodic knot.

Remark 3.4. Smith's conjecture states that a fixed point set of $\mathbb{Z}_m$ on S^3 cannot be a nontrivial knot.

If we have a periodic knot K preserved under the $\mathbb{Z}_m$-action on S^3, we define the quotient knot of knot K under this action.

Definition 3.5. see [21] A quotient knot $\bar{K}$ is the image of the knot K under the quotient map:

$$\pi : S^3 \longrightarrow S^3/\mathbb{Z}_m$$

and denoted $\bar{K} := \pi(K)$. Since the action is semi-free, $\bar{K}$ is also an embedded circle in S^3, that is, $\bar{K}$ is a knot.

To check whether a link is periodic, one may apply one of the following criteria.

Theorem 3.6 (Murasugi Conditions, see [21])**.** *Suppose we have $K \subset S^3$ a $q = p^r$-periodic knot with prime p, Δ the Alexander polynomial of K, and Δ' the Alexander polynomial of the quotient knot $\bar{K}$. Furthermore, let l be the absolute value of the linking number of K with the symmetry axis. Then*

(1) $\Delta' | \Delta$
(2) $\Delta \equiv (\Delta')^q (1 + t + \ldots + t^{l-1})^{q-1} \pmod{p}$

Example 3.7. The left-handed trefoil knot has period 3; the quotient knot is the unknot, and the linking number l is 2.

- It is obvious that the first condition is satisfied, which means $1|\Delta$.
- The Alexander polynomial of the trefoil knot is $t^2 - t + 1$. So we have

$$(1)^3(1 + t^{2-1})^{3-1} = (1 + t)^2 \equiv t^2 - t + 1 \pmod{3}.$$

This means the second condition is satisfied.

Example 3.8. For the figure eight knot, the Alexander polynomial is $-t^{-1} + 3 - t$. Since $\Delta(t) = -t^{-1} + 3 - t$ is irreducible and since $\Delta'(1)|\Delta(1)$ we deduce $\Delta' = 1$.
We have

$$-t^{-1} + 3 - t = (1 + t + \ldots + t^{l-1})^{p-1} \pmod{3}.$$

We know that the Alexander polynomial is well-defined up to multiplication by powers of t. So we take Alexander polynomial here $\Delta(t) = -1 + 3t - t^2$. Hence the polynomial on the right-hand side should have the same degree with the polynomial on the left-hand side. Hence we should have $(l - 1)(p - 1) = 2.$. We have two cases. Either $l = 3, p = 2$ or $l = 2, p = 3$. For $l = 2, p = 3$ on the right-hand side. We have $(1 + t)^2 = 1 + 2t + t^2$ but

$$1 + 3t - t^2 \neq 1 + 2t + t^2 \pmod{3}.$$

On the other hand, we have

$$1 + 3t - t^2 \neq 1 + t + t^2 \pmod{2}.$$

This shows that figure eight knot is not p-periodic for $p \geq 3$.

Theorem 3.9 (Edmonds' Criterion, see [6]). *Assume we have K, a periodic knot of period q, and $\bar{K}$, the quotient knot of K. Then there are nonnegative integers σ such that*

$$g(K) = qg(\bar{K}) + \frac{(q-1)(\sigma-1)}{2}.$$

where $g(K)$ and $g(\bar{K})$ is the Seifert genus.

Example 3.10. For a trefoil knot K, $\bar{K}$ is the unknot. The trefoil knot has genus 1, and the unknot has genus 0. If we take $\sigma = 2$, then we have $1 = 3 \cdot 0 + 2 \cdot \frac{1}{2}$.

Edmonds' Criterion is particularly useful for low genus knots. It shows in particular that the quotient knot of a genus 1 periodic knot is the unknot.

Theorem 3.11 (Naik's Criterion, see [22]). *Suppose $K \subset S^3$ is a p-periodic knot with p a prime and let $k > 1$ whereas we denote $\bar{K}$ for quotient knot of K. For $\Sigma^m(K)$ the m-fold branched cover of K suppose that $H_1(\Sigma^m(K))$ has nontrivial q-torsion part, for some prime $q \neq p$, and let l_q to be the least positive integer such that $q^{l_q} \equiv \pm 1 \pmod p$. Then there exist non-negative integers $b_1, b_2, \dots$ such that*

$$H_1(\Sigma^m(K);\mathbb{Z})_q / H_1(\Sigma^m(\bar{K});\mathbb{Z})_q = \mathbb{Z}_q^{2b_1 l_q} \oplus \mathbb{Z}_{q^2}^{2b_2 l_q} \oplus \cdots.$$

Theorem 3.12 (HOMFLYPT Criterion, see [25]). *Assume we have the unital subring R in $\mathbb{Z}[a^{\pm}, z^{\pm}]$ where $R = \langle a, a^{-1}, \frac{a+a^{-1}}{z}, z\rangle$. If a knot is p-periodic and $P(a,z)$ is its HOMFLYPT polynomial, then*

$$P(a,z) \equiv P(a^{-1}, z) \pmod{\langle p, z^p\rangle},$$

where $\langle p, z^p\rangle$ is the ideal generated by p and z^p in R.

We have Borodzik-Politarczyk criterion for periodic knots.

Theorem 3.13 (Borodzik-Politarczyk Criterion, see [4, Theorem 1.1]). *Assume we have a p^n-periodic knot K, where p is an odd prime. Suppose that $\mathbb{F} = \mathbb{Q}$ or $\mathbb{F} = \mathbb{F}_r$ for a prime r where $r \neq p$ and r has the maximal order in $\mathbb{Z}_p^n$. Here since $gcd(r,p) = 1$ any prime $r \neq p$ will have maximal order p^n. Take $c = 1$ if $\mathbb{F} = \mathbb{F}_2$ and $c = 2$ otherwise. The quantity $s(K,F)$, known as the* s-invariant *of the knot K, is derived from the Lee or Bar-Natan theory see* [2],[14] *Then*

$$\mathrm{KhP}(K,t,q) = P_0 + \sum_{n=1}^{n} (p^j - p^{j-1}) P_j,$$

Where $P_0, P_1, \dots, P_n \in \mathbb{Z}[q, q^{-1}, t, t^{-1}]$ are Laurent polynomials such that

1. *$P_0 = q^{s(K,\mathbb{F})}(q+q^{-1}) + \sum_{j=1}^{\infty}(1+tq^{2cj})S_{0j}(t,q)$, and the polynomials S_{0j} have non-negative coefficients;*
2. *$P_k = \sum_{j=1}^{\infty}(1+tq^{2cj})S_{kj}(t,q)$ and the polynomials S_{kj} have non-negative coefficients for $1 \leq k \leq n$,*
3. *$P_k(-1,q) - P_{k+1}(-1,q) \equiv P_k(-1,q^{-1}) - P_{k+1}(-1,q^{-1}) \pmod{q^{p^{n-k}} - q^{-p^{n-k}}}$;*

The criterion is rather specific, easier to implement on a computer, than to solve by hand. The following example is discussed in [4].

Example 3.14. Take the knot $15n1335221$. This knot satisfies all periodicity criteria for $p = 5$ we discussed in the thesis. In particular, it satisfies the HOMFLYPT criterion for $p = 5$. It has the Khovanov polynomial

$$\begin{aligned} &q + q^{-1} + (1+tq^4)(t^{-7}q^{-15} + 3t^{-6}q^{-13} + t^{-5}q^{-11} + 3t^{-4}q^{-9} + t^{-3}q^{-9} + 3t^{-2}q^{-7} + t^{-1}q^{-5} \\ &+ 3t^{-1}q^{-3} + q^{-3} + q^{-1} + 3tq + t^2q^3 + 3t^3q^3 + t^4q^5 + 3t^5q^7 + t^6q^9 + 4(t^{-5}q^{-11} + t^{-4}q^{-9} + 2t^{-3}q^{-7} + 2t^{-2}q^{-5} \\ &+ t^{-1}q^{-5} + t^{-1}q^{-3} + 2tq^{-1} + q^{-3} + q^{-1} + 2t'2q + t^3q^3 + t^4q^5)). \end{aligned}$$

We write KhP $= q + q^{-1} + (1+tq^4)S'_{01} + 4(1+tq^4)S'_{11}$ where

$$S'_{01} = t^{-7}q^{-15} + 3t^{-6}q^{-13} + t^{-5}q^{-11} + 3t^{-4}q^{-9} + t^{-3}q^{-9} + 3t^{-2}q^{-7} + t^{-1}q^{-5} + 3t^{-1}q^{-3} + q^{-3} + q^{-1} + 3tq + t^2q^3 + 3t^3q^3 + t^4q^5 + 3t^5q^7 + t^6q^9$$

and

$$S'_{11} = t^{-5}q^{-11} + t^{-4}q^{-9} + 2t^{-3}q^{-7} + 2t^{-2}q^{-5} + t^{-1}q^{-5} + t^{-1}q^{-3} + 2tq^{-1} + q^{-3} + q^{-1} + 2t'2q + t^3q^3 + t^4q^5$$

According to Theorem 3.13(3), $A(q) = q + q^{-1} + (1+tq^4)S'_{01}(t,q) - (1+tq^4)S'_{11}(t,q)$ and we have $A := (A(q) - A(q^{-1})) \mod q^5 - q^{-5}$. So we have $A = -10q + 5q^3 - 5q^7 + 10q^9$. Since $A \neq 0$, we need to change S'_{01} and S'_{11}. We need to satisfy Theorem 3.13(1) and (2). We must have $S'_{11} \to S'_{11} - \delta$ and $S'_{01} \to S'_{01} + 4\delta$. Here it is important that we must have non-negative coefficients for S'_{11} and S'_{01}. We have only finitely many possibilities for δ. In order to reduce the number of possibilities, we use the following argument. Take $\delta = at^iq^j$. Then, after changing S'_{01} and S'_{11}, we have $A \to A + aT_{ij}$, where $T_{ij} = (-1)^i 5(-q^{-j-4} + q^{-j} - q^j + q^{j+4}) \mod (q^5 - q^{-5})$. We deduce that $T_{ij} = (-1)R_{j'}$ with $j' = j \mod 10$ and

$$\begin{aligned} R_1 = R_5 &= 5(q - q^9), \\ R_3 &= 10(q^3 - q^7), \\ R_7 = R_9 &= 5(-q - q^3 + q^7 + q^9). \end{aligned}$$

For different δ, A will change by $-a_1R_1 - a_3R_3 - a_7R_7$. Note that coefficients change based on conditions that $S'_{11} - \delta$ must have non-negative coefficients. We must have coefficients

$$\begin{aligned} a_1 &\in \{-1, 0, 1, 2, 3, 4, 5, 6\}, \\ a_3 &\in \{-3, -2, -1, 0\}, \\ a_7 &\in \{-4, -3, -2, -1, 0, 1, 2\}. \end{aligned}$$

With these conditions, it is not possible to have $A = 0$. We deduce that a knot $15n1335221$ is not 5-periodic.

4. Webs, foams and categories

We have already studied Khovanov homology. Now, we want to define $\mathfrak{sl}_N$ homology. Actually, Khovanov homology is $\mathfrak{sl}_2$ homology, but for $\mathfrak{sl}_N$ homology, we have to use a more formalized approach. We will use webs and foams.

4.1. Webs and foams.

Definition 4.1. A trivalent graph Γ is a closed one-dimensional cell complex where three edges meet at each vertex.

Definition 4.2. In an oriented graph, the source vertex is a vertex that has zero indegree. In other words, it is a vertex where the number of incoming edges is 0. Similarly, a sink vertex is a vertex that has zero outdegree. In other words, it is a vertex where the number of outgoing edges is 0.

Definition 4.3 (N-webs)**.** A *closed N web* is a finite oriented trivalent graph V without sources and sinks properly embedded in $\mathbb{R}^2$ We call closed N-web shortly a web. Each edge is labeled by numbers $0, \ldots, N$. An edge with the 0 label can be deleted from the web, so in some papers, edge labeling starts from 1. The labelings of edges should satisfy an important condition called the *flow condition*; see Figure 3:

- If two edges with labels a and b enter a vertex, then the outgoing edge has label $a + b$. We call a vertex a *merge vertex* when the vertex has two incoming edges.
- If two edges with labels a and b exit from a vertex, then the incoming edge has label $a + b$. Similarly, we call a vertex a *split* vertex when the vertex has two outgoing edges.

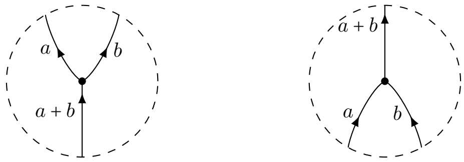


FIGURE 3. The flow condition of Definition 4.3

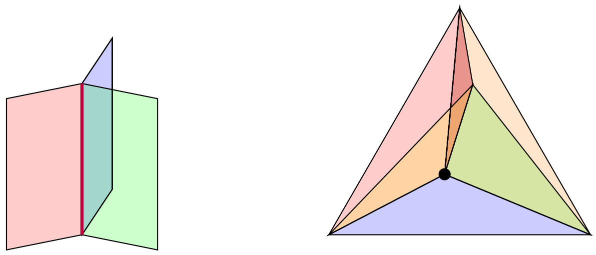


FIGURE 4. Codimension 1 and 2 singular points of a foam.

In Figure 3, the web on the left has a split vertex, and the web on the right has a merge vertex.

Remark 4.4. An empty web is just a web with no vertices and no edges.

Assume that we have two webs W_0 and W_1 in $\mathbb{R}^2$. Think of W_0 in $\mathbb{R}^2 \times \{0\}$ and W_1 in $\mathbb{R}^2 \times \{1\}$.

Definition 4.5 (foam)**.** Assume we have two webs W_0 and W_1. An N-*undecorated foam* $F\colon W_0 \to W_1$ is a compact, finite 2-dimensional CW-complex properly embedded in $\mathbb{R}^2 \times [0,1]$ such that:

- If $x \in F \setminus (W_0 \cup W_1)$, then there exists a neighborhood U of x in F homeomorphic to one of the following three models:
  - a *smooth point*: U is homeomorphic to a disk in $\mathbb{R}^2$;
  - a Y-shaped point (codimension 1 singularity): U is homeomorphic to the union of three distinct rays stemming out of a common point, times $(0,1)$;
  - a cone over a 1-skeleton of a tetrahedron (codimension 2 singularity), when x is a triple point. Compare Figure 4.
- Every *facet* F_i of F, i.e., a connected component of the set of smooth points, carries an orientation and a label by an integer $0, \dots, N$;
- a binding: compact oriented 1 dimensional manifolds. Each binding has
  - an orientation that agrees with the orientation of facets with labels a and b whereas disagrees with the orientation of facet with label $a+b$.
  - cycling ordering of the three facets around binding: when foam embedded in $\mathbb{R}^3$ this ordering must be compatible with the left-hand rule with respect to its orientation.
- Every *seam* C_i, which is a connected component of the set of Y-shaped points of F, carries an orientation;
- The orientation of every seam agrees with the orientation of precisely two adjacent facets; if these two facets are labeled by a and b, then the third facet has the label $a+b$;
- The *bottom boundary* of each facet F_i, that is $\overline{F_i} \cap (\mathbb{R}^2 \times \{0\})$, is an edge of W_0 with the same label and the orientation opposite to the orientation induced by F_i;
- The *top boundary* of each facet F_i, that is $\overline{F_i} \cap (\mathbb{R}^2 \times \{1\})$, is an edge of W_1 with the same label and the orientation agreeing with the orientation induced by F_i;

We define the composition of foams.

Definition 4.6. Assume we have webs W_0, W_1 and W_2. Furthermore, we have foams F_{01} between W_0 and W_1, F_{12} between W_1 and W_2. We define composition F_{02} of F_{01} and F_{12} as the union of F_{01} and F_{12} along W_1 where we can think of F_{01} as a subset of $\mathbb{R}^2 \times [0,1/2]$ and F_{12} as a subset of $\mathbb{R}^2 \times [1/2,1]$.

In our case, closed foams are crucial for us:

Example 4.7. A closed foam is the map from an empty web to an empty web.

4.2. **Coloring and decorations.** On webs and foams, we might have some extra structures, namely colorings and decorations. The coloring of a web is similar to the labeling.

Definition 4.8 (coloring of a web)**.** Let W be a web. A *coloring* is an assignment of a subset A_e of P=$\{1,2,\dots,N\}$ to every edge e such that $|A_e|$ = labeling of the edge. In other words, for every edge, we assign a subset of $\{1,2,\dots,N\}$. This assignment should satisfy two conditions:

- We have two edges with colorings A and B enter a vertex, then the outgoing edge should have coloring $A \cup B$ where in particular we have $A \cap B = \varnothing$.
- We have two edges with colorings A and B exiting from a vertex, then the incoming edge should have coloring $A \cup B$.

The colorings of foams are similar.

Definition 4.9 (coloring of a foam)**.** Assume we have a foam F. A *coloring* is an assignment of a subset $c(f)$ of $\{1,2,\dots,N\}$ to a face f such that $|c(f)|$ = labeling of the face f. The assignment must fulfill the following compatibility condition, which generalizes the compatibility relation for labels: at every seam where the adjacent facets f_1, f_2, and f_3 meet—assuming the orientations of f_1 and f_2 align with that of the seam—it is required that

$$c(f_3) = c(f_1) \cup c(f_2).$$

A *colored foam* is a foam with a coloring.

In addition to this structure on webs and foams, we have decorations of foams.

Definition 4.10.

- Assume we have a colored foam (F,c). We define surface $F_i(c)$ as a union of all the facets that contain $i \in P$. The restriction on orientations of facets ensures that $F_i(c)$ is also oriented.
- Assume we have a colored foam (F,c). We define surface $F_{ij}(c)$ as a union of all the facets which contain i or j but not both at the same time in their colors. The restriction on orientations of facets ensures that $F_{ij}(c)$ is also oriented.

Definition 4.11. Assume we have a colored foam (F,c) and we have $i<j$. A circle in $F_i(C) \cap F_j(C) \cap F_{ij}(C)$ is *positive* (respectively *negative*) with respect to (i,j) if it consists of positive (respectively negative) bindings. We denote the number of positive (respectively negative) by $\theta^+_{ij}(c)_F$ (respectively $\theta^-_{ij}(c)_F$). Furthermore, we have $\theta_{ij}(c) = \theta^+_{ij}(c) + \theta^-_{ij}(c)$.

4.3. **Decorations, degrees and evaluations.**

Definition 4.12 (Degree of an undecorated foam)**.** The degree $d^{un}(F)$ for a foam F is the sum of the following items.

- For a face f we have $d(f) = a(N-a)\chi(f)$ where a is the face label and χ is Euler characteristic;
- For seam i which is not a circle and is surrounded by faces with labels $a,b,a+b$ we have $d(i) = ab + (a+b)(N-a-b)$;
- For a singular point p surrounded by faces with label $a,b,c,a+b,b+c,a+b+c$ we have $d(p) = ab + bc + cm + ma + ac + bm$, where $m = N-a-b-c$.
- At the final stage we have the formula

$$d_N(F) = -\sum_{\text{f facet}} d(f) + \sum_{\text{i seam}} d(i) - \sum_{\text{p singular points}} d(p)$$

.

Another important definition for foams is the decoration.

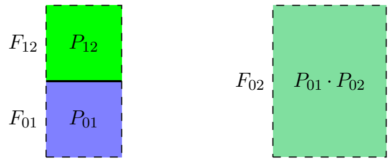


FIGURE 5. Rule for gluing decorated foams.

Definition 4.13 (Decoration of a foam)**.** Assume we have a foam F and a face f with labeling a. A *decoration* is an assignement of a symmetric homogeneous polynomial p_f in a variables to the face f.

A decorated foam is a foam together with a decoration of each face. We have the composition of foams when decoration on foams respects composition rule also. Namely, assume we have two foams F_{01} and F_{12} with decoration P_{01} on face f_1 of F_{01} and P_{12} on face f_2 of F_{12}. Assume that composition happens on faces f_1 and f_2. Then the new face should have decoration $P_{01} \cdot P_{12}$.

Remark 4.14. We fix our variables for polynomials as $X_1, X_2, \dots, X_N$, and we declare that each variable X_i has degree 2.

Definition 4.15 (Degree of decorated foam)**.** The degree $d(F)$ of a decorated foam F is defined as the sum of the undecorated degree of the foam, denoted $d^{\mathrm{un}}(F)$, and twice the total degree of the decorations on all faces. More precisely, you add $2 \cdot \deg(P_f)$ for each face f, where P_f is the polynomial decorating that face.

$$d(F) = d^{\mathrm{un}}(F) + 2 \sum_f \deg(P_f),$$

where the sum is taken over all faces f of the foam.

Definition 4.16 (Evaluation of a foam)**.** The evaluation of a closed foam involves assigning a polynomial to the foam. Assume we have a colored decorated foam (F, c). We have contributions:

$$\begin{aligned} s(F,c) &= \sum_{i=1}^{N} i \left(\frac{\chi(F_i(c))}{2} \right) + \sum_{1 \le i \le j \le N} \theta_{ij}^{+}(F,c) \\ P(F,c) &= \prod_{f \text{ face of } F} P_f(c(f)) \\ Q(F,c) &= \prod_{1 \le i \le j \le N} (X_i - X_j)^{\left(\frac{\chi(F_{ij}(c))}{2} \right)} \\ \langle F, c \rangle &= (-1)^{s(F,c)} \frac{P(F,c)}{Q(F,c)}. \end{aligned}$$

Assume we have a decorated closed foam F, we define the evaluation of a foam F:

$$\langle F \rangle = \sum_c \langle F, c \rangle,$$

where the sum runs over all colorings of F.

It is proved in [28] that $\langle F \rangle$ is a symmetric polynomial. The next observation is made in [28]; it follows promptly from the definition of $\langle F \rangle$.

Lemma 4.17. *If we have two isotopic foams F_1 and F_2 in $\mathbb{R}^2 \times [0,1]$, then $\langle F_1 \rangle = \langle F_2 \rangle$.*

4.4. **Foam categories.** We want to define $\mathfrak{sl}_N$ homology. For this, it is convenient to package webs and foams into a category theory language.

Definition 4.18. Foam$_N^*$ category is a category with objects formal shifts of N-webs denoted $q^a W$ for fixed $N \ge 2$, and morphisms between two webs $q^n W_1$ and $q^m W_2$ are decorated foams from W_1 to W_2 with degree $m - n$. Composition of two foams is defined as 4.6.

We remark that until we impose some equivalence relations on foams, the category $\mathbf{Foam}_N^*$ is huge. In particular, for the moment, isotopic foams give different morphisms.

The next category is the $\mathbb{S}\mathbf{Foam}_N^*$ category. We use the notation $\mathbb{S}_N := \mathrm{Sym}[X_1, X_2, \dots, X_N]$ as the graded ring of symmetric polynomials with complex coefficients. Recall that the variables X_i have degree 2, see 4.14.

Definition 4.19 ($\mathbb{S}\mathbf{Foam}_N^*$)**.** The category $\mathbb{S}\mathbf{Foam}_N^*$ is the $\mathbb{S}_N$ linear, $\mathbb{Z}$-graded category with

- Objects as formal shifts $q^k V$ and their direct sums, where V is a web, q is a formal variable, and $k \in \mathbb{Z}$ is grading.
- Morphisms as $\mathbb{S}_N$ linear combinations of decorated foams. Foams from $q^{a_1}W_1 \oplus q^{a_2}W_2 \cdots \oplus q^{a_k}W_k$ to $q^{b_1}W_1' \oplus q^{b_2}W_2' \cdots \oplus q^{b_l}W_l'$ is the $l \times k$ matrix whose i,j coefficient is $\mathbb{S}_N$ formal linear combination of foam between W_i and W_j' with grading $j - i$. For $p \in \mathbb{S}_N$, pF has degree $\deg(p) + \deg(F)$. In other words, $\mathrm{Hom}(q^a W_1, q^b W_2)$ is the formal linear combination of foams between W_1 and W_2. This means $\mathrm{Hom}(q^a W_1, q^b W_2)$ is freely generated $\mathbb{S}_N$ module.

We defined the $\mathbb{S}\mathbf{Foam}_N^*$ category. To understand it better, we define the evaluation functor from $\mathbb{S}\mathbf{Foam}_N^*$ to the category Sym_N^* of graded $\mathbb{S}_N$ projective modules. In this category, the modules are allowed to be infinitely generated.

