Calculation of optima and equilibria in dynamic resource extraction problems

A thesis
submitted in fulfillment of the requirements
for the award of the degree of

Doctor of Philosophy

in Mathematics

submitted by

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(October 15, 2018)

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PhD dissertation

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October 2018

Author's declaration:
I hereby declare that this dissertation is my own work.

October 15, 2018 ....

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Supervisor's declaration:
The dissertation is ready to be reviewed

October 15, 2018 ....

Dr hab. Agnieszka Wiszniewska-Matyszkiel

ACKNOWLEDGMENTS

The happiness of success lies in sharing it with all those who have helped and inspired me to achieve it. I want to express my gratitude to all people whose constant guidance and encouragement served as a beacon of light.

I feel privileged to express my heartiest thanks and a deep sense of gratitude to my supervisor *Dr hab. Agnieszka Wiszniewska-Matyszkiel*, Institute of applied mathematics, University of Warsaw, Poland, for her wisdom, vision, expertise, guidance, enthusiastic involvement and persistent encouragement during the planning and development of this research work. I also gratefully acknowledge her painstaking efforts in thoroughly going through and improving the manuscripts without which this work could not have been completed. I will always be indebted to her for affectionate support and care she provided me academically and personally. It was not easy for me to survive in a different country with a different culture. She always helped me as a guardian.

I am obliged to *Prof. Dr hab. Piotr Rybka*, Chairman of Doctoral commission and *Dr hab. Pawe Traczyk*, Director of PhD studies, Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, for providing all the facilities, help and encouragement for carrying out the research work.

I am highly obliged to *Prof. Ravindra Dhar*, *Prof. Suresh Chandra* and *Dr. hab. Pawel Morawiecki* for their invaluable advice at every critical juncture of my academic and personal life. I will forever be thankful to them for their constant encouragement, care and most importantly, teaching me how to live a life of gratitude. Their blessings are always precious.

I wish to express my appreciation to my friends *Pratima Singh*, *Rachna Mourya* and *Maria Khodos* and grateful thanks to research fellows at department for their help and motivation throughout my research work. I also would like to express my deep and sincere thanks to my friends and all other persons whose names do not appear here, for helping me either directly or indirectly in all even and odd times.

Also I would like to thank the *National Science Centre*, Krakow, Poland for providing me financial assistance by awarding me research fellowship for my Ph. D. research work. I am also thankful to the anonymous reviewers of my research publications. Their comments and suggestions were very helpful in shaping my research work.

Above all, no words are enough to pay my gratitude to the Godlike person in my life, *Ashutosh Dhar Dwivedi* for his patience, care, sacrifice, love and encouragement during this journey. A special thanks to my closest pal, *Ayush*, for always being very caring and supportive and so good to me. Thank you my sweetest adorable nice, *Piku*, for cherishing and loving me.

Last but not least, I would like to thank my family and friends.

(Rajani Singh)

ABSTRACT

Exploitation or extraction of common-property renewable resources is one of the biggest challenges in society. It encompasses a wide range of various problems among other things, the phenomenon known as *the tragedy of the commons*. Most importantly, the extraction and consumption of common natural renewable resources have a strong impact on the quality of life and well-being of both, the current and future generations. From the mathematical point of view, the only tool to deal with the whole spectrum of phenomena arising in such types of problems, in which there are at least two independent decision makers in a common resource extraction problem, are dynamic games, since both dynamic optimization methods and static games encompass only fractions of aspects of those problems.

In the dissertation, we propose several models of dynamic games and dynamic optimisation problems, modelling the *exploitation of common renewable resources* by taking into account various aspects of the problem:

- Many players in commons. Increasing number of players regarded as decomposition of the decision making structures. To be more specific, if we consider the same mass of individuals, decomposed into units of decreasing size: from *consumers*, through *North and South*, actual countries, regions etc. and finally actual decision makers.
- Relation between the Nash equilibria and the social optima and ways of solving *the tragedy of the commons* by Pigovian taxation or a tax-subsidy system.
- Taking into account information: *feedback form, closed loop*, delayed information.
- Self-enforcing environmental agreements with a delay in observation of defection.
- Completing and correcting previous results in this research field or finding counterexamples to common beliefs and methodological simplifications.

In dynamic games, the strategy of a player is a function which defines his/her behaviour at each time instant in the time interval considered in the game. Therefore, calculation of both, the social optima and the Nash equilibria requires solving the dynamic optimisation problems.

However, finding a Nash equilibrium in dynamic games requires solving a set of dynamic optimisation problems, coupled by finding a fixed point of the resulting best response correspondence in some functional space of the profiles of strategies. Due to

this coupling, the problem becomes much more complicated than the analogous dynamic optimisation problems. There are quite a few results in nonzero-sum dynamic games, and if the constraints appear (which is natural in real life problems, especially resource extraction problems), then the results are very rare. Therefore, unexpected behaviour of the solution may appear (irregularity, discontinuity, the nonexistence of equilibria of a certain type, existence of many equilibria, lack of convergence). So, we try to fill in the gaps in the simplifications of dynamic games. The dissertation also contains counterexamples to some methods and hypotheses that are regarded as correct and used to solve dynamic games.

With the strong motivation behind the chosen problems, in **Chapter 1**, we introduce the game and some preliminary knowledge of game theory, brief literature review and the mathematical optimisation tools that are used to solve the game models in the dissertation.

In **Chapter 2**, we present a discrete time, infinite horizon, a linear-quadratic dynamic game model with many players and with linear state-dependent constraints on decisions of players. In this model, players can be regarded as countries or firms. There are either finitely many players or a continuum of players. The model has an obvious application in a common fishery extraction problem where the players sell their catch at a common market.

We solve the social optimum problem for n-players and for the continuum of players. When it comes to the Nash equilibrium problem, we are only able to solve it for the continuum of players case. For n-players case, we are not able to calculate it for $n \geq 2$, only negative results can be proven: that the Nash equilibrium strategies and the value functions are not of assumed regularity with respect to the state variable and showing that presence of even a very simple and obvious constraints on strategies may result in a very complicated form of the value functions and the Nash equilibria. While looking for Nash equilibria, the social optima, we have also found a very simple counterexample to the correctness of a procedure often used in dynamic game theory literature. We also calculate the enforcement of a social optimum profile by various type of *Pigouvian tax* or a tax-subsidy system, both for n-players and for the continuum of players.

Non-existence of a symmetric feedback Nash equilibrium of assumed regularity in the linear-quadratic problem considered in **Chapter** 2 seems to be inherited from the finite time horizon truncations of the game, so in **Chapter 3**, we solve a feedback Nash equilibrium problem in a very simple 2-stage, 2-player linear-quadratic dynamic game being a truncation of the model which was studied in Chapter 2 with the infinite time horizon. As a result, we found that the presence of simple linear state-dependent constraints results in the nonexistence of a continuous symmetric feedback Nash equilibria, whereas the existence of the continuum of discontinuous symmetric feedback Nash equilibria. Our result is counter-intuitive to the common belief in the continuity of Nash equilibria for linear-quadratic dynamic games with concave payoffs.

While previous two Chapters deal with the specific value of the discount factor β, given by the so called *golden rule*, in **Chapter 4**, we solve the social optimum problem from **Chapter** 2 for more general class of linear-quadratic dynamic games with only one player, called social planner and for more general β instead of *the golden rule* β. So, we consider a discrete time linear-quadratic dynamic optimisation problem with linear state-dependent constraints. We solve the problem in the infinite time horizon and its finite horizon truncations. Although it seems simple in its linear-quadratic form, calculation of the optimal control is nontrivial.

In **Chapter 5**, we study a general class of dynamic optimization problems. We derive general rules stating what kind of errors in calculation or computation of the value function does not lead to errors in calculation or computation of optimal control. This general result concerns not only errors resulting from using the numerical methods but also errors resulting from some preliminary assumptions related to constraints on the value functions. The results are illustrated by a motivating example of discrete time Fish Wars model, proposed by Levhari and Mirman, with singularities in payoffs.

In **Chapter 6**, we study a continuous time version of the Fish Wars model with the infinite time horizon, linear state equation and state-dependent linear constraints on controls. We calculate the social optimum and a Nash equilibrium which always leads to the depletion of the resource even if the social optimum results in its sustainability. We propose two ways of solving the problems of enforcing social optimality: either by a tax-subsidy system or by an environmental agreement even if we assume that it takes time to detect any defection of a player. We also propose a general algorithm for finding the financial incentives enforcing the socially optimal profile in a large class of differential games.

Contents

List of Symbols

$\mathbb{R}$	Set of real numbers
$\bar{\mathbb{R}}$	$\mathbb{R} \cup \{-\infty, +\infty\}$
$\mathbb{R}_+$	Set of non-negative real numbers
$\mathbb{I}$	Set of players
$\mathbb{T}$	Set of time
$\mathbb{X}$	Set of possible states of the system
u	A control parameter
$\mathbb{U}$	Set of control parameters
D	Correspondence of available control parameters
U	A control (function)
$\mathcal{U}$	A set of admissible controls
X^U	Trajectory corresponding to a control U
$\mathfrak{X}$	Set of admissible trajectories
$\mathbb{D}_i$	Set of decisions of player i
D_i	Correspondences of available decisions of player i
s_i	A strategy of player i in a game or decision of player i in a dynamic game
$s_{\sim i}$	Analogously of the remaining players
s	A strategy profile in a game or profile of decisions in a dynamic game
S_i	A strategy (function) of player i in a dynamic game
$S_{\sim i}$	A strategy (function) of remaining players i in a dynamic game
S	A profile of strategies in a dynamic game
Σ	Set of strategy profiles
$\mathbb{S}_i$	Set of strategies of player i

P_i	Instantaneous or current payoff of player i
$\mathcal{P}_i\left(s_i, u^s\right)$	current payoff of player i, dependent on s_i and u^s
ϕ	State transition function
J_i	Payoff of player i in the game
β	A discount factor
ξ	Regeneration rate
$\underset{x \in G}{\text{Argmax}}\, f(x)$	The set of points at which the global maximum of f over G is attained

Chapter 1

Introduction

1.1 Historical perspective and a brief review of game theory literature

Game theory is a formal way of examining the situations of conflict and cooperation. Game is a mathematical tool to describe any situation in which there are at least two independent decision makers (called *players*), each of them has their own aim or objective (mathematically described as a *maximisation* of a certain function called *payoff*), while there is a certain interdependence between them (mathematically described as dependence of the payoff function on choices of all the players). The formal definition of game is given in Def. 1 of Section 1.2.

Formally, beginning of the game theory is dated on 1944, when the seminal book of a great mathematician Von Neumann and an economist Morgenstern [1] *Theory of Games and Economic Behavior* appeared. The book consider the *cooperative* and *non-cooperative* games of finitely many players.

Nevertheless, history of the game theory has been started even earlier in 1838, although the term *game* was not used, it appeared in the paper of Cournot [2], where a concept of equilibrium, equivalent to Nash equilibrium was introduced.

In 1921 – 1927, Borel published a series of papers (e.g., [3–5]), that firstly defined the games of strategy. In 1925, Steinhaus [6] worked on a more complicated concept of the game theory, called later the *differential game*.

However, the mathematical discipline of game theory was founded mainly by Von Neumann in 1928 (e.g., [7] where he proved the minimax theorem for the zero-sum games). Later, he extended his work to the application of game theory to economics.

In 1949, Nobel laureate John Forbes Nash wrote his doctoral dissertation named *Non-Cooperative Games*, where he introduced the concept of equilibrium point also known as *Nash equilibrium* and he proved that such equilibrium point exists. The concept of Nash equilibrium was the breakthrough in the non-cooperative non-zero-sum game theory as it was the only solution to such games extending some properties of John Von-Neumann's minimax strategies. It was published in 1950 in [8].

In 1953, another Nobel laureate Shapley provides the solution of the cooperative game theory called Shapley value [9], and he extended his work by introducing the *stochastic game* — a dynamic game with probabilistic transitions [10].

In 1954, the game in continuous time called the *differential game* was introduced by the famous game theorist Rufus Philip Isaacs [11]. It is quite probable that the discipline of differential games had been developed earlier secretly (e.g., [12,13]) because of its potential applicability in the air and the sea combat. Similarly, similar research must have been carried out in the Union of Soviet Socialist Republics (USSR) with some of the results possibly unknown to the broader scientific audience. The book [11] consisted of quite a large study in RAND, while most of Isaacs works were classified and therefore, unknown.

Beginning 1949, Bellman, an American applied mathematician, worked for many years at RAND corporation [14,15] and during that time he developed the dynamic programming techniques and founded its applications in numerous fields from economics to the aerospace engineering.

The term *dynamic programming* was firstly used in the 1957s by Bellman [16] to describe the process of solving problems backwards in order to find the optimal decision. Later, he redefined it to the modern sense — *decomposing the larger decision problems* into the smaller decision problems. The word *dynamic* represents the time-varying aspect of the optimisation problems, while the word *programming* referred to the use of the method to find an optimal program. So, dynamic programming is both a mathematical optimisation method as well as a computer programming method. Therefore, the Bellman equation is also known as a *dynamic programming equation*.

From 1948−1950, David Blackwell jointly worked with Bellman in RAND [14,15,17] on the dynamic programming techniques and given the most significant contribution to this field [18–20].

Dynamic games are the games of a special structure with dependence on time and decision made in multiple time instants. They may be of a very complicated form, and they may be with complete or incomplete information. The more formal definition of Dynamic game is defined in Subsection 1.4.1 of Section 1.4. Dynamic games are the only appropriate tool to model decision-making problems by independent but coupled players in an external environment changing in response to their decisions.

The dynamic game which we mainly considered in the dissertation belongs to the class of linear-quadratic dynamic games with constraints. Linear-quadratic dynamic games seems to be the best researched class of games (e.g., Hämäläinen [21,22] Jank and Abou-Kandil [23], Olsder et. al [24]). Both in finite and the infinite horizon, both in discrete and continuous time, Nash equilibria can be determined analytically, and the formulae are now a textbook material (e.g., Haurie et al. [25], Başar, Olsder [26] or Dockner et al. [27]), and there are in-depth monographs (e.g., **Linear-quadratic dynamic games** by Engwerda [28, 29]).

However, after imposing even a simple linear constraint, it turns out that none of those well-known results can be applied. In the latest research in the linear-quadratic dynamic game, Reddy and Zaccour [30, 31] considered the constraints in their model, and they proved two types of existence results.

Literature of the dynamic game theory can be classified into the following divisions:

- **Differential games:** are the dynamic games with continuous time. Starting from Steinhaus [6] and Isaacs [12], some of the important research work includes Başar et. al [32], Petrosyan and Zaccour [33], Carlson et. al [34], Zeeuw and Van Der Ploeg [35].

  Dockner et. al [27], Jørgensen and Zaccour [36], Balbus et. al [37, 38] work on the applications of differential games in economics.

- **Stochastic games:** are the dynamic games with probabilistic transitions of the state variable dependent on players' strategies — given a profile of strategies, the behaviour of the state variable is a stochastic process, generally with the incomplete information. Most current review of research wise Jaśkiewicz and Nowak [39, 40] and Balbus et. al [41, 42]. Genc and Zaccour [43] worked on the application of Stochastic games to economics.

- **Stopping games:** are the dynamic games with the possibility of quitting the game at any time instant, usually also stochastic games, also known as a generalisation of optimal stopping problems (e.g., Szajowski [44], Ramsey and Szajowski [45–47], Ferenstein [48]).

  Evolutionary game theory: originated as an application of the mathematical theory of the dynamic game to biological contexts (e.g., Weibull1 [49]). This is one of the recent development of the dynamic game theory. Starting from Fisher [50], there are many researches working on it (e.g., Ramsey [51,52], Broom and Křivan [53], Cressman and Apaloo [54]).

- **Games with a continuum of players:** are dynamic games with a nonatomic measure space of players (e.g., Aumann and Shapley [55], Mas-Colell [56]). In this simplest approach, the $[0, 1]$ interval with the Lebesgue measure. In the continuum of the players game, a decision is made by the players in very large populations of small interacting players. Since the problem is more complex than the dynamic game with finitely many players, quite a few scientists are working on it (e.g., Wiszniewska-Matyszkiel [57–61], Ekes [62], Wieczorek and Wiszniewska-Matyszkiel [63]).

- **Mean field game theory:** is the study of strategic decision making in very large populations of small interacting players (e.g., Caines et al. [64]). This is also one of the most recent developments of the game theory.

The real-life problems that are solved by using the tools of dynamic games and/or dynamic optimisation in the dissertation are the model of extraction of common property, renewable resource. Exploitation of an interdependent or a common-property renewable resources is one of the significant challenges of contemporaneity. This challenge encompasses a wide range of various problems, among other things the phenomenon called *the tragedy of the commons*. The term firstly originated in 1833, in an essay [65], written by a British economist Lloyd but become common knowledge to the general audience in 1968, when American philosopher and ecologist Hardin wrote an article [66] *The tragedy of the commons*. In this seminal paper, *The Tragedy of the Commons*, the *commons* is a natural resource shared by many individuals. In this context, *shared* means that each individual does not have a claim to any part of the resource, but rather to the use of a portion of it for his/her benefit. *The tragedy* means that, in the absence of regulation, each individual will tend to exploit or extract the commons to his/her advantage, typically without any limit. Under this state of affairs, the commons is depleted and eventually ruined.

The problem of extraction or common resource is investigated by many scientists including Amir and Nannerup [67], Antoniadou et. al [68], Bailey et. al [69], Long survey [70–72], Kaitala and Lindroos [73], Clemhout and Wan [74], Doyen et. al [75], Dutta and Sundaram [76], Koulovatianos [77], Başar and Olsder [26] coursebook, Dockner et. al [27], Başar et. al [78], Wiszniewska-Matyszkiel [79] and Ehtamo and Hamalainen [80, 81].

1.2 Game in normal form

1.2.1 Standard approach

Definition 1 *A game in normal or strategic form* $\mathcal{G} = \{\mathbb{I}, \{\mathbb{S}_i\}_{i\in\mathbb{I}}, \{J_i\}_{i\in\mathbb{I}}\}$ *consists of:*

1. *A set of at least two players* $\mathbb{I}$. *For finitely many players* $\mathbb{I} = \{1, \ldots, n\}$.

2. *A set of strategies* $\mathbb{S}_i$ *that are available to player* i. *If* $s_i \in \mathbb{S}_i$ *denotes the strategy chosen by player* i, *then* $s = (s_1, s_2 \ldots, s_n)$ *is called a strategy profile.*

We denote the set of all strategy profiles *by* $\Sigma = \mathbb{S}_1 \times \mathbb{S}_2 \times \cdots \times \mathbb{S}_n$*.*

3. *A set of payoff functions* $J = (J_1, J_2, \ldots, J_n)$*, where* $J_i : \Sigma \to \mathbb{R}$ *is called the payoff function of player* i*.*

Notational convention:
For a game with a set of players $\mathbb{I}$ and given a strategy s_i of player i, we introduce the notation $[s_i, s_{\sim i}]$ for a profile of strategies $s = (s_1, \ldots, s_n)$, where $s_{\sim i}$ denotes the strategy of the remaining players. So, for a strategy $\sigma \in \mathbb{S}_i$ and a profile $\tilde{s} \in \Sigma$, the symbol $(\sigma, \tilde{s}_{\sim i})$ denotes the profile $\tilde{s}$ with i-th coordinate replaced by σ.

The most important solution concept of the non-cooperative game theory is the Nash equilibrium.

A profile is called a *Nash equilibrium* if no player can benefit from unilateral deviation from it. Formally, it can be defined as follows:

Definition 2 *A strategy profile* $\bar{s}$ *is a* ***Nash equilibrium*** *iff for every player* $i \in \mathbb{I}$ *and for every strategy* $s_i \in \mathbb{S}_i$ *of player* i*,*

$$J_i\left([s_i, \bar{s}_{\sim i}]\right) \le J_i\left([\bar{s}_i, \bar{s}_{\sim i}]\right).$$

An essential property of a strategy profile, which is rarely fulfiled by Nash equilibria but considered as one of the most important properties in the case when it is assumed that the players can make the decision together, is Pareto-optimality.

A profile is called *Pareto-optimal* if there is no other profile that makes every player at least as well off and at least one player strictly better off. More formally:

Definition 3 *A strategy profile* $\bar{s}$ *is* ***Pareto-optimal*** *if there is no profile such that*

$$J_i(s) \ge J_i(\bar{s}) \text{ for all } i \in \mathbb{I} \text{ and } J_i(s) > J_i(\bar{s}) \text{ for some } i.$$

Following many papers (e.g., Levhari-Mirman [82], Singh and Wiszniewska-Matyszkiel [83, 84]), in this dissertation, we are especially interested in a special Pareto-optimal profile called the *social optimum*.

Definition 4 *A strategy profile* $\bar{s}$ *is the* ***social optimum*** *in a game with* n*-players iff*

$$\bar{s} \in \operatorname*{Argmax}_{s \in \Sigma} \sum_{i=1}^{n} J_i(s).$$

1.2.2 Continuum of players

If the number of players in a real-life game theoretic application is sufficiently large, they start behaving in such a way that is best described by the games with a continuum of players.

To define the *game in normal form* for a *continuum of players*, we need the additional measurabilty assumptions.

Definition 5 *A game in normal form* $\mathcal{G} = \{\mathbb{I}, \mathcal{L}, \lambda, (\mathbb{S}, \mathfrak{S}), \{\mathbb{S}_i\}_{i\in\mathbb{I}}, \{J_i\}_{i\in\mathbb{I}}\}$ *for the continuum of players consists of:*

1. *The* continuum of players *is the set of players* $\mathbb{I} = [0, 1]$ *with the Lebesgue measure* λ *on the* σ*-field of its Lebesgue measurable subsets* $\mathcal{L}$*. Thus, the space of players is the measure space* $(\mathbb{I}, \mathcal{L}, \lambda)$ *instead of only the set* $\mathbb{I}$*.*

2. Sets of available strategies of player i, $\mathbb{S}_i$ *are all subsets of a certain set* $\mathbb{S}$ *on which* σ*- field of its measurable subsets* $\mathfrak{S}$*, its measurability is considered, denoted by* $\mathfrak{S}$*. We assume that* $\mathbb{S}_i \in \mathfrak{S}$*.*

   For a function $s : \mathbb{I} \to \mathbb{S}$ *with* $s_i \in \mathbb{S}_i$ *(for uniformity of notation, we write* s_i *instead of* $s(i)$*), we call* strategy profiles *only such measurable function.*

   As before, Σ *denotes the* set of all strategy profiles *but now obviously the definition of profile encompasses measurability.*

3. *Payoff functions of player* i, $J_i : \Sigma \to \mathbb{R}$*. In majority of applications* J_i *are of specific form:*

   $J_i(s) = \mathcal{P}_i(s_i, u^s)$ *for a measurable function for some* $\mathcal{P}_i : \mathbb{S} \times \text{Conv}\mathbb{S} \to \bar{\mathbb{R}}$ *and* $u^s = \int_{\mathbb{I}} s_j d\lambda(j)$*, usually called* the aggregate of s*, where* Conv $\mathbb{S}$ *denotes the* convex hull *of a set* $\mathbb{S}$*.*

Definition of the *Nash equilibrium* and the *Pareto optimal* profile for continuum of players case are analogous to Def. 2 and Def. 3 with *"every i"* is replaced by *"almost every i"* and *"some i"* by *"i in a set of non-zero measure"*.

Definition 6 *A strategy profile* $\bar{s}$ *is a* ***social optimum*** *in the continuum of players game iff*

$$\bar{s} \in \operatorname*{Argmax}_{s\in\Sigma} \int_{[0,1]} J_i(s) \cdot d\lambda(i).$$

Simplifying convention:

If in a specific application, the modelled functions are independent of some arguments, we omit them in notation.

1.3 The optimal control theory and the dynamic programming method

In this section, we introduce various types of dynamic optimisation problems that we study in the dissertation and tools to solve them. Consider the following:

1. *A time set* $\mathbb{T}$, either discrete or continuous with t_0 representing the initial time. For continuous time, in most of the cases, we consider $\mathbb{T} = [0, T]$ for a *finite time horizon* T and $\mathbb{T} = [0, +\infty)$ for the *infinite time horizon.* For discrete time, we consider $\mathbb{T} = \{0, 1, 2, ..., T\}$ for the finite time horizon T and $\mathbb{T} = \mathbb{N}$ for the infinite time horizon.

2. A set of *states of the system* (*state set* for short) $\mathbb{X} \subseteq \mathbb{R}^n$. The system is characterized at each time by a *state variable* $x \in \mathbb{X}$.

3. A *potential trajectory* X of the state of the system is defined as $X : \mathbb{T} \cup \{T+1\} \to \mathbb{X}$ for discrete time with the finite time horizon T, otherwise $X : \mathbb{T} \to \mathbb{X}$ with an *initial state* of the system $X(t_0) = x_0 \in \mathbb{X}$.

4. A set of *control parameters* $\mathbb{U} \subseteq \mathbb{R}^m$.

5. There is a state dependent *constraint* given by the correspondence $D : \mathbb{X} \multimap \mathbb{U}$ with $D(x) \subseteq \mathbb{U}$, called the *correspondence of available control parameters.*

   Information structure

6. A *control* is a function that defines which control parameter u to choose at each stage, dependent on available information about the game so, it can be defined in various ways. We are interested in the form of controls which are $U : \mathbb{T} \times \mathbb{X} \to \mathbb{U}$ and they are measurable in the case of continuous time. These are called in various papers *closed loop, closed loop no-memory, feedback or Markovian.*

   It is worth mentioning that the exact meanings of closed loop and feedback are different in various works, especially in the research on optimal control problems versus dynamic games.

   In some specific cases, we consider $U : \mathbb{X} \to \mathbb{U}$, with the same ambiguous terminology.

   We use the latter form of controls only in the infinite time horizon case when the functions and the correspondences stated in the problem are not directly dependent on time.

Throughout the dissertation, we will use the term *feedback* (prevalent in most recent dynamic games theory literature). For the continuous time, there is an additional requirement, to be defined later.

7. *Behaviour or evolution of the state variable* is described by:

   a first order difference equation in a discrete time

$$X(t+1) = \phi\left(t, X(t), U(t, X(t))\right); X(t_0) = x_0, \tag{1.3.1}$$

   for the state transition function $\phi : \mathbb{T} \times \mathbb{X} \times \mathbb{U} \to \mathbb{X}$.

   a differential equation in the continuous time

$$\dot{X}(t) = \phi\left(t, X(t), U(t, X(t))\right); X(t_0) = x_0, \tag{1.3.2}$$

   for almost every t and for the state transition function $\phi : \mathbb{T} \times \mathbb{X} \times \mathbb{U} \to \mathbb{R}^n$.

   In the continuous time, some regularity assumption is additionally needed, guaranteeing that

$$\forall (t_0, x_0) \in \mathbb{T} \times \mathbb{X},\ u \in \mathbb{U}, \exists \text{a unique absolutely continuous } X \text{ which fulfils Eq. (1.3.2).} \tag{1.3.3}$$

   Obviously, this assumption encompasses joint measurability.

   Generally, in many applications, it cannot be a priori assumed that U is locally Lipschitz with respect to $\mathbb{X}$ since even discontinuous controls are often optimal. So, in order to have a general model, we do not constrain the set beside measurability and condition (1.3.4).

   The unique trajectory which solves Eq. (1.3.1) or Eq. (1.3.2) with U is called the *trajectory corresponding* to U. If we want to emphasise that X is corresponding to U, we write X^U. If we also want to emphasise the dependency on the initial condition, we write $X^U_{t_0,x_0}$ or $X^U_{x_0}$.

8. A *set of admissible controls* $\mathcal{U}$. In discrete time, it is the set of functions $U : \mathbb{T} \times \mathbb{X} \to \mathbb{U}$ which fulfil $U(t, x) \in D(x)$ while in continuous time it is a set of all measurable functions $U : \mathbb{T} \times \mathbb{X} \to \mathbb{U}$ which fulfil $U(t, x) \in D(x)$ and

$$\text{such that Eq. (1.3.2) has a unique absolutely continuous solution on the set } \mathbb{T} \cap [t_0, +\infty). \tag{1.3.4}$$

   The set of trajectories corresponding to all $U \in \mathcal{U}$ is called the *set of admissible trajectories* and it is denoted by $\mathfrak{X}$.

9. A *current or instantaneous payoff* function $P : \mathbb{T}\times\mathbb{X}\times\mathbb{U} \to \mathbb{R}\cup\{-\infty\}$, $P(t, x, u)$ defines a reward that the controller obtains when the system receives the control input u at time t and state x.

   There is also a *terminal payoff* function $G^* : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$, for the finite time horizon.

10. We consider the *discounting* of the payoffs by a *discount factor* $\beta \in (0, 1)$. For discrete time, $\beta = \frac{1}{1+r}$, while for continuous time, $\beta = e^{-r}$, for $r > 0$, called the *interest rate* in economic applications.

11. A payoff function (or performance criterion) $J : \mathbb{T} \times \mathbb{X} \times \mathbb{U} \to \bar{\mathbb{R}}$ can be defined as follows:

    The payoff function in *discrete time* fulfils:

$$J(t_0, x_0, U) = \sum_{t=t_0}^{T} \beta^{t-t_0} P\left(t, X(t), U(t, X(t))\right) + \beta^{T+1-t_0} G^*(X(T+1)) \quad (1.3.5a)$$

for the finite time horizon T

$$J(t_0, x_0, U) = \sum_{t=t_0}^{\infty} \beta^{t-t_0} P\left(t, X(t), U(t, X(t))\right) \quad (1.3.5b)$$

for the infinite time horizon

for X given by Eq. (1.3.1).

The payoff function in *continuous time* fulfils:

$$J(t_0, x_0, U) = \int_{t=t_0}^{T} \beta^{t-t_0} P\left(t, X(t), U(t, X(t))\right) dt + \beta^{T+1-t_0} G^*(X(T+1)) \quad (1.3.6a)$$

for the finite time horizon T

$$J(t_0, x_0, U) = \int_{t=t_0}^{\infty} \beta^{t-t_0} P\left(t, X(t), U(t, X(t))\right) dt \quad (1.3.6b)$$

for the infinite time horizon

for X given by Eq. (1.3.2), where we consider the Lebesgue integral.

We assume that the functions P, ϕ and G^* are measurable on $\mathbb{T} \times \mathbb{X} \times \mathbb{U}$ and $\phi(t, \cdot, \cdot)$ is Lipschtiz in (x, u).

We do not impose other direct constraints on sets or functions defined before, but we assume that $J(t_0, x_0, U)$ is *always well defined*.

We look for the optimal controls in the set $\mathcal{U}$ of admissible controls here.

Definition 7 *Given* $(t_0, x_0) \in \mathbb{T} \times \mathbb{X}$ *and for the payoff function defined either by Eq.* (1.3.5) *for discrete time or by Eq.* (1.3.6) *for continuous time, the dynamic optimization problem is*

$$\text{"find } U \in \mathcal{U} \text{ maximising } J(t_0, x_0, U)\text{"}.$$

Definition 8 ***(a)*** *A control variable* $\bar{U} \in \mathcal{U}$ *is called an optimal control in the dynamic optimization problem defined by Def. 7, if it fulfils*

$$\bar{U} \in \operatorname*{Argmax}_{U \in \mathcal{U}} J(t_0, x_0, U).$$

(b) *A control variable* $\bar{U} \in \mathcal{U}$ *is called an optimal control (for feedback information structure) for the whole class of dynamic optimization problems defined by Def. 7, if it fulfils*

$$\bar{U} \in \operatorname*{Argmax}_{U \in \mathcal{U}} J(t, x, U) \text{ for every } (t, x) \in \mathbb{T} \times \mathbb{X}.$$

Definition 9 *A function* $\bar{V} : \mathbb{T} \times \mathbb{X} \to \bar{\mathbb{R}}$ *is called the value function or the current-value function of a dynamic optimization problem given by Def. 7, if for all* $x \in \mathbb{X}$, $t \in \mathbb{T}$,

$$\bar{V}(t, x) = \sup_{U \in \mathcal{U}} J(t, x, U).$$

1.3.1 Principle of Optimality: Necessary condition

One of the most standard and useful tools for solving the dynamic optimisation problems is the Bellman equation and the Bellman's optimality principle.