Definition 4.20 (Naive evaluation functor)**.** We have functor $\widetilde{\mathcal{F}} : \mathbb{S}\mathbf{Foam}_N^* \to \mathrm{Sym}_N^*$

- For any shifted web $q^a V$, we have
$$\widetilde{\mathcal{F}}(q^a V) = \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^a V) = \bigoplus_{G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^a V)} \mathbb{S}_N\{d_N(G)\}$$
- For a morphism $F : q^a V \to q^b W$, we have the map
$$\mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^a V) \xrightarrow{\widetilde{\mathcal{F}}(F)(-) := F \circ (-)} \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^b W)$$

Remark 4.21. We note that by the second item of Definition 4.19, $\mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^a V)$ is an $\mathbb{S}_N$ module.

In our assignment for a web V, we took all foams, but it is logical to expect isotopic foams as defining the same objects, respectively the same morphisms. To overcome this problem, we need to take a suitable quotient using foam evaluation.

Suppose we have a web V and $F' \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(V, \varnothing)$, define
$$\phi_{F'} : \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V) \to \mathbb{S}_N$$
$$\phi_{F'}(F) = \langle F' \circ F \rangle$$

Now we define
$$I(V) = \bigcap_{F' \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(V, \varnothing)} \ker \phi_{F'}$$

Actually, $I(V)$ consists of all $\mathbb{S}_N$ linear combinations of foams from $\varnothing$ to V that evaluate to zero when capped with any foam from V to $\varnothing$.

As closed isotopic foams evaluate to the same polynomial, we have the following observation, which we record for a future use.

Lemma 4.22. *For any two isotopic foams F and F' from $\varnothing$ to V, $F - F'$ is in $I(V)$.*

Definition 4.23 (Evaluation functor)**.** We define a new evaluation functor $\mathcal{F} : \mathbb{S}\mathbf{Foam}_N^* \to \mathrm{Sym}_N$ where Sym_N is a category of finitely generated graded projective modules.

- For any web $q^a V$, we have $\mathcal{F}(q^a V) = \widetilde{\mathcal{F}}(q^a V)/I(q^a V) = \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, q^a V)/I(V)$.

- For morphism $G: V \to W$, we have the map

$$\mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, V)/I(V) \xrightarrow{\mathcal{F}(G)(-):=G\circ(-)} \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(\varnothing, W)/I(W)$$

It is proved in [28] that $\mathcal{F}$ is functor in the category in Sym_N. Now we define a new category where any two isotopic foams between two webs will be in the same class.

Definition 4.24. The category of $\mathbb{S}\mathbf{Foam}_N$ has the same objects as $\mathbb{S}\mathbf{Foam}_N^*$, but for morphisms, it is different: $\mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N}(V, W) := \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N^*}(V, W)/\ker \mathcal{F}$.

We need a bracket definition, and for this, we need to define a new category. In general form we have:

Definition 4.25. For an additive category A, we denote by $\mathrm{Kom}(A)$ the category of bounded cochin complexes in A and morphisms in $\mathrm{Kom}(A)$ are cochain maps.

Definition 4.26 ($\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$ category)**.** The category $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$ is defined as follows:

- objects are bounded cochain complexes in the $\mathbb{S}\mathbf{Foam}_N$ category;
- morphisms are cochain maps.

4.5. **$\mathbb{S}_N$-equivariant $\mathfrak{sl}_N$-homology.** For a link L we assign numbers between 1 and N according to its thickness. We call this labelled link. If we do not have any label on a link we assume all components are labelled with 1. We need to define the bracket $[\![D]\!] \in \mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$ for any labeled link diagram D. For this, we just need to define the bracket for a straight strand, positive and negative crossing. For any diagram, we will take the tensor product of these three diagrams.

Definition 4.27 (Bracket definition, see [7, Definition 3.3])**.**

- For a strand a bracket maps it to the corresponding web in homological degree zero.
- The bracket maps a positive crossing with a as an overstrand label and b as an understrand label, denoted as $a \geq b$, to the chain complex as in Figure 6. The differential d_k^+ can be seen in the Figure 8.
- The bracket maps a positive crossing with b as the overstrand and a as the understrand (i.e.,$b \geq a$) to the chain complex obtained by reflecting the webs and foams (from the case $a \geq b$) along the vertical axis, and swapping the labels a and b.
- For a negative crossing, the bracket maps it to the chain complex obtained from the corresponding positive crossing by reversing the q-degrees and homological degrees, and reflecting the differential foams across a horizontal plane. See Figure 7.

We create a cube of resolutions to understand the bracket definition better. Let D be a diagram with n crossings, enumerated from 1 to n and let Cr(D) denote crossing points of diagram D. At each crossing, we have labels a_i and b_i. We define c_i as the minimum of these two labels and set $C_i = \{0, \dots, c_i\}$ for the i-th positive crossings and $C_i = \{-c_i, \dots, 0\}$ for the i-th negative crossings. Similarly, we define $SC_i = [0, c_i]$ and $SC_i = [-c_i, 0]$ for positive and negative crossing. Moreover, we consider SC_i to be a CW-complex where 0-cells are the integral points, and 1-cells are the intervals. Define $\mathrm{Cube}(D) = \prod_i C_i$ and $\mathrm{SCube}(D) = \prod_i SC_i$ where SCube carries a concrete CW-complex structure.

Definition 4.28 (Immediate successor)**.** For $I, I' \in \mathrm{Cube}(D)$, we say I' is an *immediate successor* of I if I and I' agree on all coordinates except one, and this one coordinate is one larger than that of I.

Definition 4.29 (Sign assignment)**.** A *sign assignment* $\mathfrak{s}$ is an assignment of $\mathfrak{s}(I, I') \in \mathbb{F}_2$ for any pair I and I' such that I' is an immediate successor of I.

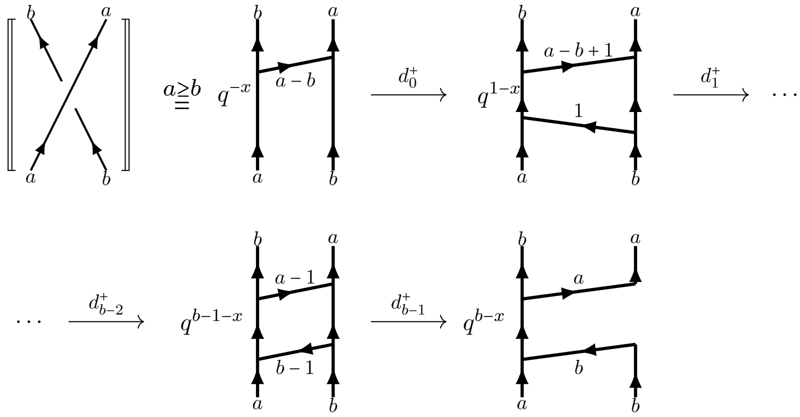


FIGURE 6. The resolution of a crossing. Here $x = b(N-b)$ and q denotes the quantum grading shift. The first term is at homological degree zero.

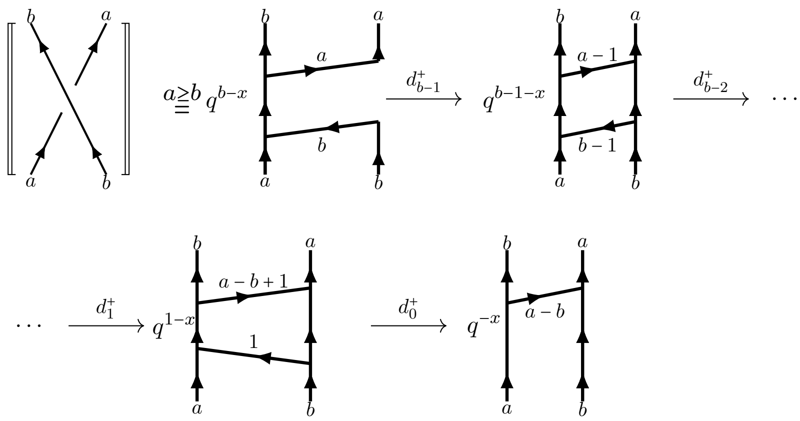


FIGURE 7. The complex associated with a negative crossing.

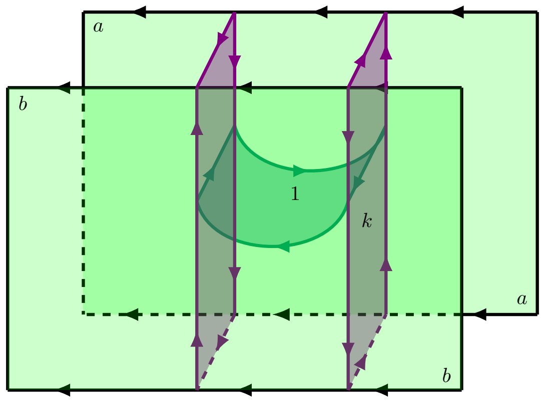


FIGURE 8. The foam that is the differential d_k^+ of the complex in Figure 6. It is decorated by constant polynomial equal 1.

.

We want $\mathscr{s}$ to satisfy the following chain condition for I, I_1, I_2, and I_{12} where $I_1 \neq I_2$, I_1 and I_2 are the immediate successors of I, and I_{12} is the immediate successor of I_1 and I_2. We have

$$\mathscr{s}(I, I_1) + \mathscr{s}(I, I_2) + \mathscr{s}(I_1, I_{12}) + \mathscr{s}(I_2, I_{12}) = 1 \in \mathbb{F}_2$$

FIGURE 9. Reidemeister one move. To the left: the source and the target of the map ϕ. To the right: the foam realizing this map (it is a product foam everywhere except near the crossing).

Remark 4.30. Algebraically, we think of $\mathscr{s}$ as a cellular 1-cochain in the cellular cochain complex $C^1_{cell}(\mathrm{SCube};\mathbb{F}_2)$ where $\delta\mathscr{s}$ is a 2-cochain with a constant value of 1.

Lemma 4.31. *For any diagram D, there exists a sign assignment $\mathscr{s}$. For any two assignments $\mathscr{s}$ and $\mathscr{s}'$, there is a coboundary such that $\mathscr{s}-\mathscr{s}'=\delta t$ where t is a cellular 0-cochain on* $\mathrm{SCube}(D)$*. Moreover, t is uniquely determined if it fixes its value on* $(0,\dots,0)$*.*

Proof. For a $c\in C^2_{cell}(\mathrm{SCube};\mathbb{F}_2)$ where c is a constant cochain with a value of 1, we have $\delta(c)=0$ because the cube has an even number of rectangles. Since we take the sum of an even number of 1's, we get 0 in $\mathbb{F}_2$. Since the cube $\mathrm{SCube}(D)$ is contractible, we have a 1-cochain $e\in C^1_{cell}(\mathrm{SCube};\mathbb{F}_2)$ such that $\delta(e)=c$. We have e as a sign assignment $\mathscr{s}$. Assume that we have two sign assignments $\mathscr{s}$ and $\mathscr{s}'$. We have $\delta(\mathscr{s}-\mathscr{s}')=\delta(\mathscr{s})-\delta(\mathscr{s}')=1-1=0$, so $\mathscr{s}-\mathscr{s}'$ is a 1-cocycle. Again, since the cube $\mathrm{SCube}(D)$ is contractible, we have t such that $\delta(t)=\mathscr{s}-\mathscr{s}'$. If we have another t' such that $\delta(t')=\mathscr{s}-\mathscr{s}'$, then we have $\delta(t-t')=0$, which means $t-t'$ is a cellular 0-cocycle. This implies that $(t-t')(a)=(t-t')(b)=0$ for any point a,b that belongs to any interval I. This means that for any points in the cube, $t-t'$ is equal to zero, so $t-t'$ is constant. □

For $I\in\mathrm{Cube}(D)$ we get web C_I by taking the $I(i)$-th resolution in Figure 6 at the i-th resolution for all crossing points i. Furthermore we define quantum degree for $I\in\mathrm{Cube}(D)$ and denote it by $Q(I)$:

$$Q(I)=\sum_{i\in\mathrm{Cr}(D)} q_i$$

where for positive crossing $q_i=I(i)-c_i(N-c_i)$ and for negative crossing $q_i=-I(i)+c_i(N-c_i)$. In addition to the quantum grading we have homological grading. We denote it by $H(I)$:

$$H(I)=\sum_{i\in\mathrm{Cr}(D)} I(i)$$

We get cochain complex $[\![\bar{D}]\!]$ by resolving D for all $I\in\mathrm{Cube}(D)$ and for the homological degree s we take the formal sum of webs D_I where $H(I)=s$. To define the differential, consider $I\in\mathrm{Cube}(D)$ and let I' be an immediate successor of I, that is, I' differs from I only at a single crossing. We define $\delta(I,I')$ as a foam given by 8, otherwise it is a product foam. The differential in the complex $[\![\bar{D}]\!]$ is defined as $(-1)^{\mathscr{s}(I,I')}\delta(I,I')$, where $\mathscr{s}(I,I')$ denotes the chosen sign assignment. By construction, there is a natural identification between $[\![\bar{D}]\!]$ and $[\![D]\!]$.

Theorem 4.32. *For any diagrams D and D' of the link L, we have $[\![D]\!]\simeq[\![D']\!]$. In other words, the complexes for these two diagrams are homotopy equivalent in* $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$*.*

Proof. The statement is well-known to the experts, with a few known proofs. To show how sign assignements work, we provide a proof in two special cases. Namely, if D' differs from D by a single Reidemeister move, and

- The case of a Reidemeister 1 move for general labelings.
- The case of a Reidemeister 2 move for diagrams with all labels equal to 1.

In this proof, the main issue will be clarifying signs. Namely, we will show how to relate sign assignments on D with sign assignments on D'. In the case of non-periodic links, we do not have a sign assignment problem, Koszul's sign rule being sufficient. We have proof of this theorem in [7, Theorem 3.5]. We will imitate [18, Section 7].

We denote the diagram obtained from D via a single Reidemeister 1 move with a positive crossing by $D\langle \ \rangle$. We assume that the strand at which the Reidemeister move is done is labeled by $a > 0$. We denote partial resolutions of $D\langle \ \rangle$ as $D\langle \ \rangle$, $D\langle \ _1\rangle, \dots, D\langle \ _{a-1}\rangle$, and $D\langle \ \rangle$. Here, by putting i we mean we label the loop which is next to the diagram by i. We can write $[\![D]\!]$ as the following bicomplex

$$0 \to [\![D\langle \ \rangle]\!] \xrightarrow{d_0^+} [\![D\langle \ _1\rangle]\!] \xrightarrow{d_1^+} \dots \xrightarrow{d_a^+} [\![D\langle \ \rangle]\!] \to 0, \tag{4.33}$$

Here d_i^+ is the identity except near the relevant crossing. The foam near the crossing is given by Figure 8. We have a chain map between $[\![D]\!]$ and $[\![D\langle \ \rangle]\!]$ given by

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & [\![D]\!] & \longrightarrow & 0 & \longrightarrow & \dots & \longrightarrow & 0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\phi} & & \downarrow & & & & \downarrow & & \\ 0 & \longrightarrow & [\![D\langle \ \rangle]\!] & \xrightarrow{d_0^+} & [\![D\langle \ _1\rangle]\!] & \xrightarrow{d_1^+} & \dots & \xrightarrow{d_a^+} & [\![D\langle \ \rangle]\!] & \longrightarrow & 0, \end{array} \tag{4.34}$$

Here ϕ is the union of the identity foams and the cup foam. It is the identity foam away from the crossing, and the cup foam when we have a resolution for an extra crossing. In general, we can say that the map between $[\![D]\!]$ and $[\![D\langle \ \rangle]\!]$ is given by $(-1)^{d(I)}\phi_I$, where $d(I)$ is a choice of a sign. The main issue with choosing appropriate sign assignments is to show that the choice $d(I) \equiv 0$ is consistent. That is, for the rest of the proof, we will deal with sign assignments on $\mathrm{Cube}(D)$ and on $\mathrm{Cube}(D\langle \ \rangle)$ so that ϕ is the chain homotopy map. We know that $\mathrm{Cube}(D\langle \ \rangle) = \mathrm{Cube}(D)\times\{0,\epsilon,2\epsilon,\dots,a\epsilon\}$. The following lemma will show us how to extend sign assignment from $\mathrm{Cube}(D)$ to $\mathrm{Cube}(D')$ where $\mathrm{Cube}(D') \cong \mathrm{Cube}(D)\times\{0,\epsilon,2\epsilon,\dots,a\epsilon\}$ where ϵ is the sign of the new crossing.

Lemma 4.35. *Suppose we have a sign assignment s for the diagram D. Assume we have the diagram D' with one more crossing compared to D, so we have $\mathrm{Cube}(D') = \mathrm{Cube}(D) \times \{0,\epsilon,2\epsilon,\dots,a\epsilon\}$ where ϵ is the sign of an additional crossing. There exists a unique sign assignment s' for D' satisfying the following conditions.*

- *The new sign assignment should be compatible with the old one, so to say for $I, I' \in \mathrm{Cube}(D)$ where I' is the immediate successor of I, we have*
$$s'((I,0),(I',0)) = s(I,I')$$
- *For $I \in \mathrm{Cube}(D)$*
$$s'((I,j),(I,j+1)) = 0$$
for all j, i.e., for positive crossing $j = 0,\dots,a-1$ and for negative crossing $j = -a,\dots,-1$. Furthermore, suppose s_1, s_2 are two sign assignments on D and $s_1 - s_2 = \delta(t)$, denote s_1' and s_2' extensions of s_1 and s_2. Now define the cellular 1-cochain t' on $\mathrm{SCube}(D')$ by $t'(1,x) = t(I)$ for any $(I,x) \in \mathrm{Cube}(D)$. Then $s_1' - s_2' = \delta t'$.

Proof. We will prove it only for a positive added crossing; the proof for the negative crossing is similar. We need to address the case of elements of the cube with the same last coordinate. We set

$$s'((I,j),(I',j)) = s(I,I') + \begin{cases} 1 & j \text{ odd} \\ 0 & j \text{ even.} \end{cases} \tag{4.36}$$

To show that the choice gives actually a sign assignement, we need to check the cochain condition. We check each case separately:

- For $I', I'_1, I'_2, I'_{12} \in \mathrm{Cube}(D')$ where these all have 0 as their last coordinate, we have
$$\begin{aligned} & s'(I', I'_1) + s'(I', I'_2) + s'(I'_1, I'_{12}) + s'(I'_2, I'_{12}) \\ &= s(I, I_1) + s(I, I_2) + s(I_1, I_{12}) + s(I_2, I_{12}) = 1. \end{aligned}$$
- For $I', I'_1, I'_2, I'_{12} \in \mathrm{Cube}(D')$ where these all have j with the condition $j \neq 0$ as their last coordinate, either we have 1 or 0 in the definition 4.36. We have
$$\begin{aligned} & s'(I', I'_1) + s'(I', I'_2) + s'(I'_1, I'_{12}) + s'(I'_2, I'_{12}) \\ &= 1 + s(I, I_1) + 1 + s(I, I_2) + 1 + s(I_1, I_{12}) + 1 + s(I_2, I_{12}) = 1 \text{ in } \mathbb{F}_2. \end{aligned}$$
- For $I, I_1 \in \mathrm{Cube}(D)$ where I_1 is the immediate successor of I, we have $I' = (I, j), I'_1 = (I_1, j), I'_2 = (I, j+1), I'_{12} = (I_1, j+1)$. For these, we have
$$\begin{aligned} & s'(I', I'_1) + s'(I', I'_2) + s'(I'_1, I'_{12}) + s'(I'_2, I'_{12}) \\ &= s(I, I_1) + 0/1 + 0 + 0 + s(I, I_1) + 1/0 = 1 \text{ in } \mathbb{F}_2. \end{aligned}$$

Now, we prove the second part. Suppose we have sign assignments s_1 and s_2 for $\mathrm{Cube}(D)$. For I and I' where I' is the immediate successor of I, we have $s_1(I, I') - s_2(I, I') = t(I) - t(I')$. Now we consider $I'_1, I'_2 \in \mathrm{Cube}(D \times \{0, \epsilon, 2\epsilon, \dots, a\epsilon\})$ such that I'_2 is the immediate successor of I'_1. We have three cases:

- $I'_1 = (I_1, j)$ and $I'_2 = (I_2, j)$. For j is even, we have
$$s'_1(I'_1, I'_2) - s'_2(I'_1, I'_2) = s_1(I_1, I_2) - s_2(I_1, I_2) = t(I_1) - t(I_2) = t'(I'_1) - t'(I'_2).$$
- $I'_1 = (I_1, j)$ and $I'_2 = (I_2, j)$. For j is odd, we have
$$s'_1(I'_1, I'_2) - s'_2(I'_1, I'_2) = 1 + s_1(I_1, I_2) - 1 - s_2(I_1, I_2) = t(I_1) - t(I_2) = t'(I'_1) - t'(I'_2).$$
- In this case, we have $I'_1 = (I, j)$ and $I'_2 = (I, j+1)$, then we have
$$s'_1(I'_1, I'_2) - s'_2(I'_1, I'_2) = 0 = t(I) - t(I) = t'(I'_1) - t'(I'_2).$$

□

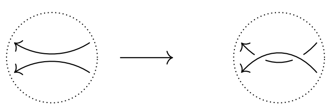
With the sign assignment from Lemma 4.35, we have the following corollary:

Corollary 4.37. *For any $I \in \mathrm{Cube}(D)$, the I-th component of the map $\phi : [\![D, s]\!] \to [\![D\langle \text{()}\circ\rangle, s']\!]$ is given by $\phi_I : D_I \to D_{(I,0)}$ without any sign correction.*

Proof. Let $I_1 \in \mathrm{Cube}(D)$ and I_2 be an immediate successor of I_1. We chose $I'_1 = (I_1, 0)$ and $I'_2 = (I_2, 0)$. The map $d' \circ \phi_{I_1}$ is the composition of the foams ϕ_{I_1} and $\delta'(I'_1, I'_2)$ with the sign $(-1)^{d(I_1) + s'(I'_1, I'_2)}$. On the other hand, we have another map which is a composition of the foams $\delta(I_1, I_2)$ and ϕ_{I_2} with the sign $(-1)^{d(I_2) + s(I_1, I_2)}$. By Lemma 4.35, $s(I_1, I_2) = s'(I'_1, I'_2)$ and we took $d(I) = 0$ for any $I \in \mathrm{Cube}(D)$, this implies ϕ is commutative with differential for any $I \in \mathrm{Cube}(D)$. □

In [18], it is proved that the map ϕ is indeed a chain homotopy equivalence. In fact, there exist explicit foams giving the inverse map. As the sign choice for D and for $D\langle \text{()}\circ\rangle$ is the same, there is no extra sign correction needed for the inverse maps either. This proves the case of Reidemeister 1 move for general labellings.

We will now sketch the proof of the Reidemeister 2a move which means the move



D $\qquad$ D'.