The philosophy behind the Bellman equation and the value function is that instead of solving the optimal control problems given some fixed (t_0, x_0), we solve it for all (t, x). So, we solve the whole class of problems instead of one, and this generalisation leads to a simplification: a dynamic optimisation problem is reduced to a sequence of static optimisation problems coupled by a difference or differential equation.

The Bellman Principle of Optimality was formulated by Bellman in [16]: *An optimal policy has the property that whatever the initial state and the initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision*". In discrete time, it can be formalised as the following necessary condition.

Theorem 1 ***Necessary condition***

Consider an arbitrary dynamic optimization problem in discrete time either for a finite or the infinite time horizon, given by Def. 7 i.e., maximisation of $J(t, x, U)$ *over* U *in the set of admissible controls* $\mathcal{U}$ *and for every* $t \in \mathbb{T}$, $x \in \mathbb{X}$.

Let $\bar{U}$ *be an optimal control for the whole class of dynamic optimization problems and* $\bar{V}$ *be the value function. Then for every* $t \in \mathbb{T}$, $x \in \mathbb{X}$,

$$\bar{V}(t, x) = \max_{u \in D(x)} P(t, x, u) + \beta \bar{V}(t + 1, \phi(t, x, u)), \tag{1.3.7}$$

called the Bellman equation *(BE), while* $\bar{U}$ *fulfils the* Bellman inclusion

$$\bar{U}(t, x) \in \operatorname*{Argmax}_{u \in D(x)} P(t, x, u) + \beta \bar{V}(t + 1, \phi(t, x, u)). \tag{1.3.8}$$

Additionally, for a finite time horizon T,

$$V(T + 1, x) = G^*(x) \text{ for all } x \in \mathbb{X}. \tag{1.3.9}$$

1.3.2 Sufficient conditions

Here we present the theorems on the sufficient condition for the feedback controls.

A sufficient condition for discrete time

Finite time horizon

The immediate sufficient condition, which is a standard textbook result is as follows:

Theorem 2 *Consider a finite time horizon* T*. If a function* $\bar{V} : \mathbb{T} \times \mathbb{R} \to \bar{\mathbb{R}}$*, satisfies the Bellman equation (BE)* (1.3.7) *and* $\bar{U} \in \mathcal{U}$ *fulfils the Bellman inclusion* (1.3.8) *and assume the terminal condition or transversality condition Eq.* (1.3.9) *holds, then* $\bar{V}$ *is the value function while* $\bar{U}$ *is an optimal control for the whole class of dynamic optimization problems.*

In a finite time horizon, the dynamic programming technique defined by Theorem 2 returns the value function by backwards induction and consequently the optimal control.

Infinite time horizon

This version of a sufficient condition for the infinite time horizon is equivalent to that proved in Stokey, Lucas and Prescott [85], Theorem 4.3 changed because of the different formulation of the optimal control problem (Stokey, Lucas and Prescott considered current and next stage state instead of current state and control in definition of the optimal control problem), it is also an immediate consequence of Theorem 4.

Theorem 3 *Consider the infinite time horizon problem. If for a function* $\bar{V} : \mathbb{T} \times \mathbb{X} \to \mathbb{R}$*, the Bellman Eq.* (1.3.7) *together with the Bellman inclusion* (1.3.8) *is fulfilled and the following* ***standard terminal condition***

$$\limsup_{t\to\infty} V(t, X(t)) \cdot \beta^t = 0, \forall X \in \mathfrak{X}, \tag{1.3.10}$$

holds, then $\bar{V}$ *is the value function while* $\bar{U}$ *is an optimal control for the whole class of dynamic optimization problems.*

The following sufficient condition is an immediate consequence of the main result of Wiszniewska-Matyszkiel [86].

Theorem 4 *Consider the infinite time horizon problem. If for a function* $\bar{V} : \mathbb{T} \times \mathbb{X} \to \mathbb{R}$*, the Bellman Eq.* (1.3.7) *together with the Bellman inclusion* (1.3.8) *is fulfilled and the following* ***weaker terminal condition***

$$(i) \limsup_{t\to\infty} V(t, X(t)) \cdot \beta^t \leq 0, \forall X \in \mathfrak{X}, \tag{1.3.11a}$$

$$(ii) \limsup_{t\to\infty} V(t, X(t)) \cdot \beta^t < 0 \Rightarrow J(t, x, U) = -\infty, \tag{1.3.11b}$$

holds for all U *such that* $X = X^U_{t,x}$*, then* $\bar{V}$ *is the value function while* $\bar{U}$ *is an optimal control for the whole class of dynamic optimization problems.*

A sufficient condition for continuous time

This is a standard textbook result (equivalent to Zabczyk [87] Theorem 1.1 for a finite time horizon and Theorem 1.2 for the infinite time).

Finite time horizon

Theorem 5 *Consider a finite time horizon* T*. If a continuously differentiable function* $\bar{V} : \mathbb{T} \times \mathbb{X} \to \mathbb{R}$*, fulfils the* Hamilton-Jacobi-Bellman *(HJB) equation*

$$r\bar{V}(t, x) - \frac{\partial \bar{V}(t, x)}{\partial t} = \max_{u \in D(x)} \left(P(t, x, u) + \langle \nabla_x \bar{V}(t, x), \phi(t, x, u) \rangle \right), \tag{1.3.12}$$

there exist $\bar{U} \in \mathcal{U}$ *such that it fulfils the inclusion*

$$\bar{U}(t, x) \in \operatorname*{Argmax}_{u \in D(x)} \left(P(t, x, u) + \langle \nabla_x \bar{V}(t, x), \phi(t, x, u) \rangle \right) \tag{1.3.13}$$

and if the following terminal condition holds

$$\bar{V}(T,x) = G^*(x) \text{ for all } x, \tag{1.3.14}$$

then $\bar{V}$ is the value function, while $\bar{U}$ is an optimal control.

Infinite time horizon

Theorem 6 *Consider the infinite time horizon. If for a continuously differentiable functions $\bar{V} : \mathbb{T} \times \mathbb{X} \to \mathbb{R}$, the HJB Eq.* (1.3.12) *together with inclusion* (1.3.13) *is fulfilled and the following terminal condition*

$$\limsup_{t\to\infty} \bar{V}(t, X(t))\beta^t = 0 \text{ for every admissible trajectory } X \in \mathfrak{X}, \tag{1.3.15}$$

holds, then $\bar{V}$ is the value function, while $\bar{U}$ is an optimal control.

Infinite time horizon, an autonomous problem

If P and ϕ are independent of time t, then $\bar{U}$ and $\bar{V}$ are also independent of t. So, we can skip t.

Theorem 7 *Consider the infinite time horizon. If a continuously differentiable functions $\bar{V} : \mathbb{X} \to \mathbb{R}$ fulfils the HJB equation*

$$r\bar{V}(x) = \max_{u\in D(x)} \left(P(x,u) + \langle \nabla \bar{V}(x) \cdot \phi(x,u)\rangle\right), \tag{1.3.16}$$

there exists $\bar{U} \in \mathcal{U}$ such that it fulfils the inclusion

$$\bar{U}(x) \in \operatorname*{Argmax}_{u\in D(x)} \left(P(x,u) + \langle \nabla \bar{V}(x), \phi(x,u)\rangle\right) \tag{1.3.17}$$

and if the following terminal condition

$$\limsup_{t\to\infty} \bar{V}(X(t))\beta^t = 0 \text{ for every admissible trajectory } X \in \mathfrak{X}, \tag{1.3.18}$$

holds, then $\bar{V}$ is the value function while $\bar{U}$ is an optimal control.

1.4 Dynamic games or multistage game

Dynamic games are the games played over a time interval in which, given strategies of the remaining players, each player faces a dynamic optimisation problem.

However, dynamic games differ from the dynamic optimization problems: In a dynamic game, controls are distributed among the set of players unlike in the optimal control problems, having only one controller. Here, each player's objective is to optimize his/her individual payoff and a fixed point of best responses is needed to find a Nash equilibrium.

1.4.1 For finitely many players

A ***dynamic game*** with n-players consists of the following:

1. A set of *finitely many players* $\mathbb{I} = \{1, \ldots, n\}$.

2. A *time set* $\mathbb{T}$: either *discrete* $\mathbb{T} = \{0, 1, \ldots, T\}$ for a finite time horizon T and $\mathbb{T} = \{0, 1, 2, \ldots\}$ for the infinite time horizon or *continuous* $\mathbb{T} = [0, T]$ for a finite time horizon and $\mathbb{T} = [0, \infty)$ for the infinite time horizon. We denote the initial time by t_0.

3. A set of possible *states of the system* (*state set* for short) $\mathbb{X} \subseteq \mathbb{R}^n$. A system is characterized at each time by a *state variable* $x \in \mathbb{X}$.

4. A *potential trajectory* X of the state of the system is defined as $X : \mathbb{T} \cup \{T+1\} \to \mathbb{X}$ for discrete time with finite time horizon T, $X : \mathbb{T} \to \mathbb{X}$ otherwise, with an *initial state* of the system $X(t_0) = x_0 \in \mathbb{X}$.

5. *The equivalent of the control parameter* in dynamic game is called the *decision or action* of player $i \in \mathbb{I}$ at time t and is denoted by s_i.

6. A set of decisions of player i is $\mathbb{D}_i \subseteq \mathbb{R}^{m_i}$ (with strategies being the set of functions $S_i : \mathbb{T} \times \mathbb{X} \to \mathbb{D}_i$, to be defined later).

   Preliminary *set of all decision profiles* is denoted by $\Delta = \mathbb{D}_1 \times \mathbb{D}_2 \times \cdots \times \mathbb{D}_n$.

7. There is a *state dependent constraint* on decisions or actions of player i, given by the correspondence $D_i : \mathbb{X} \multimap \mathbb{D}_i$ with $D_i(x) \subseteq \mathbb{D}_i$, called the *correspondence of currently available decisions.*

8. A decision profile $s \in \Delta$ available at state x, with $s_i \in D_i(x)$ is defined as $s = (s_1, \ldots, s_n)$.

   Information Structure

   Strategies that are available to players may have different information structure. Unlike in dynamic optimisation problems, it is very important to be very precise about the information structure.

We are interested in the form of strategies $S_i : \mathbb{T} \times \mathbb{X} \to \mathbb{D}_i$ that are measurable in the case of continuous time and fulfil one more condition, to be defined later. These are called in various papers *closed loop, closed loop no-memory, feedback or Markovian.*

In some specific cases, $S_i : \mathbb{X} \to \mathbb{D}_i$, with the same ambiguous terminology.

We use the later form of strategies only in the infinite time horizon case and when the functions and the correspondences stated in the problem are not directly dependent on time.

Throughout the dissertation, we will use the term *feedback* (prevalent in most recent dynamic games literature).

9. *Behaviour and evolution of the state variable*, given functions $S_i : \mathbb{T} \times \mathbb{X} \to \mathbb{D}_i$ and a strategy profile $S = (S_1, \ldots, S_n)$ is described by the following equation:

   a first order difference equation in discrete time

$$X(t+1) = \phi\left(t, X(t), S(t, X(t))\right); X(t_0) = x_0, \tag{1.4.1}$$

   for the state transition function $\phi : \mathbb{T} \times \mathbb{X} \times \Delta \to \mathbb{X}$.

   a differential equation in continuous time

$$\dot{X}(t) = \phi\left(t, X(t), S(t, X(t))\right); X(t_0) = x_0, \tag{1.4.2}$$

   for almost every t and for a state transition function $\phi : \mathbb{T} \times \mathbb{X} \times \Delta \to \mathbb{R}^n$.

   In continuous time, some regularity assumption is additionally needed for S (e.g., jointly measurable and Lipschitz in $\mathbb{X} \times \Delta$ for almost every t), guaranteeing that

$$\forall\ (t_0, x_0) \in \mathbb{T} \times \mathbb{X}\ \exists \text{ a unique X which fulfils Eq. (1.4.2).} \tag{1.4.3}$$

   The unique trajectory which solves Eq. (1.4.1) or Eq. (1.4.2) for given $S : \mathbb{T} \times \mathbb{X} \to \Delta$ is called the *trajectory corresponding to S*. If we want to emphasise that X is corresponding to S, we write X^S. If we also want to emphasise the dependency on the initial condition we write $X^S_{t_0,x_0}$ or $X^S_{x_0}$.

   Generally, it cannot be a priori assumed that S is Lipschitz with respect to X, since discontinuous strategies may appear at Nash equilibria, so we just have the condition (1.4.4).

10. In discrete time, the *set of profiles* of strategies is of the form $\Sigma = \mathbb{S}_1 \times \cdots \times \mathbb{S}_n$ ($\mathbb{S}_i$ being the set of functions $S_i : \mathbb{T} \times \mathbb{X} \to \mathbb{D}_i$, called the sets of strategies of player

i) and it is a certain set of functions $S : \mathbb{T} \times \mathbb{X} \to \Delta$ which fulfil $S_i(t, x) \in D_i(x)$, while in continuous time it is a set of all measurable function $S : \mathbb{T} \times \mathbb{X} \to \Delta$ which fulfils $S_i(t, x) \in D_i(x)$ and

such that Eq. (1.4.2) has a unique absolutely continuous solution on $\mathbb{T} \cap [t_0, +\infty)$. (1.4.4)

If Σ fulfils $\Sigma = \mathbb{S}_1 \times \cdots \times \mathbb{S}_n$, then the set of trajectories corresponding to $S \in \Sigma$ is called the *set of admissible trajectories* and is denoted by $\mathfrak{X}$.

11. *Instantaneous or current payoff* is a function $P : \mathbb{I} \times \mathbb{T} \times \mathbb{X} \times \Delta \to \mathbb{R} \cup \{-\infty\}$. We denote the function $P(i, \cdot, \cdot, \cdot)$ by P_i and it is called the *current or instantaneous payoff* of player i.

    For a finite time horizon T, we also consider the *terminal payoffs* $G_i^* : \mathbb{X} \to \mathbb{R} \cup \{-\infty\}$.

12. We consider the *discounting* of the payoffs by a *discount factor* $\beta \in (0, 1)$. For discrete time, $\beta = \frac{1}{1+r}$, while for continuous time, $\beta = e^{-r}$, for $r > 0$, called the *interest rate* in economics.

13. A payoff function $J_i : \mathbb{T} \times \mathbb{X} \times \Sigma \to \mathbb{R} \cup \{-\infty\}$ of player i is equal to his/her instantaneous payoffs, discounted and summed over time.

    For a profile S, the payoff function in *discrete time* fulfils:

$$J_i(t_0, x_0, S) = \sum_{t=t_0}^{T} \beta^{t-t_0} P_i\left(t, X(t), S(t, X(t))\right) + \beta^{T+1-t_0} G_i^*(X(T+1)) \quad (1.4.5a)$$

for the finite time horizon T

$$J_i(t_0, x_0, S) = \sum_{t=t_0}^{\infty} \beta^{t-t_0} P_i\left(t, X(t), S(t, X(t))\right) \quad (1.4.5b)$$

for the infinite time horizon

for X given by Eq. (1.4.1).

For a profile S, the payoff function in *continuous time* fulfils:

$$J_i(t_0, x_0, S) = \int_{t=t_0}^{T} \beta^{t-t_0} P_i\left(t, X(t), S(t, X(t))\right) dt + \beta^{T+1-t_0} G_i^*(X(T+1)) \tag{1.4.6a}$$

for the finite time horizon T

$$J_i(t_0, x_0, U) = \int_{t=t_0}^{\infty} \beta^{t-t_0} P_i\left(t, X(t), S(t, X(t))\right) dt \tag{1.4.6b}$$

for the infinite time horizon

for X given by Eq. (1.4.2).

We assume that the functions P_i, ϕ, G_i^* are measurable on $\mathbb{T} \times \mathbb{X} \times \Delta$ and $\phi(t, \cdot, \cdot)$ is Lipschitz continuous in $\mathbb{X} \times \mathbb{U}$.

We do not impose other direct constraints on the sets or the functions defined before, but we assume that $J_i(t_0, x_0, S)$ is *always well defined.*

1.4.2 Discounted dynamic games with continuum of players

In the dissertation, only autonomous discrete-time dynamic games with a continuum of players and infinite time horizon are being considered so, and we define only such games. In those games, we are interested only in strategies which are not directly dependent on time t.

Definition of dynamic games for the continuum of players are similar to Subsection 1.4.1 with the following changes:

1. The *space of players* $(\mathbb{I}, \mathcal{L}, \lambda)$ for a set of players $\mathbb{I} = [0, 1]$ with a Lebesgue measure λ on the σ-field of its Lebesgue measurable subsets $\mathcal{L}$.

2. The *set of decisions of player* i, $\mathbb{D}_i$ is $\mathcal{D}$ measurable subsets of a measurable space $(\mathbb{D}, \mathcal{D})$.

3. *Currently available decisions* are $D_i(x)$ for $D_i : \mathbb{X} \multimap \mathbb{D}_i$ with $D_i(x) \in \mathcal{D}$.

4. A *profile of decisions available* at state x is any measurable function $s : \mathbb{I} \to \mathbb{D}$ with $s_i \in D_i(x)$. For uniformity of notation, we write s_i instead of $s(i)$. The set of all profiles of decision is denoted by Δ.

5. The *time set* is $\mathbb{R}_+$

6. *Current payoffs* $P_i(t, X(t), s)$ are of specific form. They can be written as $\mathcal{P}_i(x, s_i, u^s)$, for some $\mathcal{P}_i : \mathbb{X} \times \mathbb{D} \times \text{Conv } \mathbb{D} \to \bar{\mathbb{R}}$, where $u^s = \int_{\mathbb{I}} s_j d\lambda(j)$, usually called *the aggregate of* s and Conv $\mathbb{D}$ denotes the convex hull of the set $\mathbb{D}$.

7. The trajectory of the state variable corresponding to a profile of strategies S is $X^S(t+1) = \varphi(X^S(t), u^S(t))$ for a function $\varphi : \mathbb{X} \times \text{Conv } \mathbb{D} \to \mathbb{X}$ and $u^S(t) = u^{S(X(t))}$ with the initial condition $X(0) = x_0$.

8. For a given profile S, the payoff function of player i is

$$J_i(x_0, S) = \sum_{t=0}^{\infty} \beta^t \mathcal{P}_i\big(X(t), S_i(X(t), u^S(t))\big).$$

1.4.3 Decomposition theorem for games with a continuum of players

Here, we cite a theorem concerning the dynamic games with a continuum of players which we use in the thesis — decomposition theorem from Wiszniewska-Matyszkiel [79].

We use a slightly reduced form of Theorem 3.2 from [79] because of the high complexity of the games considered in [79] and we cite the result restricted to the infinite time horizon case only.

To state the main theorem, we need to first define the following:

Definition 10 *Given time t and state x, the static game played at (t, x) is a game with set of players $\mathbb{I}$, set of strategies equal to the set $D_i(x)$ of currently available decisions and the current payoff function $\mathcal{P}_i(x, \cdot, \cdot)$.*

Definition 11 *Static equilibrium is a Nash equilibrium in the static game — a solution of a static game.*

Definition 12 *The continuous images of Borel sets, are called analytic sets or Souslin sets (e.g., Kuratowski [88]).*

Theorem 8 *(Wiszniewska-Matyszkiel [79] Theorem 3.2)*

(a) *If S is a profile of strategies and for all t, the profiles of decisions $S(X^S(t))$ are equilibria in the corresponding one stage games — at time t and state of the system $X^S(t)$, then S is a Nash equilibrium.*

(b) *Let the space of decisions $\mathbb{D}$ be such that the set $\{(d, d) : d \in \mathbb{D}\}$ is $\mathcal{D} \otimes \mathcal{D}$-measurable and $\mathbb{D}$ is a measurable image of a measurable space $(\mathbb{Z}, \mathcal{Z})$ that is* *an analytic subspace of a separable compact topological space $\mathbb{W}$ (with the σ-field of Borel subsets). Assume that, for almost every i and every x, the function $\mathcal{P}_i(X(t), \cdot, \cdot)$ is upper semi-continuous, for almost every i, the function $\mathcal{P}_i$ is such that the inverse images of measurable sets are $\mathcal{X} \otimes \mathcal{D} \otimes \mathcal{B}(\mathbb{U})$-analytic and the correspondence D_i has an $\mathcal{X} \otimes \mathcal{D}$-analytic graph and compact values. Every Nash equilibrium S such that, for almost every player i, the payoff $J_i(x_0, S)$ is finite, satisfies the following condition: for all t, static profiles $S(X^S(t))$ are static equilibria at the state of the system $X^S(t)$.*

1.5 Organization of the thesis

The methodology of the dissertation is based on the game theory, more specifically dynamic games and the optimal control theory. Chapter 2-5 deal with the dynamic games or a dynamic optimization problems in discrete time and Chapter 6 with continuous time. The research work presented in this dissertation is organised and structured in the form of seven Chapters, which are briefly described as follows:

- **Chapter 2** describes a linear-quadratic dynamic game model of a resource extraction problem in the infinite time horizon. In the model, there are linear state-dependent constraints on controls which makes the problem more complicated. The dynamic game is investigated for n-players and a continuum of players.

- **Chapter 3** provides a comprehensive review of a truncation of the linear-quadratic dynamic game model considered in Chapter 2 for 2-players and 2-stages. As a very important result, the existence of a continuum of discontinuous symmetric feedback Nash equilibria and non-existence of a continuous symmetric feedback Nash equilibria is proven.

- **Chapter 4** presents a dynamic optimisation problem for a more general class of linear-quadratic games with linear state-dependent constraints and more general value of the discount factor β. The problem is analysed both for the infinite time horizon and its finite time truncations.

- **Chapter 5** deals with the important theoretical aspects of the dynamic game theory. With the motivating example of Fish Wars model of Levhari and Mirman, analysed by analytical and numerical methods, the formulation of general rules when under or over-estimation of the value function results in correct optimal trajectory and the optimal strategy along with it for the dynamic optimisation problems is established.

- **Chapter 6** presents a Fish Wars model in continuous time. Besides calculating and comparing the optima and equilibria, two type of enforcement of the social optimality is analysed — a tax-subsidy system and self-enforcing environmental agreement with the assumption that there is a time delay in the observation of a default.

- **Chapter 7** concludes the thesis and the scope for future work is also mentioned.

Chapter 2

A linear-quadratic dynamic game with linear state-dependent constraints modelling exploitation of a common fishery

This chapter is mostly taken from the research article [83]. However, the results are extended to encompass interesting properties which were noticed after the publication.

We consider a discrete time linear-quadratic (LQ) dynamic game with the infinite time horizon. It is a model of extraction of a common renewable resource — a fishery — with many players — countries or firms — which sell their catch at a common market.

This LQ dynamic game model constitutes a counterexample to a simplification of the standard methodology which is regarded as correct and widely used in applied papers on dynamic games and dynamic optimization problems.

The research done in this chapter has two objectives:

1. Theoretical — Investigate LQ dynamic games in the case when there are linear state-dependent constraints on the players' decisions.

2. Applicational — Analyzing a common renewable resource problem with many players. By many players, we understand the results of a decomposition of the decision-making structure of the same mass of consumers of the resource. We also introduce a Pigouvian tax or a tax-subsidy system enforcing the social optimality.

2.1 Formulation

We consider a linear-quadratic dynamic game of exploitation of one common renewable resource — a fishery — with many players. The renewable resource that we consider is one species of fish, uniformly dispersed in a marine or deep lake fishery, equally partitioned between the players. The dynamic game considered here consists of:

1. A set of players: Either finite $\mathbb{I} = \{1, 2 \ldots, n\}$ or a continuum $\{\mathbb{I}, \mathcal{L}, \lambda\}$ with $\mathbb{I} = [0, 1]$.

2. A *time set*: Discrete and infinite time horizon $\mathbb{T} = \{0, 1, 2, \ldots\}$ with the initial time $t_0 = 0$. Subsequent t represents the subsequent periods (divided by the spawning stage).

3. The *state of the resource*: $x \in \mathbb{X} = \mathbb{R}_+$, denoting the *biomass of fish.* To make it more clear, we assume it at the beginning of time period (e.g., year after spawning).

4. The *average/aggregate extraction* of s: For n player dynamic game

$$u^s = \sum_{i=1}^{n} \frac{s_i}{n}. \tag{2.1.1}$$

   Here, we use division by n, in order to be able to compare the dynamic games with various n treated as decomposing the same set of individuals into decisive units of decreasing size (e.g., the global population decomposed into continents, countries, regions etc). At the maximal level of decomposition we have a limit game with a continuum of players for which

$$u^s = \int_{[0,1]} s_j \cdot d\lambda(j), \tag{2.1.2}$$

   as in Subsection 1.4.2.

5. The catch is sold at a common market at s dependent only on u^s. We define the price by a linear function (known as *inverse demand function* in economics),

$$A - u^s,$$

   Players face a quadratic *cost* of fishing, *identical* for every player and equal to

$$fs_i + \frac{s_i^2}{2}.$$

   So, the *current or instantaneous payoff function* of every player is equal to their revenue minus cost

$$P_i(s) = (A - u^s)s_i - fs_i + \frac{s_i^2}{2}.$$

   In economic applications, A is substantially greater than f, so, we assume $A > f$.

6. **Notational convention:** Using the aggregate $u^s \in \mathbb{R}_+$, we will use the simplifying notation $P_i(s) = \mathcal{P}(s_i, u^s)$ to emphasize both the decision of player i and the aggregate also for finitely-many-players case i.e.,

$$P(s_i, u^s) = (A - u^s)s_i - fs_i + \frac{s_i^2}{2}.$$

7. Strategies: Here, we consider the special form of the feedback strategies, dependent only on the state variable, $S_i : \mathbb{X} \to \mathbb{D}_i$.

8. The *trajectory* X of the state variable: We also use the simplifying notation with u^s. Given a strategy profile S,

$$X(t+1) = \varphi\left(X(t), u^{S(X(t))}\right); X(0) = x_0, \tag{2.1.3}$$

where the state transition function is

$$\varphi\left(X(t), u^{S(X(t))}\right) = (1+\xi)\,X(t) - u^{S(X(t))}. \tag{2.1.4}$$

9. Linear state dependent constraints on decisions: $D_i(x) = [0, cx]$.

   To make the depletion or extinction possible, we consider $c = (1+\xi)$, where $\xi > 0$ is the *regeneration rate of the resource* — natural net growth rate of the biomass of fish without fishing.

10. *Total payoff function* of player i: For choosing a strategy profile S, Eq. (1.4.5 (b)) becomes

$$J_i(x_0, S) = \sum_{t=0}^{\infty} \beta^t \cdot \left(\left(A - u^{S(X(t))}\right) S_i(X(t)) - \left(f S_i(X(t)) + \frac{S_i^2(X(t))}{2}\right)\right) \tag{2.1.5}$$

   The discount factor β from Eq. (2.1.5) measures the players' patience. From the economical point of view, more interesting solutions are the Nash equilibria at which the rates of growth of both assets — the resource and the money — are identical. When applied to renewable resources extraction, it is known as *the golden rule* (e.g., [89]). Therefore, in this chapter, we consider

$$\beta = \frac{1}{1+\xi}, \tag{2.1.6}$$

   that is $r = \xi$.

2.2 Social Optima

In the social optimum problem, n-players jointly maximize their payoffs. So, for this problem the Bellman Eq. (1.3.7) is

$$\bar{V}(x) = \max_{s_i \in [0,cx]^n} \sum_{i=1}^{n} \left((A - u^s)\, s_i - \left(f s_i + \frac{s_i^2}{2} \right) \right) + \beta \bar{V}\left((1+\xi)x - u^s \right), \qquad (2.2.1)$$

the Bellman inclusion (1.3.8) is

$$\bar{S}(x) \in \operatorname*{Argmax}_{s_i \in [0,cx]^n} \sum_{i=1}^{n} \left((A - u^s)\, s_i - \left(f s_i + \frac{s_i^2}{2} \right) \right) + \beta \bar{V}\left((1+\xi)x - u^s \right). \qquad (2.2.2)$$

We are going to extend the previously defined game by considering also single player case i.e., a dynamic optimization problem.

Lemma 9 *Consider the social optimum problem for the n-players. If the value function $\bar{V} : \mathbb{R}_+ \to \mathbb{R}$ fulfiling the Bellman Eq.* (2.2.1) *is differentiable, then the optimal solution is symmetric.*

Proof: Apply the Karush-Kuhn-Tucker first order necessary conditions to the maximum in the right hand side of Eq. (2.2.1), for given x. The constraints are:

$$s_i \geq 0, \ \forall i = 1 \ldots, n; \qquad (2.2.3a)$$

$$(1+\xi)x - s_j \geq 0, \ \forall i = 1 \ldots, n. \qquad (2.2.3b)$$

Define the adjoint variables $\mu = (\mu_1, \ldots, \mu_n) \geq 0$ for the constraints (2.2.3a) and $\nu = (\nu_1, \ldots, \nu_n) \geq 0$ for the constraints (2.2.3b) respectively.

Consider the *Lagrangian* $\mathfrak{L}(x, s, \mu, \nu) =$

$$\sum_{i=1}^{n} \left((A - u^s)s_i - \left(f s_i + \frac{s_i^2}{2} \right) + \beta V\left((1+\xi)x - u^s \right) \right) + \sum_{i=1}^{n} \mu_i s_i + \sum_{i=1}^{n} \nu_i \left((1+\xi)x - s_i \right).$$

To find a candidate for optimal control, calculate the point of zero derivative of the *Lagrangian* $\mathfrak{L}(x, s, \mu, \nu)$ with respect to s_i,

$$(A - u^s) - \frac{s_i}{n} - (f + s_i) - \left(\frac{\beta}{n} \right) V'\left((1+\xi)x - u^s \right) + \mu_i - \nu_i = 0. \qquad (2.2.4)$$

Similarly, for $j \neq i$, $\frac{\partial \mathfrak{L}(x,s,\mu,\nu)}{\partial s_j} = 0$ yields

$$(A - u^s) - \frac{s_j}{n} - (f + s_j) - \left(\frac{\beta}{n} \right) V'\left((1+\xi)x - u^s \right) + \mu_j - \nu_j = 0. \qquad (2.2.5)$$

Consider the following cases:

case 1. If both $\mu_i, \mu_j = 0$ and $\nu_i, \nu_j = 0$, then by Eq. (2.2.4) and (2.2.5), $s_i = s_j$, thus, the strategies are symmetric.

case 2. Assume two asymmetric boundary points, $s_i = 0$ and $s_j = (1+\xi)x$ with $\mu_i \neq 0$ and $\nu_j \neq 0$. Substitute $s_i = 0$ into Eq. (2.2.4) and $s_j = (1+\xi)x$ into Eq. (2.2.5) and solve for μ_i and ν_j, to get $\mu_i + \nu_j + \left(1 + \frac{1}{n}\right)(1+\xi)x = 0$. This is a contradiction, since both $\mu_i, \nu_j \geq 0$.

case 3. If $s_i = 0$ (so that $s_i \neq (1+\xi)x \implies \nu_i = 0$ and $s_j \neq 0 \implies \mu_j = 0$), then, by solving Eq. (2.2.4) and (2.2.5) for μ_i and ν_j, to get, $s_j\left(1 + \frac{1}{n}\right) + \mu_j + \nu_i = 0$. This is again a contradiction.

case 4. If $s_i = (1+\xi)x \implies \mu_i = 0$ and $s_j \neq (1+\xi)x \implies \nu_j = 0$, then by solving Eq. (2.2.4) and (2.2.5) for μ_i and ν_j, to get, $\left(1 + \frac{1}{n}\right)((1+\xi)x - s_j) + \mu_j + \nu_i = 0$. This is a contradiction, since $s_j < (1+\xi)x$.