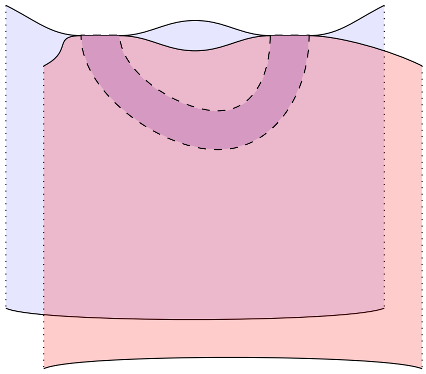


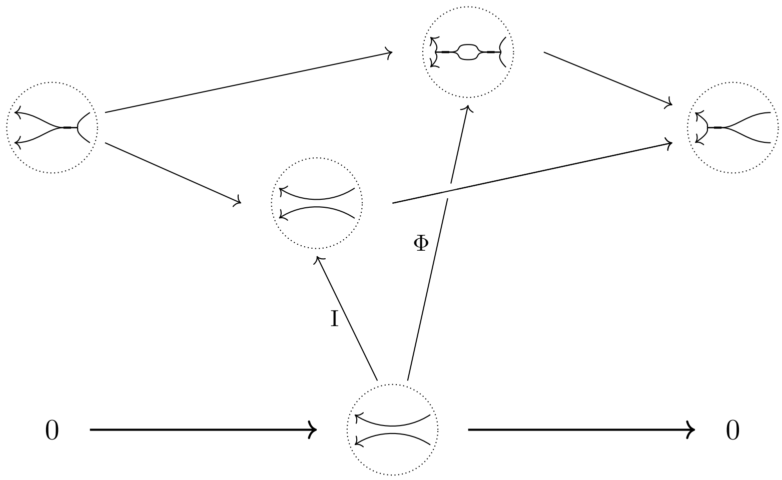
FIGURE 10. The map Φ for the Reidemeister $2a$ move in the proof of Theorem 4.32. The dashed part is the seam singularity on the foam.

does not change homotopy type of $[\![D]\!]$. Recall that we have 1 for all labels here. Furthermore we assume that the left crossing of D' is first new crossing and right crossing is the second one. We have $\mathrm{Cube}(D') = \mathrm{Cube}(D) \times \{0,1\} \times \{-1,0\}$. We have sign assignment $\mathscr{s}$ of D. We can extend this sign assignment on D' by firstly extending it on $\mathrm{Cube}(D) \times \{0,1\}$ by 4.35 and later on $\mathrm{Cube}(D') = \mathrm{Cube}(D) \times \{0,1\} \times \{-1,0\}$ by 4.35. We have the following observation.

Lemma 4.38. *The sign assignment $\mathscr{s}'$ on $\mathrm{Cube}(D) \times \{1\} \times \{-1\}$ agrees with $\mathscr{s}$.*

Proof. Let $I, I' \in \mathrm{Cube}(D)$ with I' an immediate successor of I. By (4.36), we have $\mathscr{s}_1((I,1),(I',1)) = 1 + \mathscr{s}(I,I')$. Again by (4.36) , we obtain $\mathscr{s}'((I,1,-1),(I',1,-1)) = 1 + \mathscr{s}_1((I,1),(I',1)) = \mathscr{s}(I,I')$. □

We define the following cochain map



In the figure, we have the local cochain complex of $[\![D]\!]$ at the bottom and at the top we have the local cochain complex of $[\![D']\!]$. Here I is the identity foam with the sign +1 and by Lemma 4.38, Φ is a cochain map between the complex $[\![D]\!]$ and the subcomplex of D' obtained by a $(-1,1)$-resolution of the crossing created in the Reidemeister 2a move. In [18] it can be seen that $I \oplus \Phi$ is a chain map. [18] gives also the description of the inverse map and the proof that $I \oplus \Phi\colon [\![D]\!] \to [\![D']\!]$ is a cochain homotopy equivalence.

The description of sign assignments for the Reidemeister 2b move, drawn in Figure 11, is the same. For the Reidemeister 3 move, we do not encounter problems with sign assignment, because the move preserves the crossings. The sign assignment on D induces a natural sign assignment for D'.

This implies that $[\![D]\!] \simeq [\![D']\!]$.

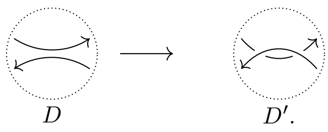


FIGURE 11. Reidemeister 2b move.

□

By Theorem 4.32, we know that we define the bracket independently of the diagram of a link.

Definition 4.39. ($\mathbb{S}_N$-equivariant Khovanov-Rozansky homology) For any diagram D of a link L, we define the $\mathbb{S}_N$-equivariant Khovanov-Rozansky homology as the homology of the chain complex $\mathcal{F}([\![D]\!])$.

5. Specialization

Recall that we have $\mathbb{S}_N$, the ring of symmetric polynomials in fixed $N \geq 0$ variables with complex coefficients. We know that $\mathbb{S}_N$ is naturally isomorphic to the ring of polynomials in N variables which are the elementary symmetric polynomials.

Theorem 5.1 (Quillen–Suslin, see [27, 30])**.** *Every finitely generated projective module over a polynomial ring is a free module.*

5.1. **Algebraic Specialization of Modules.**

Recall that Sym_N^* is the category of graded projective $\mathbb{S}_N$-modules and Sym_N is the category of finitely generated graded $\mathbb{S}_N$ projective modules. We manipulate this category and define new one.

By Theorem 5.1, an object of Sym_N is a direct sum of finitely many copies of $\mathbb{S}_N\{q^a\}$, where $\{q^a\}$ indicates a degree shift. In other words, if we write $(\mathbb{S}_N)^k$ for the k-graded part of $\mathbb{S}_N$, then $(\mathbb{S}_N\{q^a\})^k = (\mathbb{S}_N)^{k-a}$, the $(k-a)$-graded part of $\mathbb{S}_N$.

Assume we have Σ, an (unordered) N-tuple of points in $\mathbb{C}$, not necessarily distinct. We denote $P(\Sigma)$ as the evaluation of $P \in \mathbb{S}_N$ at Σ. Since P is symmetric polynomial $P(\Sigma)$ is well defined. Then, Σ specifies a left $\mathbb{S}_N$-module structure on $\mathbb{C}$, via $Pz = P(\Sigma)z$, for $P \in \mathbb{S}_N$ and $z \in \mathbb{C}$. Since $\mathbb{C}$ is left $\mathbb{S}_N$ module, for any module $M \in \mathrm{Sym}_N$, $M \otimes_{\mathbb{S}_N} \mathbb{C}$ is $\mathbb{C}$-module, that is, a vector space over $\mathbb{C}$. Furthermore, if we take $M = \mathbb{S}_N$ then $\mathbb{S}_N \otimes_{\mathbb{S}_N} \mathbb{C} = \mathbb{C}$.

Definition 5.2. (Specialization functor) We have a functor

$$\mathrm{ev}^{\Sigma} : \mathrm{Sym}_N \to \mathbf{Vect}(\mathbb{C})$$

given by $\mathrm{ev}^{\Sigma}(M) \to M \otimes_{\mathbb{S}_N} \mathbb{C}$ and for a morphism

$$F : M \to N$$

$$\mathrm{ev}^{\Sigma}(F) : M \otimes_{\mathbb{S}_N} \mathbb{C} \to N \otimes_{\mathbb{S}_N} \mathbb{C}$$

$$\mathrm{ev}^{\Sigma}(F)(m \otimes c) = F(m) \otimes c$$

We call this a *specialization functor*. If Σ consists of pairwise distinct complex numbers, then the functor is called a generic specialization functor. On the other hand, if $\Sigma = (0, \ldots, 0)$, then the functor is called a singular specialization functor.

We have $\mathbb{Z}_m$ action on $\mathrm{ev}^{\Sigma}(M)$ and on $\mathrm{ev}^{\Sigma}(F)$.

Definition 5.3. For $g \in \mathbb{Z}_m$ $g\,\mathrm{ev}^{\Sigma}(M) = gM \otimes \mathbb{C}$ and $g\,\mathrm{ev}^{\Sigma}(F)(m \otimes \mathbb{C}) = gF(m) \otimes c$.

5.2. **Algebraic Specialization of Cochain Complexes.** Let M be a graded, finitely generated free $\mathbb{S}_N$ module.

$$M = \bigoplus_{j=1}^{m} \mathbb{S}_N\{q^{a_m}\}$$

where q^{a_m} denotes the grading shift with a_m as an integer. Between two modules $\mathbb{S}_N\{q^{a_m}\}$ and $\mathbb{S}_N\{q^{a_n}\}$, we have a morphism ϕ. We say that ϕ is a degree k morphism, when it is a map $\mathbb{S}_N \to \mathbb{S}_N$ with degree $k+b-a$. Therefore ϕ is a multiplication by a homogeneous polynomial of degree $(k + a_n - a_m)/2$.
Note that we have degree 2 for variables $X_1, \dots, X_N$. Recall that the graded cochain complex is the complex that has differentials with degree zero. Assume we have the graded cochain complex C_* with graded, free $\mathbb{S}_N$ modules. Now we form two cochain complexes.

Definition 5.4 (Generic and Singular Specialization of Complexes)**.**

- For $\Sigma = (0, \dots, 0)$, we have the singular specialization C^0_*, which is obtained by applying ev^Σ to C_*.
- For Σ with the set of pairwise distinct complex numbers, we have a generic specialization C^{gen}_*, which is obtained by applying ev^Σ to C_*.

If we have $C_i = \bigoplus_{j=1}^{n_i} \mathbb{S}_N\{q^{a_{ij}}\}$, then $C^0_i = C^{gen}_i = \bigoplus_{j=1}^{n_i} \mathbb{C}\{q^{a_{ij}}\}$ because, as it was explained in Subsection 5.1, $\mathbb{S}_N \otimes_{\mathbb{S}_N} \mathbb{C} = \mathbb{C}$. The boundary maps d^0_i and d^{gen}_i are equal to $\mathrm{ev}^\Sigma(d)$, where d is the boundary map in $C_i = \bigoplus_{j=1}^{n_i} \mathbb{S}_N\{q^{a_{ij}}\}$.

Assume we have a chain complex $C_i = \bigoplus_{j=1}^{n_i} \mathbb{S}_N\{q^{a_{ij}}\}$, the differential map $d^i : C_i \to C_{i+1}$ is the sum of the maps $d_{i,kl} : \mathbb{S}_N\{q^{a_{ik}}\} \to \mathbb{S}_N\{q^{a_{i+1,l}}\}$. The map having the degree 0 is the multiplication of a homogeneous polynomial of degree $(a_{i+1,l} - a_{i,k})/2$. The singular evaluation of any homogeneous polynomial of degree at $(a_{i+1,l} - a_{i,k})/2$ can be non-zero only if $a_{i+1,l} - a_{i,k} = 0$ when we apply ev^Σ for $\Sigma = (0, \dots, 0)$. We can deduce that with the ev^Σ functor, the differential d^i_0 of the complex C^i_0 keeps the grading.

On the other hand, for the cochain complex (C^i_Σ, d^i_Σ), the situation is different. Homogeneous polynomials can be nonzero when evaluated at Σ when $a_{i+1,l} - a_{i,k} \geq 0$. This means (C^i_Σ, d^i_Σ) is filtered.

Proposition 5.5. *There exists a spectral sequence, whose first page is $H^*(C^0_*)$ and whose homology is $H^*(C^{gen}_*)$.*

Proof. The differentials $d^i : C^i \to C^{i+1}$ can be decomposed as a sum $d^{i0} + d^{i1} + \dots$, where d^{is} is given by a matrix of homogeneous polynomials of degree s. After performing a generic specialization, d^{is} becomes the map d^{is}_{gen} increasing the grading by $2s$. That is, $d^i_{gen} = d^{i0}_{gen} + d^{i1}_{gen} + \dots$. The graded part of d^i_{gen} is equal to d^{i0}_{gen}.

Specialization of d^{is} with all variables zero gives the zero map, unless $s = 0$. That is, $d^i_0 = d^{i0}_0$. The non-zero map d^{i0}_0 is equal to d^{i0}_{gen} because a degree-zero polynomial is necessarily constant. Therefore, the graded part of d^i_{gen} is equal to the differential d^i_0.

Summarizing, (C^*_{gen}, d^i_{gen}) is a filtered cochain complex, whose graded part is d^i_0. A classical argument shows the existence of the spectral sequence. □

5.3. **Geometric Specialization.** For $\Sigma \in \mathbb{C}^n$, we have the evaluation for any foam F, denoted by $\langle F \rangle_\Sigma$, which is obtained by evaluating the polynomial $\langle F \rangle$ at Σ. For any $G \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}^*_N}(V, \varnothing)$, we have

$$\Phi_{G,\sigma} : \widetilde{\mathcal{F}}(V) \to \mathbb{C}, \Phi_{G,\sigma}(F) = \langle G \circ F \rangle_\Sigma.$$

Based on this construction, we define the $\Sigma\mathbf{Foam}_N$ category.

Definition 5.6. For the category $\Sigma\mathbf{Foam}_N$, the objects are the same as the objects of $\mathbb{S}\mathbf{Foam}^*_N$. In other words, objects are webs with a formally assigned quantum grading. The morphisms are given by $\mathrm{Hom}_{\Sigma\mathbf{Foam}_N}(V, W) := \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}^*_N}(V, W)/I_\Sigma(V)$, where $I_\Sigma(V) = \cap \ker \Phi_{G,\Sigma}$.

We have a functor $\mathcal{F}_\Sigma$ from the category $\Sigma\mathbf{Foam}_N$ to the category of vector spaces.

Definition 5.7. $\mathcal{F}_\Sigma(V) = \mathrm{Hom}_{\Sigma\mathbf{Foam}_N}(\varnothing, V)$ and for a map $f : V \to W$, we have $\mathcal{F}_\Sigma(f) : \mathrm{Hom}_{\Sigma\mathbf{Foam}_N}(\varnothing, V) \to \mathrm{Hom}_{\Sigma\mathbf{Foam}_N}(\varnothing, W)$.

Taking specific Σ leads to special case. For $\Sigma = (0, \dots, 0)$ (the singular case) we have the $0\mathbf{Foam}_N$ category. If the entries of Σ are all non-zero and pairwise distinct, we speak of a generic case. Then, we will use Σ', instead of Σ. If it is needed to distinguish between singular and generic Σ in terms of $\Sigma\mathbf{Foam}_N$ category, we will call it $0\mathbf{Foam}_N$ and $\Sigma'\mathbf{Foam}_N$ category respectively.

5.4. **Geometric versus Algebraic Specialization.** We know that both $\Sigma\mathbf{Foam}_N$ and $\mathbb{S}\mathbf{Foam}_N$ are quotient categories of $\mathbb{S}\mathbf{Foam}^*_N$, but the kernel is larger in $\Sigma\mathbf{Foam}_N$ compared to the kernel in $\mathbb{S}\mathbf{Foam}_N$. This is because for $\mathbb{S}\mathbf{Foam}^*_N$, in the kernel, we have foams F such that $\langle F \rangle$ is zero, but on the other hand, for $\Sigma\mathbf{Foam}_N$, we mod out by those F, where $\langle F \rangle$ which is zero when evaluated on Σ. We have the following diagram of functors.

$$\begin{array}{ccc} \mathbb{S}\mathbf{Foam}_N & \xrightarrow{\mathcal{F}} & \mathrm{Sym}_N \\ \big\downarrow & & \big\downarrow \mathrm{ev}^\Sigma \\ \Sigma\mathbf{Foam}_N & \xrightarrow{\mathcal{F}_\Sigma} & \mathbf{Vect}_\mathbb{C} \end{array}$$

Here, $\mathbf{Vect}_\mathbb{C}$ is a category of graded vector spaces over $\mathbb{C}$.

Proposition 5.8. *The diagram above is commutative.*

Proof. This is the statement of [28, Proposition 4.1]. □

Based on these definitions, we define Khovanov-Rozansky $\mathfrak{sl}_N$-homology and Lee $\mathfrak{sl}_N$-homology.

Definition 5.9. For $\Sigma = (0, \dots, 0)$ we have a chain complex $\mathcal{F}_0([\![D]\!])$ for a link diagram D of L. We define Khovanov-Rozansky $\mathfrak{sl}_N$-homology as the cohomology space $H^k(\mathrm{ev}^\Sigma \circ \mathcal{F}([\![D]\!])) = H^k(\mathcal{F}_0([\![D]\!])) = \mathrm{KR}^{k,r}_N(L)$ of L where k is the homological grading and r is the quantum grading. Furthermore, by 5.8 we know that algebraic and geometric specialization give the same result so we can define Khovanov-Rozansky $\mathfrak{sl}_N$ homology on the cochain complex $(C^i_{(0,\dots,0)}, d^i_{(0,\dots,0)})$.

Definition 5.10. For Σ being a set of pairwise distinct N complex numbers, we have a chain complex $\mathcal{F}_\Sigma([\![D]\!])$ for a link diagram D of L. We define Lee $\mathfrak{sl}_N$-homology as the cohomology space $H^k(\mathrm{ev}^\Sigma \circ \mathcal{F}([\![D]\!])) = H^k(\mathcal{F}_\Sigma([\![D]\!])) = \mathrm{Lee}^k_N(L)$ of L where k is the homological grading. Similarly, we can define Lee $\mathfrak{sl}_N$ homology on the cochain (C^i_Σ, d^i_Σ) for generic Σ.

Theorem 5.11 (Lee-Gornik spectral sequence)**.** *Let D be a link diagram. There is a spectral sequence whose first page is* $\mathrm{KR}^{k,r}_N(L)$ *abutting to* $\mathrm{Lee}^k_N(L)$.

Proof. Here take $C_* = \mathcal{F}([\![D]\!])$ over $\mathbb{S}_N$. The cochain $C^0_* = \mathcal{F}_0([\![D]\!])$ and $C^{gen}_* = \mathcal{F}_0([\![D]\!])$ are the specialization of C_*. The statement follows from Proposition 5.5. □

Now assume we have a link L and its mirror L'. For Khovanov homology we have $\mathrm{Kh}^{i,j}(L) \cong \mathrm{Kh}^{-i,-j}(L')$. For Khovanov-Rozansky homology we have a similar relation.

Proposition 5.12. *For Khovanov-Rozansky $\mathfrak{sl}_N$ homology we have an isomorphism* $\mathrm{KR}^{k,r}_N(L) \cong \mathrm{KR}^{-k,-r}_N(L')$.

Proof. Assume we have a diagram D of link L with n crossings. Enumerate $\mathrm{Cr}(D) = \{1, \dots, n\}$. For each vertex I, we associate $\mathcal{F}_0(D_I)$. Now assume we have a mirror diagram D'. For $I \in \mathrm{Cube}(D)$, $I = (i_1, \dots, i_n)$, denote by I' the dual resolution $(-i_1, \dots, -i_n) \in \mathrm{Cube}(D')$. The webs $D'_{I'}$ and D_I are isomorphic because D and D' are mirrors to each other. We have a map $i : \mathcal{F}_0([\![D]\!]) \to \mathcal{F}_0([\![D']\!])$. In other words, we have $i : \mathcal{F}_0([\![D_I]\!]) \to \mathcal{F}_0([\![D'_{I'}]\!])$.

The differentials in the mirror complex are dualized. For example, if we have a differential from $\mathcal{F}_0([\![D_{I_1}]\!])$ to $\mathcal{F}_0([\![D_{I_2}]\!])$ then for the mirror complex we have a differential from $\mathcal{F}_0([\![D'_{I'_2}]\!])$ to $\mathcal{F}_0([\![D'_{I'_1}]\!])$; and if the first differential is given by matrix A, then the second differential in the mirror complex is given by A^T. Now fix the basis of $\mathcal{F}_0([\![D]\!])$. We have just showed that $\mathcal{F}_0([\![D]\!])$ and $\mathcal{F}_0([\![D']\!])$ have the same basis. If we send the basis of $\mathcal{F}_0([\![D']\!])$ to its dual basis, that is, the basis of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_0([\![D]\!]),\mathbb{C})$ we get an isomorphism between $\mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_0([\![D]\!]),\mathbb{C})$ and $\mathcal{F}_0([\![D]\!])$. In other words, we have

$$\mathcal{F}_0([\![D']\!]) \cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_0([\![D]\!]),\mathbb{C})$$

with underlying gradings reversed. By the universal coefficient theorem, since we work over the field $\mathbb{C}$, we obtain

$$H^{-k,-r}(\mathrm{Hom}_{\mathbb{C}}(\mathcal{F}_0([\![D]\!]),\mathbb{C})) = H^{k,r}(\mathcal{F}_0([\![D]\!]))$$

so we get

$$H^{k,r}([\![D]\!]) = H^{-k,-r}([\![D']\!])$$

□

5.5. **Computation of Lee-Gornik homology.** Recall that the decoration of a foam F is an assignment of a symmetric polynomial to every face of a foam F according to specific rules.

Definition 5.13 (Algebra of decorations)**.** Let F be a foam and f be its face. The algebra of decorations A_f is an algebra that is generated by all possible decorations on the face f modulo all decorations that make F a zero map in $\Sigma'\mathbf{Foam}_N$.

Theorem 5.14. *Let f be the foam facet with label a. The algebra of decorations is the direct sum of one-dimensional algebras indexed by the subsets of Σ with cardinality a. In each summand, we have a generator 1_A, which is an idempotent in A_f. Furthermore, this algebra for faces should satisfy the admissibility condition of coloring of foams in Definition 4.9. Namely, at every seam where the adjacent facets f_1, f_2, and f_3 meet—assuming the orientations of f_1 and f_2 align with that of the seam—it is required that*

$$A(f_3) = A(f_1) \cup A(f_2).$$

Proof. [29, Lemma 4.2], [7, Lemma 2.28]

□

We define an algebra associated to a foam F. Assume we have a web W and a foam F from W to W. Then the algebra A_F is generated by all possible decorations on the foam F modulo the decorations that evaluate to zero under F_Σ.

Theorem 5.15 (See [7],Lemma 3.10)**.** *The algebra A_F can be written as a direct sum of one-dimensional summands. The summands are in bijection with colorings of all facets by a subset of Σ as in Theorem 5.14.*

To compute link homologies more effectively, we pass to the Karoubi envelope of the $\Sigma'\mathbf{Foam}_N$ category. The key reason for this is that the original chain complexes assigned to link diagrams are generally not easy to study and with using [3, Proposition 3.3] we get chain complex in $\Sigma'\mathbf{Foam}$ category which homology is easy to calculate.

Definition 5.16 (Karoubi envelope)**.** Assume we have a category C. The Karoubi envelope of C is the category obtained by formally splitting all idempotents of C. More precisely, the category $\mathbf{Kar}(C)$ has objects as pairs (O,e) where O is an object in C and $e: O \to O$ is an idempotent. A morphism between (O,e) and (O',e') is a map $f \in \mathrm{Mor}(O,O')$ such that $f \circ e = e' \circ f$.

Consider the category $\Sigma'\mathbf{Foam}_N$ and consider the identity foam F on W. As mentioned in Theorem 5.15, a decoration on F induces a coloring on W.

Definition 5.17. The category $\mathbf{Kar}^0(\Sigma'\mathbf{Foam}_N)$ is the full subcategory of the Karoubi envelope of $\Sigma'\mathbf{Foam}_N$ whose objects are (W, F_W) where W is a web and F_W is an identity foam colored by subsets of the set of entries of Σ'.

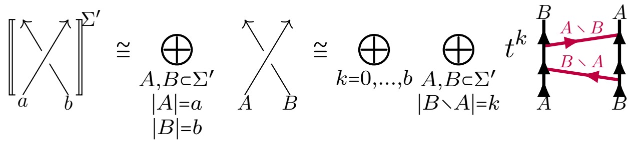
Example 5.18 (See [26, Corollary 3.19])**.** We depict any web W in $\mathbf{Kar}^0(\Sigma'\mathbf{Foam}_N)$ as a direct sum of its decorations:

$$W \equiv \sum_D (W, D)$$

where D runs through all admissible decorations.

Theorem 5.19. *Let D be a diagram of a link with chain complex $[\![D]\!]_{\Sigma'}$ in $\mathbf{Kar}^0(\Sigma'\mathbf{Foam}_N)$. In the category $\mathbf{Kar}^0(\Sigma'\mathbf{Foam}_N)$, the complex is isomorphic to the complex with trivial differentials. Locally, we write:*

(5.20)



Proof. [7, Lemma 3.13], and [28, Lemma 5.9] □

As we know, when we apply $F_{\Sigma'}$ to $[\![D]\!]_{\Sigma'}$, we get the $\mathfrak{sl}_N$ Lee homology. Therefore, we can compute the Lee homology of labeled links with this formula.

Theorem 5.21. *Let L be a link with labels equal to 1. The Lee homology of L is isomorphic to $\mathbb{C}^{N^{\#L}}$. Furthermore, for each map $\Phi : \{\text{components of } L\} \to \{1, \ldots, N\}$, we can assign a class $\ell_\phi \in Lee_N(L)$ of homological degree*

$$\deg(\ell_\phi) = \sum_{a \neq b, a, b \in \{1,\ldots,n\}} \mathrm{lk}(\phi^{-1}(a), \phi^{-1}(b))$$

These classes generate $\mathrm{Lee}_N(L)$.