Therefore, the strategies are symmetric. ■

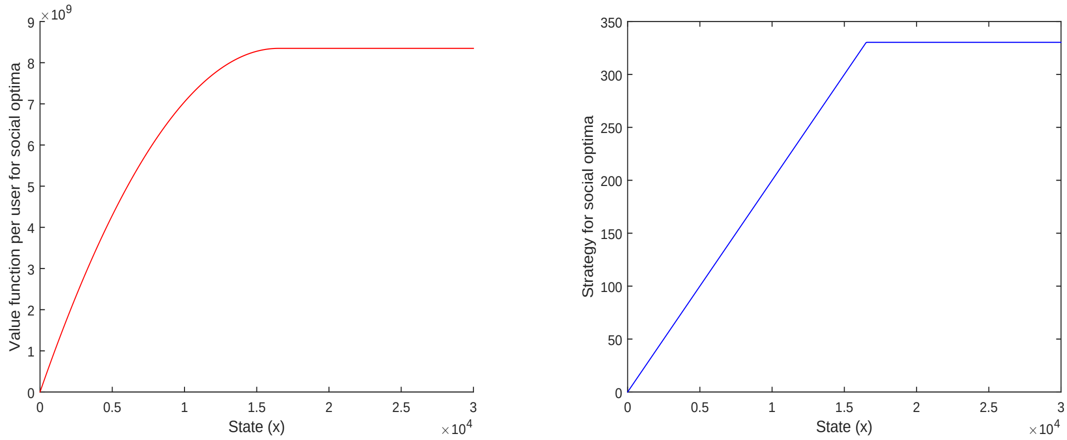
Theorem 10 *Consider the social optimum problem. For* $\beta = \frac{1}{1+\xi}$*, corresponding to the* golden rule *interest rate*
(a) *The value function for n-players with* $n \geq 1$ *is,*

$$\bar{V}^{\mathrm{SO}}(x) = \begin{cases} \hat{g} \cdot x + \frac{\hat{h}}{2} \cdot x^2, & \text{if } 0 < x < \tilde{x}, \\ \tilde{k}, & \text{if } x \geq \tilde{x}, \end{cases} \tag{2.2.6}$$

for $\hat{s} = \frac{A-f}{3}$, $\tilde{x} = \frac{\hat{s}}{\xi}$, $\hat{h} = -3n\xi\,(1+\xi)$, $\hat{g} = n(A-f)(1+\xi)$, *and* $\tilde{k} = \frac{(A-f)^2(1+\xi)n}{6\xi}$.
(b) *The value function of player i for n-players with* $n \geq 1$ *is* $\bar{V}_i^{\mathrm{SO}}(x) = \frac{\bar{V}^{\mathrm{SO}}(x)}{n}$*, and it is independent of number of players n.*
(c) *The value function for the continuum of players is the same as the value function of player i for n-players i.e.,* $\bar{V}_i^{\mathrm{SO}}(x)$.
(d) *The unique social optimum both for the n-players and the continuum of players is*

$$\bar{S}_i^{\mathrm{SO}}(x) = \begin{cases} \xi x, & \text{if } 0 < x < \tilde{x}, \\ \hat{s}, & \text{if } x \geq \tilde{x} \end{cases} \tag{2.2.7}$$

The results of Theorem 10 are presented in Fig. 2.1a and 2.1b, for the specific values of the parameters $A = 1000$, $f = 9$ and $\xi = 0.02$.



(a) The value function **(b)** The social optimum strategy

Figure 2.1: Social optimum problem for arbitrary number of players

Proof:

The theorem can be easily proved just by substituting $\bar{V}^{\text{SO}}$ and $\bar{V}^{\text{SO}}$ to the Bellman Eq. (2.2.1) and the Bellman inclusion (2.2.2). Nevertheless, it is not the way in which such results are obtained. Therefore, we present the whole path which leads us to a solution.

(a) By Lemma 9, if V fulfiling the Bellman Eq. (2.2.1) and terminal condition is differentiable, then the optimal solution is symmetric. We can easily check that the function $\bar{V}^{\text{SO}}$ is differentiable, with derivative at $\tilde{x}$ equal to 0. So, for simplicity, consider symmetry a priori and assume $S_i \equiv S$ to maximize

$$n \sum_{t=0}^{\infty} (A - S(X(t)) \, S(X(t)) - \left(f S(X(t)) + \frac{S^2(X(t))}{2}) \right) \beta^t$$

over the set of feedback controls.

The Bellman Eq. (2.2.1) reduces to

$$\bar{V}(x) = \max_{s \in [0,(1+\xi)x]} n \left[(A - s)\, s - \left(fs + \frac{s^2}{2} \right) \right] + \beta \bar{V} \left((1+\xi)x - s \right). \tag{2.2.8}$$

Assume that the value function $\bar{V}(x)$ is of quadratic form: $\bar{V}(x) = k + gx + \frac{hx^2}{2}$ and look for a solution of the Eq. (2.2.8) in this class of functions.

The first order condition for s to be the optimal solution is

$$s = \frac{-n\,(1+\xi)\,(A-f) + g + h(1+\xi)x}{(h - 3n(1+\xi))}. \tag{2.2.9}$$

Substitute this s into the Bellman Eq. (2.2.8) to calculate the values of the constants for which Eq. (2.2.9) is fulfilled. Obtained sets of values of the constants are as follows,

$$\hat{h} = -3n\xi(1+\xi), \;\; \hat{g} = n(1+\xi)(A-f), \;\; \hat{k} = 0; \tag{2.2.10a}$$

$$g = h = 0, \;\; \tilde{k} = \frac{n(A-f)^2}{6(1-\beta)}; \tag{2.2.10b}$$

$$h = 0, \quad \text{arbitrary } g \neq 0, \;\; k(g) = \frac{(n(A-f)-\beta g)^2}{6n(1-\beta)}. \tag{2.2.10c}$$

Notice that $h \leq 0$ for all such sets of constants. So, the maximized function is strictly concave, which implies that s from Eq. (2.2.9), is the unique global maximizer if $s \in [0, cx]$. Consider the following cases.

case 1 If k, g and h are as in (2.2.10a).

Then the zero derivative point is ξx, which is less than cx. Therefore, it defines the unique maximizer in this case. However, the function

$$\bar{V}_1(x) = \hat{g}x + \frac{\hat{h}x^2}{2} \tag{2.2.11}$$

does not fulfil the terminal condition (1.3.10), since $\lim_{t\to+\infty} \beta^t \bar{V}_1(X^0(t)) = -\infty$, where X^0 is the trajectory corresponding to the profile $S \equiv 0$.

Now, consider the weaker terminal condition from Eq. (1.3.11). It also does not solve the problem as the payoff for $S \equiv 0$ is 0. This means that the sufficient condition is not fulfiled, so at this stage of the proof, it cannot be checked whether $\bar{V}_1$ is the value function or not.

case 2 If k and h are as in (2.2.10b), then the candidate for the value function is

$$\bar{V}_2(x) = \tilde{k} \tag{2.2.12}$$

and the terminal condition is obviously fulfilled, since $\bar{V}_2$ is constant. The Bellman Eq. (2.2.1) has the form $\bar{V}_2(x) = \max_{s\in[0,cx]^n} \sum_{i=1}^{n} P_i(s) + \beta\tilde{k}$.

Therefore, the candidate for the optimal strategy of each player is $\hat{s} = \frac{A-f}{3}$, which is independent of x.

Note that for x close to 0, $\hat{s} > (1+\xi)x$, so, $\hat{s}$ cannot be the social optimum for these x and the Bellman Eq. (2.2.1) is not fulfilled. Hence, $\bar{V}_2(x) = \tilde{k}$ cannot be the value function of the problem, since Eq. (2.2.8) is also a necessary condition.

case 3 If k, g and h are as in (2.2.10c).

Then, $\lim_{t\to+\infty} \beta^t \left(gX(t) + k(g)\right) \neq 0$ for the trajectory X^0 which violated the terminal condition in **case 1**. Besides, $k \neq= 0$, the candidate for optimal s, given by Eq. (2.2.9) is constant. So, either $S \equiv 0$ or $S(0) \neq 0$. In the first case $V \equiv 0$, which is not true. In the second case $S(0) \in [0, c \cdot 0]$. So, such function cannot be the value function for our model.

case 4 Consider **case 1** and **case 2** together.

Consider the only continuous combination of $\bar{V}_1$ and $\bar{V}_2$ which makes sense, i.e., with $\bar{V}(0) = 0$ with one switching point and continuous. Then the candidate for the value function is

$$\bar{V}(x) = \begin{cases} \bar{V}_1(x), & \text{if } 0 \leq x \leq \tilde{x}, \\ \bar{V}_2(x), & \text{if } x > \tilde{x}. \end{cases}$$

for $\bar{V}_1(x)$ and $\bar{V}_2(x)$ from Eq. (2.2.11)–(2.2.12). So, $\bar{V}(x) = \bar{V}^{\text{SO}}(x)$.

Note that $\bar{V}^{\text{SO}}$ is continuous as well as differentiable. The corresponding candidate for the optimal profile is $\bar{S}^{\text{SO}}(x)$.

After finding the candidates for the value function and the optimal profile, it is also needed to be proved that the Bellman Eq. (2.2.8) is really fulfilled by this piecewise defined function.

For brevity of notations, given a state x and a decision s (by symmetry, the aggregate extraction will be also equal to s), denote the next stage state by $x_{\text{next}}(x, s)$,

$$x_{\text{next}}(x, s) = (1 + \xi)x - s.$$

The set of s for which $x_{\text{next}}(x, s) \leq \tilde{x}$, is denoted by S_{I} and it is written as

$$S_{\text{I}} = [s_{\text{Bd}}, (1 + \xi)x],$$

while the set of the remaining s,

$$S_{\text{II}} = [0, s_{\text{Bd}}),$$

where s_{Bd} denotes s for which $x_{\text{next}}(x, s) = \tilde{x}$, i.e., $\tilde{x} = (1 + \xi)x - s_{\text{Bd}}$ whenever it is non-negative, otherwise take $s_{\text{Bd}} = 0$ (this holds for some $x < \tilde{x}$; then $S_{\text{II}} = \emptyset$).

If for some x, $s_{\text{Bd}} = 0$, which may hold only for $x \leq \tilde{x}$, then for this x, the Bellman Eq. (2.2.8) reduces to $\bar{V}_1(x) = \max_{s \in [0,(1+\xi)x]} \sum_{i=1}^{n} (\mathcal{P}(s, s)) + \beta \bar{V}_1 (x_{\text{next}}(x, s))$, which was solved during the calculation of constants in **case 1**.

So, consider $s_{\text{Bd}} > 0$, then $S_{\text{I}} \neq \emptyset$ and $S_{\text{II}} \neq \emptyset$. This situation can be decomposed into two cases.

(I) For $x \le \tilde{x}$, the Bellman Eq. (2.2.8) can be rewritten as
$\bar{V}_1(x) = \max\{\max_{s\in S_{\rm I}} n\mathcal{P}(s,s) + \beta\bar{V}_1(x_{\rm next}(x,s)), \max_{s\in S_{\rm II}} n\mathcal{P}(s,s) + \beta\bar{V}_2(x_{\rm next}(x,s))\}$.

Since $\max_{s\in S_{\rm I}} n\mathcal{P}(s,s)+\beta\bar{V}_1(x_{\rm next}(x,s))$ is attained at $\xi x \in S_{\rm I}$, while $\max_{s\in S_{\rm II}} n\mathcal{P}(s,s) + \beta\bar{V}_2(x_{\rm next}(x,s))$ is attained at $s_{\rm Bd} \in S_{\rm I}$, but not to $S_{\rm II}$, and $s_{\rm Bd}$ does not optimize $n\mathcal{P}(s,s) + \beta V_2(x_{\rm next}(x,s))$ on $S_{\rm I}$, the Bellman Eq. (2.2.8) is fulfilled.

(II) If $x > \tilde{x}$, then the Bellman Eq. (2.2.8) can be rewritten as
$\bar{V}_2(x) = \max\{\max_{s\in S_{\rm I}} n\mathcal{P}(s,s) + \beta\bar{V}_1\left(x_{\rm next}(x,s)\right), \max_{s\in S_{\rm II}} n\mathcal{P}(s,s) + \beta\bar{V}_2\left(x_{\rm next}(x,s)\right)\}$. First, consider the optimization over $S_{\rm I}$.

The first order condition for maximization of $\frac{\partial(n\mathcal{P}(s,s)+\beta\bar{V}_1(x_{\rm next}(x,s))}{\partial s} = 0$, is attained at $\xi x \notin S_{\rm I}$. So, the supremum over $S_{\rm I}$ is attained at $s_{\rm Bd} \in$ Closure $(S_{\rm II})$.

Since $n\mathcal{P}(s,s)+\beta\bar{V}_2\left(x_{\rm next}(x,s)\right)$ is strictly concave and $\frac{A-f}{3}$ is its only global maximum, $\mathcal{P}\left(\frac{A-f}{3},\frac{A-f}{3}\right) > \mathcal{P}\left(s_{\rm Bd}, s_{\rm Bd}\right)$. Since V is continuous, $\frac{A-f}{3}$ is the global maximum over $[0,(1+\xi)x]$.

Therefore, in **case 4**, the Bellman Eq. (2.2.8) is fulfilled.

The terminal condition given by Eq. (1.3.9) is obvious, since $\bar{V}^{\rm SO}$ is bounded.

(b) Immediate.

(c) For the continuum of players, the social optimum is defined by the inclusion,

$$\bar{S} \in \operatorname*{Argmax}_{S\in\Sigma} \int_0^1 \sum_{t=0}^{\infty} \beta^t \mathcal{P}\left(S_i(X(t)), u^{S(X(t))}\right) d\lambda(i).$$

Notice that along the optimal profile $\bar{S}$, $\mathcal{P}$ is non-negative, since, otherwise, at t for which $\mathcal{P}\left(\bar{S}_i(X(t)), u^{\bar{S}(X(t))}\right)$ is negative, replace $\bar{S}_i(X(t))$ by 0 and increase the aggregate payoff.

Since $\bar{S}$ is a profile, $\bar{S}(X(t))$ is measurable, so, $\beta^t\mathcal{P}\left(\bar{S}_i(X(t)), u^{\bar{S}(X(t))}\right)$ is integrable. As $\mathcal{P}$ is bounded on the set on which it is non-negative, along the optimal profile, the series is absolutely convergent, so,
$\int_0^1 \sum_{t=0}^{\infty} \beta^t\mathcal{P}\left(\bar{S}_i(X(t)), u^{\bar{S}(X(t))}\right) d\lambda(i) = \sum_{t=0}^{\infty}\beta^t \int_0^1 \mathcal{P}\left(\bar{S}_i(X(t)), u^{\bar{S}(X(t))}\right) d\lambda(i)$.
Since $\mathcal{P}(s_i,u)$ is concave in s_i, by the Jensen inequality

$$\sum_{t=0}^{\infty}\beta^t \int_0^1 \mathcal{P}\left(\bar{S}_i(X(t)), u^{\bar{S}(X(t))}\right) d\lambda(i) \le \sum_{t=0}^{\infty}\beta^t\mathcal{P}\left(\int_0^1 \bar{S}_i(X(t))d\lambda(i), u^{\bar{S}(X(t))}\right).$$

The right hand side of this expression is equal to,

$$\max_{S\in\Sigma}\sum_{t=0}^{\infty}\beta^t\mathcal{P}\left(u^{S(X(t))}, u^{S(X(t))}\right) = \max_{S\in\mathbb{S}}\sum_{t=0}^{\infty}\beta^t\mathcal{P}\left(S(X(t)), S(X(t))\right),$$

which reduces the continuum of players problem to the social optimum problem for n-players with $n = 1$.

(d) The function $\bar{V}^{\mathrm{SO}}(x)$ is the value function, so, the function $\bar{S}^{\mathrm{SO}}$ is the social optimum for any finite $n \geq 1$. The result for the continuum of players case is immediate, by reduction of the social optimum problem to the social optimization problem for $n = 1$.

Therefore, the social optimum both for n players and the continuum of players is the profile $\bar{S}_i^{\mathrm{SO}}$ defined by Eq. (2.2.7) . Moreover, the optimal profile is unique, since maximisation of the Bellman equation is also the necessary condition for a control to be optimal by (Theorem 1). ■

Corollary 11 *The value function for the social optimum problem is continuous, differentiable and strictly increasing for $x < \tilde{x}$, non-decreasing globally, while the social optimum leads to the sustainability of the resource.*

Remark 1 *The result is independent of the number of players which implies that our game properly models the situation in which increasing the number of players represents considering a more decomposed decision-making structure without introducing additional fishermen as new players.*

Corollary 12 ***(a)*** *The value function and the social optimum do not change if we change the dynamics for $x \geq \tilde{x}$ and consider the state transition equation*

$$X(t+1) = \varphi(X(t), u^{S(X(t))}) \text{ with } X(0) = x_0, \tag{2.2.13}$$

for the function $\varphi(x, u) = (1 + \xi)x - u$ for $x < \tilde{x}$ and such that the interval $[\tilde{x}, +\infty)$ is invariant under Eq. (2.2.13) given $S = \bar{S}^{\mathrm{SO}}$.

(b) *If we replace the golden rule β by any $\beta \in (0, 1)$, then the social optimum remains unchanged at $[\tilde{x}, +\infty)$, while the value function on this interval is $\frac{(A-f)^2 n}{6(1-\beta)}$.*

Proof: In both cases, for $x > \tilde{x}$ we have the unconstrained global maximum at $\hat{s}$ (since, the discounted sum of global maxima at each time instant), which is feasible in those cases.

The value function in **(b)** is, therefore, equal to $\frac{\mathcal{P}(\hat{s},\hat{s})}{1-\beta}$. ■

2.2.1 A counterexample — a general conclusion for solving the dynamic optimisation and a Nash equilibrium problem in dynamic games

While proving Theorem 10, we have obtained the following result which can be used as a counterexample for a common simplification used in the calculation of the optimal controls and the Nash equilibria in infinite time horizon problems.

Corollary 13 *The unique quadratic solution of the Bellman Eq.* (2.2.1) *is* $\bar{V}_1$ *given by Eq.* (2.2.11). *For this* $\bar{V}_1$ $\exists!$ S *fulfiling the Bellman inclusion* (2.2.2) *with* $\bar{V}_1$ *and it is linear and the corresponding trajectory is constant.*

Nevertheless, the function $\bar{V}_1$ *is* not *the value function for the social optimum problem, while this unique* S *is* not *the social optimum.*

Proof: The value function $\bar{V}^{\mathrm{SO}} \neq \bar{V}_1$ and the unique social optimum $\bar{S}^{\mathrm{SO}} \neq S$. ■

This result is a simple **counterexample** showing that skipping checking the terminal condition while looking for the optimal control in the feedback form or a feedback Nash equilibrium, which often appears in literature (e.g., most of the papers in the Fish Wars thread; terminal condition in the infinite time horizon is sometimes also omitted in the textbooks), may lead to finding the wrong results. Although we are conscious that some counterexamples already exist (e.g., [90]) they are very elaborate, while this problem is simple and well motivated by its applicability and the fact that the existence of a quadratic value function for a linear-quadratic dynamic game or a dynamic optimization problems is a kind of *folk theory* in the field.

This misleading belief obviously comes from unconstrained problems. So, an author who is not checking the terminal condition is likely to regard the unique quadratic solution of the Bellman Eq. as an obvious candidate for the value function.

Moreover, this is also a counterexample for the uniqueness of the solution of the Bellman Eq. (2.2.8) in such a simple class of dynamic optimisation problems.

2.3 Nash equilibria

2.3.1 Nash Equilibria for the Continuum of Players Case

Next, we solve the problem of *Nash equilibrium*. We start from the game with a continuum of players.

Theorem 14 *Consider the game with a continuum of players* $(\mathbb{I}, \mathcal{L}, \lambda)$ *with* $\mathbb{I} = [0, 1]$.

(a) *The only feedback Nash equilibrium profile (up to equivalence almost everywhere) is*

$$\bar{S}_i^{\mathrm{NE}}(x) = \begin{cases} (1+\xi)x, & \text{if } x \le \hat{x}_1, \\ \frac{A-f}{2}, & \text{otherwise,} \end{cases}$$

for $\hat{x}_1 = \frac{A-f}{2(1+\xi)}$.

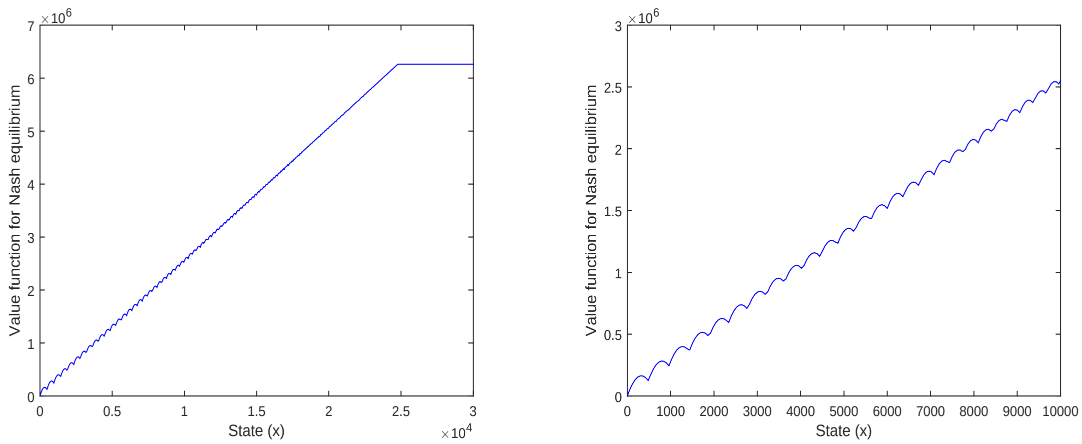
(b) *The value function of the considered problem is*

$$\bar{V}_i^{\mathrm{NE}}(x) = \begin{cases} \mathcal{P}_{\mathrm{depl}}(x) := \mathcal{P}\left((1+\xi)x, (1+\xi)x\right), & \text{if } x \le \hat{x}_1 \\ \sum_{k=1}^{N} \frac{(A-f)^2 \beta^{k-1}}{8} + \beta^N \mathcal{P}_{\mathrm{depl}}\left((1+\xi)^N x - \frac{(A-f)}{2} \sum_{k=1}^{N} (1+\xi)^{k-1}\right), & \text{if } \hat{x}_N < x < \hat{x}_{N+1} \\ \frac{(A-f)^2}{8(1-\beta)}, & \text{otherwise,} \end{cases}$$

where $\mathcal{P}_{\mathrm{depl}}(x) = \left(A - f - \frac{3}{2}(1+\xi)x\right)(1+\xi)x$ *is the payoff resulting from immediate depletion of the resource and* $\hat{x}_N = \sum_{k=1}^{N} \frac{A-f}{2(1+\xi)^{k-1}}$ *for* $N \ge 1$.

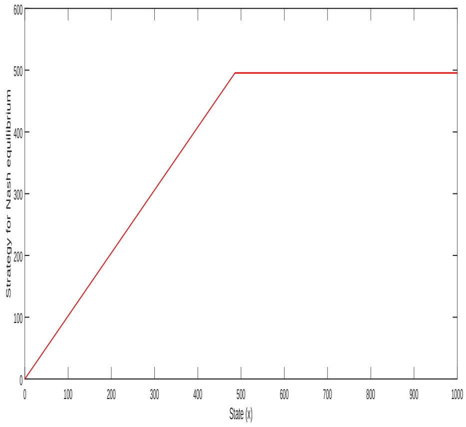
(c) *For* $x \in (\hat{x}_N, \hat{x}_{N+1}]$ *with* $\hat{x}_0 = 0$, *the resource will be depleted in* $N+1$ *stages, while for* $x \ge \hat{x}_\infty = \lim_{N\to\infty} \hat{x}_N$, *the resource will never be depleted.*

The results of Theorem 14 are illustrated in Fig. 2.2a-2.3b for $A = 1000,\ f = 9,\ \xi = 0.02$. Because drawing a piecewise continuous function with infinitely many pieces is impossible, we draw the accurate graph of $\bar{V}_i^{\mathrm{NE}}$ over $x \in [o, x_N]$ and $[\tilde{x}, +\infty)$ for $N = 1000$.

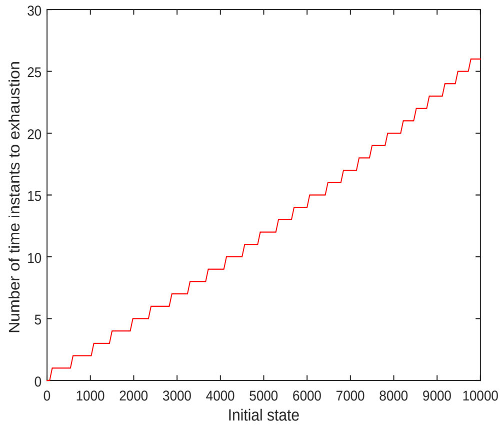


(a) The value function **(b)** Zoomed — value function

Figure 2.2: Nash equilibrium problem for the continuum of players — The value function



(a) The Nash equilibrium strategy



(b) The number of time instants to resource exhaustion

Figure 2.3: Nash equilibrium problem for the continuum of players–the strategy and time to resource exhausion

Proof: **(a)** By Theorem 8, for a dynamic game with the continuum of players, *a profile is a Nash equilibrium if and only if it is a sequence of Nash equilibria in one stage games*. So, for such games, at each stage with state x, the Nash equilibrium is a profile of decisions $\bar{s}$ such that for almost every i,

$$\bar{s}_i \in \operatorname*{Argmax}_{s_i \in [0,(1+\xi)x]} \mathcal{P}(s_i, u^{\bar{s}}).$$

Consider any $u^{\bar{s}}$ and note that the influence of any single player on $u^{\bar{s}}$ is negligible.

Since for given $u^{\bar{s}}$, every player faces the same decision making problem with unique solution, all the profiles are symmetric. So, $u^{\bar{s}} = s_i$. Consider

$$\bar{S}^{\mathrm{NE}}(x) = \begin{cases} (1+\xi)\,x, & \text{if } x \leq \hat{x}_1, \\ \frac{(A-f)}{2}, & \text{otherwise.} \end{cases}$$

Now, assume that in the static game considered, for some x, a static profile at x gives some other aggregate $u \neq S(x)$.

case 1 If $x \leq \hat{x}_1$ and $u < (1+\xi)x$, then the best response of every player i is $(1+\xi)x \implies u^s > u$, which is a contradiction.

case 2 If $x > \hat{x}_1$ and $u < \frac{A-f}{2}$, then the best response of every player i is $s_i > \frac{A-f}{2} \implies u^s > u$, a contradiction.

case 3 If $x > \hat{x}_1$ and $u > \frac{A-f}{2}$, then the best response of every player i is

$s_i < \frac{A-f}{2} \implies u^s < u$, again a contradiction.

case 4 Consider that $u = \bar{S}^{\mathrm{NE}}(x)$.

(i) If $X(t) = x \leq \hat{x}_1$, then at this stage, $\forall\, i$, $\underset{s_i \in [0,(1+\xi)x]}{\text{Argmax}}\ \mathcal{P}(s_i, u) = (1+\xi)x$.

At the next stage, $X(t+1) = 0$.

(ii) If $X(t) = x > \hat{x}_1$, then at this stage, $\forall\, i$, $\underset{s_i \in [0,(1+\xi)x]}{\text{Argmax}}\ \mathcal{P}(s_i, u) = \frac{A-f}{2}$.

Therefore, for every state x, $\bar{S}^{\mathrm{NE}}(x)$ is the static Nash equilibrium at x.

(b) and **(c)** are proved together. Consider the following cases.

case 1 If $x \leq \hat{x}_1$, then the optimal decision is $(1+\xi)x$, meaning that the resource depletes immediately. The value function is given by

$$V_i^{\mathrm{NE}}(x) = \mathcal{P}_{\mathrm{depl}}(x) := \left(A - f - \frac{3}{2}\,(1+\xi)\,x\right)(1+\xi)\,x.$$

case 2 If $\hat{x}_1 < x < \hat{x}_\infty$, then the optimal choice for every player is $\frac{A-f}{2}$.

Define $\hat{x}_2$ such that $\hat{x}_1 = (1+\xi)\hat{x}_2 - \frac{A-f}{2}$, then $\forall\, x \in (\hat{x}_1, \hat{x}_2]$, the state in the next stage belongs to $(0, \hat{x}_1]$. Further, prove recursively and assume that $\hat{x}_{N+1}$ such that $\hat{x}_N = (1+\xi)\hat{x}_{N+1} - \frac{A-f}{2}$ and $\forall\, x \in (\hat{x}_N, \hat{x}_{N+1}]$, the state in the next stage belongs to $(\hat{x}_{N-1}, \hat{x}_N]$. Moreover, assume the resource will be depleted in N stages.

Solve above recurrence equation for $\hat{x}_{N+1}$, to get $\hat{x}_{N+1} = \frac{\hat{x}_N}{1+\xi} + \frac{A-f}{2(1+\xi)} = \sum_{k=1}^{N} \frac{A-f}{2(1+\xi)^{k-1}}$.

For $N \geq 1$ and $\forall\, x \in (\hat{x}_{N+1}, \hat{x}_{N+2}]$, the resource will be depleted in $N+2$ stages.

The limit $\hat{x}_\infty$ of the sequence $\hat{x}_N$ is

$$\hat{x}_\infty = \lim_{N\to\infty} \hat{x}_N = \left(\frac{A-f}{2\xi}\right).$$

Since for all $x \geq \hat{x}_\infty$, $(1+\xi)x - \frac{A-f}{2} > x$, the resource will never be depleted.

For all $x \in (\hat{x}_1, \hat{x}_\infty)$, the recurrence equation of the value function is

$\bar{V}_i^{\mathrm{NE}}(x) = \mathcal{P}(\frac{A-f}{2}, \frac{A-f}{2}) + \beta \bar{V}_i^{\mathrm{NE}}\left((1+\xi)x - \frac{A-f}{2}\right)$.

For $X(t) \in (\hat{x}_1, \hat{x}_2]$, at the next stage, $X(t+1) \in [0, \hat{x}_1] \implies X(t+2) = 0$. So, the value function is $\bar{V}_i^{\mathrm{NE}}(x) = \mathcal{P}\left(\frac{A-f}{2}, \frac{A-f}{2}\right) + \beta \mathcal{P}_{\mathrm{depl}}\left((1+\xi)x - \frac{A-f}{2}\right), \forall\, x \in (\hat{x}_1, \hat{x}_2]$.

Proceed inductively in the same manner to get,

$$\bar{V}_i^{\mathrm{NE}}(x) = \sum_{k=1}^{N} \frac{(A-f)^2\beta^{k-1}}{8} + \beta^N \mathcal{P}_{\mathrm{depl}}\left((1+\xi)^N x - \frac{(A-f)\sum_{k=1}^{N}(1+\xi)^{k-1}}{2}\right), \forall\, x \in [\hat{x}_N, \hat{x}_{N+1}].$$

case 3 Finally, if $x \geq \hat{x}_\infty$, then $\bar{V}_i^{\mathrm{NE}}(x) = \sum_{t=0}^{\infty} \beta^t \mathcal{P}\left(\frac{A-f}{2}, \frac{A-f}{2}\right) = \frac{(A-f)^2(1+\xi)}{8\xi}$. ■

Remark 2 *The form of the value function in Theorem 14 is very unusual for linear-quadratic games, and this strange shape appears because of the constraints and the possibility of extinction of the exploited species.*

Corollary 15 *Let us change the dynamics for $x \geq \hat{x}_\infty$ and consider the state trajectory*

$$X(t+1) = \varphi(X(t), u^{S(X(t))});\ X(0) = x_0, \tag{2.3.1}$$

for $\varphi(x,u) = (1+\xi)x - u$ and for $x < \hat{x}_\infty$ while for $x \geq \hat{x}_\infty$ any t such that the interval $[\hat{x}_\infty, +\infty)$ is invariant under Eq. (2.3.1) given $S = \bar{S}^{\mathrm{NE}}$, the value function and the Nash equilbrium profile will not change.

Proof: For $x > \tilde{x}$, for each player i, given the strategies of the other players $\bar{S}^{\mathrm{NE}}_{\sim i}$, $\bar{S}^{\mathrm{NE}}$ is the unconstrained global maximum (as the discounted sum of global maxima at each time instant), which is feasible. ■

Corollary 16 *The value function for the Nash equilibrium in the case of the continuum of players is continuous, but not differentiable. For some values of the constants, it is also not monotone.*

Proof: Immediate for $x < \hat{x}_\infty$.

For $x \geq \hat{x}_\infty$, $\lim_{x \to x_\infty^-} \bar{V}_i^{\mathrm{NE}}(x) = \lim_{N \to \infty} \beta^N \mathcal{P}_{\mathrm{depl}} \left((1+\xi)^N x - \frac{(A-f) \sum_{k=1}^{N} (1+\xi)^{k-1}}{2} \right)$
$+ \sum_{k=1}^{N} \beta^k \mathcal{P}\left(\frac{A-f}{2}, \frac{A-f}{2}\right) = 0 + \frac{(A-f)^2}{8(1-\beta)} = \bar{V}_i^{\mathrm{NE}}(x_\infty) = \lim_{x \to x_\infty^+} \bar{V}_i^{\mathrm{NE}}(x)$. ■

Corollary 17 *For every $x > 0$ and almost every $i \in \mathbb{I}$,*

(a) $\frac{\bar{V}^{\mathrm{SO}}(x)}{n} > \bar{V}_i^{\mathrm{NE}}(x)$.

(b) $\bar{S}_i^{\mathrm{SO}}(x) < \bar{S}_i^{NE}(x)$.

Proof: Immediate by comparison of the calculated results. ■

2.3.2 Nash equilibria for finitely many players

In this subsection, the problem of finding the Nash equilibrium for n-players is solved.

Moreover, it is not possible to calculate the Nash equilibrium either by using the undetermined coefficient method/ (Ansatz method) with assuming the quadratic form of the value function or by the decomposition method specific to the large games.

Theorem 18 *Consider the n player Nash equilibrium problem for the* golden rule $\beta = \frac{1}{1+\xi}$. *For $n \geq 2$, if there exists a symmetric feedback Nash equilibria, then the symmetric feedback Nash equilibrium strategy is not piecewise linear with the quadratic value function for less than three intervals of the constant coefficients.*

Proof: From the necessary condition (given by Theorem 1) given from the Eq.(1.3.7)–(1.3.8) for the unbounded payoffs.

The Bellman Eq. (1.3.7) for player i, given the strategies of the others $S_{\sim i}(x)$ is

$$\bar{V}_i(x) = \max_{s_i\in[0,cx]} P_i(s_i, S_{\sim i}(x)) + \beta \bar{V}_i\left((1+\xi)x - \frac{s_i}{n} - \frac{\sum\limits_{j\neq i} S_j(x)}{n}\right), \tag{2.3.2}$$

the Bellman inclusion (1.3.8) is

$$S_i(x) \in \operatorname*{Argmax}_{s_i\in[0,cx]} P_i(s_i, S_{\sim i}(x)) + \beta \bar{V}_i\left((1+\xi)x - \frac{s_i}{n} - \frac{\sum\limits_{j\neq i} S_j(x)}{n}\right). \tag{2.3.3}$$

To calculate the piecewise linear Nash equilibria, assume that the strategies of the others are of the form $S_{\sim i}(x) = (ax+b, \ldots, ax+b)$.

Note that, if the strategies of all the players are linear, then the payoff of player i is quadratic which implies that the right-hand side of the Bellman Eq. (2.3.2) is also quadratic.

So, assume that at the Nash equilibrium, the value function of player i has the quadratic form: $k + gx + \frac{hx^2}{2}$, while the Nash equilibrium strategy of player i has the linear form in the state variable.

First order condition for s_i to be optimal is

$$s_i = \frac{-n^2(1+\xi)(A-f-ax-b)^2 + n\left((h-a)x-b\right)\xi + \left((1-a)h-a\right)x - b - hb + g) + h(ax+b)}{h-(n^2+2n)(1+\xi)}.$$

For the symmetry assumption, substitute $s_i = ax+b$, to get

$$a = \frac{h(1+\xi)}{h-(2n+1)(1+\xi)}, \quad b = \frac{g-n(1+\xi)(A-f)}{h-(2n+1)(1+\xi)}.$$

Substitute the values of a, b and s into the Bellman Eq. (2.3.2) in order to calculate the coefficients h, g and k by equating the both side coefficients at x^2, x and the constants. There are three possible values of h: positive h^+, negative h^- and 0.

$$h^+ = 1/2\left(-3\,\xi + 4\,n - 1 + 4\sqrt{(1+\xi)\left(\frac{9}{16}\,\xi + (n-1/4)^2\right)}\right)(1+\xi),$$

$$h^- = -1/2\left(3\,\xi - 4\,n + 1 + 4\sqrt{(1+\xi)\left(\frac{9}{16}\,\xi + (n-1/4)^2\right)}\right)(1+\xi).$$

The corresponding value of g is

$$g = \frac{(1+\xi)\left((-1-2\,n^2)\,\xi - 1 - 2\,n^2 + h\right)(A-f)}{((1-4\,n)\xi + h - 4\,n + 1)},$$

whenever $h \neq 0$, otherwise g is arbitrary. For given h and g, value of the real constant k is unique.

For h^- and the resultant g^-, the value of the constant $k = 0$.

For h^+, the right hand side of the Bellman Eq. (2.3.2) is strictly convex, so the zero derivative point is not a maximizer. Therefore, consider the cases of negative h^- and $h = 0$. After substitution, unique value of a and b can be obtained, for each of those sets of constants.

Exclude the case $h = 0$ and $g \neq 0$. In this case, the resultant value of s_i, which solves the Bellman Eq. (2.3.2), is constant and for this s_i the candidate for the value function is also a constant with $g = 0$, which is a contradiction.

Therefore, at this moment, only the necessary condition for the value function of each of the players at any symmetric Nash equilibrium has been proven. The analysis of the problem is without solving the Nash equilibrium problem explicitly.

Lemma 19 *At a symmetric feedback Nash equilibrium, the value function $\bar{V}_i$ of player i fulfils*

(a) $\bar{V}_i(0) = 0$.

(b) $\bar{V}_i(x) \geq 0, \ \forall x$.

(c) $\bar{V}_i(x) \leq \frac{\bar{V}_i^{SO}(x)}{n}, \ \forall x \neq 0$.

Proof: **(a)** Immediate.

(b) Since 0 strategy is always available to player i.

(c) The value function for the social optimum problem is $\bar{V}^{\mathrm{SO}}(x) = \max_{S \in \mathbb{S}^n} \sum_{i=1}^{n} J_i(x, S)$, while for Nash equilibrium problem, $\bar{V}_i(x) = \max_{S_i \in \mathbb{S}} J_i(x, S)$; $i = 1, \ldots, n$.

It can be easily checked that the social optimum profile $\bar{S}^{\mathrm{SO}}$ is not a Nash equilibrium.

Since $\frac{\bar{V}^{\mathrm{SO}}(x)}{n} = \frac{1}{n} \max_{S \in \mathbb{S}^n} \sum_{j=1}^{n} J_j(x, S) \geq \frac{1}{n} \sum_{j=1}^{n} J_j(x, S), \ \forall S$ with strict inequality whenever $S \neq \bar{S}^{\mathrm{SO}}$, which holds for a symmetric social optimum profile. Analogously, for a symmetric Nash equilibrium, $\frac{1}{n} \sum_{j=1}^{n} J_j(x, S) = \frac{1}{n} \sum_{j=1}^{n} \bar{V}_j(x)$.

Since by the symmetry, all $J_j(x, S)$ are equal, so, $\bar{V}_i(x) = \frac{1}{n} \sum_{j=1}^{n} \bar{V}_j(x) < \frac{\bar{V}^{\mathrm{SO}}(x)}{n}$. ■

Note that none of the obtained functions $k + gx + \frac{1}{2}hx^2$, fulfils both (a) and (b), so consider the combination of both.

The only such combination which fulfils **(a)** and **(b)** is

$$\bar{V}_i^{\text{cand1}} = \begin{cases} g^- x + \frac{h^- x^2}{2}, & \text{if } x \le \bar{x}, \\ \tilde{k}, & \text{otherwise,} \end{cases}$$

where $\tilde{k} = \frac{\left(A-f-\frac{3\hat{s}}{2}\right)\hat{s}}{1-\beta}$ is the positive real constant while $h = 0$ and $g = 0$, $\hat{s} = \frac{(A-f)n}{(2n+1)}$ (equal to the Nash equilibrium in a one stage game) and for some $\bar{x} > 0$ (since otherwise, the Bellman Eq. (2.3.2) does not hold). The resultant candidate for the Nash equilibrium strategy is

$$\bar{S}_i^{\text{cand1}} = \begin{cases} ax + b, & \text{if } x \le \bar{x}, \\ \hat{s}, & \text{otherwise.} \end{cases}$$

However, whether $ax + b \le (1 + \xi)x$, remains to be checked. It does not hold for x close to 0, since $b > 0$.

Denote the point at which $ax + b = (1 + \xi)x$ by $\tilde{x}$. Then $\tilde{x} \in (0, \frac{n(A-f)}{(2n+1)\xi})$.

For $x \le \tilde{x}$, the calculated $ax + b > (1 + \xi)x$. So, if $\tilde{x} < \bar{x}$, then the candidate for symmetric feedback Nash equilibrium strategy has at least three pieces:

$$\bar{S}_i^{\text{cand2}} = \begin{cases} (1+\xi)x, & \text{if } x \le \tilde{x}, \\ ax + b, & \text{if } \tilde{x} < x \le \bar{x}, \\ \hat{s}, & \text{otherwise,} \end{cases}$$

and the corresponding candidate for the value function is

$$\bar{V}_i^{\text{cand2}} = \begin{cases} \left(A - f - \frac{3}{2}(1+\xi)x\right)(1+\xi)x, & \text{if } x \le \tilde{x}, \\ g^- x + \frac{h^- x^2}{2}, & \text{if } \tilde{x} < x \le \bar{x}, \\ \tilde{k}, & \text{otherwise.} \end{cases}$$

Note that, for $x = \tilde{x} + \epsilon$ for small $\epsilon > 0$, the Bellman Eq. (2.3.2) does not hold, since $(1 + \xi)x - (ax + b) < \tilde{x}$.

If $\tilde{x} \ge \bar{x}$, then

$$\bar{S}_i^{\text{cand3}} = \begin{cases} (1+\xi)x, & \text{if } x \le \bar{x}, \\ \hat{s}, & \text{otherwise,} \end{cases}$$

and the corresponding candidate for the value function is

$$\bar{V}_i^{\text{cand3}} = \begin{cases} \left(A - f - \frac{3}{2}(1+\xi)x\right)(1+\xi)x, & x \le \bar{x}, \\ \tilde{k}, & \text{otherwise.} \end{cases}$$

Note that for $x > \frac{\hat{s}(n-1)}{n(1+\xi)}$, $(1+\xi)x$ is not the best response to $S_{\sim i}(x) \equiv (1+\xi)x$. So, if $\bar{x} \le \frac{\hat{s}(n-1)}{n(1+\xi)}$, then $\bar{x} < \frac{\hat{s}}{\xi}$ — the critical value below which the constant $\hat{s}$ is not feasible. Thus, the value function cannot be $\tilde{k}$ for $x \in [\bar{x}, \infty)$. Therefore, in this case the necessary conditions (2.3.2) and (2.3.3) are not fulfilled. ■

Remark 3 *The non-existence of a Nash equilibrium of assumed regularity seems to be inherited from the finite time horizon truncations of the game, which are studied in the next chapter.*

2.4 Enforcing social optimality by a tax system

In this section, enforcement of the social optimum profile by a *Pigouvian tax system* has been studied. Formally, *introduction of a tax* or a *tax-subsidy system* is a modification of the game by changing the current payoffs. To be more specific, we consider a function $\mathcal{T}\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ called a *tax-subsidy system.* When $\mathcal{T}(s_i, x) \ge 0$ for every (s_i, x) then we call it a *tax system.* A tax-subsidy system is substracted from the current payoff. So, the current payoff of player i in the modified game becomes

$$\mathcal{P}_i(s_i, u^s) - \mathcal{T}(s_i, x). \tag{2.4.1}$$

The first function we consider is a tax-subsidy system linear in player's strategy: $\mathcal{T}(s_i, x) = \tau(x)s_i + \tau_0$. In such a case $\tau(x)$ is called the *tax rate.*

Definition 13 *A* tax system *or a* tax-subsidy system enforces *a profile $\bar{S}$ if $\bar{S}$ is the Nash equilibrium in the game modified by a tax-subsidy system with payoff defined by Eq.* (2.4.1).

For the continuum of players game, it is not possible to enforce the social optimality for all states by τ constant in x. However, such a constant τ can be calculated for $x \ge \frac{A-f}{3\xi}$.

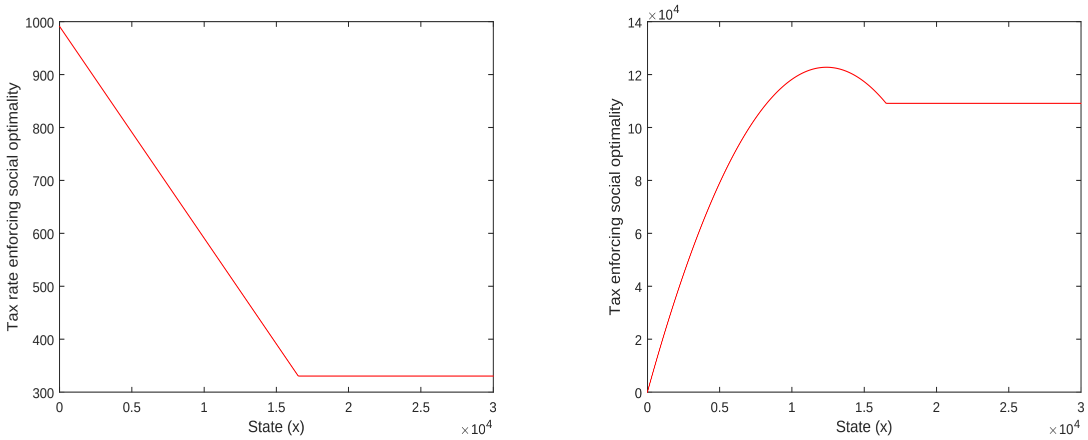
Consider the variable tax rate $\tau(x)$, in order to enforce the social optimum profile for all levels of the biomass of fish.

Proposition 20 *Consider the game with the continuum of players.*

(a) *The tax-rate enforcing the socially optimal profile* $\bar{S}^{\mathrm{SO}}$ *from Theorem 10d) is*

$$\tau(x) = \begin{cases} A - f - 2\xi x, & \text{if } x \leq \frac{A-f}{3\xi}, \\ \frac{A-f}{3}, & \text{otherwise}, \end{cases} \tag{2.4.2}$$

while τ_0 *is arbitrary.* ***(b)*** *The tax-rate given in* ***(a)*** *enforces the socially optimal profile* $\bar{S}^{\mathrm{SO}}$ *from Theorem 10d) for any* $\beta \in (0,1)$*. Moreover, it guarantees the sustainability of the resource.*



(a) The rate of tax **(b)** The tax

Figure 2.4: Problem of enforcing $\bar{S}^{\mathrm{SO}}$ and guaranteeing sustainability for the continuum of players

Proof: **(a)** To calculate a linear tax enforcing the social optimum profile with the tax-rate τ, modify the current payoff function by subtracting $\tau(x)s_i$ and then equate the Nash equilibrium of this modified game with the social optimum of the original game.

Note that the result of introducing a linear tax of rate $\tau(x)$ is mathematically equivalent to increasing the constant f by $\tau(x)$.

case 1 If $x > \frac{A-f}{3\xi}$ then $\bar{S}^{\mathrm{SO}}(x) = \frac{A-f}{3}$. So,

$$\left(\frac{A-f}{3}\right) = \left(\frac{A-f-\tau(x)}{2}\right).$$

Solve this expression for $\tau(x)$ to get $\tau(x) = \frac{A-f}{3}$.

case 2 If $x \leq \frac{A-f}{3\xi}$ then $\bar{S}^{\mathrm{SO}}(x) = \xi x$.

Put the new Nash equilibrium strategy equal to ξx to get,

$$\xi x = \left(\frac{A-f-\tau(x)}{2}\right),$$

Solve this expression for $\tau(x)$ to get $\tau(x) = A - f - 2\xi x$.

(b) Analogously, only without using the social optimality of the profile $\bar{S}^{\mathrm{SO}}$. Sustainability of $\bar{S}^{\mathrm{SO}}$ has already been noticed in Corollary 11. ■

Enforcement of the social optimum profile problem can also be solved by using the different types of tax or tax-subsidy systems.

Proposition 21 ***(i)*** *Consider the game with a continuum of player and a tax-subsidy system with*

$$\mathcal{T}(s_i, x) = \tau(x)(s_i - S^{\mathrm{SO}}),$$

then the results are equivalent. Moreover, the tax rate τ given by Proposition 20 enforces the profile S_i^{SO}, but there is no tax to be paid (i.e., the tax is purely regulatory). ***(ii)*** *Consider the tax rate τ from Proposition 20 but the tax is introduced only for over-exploiting the resource over the $\bar{S}_i^{\mathrm{SO}}$ constraint, i.e.,*

$$\mathcal{T}(s_i, x) = \tau(x)(s_i - \bar{S}_i^{\mathrm{SO}})^+,$$

then $\tau(x)$ enforces the social optimal profile for every number of players including finitely many players n.

Proof: **(i)** Immediate.
(ii) For continuum of players immediate.
For n players, the Bellman Eq. (1.3.7) for player i, given strategies $\bar{S}_j$ of the other players is

$$\bar{V}_i(x) = \max_{s_i \in [0, cx]} \mathcal{P}(s_i, S_{\sim i}(x)) - \tau(x)\big(s_i - \bar{S}_i^{\mathrm{SO}}\big)^+ + \beta \bar{V}_i\left((1+\xi)x - \frac{s_i}{n} - \frac{\sum_{j \neq i} \bar{S}_j(x)}{n}\right), \tag{2.4.3}$$

the inclusion (1.3.8) is

$$\bar{S}_i(x) \in \operatorname*{Argmax}_{s_i \in [0, cx]} \mathcal{P}(s_i, S_{\sim i}(x)) - \tau(x)\big(s_i - \bar{S}_i^{\mathrm{SO}}\big)^+ + \beta \bar{V}_i\left((1+\xi)x - \frac{s_i}{n} - \frac{\sum_{j \neq i} \bar{S}_j(x)}{n}\right). \tag{2.4.4}$$

The proof is immediate by substituting $S_i = \bar{S}_i^{\mathrm{SO}}$ and $V_i = \frac{\bar{V}^{\mathrm{SO}}}{n}$ for $\bar{S}_i^{\mathrm{SO}}$ from Eq. (2.2.7) and $\bar{V}^{\mathrm{SO}}$ from Eq. (2.2.6) into the Bellman Eq. (2.4.3) in the modified game. ■

It is worth emphasizing that in the n player game, we were not able to find a Nash equilibrium, we have only proved that it is not in a certain class of functions, but calculation of a tax system which enforces the social optimum profile is relatively simple, and we are able to find the Nash equilibrium of the resultant modified game. So, the results for the abstract concept of the game with a continuum of players turned out to be useful in usual n player games.

Chapter 3

A counterexample to common belief of regularity of Nash equilibrium in linear-quadratic games considered in Chapter 2

The results of this chapter appeared in paper [91], however, here we give additional reasoning and explanations which make the proofs and the results easier to understand.

In [91], we have obtained non-existence of continuous symmetric feedback Nash equilibrium, while the existence of a continuum of discontinuous symmetric feedback Nash equilibria in a finite horizon truncation of a linear-quadratic game with linear state-dependent constraints, being a truncation of the game of Chapter 2.

We want to describe the history of this research slightly, including same previous directions which turned out to be the dead ends.

In all the research, we have looked for symmetric feedback Nash equilibria, since open loop equilibria are less realistic and usually they do not coincide with the feedback Nash equilibria. We were interested only in symmetric solutions which seems obvious in this class of games with concave payoffs, linear dynamics and symmetric players. The research ending by this result was the continuation of the research from our paper [83], more specifically, the n-players Nash equilibrium problem.

First, we have checked different possible candidates for Nash equilibria, expecting some irregularity at the points where the number of time instant to extinction changes, but initially expecting continuity.

We have noticed that it is impossible to have both S_i and V_i continuous at those points. First, we have relaxed the assumption of continuity of S_i, then we have tried various assumptions.

- S_i discontinuous but V_i continuous.

- S_i continuous but V_i discontinuous is impossible, so, we have rejected it.

- We have switched to the continuity of some functions related to the dynamics of the system, allowing both S_i and V_i to be discontinuous.

All of those attempts have been unsuccessful. Finally, we have noticed that the irregularity is inherited from the finite time truncations of the game. Therefore, in this chapter, we study a simple 2-player and 2-stage linear-quadratic dynamic game with linear state-dependent constraints.

3.1 The model

To simplify the complex calculations, we slightly change the dynamic game. The game studied here is equivalent to the game from Chapter 2 with the time set $\mathbb{T} = \{1, 2\}$ and $f = 0$, or if we introduce A of this chapter as the $A - f$ from the Chapter 2.

We again consider the *golden rule* $\beta = \frac{1}{1+\xi}$.

For this 2-player and 2-stage game model, we simplify the notation and introduce a simpler term of writing the *current payoff*:

$$P(s_i, s_{\sim i}) := P_i(s) = \left(A - \frac{s_i + s_{\sim i}}{2}\right) s_i - \frac{s_i^2}{2}. \tag{3.1.1}$$

Therefore, the Bellman Eq. (1.3.7) becomes

$$\bar{V}(t, x) = \max_{s_i \in [0, (1+\xi)x]} P(s_i, s_{\sim i}) + \beta \bar{V}\left((1+\xi)x - \frac{(s_i + s_{\sim i})}{2}\right), \tag{3.1.2}$$

the Bellman inclusion (1.3.8) becomes

$$\bar{S}(t, x) \in \operatorname*{Argmax}_{s_i \in [0, (1+\xi)x]} P(s_i, s_{\sim i}) + \beta \bar{V}\left((1+\xi)x - \frac{(s_i + s_{\sim i})}{2}\right), \tag{3.1.3}$$

and since the game is a two stage truncation with no terminal payoff, the terminal condition (1.3.9) becomes

$$\bar{V}_i(3, x) = 0. \tag{3.1.4}$$

3.2 Calculation of the feedback Nash equilibrium

To calculate the Nash equilibrium profile, we solve the game backwards from the terminal time. From Eq. (3.1.2)–(3.1.3), if the state at $t = 2$ is x, then the best choice of a player at $t = 2$ given a strategy of their opponent depends only on the current opponent's decision and state, whatever the previous decisions were. So, we can consider a static game that is played at time 2.

Theorem 22 *Consider the Nash equilibrium problem in the one stage game played at time* 2 *for given* x.

The unique Nash equilibrium strategy profile $(\bar{s}_1, \bar{s}_2)$ *at terminal time* $t = 2$ *is given by*

$$\bar{s}_i = \bar{S}_i(2, x) := \begin{cases} (1+\xi)x, & \text{if } x \leq \hat{x}_1, \\ \hat{s}, & \text{if } x \geq \hat{x}_1, \end{cases} \tag{3.2.1}$$

for $i \in \{1, 2\}$ *and*

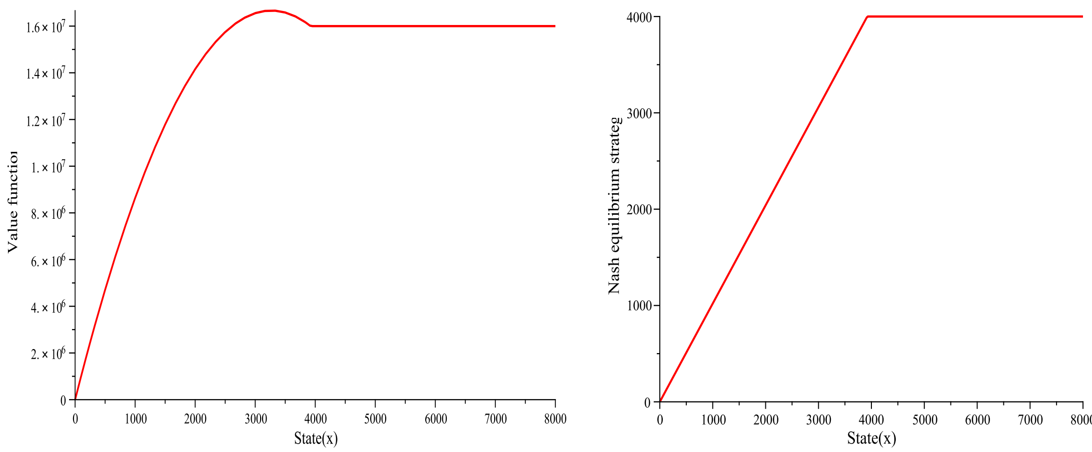
$$\hat{s} = \frac{2A}{5}, \ \hat{x}_1 = \frac{\hat{s}}{1+\xi}. \tag{3.2.2}$$

For every Nash equilibrium, for the original two stage game, at terminal time 2, players' strategies fulfil

$$S_i(2, x) = \bar{S}_i(2, x)$$

for $\bar{S}_i(2, x)$ *from Eq.* (3.2.1)*, while the value functions at* $t = 2$*, given the opponent's strategy* $\bar{S}_{\sim i}(2, x)$*, are given by*

$$\bar{V}_i(2, x) := \begin{cases} \left(A - \frac{3}{2}(1+\xi)x\right)(1+\xi)x, & \text{if } x \leq \hat{x}_1, \\ \left(A - \frac{3}{2}\hat{s}\right)\hat{s}, & \text{if } x \geq \hat{x}_1. \end{cases} \tag{3.2.3}$$



(a) The value function at stage 2, depending on the initial state

(b) A Nash equilibrium strategy at stage 2, depending on the initial state

Figure 3.1: Nash equilibrium problem for 2 players

Proof: Since the Nash equilibrium strategy of player i has to fulfil Eq. (3.1.3). Calculate the zero derivative point of the right hand side of Eq. (3.1.2) with respect to s_i to get the first order condition, $s_i = \frac{2A - s_{\sim i}}{4}$.

Note that if $\frac{2A - s_{\sim i}}{4} \geq (1+\xi)x$, then the maximum of the right hand side of Eq. (3.1.2) is attained at $(1+\xi)x$.

Therefore, the unique symmetric feedback Nash equilibrium profile and the players' value functions are given by Eq. (3.2.1) and Eq. (3.2.3). ■

Corollary 23 *$\bar{S}_i(2,x)$ is non-decreasing, while $\bar{V}_i(2,x)$ is not concave, non-monotone and not differentiable at $\hat{x}_1$. $\bar{V}_i(2,x)$ is increasing at the interval $[0, \frac{A}{3(1+\xi)}]$.*

Proof: Immediate. Since $(A - \frac{3(1+\xi)x}{2})(1+\xi)x$ is concave and its maximum is attained at $\frac{A}{3(1+\xi)} < \hat{x}_1$. ■

Now, given Nash equilibrium strategies and the value functions at time 2 from Theorem 22, proceed backwards in order to solve the problem at time $t = 1$.

Lemma 24 *Consider $x \geq 0$ in the game with $t = 1$ and the terminal payoff $\bar{V}_i(2,x)$ from Eq.* (3.2.3)*. With notation*

$$x_{\text{next}}(x, s_i, s_{\sim i}) = (1+\xi)x - \frac{(s_i + s_{\sim i})}{2}, \tag{3.2.4}$$

denote by $s_{\text{Bd}}(s_{\sim i})$, the point s at which $x_{\text{next}}(x, s_i, s_{\sim i}) = \hat{x}_1$ for $\hat{x}_1$ from Eq. (3.2.2)*.*
(a) *Given $s_{\sim i}$, the best response of player i belongs to the set*

$$\{0,\ d_{\text{I}}(s_{\sim i}),\ d_{\text{II}}(s_{\sim i}),\ s_{\text{Bd}}(s_{\sim i}),\ (1+\xi)x\}$$

$$\text{for } d_{\text{I}}(s_{\sim i}) = \frac{6(1+\xi)^2 x + 2A - s_{\sim i}(5+3\xi)}{11+3\xi}, \tag{3.2.5a}$$

$$d_{\text{II}}(s_{\sim i}) = \frac{2A - s_{\sim i}}{4}, \tag{3.2.5b}$$

$$s_{\text{Bd}}(s_{\sim i}) = 2(1+\xi)x - 2\hat{x}_1 - s_{\sim i}. \tag{3.2.6}$$

Moreover, the best response is at most $\frac{A}{2}$.
(b) *For every symmetric feedback Nash equilibrium and for every x, the feedback Nash equilibrium strategy belongs to the set $\{(1+\xi)x,\ s_{\text{I}}(x),\ \hat{s},\ s^{\text{sym}}_{\text{Bd}}(x)\}$, for*

$$s_{\text{I}}(x) = \frac{A + 3(1+\xi)^2 x}{8+3\xi}, \quad \hat{s} \text{ from Eq. (3.2.2) and} \tag{3.2.7}$$

$$s^{\text{sym}}_{\text{Bd}}(x) = (1+\xi)x - \hat{x}_1. \tag{3.2.8}$$

Proof: **(a)** Consider the state x, a fixed player i and their opponent's strategy $s_{\sim i}$. For brevity of notation, given a strategy profile $(s_i, s_{\sim i})$ and state x, abbreviate further $x_{\text{next}}(x, s_i, s_{\sim i})$ from Eq. (3.2.4) as x_{next}, whenever it does not lead to confusion.

So, for $t = 1$, given x and $s_{\sim i}$, the Bellman Eq. (3.2.11) becomes

$$\bar{V}_i(1, x) = \max_{s_i \in [0,(1+\xi)x]} \mathrm{RBE}(s_i), \tag{3.2.9}$$

where

$$\mathrm{RBE}(s_i) := \begin{cases} \left(A - \frac{s_{\sim i}}{2} - s_i\right) s_i + \beta \hat{s} \left(A - \frac{3}{2}\hat{s}\right), & \text{if } x_{\text{next}} \geq \hat{x}_1, \\ \left(A - \frac{s_{\sim i}}{2} - s_i\right) s_i + \left(A - \frac{3(1+\xi)x_{\text{next}}}{2}\right) x_{\text{next}}, & \text{if } 0 \leq x_{\text{next}} \leq \hat{x}_1, \end{cases} \tag{3.2.10}$$

while the Nash equilibrium strategy has to fulfil

$$\bar{S}_i(1, x) \in \operatorname*{Argmax}_{s_i \in [0,(1+\xi)x]} \mathrm{RBE}(s_i). \tag{3.2.11}$$

Calculate the point of zero derivative points to get $d_{\mathrm{I}}(x)$ and $d_{\mathrm{II}}(x)$. It cannot be a priori excluded that the maximum can also be attained at the boundary points: either 0 or $(1+\xi)x$ or at the switching point $s_{\mathrm{Bd}}(s_{\sim i})$, at which the function is non-differentiable. Those are all the possible candidates.

$d_{\mathrm{II}}(s_{\sim i})$ is always less than $\frac{A}{2}$ since $d_{\mathrm{II}}(s_{\sim i}) \leq d_{\mathrm{II}}(0) = \frac{A}{2}$. So, if $d_{\mathrm{II}}(s_{\sim i})$ is the best response then the best response is at most $\frac{A}{2}$.

Next, consider the case when $d_{\mathrm{I}}(s_{\sim i})$ is the best response and show that it has to fulfil $d_{\mathrm{I}}(s_{\sim i}) \leq \frac{A}{2}$.

$d_{\mathrm{I}}(s_{\sim i})$ can be the best response only if $x_{\text{next}}(x, d_{\mathrm{I}}(s_{\sim i}), s_{\sim i}) \leq \hat{x}_1$, which implies that $x_{\text{next}}(x, d_{\mathrm{I}}(s_{\sim i}), s_{\sim i}) \leq \frac{A}{3(1+\xi)}$. So, $V(2, \cdot)$ is strictly increasing in a neighbourhood of $x_{\text{next}}(x, d_{\mathrm{I}}(s_{\sim i}), s_{\sim i})$.

Now assume that the best response s_i is equal to $d_{\mathrm{I}}(s_{\sim i})$ and that it is greater than $\frac{A}{2}$. Note that, the global maximum of $P(\cdot, s_{\sim i})$ is attained at $s_i = \frac{A}{2} - \frac{s_{\sim i}}{4} \leq \frac{A}{2}$. Since the function $V(2, \cdot)$ is strictly increasing in a neighbourhood of $x_{\text{next}}(x, d_{\mathrm{I}}(s_{\sim i}), s_{\sim i})$, player i can increase their current payoff by reducing s_i by ϵ and the other component of $\mathrm{RBE}(s_i)$ from Eq. (3.2.10) does not decrease, which contradicts that s_i is a best response.

$s_{\mathrm{Bd}}(s_{\sim i})$ can be the best response only if $d_{\mathrm{I}}(s_{\sim i}) \leq s_{\mathrm{Bd}}(s_{\sim i})$ and $d_{\mathrm{II}}(s_{\sim i}) \geq s_{\mathrm{Bd}}(s_{\sim i})$. So, $s_{\mathrm{Bd}}(s_{\sim i}) \leq d_{\mathrm{II}}(s_{\sim i}) \leq \frac{A}{2}$.