Proof. [10, Theorem 2] □

5.6. **$\mathfrak{sl}_N$-homology for periodic links.** In this section, we study group action on homology. For that, we take $G = \mathbb{Z}_m$. We have a group action on $\mathbb{R}^2 \times \mathbb{R}$ by rotating about the axis $(0,0) \times \mathbb{R}$. Take m-periodic link L and D periodic link diagram of L.

5.6.1. *Group actions on* $[\![D]\!]$. We want to construct the $\mathbb{Z}_m$-equivariant $\mathfrak{sl}_N$-homology of a periodic link. For this, we need to prove:

- Existence of an $\mathbb{Z}_m$ action on $[\![D]\!]$
- Equivariance of the evaluation functor $\mathcal{F}$, implying the existence of a $\mathbb{Z}_m$-action on $\mathcal{F}([\![D]\!])$
- Independence of the action on the diagram.

Proposition 5.22. *Assume we have a link diagram D. We have an action of $\mathbb{Z}_m$ on $[\![D]\!]$ by rotating resolution diagrams.*

Proof. For the proof we need two lemmas.

Lemma 5.23. *Let W be a web and set*

$$\mathcal{G}(W) = \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N}(W, \varnothing).$$

(1) Consider the bilinear map

$$Q_W : \mathcal{G}(W) \times \mathcal{F}(W) \longrightarrow S_N, \quad Q_W(B, A) = \langle B \circ A \rangle,$$

where $\mathcal{F}$ is the evaluation functor and $\langle - \rangle$ denotes the evaluation of closed foams, as described in 4.16. The map Q_W is nondegenerate; that is, the associated adjoint map

$$Q_W^{\mathrm{ad}}: \mathcal{F}(W) \longrightarrow \mathrm{Hom}_{S_N}(\mathcal{G}(W), S_N), \quad Q_W^{\mathrm{ad}}(A)(B) = \langle B \circ A \rangle$$

is a monomorphism.

(2) *Let W_1, W_2 be webs and let $F, F': W_1 \to W_2$ be foams. Then F and F' are equal in $\mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N}(W_1, W_2)$ if and only if, for every $A \in \mathcal{F}(W_1)$ and $B \in \mathcal{G}(W_2)$, we have*

$$\langle B \circ \mathcal{F}(F) \circ A \rangle = \langle B \circ \mathcal{F}(F') \circ A \rangle.$$

Proof. A direct proof is given in [5, Lemma 2.20]. □

Lemma 5.24. *Let $\alpha: I \to \mathrm{Diff}(\mathbb{R}^2)$ be a loop of diffeomorphisms that is constant on some neighborhood of ∂I and satisfies $\alpha_0 = \alpha_1 = \mathrm{id}$. For any web W, define*

$$H_\alpha: \mathbb{R}^2 \times I \to \mathbb{R}^2 \times I, \quad H_\alpha(x,t) = \big(\alpha_t(x), t\big),$$

and set

$$\Sigma_\alpha(W) \coloneqq H_\alpha(W \times I): W \to W.$$

Then $\Sigma_\alpha(W) \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N}(W, W)$ is an identity in the foam category.

Proof. To show that $\Sigma_\alpha(W)$ acts as the identity map, it suffices to verify that for every $A \in \mathcal{F}(W)$ and every $B \in \mathcal{G}(W)$, the equality

$$\langle B \circ \Sigma_\alpha(W) \circ A \rangle = \langle B \circ A \rangle$$

holds. Consider the closed foam

$$G = B \circ \Sigma_\alpha(W) \circ A$$

in $\mathbb{R}^2 \times I$, embedded so that

$$A \subset \mathbb{R}^2 \times \big[0, \tfrac{1}{3}\big], \quad \Sigma_\alpha(W) \subset \mathbb{R}^2 \times \big[\tfrac{1}{3}, \tfrac{2}{3}\big], \quad B \subset \mathbb{R}^2 \times \big[\tfrac{2}{3}, 1\big].$$

Define the diffeomorphism

$$H: \mathbb{R}^2 \times I \longrightarrow \mathbb{R}^2 \times I$$

by

$$H(x,t) = \begin{cases} (x,t), & t \le \frac{1}{3} \text{ or } t \ge \frac{2}{3}, \\ \big(\alpha'_{3t-1}(x),\, t\big), & \frac{1}{3} \le t \le \frac{2}{3}, \end{cases}$$

where α' is the reparametrization of α defined on I. The diffeomorphism H carries the foam

$$G' = B \circ (W \times I) \circ A$$

onto G. The evaluation of closed foams, as described in [28, Definition 2.12] is carried out in a combinatorial manner, with the resulting value determined by the Euler characteristics of the faces and the combinatorial data of their colorings. In particular, foams that are related by a diffeomorphism have identical evaluations; that is,

$$\langle G' \rangle = \langle G \rangle.$$

Moreover,

$$\langle B \circ A \rangle = \langle G' \rangle = \langle G \rangle.$$

By Lemma 5.23, this implies that $\Sigma_\alpha(W)$ acts as the identity morphism in $\mathrm{Hom}_{\mathbb{S}\mathbf{Foam}_N}(W, W)$. □

We continue the proof of Proposition 5.22. The proof follows a similar approach to that in [24]. Let us fix a generator $g \in \mathbb{Z}_m$ and denote by

$$\rho_g : D \to D$$

a cobordism representing the rotation by g. We choose a specific ρ_g obtained as the trace of a continuous family of rotations in $\mathbb{R}^3$, where the rotation angle increases linearly from 0 to $\frac{2\pi}{m}$.

More precisely, define a path

$$\alpha : I \to \mathrm{Diff}(\mathbb{R}^3)$$

such that α_t is the rotation about the axis of D by the angle $\frac{2\pi t}{m}$. Consider the map

$$H_\alpha : \mathbb{R}^3 \times I \to \mathbb{R}^3 \times I$$

given by

$$H_\alpha(x,t) = (\alpha_t(x), t),$$

and define

$$\rho_g = H_\alpha(D \times I).$$

Clearly,

$$\rho_g : D \to \alpha_1(D) = D.$$

We have the action ρ_g of $\mathbb{Z}_m$ on D, and this action induces on crossing points of link L. Thus it reduces action on Cube(D) denoted $(g,I) \mapsto gI$. Furthermore, we can define an action for D_I: We have $gD_I = D_{gI}$, where g acts on D_I by rotation. Let $\rho_{g,I} : D_I \to D_{gI}$ denote the foam realizing the rotation. We need a sign assignment to construct an action on $[\![D]\!]$. The sign assignment needs to satisfy some invariance property. We define the action of g on sign assignments via $\jmath \mapsto g\jmath$, where $g\jmath(gI, gI') = \jmath(I,I')$. The sign assignment $g\jmath$ does not need to be equal to $\jmath$. But we must have $g\jmath - \jmath = \delta t$ for some 0-cochain t. Define $\mathcal{G}_g = [\![\rho_g, t]\!]$ The map $\mathcal{G}_g$ depends on t, but by Lemma 4.31, we know that for two different t_1 and t_2, we have $t_1 = t_2 + a$ where a is constant. There are two options. Either we fix t by requiring that $t(0,\dots,0) = 0$, or we emphasize the dependence of t by writing $\mathcal{G}_g = [\![\rho_g, t]\!]$. Unless specified explicitly otherwise, we adopt the first convention. Furthermore, to be more clear for each I in Cube(D), the I-th component of the map $\mathcal{G}_g$ is given by

$$\mathcal{G}_{g,I} = (-1)^{t(I)} \rho_{g,I}.$$

We need to prove that our map $\mathcal{G}_g$ is actually cochain map which means if d is a differential on $[\![D]\!]$, then $d\mathcal{G}_g = \mathcal{G}_g d$. Take resolutions $I, I' \in \mathrm{Cube}(D)$ such that I' is an immediate successor of I. Let us have ϕ as the foam which gives the component of the differential from D_I to $D_{I'}$. We have the following diagram:

$$\begin{array}{ccc} D_I & \xrightarrow{(-1)^{t(I)}\rho_{g,I}} & D_{gI} \\ {\scriptstyle (-1)^{\jmath(I,I')}\phi}\downarrow & & \downarrow{\scriptstyle (-1)^{\jmath(gI,gI')}g\phi} \\ D_{I'} & \xrightarrow{(-1)^{t(I')}\rho_{g,I'}} & D_{gI'}. \end{array}$$

Here the vertical maps are differentials, and the horizontal maps are given by $\mathcal{G}_{g,I}$ and $\mathcal{G}_{g,I'}$. The foams $\rho_{g,I'} \circ \phi$ and $g\phi \circ \rho_{g,I}$ are isotopic. By definition the coboundary of the 0-cochain t is $(\delta t)(I,I') = t(I') - t(I)$ and by the property $\delta t = g\jmath - \jmath$ we have $t(I') - t(I) = g\jmath(I,I') - \jmath(I,I') = \jmath(gI,gI') - \jmath(I,I')$., this shows the diagram commutes. This means $\mathcal{G}_g$ is a chain map in Kom($\mathbb{S}\mathbf{Foam}_N$). Lastly, we need to prove that $\mathcal{G}_g$ generates an action of G. In other words, we need to show that $(\mathcal{G}_g)^m = \mathrm{Id}$. For $m = 2$, we have $(\mathcal{G}_g)^2(D_I) = (-1)^{t(I)+t(gI)}\rho_{g,gI} \circ \rho_{g,I}$. Now for general m, we have

$$\mathcal{G}_g^m(D_I) = (-1)^{t(I)+\cdots+t(g^{m-1}I)}\rho_{g,g^{m-1}I} \circ \rho_{g,gI} \circ \rho_{g,I}$$

Define $t'(I) = t(I) + \cdots + t(g^{m-1}I)$. We have $\delta(t') = \delta(t) + \delta(tg) + \cdots \delta(t(g^{m-1})) = g\jmath - \jmath + g^2\jmath - g\jmath + \cdots + g^m\jmath - g^{m-1}\jmath = 0$ by telescope sum. Since $\delta(t') = 0$, we deduce that t' is a constant function. For $I = (0,\dots,0)$, we have $t'(I) = t(I) + \cdots + t(g^{m-1}I)$, but since $(0,\dots,0)$ is fixed in any action, we have $t'(I) = 0 + \cdots + 0 = 0$. Hence, we have $\mathcal{G}_g^m(D_I) = \rho_{g,g^{m-1}I} \circ \rho_{g,gI} \circ \rho_{g,I}$. Moreover, Lemma 5.24 shows that

$$\rho_{g,g^{m-1}I} \circ \cdots \circ \rho_{g,gI} \circ \rho_{g,I} = \mathrm{id}_{D_I}.$$

Indeed, let $\beta : I \to \mathrm{Diff}(\mathbb{R}^3)$ be the path of rotations, where β_t rotates $\mathbb{R}^3$ around the axis of D by an angle of $2\pi t$. Using the notation of Lemma 5.24 we have

$$\rho_{g,g^{m-1}I} \circ \cdots \circ \rho_{g,gI} \circ \rho_{g,I} = \Sigma\beta(D_I).$$

Therefore, Lemma 5.24 implies that

$$\rho_{g,g^{m-1}I} \circ \cdots \circ \rho_{g,gI} \circ \rho_{g,I} = \mathrm{id}_{D_I},$$

as desired. Consequently,

$$(\mathcal{G}_g)^m = \mathcal{G}_{g^m} = \mathrm{id}_{D_I},$$

which completes the proof. □

Remark 5.25. The proof that $\mathcal{G}^m_{g,I}$ is the identity uses the fact that $t(0,\ldots,0) = 0$. Another choice $t(0,\ldots,0) = 1$, if m is odd, leads to an action such that $\mathcal{G}^m_{g,I}$ is minus the identity.

Note that if M is a $\mathbb{S}_N$-module, and $\mathbb{Z}_m$ acts on M, then we can regard M as a $\mathbb{S}_N[\mathbb{Z}_m]$-module.

Proposition 5.26. *Suppose D is a periodic diagram; then, the functor $\mathcal{F}$ extends to a $\mathbb{Z}_m$-equivariant functor with values in the category of graded $\mathbb{S}_N[\mathbb{Z}_m]$-modules that are free as $\mathbb{S}_N$ modules.*

Proof. Assume we have a web V and $g \in \mathbb{Z}_m$. We want to show that $g\mathcal{F}[\![V]\!] = \mathcal{F}[\![gV]\!]$ for that firstly we show $g\widetilde{\mathcal{F}}[\![V]\!] = \widetilde{\mathcal{F}}[\![gV]\!]$. The web gV is obtained by rotating the web V. For the functor $\widetilde{\mathcal{F}}$, by the Definition 4.15 the degree is preserved by the group action for a foam F so $d_N(F) = d_N(gF)$. We have:

$$g\widetilde{\mathcal{F}}[\![V]\!] = \bigoplus_{F \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}^*_N}(\varnothing, V)} g\mathbb{S}_N\{d_N(F)\} = \bigoplus_{F \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}^*_N}(\varnothing, V)} \mathbb{S}_N\{d_N(gF)\}$$

The second equality in the above equation is the definition of the G-action on the category of $\mathbb{S}_N$-modules. Setting $F' = gF$, $F' \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}^*_N}(\varnothing, gV)$ we obtain

$$\bigoplus_{F \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}^*_N}(\varnothing, V)} \mathbb{S}_N\{d_N(gF)\} = \bigoplus_{A \in \mathrm{Hom}_{\mathbb{S}\mathbf{Foam}^*_N}(\varnothing, gV)} \mathbb{S}_N\{d_N(A)\} = \widetilde{\mathcal{F}}[\![gV]\!],$$

so indeed $g\widetilde{\mathcal{F}}[\![V]\!] = \widetilde{\mathcal{F}}[\![gV]\!]$. We conclude that $\widetilde{\mathcal{F}}$ is G-equivariant on objects. For functor $\mathcal{F}$, we need to show that $gI(V) = I(gV)$ for a web V. For F' and F, we have

$$\phi_{gF'}(gF) = \langle gF' \circ gF \rangle = \langle F' \circ F \rangle = \phi_{F'}(F)$$

This means that $g \ker \phi_{F'} = \ker \phi_{gF'}$ which implies $gI(V) = I(gV)$. As we know, $\mathcal{F}(V) = \widetilde{\mathcal{F}}(V)/I(V)$, since $\widetilde{\mathcal{F}}$ is G-equivariant and $gI(V) = I(gV)$ we deduce that $\mathcal{F}$ is G-equivariant on objects. Next, we need to prove $\mathcal{F}$ is G-equivariant on foams. Namely, for foam $F : V \to W$ we should have $\mathcal{F}(gF) = g\mathcal{F}(F)$.

$\mathcal{F}(gF)$:

$$F : V \to W$$

$$gF : gV \to gW$$

$$\mathcal{F}(gF) : \mathrm{Hom}(\varnothing, gV)/I(gV) \xrightarrow{\mathcal{F}(gF)(-):=gF\circ(-)} \mathrm{Hom}(\varnothing, gW)/I(gW)$$

On the other hand $g\mathcal{F}(F)$:

$$F : V \to W$$

$$\mathcal{F}(F) : \mathrm{Hom}(\varnothing, V)/I(V) \xrightarrow{\mathcal{F}(F)(-):=F\circ(-)} \mathrm{Hom}(\varnothing, W)/I(W)$$

$$g\mathcal{F}(F) : g(\mathrm{Hom}(\varnothing, V)/I(V)) \xrightarrow{g\mathcal{F}(F)(-):=F\circ(-)} g(\mathrm{Hom}(\varnothing, W)/I(W))$$

$$gF(F) : \mathrm{Hom}(\varnothing, gV)/I(gV) \xrightarrow{\mathcal{F}(gF)(-):=gF\circ(-)} (\mathrm{Hom}(\varnothing, gW)/I(gW))$$

By the above equations we see that $\mathcal{F}(gF) = g\mathcal{F}(F)$. This proves that $\mathcal{F}$ is G-equivariant on foams. Hence $\mathcal{F}$ is G-equivariant.

□

Definition 5.27. (see [5]) Suppose L is an m-periodic link, and D_1 and D_2 are two m-periodic diagrams representing L. We say that D_1 and D_2 are $\mathbb{Z}_m$-equivariant periodic diagrams if they are related by an *equivariant isotopy* — that is, an isotopy that respects the $\mathbb{Z}_m$-action at every stage.

Theorem 5.28. *Suppose we have two different $\mathbb{Z}_m$-equivalent m−periodic link diagrams D and D' of an m−periodic link L, then there is a chain homotopy equivalence between $[\![D]\!]$ and $[\![D']\!]$ in the category* $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$ *and induced quasi-isomorphism between $\mathcal{F}([\![D]\!])$ and $\mathcal{F}([\![D']\!])$ in the category* $\mathrm{Kom}(\mathrm{Sym}_N[\mathbb{Z}_m])$.

Proof. We know that D and D' are connected by equivariant Reidemeister moves. We need to use the theorem below to prove Theorem 5.28. □

Theorem 5.29. *Suppose D' is obtained from D by a single equivariant Reidemeister move. Then this move induces a chain homotopy equivalence*

$$\phi : [\![D]\!] \to [\![D']\!]$$

in a $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$

$$\mathcal{F}(\phi) : \mathcal{F}([\![D]\!]) \to \mathcal{F}([\![D']\!]),$$

where $\mathcal{F}(\phi)$ is a quasi-isomorphism in the category $\mathrm{Kom}(\mathrm{Sym}_N[\mathbb{Z}_m])$.

Proof. We will prove this theorem later in Section 6 □

5.7. **Equivariant $\mathfrak{sl}_N$-homology.** By Theorem 5.28, we have the quasi-isomorphic chain complex for $\mathcal{F}([\![D]\!])$ and $\mathcal{F}([\![D']\!])$. We define the cohomology of $\mathcal{F}([\![D]\!])$ as an $\mathbb{S}_N[\mathbb{Z}_m]$-module and denote this cohomology by $\mathrm{KR}^*_{\mathbb{S}_N[\mathbb{Z}_m]}(L)$.

Proposition 5.30. *The $\mathbb{S}_N[\mathbb{Z}_m]$-module structure on $\mathrm{KR}^*_{\mathbb{S}_N[\mathbb{Z}_m]}(L)$ induces a $\mathbb{C}_N[\mathbb{Z}_m]$-module structure on $\mathrm{KR}^*_N(L)$ and $\mathrm{Lee}^*_N(L)$. The Lee-Gornik spectral sequence exists in the category of finitely generated $\mathbb{C}[\mathbb{Z}_m]$-modules.*

Proof. Suppose D and D' are m-periodic $\mathbb{Z}_m$-equivalent link diagrams of L. We know that D and D' are related to each other by a sequence of equivariant Reidemeister moves. Hence, we have a chain homotopy equivalence $h : [\![D]\!] \to [\![D']\!]$ in the category of complexes $\mathbb{S}_N$-modules. By Theorem 5.28, we have a quasi-isomorphism $\mathcal{F}(h)$ between $\mathcal{F}([\![D]\!])$ and $\mathcal{F}([\![D']\!])$, and this quasi-isomorphism is $\mathbb{S}_N[\mathbb{Z}_m]$-equivariant.

Now choose Σ a set of N complex numbers. We apply the ev^Σ evaluation functor. Since ev^Σ is an additive functor the map $h_\Sigma = \mathrm{ev}^\Sigma(\mathcal{F}(h))$ is a chain homotopy equivalence between $\mathrm{ev}^\Sigma(\mathcal{F}([\![D]\!]))$ and $\mathrm{ev}^\Sigma(\mathcal{F}([\![D']\!]))$. Specifically, h_Σ induces an isomorphism between cohomology spaces of $\mathrm{ev}^\Sigma(\mathcal{F}([\![D]\!]))$ and $\mathrm{ev}^\Sigma(\mathcal{F}([\![D']\!]))$.

By Definition 5.3 for any $g \in \mathbb{Z}_m$, $g\,\mathrm{ev}^\Sigma(\mathcal{F}(W)) = \mathrm{ev}^\Sigma(g\mathcal{F}(W))$ for $W \in \mathbb{S}\mathbf{Foam}_N$ and $g\,\mathrm{ev}^\Sigma(\mathcal{F}(F)) = \mathrm{ev}^\Sigma(g\mathcal{F}(F))$ for a foam $F : W \to W'$. These imply $g\,\mathrm{ev}^\Sigma(\mathcal{F}([\![D']\!])) = \mathrm{ev}^\Sigma(g\mathcal{F}([\![D']\!]))$. This shows that h_Σ is $\mathbb{Z}_m$-equivariant. Hence $[h_\Sigma]$ is $\mathbb{Z}_m$ equivariant. We deduce that the $\mathbb{Z}_m$ equivariant isomorphism of vector spaces is an isomorphism of $\mathbb{C}[\mathbb{Z}_m]$ modules. In other words, h_Σ is a quasi-isomorphism in the category of chain complexes of $\mathbb{C}[\mathbb{Z}_m]$ modules.

We have a $\mathbb{Z}_m$ action on $\mathcal{F}[\![D]\!]$, we have $\mathbb{Z}_m$-action on $\mathrm{ev}^\Sigma(\mathcal{F}([\![D']\!]))$. For any $g \in \mathbb{Z}_m$, $g\,\mathrm{ev}^\Sigma(\mathcal{F}([\![D']\!] = \mathrm{ev}^\Sigma(g\mathcal{F}([\![D']\!]))$. This shows that we have a $\mathbb{C}[\mathbb{Z}_m]$ structure on $\mathrm{KR}^*_N(L)$ and on $\mathrm{Lee}^*_N(L)$. □

Definition 5.31. Assume we have an m-periodic link. The equivariant Khovanov-Rozansky $\mathfrak{sl}_N$-homology $\mathrm{EKR}_N^{k,r}$ is the group which inherits its $\mathbb{C}[\mathbb{Z}_m]$ module structure from the action of $\mathbb{Z}_m$ on $\mathrm{KR}_N^*(L)$. Similarly, the equivariant Lee $\mathfrak{sl}_N$- homology ELee_N^k is the group Lee^k with its $\mathbb{C}[\mathbb{Z}_m]$ module structure which comes from $\mathbb{Z}_m$ action on $\mathrm{Lee}_N^*(L)$.

We have a mirror property in link diagrams at this equivariant homologies also.

Proposition 5.32. *Suppose L is the m-periodic link with its periodic link diagram D, and suppose L' is the mirror image of L. Then for any k,r, there is a map of $\mathbb{C}[\mathbb{Z}_m]$-modules*

$$\mathrm{EKR}_N^{k,r}(L) \cong \mathrm{EKR}_N^{-k,-r}(L')$$

Proof. We already set an isomorphism in Proposition 5.12 at the level of vector spaces over $\mathbb{C}$. Now we need to show this isomorphism is $\mathbb{Z}_m$-equivariant. We have a $\mathbb{Z}_m$ action on $\mathrm{Cube}(D)$ and on $\mathrm{Cube}(D')$. For $g \in \mathbb{Z}_m$, we have $(gI)' = gI'$. Furthermore, taking a mirror of resolution commutes with the action. We have $gD_I = D_{gI}$ and $gD'_{I'} = D'_{gI'}$. We define $i : \mathcal{F}_0([\![D]\!]) \to \mathcal{F}_0([\![D']\!])$, $i(\mathcal{F}_0(D_I)) = \mathcal{F}_0(D'_{I'})$. Now we show that i map commutes with the group action. For $g \in \mathbb{Z}_m$

$$gi(\mathcal{F}_0(D_I)) = g\mathcal{F}_0(D'_{I'}) = \mathcal{F}_0(gD'_{I'}) = \mathcal{F}_0(D'_{(gI)'})$$

$$\mathcal{F}_0(D'_{(gI)'}) = \mathcal{F}_0(D'_{(gI)'}) = i(\mathcal{F}_0(D_{gI})) = i(\mathcal{F}_0 g(D_I)) = i(g\mathcal{F}_0(D_I))$$

We have $\mathcal{F}_0([\![D]\!])$ as $\mathbb{C}[\mathbb{Z}_m]$- module. Then obviously $\mathcal{F}_0([\![D']\!])$ has the same basis. On this basis, we can write

$$\Phi : \mathcal{F}_0([\![D']\!]) \to \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_m]}(\mathcal{F}_0([\![D]\!]), \mathbb{C}[\mathbb{Z}_m])$$

where Φ sends the basis of $\mathcal{F}_0([\![D']\!])$ to the dual basis of $\mathcal{F}_0([\![D]\!])$. This is an isomorphism. With the choice of basis, the differential in the chain complex $\mathcal{F}_0([\![D']\!])$ is the transpose of the differential on $\mathcal{F}_0([\![D]\!])$. Actually, $\mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_m]}(\mathcal{F}_0([\![D]\!]), \mathbb{C}[\mathbb{Z}_m])$ has the same differential as $\mathcal{F}_0([\![D']\!])$ so Φ is actually an isomorphism of chain complexes. □

5.8. **Decomposition of $\mathfrak{sl}_N$-homology.** We note that $\mathrm{EKR}_N^{k,r}$ and ELee_N^k are $\mathbb{C}[\mathbb{Z}_m]$-modules. Since the group algebra $\mathbb{C}[\mathbb{Z}_m]$ is semisimple, we have a decomposition:

$$\mathbb{C}[\mathbb{Z}_m] = \bigoplus_{i=0}^{m-1} \mathbb{C}_{\xi_m^i},$$

where $\mathbb{C}_{\xi_m^j}$ denotes the ξ_m^j-eigenspace of $\mathbb{C}[\mathbb{Z}_m]$, and $\xi_m = \exp(\frac{2\pi\sqrt{-1}}{m})$ for $j = 0, 1, \ldots, m-1$. We express this decomposition using pairwise orthogonal idempotents, denoted as $e_0, e_1, \ldots, e_{m-1}$, where $e_j e_k = \delta_{jk} e_j$. Moreover, we have $g.e_j = \xi_m^j e_j$, and

$$1 = \sum_{j=0}^{m-1} e_j.$$

Similarly, we can decompose any $\mathbb{C}[\mathbb{Z}_m]$ module M:

$$M = \bigoplus_{i=0}^{m-1} M_{\xi_m^i},$$

where $M_{\xi_m^i} \coloneqq e_i M$ is the ξ_m^i-eigenspace of M for $i = 0, 1, \ldots, m-1$.

Theorem 5.33. *For any finitely generated $\mathbb{C}[\mathbb{Z}_m]$-module M, we have*

$$\mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_m]}(M_{\xi_m^j}, M_{\xi_m^k}) = 0$$

unless $j = k$.

Proof. Assume we have a homomorphism $\Phi : M_{\xi_m^j} \to M_{\xi_m^k}$.Then, for any morphism $A : M \to M$, we have

$$\begin{array}{ccc} M_{\xi_m^j} & \xrightarrow{A} & M_{\xi_m^j} \\ \downarrow \Phi & & \downarrow \Phi \\ M_{\xi_m^k} & \xrightarrow{A} & M_{\xi_m^k}. \end{array}$$

$A\Phi = \Phi A$, $e_k A\Phi = e_k \Phi A$, $\sigma_k e_k \Phi = e_k \Phi A$, $\sigma_k \Phi = e_k \Phi A$, $\sigma_k \Phi = \Phi A$, $\sigma_k \Phi = \Phi \sigma_j$, from here we deduce $\sigma_k = \sigma_j$, which means $k = j$. We write the third equality above because we have $Ae_k = \sigma_k e_k$ and for the fourth equation e_k behaves like Id because the projection of $M_{\xi_m^k}$ to itself is an identity. Similarly, for the fifth equation, e_k is again an identity. For the sixth equation, we know A behaves like multiplication on the eigenspace $M_{\xi_m^j}$, so we can write σ_j instead of A. □

Similarly, we can apply this decomposition to $\mathbb{S}_N[\mathbb{Z}_m]$, as $\mathbb{S}_N$ is a ring of complex polynomials. For a chosen generator g, we have:

$$\mathbb{S}_N[\mathbb{Z}_m] = \bigoplus \mathbb{S}_{N,\xi_m^i}, \ \mathbb{S}_{N,\xi_m^i} := e_j \mathbb{S}_N[\mathbb{Z}_m],$$

where $\mathbb{S}_{N,\xi_m^i}$ is the ideal consisting of ξ_m^i. Consequently, for any finitely generated $\mathbb{S}_N[\mathbb{Z}_m]$ module M, we have the decomposition:

$$M = \bigoplus_{i=0}^{m-1} M_{\xi_m^i},$$

where $M_{\xi_m^i} := e_i M$. Moreover, for any finitely generated $\mathbb{S}_N[\mathbb{Z}_m]$ module M, we have:

$$\mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_m]}(M_{\xi_m^j}, M_{\xi_m^k}) = 0$$

unless $j = k$.

Now assume we have an m-periodic link diagram D. The $\mathbb{S}_N$-equivariant Khovanov-Rozansky homology of D admits a decomposition into the eigenspaces of the action $\mathbb{Z}_m$:

$$H^{k,r}(\mathcal{F}([\![D]\!])) = \bigoplus_{i=0}^{m-1} H^{k,r}_{\xi_m^i}(\mathcal{F}([\![D]\!])).$$

In particular, we have a decomposition at the level of the cochain complex:

$$\mathcal{F}([\![D]\!]) = \bigoplus_{i=0}^{m-1} (\mathcal{F}_{\xi_m^i}([\![D]\!])).$$

We can continue the decomposition by grouping $i < m$ such that we will have another composition. Namely, for any d dividing m, we define

$$M_d = \bigoplus_{\substack{0 \le i < m \\ \gcd(i,m)=m/d}} M_{\xi_m^i} = \bigoplus_{\substack{0 \le i < m \\ \gcd(i,d)=1}} M_{\xi_d^i}.$$

We now define $\mathrm{EKR}_N^{*,*}(L,d)$ for each d dividing m as follows:

$$\mathrm{EKR}_N^{*,*}(L,d) := \mathrm{Hom}_{C[\mathbb{Z}_m]}\left(C[\mathbb{Z}_m]^d, \mathrm{EKR}_N(L)\right) \cong H^*\left(C[\mathbb{Z}_m]^d, \mathrm{EKR}_N(L)\right).$$

This definition leads to the following decomposition for $\mathrm{EKR}_N(L)$:

$$\mathrm{EKR}_N^{k,r}(L) = \bigoplus_{d|m} \mathrm{EKR}_N^{k,r,d}(L,d). \tag{5.34}$$

We have this isomorphism because $\mathbb{C}[\mathbb{Z}_m]$ is semisimple, and so $\mathrm{Ext}^i_{\mathbb{C}[\mathbb{Z}_m]}(M,N) = 0$ for $i > 0$ for any $\mathbb{C}[\mathbb{Z}_m]$-modules M, N. We write a similar decomposition for Lee homology. We know that $\mathrm{Lee}_N^k(L)$ depends only on the linking numbers of components of L. Since $\mathrm{ELee}_N^k(L)$ depends on the action on $\mathrm{Lee}_N^k(L)$, we need to understand the action on components of L.

Recall that $\mathrm{Lee}_N^k(L)$ was generated by classes l_ψ where $\psi : \{\text{components of L}\} \to \{1, \dots, N\}$ is any coloring. $\mathbb{Z}_m$ acts on S^3 preserving L and acts on the components of L. Specifically, there exists an action $g \in \mathbb{Z}_m$ on the set of all colorings of components of L. We denote this action $(g, \psi) \to g\psi$. We call an order of coloring the minimum number i such that $g^i \psi = \psi$ for all g. We denote the order of coloring as $\theta(\psi)$. We can see l_ψ as a vector, and we can see $\mathrm{ELee}_N^k(L, \theta(\psi))$ as an eigenspace which is generated by the coloring with the order $\theta(\psi)$. As a result, we have the decomposition:

$$\mathrm{ELee}_N^k(L) = \bigoplus_{d|m} \mathrm{ELee}_N^k(L, d).$$

Lemma 5.35. *Suppose the group $\mathbb{Z}_m$ acts trivially on the components of an unlabeled L. Then, $\mathrm{ELee}_N^k(L, d)$ is trivial unless $d = 1$.*

Proof. Since $d = 1$, there are no other components in the decomposition. □

6. Proof of Theorem 5.29

Proof. In this proof, we have $G = \mathbb{Z}_m$, which acts on $\mathbb{R}^2$ by rotating the angle $e^{2\pi i/m}$. Since Reidemeister moves 1, 2 change the number of crossings, we can assume that D' has no fewer crossings than D. We construct ϕ as a family of foams $\phi_{I,J}$ for $(I, J) \subset \mathrm{Cube}(D) \times \mathrm{Cube}(D')$ and signs $\mathscr{d}(I, J)$ so that the component of ϕ from ϕ_I to ϕ_J is $(-1)^{\mathscr{d}(I,J)} \phi_{I,J}$.

We need to deal with two problems: a geometric one and an algebraic one.

- Geometrical problem: The group G acts on $\mathrm{Cube}(D)$ and on $\mathrm{Cube}(D')$. The action is the permutation of crossings. We need to form foams $\phi_{I,J}$ such that $\phi_{gI,gJ}$ is isotopic to foam $g\phi_{I,J}$ between gD_I and gD_J.
- Algebraical problem: We need to show that the sign assignment on D induces a compatible sign assignment on D'. Specifically, we need to show that the following diagram commutes.

$$\begin{array}{ccc} D_I & \xrightarrow{(-1)^{t(I)}\rho_{g,I}} & D_{gI} \\ {\scriptstyle (-1)^{\mathscr{d}(I,J)}\Phi_{I,J}}\big\downarrow & & \big\downarrow{\scriptstyle (-1)^{\mathscr{d}(gI,gJ)}\Phi_{gI,gJ}} \\ D'_J & \xrightarrow{(-1)^{t'(J)}\rho'_{g,J}} & D'_{gJ}, \end{array} \tag{6.1}$$

where t is the cochain on $\mathrm{SCube}(D)$ defined by the property $\delta t = \mathscr{s} - g\mathscr{s}$. We give a proof in three steps:

- We prove all details for a positive Reidemeister 1 move for $G = \mathbb{Z}_2$;
- We prove the algebra part of the Reidemeister 1 move for $G = \mathbb{Z}_m$;
- We sketch the geometry part and prove the algebra part of a Reidemeister 2a move and $G = \mathbb{Z}_2$;

The proofs of the Reidemeister 2a move for any m and the Reidemeister 2b move are analogous, so we do not provide them again. For the case of the Reidemeister 3 move, we have a natural bijection between crossings of D and D', so we do not need to extend our sign assignment. Only a geometric part is needed, but it is similar to the discussion of the geometric part for the first move; we omit the details.

6.1. **Positive Reidemeister 1 move, $\mathbb{Z}_m$ action for $m = 2$.** We have a diagram D', which is the diagram obtained by applying an equvariant Reidemeister move, that is, two Reidemeister 1 moves to diagram D, denoted by $D' = D\langle \circlearrowleft \circlearrowleft \rangle$. Furthermore, we have $D = D\langle \uparrow \uparrow \rangle$ and the diagram D with one Reidemeister move applied to one crossing is denoted by $D\langle \uparrow \circlearrowleft \rangle$ and for the other crossing, it is denoted by $D\langle \circlearrowleft \uparrow \rangle$. Since the two Reidemeister one moves are symmetric, $\mathbb{Z}_2$ takes $D\langle \uparrow \circlearrowleft \rangle$ to $D\langle \circlearrowleft \uparrow \rangle$ and vice versa. We want to prove that $[\![D']\!] \simeq [\![D]\!]$.

For the cochain complex $[\![D\langle\cdots\rangle]\!]$ we have

$$[\![D\langle\cdots\rangle]\!] = \{0 \to [\![D\langle\cdots\rangle]\!] \xrightarrow{d} [\![D\langle\cdots\rangle]\!] \to 0\},$$

and for $[\![D\langle\cdots\rangle]\!]$ we have

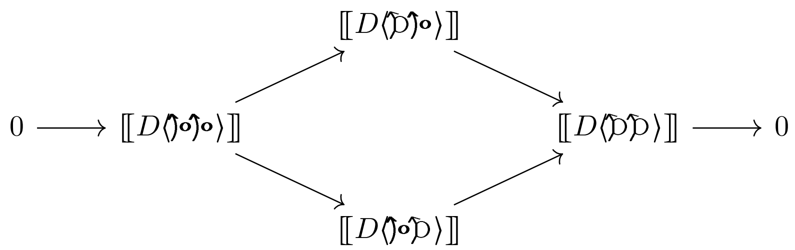
$$[\![D\langle\cdots\rangle]\!] = \{0 \to [\![D\langle\cdots\rangle]\!] \xrightarrow{d} [\![D\langle\cdots\rangle]\!] \to 0\},$$

In terms of Cube notation, we have relations

(6.2) $$\mathrm{Cube}(D\langle\cdots\rangle) \cong \mathrm{Cube}(D\langle\cdots\rangle) \times \{0,1\}^2$$

$$\mathrm{Cube}(D\langle\cdots\rangle) = \bigcup_{x=0,1} \mathrm{Cube}(D\langle\cdots\rangle) \times \{(0,x)\}$$
$$\mathrm{Cube}(D\langle\cdots\rangle) = \bigcup_{x=0,1} \mathrm{Cube}(D\langle\cdots\rangle) \times \{(x,0)\}.$$

Here on the right-hand side of the equation, we label extra crossing points by $\{0,1\}$ and $\{0,x\},\{x,0\}$. We split the cochain complex $[\![D\langle\cdots\rangle]\!]$. Namely, we have

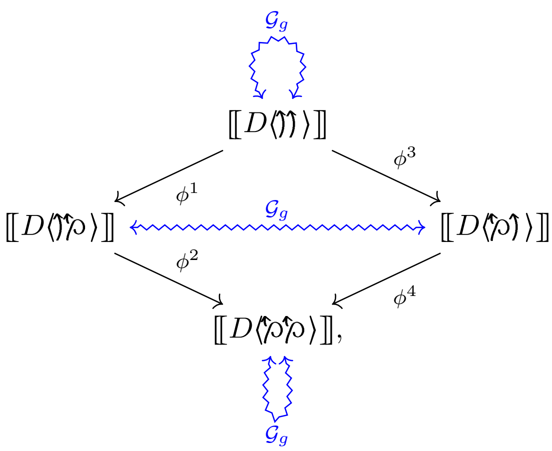


We have the following maps corresponding to the non-equivariant Reidemeister moves:

$$\phi^1 \colon [\![D\langle\cdots\rangle]\!] \to [\![D\langle\cdots\rangle]\!] \qquad \phi^2 \colon [\![D\langle\cdots\rangle]\!] \to [\![D\langle\cdots\rangle]\!]$$
$$\phi^3 \colon [\![D\langle\cdots\rangle]\!] \to [\![D\langle\cdots\rangle]\!] \qquad \phi^4 \colon [\![D\langle\cdots\rangle]\!] \to [\![D\langle\cdots\rangle]\!]$$

With these maps, we have the following diagram in $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$

(6.3)



Here the blue arrows mean g action on the cochain complex. For example, the blue arrow in the middle means a 180-degree rotation of diagram $D\langle\cdots\rangle$. [1] In order to understand this diagram better, we specify to a single resolution I of D.

[1]The notation $D\langle\cdots\rangle$ might suggest that there is a problem of the orientation, but this is not the case. The notation specifies which resolution is taken for the diagram, but not its actual position in $\mathbb{R}^2$.

Lemma 6.4. *For any $I \in \mathrm{Cube}(D)$, the diagram below is commutative in $\mathbb{S}\mathbf{Foam}_N$.*

$$
\text{(6.5)}\qquad
\begin{array}{ccc}
D\langle \uparrow\uparrow \rangle_I & \xrightarrow{\rho_{g,I}} & D\langle \uparrow\uparrow \rangle_{gI} \\
\downarrow \phi^1_I & & \downarrow \phi^3_I \\
D\langle \uparrow\wp \rangle_{(I,0)} & \xrightarrow{\rho_{g,(I,0)}} & D\langle \wp\uparrow \rangle_{(gI,0)} \\
\downarrow \phi^2_{(I,0)} & & \downarrow \phi^4_{(I,0)} \\
D\langle \wp\wp \rangle_{(I,0,0)} & \xrightarrow{\rho_{g,(I,0,0)}} & D\langle \wp\wp \rangle_{(gI,0,0)}.
\end{array}
$$

In this diagram, the map $\rho_{g,I}$ is the foam from the diagram with resolution in I to the diagram with resolution in gI. We have a similar definition for $\rho_{g,(I,0)}$ and $\rho_{g,(I,0,0)}$.

Proof. For the above square, we consider $\rho_{g,(I,0)} \circ \phi^1_I$. This foam arises from applying ϕ^1_I foam and we rotate the upper side of it. On the other hand, for $\phi^3_I \circ \rho_{g,I}$ we have the foam where we take foam ϕ^3_I and rotate its lower part. These foams are isotopic rel boundary, so they are equivalent in the $\mathbb{S}\mathbf{Foam}_N$ category. The second square is similar, which means that the diagram commutes. This proves the geometric part of our proof. □

For the algebraic part for a positive Reidemeister 1 move for $\mathbb{Z}_2$, we fix the sign assignment $\mathfrak{s}$ on D and take $g \in \mathbb{Z}_2$ as a generator from $\mathbb{Z}_2$. By Lemma 4.31, we have $g\mathfrak{s} = \mathfrak{s} + \delta\mathfrak{t}$ where $\mathfrak{t}$ is the 0-cochain on $\mathrm{SCube}(D)$ with the property $\mathfrak{t}(0,\dots,0) = 0$. We can extend $\mathfrak{s}$ to the diagram $D\langle \wp\wp \rangle$ in two ways. The first one is extending $\mathfrak{s}$ on the diagram $D\langle \uparrow\wp \rangle$ and then to $D\langle \wp\wp \rangle$. The second one is extending $\mathfrak{s}$ on the diagram $D\langle \wp\uparrow \rangle$ and then to $D\langle \wp\wp \rangle$. We have a relation between these sign assignments. We have $\mathfrak{s}$ on D and $\mathfrak{s}_1$ on $D\langle \wp\uparrow \rangle$. Write $\mathfrak{s}_2 = g\mathfrak{s}_1$, $\mathfrak{s}_2$ is the sign assignment on $D\langle \uparrow\wp \rangle$. We extend $\mathfrak{s}_1$ on $D\langle \wp\wp \rangle$ by Lemma 4.35 we denote the new sign assignment $\mathfrak{s}_3$. Similarly, we can extend $\mathfrak{s}_2$ on $D\langle \wp\wp \rangle$, and denote this new sign assignment $\mathfrak{s}_4$. In this way, we have constructed two sign assignments on $D\langle \wp\wp \rangle$. For these two assignments, there exists $\mathfrak{t}'$ such that we have $\mathfrak{s}_3(I,I') - \mathfrak{s}_4(I,I') = \mathfrak{t}'(I) - \mathfrak{t}'(I')$ for any $I, I' \in \mathrm{Cube}(D\langle \wp\wp \rangle)$ where I' is an immediate successor of I.

Lemma 6.6. *(a) We have $g\mathfrak{s}_3 = \mathfrak{s}_4$.*

(b) If $\mathfrak{t}$ is a 0-cochain on $\mathrm{SCube}(D)$ such that $\mathfrak{s} - g\mathfrak{s} = \partial\mathfrak{t}$, then the 0-cochain $\mathfrak{t}'$ on $\mathrm{SCube}(D\langle \wp\wp \rangle)$ defined by

$$\mathfrak{t}'((I,x,y)) = xy + \mathfrak{t}(I) \in \mathbb{F}_2,$$

satisfies $\mathfrak{s}_3 - \mathfrak{s}_4 = \partial\mathfrak{t}'$.

Proof. Let $I'_1, I'_2 \in \mathrm{Cube}(D\langle \wp\wp \rangle)$, where I'_2 is an immediate successor of I'_1. The action of g switches the last two crossings. We write $I'_k = (I_k, x_k, y_k)$ for $k = 1,2$ with $I_k \in \mathrm{Cube}(D)$, $x_k, y_k \in \{0,1\}$.

For the action g, we have $gI'_k = (gI_k, y_k, x_k)$. By Lemma 4.35, we extend $\mathfrak{s}_3$ and have

$$\mathfrak{s}_3((I_1,x_1,y_1),(I_2,x_2,y_2)) = \begin{cases} 0 & \text{if } y_2 = y_1 + 1 \\ \mathfrak{s}_1((I_1,x_1),(I_2,x_2)) + y_1 & \text{if } y_1 = y_2. \end{cases}$$

Similarly,

$$\mathfrak{s}_4((I_1,x_1,y_1),(I_2,x_2,y_2)) = \begin{cases} 0 & \text{if } x_2 = x_1 + 1 \\ \mathfrak{s}_2((I_1,y_1),(I_2,y_2)) + x_1 & \text{if } x_1 = x_2. \end{cases}$$

More precisely,

$$\text{(6.7)}\qquad \mathfrak{s}_3((I_1,x_1,y_1),(I_2,x_2,y_2)) = \begin{cases} 0 & y_2 = y_1 + 1 \\ y_1 & y_1 = y_2 \text{ and } x_2 = x_1 + 1 \\ \mathfrak{s}(I_1,I_2) + x_1 + y_1 & y_1 = y_2 \text{ and } x_1 = x_2. \end{cases}$$

and

$$\mathfrak{s}_4((I_1,x_1,y_1),(I_2,x_2,y_2)) = \begin{cases} 0 & x_2 = x_1+1 \\ x_1 & x_1 = x_2 \text{ and } y_2 = y_1+1 \\ g\mathfrak{s}(I_1,I_2)+x_1+y_1 & y_1 = y_2 \text{ and } x_1 = x_2. \end{cases} \tag{6.8}$$

By equations 6.7 and 6.8, we have $g\mathfrak{s}_3 = \mathfrak{s}_4$. □

For the second part of the proof, we observe

$$\begin{aligned} &\mathfrak{s}_3((I_1,x_1,y_1),(I_2,x_2,y_2)) - \mathfrak{s}_4((I_1,x_1,y_1),(I_2,x_2,y_2)) = \\ &= \begin{cases} x_1 & x_1 = x_2 \text{ and } y_2 = y_1+1 \\ y_1 & y_1 = y_2 \text{ and } x_2 = x_1+1 \\ \mathfrak{s}(I_1,I_2) - g\mathfrak{s}(I_1,I_2) & x_1 = x_2,\ y_1 = y_2. \end{cases} \end{aligned} \tag{6.9}$$

In (6.9), we have exhausted all possibilities for x_1, x_2, y_1, y_2. Namely, we cannot have the case $x_2 = x_1+1$ and $y_2 = y_1+1$. In addition to this equation with the definition $\mathfrak{t}'(I,x,y) = \mathfrak{t}(I)+xy$, we have:

$$\delta\mathfrak{t}' = \mathfrak{t}'(I_1,x_1,y_1) - \mathfrak{t}'(I_2,x_2,y_2) = x_1y_1 + x_2y_2 + \mathfrak{t}(I_1) - \mathfrak{t}(I_2). \tag{6.10}$$

We want to show $\mathfrak{s}_3 - \mathfrak{s}_4 = x_1y_1 + x_2y_2 + \mathfrak{t}(I_1) - \mathfrak{t}(I_2)$. We have two cases

- *First case* $I_1 = I_2$: Note that I_2' is an immediate successor of I_1'. Since we have $I_1' \neq I_2'$, we cannot have $x_1 = x_2$ and $y_1 = y_2$ so we can have $(x_1,y_1) = (0,0), (x_2,y_2) = (0,1)$, $(x_1,y_1) = (0,0), (x_2,y_2) = (1,0)$, $(x_1,y_1) = (0,1), (x_2,y_2) = (1,1)$ or $(x_1,y_1) = (1,0), (x_2,y_2) = (1,1)$. For the cases where we have $(x_2,y_2) = (1,1)$, $\mathfrak{s}_3 - \mathfrak{s}_4 = 1$ and $x_1y_1 + x_2y_2 + \mathfrak{t}(I_1) - \mathfrak{t}(I_2) = 1$. For the cases where we have $(x_1,y_1) = (0,0)$, $\mathfrak{s}_3 - \mathfrak{s}_4 = 0$ and $x_1y_1 + x_2y_2 + \mathfrak{t}(I_1) - \mathfrak{t}(I_2) = 0$.
- *Second case* $I_1 \neq I_2$: In this case, we have $x_1 = x_2$ and $y_1 = y_2$ because I_2' is an immediate successor of I_1'. We have $\mathfrak{s}_3 - \mathfrak{s}_4 = \mathfrak{s} - g\mathfrak{s} = \delta\mathfrak{t} = \mathfrak{t}(I_1) - \mathfrak{t}(I_2) = x_1y_1 + x_2y_2 + \mathfrak{t}(I_1) - \mathfrak{t}(I_2)$.