$(1+\xi)x$ is the best response iff $(1+\xi)x \leq d_{\mathrm{I}}(x)$ and $x_{\text{next}}(x, (1+\xi)x, s_{\sim i}) \leq \hat{x}_1$, so a reasoning analogous to that for $d_{\mathrm{I}}(s_{\sim i})$ applies.

(b) Since, for a symmetric Nash equilibrium, the strategy of player i has to be equal to the best response to it.

So, by **(a)**, possible candidates for the symmetric optimal strategy of player i are:

$\{0, (1+\xi), s_{\rm I}(x), s_{\rm II}(x), s_{\rm Bd}^{\rm sym}(x)\}$ where,

$$s_{\rm I}(x) : d_{\rm I}(s) = s,\ \ \hat{s} : d_{\rm II}(s) = s,\ \ s_{\rm Bd}^{\rm sym}(x) : s_{\rm Bd}(s) = s.$$

For 0, the best response is greater than 0. ■

Lemma 25 *Consider the game from Lemma 24. For any state x and a strategy $s^*_{\sim i} \in \{s_{\rm I}(x), \hat{s}, s_{\rm Bd}^{\rm sym}(x), (1+\xi)x\} \cap [0, \frac{A}{2}]$, the best response fulfils*

$$\mathrm{BR}_i(s^*_{\sim i}) \subseteq \begin{cases} \{(1+\xi)x\} & \text{if } x \leq \hat{y}_1(s^*_{\sim i}), \\ \{d_{\rm I}(s^*_{\sim i})\} & \text{if } \hat{y}_1(s^*_{\sim i}) < x \leq \hat{x}_2(s^*_{\sim i}), \\ \{d_{\rm I}(s^*_{\sim i}), d_{\rm II}(s^*_{\sim i})\} & \text{if } \hat{x}_2(s^*_{\sim i}) < x < \hat{y}_2(s^*_{\sim i}), \\ \{d_{\rm II}(s^*_{\sim i})\} & \text{if } x \geq \hat{y}_2(s^*_{\sim i}), \end{cases} \tag{3.2.12}$$

for $d_{\rm I}$, $d_{\rm II}$ from Eq. (3.2.5) *and*

$$\hat{y}_1(s^*_{\sim i}) = \frac{2A - s^*_{\sim i}(5+3\xi)}{5+2\xi-3\xi^2}, \tag{3.2.13}$$

$$\hat{x}_2(s^*_{\sim i}) = \frac{(15(1+\xi)s^*_{\sim i} + 2A(13+5\xi))}{40(1+\xi)^2}, \tag{3.2.14}$$

$$\hat{y}_2(s^*_{\sim i}) = \frac{(15(1+\xi)s^*_{\sim i} + A(27+11\xi))}{40(1+\xi)^2}. \tag{3.2.15}$$

Proof: Note that the function $\mathrm{RBE}(s_i)$ from Eq. (3.2.10) is piecewise concave with at most two pieces. The switching points $\hat{y}_1$, $\hat{y}_2$ and $\hat{x}_2$ are defined as x for which,

$$\begin{aligned} \hat{y}_1 &\text{ is such that } d_{\rm I}(s^*_{\sim i}) = (1+\xi)x, \\ \hat{y}_2 &\text{ is such that } d_{\rm I}(s^*_{\sim i}) = s_{\rm Bd}(s^*_{\sim i}), \\ \hat{x}_2 &\text{ is such that } d_{\rm II}(s^*_{\sim i}) = s_{\rm Bd}(s^*_{\sim i}). \end{aligned}$$

For given $s^*_{\sim i}$ and x, the optimization problem from Eq. (3.2.9) can be decomposed into two optimization problems of strictly concave and differentiable functions:

RBE_1 over the interval $[s_{\rm Bd}(s^*_{\sim i}), (1+\xi)x] \cap [0, (1+\xi)x]$ and RBE_2 over the interval $[0, s_{\rm Bd}(s^*_{\sim i})] \cap [0, (1+\xi)x]$ for RBE_1 and RBE_2 defined as follows:

$$\mathrm{RBE}_1(s_i) := \left(A - \frac{s^*_{\sim i}}{2} - s_i\right) s_i + \beta\hat{s}\left(A - \frac{3}{2}\hat{s}\right), \tag{3.2.16}$$

$$\mathrm{RBE}_2(s_i) := \left(A - \frac{s^*_{\sim i}}{2} - s_i\right) s_i + x_{\text{next}} \left(A - \frac{3(1+\xi)x_{\text{next}}}{2}\right). \tag{3.2.17}$$

By Lemma 24 **(a)**, the maximum of $\mathrm{RBE}(s_i)$ from Eq. (3.2.10) can be attained at one of the points: $0,\ (1+\xi)x,\ s_{\mathrm{Bd}}(s^*_{\sim i}),\ d_{\mathrm{I}}(s^*_{\sim i}),\ d_{\mathrm{II}}(s^*_{\sim i})$.
If $s^*_{\sim i} \le \frac{A}{2}$, then $d_{\mathrm{I}}(s^*_{\sim i}) > 0$ and $d_{\mathrm{II}}(s^*_{\sim i}) > 0$, so 0 is not the best response.

Since the maximised function is strictly concave, the fact whether the zero derivative point is within the interval considered, left or right to it determines the global maximum on this interval.

By calculating partial derivatives and checking their signs, observe that for all x, at any $s^*_{\sim i}$ which can appear at a symmetric Nash equilibrium i.e., for every $s^*_{\sim i} \in \{s_{\mathrm{I}}(x),\ \hat{s},\ s^{\mathrm{sym}}_{\mathrm{Bd}}(x),\ (1+\xi)x\}$, all the functions $d_{\mathrm{II}}(s^*_{\sim i}) - s_{\mathrm{Bd}}(s^*_{\sim i})$, $d_{\mathrm{I}}(s^*_{\sim i}) - s_{\mathrm{Bd}}(s^*_{\sim i})$, $d_{\mathrm{II}}(s^*_{\sim i}) - (1+\xi)x$, $d_{\mathrm{I}}(s^*_{\sim i}) - (1+\xi)x$ and $-s_{\mathrm{Bd}}(s^*_{\sim i})$ are strictly decreasing in x. So, to prove that e.g., $d_{\mathrm{II}}(s^*_{\sim i}) \ge s_{\mathrm{Bd}}(s^*_{\sim i})$ on some interval of state variables, it is enough to check it at the upper bound of the interval only. Similarly, the function $(1+\xi)x - s_{\mathrm{Bd}}(s^*_{\sim i})$ is either strictly decreasing in x for $s^*_{\sim i} = s_{\mathrm{I}}(x)$ and $s^*_{\sim i} = \hat{s}$ or it is a positive constant for $s^*_{\sim i} = (1+\xi)x$ and $s^*_{\sim i} = s^{\mathrm{sym}}_{\mathrm{Bd}}(x)$.

For brevity, write s_{Bd} instead of $s_{\mathrm{Bd}}(s_{\sim i})$ from Eq. (3.2.6) and consider the following four cases for the division of the state set.

case 1: If $x \le \hat{y}_1(s^*_{\sim i})$, then $d_{\mathrm{I}} \ge (1+\xi)x$ and $d_{\mathrm{II}} > s_{\mathrm{Bd}}$. So, the maximum of (3.2.17) over $[0, s_{\mathrm{Bd}}]$ (if it is non-empty) is at s_{Bd} while (3.2.16) is strictly increasing on $[s_{\mathrm{Bd}}, (1+\xi)x]$. So, the maximum of $\mathrm{RBE}(s_i)$ is attained at $(1+\xi)x$.
Also note that for $x = \hat{y}_1(s^*_{\sim i})$, $d_{\mathrm{I}}(s^*_{\sim i}) = (1+\xi)x$.

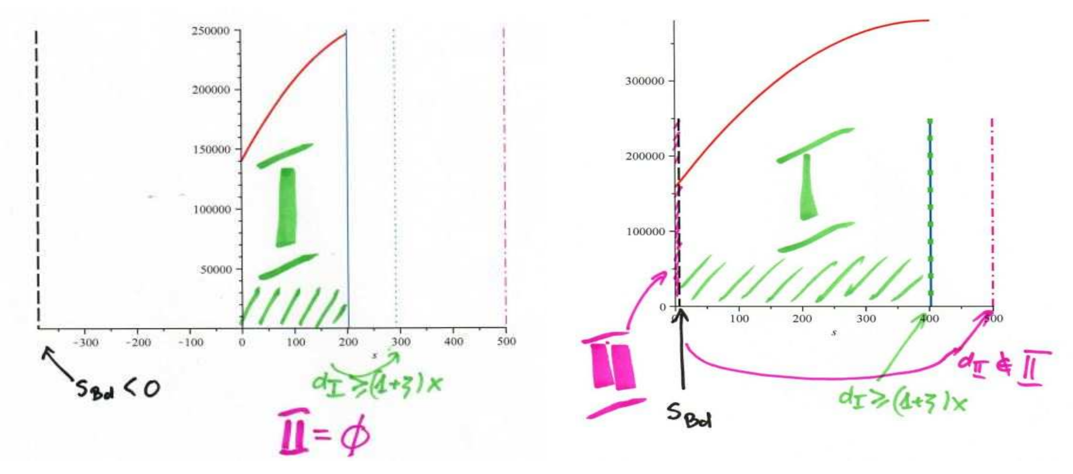


Figure 3.2: case 1: The maximized function for $x \le \hat{y}_1(s^*_{\sim i})$, the sets denoted by I and II are the set on which RBE equals RBE_1 and RBE_2 respectively

case 2: If $\hat{x}_2(s^*_{\sim i}) < x < \hat{y}_2(s^*_{\sim i})$, then both $d_{\mathrm{I}}(s^*_{\sim i}) \in (s_{\mathrm{Bd}}, (1+\xi)x]$ and $d_{\mathrm{II}}(s^*_{\sim i}) \in (0, s_{\mathrm{Bd}})$ and $s_{\mathrm{Bd}} \in (0, (1+\xi)x)$. Therefore, the supremum of (3.2.17) on $[0, s_{\mathrm{Bd}}]$ is attained at $d_{\mathrm{II}}(s^*_{\sim i})$, while the supremum of (3.2.16) on $[s_{\mathrm{Bd}}, (1+\xi)x]$ is attained at $d_{\mathrm{I}}(s^*_{\sim i})$. So, the supremum of $\mathrm{RBE}(s_i)$ can be attained either at $d_{\mathrm{I}}(s^*_{\sim i})$ or $d_{\mathrm{II}}(s^*_{\sim i})$, depending on whether $\mathrm{RBE}_1(d_{\mathrm{I}}(s^*_i))$ or $\mathrm{RBE}_2(d_{\mathrm{II}}(s^*_i))$ is greater. So, only $d_{\mathrm{I}}(s^*_{\sim i})$ and $d_{\mathrm{II}}(s^*_{\sim i})$ can be in the best response.

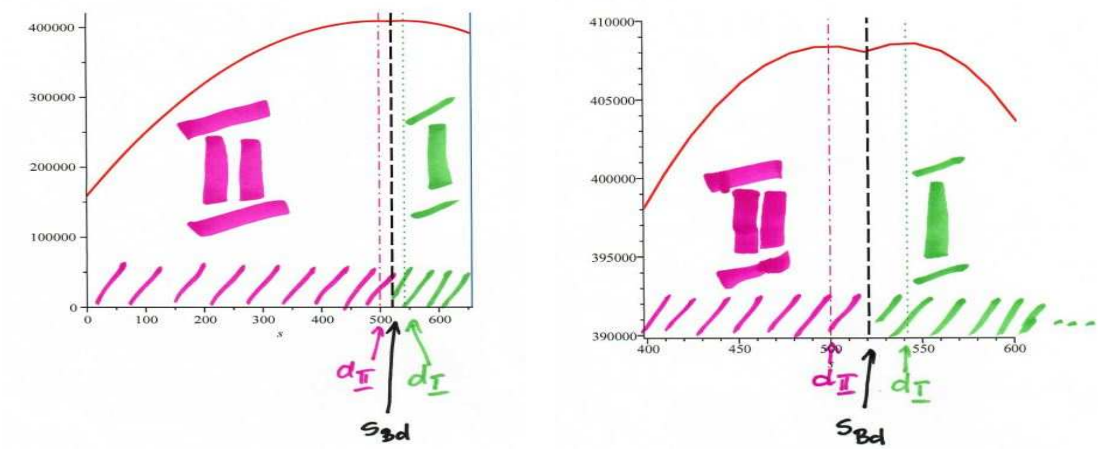


Figure 3.3: case 2: The maximized function for $x \in [\hat{x}_2((s^*_{\sim i})), \hat{y}_2((s^*_{\sim i})]$ normal and zoomed view

case 3: If $\hat{y}_1(s^*_{\sim i}) < x \leq \hat{x}_2(s^*_{\sim i})$, then $d_{\mathrm{II}}(s^*_{\sim i}) \geq s_{\mathrm{Bd}}$ and $s_{\mathrm{Bd}} \in [0, (1+\xi)x]$. So, the supremum of (3.2.17) on $[0, s_{\mathrm{Bd}}]$ is attained at s_{Bd}. Since the zero derivative point of (3.2.16), $d_{\mathrm{I}}(s^*_{\sim i}) \in [s_{\mathrm{Bd}}, (1+\xi)x]$, the maximum of $\mathrm{RBE}(s_i)$ is attained at $d_{\mathrm{I}}(s^*_{\sim i})$. So, $d_{\mathrm{I}}(s^*_{\sim i})$ is the unique best response.

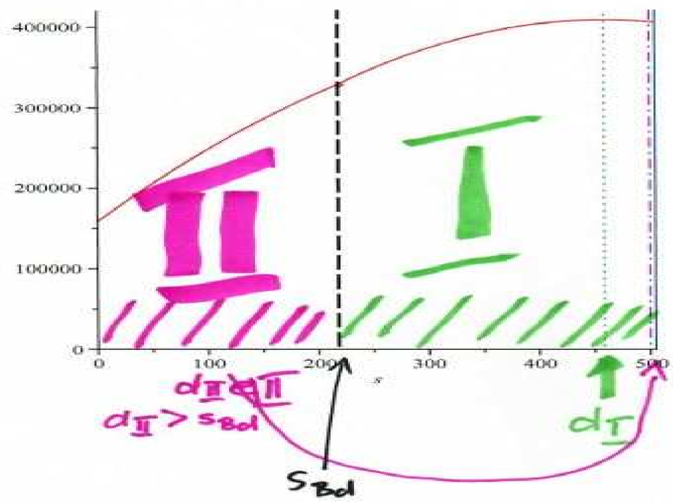


Figure 3.4: case 3: The maximized function for $\hat{y}_1(s^*_{\sim i}) < x < \hat{x}_2(s^*_{\sim i})$

case 4: If $x \geq \hat{y}_2(s^*_{\sim i})$, then $d_{\mathrm{I}}(s^*_{\sim i}) \leq s_{\mathrm{Bd}}$ while $d_{\mathrm{II}}(s^*_{\sim i}) \in [0, \min\{s_{\mathrm{Bd}}, (1+\xi)x\}]$. So, the maximum of (3.2.16) on $[s_{\mathrm{Bd}}, (1+\xi)x)]$ is either at s_{Bd} and s_{Bd} is not the maximum of (3.2.17) on $[0, s_{\mathrm{Bd}}]$, or the interval $[s_{\mathrm{Bd}}, (1+\xi)x]$ is empty. Therefore, the maximum of $\mathrm{RBE}(s_i)$ over $[0,(1+\xi)x]$ is attained at $d_{\mathrm{II}}(s^*_{\sim i})$. So, $d_{\mathrm{II}}(s^*_{\sim i})$ is the unique best response.

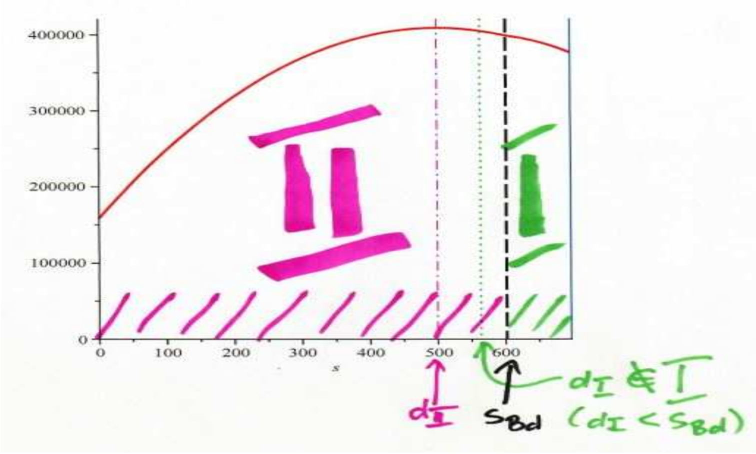


Figure 3.5: case 4: The maximized function for $x > \hat{y}_2(s^*_{\sim i})$

It is worth emphasizing that Fig. 3.2–3.5 for **cases** $1-4$ studied in the proof are not only illustrations for some values of parameter — the inequalities between d_{I} or d_{II} and s_{Bd} , 0, and $(1+\xi)x$ remain unchanged as long as $s^*_{\sim i} \leq \frac{A}{2}$.

The only thing not exactly illustrated in Figures are sub-case of **case 4**. In this case, for large x and small $s^*_{\sim i}$, s_{Bd} can be larger than $(1+\xi)x$, but this does not change the substantial fact that $d_{\mathrm{II}} \leq \min\{(1+\xi)x, s_{\mathrm{Bd}}\}$. ■

Lemma 26 *Consider any $x \geq 0$ in the game from Lemma 25.*

The best response correspondence $\mathrm{BR}_i : [0,(1+\xi)x] \to [0,(1+\xi)x]$, restricted to the strategies $s^*_{\sim i} \in \{s_{\mathrm{I}}(x), \hat{s}, s^{\mathrm{sym}}_{\mathrm{Bd}}(x), (1+\xi)x\}$ with $s^*_{\sim i} \leq \frac{A}{2}$ is given by

$$\mathrm{BR}_i(s^*_{\sim i}) = \begin{cases} \{(1+\xi)x\} & \text{if } x \leq \hat{y}_1(s^*_{\sim i}), \\ \{d_{\mathrm{I}}(s^*_{\sim i})\} & \text{if } \hat{y}_1(s^*_{\sim i}) < x < y_{\mathrm{bd}}(s^*_{\sim i}), \\ \{d_{\mathrm{I}}(s^*_{\sim i}), d_{\mathrm{II}}(s^*_{\sim i})\} & \text{if } x = y_{\mathrm{bd}}(s^*_{\sim i}), \\ \{d_{\mathrm{II}}(s^*_{\sim i})\} & \text{if } x > y_{\mathrm{bd}}(s^*_{\sim i}) \end{cases} \tag{3.2.18}$$

where, besides the constants from Lemma 25,

$$y_{\mathrm{bd}}(s^*_{\sim i}) = \frac{45(1+\xi)s^*_{\sim i} + 2A(35 + 15\xi + \sqrt{2(11+3\xi)})}{120(1+\xi)^2}. \tag{3.2.19}$$

Proof: First, the best response is non-empty (as a maximum of a continuous function over a compact set). By Lemma 25, the best response is known besides the interval $[\hat{x}_2(s^*_{\sim i}), \hat{y}_2(s^*_{\sim i})]$.
So, consider $x \in [\hat{x}_2(s^*_{\sim i}), \hat{y}_2(s^*_{\sim i})]$. If $x < y_{\mathrm{bd}}(s^*_{\sim i})$, then $\mathrm{RBE}(d_{\mathrm{I}}(s^*_{\sim i})) > \mathrm{RBE}(d_{\mathrm{II}}(s^*_{\sim i}))$,

while if $x > y_{\rm bd}(s^*_{\sim i})$, then $\mathrm{RBE}(d_{\rm II}(s^*_{\sim i})) > \mathrm{RBE}(d_{\rm I}(s^*_{\sim i}))$. Finally , if $x = y_{\rm bd}(s^*_{\sim i})$, then $\mathrm{RBE}(d_{\rm I}(s^*_{\sim i})) = \mathrm{RBE}(d_{\rm II}(s^*_{\sim i}))$. ■

3.3 Multiple discontinuous symmetric feedback Nash equilibria

Theorem 27 *Consider any profile S with $S_i(1,x) = \bar{S}_i^{\rm L}(1,x)$ or $S_i(1,x) = \bar{S}_i^{\rm R}(1,x)$, where*

$$\bar{S}_i^{\rm L}(1,x) = \begin{cases} (1+\xi)x & \textit{if } x \le Y_1, \\ s_{\rm I}(x) & Y_1 < x \le Y_2, \\ \hat{s} & \textit{if } x > Y_2, \end{cases} \tag{3.3.1}$$

$$\bar{S}_i^{\rm R}(1,x) = \begin{cases} (1+\xi)x & \textit{if } x \le Y_1, \\ s_{\rm I}(x) & Y_1 < x < Y_2, \\ \hat{s} & \textit{if } x > Y_2, \end{cases} \tag{3.3.2}$$

for Y_1 being the unique solution of equation $\hat{y}_1((1+\xi)x) = \hat{y}_1(s_{\rm I}(x))$ and an arbitrary $Y_2 \in [y_{\rm bd}(\hat{s}), y_{\rm bd}(s_{\rm I}(Z))]$ for Z being the unique solution of $y_{\rm bd}(s_{\rm I}(Z)) = Z$ and for $S_i(2,x) = \bar{S}_i(2,x)$ from Eq. (3.2.1). Then S is a symmetric feedback Nash equilibrium profile for 2 player 2 stage LQ dynamic game and only such profiles can be symmetric feedback Nash equilibria.

Fig. 3.6–3.7 are for the values of the parameters: $A = 10000$ and $\xi = 0.02$.

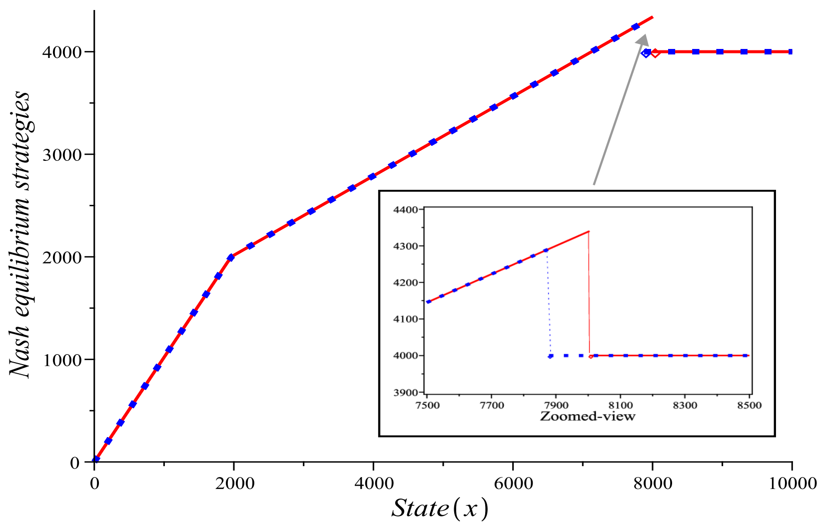


Figure 3.6: Two symmetric Nash equilibria — the decision at stage 1 depending on the initial state

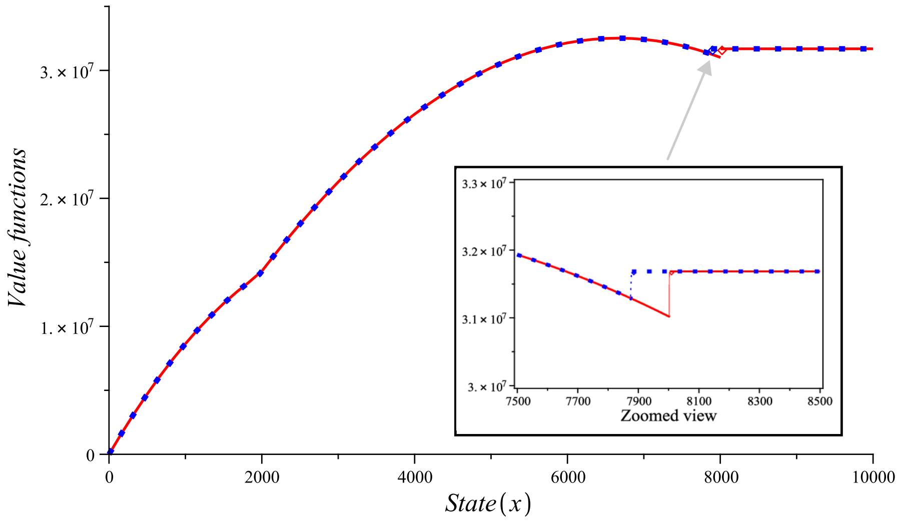


Figure 3.7: The value functions at stage 1 for Nash equilibria from Fig. 3.6, depending on the initial state

Proof: (of Theorem 27) By Corollary 23, for every Nash equilibrium, its profile of decisions at time 2 coincides with the Nash equilibrium of one stage game with identical Bellman equations. So, the value functions for Nash equilibria at stage 2 are equal to $\bar{V}_i(2,x)$. Hence, the Nash equilibrium problem at stage 1 is equivalent to the Nash equilibrium problem in a one stage game from Lemmas 24 and 26.

The maximal set of x on which a symmetric feedback Nash equilibrium can be equal to $(1+\xi)x$ is $[0, Y_1]$.

The maximal set of x on which a symmetric feedback Nash equilibrium can be equal to $s_{\rm I}(x)$ is $[Y_1, y_{\rm bd}(s_{\rm I}(x))]$.

The maximal set of x on which a symmetric feedback Nash equilibrium can be equal to $\hat{s}$ is $[y_{\rm bd}(\hat{s}), +\infty)$.

Note that $y_{\rm bd}(s_{\rm I}(x)) > y_{\rm bd}(\hat{s})$, since $s_{\rm I}(x) > \hat{s}$ for $x \geq y_{\rm bd}(\hat{s})$. Consider any $Y_2 \in [y_{\rm bd}(\hat{s}), y_{\rm bd}(s_{\rm I}(Z))]$ and a profile S with $S_i(2,x) = \bar{S}_i(2,x)$ and $S_i(1,x) = \bar{S}_i^{\rm L}(1,x)$. The Bellman equation and the Bellman inclusion at stage 2 has been checked in Theorem 22, is fulfilled and for every x, $\bar{S}_i^{\rm L}(1,x) \in {\rm BR}_i(\bar{S}_i^{\rm L}(1,x))$.

Analogously, it can be proved for $S_i(1,x) = \bar{S}_i^{\rm R}(1,x)$. ■

Corollary 28 *For the considered 2-player, 2-stage linear-quadratic dynamic game, there is a continuum of discontinuous symmetric feedback Nash equilibria, whereas no continuous symmetric feedback Nash equilibrium with respect to x. These equilibria are functions differing by the state Y_2 at which the jump appears at stage 1 and whether the function is left or right continuous.*

The value functions corresponding to these symmetric feedback Nash equilibria are also discontinuous at stage 1 at the same points Y_2.

Chapter 4

A linear-quadratic dynamic optimization problem with linear state dependent constraints in discrete time

The idea behind this chapter is to investigate the social optimum problem from Chapter 2 for $n = 1$, but for more general β instead of the *golden rule* β only.

In this chapter, we consider a discrete time linear-quadratic dynamic optimisation problem with more general payoff function and with linear state-dependent constraints. Our problem is equivalent to the social optimum problem considered in Chapter 2, but only with single social planner i.e., $n = 1$. We study the optimisation problem in the infinite time horizon and finite time truncations of the problem with the horizon T.

Our model is similar to the model considered before in chapter 2, but with slightly modified *current payoff function*

$$P(s) = \left(A - \frac{Bs}{2}\right) s$$

for the constants $A > 0$ and $B > 0$ and the discount factor $\beta = \frac{1}{1+\xi} - \epsilon$ for the constant $\epsilon \in \left(0, \frac{1}{1+\xi}\right)$. In the infinite horizon, we study also the golden rule $\beta = \frac{1}{1+\xi}$ for comparison.

For consistency of notations with the previous Chapters, we denote the social optimum $\bar{U}$ from Chapter 2 by $\bar{S}$.

Therefore, for our dynamic optimization problem, the Bellman Eq. (1.3.7) is

$$\bar{V}(t, x) = \sup_{s \in [0,(1+\xi)x]} \left(\left(A - \frac{Bs}{2}\right) s + \beta \bar{V}\left(t+1, (1+\xi)x - s\right) \right), \tag{4.0.1}$$

while the Bellman inclusion (1.3.8) is

$$\bar{S}(t, x) \in \operatorname*{Argmax}_{s \in [0,(1+\xi)x]} \left(\left(A - \frac{Bs}{2}\right) s + \beta \bar{V}\left(t+1, (1+\xi)x - s\right) \right). \tag{4.0.2}$$

and the terminal condition for finite time horizon is

$$V(T + 1, x) = 0, \tag{4.0.3}$$

while $\limsup_{t\to\infty} V(t, X(t))\beta^t = 0 \; \forall$ admissible trajectory X. Generally, in dynamic programming techniques (based on the Bellman Equation), the optimal solution is calculated by backwards induction in finite horizon optimisation problems, while for the infinite time horizon, it is usually calculated by the undetermined coefficient or Ansatz method assuming a regular form of the value function. However, our model turned out to be a counterexample to such a way of solving infinite time horizon optimisation problems. Therefore, we choose a different approach to solve the problem in the infinite time.

4.1 Finite horizon truncations of the initial dynamic optimization problem

In this section, we consider truncations of the dynamic optimization problem with horizon N, and for brevity of notation, we denote the optimal control $\bar{U}$ by S^N, the value function $\bar{V}$ by V^N, the total payoff J by J^N and it is defined by

$$J^N(t_0, x_0, S) = \sum_{t=t_0}^{N} \beta^{t-t_0} P(S(t, X(t)), t). \tag{4.1.1}$$

So, the terminal condition (4.0.3) becomes

$$V^N(N+1) = 0. \tag{4.1.2}$$

Theorem 29 *Consider the following functions*

$$V^N(0,x) := \begin{cases} V_0(x) := K_0 + G_0 x + \frac{H_0}{2}x^2 & \text{if } \hat{x_0} \le x < \hat{x_1}, \\ \vdots & \\ V_{N-1}(x) := K_{N-1} + G_{N-1}x + \frac{H_{N-1}}{2}x^2 & \text{if } x\hat{_{N-1}} \le x < \hat{x_N}, \\ V_N(x) := K_N + G_N x + \frac{H_N}{2}x^2 & \text{if } \hat{x}_N \le x < \hat{y}_N, \\ U_N(x) := \sum_{i=0}^{N} \beta^i P(\hat{s}) & \text{if } x \ge \hat{y}_N; \end{cases} \tag{4.1.3}$$

with $V^N(t,x) = V^{N-t}(0,x)$ *for* $t \le N$ *and* $V^N(N,x) = 0$*, where the constants are* $H_0 = -B(1+\xi)^2$*,* $G_0 = A(1+\xi)$*,* $K_0 = 0$*,* $\hat{x}_0 = 0$ *and*

$$\hat{y}_N = \frac{\hat{s}}{(1+\xi)} \sum_{i=0}^{N-1} \frac{1}{(1+\xi)^i}. \tag{4.1.4}$$

The number i corresponds to time to resource exhaustion *for $\hat{x}_{i-1} < x < \hat{x}_i$. So, V_i corresponds to time to resource exhaustion $i+1$.*

$$S^N(0,x) = \begin{cases} S_0(x) := a_0 x + b_0 & \text{if } \hat{x_0} \leq x < \hat{x_1}, \\ \vdots & \\ S_{N-1}(x) := a_{N-1}x + b_{N-1} & \text{if } x\hat{_{N-1}} \leq x < \hat{x_N}, \\ S_N(x) := a_N x + b_N & \text{if } \hat{x}_N \leq x < \hat{y}_N, \\ \hat{s} = \frac{A}{B} & \text{if } x \geq \hat{y}_N \end{cases} \tag{4.1.5}$$

with $S^N(t,x) = S^{N-t}(0,x)$ for $t \leq N$.
Values of the constants in Eq. (4.1.3)-(4.1.5) *are given by the recurrence equations,*

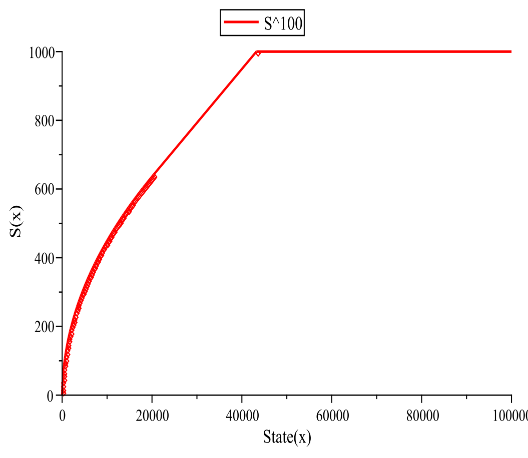
$$H_{i+1} = \frac{\beta B H_i (1+\xi)^2}{B - \beta H_i}, G_{i+1} = \frac{\beta(1+\xi)(BG_i - AH_i)}{B - \beta H_i}, K_{i+1} = \beta K_i + \frac{(A - \beta G_i)^2}{2(B - \beta H_i)} \tag{4.1.6}$$

$$a_{i+1} = \frac{-\beta H_i (1+\xi)}{B - \beta H_i}, \quad b_{i+1} = \frac{A - \beta G_i}{B - \beta H_i}, \quad \hat{x}_{i+1} = \frac{b_i - b_{i+1}}{a_{i+1} - a_i}, \quad \hat{y}_{i+1} = \frac{\hat{y}_i + \hat{s}}{1+\xi}. \tag{4.1.7}$$

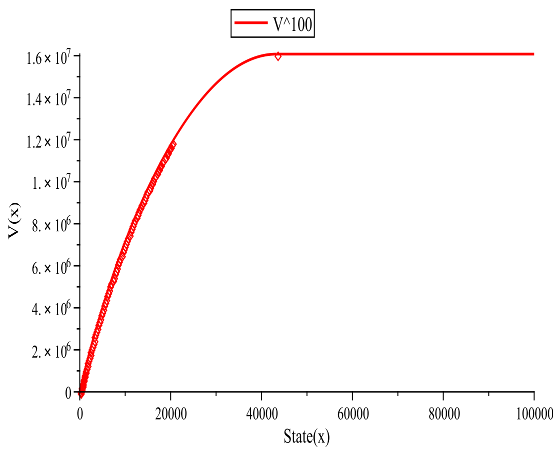
Equivalently, a_i and b_i can be rewritten as

$$a_i = -\frac{H_i}{B(1+\xi)}, \quad b_i = \frac{A(1+\xi) - G_i}{B(1+\xi)}. \tag{4.1.8}$$

The function V^N is the value function, while S^N is the optimal control for N time horizon truncations of the initial problem.