Continuing the proof of the algebraic part, we claim that the following diagram is commutative in $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$.

$$\begin{array}{ccc} [\![D\langle \uparrow\uparrow\rangle, \mathfrak{s}]\!] & \xrightarrow{[\![\rho_g,\mathfrak{t}]\!]} & [\![D\langle \uparrow\uparrow\rangle, \mathfrak{s}]\!] \\ \downarrow \phi^1 & & \downarrow \phi^3 \\ [\![D\langle \uparrow\curvearrowleft\rangle, \mathfrak{s}_1]\!] & & [\![D\langle \curvearrowleft\uparrow\rangle, \mathfrak{s}_2]\!] \\ \downarrow \phi^2 & & \downarrow \phi^4 \\ [\![D\langle \curvearrowleft\curvearrowleft\rangle, \mathfrak{s}_3]\!] & \xrightarrow{[\![\rho_g,\mathfrak{t}']\!]} & [\![D\langle \curvearrowleft\curvearrowleft\rangle, \mathfrak{s}_4]\!]. \end{array} \tag{6.11}$$

Note that $\rho_g, \mathfrak{t}$ are the same as defined in Lemma 6.4. For any $I \in \mathrm{Cr}(D)$, we show in Lemma 6.4 that the diagram is commutative in $\mathbb{S}\mathbf{Foam}$. We can generalize it in $\mathrm{Kom}(\mathbb{S}\mathbf{Foam})$ without a sign. We just need to show the sign that makes no problem for commutativity. By the definition of $\mathcal{G}_{\rho,I}$, the sign we get from D_I starting with $\phi^2 \circ \phi^1$ and then through $[\![\rho_g,\mathfrak{t}']\!]$ is $(-1)^{\mathfrak{t}'((I,0,0))}$. Similarly, when we start with $[\![\rho_g,\mathfrak{t}]\!]$ and then by $\phi^4 \circ \phi^3$ gives the sign of $(-1)^{\mathfrak{t}(I)}$. By the definition of $\mathfrak{t}'$, $(-1)^{\mathfrak{t}'((I,0,0))} = (-1)^{\mathfrak{t}(I)}$. □

Lemma 6.12. *The compositions $\phi^4 \circ \phi^3$ and $\phi^2 \circ \phi^1$ are equal as maps in* $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$.

Proof. For any $I \in \mathrm{Cr}(D)$, the map $\phi_I^4 \circ \phi_I^3$ is given by the foams that start with a Reidemeister move for the first crossing and then for the second crossing, i.e., $\uparrow\uparrow \to \curvearrowleft\uparrow \to \curvearrowleft\curvearrowleft$. Similarly, the other foam $\phi_I^2 \circ \phi_I^1$ is given by the foams that start with a Reidemeister move for the second crossing and then for the first crossing, i.e., $\uparrow\uparrow \to \uparrow\curvearrowleft \to \curvearrowleft\curvearrowleft$. All the foams $\phi_I^1, \ldots, \phi_I^4$ are product foams of the identity except for the relevant crossings. □

Denote ϕ as the composition $\phi_I^4 \circ \phi_I^3 = \phi_I^2 \circ \phi_I^1$. It is induced by a composition of individual, non-equivariant Reidemeister moves. Specifically, ϕ is a (nonequivariant) chain homotopy equivalence. The horizontal maps in 6.11 are group actions on $[\![D]\!]$ and $[\![D\langle \uparrow\!\circlearrowright \uparrow\!\circlearrowright \rangle]\!]$. The commutativity of 6.11 implies that ϕ commutes with the group action. This proves the first part of 5.29 for the specific case of Reidemeister move 1 and $\mathbb{Z}_2$.

For the proof of the second part, we apply the evaluation functor $\mathcal{F}$ from the category $\mathrm{Kom}(\mathbb{S}\mathbf{Foam}_N)$ to the category $\mathrm{Kom}(\mathrm{Sym}_N)$. The map $\mathcal{F}(\phi) : \mathcal{F}([\![D]\!]) \to \mathcal{F}([\![D']\!])$ is a chain homotopy equivalence. More specifically, it is a quasi-isomorphism in $\mathrm{Kom}(\mathrm{Sym}_N)$. By Proposition 5.26, $\mathbb{Z}_m$ acts on $\mathcal{F}([\![D]\!])$ and on $\mathcal{F}([\![D']\!])$. By 6.11 and 6.12, $\mathcal{F}(\phi)$ commutes with the $\mathbb{Z}_m$ action. A $\mathbb{Z}_m$-equivariant quasi-isomorphism is a quasi-isomorphism in $\mathrm{Kom}(\mathrm{Sym}_N[\mathbb{Z}_m])$. We have now completed the proof of Step 1.

6.2. **Positive Reidemeister one move, $\mathbb{Z}_m$ action for general m.** This step is similar to the previous one. Let D be a periodic link diagram, and D' be the link diagram obtained by applying the Reidemesiter one move. We again identify $\mathrm{Cube}(D') \equiv \mathrm{Cube}(D) \times \{0,1\}^m$. For any $I \in \mathrm{Cr}(D)$ and a generator $g \in Z_m$ with $x_1, x_2, \ldots, x_m \in \{0,1\}$, we have

$$g(I, x_1, x_2, \ldots, x_m) = (gI, x_2, x_3, \ldots, x_m, x_1)$$

We define two maps ϕ_I^A and ϕ_I^B, $\phi_I^A = \phi_I^m \circ \phi_I^{m-1} \circ \cdots \circ \phi_I^1$ where ϕ_I^i is the foam that realizes the i-th Reidemeister move as in Figure 9. For any $I \in \mathrm{Cr}(D)$, we have the following diagram.

$$\begin{array}{ccc} D\langle \uparrow \ldots \uparrow \rangle_I & \xrightarrow{\ \rho_{g,I}\ } & D\langle \uparrow \ldots \uparrow \rangle_{gI} \\ \downarrow \phi_I^A & & \downarrow \phi_I^B \\ D\langle \uparrow\!\circlearrowright \ldots \uparrow\!\circlearrowright \rangle_{(I,0,\ldots,0)} & \xrightarrow{\ \rho_{g,(I,0,\ldots,0)}\ } & D\langle \uparrow\!\circlearrowright \ldots \uparrow\!\circlearrowright \rangle_{(gI,0,\ldots,0)}, \end{array} \tag{6.13}$$

This diagram is a generalization of the diagram (6.5). The geometric part of this step is proved in the same way as in Step 1. We omit the details. We pass to the algebraic part directly. Take s sign assignment for the diagram D, and let t be such that $s - gs = \delta t$. We get the sign assignment s' on $\mathrm{Cube}(D')$ by Lemma 4.35.

Lemma 6.14. *Assume $I_1', I_2' \in \mathrm{Cube}(D')$ and I_2' is an immediate successor of I_1'. Write $I_1' = (I_1, x_1, \ldots, x_m)$, $I_1' = (I_1, y_1, \ldots, y_m)$ where $I_1, I_2 \in \mathrm{Cube}(D)$. If $x_k \neq y_k$ for some k, then*

$$s'(I_1', I_2') = x_{k+1} + \cdots + x_m.$$

If $x_k = y_k$ for all k, then

$$s'(I_1', I_2') = x_1 + \cdots + x_m + s(I_1, I_2).$$

Proof. Define the sign assignment s_l' on diagram D_l', which is the sign assignment obtained after the first l Reidemeister moves. Assume $x_k \neq y_k$ for some k. By 4.35, we have:

$$s_k'((I_1, x_1, \ldots, x_k), (I_2, y_1, \ldots, y_k)) = 0.$$

We continue to apply inductively for $j = k+1, \ldots, m$, and by either 4.35 (if $x_j = 0$) or (4.36) (if $x_j = 1$), we get

$$s_k'((I_1, x_1, \ldots, x_k), (I_2, y_1, \ldots, y_k)) = x_{k+1} + \cdots + x_j.$$

This leads to the result $s'(I_1', I_2') = x_{k+1} + \cdots + x_m$. Similarly, if we take $x_1 = y_1, \ldots, x_m = y_m$, then we can again apply induction. If $x_i = y_i = 0$ by 4.35, we have the result; if $x_i = y_i = 1$, then by (4.36), we have the result. □

We have a generalization of Lemma 6.6.

Lemma 6.15. *Assume $s - gs = \delta t$. Define the 0-cochain on $\mathrm{SCube}(D')$ defined by $t'(I, x_1, \ldots, x_m) = t(I) + x_1(x_2 + \cdots + x_m)$. Then for any $I_1', I_2' \in \mathrm{Cube}(D')$ where I_2' is an immediate successor of I_1', we have*

$$s'(I_1', I_2') - s'(gI_1', gI_2') = t'(I_1') - t(I_2'). \tag{6.16}$$

Proof. We have two cases:

- Suppose $I_1 = I_2$ and $x_i = y_i$ except for k, $x_k = 0$, and $y_k = 1$. By Lemma 6.14, we have
$$\mathscr{s}'(I_1', I_2') = x_{k+1} + \cdots + x_m.$$
In addition to that, we have
$$\mathscr{s}'(gI_1', gI_2') = x_{k+1} + \cdots + x_m.$$
Thus,
$$\mathscr{s}'(I_1', I_2') - \mathscr{s}'(gI_1', gI_2') = \begin{cases} x_1 & k > 1 \\ x_2 + \cdots + x_m & k = 1. \end{cases}$$
On the other hand, for $k > 1$, we have
$$\mathscr{t}'(I_1') - \mathscr{t}'(I_2') = \mathscr{t}(I_1') + x_1(x_2 + \cdots + x_m) - (\mathscr{t}(I_2') + x_1(x_2 + \cdots + x_m + 1)) = x_1.$$
For $k = 1$, we have
$$\mathscr{t}'(I_1') - \mathscr{t}'(I_2') = \mathscr{t}(I_1') + x_1(x_2 + \cdots + x_m) - (\mathscr{t}(I_2') + (x_1 + 1)(x_2 + \cdots + x_m)) = x_2 + \cdots + x_m.$$
- Suppose $I_1 \neq I_2$, then $x_k = y_k$ for all k. Thus, we have
$$\mathscr{s}'(I_1', I_2') - \mathscr{s}'(gI_1', gI_2') = \mathscr{s}(I_1, I_2) - \mathscr{s}(gI_1, gI_2).$$
$$\mathscr{s}(I_1, I_2) - \mathscr{s}(gI_1, gI_2) = \mathscr{t}(I_1) - \mathscr{t}(I_2)$$
$$\mathscr{t}'(I_1') - x_1(x_2 + \cdots + x_m) - (\mathscr{t}'(I_2') - x_1(x_2 + \cdots + x_m)) = \mathscr{t}'(I_1') - \mathscr{t}'(I_2').$$

The remaining part of the step is similar to part $m = 2$. In short, we repeat the proof of Lemma 6.12 to show that ϕ_I^A and ϕ_I^B induce the same map
$$\Phi\colon [\![D\langle \uparrow, \ldots, \uparrow \rangle]\!] \xrightarrow{\simeq} [\![D\langle \uparrow\!\rho, \ldots, \uparrow\!\rho \rangle]\!].$$
The corresponding diagram of 6.11 is
$$\begin{array}{ccc} [\![D\langle \uparrow, \ldots, \uparrow \rangle, \mathscr{s}]\!] & \overset{[\![\rho_g, \mathscr{t}]\!]}{\rightsquigarrow} & [\![D\langle \uparrow, \ldots, \uparrow \rangle, \mathscr{s}]\!] \\ \downarrow \Phi & & \downarrow \Phi \\ [\![D\langle \uparrow\!\rho, \ldots, \uparrow\!\rho, \mathscr{s}' \rangle]\!] & \overset{[\![\rho_g, \mathscr{t}']\!]}{\rightsquigarrow} & [\![D\langle \uparrow\!\rho, \ldots, \uparrow\!\rho \rangle, g\mathscr{s}']\!]. \end{array}$$

The same argument as in the previous step implies that this diagram is commutative. Specifically, $\mathcal{F}(\phi)$ induces a $\mathbb{Z}_m$-equivariant chain homotopy equivalence, which means $\mathcal{F}(\phi)$ is a quasi-isomorphism in the category $\mathrm{Kom}(\mathrm{Sym}_N[\mathbb{Z}_m])$. □

6.3. **Step 3: Reidemeister 2a move, $\mathbb{Z}_m$ action for $m = 2$.** Let D be a periodic link diagram, and D' be the link diagram obtained by applying the equivariant Reidemesiter 2a move. We have the identification
$$\mathrm{Cube}(D') \cong \mathrm{Cube}(D) \times \{0, 1\} \times \{-1, 0\} \times \{0, 1\} \times \{-1, 0\}.$$
For $I \in \mathrm{Cr}(D)$, denote
$$I_1' = (I, 0, 0, 0, 0), \quad I_2' = (I, 1, -1, 0, 0), \quad I_3' = (I, 0, 0, 1, -1), \quad I_4' = (I, 1, -1, 1, -1).$$
There are four different foams for I_1', I_2', I_3', I_4'. These foams are part of the I-th component of the map $\phi\colon [\![D]\!] \to [\![D']\!]$, where we define $\phi_I := \phi_{I,1} + \phi_{I,2} + \phi_{I,3} + \phi_{I,4}$.

These four foams are as follows:

- $\phi_{I,1}$ is the identity foam;
- $\phi_{I,2}$ is the foam from Figure 10 at the first place where the Reidemeister move is applied, followed by the identity foam;
- $\phi_{I,3}$ is the identity foam followed by the foam from Figure 10 for the second Reidemeister move;
- $\phi_{I,4}$ is the foam from Figure 10 for the first Reidemeister 2a move, followed by the foam from Figure 10 for the second move.

We have $g \in \mathbb{Z}_2$, where the action involves switching pairs of points. For example, $gI_2 = I_3$ because g sends $(1,-1)$ to $(0,0)$ and $(0,0)$ to $(1,-1)$. Therefore, we have

$$g\phi_{I,1} = \phi_{gI,1},\ g\phi_{I,2} = \phi_{gI,3},\ g\phi_{I,3} = \phi_{gI,2},\ g\phi_{I,4} = \phi_{gI,4}. \tag{6.17}$$

This implies $g\phi_I = \phi_{gI}$. Thus, g commutes with $\Phi\colon [\![D]\!] \to [\![D']\!]$ up to sign. This proves the geometric part of step 3.

Let $\jmath$ be a sign assignment on diagram D. We extend $\jmath$ to a sign assignment $\jmath'$ on D' by adding crossings and applying Lemma 4.35. We add x_1, then x_2 and x_3, x_4. The analogy of Lemma 6.6 is as follows:

Lemma 6.18. *Assume $\jmath - g\jmath = \partial t$. Define the 0-cochain on* $\mathrm{SCube}(D')$ *as $t'(I, x_1, \dots, x_4) = t(I) + (x_1 + x_2)(x_3 + x_4)$. Then, $\jmath' - g\jmath' = \partial t'$.*

Proof. Take $I_1', I_2' \in \mathrm{Cube}(D')$ such that I_2' is an immediate successor of I_1'. Write $I_s' = (I_s, x_{1s}, x_{2s}, x_{3s}, x_{4s})$. By Lemma 6.14, we have

$$\jmath'(I_1', I_2') = \begin{cases} x_{j+1,1} + \dots + x_{41} & x_{j1} \neq x_{j2} \\ \jmath(I_1, I_2) & x_{j1} = x_{j2} \text{ for all } j. \end{cases}$$

We know that if $I' = (I, x_1, \dots, x_4) \in \mathrm{Cube}(D')$, then $gI' = (gI, x_3, x_4, x_1, x_2)$. Thus, we have

$$\jmath'(I_1', I_2') - \jmath'(gI_1', gI_2') = \begin{cases} x_{31} + x_{41} & x_{11} \neq x_{12} \text{ or } x_{21} \neq x_{22} \\ x_{11} + x_{21} & x_{31} \neq x_{32} \text{ or } x_{41} \neq x_{42} \\ t(I_1) - t(I_2) & x_{j1} = x_{j2} \text{ for all } j. \end{cases}$$

The proof is the same as in Lemma 6.15. In order to finish the proof of 5.29 at step 3, consider the diagram:

$$\begin{array}{ccc} [\![D, \jmath]\!] & \xrightarrow{[\![\rho_g, t]\!]} & [\![D, \jmath]\!] \\ \downarrow \Phi & & \downarrow \Phi \\ [\![D, \jmath']\!] & \xrightarrow{[\![\rho_g, t']\!]} & [\![D, g\jmath']\!]. \end{array}$$

We have already showed that the diagram above is commutative up to sign, and now by Lemma 6.18, we conclude that this diagram is commutative. This shows that Φ is $\mathbb{Z}_m$-equivariant. By Theorem 4.32, we know that Φ is a chain homotopy equivalence. Similarly to Steps 1 and 2, we conclude that $\mathcal{F}(\Phi)$ is a quasi-isomorphism in the $\mathrm{Kom}(\mathrm{Sym}_N[\mathbb{Z}_m])$ category. $\square$

7. The skein spectral sequence

7.1. **Review of the Ind and Res Functors.** We review the Ind and Res functors before constructing the spectral sequence. For a finite group G, we denote BG as the category with a single object $*$ and $\mathrm{Hom}_{BG}(*, *) = G$. If $\mathcal{B}$ is an additive category, we denote by $\mathcal{B}[G] = \mathrm{Fun}(BG, \mathcal{B})$ the category of G-objects in $\mathcal{B}$. For a subgroup H of G, we have a canonical inclusion of categories $BH \subset BG$, leading to the restriction functor $\mathrm{Res}^G_H\colon \mathcal{B}[G] \to \mathcal{B}[H]$. We also have the functor $\mathrm{Ind}^G_H\colon \mathcal{B}[H] \to \mathcal{B}[G]$, the biadjoint functor of Res^G_H. For $C \in B[H]$ and $D \in B[G]$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{B}[G]}(\mathrm{Ind}^G_H(C), D) &\cong \mathrm{Hom}_{\mathcal{B}[H]}(C, \mathrm{Res}^G_H(D)), \\ \mathrm{Hom}_{\mathcal{B}[G]}(C, \mathrm{Ind}^G_H(D)) &\cong \mathrm{Hom}_{\mathcal{B}[H]}(\mathrm{Res}^G_H(C), D). \end{aligned} \tag{7.1}$$

Assuming $G/H = \{g_1H, g_2H, \dots, g_kH\}$, then $\mathrm{Ind}^G_H(C)$ can be written as the direct sum

$$\mathrm{Ind}^G_H(C) = \bigoplus_{i=1}^{k} g_i C, \tag{7.2}$$

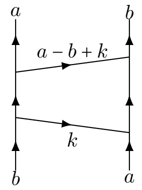


FIGURE 12. The k-smoothing of a positive crossing, for $0 \le k \le b \le a$. In order to obtain the k-smoothing of a negative crossing, reflect the above picture about the vertical line and switch labels.

For $g \in G$, we can write $g = g_i h$. If we write $g = g_j h'$ we have $g_i g_j \in H$ but g_i and g_j can not be in the same coset unless $g_i = g_j$. Thus for $g \in G$ there is a unique way to write $g = g_i h$. We have

$$g \cdot (-)\colon g_j C \to g_k C, \quad x \mapsto (h' g_j^{-1} h g_j) \cdot x,$$

where $g_k = g_i \cdot g_j \cdot h'$, with $h' \in H$ and g_k representing the coset of $g_i \cdot g_j$.

7.2. **Construction of the Spectral Sequence.** The initial construction will be done for the general link diagram, and later, we will focus on periodic link diagrams. Let D be a labeled link diagram, where each link component is labeled by $c \in \{1, 2, \dots N\}$. Recall that $\mathrm{Cr}(D)$ is the set of crossings.

We define the extended cube of resolutions $\mathrm{Cube}^+(D)$. For a crossing $i \in \mathrm{Cr}(D)$, we have $C_i = \{0, \dots, c_i\}$ where $c_i = \min(a_i, b_i)$, a_i and b_i are two labels at the crossing point i, if the crossing is positive, and $C_i = \{-c_i, \dots 0\}$ if the crossing is negative. We extend C_i by the definition $\hat{C}_i = C_i \cup \{*\}$. $\mathrm{Cube}^+(D)$ is the product of the $\hat{C}_i$.

For $\hat{I} \in \mathrm{Cube}^+(D)$, we define the resolution diagram $D_{\hat{I}}$. If the i-th crossing in $\hat{I}$ is equal to $*$, we do not resolve the crossing. Otherwise, we resolve the crossing as in the standard case. The resolutions are depicted in Figure 12, see also the skein relation in Figure 6. For $\hat{I} \in \mathrm{Cube}^+$, we define $\mathrm{supp}\, \hat{I}$ to be the set of crossings $i \in \mathrm{Cr}(D)$ where $\hat{I}(i) \neq *$. If $\hat{I}, \hat{J} \in \mathrm{Cube}^+$ and $\mathrm{supp}\, \hat{I} \cap \mathrm{supp}\, \hat{J} = \varnothing$, we define $\hat{I} \cup \hat{J}$ to be the resolution such that

$$(\widehat{I} \vee \widehat{J})(i) = \begin{cases} \widehat{I}(i) & i \in \mathrm{supp}\ \widehat{I} \\ \widehat{J}(i) & i \in \mathrm{supp}\ \widehat{J} \\ * & \text{otherwise.} \end{cases}$$

For $\hat{I}$ with support X, we define $[\![D_{\widehat{I}}]\!]$ as a cochain complex generated by those D_I for which I and $\hat{I}$ coincide on X. Furthermore, the differential is given by foams of Figure 8 with the sign assignment $\jmath_I$ inherited from the sign assignment $\jmath$ on D. We also define the degree for $\hat{I}$ as

$$\deg \widehat{I} = \sum_{i \in \mathrm{supp}\ \widehat{I}} I(i).$$

For a subset $X \subset \mathrm{Cr}(D)$, we let

$$A(X) = \{\widehat{I} \in \mathrm{Cube}^+(D) \colon \mathrm{supp}\ \widehat{I} = X\}, A_k(X) = \{\widehat{I} \in A(X) \colon \deg \widehat{I} = k\}$$

Let $X \subset \mathrm{Cr}(D)$ be a subset of positive crossings (for negative crossings, the discussion will be similar). Set $Y = \mathrm{Cr}(D) - X$. We let $\mathrm{Cube}(X)$, $\mathrm{Cube}(Y)$ be the cubes of resolution for X and Y. In other words, we have $\mathrm{Cube}(X) = \prod_{i \in X} C_i$, $\mathrm{Cube}(Y) = \prod_{i \in Y} C_i$. For $\mathrm{Cube}(D)$ we have $\mathrm{Cube}(D) = \mathrm{Cube}(X) \times \mathrm{Cube}(Y)$. For $I \in \mathrm{Cube}(D)$, we denote I_X, I_Y its projections on $\mathrm{Cube}(X)$ and on $\mathrm{Cube}(Y)$ respectively.