(a) The optimal control



(b) The value function

Figure 4.1

The figures are for the values of the parameters: $A = 1000,\ B = 1,\ \xi = 0.02,\ \epsilon = 0.01$ and $T = 100$.

To prove Theorem 29, the following sequence of Lemmata are needed.

Lemma 30 ***(a)*** *For every i, the constants $H_i < 0$ and $H_{i+1} > H_i$.*
(b) *For every i, the constant $(1+\xi)x \geq a_i > 0$ and $a_{i+1} < a_i$.*
(c) *For every i, the constant $G_i > 0$.*

Proof: **(a)** It can be easily verified that $|H_1| > |H_2|$ with $H_0,\ H_1 < 0$.

Now, assume that $H_i < 0,$ and $H_{i-1} < 0$ with $|H_i| < |H_{i-1}|$.

It is immediate from the definition that $H_{i+1} < 0$.

Now consider $\frac{|H_{i+1}|}{|H_i|} = \frac{|H_i|}{|H_{i-1}|}\frac{|B-\beta H_{i-1}|}{|B-\beta H_i|} < 1$. Since both the numerator and denominator are positive so,
$\left(\frac{B-\beta H_{i-1}}{B-\beta H_i}\right) < 1\left(\frac{B-\beta H_i}{B-\beta H_i}\right) = 1$, which gives $|H_{i+1}| < |H_i|$.
(b) Consider, $a_i = \frac{H_i}{B(1+\xi}$ from Eq. (4.1.8). The proof is immediate by **(a)** as the sign of a_i is opposite to H_i.
(c) It can be easily verified that $G_0,\ G_1 > 0$.

Now, assume that $G_i > 0$. By **(a)**, $H_i < 0$ for all i. So, it is immediate from the definition that $G_{i+1} > 0$. ■

Lemma 31 *Define the constant $F_i := \frac{G_i}{H_i}$.*

F_i is given by the recurrence relation

$$F_{i+1} = \frac{F_i}{(1+\xi)} - \frac{A}{B(1+\xi)}, \tag{4.1.9}$$

and for every i, $F_i < 0$. Moreover, $F_{i+1} < F_i$.

Proof: Consider the recurrence equation $F_{i+1} = \frac{G_{i+1}}{H_{i+1}}$.

$F_i < 0$ for all i is an immediate consequence of Lemma 30.

It can be easily verified that $F_0 > F_1$.

Now, assume that $F_{i-1} > F_i$ and check for $i+1$.

Consider $F_{i+1} - F_i = \frac{F_i - F_{i-1}}{(1+\xi)}$, which gives $F_i > F_{i+1}$. ■

Lemma 32 *Define x_{next} as*

$$x_{\text{next}}(x, S) = (1+\xi)x - S. \tag{4.1.10}$$

For all $i < N$, $\hat{x}_i$ fulfils

$$\hat{x}_i = x_{\text{next}}(\hat{x}_{i+1}, S_{i+1}) \tag{4.1.11}$$

Proof: It can be easily verified that $\hat{x}_0 = x_{\text{next}}(\hat{x}_1, S_1)$.

Now assume it for i, $\hat{x}_{i-1} = (1+\xi)\hat{x}_i + S_2(\hat{x}_i)$, which gives

$$\hat{x}_i = \frac{\hat{x}_{i-1}(\beta H_{i-1} - 3) + \beta G_{i-1} - A}{-3(1+\xi)}. \tag{4.1.12}$$

Check for $\hat{x}_{i+1} = \frac{\hat{x}_i + b_{i+1}}{1+\xi-a_{i+1}}$.

Substitute $\hat{x}_i$ from Eq. (4.1.12) to get $\hat{x}_{i+1} = \frac{\hat{x}_i(\beta H_i - 3) + \beta G_i - A}{-3(1+\xi)}$. ■

Lemma 33 ***(a)*** *For every i, $\hat{y}_{i+1} > \hat{y}_i$.*
(b) *$\hat{y}_i = -F_i$.*

Proof: **(a)** It can be easily checked that $\hat{y}_1 > \hat{y}_0$.

Now assume that $\hat{y}_i > \hat{y}_{i-1}$.

To check for $i+1$, consider $\hat{y}_{i+1} - \hat{y}_i = \frac{y_i + \hat{s}}{1+\xi} - \frac{y_{i-1}+\hat{s}}{1+\xi}$.

Simplify further and use the fact that $\hat{y}_i > \hat{y}_{i-1}$, to get $\hat{y}_{i+1} - \hat{y}_i > 0$.

(b) It can be easily checked that $\hat{y}_0 = -F_0$.

Now assume that $\hat{y}_i = -F_i$.

To check for $i+1$, consider $\hat{y}_{i+1} = \frac{y_i+\hat{s}}{1+\xi}$. Substitute $\hat{y}_i = -F_i$ and simplify to get $\hat{y}_{i+1} = -F_{i+1}$. ■

Lemma 34 *The function $S^N(0,x)$ is continuous, i.e., its components fulfil*
(a) *For all $1 \le i \le N$, $S_{i-1}(\hat{x}_i) = S_i(\hat{x}_i)$.*
(b) *$S_N(\hat{y}_N) = \hat{s}$.*

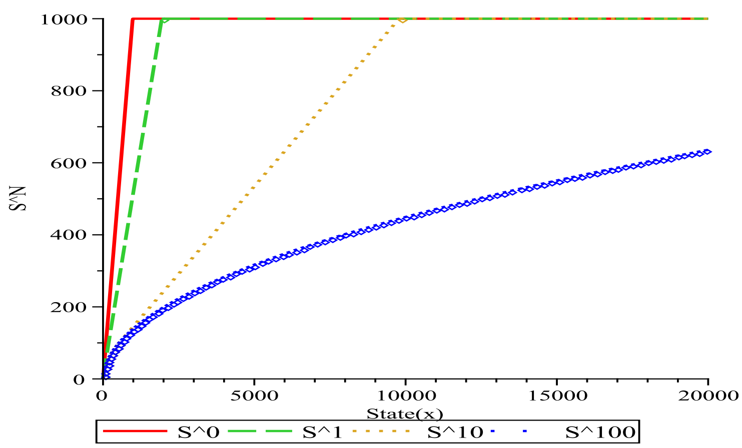


Figure 4.2: The optimal control for various time horizons

Proof: **(a)** Since $S_0(\hat{x}_1) = S_1(\hat{x}_1)$, so, $S^N(0,x)$ is continuous in x at $\hat{x}_1$.

Now, assume that $S^N(0,x)$ is continuous in x at $\hat{x}_i$.

So, $\hat{x}_i$ is a root of $S_i(x) - S_{i-1}(x) = 0$ i.e.,

$$S_i(\hat{x}_i) - S_{i-1}(\hat{x}_i) = 0. \tag{4.1.13}$$

Next, check that this holds for $i+1$. Consider the left hand side: $S_{i+1}(\hat{x}_{i+1}) - S_i(\hat{x}_{i+1}) = (a_{i+1} - a_i)\hat{x}_{i+1} + (b_{i+1} - b_i)$.

Substitute $\hat{x}_{i+1} = \frac{b_i - b_{i+1}}{a_{i+1} - a_i}$ and simplify further to get

$(a_{i+1} - a_i)\hat{x}_{i+1} + (b_{i+1} - b_i) = 0$, equivalently, $S_{i+1}(\hat{x}_{i+1}) - S_i(\hat{x}_{i+1}) = 0$.

Therefore, $S^N(0, x)$ is continuous in x at $\hat{x}_{i+1}$.

(b) It can be easily verified that $S_0(\hat{y}_0) = \hat{s}$.

Now, assume that $S_i(\hat{y}_i) = \hat{s}$ and check for $i + 1$.

Consider the left hand side limit: it is $S_{i+1}(\hat{y}_{i+1}) = a_{i+1}\hat{y}_{i+1} + b_{i+1}$.

By Lemma 33 **(b)**, $\hat{y}_{i+1} = -F_{i+1}$. Substitute a_{i+1}, b_{i+1} from Eq. (4.1.8) and simplify further to get $a_{i+1}\hat{y}_{i+1} + b_{i+1} = \hat{s}$. ■

Lemma 35 ***(a)*** *$x_{\text{next}}(x, S_N)$ is non-increasing in x for all N and strictly increasing in x for $N > 1$.*

(b) *If $x \geq \hat{x}_{N-1}$, then $x_{\text{next}}(x, S_N) \geq \hat{x}_{N-2}$ and if $x \leq \hat{x}_N$, then $x_{\text{next}}(x, S_N) \leq \hat{x}_{N-1}$.*

Proof: **(a)** Since, $x_{\text{next}}(x, S_N) = ((1 + \xi) - a_N)x - b_N$, it is true by the fact that $a_N \leq (1 + \xi)$ and $a_N < (1 + \xi)$ for $N > 1$ resulting from Lemma 30 **(b)**.

(b) By the fact that $x_{\text{next}}(\hat{x}_N, S_N) = \hat{x}_{N-1}$ (Lemma 32 **(a)**), continuity and monotonicity of $S_N(0, x)$ in x (Lemma 34 **(a)**). ■

Lemma 36 *The function $V^N(0, x)$ is continuous, i.e.,*

(a) *For all $2 \leq i \leq N$, $V_{i-1}(\hat{x}_i) = V_i(\hat{x}_i)$.*

(b) *$V_N(\hat{y}_N) = U_N$.*

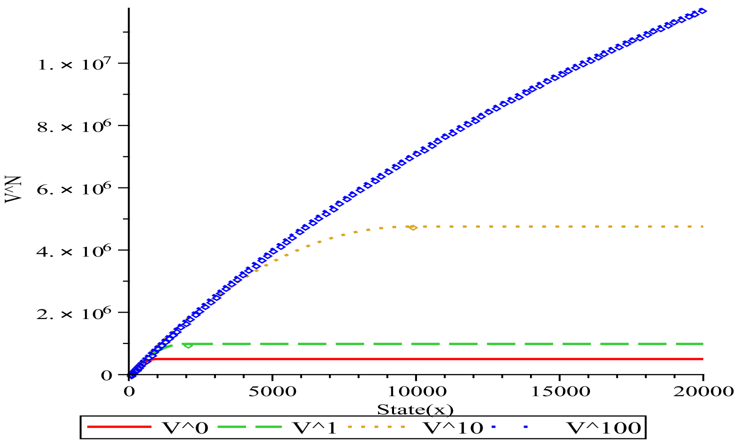


Figure 4.3: The value function for various time horizons

Proof: **(a)** Since $V_0(\hat{x}_1) = V_1(\hat{x}_1)$, so, $V^N(0, x)$ is continuous in x at $\hat{x}_1$.

Now, assume that $V^N(0, x)$ is continuous in x at $\hat{x}_i$ i.e.,

$$V_{i-1}(\hat{x}_i) = V_i(\hat{x}_i). \tag{4.1.14}$$

By Lemma 35, if $x \in [\hat{x}_{i+1}, \hat{x}_{i+2}]$, then $x_{\text{next}}(x, S^N) \in [\hat{x}_i, \hat{x}_{i+1}]$

To check for $i + 1$, consider the left-hand side limit

$\lim_{\delta \to 0^-} V_i(\hat{x}_{i+1} + \delta) = \lim_{\delta \to 0^-} P(S_i(\hat{x}_{i+1} + \delta)) + \beta V_{i-1}(x_{\text{next}}(\hat{x}_{i+1} + \delta, s_i))$.

By Lemma 34 and continuity of V_{i-1} and S_i, $\lim_{\delta \to 0^-} V_i(\hat{x}_{i+1} + \delta) = P(S_i(\hat{x}_{i+1})) + \beta V_{i-1}(\hat{x}_i)$

Now, consider the right-hand side limit

$\lim_{\delta \to 0^+} V_{i+1}(\hat{x}_{i+1} + \delta) = \lim_{\delta \to 0^+} P(S_{i+1}(\hat{x}_{i+1} + \delta)) + \beta V_i(x_{\text{next}}(\hat{x}_{i+1} + \delta, s_{i+1}))$.

By Lemmas 34 – 32, continuity of V_i and S_{i+1} and by using Eq. (4.1.14),

$\lim_{\delta \to 0^+} V_{i+1}(\hat{x}_{i+1} + \delta) = P(S_{i+1}(\hat{x}_{i+1})) + \beta V_i(\hat{x}_{i-1}) = \lim_{\delta \to 0^-} V_i(\hat{x}_{i+1} + \delta)$, which shows that $V^N(x, 0)$ is continuous in x at $\hat{x}_{i+1}$.

Therefore, $V^N(0, x)$ is continuous in x at $\hat{x}_i$ for all $i \leq N$.

(b) It can be easily verified that $V_0(\hat{y}_0) = U_0(\hat{y}_0)$.

Now, assume that $V_i(\hat{y}_i) = U_i(\hat{y}_i)$.

To check for $i + 1$, consider the left-hand side limit

$\lim_{\delta \to 0^-} V_{i+1}(\hat{y}_{i+1} + \delta) = \lim_{\delta \to 0^-} P(S_{i+1}(\hat{y}_{i+1} + \delta)) + \beta V_i(x_{\text{next}}(\hat{y}_{i+1} + \delta, s_{i+1}))$.

By Lemma 34 **(b)**, $\lim_{\delta \to 0^-} V_{i+1}(\hat{y}_{i+1} + \delta) = P(S_{i+1}(\hat{y}_{i+1})) + \beta V_i(\hat{y}_i)$.

By Eq. (4.1.7) and since $V_i(\hat{y}_i) = U_i(\hat{y}_i)$,

so $\lim_{\delta \to 0^-} V_{i+1}(\hat{y}_{i+1} + \delta) = P(\hat{s}) + \beta U_i(\hat{y}_i) = U_{i+1}(\hat{y}_{i+1})$.

Now, consider the right-hand side limit

$\lim_{\delta \to 0^+} V_{i+1}(\hat{y}_{i+1} + \delta) = \lim_{\delta \to 0^+} P(S_{i+1}(\hat{y}_{i+1} + \delta)) + \beta V_i(x_{\text{next}}(\hat{y}_{i+1} + \delta, S_{i+1})$.

From Lemma 32 and 34 **(b)**, and since $V_i(\hat{y}_i) = U_i(\hat{y}_i)$, so

$\lim_{\delta \to 0^+} V_{i+1}(\hat{y}_{i+1} + \delta) = P(S_{i+1}(\hat{y}_{i+1})) + \beta V_i(\hat{y}_i) = U_{i+1}(\hat{y}_{i+1})$.

Therefore, $V_{i+1}(\hat{y}_{i+1}) = U_{i+1}(\hat{y}_{i+1})$. ■

Lemma 37 ***(a)*** *The function $V^N(0, x)$ is differentiable in x for $x < \hat{y}_N$ and for all i, its derivative is continuous at $x = \hat{x}_i$, i.e., $V'_{i-1}(\hat{x}_i) = V'_i(\hat{x}_i)$.*
(b) *$V^N(x, 0)$ is differentiable at $\hat{y}_N$ and $V'_N(\hat{y}_N) = 0$.*

Proof: **(a)** Since $V'_0(\hat{x}_1) = V'_1(\hat{x}_1)$, $V^N(x, 0)$ is differentiable in x at $x = \hat{x}_1$ and the derivative is continuous.

Now, assume that $V^N(0, x)$ is differentiable in x at $\hat{x}$ and the derivative is continuous which implies that

$$G_i - G_{i-1} + (H_i - H_{i-1})\hat{x}_i = 0. \tag{4.1.15}$$

To check for $i+1$, consider $V'_{i+1}(\hat{x}_{i+1}) - V'_i(\hat{x}_{i+1}) = 0$ or equivalently, $(G_{i+1} - G_i) + (H_{i+1} - H_i)\hat{x}_{i+1} = 0$.

Rewrite Eq. (4.1.8) for $i+1$ and i and subtract to get,

$G_{i+1} - G_i = \beta(1+\xi)(b_{i+1} - b_i)$ and $H_{i+1} - H_i = -B(1+\xi)(a_{i+1} - a_i)$.

Notice that $\frac{G_{i+1}-G_i}{H_{i+1}-H_i} = -\frac{b_{i+1}-b_i}{a_{i+1}-a_i} = \hat{x}_{i+1}$, implies that $\hat{x}_{i+1}$ is a root of the equation $V'_{i+1}(x) - V'_i(x) = 0$. So, $V^N(x,0)$ is differentiable in x at $\hat{x}_{i+1}$.

(b) It can be easily verified that $V'_0(\hat{y}_0) = 0$,

Now assume that $V'_i(\hat{y}_i) = 0$.

To check for $i+1$, consider $V'_{i+1}(\hat{y}_{i+1}) = G_{i+1} + \hat{y}_{i+1}H_{i+1}$.

Use Lemma 31 and 33 **(b)** and simplify to get, $V'_{i+1}(\hat{y}_{i+1}) = 0$. ■

Lemma 38 *For all x, V^N is concave in x and it is strictly concave for $x < \hat{y}_N$ and it is differentiable for all x.*

Proof: Since by Lemma 30, $H_i < 0$, V_i are strictly concave and differentiable. Note that U_i is constant for every i, so, it is also concave.

Since V_i is strictly concave and by Lemma 36 **(a)**, V^N is continuous, $\frac{\partial V_i}{\partial x}$ is strictly decreasing and, since by Lemma 37, $V'_i(\hat{x}_i) = V'_{i+1}(\hat{x}_i)$, so, $V^N(\cdot,0)$ is differentiable for $x < \hat{y}_N$ and its derivative is strictly decreasing on $(0, \hat{y}_N)$.

Since, $\frac{\partial V^N(0,\cdot)}{\partial x}$ is strictly decreasing for $x \leq \hat{y}_N$, $V^N(\cdot,0)$ is strictly concave on the interval $[0, \hat{y}_N)$.

Since by Lemma 37 **(b)**, $\frac{\partial V_N(\hat{y}_N)}{\partial x} = 0 = \frac{\partial U^N(\hat{y}_N)}{\partial x}$, $\frac{\partial V^N(0,x)}{\partial x} = 0$ for $x \geq y_N$.

Since $\frac{\partial V^N(0,\cdot)}{\partial x}$ is non-increasing, $V^N(0,\cdot)$ is concave on the whole domain. ■

Lemma 39 ***(a)*** *For any N, $P(s) + \beta V^N((1+\xi)x - s)$ is strictly concave and differentiable in S and the supremum in the right hand side of the Bellman Eq.* (4.0.1) *is attained.*

(b) *If for some $s \in [0, (1+\xi)x]$, $\frac{\partial(P(s)+\beta V^N(x_{\text{next}}(x,s))}{\partial s} = 0$, then s is the unique optimum of the right hand side of Bellman equation.*

Proof: **(a)** Immediately by Lemma 38 and boundedness of P and V^N.

(b) Note that for $x > \hat{y}_N$, the function is strictly decreasing, so, zero derivative cannot be attained for $x > \hat{y}_N$. If a point fulfils first order condition for optimization of a strictly concave function then it is the unique optimum. ■

Proof: (of Theorem 29)

The proof will be done inductively in two ways: by forward induction with respect to the horizon N and within the fixed horizon N, by backward induction corresponding to the dynamic programming techniques, which is rewritten to forward induction with respect to time to resource exhaustion.

For $N=0$ it can be easily verified that the value function

$$V^0(0,x) = \begin{cases} V_0(x) := (A - \frac{B}{2}(1+\xi)x)(1+\xi)x & \hat{x}_0 < x < \hat{y}_0, \\ U_0(x) := \frac{A^2}{2B} & x \geq \hat{y}_0, \end{cases}$$

fulfils the Bellman Eq. (4.0.1) and there is a unique optimal control

$$S^0(0,x) = \begin{cases} S_0(x) := (1+\xi)x & \hat{x}_0 < x < \hat{y}_0, \\ \hat{s} & x \geq \hat{y}_0, \end{cases}$$

which fulfils the Bellman inclusion (4.0.2).

Assume that the value function and the optimal control are given by Eq. (4.1.5)–(4.1.3) for N and prove it for $N+1$.

The Bellman Eq. (4.0.1) has the form

$$V^{N+1}(t,x) = \sup_{s \in [0,(1+\xi)x]} P(s) + \beta V^{N+1}(t+1, x_{\text{next}}(x,s)) \text{ for all } t \leq N, \tag{4.1.16}$$

while the Bellman inclusion — necessary and sufficient condition for a control to be optimal is

$$S^{N+1}(t,x) \in \operatorname*{Argmax}_{s \in [0,(1+\xi)x]} P(s) + \beta V^{N+1}(t+1, x_{\text{next}}(x,s)) \text{ for all } t \leq N. \tag{4.1.17}$$

By the Bellman principle of optimality, at time $t+1$, the solution has to coincide with the optimal solution of the N horizon problem with the state resulting from the first decision. Since the only dependence on time in the functions of the model is by discounting, so, $V^{N+1}(1,x) = V^N(0,x)$ and $S^{N+1}(1,x) = S^N(0,x)$. By analogous reasoning, $V^{N+1}(t+1,x) = V^N(t,x)$ and $S^{N+1}(t+1,x) = S^N(t,x)$ for all $t \leq N$. Thus, Eq. (4.1.16)–(4.1.17) is needed to be checked only for $t=0$.

Eq. (4.1.16)–(4.1.17) can be rewritten as

$$V^{N+1}(0,x) = \sup_{s \in [0,(1+\xi)x]} P(s) + \beta V^N(0, x_{\text{next}}(x,s)) \text{ for all } t \leq N, \tag{4.1.18}$$

$$S^{N+1}(0,x) \in \operatorname*{Argmax}_{s \in [0,(1+\xi)x]} P(s) + \beta V^N(0, x_{\text{next}}(x,s)) \text{ for all } t \leq N. \tag{4.1.19}$$

The maximum of the right hand side of Eq. (4.1.18) exists, it is unique by Lemma 39 and whenever there exists a point in $[0,(1+\xi)x]$ at which the derivative of the right hand side of Eq. (4.1.18) is 0, it is the maximum, while if this zero derivative point is greater than $(1+\xi)x$, the maximum is attained at $(1+\xi)x$.

Now find the maximum depending on the interval in which x belongs to. If $x \in [x_0, x_1]$, then $x_{\text{next}}(x, S_0) = 0$, so $V^N(x_{\text{next}}(x, S_0)) = 0$.

By Lemma 35, if $x \in [\hat{x}_{k+1}, \hat{x}_{k+2})$, then $V^N\left(0, x_{\text{next}}(x, S^N)\right) = V_k\left(x_{\text{next}}(x, S_{k+1})\right)$.

So, if $S_{k+1}(x)$ maximizes the right hand side of

$$V_{k+1}(x) = \sup_{s \in [0,(1+\xi)x]} P(s) + \beta V_k(x_{\text{next}}(x, s)), \tag{4.1.20}$$

then for this x, Eq. (4.1.18) reduces to Eq. (4.1.20). So, what remains to be proven is the fact that S_{k+1} is really the maximizer of the right hand side of Eq. (4.1.20) and that this equation is fulfilled. It is done by induction with respect to k.

For $k = 0$ it is immediate by substituting the auxiliary $V_{-1} \equiv 0$. Now assume that it is fulfilled for k and prove it for $k+1$.

The first order condition for s to be optimal is

$$A - Bs - \beta G_k - \beta H_k((1+\xi)x - s) = 0$$

Solve this equation for s to get the optimal S_{k+1}

$$S_{k+1}(x) = a_{k+1}x + b_{k+1} = \frac{\beta H_k(1+\xi)x + \beta G_k - A}{\beta H_k - 3} \tag{4.1.21}$$

with the constants $a_{k+1} = \frac{\beta H_k(1+\xi)}{\beta H_k - B}$ and $b_{k+1} = \frac{\beta G_k - A}{\beta H_k - B}$.

Substitute this $S_{k+1}(x)$ into Eq. (4.1.20), to get $V_{k+1}(x) = K_{k+1} + G_{k+1}x + \frac{H_{k+1}}{2}x^2$, with the recurrence equation for the constants as in Eq. (4.1.6).

So, two cases remains to be proven: $x \in [\hat{x}_{N+1}, \hat{y}_{N+1})$ and $x \geq \hat{y}_{N+1}$.

In the latter case obviously $x_{\text{next}}(x, S_{N+1}) \geq \hat{y}_N$, the Bellman Eq. (4.1.18) reduces to $U_{N+1}(x) = \sup_{s \in [0,(1+\xi)x]} P(s) + \beta U_N((x, s))$ and it is fulfilled with $S(x) = \hat{s}$.

In the former case there are two sub-cases:

(i) If $x_{\text{next}}(x, S_{N+1}) \in [\hat{x}_N, \hat{y}_N)$, then the Bellman Eq. (4.1.18) reduces to

$V_{N+1}(x) = \sup_{s \in [0,(1+\xi)x]} P(s) + \beta V_N(x_{\text{next}}(x, s))$, then the reasoning is the same as for $x \in [\hat{x}_k, \hat{x}_{k+1})$ for $k < N$.

(ii) If $x_{\text{next}}(x, S_{N+1}) \geq \hat{y}_N$, then the Bellman Eq. (4.1.18) reduces to
$V_{N+1}(x) = \sup_{s \in [0,(1+\xi)x]} P(s) + \beta U_N(x_{\text{next}}(x, s))$ and it is fulfilled with $S(x) = \hat{s}$. ■

4.2 Limit properties of the finite horizon truncations of the problem

Proposition 40 ***(a)*** *The limit of H_i is given by*

$$\lim_{i\to\infty} H_i = \begin{cases} \frac{B\left(1-\beta(1+\xi)^2\right)}{\beta} & \text{for } \beta(1+\xi)^2 > 1, \\ 0 & \text{for } \beta(1+\xi)^2 \le 1. \end{cases}$$

(b) *The limit of a_i is given by*

$$\lim_{i\to\infty} a_i = \begin{cases} \frac{\beta(1+\xi)^2-1}{\beta(1+\xi)} & \text{for } \beta(1+\xi)^2 > 1, \\ 0 & \text{for } \beta(1+\xi)^2 \le 1. \end{cases}$$

Proof: **(a)** Consider the recurrence relation for H_i given by Eq. (4.1.6).

Calculate the fixed point to get the values: 0 and $\frac{B(1-\beta(1+\xi)^2)}{\beta}$.

By Lemma 30, H_i is increasing and bounded from above by 0. So the limit exists, and it is non-positive. Consider the following cases.

case 1 If $\beta(1+\xi)^2 > 1$, then $H_1 < \frac{B\left(1-\beta(1+\xi)^2\right)}{\beta}$.

Consider any auxiliary sequence given by Eq. (4.1.6) without predetermined initial condition and denote it by $\{h_i\}$.

This h_i is increasing if $h_1 < \frac{B\left(1-\beta(1+\xi)^2\right)}{\beta}$ and decreasing if $\frac{B\left(1-\beta(1+\xi)^2\right)}{\beta} < h_1 < 0$.

So, 0 cannot be the limit of H_i. Therefore, in this case $\lim_{i\to\infty} H_i = \frac{B(1-\beta(1+\xi)^2)}{\beta}$.

case 2 If $\beta(1+\xi)^2 \le 1$, then $\frac{B\left(1-\beta(1+\xi)^2\right)}{\beta} \ge 0$.

So, either the limit is positive, and hence it cannot be the limit of H_i, or it is 0.

Therefore, in this case $\lim_{i\to\infty} H_i = 0$.

(b) Immediate by substitution of the limit of H_i from **(a)** into $a_i = \frac{-H_i}{B(1+\xi)}$. ■

Proposition 41 ***(a)*** *The limit of F_i is given by $\lim_{i\to\infty} F_i = -\frac{A}{B\xi}$.*
(b) *The limit of G_i is given by*

$$\lim_{i\to\infty} G_i = \begin{cases} \frac{A(\beta(1+\xi)^2-1)}{\beta\xi}, & \text{for } \beta(1+\xi)^2 > 1, \\ 0, & \text{for } \beta(1+\xi)^2 \le 1. \end{cases}$$

Proof: **(a)** Calculate the fixed point of F_i which is $\frac{-\hat{s}}{\xi}$. By Lemma 31, F_i is decreasing.

Now consider any sequence given by Eq. (4.1.9) without predetermined initial condition and denote it by $\{f_i\}$.

If $f_1 > \frac{-\hat{s}}{\xi}$, then f_i is decreasing, while if $f_1 < \frac{-\hat{s}}{\xi}$, then f_i is increasing.

Therefore $\lim_{i\to\infty} F_i = \frac{-\hat{s}}{\xi}$.

(b) $\lim_{i\to\infty} G_i = \left(\lim_{i\to\infty} H_i\right) \cdot \left(\lim_{i\to\infty} F_i\right)$ since both H_i and F_i are convergent, it is immediate by Prop. 41 **(a)** and 40 **(a)**. ■

Proposition 42 *For every i, $\lim_{i\to\infty} \hat{y}_i = \lim_{i\to\infty} \hat{x}_i = \frac{\hat{s}}{\xi} := \tilde{x}$.*

Proof: Immediate by the definition of $\hat{y}_i$ given in Eq. (4.1.4). ■

4.3 The infinite time horizon

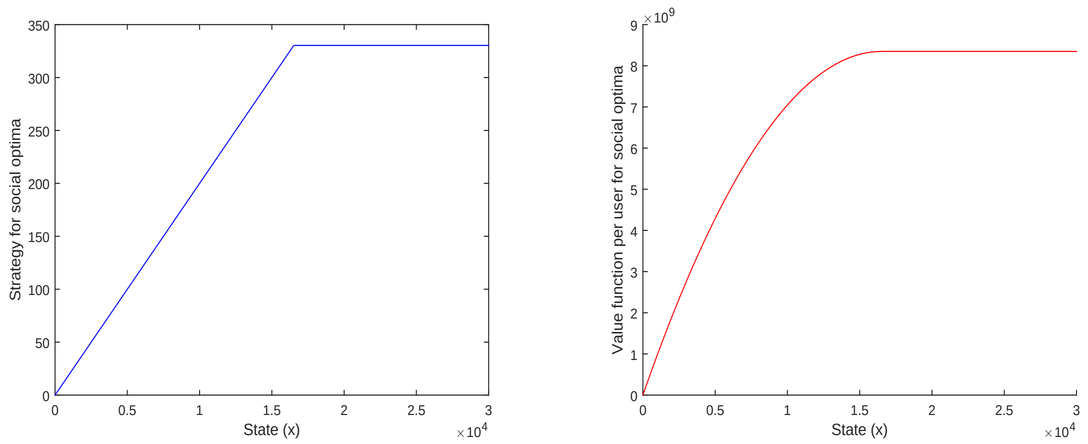
In this subsection we solve the infinite horizon problem.

Theorem 43 *Consider $\beta = \frac{1}{1+\xi}$, for $\epsilon = 0$. The value function is*

$$\bar{V}(x) = \begin{cases} \hat{G} \cdot x + \frac{\hat{H}}{2} \cdot x^2 & \text{if } x \in (0, \tilde{x}), \\ \tilde{k} & \text{otherwise}, \end{cases}$$

for $\hat{s} = \frac{A}{B}$, $\tilde{x} = \frac{\hat{s}}{\xi}$, $\hat{H} = -B\xi\,(1+\xi)$, $\hat{G} = A(1+\xi)$, and $\tilde{k} = \frac{A^2(1+\xi)}{2B\xi}$, while the unique optimal control is given by,

$$\bar{U}(x) = \begin{cases} \xi x, & \text{for } x \in (0, \tilde{x}), \\ \hat{s} & \text{otherwise}. \end{cases}$$



(a) The optimal control **(b)** The value function

Figure 4.4: Optimal control and the vallue function for golden rule β

Proof: The proof follow the same lines as in Theorem 2.2 from Chapter 2 with substituting $n = 1$. ■

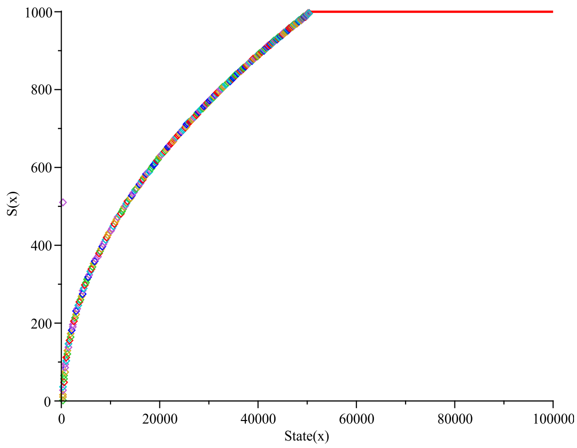
Theorem 44 *Consider* $\beta = \frac{1}{1+\xi} - \epsilon$. *The value function is given by,*

$$\bar{V}(x) = \begin{cases} \tilde{k} & \text{if } x \geq \tilde{x}, \\ V_N(x) & \hat{x}_N \leq x < \hat{x}_{N+1}, \end{cases} \tag{4.3.1}$$

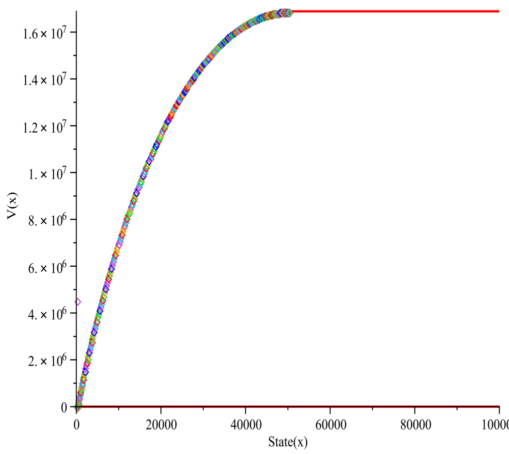
for $\tilde{k} = \frac{P(\hat{s})}{1-\beta}$, $\hat{x}_0 = 0$ *and* $\hat{x}_N$ *defined by Eq.* (4.1.7) *and* V_N *is given by Eq.* (4.1.3), *while the optimal control is*

$$\bar{S}(x) = \begin{cases} \hat{s} & \text{if } x > \tilde{x}, \\ S_N(x) & \hat{x}_N \leq x < \hat{x}_{N+1}, \end{cases} \tag{4.3.2}$$

where S_N *are given by Eq.* (4.1.5).



(a) Optimal control



(b) Value function

Figure 4.5: Optimal control and the value function for dynamic optimization problem with infinite time horizon

The figures are drawn for the values of the parameters: $T = 1000$, $\epsilon = 0.01$, $A = 1000$ $B = 1$, $\xi = 0.02$. **Proof:** First, check the interval $[\tilde{x}, +\infty)$. For x in this interval, the global optimum $\hat{s}$ of the objective function $P(s)$ is available and the resulting next stage state remains in this interval, so, the control $S \equiv \hat{s}$, being the global maximizer of $J(x, \cdot)$ is available. So, $S \equiv \hat{s}$ is the optimal control and $\bar{V}(x) = \tilde{k}$.

$\bar{V}$ is continuous, differentiable and concave. By Prop. 42, $\hat{x}_N$ and $\hat{y}_N$ converge to $\tilde{x}$. So, the right hand side of the Bellman Eq. (4.0.1) has a unique solution given by the zero derivative point or equal on $s = (1 + \xi)x$. It has already been checked that when $x \in [x_N, x_{N+1})$, then the zero derivative point is SN, while proving the Theorem 44. If $x > \tilde{x}$, then it is at $\hat{s}$. So, $\bar{V}$ fulfils the Bellman Eq. (4.0.1), while $\bar{S}$ fulfils the inclusion (4.0.2). The terminal condition (4.1.2) is obviously fulfilled, since $\bar{V}$ is bounded. Therefore, $\bar{V}$ is the value function while $\bar{S}$ is the optimal control. ■

Corollary 45 *$\forall x < \tilde{x}$, $\exists N$ such that $\bar{V} = \bar{V}^N(x, 0)$.*

$\forall x < \tilde{x}$, $\exists N_x \forall N > N_x$ such that $\bar{V}^N(x, 0) = \bar{V}(x)$.

Corollary 46 *For each N, $\bar{V}|_{[0,\hat{x}_N]} = V^N(x, 0)$ and $\bar{S}|_{[0,\hat{x}_N]} = S^N(x, 0)$. So, for any $x < \tilde{x}$, the solution of the infinite horizon problem coincides with the solution of its finite horizon truncation.*

4.4 An important methodological issue — how not to solve the infinite horizon problem

If we try to solve the infinite horizon problem for $\epsilon > 0$, by the undetermined coefficient method/ Ansatz method, starting from writing a quadratic value function and finding s which fulfils them, we obtain, two candidates for h: negative or 0, and the unique g and k for each h, then the Bellman Eq. (4.0.1) and inclusion (4.0.2) are fulfilled besides a small interval $[0, x_{\min}(\epsilon)]$ with $\lim_{\epsilon \to 0} x_{\min}(\epsilon) \to 0$.

So, for arbitrary small $\eta > 0$, there exists a $\epsilon > 0$ such that the sufficient condition for the infinite time horizon optimisation problem for $\beta = \frac{1}{1+\xi} - \epsilon$ is fulfilled besides an interval of length less than η.

The consequence of this error on such a small interval is the fact that the value function and the optimal control is incorrectly calculated on the whole interval $\left(0, \frac{\hat{s}}{\xi}\right)$.

Proposition 47 *Consider a function*

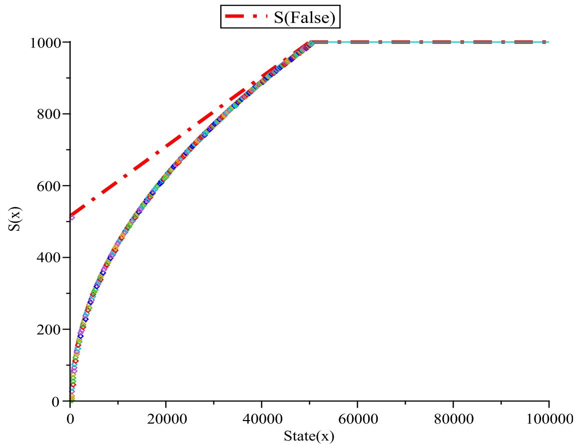
$$S^{\text{False}} = \begin{cases} a^{\text{F}} x + b^{\text{F}} & \text{for } x < \tilde{x}, \\ \hat{s} & \text{for } x \geq \tilde{x}, \end{cases}$$

for $a^{\text{F}} = \frac{\left((1+\xi)^2\epsilon - \xi\right)}{(1+\xi)\epsilon - 1}$, $b^{\text{F}} = \frac{-A\epsilon(1+\xi)}{B\xi((1+\xi)\epsilon - 1)}$ *and*

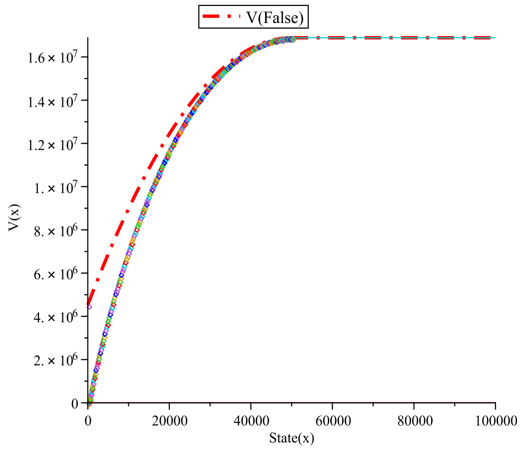
$$V^{\text{False}} = \begin{cases} \frac{h^{\text{F}} x^2}{2} + g^{\text{F}} x + k^{\text{F}} & \text{for } x < \tilde{x}, \\ \tilde{k} & \text{for } x \geq \tilde{x}, \end{cases}$$

for $h^{\text{F}} = \frac{-B(1+\xi)\left((1+\xi)^2\epsilon - \xi\right)}{((1+\xi)\epsilon - 1)}$, $g^{\text{F}} = \frac{A(1+\xi)\left((1+\xi)^2\epsilon - \xi\right)}{\xi((1+\xi)\epsilon - 1)}$, $k^{\text{F}} = \frac{-A^2\epsilon^2(1+\xi)^4}{2B\xi^2((1+\xi)\epsilon - 1)((1+\xi)\epsilon + \xi)}$, *then V^{False} fulfils the Bellman Eq.* (4.0.1) *while S^{False} fulfils the Bellman inclusion* (4.0.2) *on the set $[x_{\min}, +\infty)$ for $x_{\min} = \frac{A\epsilon(1+\xi)}{B\xi}$. The terminal condition is also fulfilled.*

Such V^{False} is the only function that fulfils the Bellman Eq. (4.0.1) *and terminal condition* (4.1.2) *on* $[x_{\min}, +\infty)$ *in the class of piecewise quadratic functions with at most two pieces.*



(a) Optimal control



(b) Value function

Figure 4.6: The actual optimal control and value function for the infinite horizon compared to the result of the Anstaz method restricted to $[x_{\min}, +\infty)$

Proof: The proof follows the same lines as the proof of Theorem 10 from Chapter 2, although the function is different and here the Bellman Eq. (4.0.1) is fulfilled for $x \geq x_{\min}$. The terminal condition (4.1.2) holds since the function is bounded. ■

Chapter 5

Can using inaccurate methods of calculation of value functions result in correct optima and equilibria?

Most of the results of this chapter are from Wiszniewska-Matyszkiel and Singh [92].

In most of the real-life decision-making problems of resource extraction, either simple dynamic optimisation problems or more compound dynamic games, the most important question is what to do in a specific time instant, i.e., the optimal control or the Nash equilibrium strategy at the corresponding trajectory.

One of the most extensively studied models of extraction of a common renewable resource is the *fish wars* model by Levhari and Mirman and its extensions *with singularities in payoffs*.

Although the solution of Levhari and Mirman model can be calculated analytically, this does not have to be true for its modifications. In such a case, numerical or approximated solution may be considered instead. The objective of the research done in this chapter is an answer the question whether at least the most important question stated before — what to do in a specific time instant can be answered approximately correctly for reasonable initial conditions in spite of the singularities in payoffs, and consequently, possibility of substantial errors in the value functions. As a motivating example, we consider Levhari and Mirman fish wars model in discrete time with finite time horizon.

We found that in spite of the substantial error in the calculation of the value function on some sets, we have obtained very high accuracy of social optimum and Nash equilibrium along the optimal trajectory.

5.1 Formulation of the motivating Levhari-Mirman Fish Wars model

The dynamic game considered in this chapter as a motivating example consists of:

1. A set of finitely many players $\mathbb{I} = \{1, 2 \ldots, n\}$.

2. A *time set* $\mathbb{T}$ is discrete with finite time horizon T and the initial time $t_0 = 0$.

3. The *state of the resource* is $x \in \mathbb{X} = [0,1]$, denoting the *biomass of fish.*

4. At each time instant, country i extracts or consume $s_i := c_i x$, where s_i denotes the *catch* or *consumption* for c_i being the *catch rate* or *consumption coefficient.*

5. We assume that the fish is uniformly distributed over the sea and each country can fish only in its Exclusive Economic Zone, identical for each country. So, at state x, $s_i(t,x) \in D_i(x)$ for $D_i(x) = [0, \frac{x}{n}]$.

6. The *current or instantaneous payoff function* of country i for given s_i is $P_i(t,x,s) = \ln s_i$ with $\ln(0) = -\infty$.

7. The *terminal payoff* is $\frac{\ln(x)}{n}$, which implies that countries divide the remaining biomass equally after termination of the game.

8. Payoffs are discounted by a *discount factor* $\beta \in (0,1)$.

9. We are interested in calculating the feedback strategies, $S_i : \mathbb{T} \times \mathbb{X} \to \mathbb{D}_i$ for $S_i(t, X(t)) := C_i(t, X(t))X(t)$.

10. The *trajectory* X of the state variable given a strategy profile S is

$$X(t+1) = \phi\left(t, X(t), S(t, X(t))\right); \; X(0) = x_0, \tag{5.1.1}$$

for some constant $\alpha \in (0,1)$ and for the state transition function

$$\phi\left(t, X(t), S(t, X(t))\right) = \left(X(t) - \sum_{i=1}^{n} S_i(t, X(t))\right)^{\alpha}. \tag{5.1.2}$$

11. The *total payoff function* of player i given strategy profile S in the game is

$$J_i\left(t_0, x_0, S\right) = \sum_{t=t_0}^{T} \beta^{(t-t_0)} \ln\left(S_i(t, X(t)\right) + \beta^{(T+1-t_0)} \ln\left(\frac{X(T+1)}{n}\right). \tag{5.1.3}$$

5.1.1 Analytic solutions

Calculation of social optimum

For the *social optimum* problem, the Bellman Eq. (1.3.7) is

$$\bar{V}(t,x) = \max_{s \in [0, \frac{x}{n}]^n} \sum_{i=1}^{n} \ln s_i + \beta \bar{V}\left(t+1, \left(x - \sum_{i=1}^{n} s_i\right)^{\alpha}\right), \tag{5.1.4}$$

the Bellman inclusion (1.3.8) is,

$$\bar{S}(t,x) \in \underset{s\in\left[0,\frac{x}{n}\right]^n}{\operatorname{Argmax}} \sum_{i=1}^{n} \ln s_i + \beta \bar{V}\left(t+1, \left(x - \sum_{i=1}^{n} s_i\right)^{\alpha}\right), \tag{5.1.5}$$

while the *terminal condition* (1.3.9) is

$$V(T+1,x) = n \ln\left(\frac{x}{n}\right). \tag{5.1.6}$$

Proposition 48 *Consider the social optimum problem for n-players.*
The unique feedback social optimum profile S^{SO} is given by

$$S_i^{\mathrm{SO}}(t,x) = c_t^{\mathrm{SO}} x, \quad \textit{for } c_t^{\mathrm{SO}} = \frac{1}{nB_t^{\mathrm{SO}}}, \tag{5.1.7}$$

where $B_t^{\mathrm{SO}} = \sum_{i=0}^{N+1} (\alpha\beta)^i$ *for* $N = T - t$,
while the value function at social optimum is given by

$$V^{\mathrm{SO}}(t,x) = n(A_t^{\mathrm{SO}} + B_t^{\mathrm{SO}} \ln x), \tag{5.1.8}$$

for $A_t^{\mathrm{SO}} = \beta^{N+1} \ln \frac{1}{n} + \sum_{i=0}^{N} \beta^{N-i} \left[(B_{T-i}^{\mathrm{SO}} - 1) \ln\left(\frac{B_{T-i}^{\mathrm{SO}} - 1}{B_{T-i}^{\mathrm{SO}}}\right) + \ln\left(\frac{1}{nB_{T-i}^{\mathrm{SO}}}\right)\right]$.

Proof: The proof follows the backward induction with respect to time t.

It can be easily verified for $t = T+1$ that the value function and social optimum strategy are as assumed.

Now, assume that the value function and social optimum strategy are as assumed for $t = k+1 \le T+1$ and check for time $t = k$.

At time $t = k$, the value function has to fulfil the Bellman Eq. (5.1.4). Calculate the derivative of the right hand side of Bellman Eq. (5.1.4) with respect to s_i and substitute $V^{\mathrm{SO}}\left(k+1, (x - \sum_{i=1}^{n} s_i)^{\alpha}\right) = n\left(A_{k+1}^{\mathrm{SO}} + B_{k+1}^{\mathrm{SO}} \ln\left(\left(x - \sum_{i=1}^{n} s_i\right)^{\alpha}\right)\right)$ to get,

$$\frac{1}{s_i} - \frac{n\alpha\beta B_{k+1}^{\mathrm{SO}}}{x - \sum_{j=1}^{n} s_j} = 0, \tag{5.1.9}$$

This is identical for all i, which implies that the optimal s_i is unique and symmetric for all i. So, solve Eq. (5.1.9) for the symmetric strategy $s_i = s$ to get

$$S_i^{\mathrm{SO}}(k,x) = \frac{x}{n(1+\alpha\beta B_{k+1}^{\mathrm{SO}})} = \frac{x}{nB_k^{\mathrm{SO}}}, \tag{5.1.10}$$

$$x - nS_i^{\text{SO}} = \left(\frac{x(B_k^{\text{SO}} - 1)}{B_k^{\text{SO}}} \right). \tag{5.1.11}$$

Substitute the values from Eq. (5.1.10)–(5.1.11) into Eq. (5.1.4) to get, $V^{\text{SO}}(k,x) = n(A_k^{\text{SO}} + B_k^{\text{SO}} \ln x)$, for the constants B_k^{SO} and A_k^{SO} as assumed. ■

Calculation of Nash equilibrium

Consider the Nash equilibrium problem. Note that the best response, instead of being a function of $\bar{s}_{\sim i}$, can be reduced to best response to its sum $O(t,x) = \sum_{j \neq i} s_j(t,x)$ only.

For the Nash equilibrium problem, given the sum of other player's strategies $O(t,x)$, the Bellman Eq. (1.3.7) is

$$V_i(t,x) = \max_{s_i \in \left[0, \frac{x}{n}\right]} \ln s_i + \beta V_i \left(t+1, (x - s_i - O(t,x))^\alpha\right), \tag{5.1.12}$$

the Bellman inclusion (1.3.8) is

$$S_i(t,x) \in \operatorname*{Argmax}_{s_i \in \left[0, \frac{x}{n}\right]} \ln s_i + \beta V_i \left(t+1, (x - s_i - O(t,x))^\alpha\right), \tag{5.1.13}$$

while the terminal condition (1.3.9) is

$$V_i(T+1,x) = \ln\left(\frac{x}{n}\right). \tag{5.1.14}$$

Proposition 49 *Consider the Nash equilibrium problem for n-players.*

The unique feedback Nash equilibrium profile S^{NE} is given by

$$S_i^{\text{NE}}(t,x) = c_t^{\text{NE}} x, \quad \textit{for } c_t^{\text{NE}} = \frac{1}{n + B_t^{\text{NE}}}, \tag{5.1.15}$$

where $B_t^{\text{NE}} = \sum_{i=1}^{N+1} (\alpha\beta)^i$ for $N = T - t$,

while the value function at a Nash equilibrium is given by

$$V_i^{\text{NE}}(t,x) = A_t^{\text{NE}} + \left(B_t^{\text{NE}} + 1\right) \ln x, \tag{5.1.16}$$

for $A_t^{\text{NE}} = \beta^{N+1} \ln\left(\frac{1}{n}\right) + \sum_{i=0}^{N} \beta^{N-i} \left[B_{T-i}^{\text{NE}} \ln\left(\frac{B_{T-i}^{\text{NE}}}{n + B_{T-i}^{\text{NE}}}\right) + \ln\left(\frac{1}{n + B_{T-i}^{\text{NE}}}\right)\right]$.

Proof: The proof follows backward induction with respect to time t and by finding the fixed point — a profile S such that each S_i is in the best response to $\sum_{j \neq i} S_j := o$.

It can be easily verified for $t = T+1$ that the value function and Nash equilibrium strategies are as assumed.

Now, assume that the value functions and and Nash equilibrium strategies are as assumed for $t = k+1 \leq T$ and check for time $t = k$.

Given the current value of the strategy of the others, define an auxiliary function of the value function for a player i as $V_i^{\mathrm{B}}(t, x, o)$ for $o = O(t, x)$ and assume that the other players will always use their Nash equilibrium strategies.

Then, $V_i^{\mathrm{NE}}(k, x) = V_i^{\mathrm{B}}(t, x, \sum_{j \neq i} S_j^{\mathrm{NE}}(k, x))$. So, the Bellman Eq. (5.1.12) becomes

$$V_i^{\mathrm{B}}(k, x, o) = \max_{s_i \in \left[0, \frac{x}{n}\right]} \ln s_i + \beta V_i^{\mathrm{NE}}(k+1, (x - s_i - o)^{\alpha}), \tag{5.1.17}$$

Since at time $t = k$, the value function has to fulfil the Bellman Eq. (5.1.17). Calculate the point of zero derivative of the right hand side of Bellman Eq. (5.1.17) with respect to s_i and substitute $V_i^{\mathrm{NE}}(k+1, (x - s_i - o)^{\alpha}) = A_{k+1}^{\mathrm{NE}} + (B_{k+1}^{\mathrm{NE}} + 1) \ln (x - s_i - o)^{\alpha}$ to get,

$$\frac{1}{s_i} - \frac{\alpha\beta(B_{k+1}^{\mathrm{NE}} + 1)}{x - s_i - o} = 0, \tag{5.1.18}$$

which simplifies to

$$B_k^{\mathrm{N}} s_i = x - \sum_{j=1}^{n} s_j. \tag{5.1.19}$$

This is identical for all players, which implies that s_i is unique and symmetric for all i. So, solve Eq. (5.1.19) for symmetric s_i with $o = (n-1)s_i$ to get,

$$S_i^{\mathrm{NE}}(k, x) \equiv \frac{x}{n + B_k^{\mathrm{NE}}} \tag{5.1.20}$$

$$x - nS_i^{\mathrm{NE}} \equiv \frac{xB_k^{\mathrm{NE}}}{n + B_k^{\mathrm{NE}}}. \tag{5.1.21}$$

Substitute the values from Eq. (5.1.20)–(5.1.21) into Eq. (5.1.17) to get,

$V_i^{\mathrm{N}}(k, x) = A_k^{\mathrm{NE}} + (B_k^{\mathrm{NE}} + 1) \ln x$, for the constants B_k^{NE} and A_k^{NE} as assumed. ■

5.1.2 Numerical solutions

Generally, the same method of dynamic programming or Bellman equation is used both in analytic and numerical approaches. However, in numerical analysis, we restrict *a priori* to symmetric solutions only.

In the numerical approach, like in the analytic approach, we use the Bellman equation to find the value function first, then the optimal solution using it either in the social optimum or the best response to the other players' strategies for the Nash equilibrium, starting from terminal time T. We do it recursively stage by stage.

Purposely, we do not use any information that the value function has a particular form (in the opposite case, it is enough to find the value of unknown coefficients numerically). We assume some theoretical assumptions in solving the problem numerically a priori that the unique solution exists, and it is symmetric. For the Nash equilibrium, we also assume monotonicity of the best responses.

For computing Nash equilibrium, starting from the terminal time, we first calculate an approximate of the value function of a player for optimization given the sum of decisions of the remaining players at this stage — which simplifies to knowing the sum of current decisions of the other players o and the best response to o, then we look for a fixed point at this stage. Subsequently, with the fixed point and the value function for the equilibrium at this stage, we switch to the previous period. This reduces the complexity. However, any method of finding a Nash equilibrium using Bellman equation is costly, since it requires solving the Bellman equation given the strategies of the others at each time instant at least the whole set of the values of the state variable that can be reached from the initial state. So, any further constraining of this set which does not spoil the solution is very welcome.

In the calculation of the value function for each stage, we approximate the continuous state space by a finite grid. In the case of calculation of equilibrium, we also need a grid for consumption of the other players o, which is also continuous.

From the fact that the value function tends to $-\infty$ as x tends to 0, as well as instantaneous payoff tends to $-\infty$ as s tends to 0, a finer grid for o is needed for small x. So, s in the social optimisation and both c and o in the computation of the Nash equilibrium are written in a form cx.

Optimization is taken for c over fixed interval $\left[0, \frac{1}{n}\right]$ or $\left[0, \frac{n-1}{n}\right]$, respectively.

Computation of social optimum

To reduce the complexity of computation, assume that all c_i are identical *a priori*, which reduces the computation of the maximum at each stage to one dimensional.

So, Eq. (5.1.4) and (5.1.5) reduces to

$$V(t,x) = \max_{c\in\left[0,\frac{1}{n}\right]} n\ln(cx) + \beta V\left(t+1, (x-ncx)^{\alpha}\right), \tag{5.1.22}$$

$$C_i(t,x) := C(t,x) \in \operatorname*{Argmax}_{c\in\left[0,\frac{1}{n}\right]} n\ln(cx) + \beta V\left(t+1, (x-ncx)^{\alpha}\right). \tag{5.1.23}$$

Take a grid for the state variable $x \in [0,1]$. The grid is not uniform — it is refined on an sub-interval and same applies to a Nash equilibrium case. The algorithm is as follows:

- From Eq. (5.1.4), compute $V^{\mathrm{SO}}(T+1,x)$ for all x in the grid of x.
- Starting from $t = T$ backwards to $t = 1$, compute $V^{\mathrm{SO}}(t,x)$ from Eq. (5.1.4) and $C_i^{\mathrm{SO}}(t,x)$ from Eq. (5.1.5) for all the grid points of x, using computed $V^{\mathrm{SO}}(t+1,\cdot)$.
- Since $(x - nxc_i)^\alpha$ is usually not a grid point of x, use interpolation: cubic (preferable) or linear. Those two are calculated in the same operation, using *fminbnd* function in Matlab.
- Using computed $C^{\mathrm{SO}}(t,x)$, starting from $t = 1$ to T, calculate $X(t)$ and $C^{\mathrm{SO}}(t,X(t))$.

Computation of Nash equilibria

Consider the symmetric equilibria only, which reduced the problem to one V and one $c_i := c$ for all players and finding c being the best response to o chosen by the others. So, Eq. (5.1.12)–(5.1.13) reduces to

$$V(t,x) = \max_{c\in\left[0,\frac{1}{n}\right]} \ln(cx) + \beta V\left(t+1,(x-cx-o)^\alpha\right), \tag{5.1.24}$$

$$C(t,x) \in \operatorname*{Argmax}_{c\in\left[0,\frac{1}{n}\right]} \ln(cx) + \beta V\left(t+1,(x-cx-o)^\alpha\right). \tag{5.1.25}$$

Besides grid for the state variable, a grid for the sum of decisions of the other players $o \in \left[0,\frac{n-1}{n}\right]$, is needed. The initial grid for o is not very fine, as its size is the main component of the cost — refine it only on small subsets. The algorithm is as below:

- From Eq. (5.1.14), compute $V^{\mathrm{NE}}(T+1,x)$ for all x in the grid of x.
- Starting from $t = T$ backwards to $t = 1$ for every grid point of x
  1. Compute an auxiliary $V^{\mathrm{B}}(t,x,o)$ from Eq. (5.1.24) and $C^{\mathrm{B}}(t,x,o)$ from Eq. (5.1.25). Do it for all grid points of o, using computed $V^{\mathrm{NE}}(t+1,\cdot,o)$.
  2. Since $(x-cx-o)^\alpha$ is usually not a grid point of x, use interpolation: cubic (preferable, since more accurate) or linear. Those two are calculated in the same operation, using *fminbnd* function in Matlab.
  3. Find $\hat{o}$ in the grid minimizing $\mid o - C^{\mathrm{B}}(t,x,o)(n-1) \mid$.

4. Take the interval of two neighbouring points of $\hat{o}$ unless $\hat{o}$ is the first or the last point of the grid — in this case, take the interval with the end in $\hat{o}$ and its neighbour in the grid. This is the new set of possible o. Divide it into a finer grid and repeat step 1 and 3 on this grid. Repeat step 1–4 with the refined grid on the new set until the distance of points in the sub-grid is of required accuracy.
5. Substitute $V^{\mathrm{NE}}(t,x) = V^{\mathrm{B}}(t,x,\hat{o})$ and $C^{\mathrm{NE}}(t,x) = C^{\mathrm{B}}(t,x,\hat{o})$.

- By using the computed $C^{\mathrm{NE}}(t,x)$, starting from $t=1$ to T, calculate $X(t)$ and $C^{\mathrm{NE}}(t,X(t))$.

5.1.3 Comparison of analytic and numerical results

Here we compare the actual results, calculated in Section 5.1.1 with the results of numerical computation according to the algorithms given in Subsection 5.1.2.

Figures are for the values of the parameters: $n=2$, $\alpha=0.6$, $\beta=\frac{1}{1.02}$, $T=10$, $t_0=1$, $x_0=0.025x^*$, for $x^*=(\alpha\beta)^{\frac{\alpha}{1-\alpha}}$ being the steady state of the infinite horizon social optimum problem. However, due to the reduction of the initial problem to Eq. (5.1.22)–(5.1.23) for the social optimum problem and to Eq. (5.1.24)–(5.1.25) for the Nash equilibrium problem, increasing n increases neither the complexity, nor the errors.

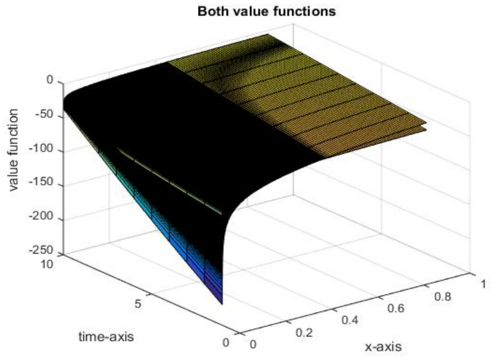
We compare the actual results to the numerical results with initial uniform grid for x of 100 points refined on the interval $\left[0,\frac{1}{2}\right]$ to about 10^4 points. For the Nash equilibrium, the number of grid points for o for each iteration is 21 while the number of iterations is 4. Intentionally, we do not increase further the number of points in the grid for state variable for very small x or $x>\frac{1}{2}$.

The social optimum

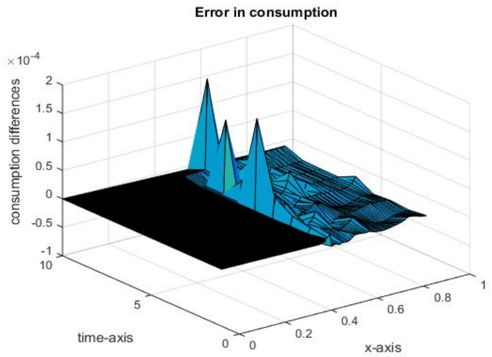
There is a substantial difference between the actual and numerical value functions (see Fig. 5.1a) for two regions of the set of states x: close to 0, the point of singularity of the actual value function, and the interval $(\frac{1}{2},1]$, at which the grid is rare.

Despite these differences, both numerical and actual consumptions are the same with an error of rank 10^{-4}, (see Fig. 5.1b, mainly at the region with rare grid).