We introduce one more piece of notation. Assume we have $I \in \mathrm{Cube}(D)$. Let $\hat{I} \in \mathrm{Cube}^+(D)$ be obtained by taking I_X and extending it by putting $*$ for all crossings in Y. This means that crossings in X are already resolved so $D_{\hat{I}}$ has a set of crossings Y. This means $(D_{\hat{I}})_{I_Y} = D_I$. We can write $[\![D]\!]$ as the following bicomplex.

$$(7.3)\qquad 0\to \bigoplus_{\widehat{I}\in A_0(X)} [\![D_{\widehat{I}}]\!]\{q^{-b(N-b)|X|}\} \xrightarrow{\pm d_0} \bigoplus_{\widehat{I}\in A_1(X)} [\![D_{\widehat{I}}]\!]\{q^{1-b(N-b)|X|}\} \xrightarrow{\pm d_1} \dots.$$

Here q is the grading shift. The differentials d_i are defined as follows. Suppose I, $J\in \mathrm{Cube}(D)$ are such that J is an immediate successor of I. We have two cases

- Assume $I_X = J_X$, the part of the differential on $[\![D]\!]$ from I to J contributes to the differential on $[\![D_{\hat{I}}]\!]$. It goes from $(D_{\hat{I}})_{I_Y}$ to $(D_{\hat{I}})_{J_Y}$ with the sign $(-1)^{\mathcal{s}(I,J)}$. We call this differential part the internal differential or horizontal differential.
- Assume $I_Y = J_Y$, we set $s = \deg \widehat{I}$ the part of the differential on $[\![D]\!]$ that contributes to the differential d_s going from $[\![D_{\widehat{I}}]\!]$ to $[\![D_{\widehat{J}}]\!]$. In particular, it goes from $(D_{\hat{I}})_{I_Y}$ to $(D_{\hat{J}})_{J_Y}$ with sign $\mathcal{s}(I,J)$. We call this differential part the external differential or vertical differential.

The sum of these two differentials is equal to the differential on $[\![D]\!]$. Therefore, we have the following result.

Lemma 7.4. *The total complex* (7.3) *is equal to* $[\![D]\!]$.

In general, a bicomplex leads to a spectral sequence. To obtain a bicomplex that gives rise to a spectral sequence, we apply functor F to (7.3) so that it operates in an Abelian category. To be more precise, we define the triply graded bicomplex

$$M(D,X)^{k,\ell,h} = \bigoplus_{\widehat{I}\in A_k(X)} \mathcal{F}([\![D_{\widehat{I}}]\!])\{q^{k-|X|b(N-b)}\}_{\ell,h}.$$

Here ℓ is the homology grading and q is the quantum grading. If X is a subset of negative crossings, we define

$$M(D,X)^{k,\ell,h} = \bigoplus_{\widehat{I}\in A_k(X)} \mathcal{F}([\![D_{\widehat{I}}]\!])\{q^{-k+|X|b(N-b)}\}_{\ell,h}.$$

In the bicomplex $M(D,X)^{\bullet,\ell,h}$, we have an internal (horizontal) differential and the external (vertical) differential going from $M(D,X)^{\bullet,\ell,h}$ to $M(D,X)^{\bullet+1,\ell,h}$.

Lemma 7.5. *The cohomology of the total complex* $\mathrm{Tot}^{r,h} M(D,X) = \oplus_{k+\ell=r} M(D,X)^{k,\ell,h}$ *is the* $\mathbb{S}_N$*-valued Khovanov-Rozansky homology of the link.*

Proof. The statement is tautological. By construction $\mathrm{Tot}^{r,h}$ is the chain complex whose module structure is the same as the $\mathbb{S}_N$-valued $\mathfrak{sl}_N$ chain complex associated with D. The total differential (the sum of the horizontal differential and the vertical differential) is the $\mathfrak{sl}_N$-differential. □

Assume that D is a $\mathbb{Z}_m$ periodic link diagram. Our primary focus will be on the case when X is an orbit of crossings. In this case, $\mathbb{Z}_m$ acts on $\mathrm{Cr}(D)$ and it preserves X. For any k, this action can be induced on $A_k(X)$. For $\hat{I}\in A_k(X)$, define the isotropy group of $\hat{I}$ $\mathrm{Iso}(\widehat{I}) = \{g\in \mathbb{Z}_m\colon \widehat{I}\circ g = \widehat{I}\}$. For any $d|m$ define

$$(7.6)\qquad A_k^d(X) = \{\widehat{I}\in A_k(X)\colon \mathrm{Iso}(\widehat{I}) = \mathbb{Z}_d\}$$

and denote by $\overline{A}_k^d(X)$ the quotient of $A_k^d(X)$ by the action of $\mathbb{Z}_m$. Notice that for $\hat{I}\in A_k^d(X)$, the diagram $D_{\widehat{I}}$ is d-periodic. Furthermore, for any $g\in G$ with the group action on $[\![D]\!]$, we have a map $\mathcal{G}_{g,\widehat{I}}\colon [\![D_{\widehat{I}}, \mathcal{s}_{\widehat{I}}]\!] \to [\![D_{g\widehat{I}}, \mathcal{s}_{g\widehat{I}}]\!]$, where $\mathcal{s}_{\widehat{I}}$ and $\mathcal{s}_{g\widehat{I}}$ denote restrictions of the sign assignment $\mathcal{s}$ on $\mathrm{Cube}(D)$ to $\mathrm{Cube}(D_{\widehat{I}})$ and $\mathrm{Cube}(D_{g\widehat{I}})$, respectively.

Lemma 7.4 can be generalized for an equivariant setting. Assume X is a set of crossings in which either all crossings in X are positive or all crossings in X are negative. Additionally, suppose X is $\mathbb{Z}_m$ invariant. Note that $D_{\hat{I}}$ is d–periodic diagram for any $\hat{I}\in A_k^d(X)$. We have the natural $\mathbb{Z}_d$-action on $[\![D_{\widehat{I}}]\!]$ and $\mathcal{F}([\![D_{\widehat{I}}]\!])$ as defined in Proposition 5.22. We define the equivariant version of the bicomplex $M(D,X)^{k,\ell,h}$ by

(7.7)
$$\mathrm{EM}(D,X)^{k,l,\bullet} = \begin{cases} \bigoplus_{d|k} \bigoplus_{\widehat{I}\in \overline{A}^d_k(X)} \mathrm{Ind}^{\mathbb{Z}_m}_{\mathbb{Z}_d} \left(\mathcal{F}([\![D_{\widehat{I}}]\!])\{q^{-|X|b(N-b)+k}\} \otimes \mathbb{C}_{s(m,d,\widehat{I})}\right) & X \text{ is positive,} \\ \bigoplus_{d|k} \bigoplus_{\widehat{I}\in \overline{A}^d_k(X)} \mathrm{Ind}^{\mathbb{Z}_m}_{\mathbb{Z}_d} \left(\mathcal{F}([\![D_{\widehat{I}}]\!])\{q^{|X|b(N-b)-k}\} \otimes \mathbb{C}_{s(m,d,\widehat{I})}\right) & X \text{ is negative.} \end{cases}$$

where $s(m,d,\widehat{I}) \in \mathbb{F}_2$:

$$s(m,d,\widehat{I}) = t(I_0) + t(gI_0) + \cdots + t(g^{m/d-1}I_0) \tag{7.8}$$

Here, we take the tensor product over the ring $\mathbb{C}[\mathbb{Z}_d]$ and we think $\mathrm{Sym}_N[\mathbb{Z}_d]$-module $\mathcal{F}([\![D_{\widehat{I}}]\!])$ as a right $\mathbb{C}[\mathbb{Z}_d]$-module with the standard action of $\mathbb{C}$ on Sym_N. On the one-dimensional complex vector space $\mathbb{C}_j$, $\mathbb{Z}_d$ acts either trivially if $j=0$ or it acts as the sign action, i.e., the generator of $\mathbb{Z}_d$ acts on $\mathbb{C}$ by multiplication by -1, if $j=1$. Also, here $I_0 = \widehat{I} \vee J_0$ for $J_0 = (0,\dots,0) \in \mathrm{Cube}(D_{\widehat{I}})$ and t a 0-cochain on $\mathrm{SCube}(D)$ satisfying $g\jmath - \jmath = \partial t$, $t((0,\dots,0)) = 0$.

Lemma 7.9. *We have an isomorphism* $\mathrm{EM}(D,X) \cong M(D,X)$ *as complexes of* $\mathbb{S}_N$*-modules.*

Proof. Since we need to show they are isomorphic as $\mathbb{S}_N$-modules, we do not care about the action of $\mathbb{C}_{s(m,d,\widehat{I})}$. It is enough to show that both sides have the same $\mathcal{F}([\![D_{\widehat{I}}]\!])$. For any $\widehat{I} \in A_k(X)$, this $\widehat{I}$ must be in one of $A^d_k(X)$ for $d|k$. Furthermore, we can get this $\widehat{I}$ from $\widehat{J} \in \overline{A}^d_k(X)$ such that $g\widehat{J} = \widehat{I}$ where $g \in \mathbb{Z}_m/\mathbb{Z}_d$. For any $\widehat{I}$ for $\widehat{I} \in A_k(X)$, we have $\mathcal{F}([\![D_{\widehat{I}}]\!] = \mathcal{F}([\![D_{g\widehat{J}}]\!])$ for $J \in \overline{A}^d_k(X)$ and for $g \in \mathbb{Z}_m/\mathbb{Z}_d$. □

Lemma 7.10. *We have an isomorphism between the total complex of* $\mathrm{EM}(D,X) \cong M(D,X)$ *as complexes of* $\mathbb{S}_N[\mathbb{Z}_m]$*-modules.*

Proof. By Lemma 7.9, we need to show that the isomorphism between $\mathrm{EM}(D,X)$ and $\mathcal{F}([\![D]\!])$ as $\mathbb{S}_N$-modules is $\mathbb{Z}_m$-equivariant. Recall that we have g as a generator of $\mathbb{Z}_m$ acting on the plane by rotation by the angle $\frac{2\pi}{m}$. Fix a sign assignment $\jmath$ on D, and let t be the 0-cochain satisfying $\partial t = g\jmath - \jmath$, $t((0,\dots,0)) = 0$. For a divisor d of m set $h = g^{m/d}$ to be a generator of $\mathbb{Z}_d \subset \mathbb{Z}_m$. Take $\widehat{I} \in \overline{A}^d_k(X)$ and consider the partial resolution $D_{\widehat{I}}$. Define $\jmath_{\widehat{I}}$ to be the sign assignment on $\mathrm{Cube}(D_{\widehat{I}})$, defined as $\jmath_{\widehat{I}}(J,J') = \jmath(\widehat{I}\vee J, \widehat{I}\vee J')$. Since $D_{\widehat{I}}$ is a d-periodic diagram, by Proposition 5.22 we can define an action of $\mathbb{Z}_d$ on $[\![D_{\widehat{I}}]\!]$. Specifically, we let $t_{\widehat{I}}$ be the 0-cochain on $\mathrm{Cube}(D_{\widehat{I}})$ such that $h\jmath_{\widehat{I}} - \jmath_{\widehat{I}} = t_{\widehat{I}}$ and $t_{\widehat{I}}(0,\dots,0) = 0$. Corresponding to the action of h (i.e. rotation by the angle $\frac{2\pi}{d}$) we have the map $\mathcal{H}_h\colon [\![D_{\widehat{I}}]\!] \to [\![D_{\widehat{I}}]\!]$.

There are two maps that are induced by the action of h on $[\![D_{\widehat{I}}]\!]$. The external one is $(\mathcal{G}_g)^{m/d}$, where $\mathcal{G}_g$ is the action constructed in Proposition 5.22 for $[\![D]\!]$. The other map is $\mathcal{H}_h$. Since these two maps are obtained from the same sets of foams, these two maps are actually equal up to a sign choice. To complete the proof of Lemma 7.10, we need to compare $t_{\widehat{I}}(J)$ and $t(\widehat{I}\vee\widehat{J})$ for $J \in \mathrm{Cube}(D_{\widehat{I}})$. We know for any two 0-cochains t_1, t_2 such that $h\jmath_{\widehat{I}} - \jmath_{\widehat{I}} = \partial t_1 = \partial t_2$. We can say $J \mapsto t_{\widehat{I}}(J)$ and $J \mapsto t(\widehat{I}\vee J)$ are either equal or differ by an overall sign. To understand this sign issue, let $J_0 = (0,\dots,0) \in \mathrm{Cube}(D_{\widehat{I}})$, set $I_0 = J_0 \vee \widehat{I}$. Suppose that $\rho_g\colon D_{I_0} \to D_{gI_0}$ is the foam realizing the rotation of D_{I_0} by $g \in \mathbb{Z}$, i.e., the I_0-th component of $\mathcal{G}_g$ is equal to $(-1)^{t(I_0)}\rho_g$. Let $\rho_h = \rho_{g^{m/d-1}I_0} \circ \cdots \circ \rho_{gI_0} \circ \rho_{I_0}$ and $t_h(I_0) = t(I_0) + t(gI_0) + \cdots + t(g^{m/d-1}I_0)$, then the I_0-th component of $\mathcal{H}_h$ is equal to $(-1)^{t_h(I_0)}\rho_h$. By the proof of Proposition 5.22, $t_{\widehat{I}}(0,\dots,0) = 0$. In other words, the J_0-th component of $\mathcal{H}_h$ is equal to ρ_h. Therefore $s(m,d,\widehat{I}) = t_h$. We conclude by (7.8). □

Proposition 7.11 (Skein spectral sequence)**.** *Let D be an $m = p^\ell$-periodic labeled link diagram, with p an odd prime and $\ell \geq 1$. Let $X \subset \mathrm{Cr}(D)$ be an orbit of crossings between an a-labeled* *overstrand and a b-labeled understrand, where $a \geq b$. If $0 \leq u \leq \ell$ and X is a set of positive crossings, we obtain, for any $1 \leq s \leq |X|b$, a spectral sequence with*

$$E_1^{k,l,\bullet}(D,X,p^{\ell-u}) = \bigoplus_{p^s|k}\bigoplus_{\widehat{I}\in\overline{A}_k^s(D,X)} \mathrm{EKR}_N^{\bullet,\bullet}(D_{\widehat{I}},\kappa(u,s))^{\oplus\lambda(u,s)}t^k q^{-|X|b(N-b)-k}, \tag{7.12}$$

with $0 \leq k \leq p^\ell b$ and

$$\kappa(u,s) = \begin{cases} 1, & u \geq s, \\ p^{s-u}, & \text{otherwise}, \end{cases} \qquad \lambda(u,s) = \begin{cases} \phi(p^{\ell-u}), & u \geq s, \\ p^{\ell-s}, & u < s, \end{cases}$$

converging to $\mathrm{EKR}_N^{\bullet,\bullet}(D,p^{\ell-u})$. On the other hand, if X is the set of negative crossings, we obtain a spectral sequence with

$$E_1^{k,l,\bullet}(D,X,p^{\ell-u}) = \bigoplus_{p^s|k}\bigoplus_{\widehat{I}\in\overline{A}_{p^\ell+k}^s(D,X)} \mathrm{EKR}_N^{\bullet,\bullet}(D_{\widehat{I}},\kappa(u,s))^{\oplus\lambda(u,s)}t^k q^{|X|b(N-b)-k},$$

where $-p^\ell b \leq k \leq 0$.

Proof. We prove this proposition only in the positive case. Note that the total complex of $\mathrm{EM}(D,X)$ is the complex of $\mathcal{F}([\![D]\!])$ by Lemma 7.10. We will denote the singular specialization of $\mathrm{EM}(D,X)$ by $\mathrm{EM}_0(D,X)$. We fix $0 \leq u \leq \ell$ and consider the bicomplex derived from $\mathrm{EM}(D,X)$:

$$\mathrm{EM}^{k,l,*}(D,X,p^{\ell-u}) := \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}, \mathrm{EM}_0^{k,l,*}(D,X)).$$

On considering separately the internal (vertical) and the external (horizontal) differentials in $\mathrm{EM}(D,X,p^{\ell-u})$, we obtain a spectral sequence of $\mathbb{C}[\mathbb{Z}_m]$-modules converging to $\mathrm{EKR}_N^{*,*}(D,p^{\ell-u})$, whose E_1-page is given by

$$\begin{aligned} E_1^{k,l,*}(D,X,p^{\ell-u}) &= H^{k,*}(\mathrm{EM}_0^{*,l,*}(D,X,p^{\ell-u}),d_{\mathrm{vert}}) \\ &\cong \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}, H^{k,*}(\mathrm{EM}_0^{*,l,*}(D,X),d_{\mathrm{vert}})). \end{aligned}$$

i.e., we take the vertical homology of $\mathrm{EM}(D,X,p^{\ell-u})$. The aim of the proof is to show that this page is isomorphic to (7.12). Consider the decomposition of the group algebra $\mathbb{C}[\mathbb{Z}_{p^\ell}]$. Recall from Section 5.8 that

$$\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}} = \bigoplus_{\substack{0\leq i<p^\ell \\ \gcd(i,p^\ell)=p^u}} \mathbb{C}_{\xi_{p^\ell}^i}.$$

Observe that for any $0 \leq s \leq \ell$ we have

$$\mathrm{Res}_{\mathbb{Z}_{p^s}}^{\mathbb{Z}_{p^\ell}}(\mathbb{C}_{\xi_{p^{\ell-u}}^j}) = \mathbb{C}_{(\xi_{p^{\ell-u}}^j)^{p^{\ell-s}}} = \begin{cases} \mathbb{C}_1, & s \leq u, \\ \mathbb{C}_{\xi_{p^{s-u}}^j}, & s > u. \end{cases}$$

Therefore,

$$\mathrm{Res}_{\mathbb{Z}_{p^s}}^{\mathbb{Z}_{p^\ell}}(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}) = \begin{cases} \mathbb{C}_1^{\phi(p^{\ell-u})}, & s \leq u, \\ \mathbb{C}[\mathbb{Z}_{p^s}]_{p^{s-u}}^{p^{\ell-s}}, & s > u. \end{cases} \tag{7.13}$$

By the definition of $\mathrm{EM}_0(D,X)$, we obtain

$$\begin{aligned} E_1^{k,l,\bullet}(D,X,p^{\ell-u}) &= \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}, H^{k,*}(\mathrm{EM}_0^{*,l,*}(D,X),d_{\mathrm{vert}})) \\ &\cong \bigoplus_{p^s|k}\bigoplus_{\widehat{I}\in\overline{A}_k^s(D,X)} \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}\Big(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}, \mathrm{Ind}_{\mathbb{Z}_{p^s}}^{\mathbb{Z}_{p^\ell}}\mathrm{EKR}(D_{\widehat{I}})t^k q^{-|X|b(N-b)+k}\Big). \end{aligned}$$

Consider the right-hand side of the above equation:

$$\begin{aligned}
&\mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}\left(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}, \mathrm{Ind}_{\mathbb{Z}_{p^s}}^{\mathbb{Z}_{p^\ell}} \mathrm{EKR}(D_{\widehat{T}})t^k q^{-|X|b(N-b)+k}\right)\\
&\overset{(7.1)}{\cong} \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^s}]}\left(\mathrm{Res}_{\mathbb{Z}_{p^s}}^{\mathbb{Z}_{p^\ell}}(\mathbb{C}[\mathbb{Z}_{p^\ell}]_{p^{\ell-u}}), \mathrm{EKR}(D_{\widehat{T}})t^k q^{-|X|b(N-b)+k}\right)\\
&\overset{(7.13)}{=} \begin{cases} \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}(\mathbb{C}_1^{\phi(p^{\ell-u})}, \mathrm{EKR}(D_{\widehat{T}})t^k q^{-|X|b(N-b)+k}), & s \le u,\\ \mathrm{Hom}_{\mathbb{C}[\mathbb{Z}_{p^\ell}]}(\mathbb{C}[\mathbb{Z}_{p^s}]_{p^{s-u}}^{p^{\ell-s}}, \mathrm{EKR}(D_{\widehat{T}})t^k q^{-|X|b(N-b)+k}) & s > u \end{cases}\\
&= \begin{cases} \mathrm{EKR}^{*,*}(D_{\widehat{T}}, 1)^{\oplus\phi(p^{\ell-u})} t^k q^{-|X|b(N-b)+k}, & s \le u\\ \mathrm{EKR}^{*,*}(D_{\widehat{T}}, p^{s-u})^{\oplus p^{\ell-s}} t^k q^{-|X|b(N-b)+k}, & s > u \end{cases}\\
&= \mathrm{EKR}^{*,*}(D_{\widehat{T}}, \kappa(u,s))^{\lambda(u,s)} t^k q^{-|X|b(N-b)+k}.
\end{aligned}$$

The proposition follows. □

8. Polynomial Invariants

8.1. **Poincare polynomials of $\mathfrak{sl}_N$ and Lee homology.** First, we remind a common construction.

Definition 8.1. Let L be a link. The LeeP_N polynomial is

$$\mathrm{LeeP}_N(L) = \sum_{k,r} \dim_{\mathbb{C}} \mathrm{Gr}^r \mathrm{Lee}_N^k(L) t^k q^r,$$

where Gr^r is the r-th graded part of the filtered Lee_N homology, and the Khovanov–Rozansky polynomial $\mathrm{KRP}_N(K)$ is the Poincaré polynomial of $\mathfrak{sl}_N$-homology:

$$\mathrm{KRP}_N(L) = \sum_{k,r} t^k q^r \dim_{\mathbb{C}} \mathrm{KR}_N^{k,r}(L).$$

For an m-periodic link, we modify the definition above and generalize the approach of [23].

Definition 8.2. Assume we have an m-periodic link L and let $d|m$. The *equivariant Khovanov–Rozansky polynomial*, for $\mathfrak{sl}_N$-homology, is

$$\mathrm{KRP}_{N,d}(L) = \sum_{k,r} t^k q^r \dim_{\mathbb{C}_d} \mathrm{EKR}_N^{k,r,d}(L). \tag{8.3}$$

The *equivariant Lee polynomial* is:

$$\mathrm{LeeP}_{N,d}(L) = \sum_{k,r} \dim_{\mathbb{C}_d} \mathrm{Gr}^r \mathrm{ELee}_N^{k,d}(L) t^k q^r,$$

We have the following relation between the Khovanov–Rozansky polynomial and the equivariant Khovanov–Rozansky polynomial.

$$\mathrm{KRP}_N(L) = \sum_{d|m} \phi(d) \mathrm{KRP}_{N,d}(L), \tag{8.4}$$

where $\phi(d) = \#\{1 \le i \le d : \gcd(i,d) = 1\}$ is Euler's totient function.

We can compute Lee homology from Proposition 5.21. For the precise formula for the knot, we refer to [15, Proposition 2.6]. Other references include [10, 16, 17, 29, 33].

Lemma 8.5. *For any knot K, we have* $\mathrm{LeeP}_N(K) = q^{s_N(K)}(q^{-N+1} + q^{-N+3} + \cdots + q^{N-1})$*, where $s_N(K)$ is the Lewark's s_N-invariant; see* [15].

We have the following statement as a consequence of Lemma 5.35.

Lemma 8.6. *If the action of $\mathbb{Z}_m$ on the components of L is trivial, then* $\mathrm{LeeP}_{N,d}$ *is equal to* LeeP_N *if $d = 1$, and* $\mathrm{LeeP}_{N,d}$ *is equal to* 0 *otherwise.*

The following proposition shows the relation between polynomials KRP and LeeP. Its proof is the same as in the Khovanov case, see [4, Proposition 2.17]. See also [8, Theorem 5.1] and [15, Proposition 5.2].

Proposition 8.7. *For a link L, there are polynomials $R_1, R_2, \dots$ with non-negative coefficients such that*

$$\mathrm{KRP}_N(L) = \mathrm{LeeP}_N(L) + (1 + tq^{2N})R_1 + (1 + tq^{4N})R_2 + \dots.$$

Furthermore, for an m-periodic link L where $d|m$, we have

$$\mathrm{KRP}_{N,d}(L) = \mathrm{LeeP}_{N,d}(L) + (1 + tq^{2N})R_1^d + (1 + tq^{4N})R_2^d + \dots$$

for polynomials $R_1^d, R_2^d, \dots$ with non-negative coefficients.

8.2. **The Reshetikhin-Turaev RT_N polynomials.** We recall that for a link L, the HOMFLYPT polynomial $X(a,b)$ is defined by its value on the unknot and skein relation.

(8.8) $$aX_{L_+}(a,b) - a^{-1}X_{L_-}(a,b) = bX_{L_0}(a,b),$$

where L_0 is the 0 resolution, L_+ is the positive crossing, and L_- is the negative crossing.

Reshetikhin-Turaev is a specific case of the HOMFLYPT polynomial. For $N \geq 0$ Reshetikhin-Turaev is

(8.9) $$\mathrm{RT}_N(q) = X(q^N, q - q^{-1})$$

The normalization of this polynomial is

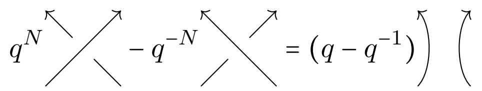
$$\mathrm{RT}_N(\text{unknot}) = \frac{q^N - q^{-N}}{q - q^{-1}}.$$

For $N = 0$, RT_0 is the Alexander polynomial, and for $N = 1$, $RT_1 \equiv 1$, and for $N = 2$, we have the Jones polynomial which categorifies Khovanov homology. For $N > 2$, we call these polynomials as $\mathfrak{sl}_N$ polynomials of L. In [12,13] it was proved that $\mathfrak{sl}_N$ homology categorifies the $\mathfrak{sl}_N$ polynomial.

Lemma 8.10. *For a link L and for $\mathrm{KR}_N^{k,r}(L)$ its $\mathfrak{sl}_N$-homology, we have*

$$\mathrm{RT}_N(L) = \sum_{k,r}(-1)^k q^r \dim \mathrm{KR}_N^{k,r}(L) = \mathrm{KRP}_N|_{t=-1}.$$

The skein relation for RT_N polynomial is a particular version of the skein relation of the HOMFLYPT polynomial.

(8.11)


8.3. **Difference polynomials.** Fix $m = p^l$ for a prime p, and let D be an m-periodic diagram of an m-periodic link L. The $\mathfrak{sl}_N$ homology of L decomposes as in (5.34). We have

$$\mathrm{RT}_{N,j} = \mathrm{KRP}_{N,p^j}|_{t=-1},$$

where $\mathrm{KRP}_{N,d}$ is as in (8.3).

We have the corollary that will be used in the future.

Corollary 8.12. *Assume L is a p^n periodic link for the prime p, and assume L' is its mirror; then $\mathrm{RT}_{N,j}(L)(q) = \mathrm{RT}_{N,j}(L')(q^{-1})$.*

Proof. We know from Proposition 5.32 we have an isomorphism of $\mathbb{C}$-vector spaces $\mathrm{EKR}^{k,r,m}(L) = \mathrm{EKR}^{-k,-r,m}(L')$. Thus, by (8.3), we have

$$RT_{N,j}(L)(q) = \sum_{k,r}(-1)^k q^r \dim \mathrm{EKR}_N^{k,r,p_j}(L) = \sum_{k,r}(-1)^k q^r \dim \mathrm{EKR}_N^{-k,-r,p_j}(L')$$

Make a change of variables: set $k' = -k$, $r' = -r$ then we have

$$RT_{N,j}(L)(q) = \sum_{k',r'}(-1)^{-k'}q^{-r'}\dim \mathrm{EKR}_N^{k',r',p_j}(L') = \sum_{k',r'}(-1)^{k'}(q^{-1})^{r'}\dim \mathrm{EKR}_N^{k',r',p_j}(L')$$

$$\sum_{k',r'}(-1)^{k'}(q^{-1})^{r'}\dim \mathrm{EKR}_N^{k',r',p_j}(L') = RT_{N,j}(L)(q^{-1})$$

we have

$$RT_{N,j}(L')(q) = RT_{N,j}(L)(q^{-1})$$

□

For Reshetikhin-Turaev, we have difference $\mathfrak{sl}_N$ polynomials.

Definition 8.13. $\mathrm{DRT}_{N,j}(D) = \begin{cases}\mathrm{RT}_{N,p^j}(D) - \mathrm{RT}_{N,p^{j+1}}(D) & 0 \le j < \ell \\ \mathrm{RT}_{N,p^\ell}(D) & j = \ell.\end{cases}$

Proposition 8.14. $\mathrm{DRT}_{N,j}(D)$ *polynomials have the following relations between each other.*

(1) For $j = 0$ we have

$$q^{mN}\,\mathrm{DRT}_{N,0}(L_+) - q^{-mN}\,\mathrm{DRT}_{N,0}(L_-) = (q^{-m} - q^m)\,\mathrm{DJ}_{N,0}(L_0).$$

(2) For any $0 \le j < \ell$, we have

$$q^{mN}\,\mathrm{DRT}_{N,\ell-j}(L_+) - q^{-mN}\,\mathrm{DRT}_{N,\ell-j}(L_-) \equiv (q^{-m} - q^m)\,\mathrm{DJ}_{N,\ell-j}(L_0) \pmod{q^{p^j} - q^{-p^j}}.$$

Proof. We use [23, Theorem 3.6]. Assume that $\{E_r^{*,*}, d_r\}_{r\ge1}$ is a spectral sequence of graded finite-dimensional $\mathbb{C}$-vector spaces which converge to a double-graded $\mathbb{C}$-vector space $H^{*,*}$. Moreover, assume the spectral sequence collapses at a finite stage. Consider the Poincaré polynomials of the page $E_r^{*,*}$:

$$P(E_r^{*,*}) = \sum_{i,j} t^{i+j}\,\mathrm{qdim}_{\mathbb{C}}\,E_r^{i,j}.$$

For a graded $\mathbb{C}$-vector space V^*, we have

$$\mathrm{qdim}_{\mathbb{C}}\,V^* = \sum_i q^i \dim_{\mathbb{C}} V^i.$$

By [19, Exercise 1.7], we conclude that for any $r \ge 1$,

$$P(E_r^{*,*})(-1,q) = P(E_\infty^{*,*})(-1,q) = \sum_{i,j}(-1)^i\,\mathrm{qdim}_{\mathbb{C}}\,H^{i,*}. \tag{8.15}$$

For a fixed p^l-periodic diagram D we apply (8.15) to spectral sequences constructed in Proposition 7.11. We get

$$P(E_1^{*,*}(p^{\ell-u}))(-1,q) = P(E_\infty^{*,*})(-1,q) = \mathrm{RT}_{N,\ell-u}(D).$$

Recall that $E_1^{*,*}(p^{\ell-u})$ is the first page of the homology of a diagram which is invariant under the action of a subgroup of order p^t for $t \le \ell - u$ smaller order. The description of $E_1^{*,*}(p^{\ell-u})$ implies that $P(E_1^{*,*})(-1,q)$ is a linear combination of polynomials $\mathrm{RT}_{N,j}(D_{\widehat{I}})$, where $\widehat{I} \in A_k(X)$ and appropriate j. Consequently,

$$\begin{aligned} DP_{N,\ell-u}(D) &= \mathrm{RT}_{N,\ell-u}(D) - \mathrm{RT}_{N,\ell-u+1}(D) = \\ &= P(E_1^{*,*}(p^{\ell-u}))(-1,q) - P(E_1^{*,*}(p^{\ell-u+1}))(-1,q). \end{aligned} \tag{8.16}$$

we apply formula (8.16) to $DP_{N,\ell-u}(L_+)$ and $DP_{N,\ell-u}(L_+)$. We get

$$DP_{N,\ell-u}(L_+) = \sum_{k=0}^{p^\ell}\sum_{s=u}^{\ell}\sum_{\widehat{I}\in A_k^s(L_+,X)}(-1)^k q^{-p^\ell(N-1)-k}DP_{N,s-u}(D_{\widehat{I}}),$$

$$DP_{N,\ell-u}(L_-) = \sum_{k=-p^\ell}^{0}\sum_{s=u}^{\ell}\sum_{\widehat{I}\in A^s_{p^\ell+k}(L_-,X)}(-1)^k q^{p^\ell(N-1)+p^\ell-k}DP_{N,s-u}(D_{\widehat{I}}).$$

By $A_k(L_+,X) = A_{p^\ell-k}(L_-,X)$, we get

$$q^{p^\ell N}DP_{N,\ell-u}(L_+) - q^{-p^\ell N}DP_{N,\ell-u}(L_-) = \sum_{k=0}^{p^\ell}\sum_{s=u}^{\ell}\sum_{\widehat{I}\in A_k^s(L_+,X)}(-1)^k(q^{p^\ell-k} - q^{-p^\ell+k})DP_{N,s-u}(D_{\widehat{I}}).$$

We know $A_k^s(L_+,X)$ is empty unless p^s divides k. In the above equation observe that for $k=0$ we have $(q^{p^\ell} - q^{-p^\ell})\,\mathrm{DRT}_{N,\ell-u}(L_0)$ and for $k=p^\ell$ the sum is zero Hence we have

$$q^{p^\ell N}\,\mathrm{DRT}_{N,\ell-u}(L_+) - q^{-p^\ell N}\,\mathrm{DRT}_{N,\ell-u}(L_-) - (q^{p^\ell} - q^{-p^\ell})\,\mathrm{DRT}_{N,\ell-u}(L_0) =$$
$$\sum_{k=1}^{p^\ell-1}\sum_{s=u}^{\ell}\sum_{\widehat{I}\in A_k^s(L_+,X)}(-1)^k(q^{p^\ell-k} - q^{-p^\ell+k})\,\mathrm{DRT}_{N,s-u}(D_{\widehat{I}}).$$

For $u=\ell$, since $s=u$ and $u=\ell$ we have $\ell=s$ which implies $k\le p^s-1$ so p^s can not divide k. Hence the right-hand side is zero. We have

$$q^{p^\ell N}DP_{N,0}(L_+) - q^{-p^\ell N}DP_{N,0}(L_-) = (q^{p^\ell} - q^{-p^\ell})DP_{N,0}(L_0),$$

as we want.

For $0\le u<\ell$ and for $u\le s\le\ell$ and k divisible by p^s, we write $k=k'p^s$.

$$p^\ell - k = p^\ell - k'p^s = p^s(p^{\ell-s}-k')$$

Set $p^{\ell-s}-k' = A$. We have

$$q^{p^\ell-k} - q^{-p^\ell+k} = q^{p^sA} - q^{-p^sA} = q^{-p^sA}(q^{2p^sA}-1)$$

Since $q^{p^u} - q^{-p^u} \equiv 0$ (mod $q^{p^u}-q^{-p^u}$) , we have $q^{2p^u}\equiv 1$ (mod $q^{p^u}-q^{-p^u}$) Hence

$$q^{p^\ell-k} - q^{-p^\ell+k} = q^{p^sA} - q^{-p^sA} = q^{-p^sA}(q^{2p^sA}-1) \equiv 0 (mod q^{p^u} - q^{-p^u})$$

We deduce by the above equations

$$(q^{p^\ell-k} - q^{-p^\ell+k})\,\mathrm{DRT}_{N,s-u}(D_{\widehat{I}}) \equiv 0 \pmod{q^{p^u}-q^{-p^u}}.$$

Consequently,

$$q^{p^\ell N}\,\mathrm{DRT}_{N,0}(L_+) - q^{-p^\ell N}\,\mathrm{DRT}_{N,0}(L_-) \equiv (q^{p^\ell} - q^{-p^\ell})\,\mathrm{DRT}_{N,0}(L_0) \pmod{q^{p^u}-q^{-p^u}}.$$

□

8.4. **Periodicity criterion.** The result in this section ports the periodicity criterion of [4] to the case of $\mathfrak{sl}_N$-homology.

Theorem 8.17. *Assume L is an $m=p^\ell$ periodic knot with p a prime. Then, there exist polynomials $\mathcal{P}_0,\mathcal{P}_1,\dots$ such that*

$$\mathrm{KRP}_N = \mathcal{P}_0 + \sum_{j=1}^{\ell}(p^j - p^{j-1})\mathcal{P}_j.$$

In this equation $\mathcal{P}_0,\dots$ are Laurent polynomials in t,q such that

(P-1) The Laurent polynomial $\mathcal{P}_0$ *can be presented as*

$$\mathcal{P}_0 = q^{s_N(L)}(q^{1-N} + q^{3-N} + \cdots + q^{N-1}) + \sum_{j=1}^{\infty}(1 + tq^{Nj})\mathcal{S}_{0j}(t,q),$$

while the Laurent polynomials $\mathcal{P}_k$*,* $k > 0$*, can be presented as*

$$\mathcal{P}_k = \sum_{j=1}^{\infty}(1 + tq^{Nj})\mathcal{S}_{kj}(t,q).$$

(P-2) The Laurent polynomials $\mathcal{S}_{kj}$*,* $k \geq 0$*, from item (P-1) have non-negative coefficients.*
(P-3) The polynomials $\mathcal{P}_k$*,* $k \geq 0$*, satisfy the following congruence relation:*

$$\mathcal{P}_k(-1,q) - \mathcal{P}_{k+1}(-1,q) \equiv \mathcal{P}_k(-1,q^{-1}) - \mathcal{P}_{k+1}(-1,q^{-1}) \pmod{q^{p^{\ell-k}} - q^{-p^{\ell-k}}}.$$

Proof. For integral k,r, we have $\mathrm{KR}_N^{k,r}(L) = \mathrm{EKR}_N^{k,r}(L)$ as vector spaces. The latter have decomposition as in (5.34):

$$\mathrm{EKR}_N^{k,r}(L) = \bigoplus_{d|m}\mathrm{EKR}_N^{k,r,d}(L).$$

We have $m = p^\ell$, and we have $\mathcal{P}_j = P_{N,p^j}$ as the Poincaré polynomial of $\mathrm{EKR}_N^{\bullet,\bullet,p^j}(L)$. By the (8.4), we have

$$\mathrm{KRP}_N(L) = \sum_{j=0}^{\infty}(p^j - p^{j-1})\mathcal{P}_j,$$

where $p^j - p^{j-1}$ is the Euler's totient function for p^j. In this equation, $\mathcal{P}_j$ is equal to the $\mathrm{KRP}_{N,d}$ in Proposition 8.7. The sum above is finite because E_1 page has modules of finite dimension over $\mathbb{C}$. Since E_1 is a finite spectral sequence that degenerates in a finite page, so the Poincaré polynomial of the page gets zero. Hence, write $\mathcal{S}_{jk} = R_k^{p^j}$ we have

$$\mathcal{P}_j = \mathrm{LeeP}_{N,p^j}(L) + \sum_{k=1}^{\infty}(1 + tq^{2Nk})\mathcal{S}_{jk}.$$

By Proposition 8.7, we know $\mathcal{S}_{jk}$ is non-negative. The computation of ELee in Lemma 8.6, together with Lemma 8.5, gives

$$\mathrm{LeeP}_{N,p^0}(L) = q^{s_N(L)}(q^{-N+1} + q^{-N+3} + \cdots + q^{N-1}),$$

while $\mathrm{LeeP}_{N,p^j}(L) = 0$ for $j > 0$. This proves (P-1) and (P-2).

For (P-3), we use Proposition 8.14. Specifically, we have

$$(\mathcal{P}_j - \mathcal{P}_{j+1})|_{t=-1} = \mathrm{DRT}_{N,j}$$

where $\mathrm{DRT}_{N,j}$ is a difference polynomial. Proposition 8.14 implies that changing an orbit of crossings on a diagram does not affect $\mathrm{DRT}_{N,j}$ modulo the ideal generated by $q^{p^{n-j}} - q^{-p^{n-j}}$. We get a mirror of the link by changing all orbits of crossings. Since changing the orbit of crossing does not affect $\mathrm{DRT}_{N,j}$ modulo the ideal generated by $q^{p^{n-j}} - q^{-p^{n-j}}$, we stay in the same relation after the first change, i.e., changing the orbit of the first crossing. By Corollary 8.12, we get the result. □

8.5. **Periodicity 3 and 4.** Now we will show that the periodicity criteria cannot hinder a knot from being 3 or 4 periodic. We begin with the following result.

Theorem 8.18 ([9]). *If* K *is a knot and* X *is its HOMFLY-PT polynomial, then* $X(a,b) = T(a,b)q(a,b) + 1$*, where* $q(a,b)$ *is a Laurent polynomial with integer coefficients and* $T(a,b) = a^4 - 2a^2 + 1 - a^2b^2$ *is the HOMFLY-PT polynomial for the trefoil.*

The following result is deduced from Theorem 8.18 and (8.9). Before stating it, we introduce the following notation:

$$T_N = q^{4N} - 2q^{2N} + 1 - q^{2N}(q - q^{-1})^2.$$

Corollary 8.19. *For a knot K, the* RT_N *polynomial has the form*

$$\mathrm{RT}_N(q) = A(q)T_N + 1$$

where $A(q)$ is a Laurent polynomial with integer coefficients.

Lemma 8.20.

- *If ζ_6 is a root of unity of order 6, then $T_N(\zeta_6) = 0$ unless $3|N$;*
- *If ζ_8 is a root of unity of order 8 and N is odd, then $T_N(\zeta_8) = 0$.*

Proof. We prove the ζ_6-part for $N \equiv 1 \pmod 3$ and $N \equiv 2 \pmod 3$.

For $N = 1$, we have

$$\begin{aligned} T_N &= q^4 - 2q^2 + 1 - q^2(q - q^{-1})^2 \\ &= (q^2 - 1)^2 - (q^3 - q)(q - q^{-1}) \\ &= (q^2 - 1)(q^2 - 1 - q^2 + 1) = 0 \end{aligned}$$

For $N \equiv 1 \pmod 3$, we write $N = 3k + 1$. We have

$$\begin{aligned} T_N &= q^{4(3k+1)} - 2q^{2(3k+1)} + 1 - q^{2(3k+1)}(q - q^{-1})^2 \\ &= q^{12k}q^4 - 2q^{6k}q^2 + 1 - q^{6k}q^2(q - q^{-1})^2 \end{aligned}$$

Since $(\zeta_6)^6 = 1$, we have

$$T_N(\zeta_6) = (\zeta_6)^4 - 2(\zeta_6)^2 + 1 - (\zeta_6)^2((\zeta_6) - (\zeta_6)^{-1})^2 = 0$$

For $N = 2$, we have

$$T_N = q^8 - 2q^4 + 1 - q^4(q - q^{-1})^2$$

Since $(\zeta_6)^6 = 1$, we have

$$\begin{aligned} T_N(\zeta_6) &= (\zeta_6)^8 - 2(\zeta_6)^4 + 1 - (\zeta_6)^4((\zeta_6) - (\zeta_6)^{-1})^2 \\ &= (\zeta_6)^2 - 2(\zeta_6)^4 + 1 - (\zeta_6)^4((\zeta_6)^2 - 2 + (\zeta_6)^{-2}) \\ &= (\zeta_6)^2 - 2(\zeta_6)^4 + 1 - 1 + 2(\zeta_6)^4 - (\zeta_6)^2 = 0 \end{aligned}$$

For $N \equiv 2 \pmod 3$, we write $N = 3k + 2$. We have

$$\begin{aligned} T_N(\zeta_6) &= (\zeta_6)^{4(3k+2)} - 2(\zeta_6)^{2(3k+2)} + 1 - (\zeta_6)^{2(3k+2)}((\zeta_6) - (\zeta_6)^{-1})^2 \\ &= (\zeta_6)^{12k}(\zeta_6)^8 - 2(\zeta_6)^{6k}(\zeta_6)^4 + 1 - (\zeta_6)^{6k}(\zeta_6)^4((\zeta_6) - (\zeta_6)^{-1})^2 \end{aligned}$$

Since $(\zeta_6)^6 = 1$, the last terms simplifies to

$$(\zeta_6)^8 - 2(\zeta_6)^4 + 1 - (\zeta_6)^4((\zeta_6) - (\zeta_6)^{-1})^2 = 0$$

The proof for ζ_8 is essentially the same. Firstly, we show for $N = 1$ and later for $N = 2k + 1$. We omit the details. □

Corollary 8.21. *For any knot K, we have the congruences* $\mathrm{RT}_N(q) - \mathrm{RT}_N(q^{-1}) \equiv 0 \bmod q^3 - q^{-3}$, $\mathrm{RT}_N(q) - \mathrm{RT}_N(q^{-1}) \equiv 0 \bmod q^4 - q^{-4}$.

Proof. We start with the first part. This congruence is the same as saying that for any root of unity ζ_6 of order 6, it holds $\mathrm{RT}_N(\zeta_6) - \mathrm{RT}_N(\zeta_6^{-1}) = 0$. We have two cases here. The first case, suppose that N is not a multiple of 3. By Corollary 8.19 and Lemma 8.20, we have $\mathrm{RT}_N(\zeta_6) = 1$, so $\mathrm{RT}_N(\zeta_6) - \mathrm{RT}_N(\zeta_6^{-1}) = 0$. The second case, suppose $3|N$. We write the Khovanov-Rozansky polynomial as follows, see Proposition 8.7.

$$\mathrm{KRP}_N(t,q) = q^s(q^{1-N} + q^{3-N} + \cdots + q^{N-1}) + \sum_j (1 + tq^{2Nj})R_j(t,q).$$

and we have $\mathrm{RT}_N(q) = \mathrm{KRP}_N(-1,q)$. For the term $(1 + tq^{2Nj})$ for $t = -1$ and $q = \zeta_6$ is equal to zero because $(\zeta_6)^6 = 0$. At the same time, we have

$$q^{1-N} + q^{3-N} + \cdots + q^{N-1} = \frac{q^N - q^{-N}}{q - q^{-1}}.$$

The latter expression is zero when evaluated at a root of unity of order dividing $2N$. That is to say

$$\mathrm{RT}_N(\zeta_6) = \mathrm{KRP}_N(-1,\zeta_6) = 0.$$

For the second part, first assume that N is odd. Then, $\mathrm{RT}_N(\zeta_8) = 1$ by the same argument combining. Again, we have $\mathrm{RT}_N(\zeta_6) = 1$ and

$$\mathrm{RT}_N(\zeta_6) = \mathrm{KRP}_N(-1,\zeta_6) = 0.$$

Now assume N is even. Assume that $4|N$ then as the same argument above we have

$$\mathrm{RT}_N(\zeta_8) = \mathrm{KRP}_N(-1,\zeta_8) = 0.$$

We have only one case, namely when $N = 4k + 2$. Assume we split this case into two cases. For some k we can write $N = 4k + 2 = 8m + 2$, and for some k we can write $N = 4k + 2 = 8m - 2$. For $N = 8k + 2$, take ζ_8 such that $\zeta_8^4 = 1$. From the formula of HOMFLYPT polynomial $X(a,b)$ we have $\mathrm{RT}_N(q) = X(q^N, q - q^{-1})$. Since $\zeta_8^4 = 1$, $\mathrm{RT}_N(\zeta_8) = X(\zeta_8^{4k+2}, \zeta_8 - \zeta_8^{-1}) = X(\zeta_8^2, \zeta_8 - \zeta_8^{-1}) = \mathrm{RT}_2(\zeta_8)$.

Now, RT_2 is the Jones polynomial. It was proved in [4, Section 4.6] that $\mathrm{RT}_2(\zeta_8) - \mathrm{RT}_2(\zeta_8^{-1}) = 0$. The same proof is valid for when $\zeta_8^4 = -1$. The remaining case is when $N = 8k - 2$ and $\zeta_8^4 = -1$. Write $X(a,b) = \sum \alpha_{ij} a^i b^j$. Since $\mathrm{RT}_N(q) = X(q^N, q - q^{-1})$, we have

$$\mathrm{RT}_N(\zeta_8) - \mathrm{RT}_N(\zeta_8^{-1}) = \sum \alpha_{ij}\zeta_8^{Ni}(\zeta_8 - \zeta_8^{-1})^j - \zeta_8^{-Ni}(-\zeta_8 + \zeta_8^{-1})^j =$$
$$\sum \alpha_{ij}(\zeta_8^{-2i} - \zeta_8^{2i})(\zeta_8 - \zeta_8^{-1})^j = -\mathrm{RT}_2(\zeta_8) + \mathrm{RT}_2(\zeta_8^{-1}).$$

After all, RT_2 is the Jones polynomial. It was proved in [4, Section 4.6] that $\mathrm{RT}_2(\zeta_8) - \mathrm{RT}_2(\zeta_8^{-1}) = 0$. The same proof is valid for when $\zeta_8^4 = 1$. □

Corollary 8.22. *Assume K is a knot. Set $\mathcal{P}_0 = \mathrm{KRP}_N$, $\mathcal{S}_{0j} = R_j$, where R_j is as in Proposition 8.7. Then $\mathcal{S}_{0j}, \mathcal{P}_0$ satisfy the statement of Theorem 8.17 regardless of whether K is 3 or 4-periodic.*

Proof. We prove this corollary just for 3-periodic knots. The proof for 4-periodic knots is similar. Item (P-1) is satisfied by definition. By Proposition 8.7, $\mathcal{S}_{0j}$ has non-negative coefficients. The congruence (P-3) is a direct consequence of Corollary 8.21. □

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