Similarly, the optimal trajectory as well as the optimal consumption along with it, are identical (Fig. 5.2a and 5.3a). In this case the rank of error decreases to 10^{-6} (see Fig. 5.2b and 5.3b). So, there is no need to refine the grid for x at the interval $(\frac{1}{2},1]$, at which it is rare, as well as on the set of points close to singularity at 0, since neither numerical nor analytic optimal trajectory has a nonempty intersection with those sets (see Prop. 48).

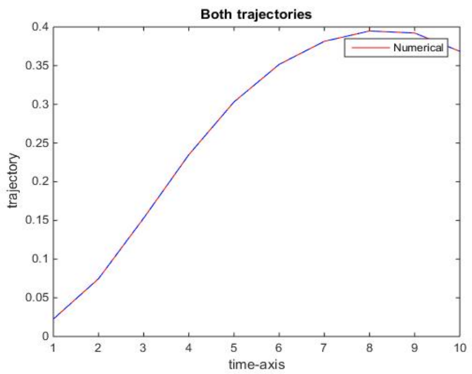


(a) Numerical and actual value functions, $V^{\mathrm{SO}}(t, x)$, for the social optimum

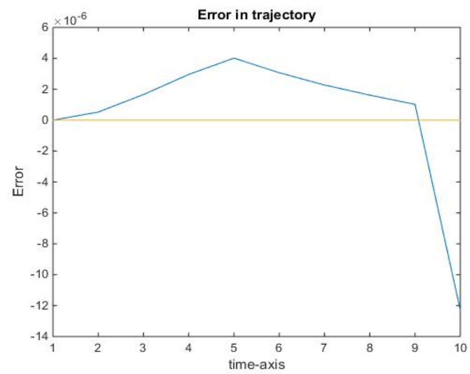


(b) The error in $S^{\mathrm{SO}}(t, x)$ (the difference between numerical and actual values) for the social optimum

Figure 5.1:

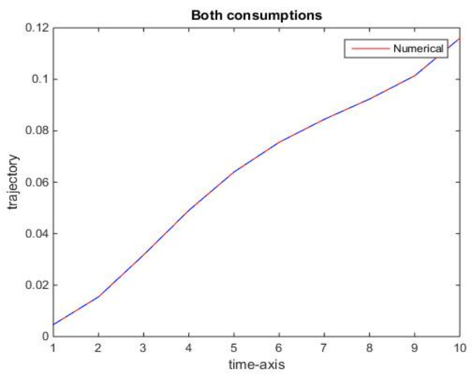


(a) The trajectories of numerical (red) and actual (blue dashed) state variable, $X(t)$, for the social optimum

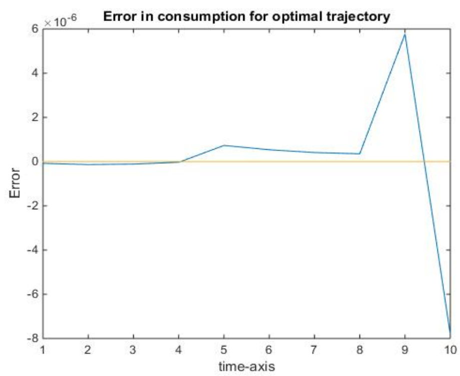


(b) The error in the trajectory of state variable, $X(t)$, for the social optimum

Figure 5.2:



(a) The optimal consumption path, $S^{\mathrm{SO}}(t, X(t))$, for the social optimum: numerical (red) and actual (blue dashed)



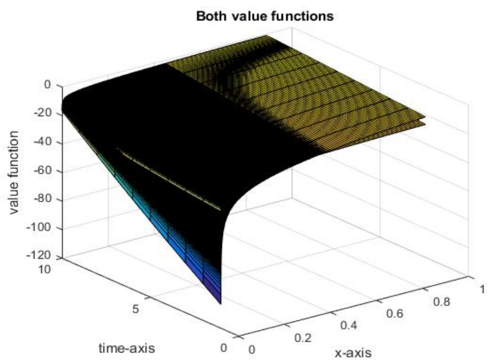
(b) The error in the optimal consumption path, $S^{\mathrm{SO}}(t, X(t))$, for the social optimum

Figure 5.3:

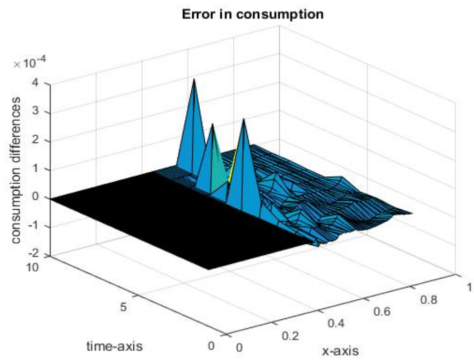
Nash equilibrium

Analogously, for the Nash equilibrium, when we compare the actual results with the results of numerical analysis, we have the same observations: the difference between the value functions (Fig. 5.4a) is large on the same regions as for the social optimum, with apparently equal consumptions, $S^{\mathrm{NE}}(t,x)$ of error of rank 10^{-4} (Fig. 5.4b), with two ranks better accuracy of Nash equilibrium consumption and state trajectories (Fig. 5.5a-5.6a) with errors of rank 10^{-6} (Fig. 5.5b and 5.6b).

Note that the rank of accuracy for computing equilibria is same as for the less complex problem of computing social optima. It means that accuracy of finding a fixed point was very high. Thanks to iterative procedure of refining the grid on a small interval, a point which we know that it contains the equilibrium, it was at a reasonable time cost (see Proposition 49).

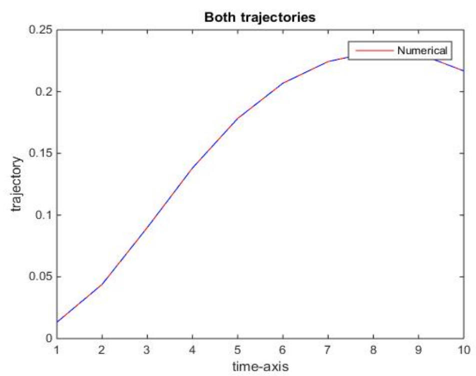


(a) Numerical and actual value functions, $V^{\mathrm{NE}}(t,x)$, for the Nash equilibrium

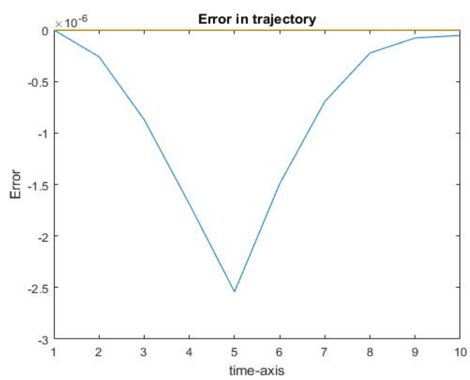


(b) The error in $S^{\mathrm{NE}}(t,x)$ (difference between numerical and actual values) for the Nash equilibrium

Figure 5.4:

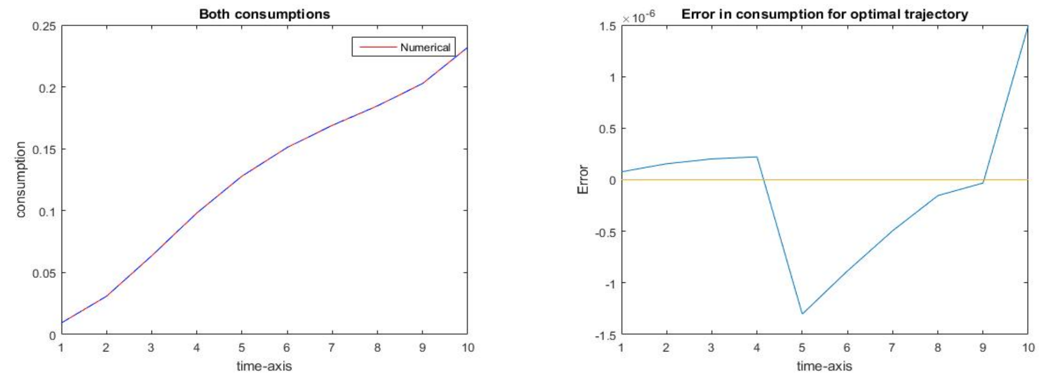


(a) The trajectories of numerical (red) and actual (blue dashed) state variable, $X(t)$, for the Nash equilibrium



(b) The error in the trajectory of the state variable, $X(t)$, for the Nash equilibrium

Figure 5.5:



(a) The optimal consumption path, $S^{\mathrm{NE}}(t, X(t))$, numerical (red) and actual (blue dashed) for the Nash equilibrium

(b) The error in the optimal consumption path, $S^{\mathrm{NE}}(t, X(t))$, for the Nash equilibrium

Figure 5.6:

5.1.4 General conclusions from the analysis of the Fish Wars example

In both cases: the social optimum and the Nash equilibrium, we obtained a substantial error in the value functions on some intervals and high accuracy of the optimum/equilibrium trajectory and consumption path. This situation is somewhat unusual.

It is worth emphasising that it is not only for specific parameters and grids which we present here graphically, but a more general rule. Even the first results obtained for the procedure of finding social optima, with quite a few points of the grid, assumed to be used to test the program, before refinement of the grid on certain subsets of the set of states was introduced, revealed the same apparent paradox. While there was a considerable error in the value function, especially at the initial time and the regions close to boundaries of the set of states, the social optimum consumption path $S(t, X(t))$ as well as $X(t)$ was computed with unexpected (for this inaccuracy of V) accuracy.

A similar paradox took place while computing the Nash equilibrium. In spite of inaccurate computation of the value function close to the boundaries, the accuracy of computing the equilibrium path was comparable to the distance between the grid points for $\tilde{s}$ — the maximal precision that can be expected.

The only information that has been used in numerical computation is the fact that the value function is continuous and increasing in x, $V(t, 0) = -\infty$, the optimal trajectory remains below a certain level, (we took $\frac{1}{2}$) whenever the initial condition x_0 is below this level, and it is over some small $\epsilon > 0$ whenever x_0 is, while in computation of equilibrium, we also use the facts that the best response of a player is a decreasing function of joint consumption of the others, the equilibrium exists and it is unique.

5.2 Extension to arbitrary discrete time dynamic optimization problems and dynamic games

Here we prove the theorems, which allow, in a very general environment of dynamic optimization problems, to assess whether a certain kind of error in approximation of the value function — either resulting from using numerical computation with low accuracy on some sets, or from replacing the actual value function by some a priori estimation of it on some sets.

Consider any discrete time dynamic optimisation problem from Chapter 1 either in a finite time horizon T or infinite time horizon.

We assume that the optimisation problem is such that J is always well defined although it may be $\pm\infty$ and it is bounded from above.

In the infinite time horizon version of motivating example, the standard terminal condition (1.3.10) does not hold, so it has to be replaced by a weaker one (1.3.11). We introduced the following notations for the dynamic optimization problem:

- V — the actual value function;
- V^{approx} — another function regarded as an approximation of V; it may be either a solution of a numerical procedure or the actual V with values on certain subsets replaced by other values, e.g. some constraints known *a priori*.
- $\text{RHS}_{t,x}(s) = P(t,x,s) + \beta V(t+1, \phi(t,x,s))$ — the maximized function in the right hand side of the Bellman Eq. (1.3.7).
- $\text{RHS}^{\text{approx}}_{t,x}(s) = P(t,x,s) + \beta V^{\text{approx}}(t+1, \phi(t,x,s))$ — the maximized function in the right hand side of the Bellman Eq. (1.3.7) with V replaced by V^{approx}
- OPT — the set of optimal controls; we assume that it is nonempty.
- $\text{OPT}^{\text{approx}}$ — the set of controls $\tilde{S} \in \mathcal{U}$ such that $\tilde{S}(t,x) \in \operatorname*{Argmax}_{s\in\mathbb{U}} \text{RHS}^{\text{approx}}_{t,x}(s)$; we assume that it is nonempty.
- $\Omega = \{(t,x) : X^{\bar{S}}_{t_0,x_0}(t) = x \text{ for some } \bar{S} \in \text{OPT and } x_0 \in \mathbb{X}_0\}$.
- For $\bar{S} \in \mathcal{U}$, $\Omega^{\bar{S}} = \{(t,x) : X^{\bar{S}}_{x_0,S}(t) = x \text{ for some } x_0 \in \mathbb{X}_0\}$.
- $\Omega^{\text{approx}} = \{(t,x) : X^{\tilde{S}}_{t_0,x_0}(t) = x \text{ for some } \tilde{S} \in \text{OPT}^{\text{approx}} \text{ and } x_0 \in \mathbb{X}_0\}$.

The following theorem explains the apparent paradox of low errors in the computation of the optimal control path despite substantial errors in the computation of the value function in our motivating example.

Theorem 50 *Assume that either the horizon is finite or the terminal condition* (1.3.11) *holds for* V*.*

Assume also one of the following assumption holds:

(i) $V(t,x) = V^{\text{approx}}(t,x)$ *for all* $(t,x) \in \Omega^{\text{approx}}$*, and* $V(t,x) \leq V^{\text{approx}}(t,x)$ *for all* $(t,x) \in \mathbb{T} \times \mathbb{X}$*;*

(ii) $V(t,x) = V^{\text{approx}}(t,x)$ *for all* $((t,x)) \in \Omega$*, and* $V(t,x) \geq V^{\text{approx}}(t,x)$ *all* $(t,x) \in \mathbb{T} \times \mathbb{X}$*;*

(iii) $V(t,x) = V^{\text{approx}}(t,x)$ *for all* $(t,x) \in \Omega \cup \Omega^{\text{approx}}$*.*

Then $\Omega = \Omega^{\text{approx}}$*, for every* $\bar{S} \in \mathrm{OPT}$ *there exist* $\tilde{S} \in \mathrm{OPT}^{\text{approx}}$ *such that* $\tilde{S}\mid_{\Omega} = \bar{S}\mid_{\Omega}$ *and for every* $\tilde{S} \in \mathrm{OPT}^{\text{approx}}$ *there exist* $\bar{S} \in \mathrm{OPT}$ *such that* $\tilde{s}\mid_{\Omega} = \bar{c}\mid_{\Omega}$ *.*

To prove Theorem 50, following lemmas first needed to be proven.

Lemma 51 *Consider an arbitrary set* $\mathbb{U}$ *and two functions* $f, g : \mathbb{U} \to \bar{\mathbb{R}}$ *with* $f(s) = g(s)$ *for all* $s \in \operatorname{Argmax} f \neq \emptyset$ *and* $f(s) \geq g(s)$ *otherwise.*

Then $\operatorname{Argmax} f = \operatorname{Argmax} g$*.*

Proof: Take $\bar{s} \in \operatorname{Argmax} f$ and any other s.

By the assumptions, $g(s) \leq f(s) \leq f(\bar{s}) = g(\bar{s})$. So, $\bar{s} \in \operatorname{Argmax} g$.

Next, consider $\tilde{s} \in \operatorname{Argmax} g$ and assume that $\tilde{s} \notin \operatorname{Argmax} f$. Take $\bar{s} \in \operatorname{Argmax} f$.

By the assumptions, $g(\bar{s}) \leq g(\tilde{s}) \leq f(\tilde{s}) < f(\bar{s}) = g(\bar{s})$, which is a contradiction. ■

Lemma 52 *Consider an arbitrary set* $\mathbb{U}$ *and two functions* $f, g : \mathbb{U} \to \bar{\mathbb{R}}$ *with* $f(s) = g(s)$ *for all* $s \in \operatorname{Argmax} f \cup \operatorname{Argmax} g$ *with both* $\operatorname{Argmax} f, \operatorname{Argmax} g \neq \emptyset$*.*

Then $\operatorname{Argmax} f = \operatorname{Argmax} g$*.*

Proof: Consider $\tilde{s} \in \operatorname{Argmax} g$ and $\bar{s} \in \operatorname{Argmax} f$.

By the assumptions, $f(\tilde{s}) \leq f(\bar{s}) = g(\bar{s}) \leq g(\tilde{s}) = f(\tilde{s})$, which implies that $f(\tilde{s}) = f(\bar{s})$ and $g(\bar{s}) = g(\tilde{s})$. So, $\tilde{s} \in \operatorname{Argmax} f$ and $\bar{s} \in \operatorname{Argmax} g$. ■

Proof: (of Theorem 50)

First, note that

$$\text{if } (t,x) \in \Omega, \text{ then for all } s \in \operatorname{Argmax} \mathrm{RHS}_{t,x}(t+1, \phi(t,x,s)) \in \Omega \tag{5.2.1}$$

and if $(t,x) \in \Omega^{\text{approx}}$, then for all $s \in \operatorname{Argmax} \mathrm{RHS}_{t,x}^{\text{approx}}(t+1, \phi(t,x,s)) \in \Omega^{\text{approx}}$. (5.2.2)

The proof follows the induction over t.

Together with $\Omega = \Omega^{\text{approx}}$, it is proved that $\operatorname{Argmax} \mathrm{RHS}_{t,x} = \operatorname{Argmax} \mathrm{RHS}_{t,x}^{\text{approx}}$ for all $(t,x) \in \Omega$ (and, consequently, for all $(t,x) \in \Omega^{\text{approx}}$).

Consider $t = t_0$ and arbitrary $x \in \mathbb{X}_0$.

In this case, $\Omega \cap \{(t,y) \in \mathbb{T} \times \mathbb{X} : t \le t_0\} = \{t_0\} \times \mathbb{X}_0 = \Omega^{\text{approx}} \cap \{(t,x) \in \mathbb{T} \times \mathbb{X} : t \le t_0\}$ in all the cases (i)–(iii).

Next, consider any $t \ge t_0$ and any x such that $(t,x) \in \Omega$.

Assume that $\Omega \cap \{(k,y) \in \mathbb{T} \times \mathbb{X} : k \le t\} = \Omega^{\text{approx}} \cap \{(k,y) \in \mathbb{T} \times \mathbb{X} : k \le t\}$.

Use Lemma 51 applied to the functions $\mathrm{RHS}_{t,x}^{\text{approx}}$ and $\mathrm{RHS}_{t,x}$, and Eq. (5.2.2)–(5.2.1), respectively to get,

$\operatorname{Argmax} \mathrm{RHS}_{t,x} = \operatorname{Argmax} \mathrm{RHS}_{t,x}^{\text{approx}}$ in cases (i) and (ii).

Again, use Lemma 52 applied to the functions $\mathrm{RHS}_{t,x}^{\text{approx}}$ and $\mathrm{RHS}_{t,x}$, and any of Eq. (5.2.2) or Eq. (5.2.1) to get,

$\operatorname{Argmax} \mathrm{RHS}_{t,x} = \operatorname{Argmax} \mathrm{RHS}_{t,x}^{\text{approx}}$ in case (iii).

Consequently, in all the cases (i)–(iii), $\Omega \cap \{(k,y) \in \mathbb{T} \times \mathbb{X} : k \le t+1\} = \Omega^{\text{approx}} \cap \{(k,y) \in \mathbb{T} \times \mathbb{X} : k \le t+1\}$.

This ends the proof that $\Omega = \Omega^{\text{approx}}$ and that $\operatorname{Argmax} \mathrm{RHS}_{t,x} = \operatorname{Argmax} \mathrm{RHS}_{t,x}^{\text{approx}}$ for all $(t,x) \in \Omega$.

Take any $\tilde{S} \in \mathrm{OPT}^{\text{approx}}$. Define $\bar{S} \in \mathcal{U}$ such that $\bar{S}(t,x) = \tilde{S}(t,x)$ for all $(t,x) \in \Omega$ and $\bar{S}(t,x)$ being any selection from $\operatorname{Argmax} \mathrm{RHS}_{t,x}$ otherwise.
The terminal condition either (1.3.10) or (1.3.11) is fulfilled by assumption.

By the fact that $\operatorname{Argmax} \mathrm{RHS}_{t,x} = \operatorname{Argmax} \mathrm{RHS}_{t,x}^{\text{approx}}$, also the Bellman equation is fulfilled, so $\bar{S} \in \mathrm{OPT}$.

Next, take any $\bar{S} \in \mathrm{OPT}$. Define $\tilde{S} \in \mathcal{U}$ such that $\tilde{S}(t,x) = \bar{S}(t,x)$ for all $(t,x) \in \Omega$ and $\tilde{S}(t,x)$ being any selection from $\operatorname{Argmax} \mathrm{RHS}_{t,x}^{\text{approx}}$ otherwise. Since $\operatorname{Argmax} \mathrm{RHS}_{t,x} = \operatorname{Argmax} \mathrm{RHS}_{t,x}^{\text{approx}}$, by the definition, $\bar{S} \in \mathrm{OPT}^{\text{approx}}$. ■

Theorem 53 *Assume that the terminal condition* (1.3.11) *holds for V.*

Consider a control $S \in \mathcal{U}$.

Assume also that one of the following assumption holds:

(i) If $S \in \mathrm{OPT}^{\text{approx}}$ and $V = V^{\text{approx}}$ for all $(t,x) \in \Omega^S$, and $V \le V^{\text{approx}}$ for all $(t,x) \in \mathbb{T} \times \mathbb{X}$, then there exist $\bar{S} \in \mathrm{OPT}$ such that $\Omega^{\bar{S}} = \Omega^S$ and $S|_{\Omega^S} = \bar{S}|_{\Omega^S}$.

(ii) If $S \in \mathrm{OPT}$ and $V = V^{\text{approx}}$ for all $(t,x) \in \Omega^S$, and $V \ge V^{\text{approx}}$ for all $(t,x) \in \mathbb{T} \times \mathbb{X}$, then there exist $\tilde{S} \in \mathrm{OPT}^{\text{approx}}$ such that $\Omega^{\tilde{S}} = \Omega^S$ $\tilde{S}|_{\Omega} = S|_{\Omega}$.

Proof: Analogous to the proof of Theorem 50, with concentrating only on a single S from OPT or $\mathrm{OPT}^{\mathrm{approx}}$. ■

5.2.1 Illustration of usefulness of Theorems 50 and 53 by examples

Proposition 54 *Consider the Fish Wars game from section 5.1 but with $T = +\infty$ and strategies not directly dependent on t.*
(a) The social optimum strategy is given by $\bar{S}^{\mathrm{SO}}(x) = \frac{x(1-\alpha\beta)}{n}$,
with the value function $\bar{V}^{\mathrm{SO}}(x) = n(\bar{A}^{\mathrm{SO}} + \bar{B}^{\mathrm{SO}}\ln(x))$, for
$\bar{A}^{\mathrm{SO}} = \frac{1}{1-\beta}\left(\frac{\alpha\beta}{1-\alpha\beta}\ln(\alpha\beta) + \ln(1-\alpha\beta) - \ln(n)\right)$ *and* $\bar{B}^{\mathrm{SO}} = \frac{1}{1-\alpha\beta}$.
(b) The Nash equilibrium strategy is given by $\bar{S}^{\mathrm{NE}}(x) = \left(\frac{1-\alpha\beta}{n(1-\alpha\beta)+\alpha\beta}\right)$,
with the value function $\bar{V}^{\mathrm{NE}}(x) = \bar{A}^{\mathrm{NE}} + \bar{B}^{\mathrm{NE}}\ln(x)$, for
$\bar{A}^{\mathrm{NE}} = \frac{1}{1-\beta}\left(\frac{\alpha\beta}{1-\alpha\beta}\ln\left(\frac{\alpha\beta}{n(1-\alpha\beta)+\alpha\beta}\right) + \ln\left(\frac{1-\alpha\beta}{n(1-\alpha\beta)+\alpha\beta}\right)\right)$ *and* $\bar{B}^{\mathrm{NE}} = \frac{1}{1-\alpha\beta}$.

Proof: The formulae have been proposed by Levhari, Mirman [82] and Okuguchi [93]. It can be easily checked by substitution that the Bellman Eq. (1.3.7) holds which was not done in [82] and [93]. The proofs in [82] and [93] also lack checking the terminal condition.

So, to complete the proof, check the weaker terminal condition given by Eq. (1.3.11). one part is immediate by the fact that $\ln(s_i x) \le 0$.

To prove another part for the social optimum, consider a profile of strategies S for which $\limsup_{t\to\infty} \beta^t \bar{V}^{\mathrm{SO}}(X^S(t)) < 0$. So, there exists a sub-sequence t_k such that $\lim_{k\to\infty} \beta^{t_k}\bar{V}^{\mathrm{SO}}(X^S(t_k)) < 0$. So, $\lim_{k\to\infty} \beta^{t_k}\ln(X^S(t_k)) < 0$.
$\sum_{i=1}^{n} J_i(S) = \sum_{i=1}^{n}\sum_{t=t_0}^{\infty} \beta^{t-t_0}\ln(S_i(X^S(t))X^S(t)) \le \sum_{i=1}^{n}\sum_{k=0}^{\infty} \beta^{t_k-t_0}\ln(S_i(X^S(t_k))X^S(t_k))$
$\le \sum_{i=1}^{n}\sum_{k=0}^{\infty} \beta^{t_k-t_0}\ln(X^S(t_k)) \to -\infty$.
The proof for the Nash equilibrium is analogous. ■

Example 1 *Assume that some preliminary analysis done for the problem resulted in finding an $\epsilon < x_0$ for which we know that the optimal trajectory $X(t) > \epsilon$ for all t (such an ϵ obviously exists).*

Changing V by assigning $V^{\mathrm{approx}} = -\infty$ for all $x < \frac{\epsilon}{2}$ (see Fig. 5.7) changes neither the optimal trajectory nor the optimal control path.

So, if we want to compute the optimal control, this substitution allows us to look for the social optimum and the value function for $x \ge \frac{\epsilon}{2}$ only and to avoid problems resulting from inaccuracies resulting from closeness to the actual singularity.

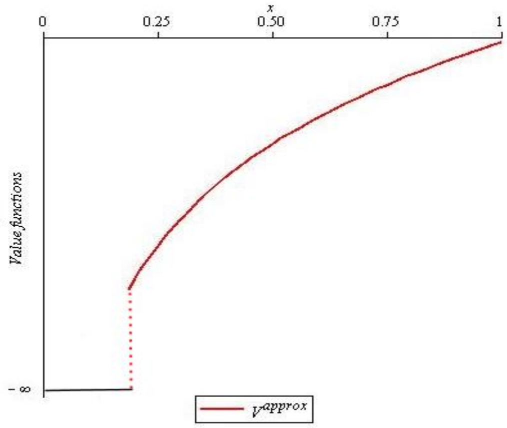


Figure 5.7: Intentional underestimation of the value function on some interval not influencing the optimal state and consumption trajectories in Example 1

Example 2 *Assume again that some preliminary analysis done for the problem resulted in finding an $\epsilon < x_0$ for which we know that the optimal trajectory $X(t) > \epsilon$ for all t.*

Since $V(x) \geq J(t_0, x, S)$ for every control S, changing V by assigning $V^{\text{approx}}(x) = J(t_0, x, \bar{S})$ for all $x < \frac{\epsilon}{2}$, for any control $\bar{S}$ changes neither the optimal trajectory nor the optimal control path.

So, if we want to compute the optimal control, this substitution allows us to look for the value function for $x \geq \frac{\epsilon}{2}$ only (and the resulting optimal control).

Example 3 *Assume that preliminary analysis done for the problem resulted in finding constants a and b for which we know that $ax+b$ is an upper bound for the value function for $x > \frac{1}{2}$ (see Fig. 5.8) and in discovering the fact that if $x_0 \leq \frac{1}{2} - \epsilon$, then for all t, the optimal trajectory $X^{\bar{S}}_{t_0,x_0}(t) \leq \frac{1}{2} - \epsilon$.*

If we change V by assigning $V^{\text{approx}}(x) = ax + b$ for all $x > \frac{1}{2}$, and calculate the maxima $\tilde{S}(t,x)$ of the right hand side of the Bellman equation with V^{approx}, then if the trajectory $X^{\tilde{S}}_{t_0,x_0}(t) < \frac{1}{2} - \epsilon$ for all t, then $\tilde{S}(t, X^{\tilde{S}}_{t_0,x_0}(t))$ is the accurate solution path.

So, if we want to compute the optimal control, this substitution allows us to restrict the computation of the value function for $x < \frac{1}{2} - \epsilon$ with one restriction: our results are really the optimal control only when for all $x_0 < \frac{1}{2} - \epsilon$, the computed $\tilde{S}$ is such that $X^{\tilde{S}}_{t_0,x_0}(t) < \frac{1}{2} - \epsilon$ for all t.

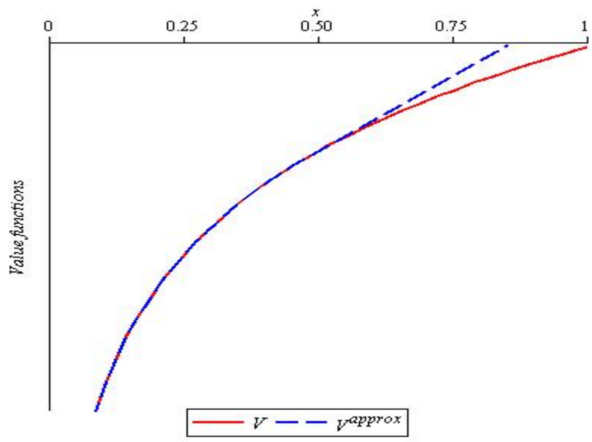


Figure 5.8: Intentional overestimation of the value function on some interval not influencing the optimal state and consumption trajectories in Example 3

Chapter 6

Fish Wars model in the continuous time — Optima, equilibria, enforcement and delayed observation

In this chapter, we consider the Fish War model similar to Chapter 5, but in continuous time and with the linear dynamics. We assume that the game does not directly depend on time, so, we skip t whenever it does not lead to the confusion. So, the differential game consists of:

1. A set of finitely many players $\mathbb{I} = \{1, 2 \ldots, n\}$.

2. A *time set* $\mathbb{T} = \mathbb{R}_+$ is continuous with the infinite time horizon and the initial time $t_0 = 0$.

3. The *state of the resource* is $x \in \mathbb{X} = \mathbb{R}_+$, denoting the *biomass of fish*.

4. At each time instant, country i extracts or consume $s_i := c_i x$, where s_i denotes the *cath* or *consumption* for c_i being the *cath rate* or *consumption coefficient*.

5. The set of decisions of each player is $\mathbb{D} = \mathbb{R}_+$.

6. There are linear state dependent constraints on decisions. So, at state x, $S_i(t, x) \in D_i(x)$ for $D_i(x) = [0, Mx]$ for a constant $M > 0$.

7. The *current or instantaneous payoff function* of country i for given s_i is

$$P_i(x, s_i) := \ln(s_i) = \ln(c_i x), \tag{6.0.1}$$

   with $\ln(0) = -\infty$.

8. Payoffs are discounted by a *discount factor* $\beta = \exp^{-r}$ for the constant $r \in (0, 1)$.

9. We are interested in calculating the feedback strategies, $S_i : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{D}_i$ for $S_i(X(t)) := C_i(X(t))X(t)$. Then $S : \mathbb{R}_+ \to \mathbb{R}^n_+$ defines a profile of strategies. The symbol $\mathcal{U}$ denotes the set of controls of player i, so, $\mathcal{U}^n$ is the set of profiles.

10. Since for feedback strategies, the results for the optimization over $S_i(x)$ are equivalent to the dynamic optimization over $C_i(x) = \frac{S_i(x)}{x}$, for $x \neq 0$ and arbitrary for $x = 0$, for simplicity of calculation, we consider the control parameter c_i instead of s_i throughout in this chapter.

11. The *trajectory* X of the state variable, given a strategy profile S is,

$$\dot{X}(t) = \phi\left(X(t), S(X(t))\right);\ X(0) = x_0, \tag{6.0.2}$$

for $\xi > 0$ and for the state transition function

$$\phi(X(t), S) := \left(\xi x - \sum_{i=1}^{n} s_i\right) = \left(\xi - \sum_{i=1}^{n} c_i\right) x. \tag{6.0.3}$$

12. *Total payoff function* of player i for a strategy profile S in the game is

$$J_i\left(x_0, S\right) = \int_{t=0}^{\infty} e^{-rt} P_i(X(t), S_i(X(t)))dt \text{ for } i = 1, 2, \cdots n. \tag{6.0.4}$$

and for X given by Eq. (6.0.2).

Analogously, $J_i\left(x, S\right)$ can be defined for arbitrary initial x.

6.1 Calculation of optima and equilibria

6.1.1 Social Optimum

First we consider the social optimum problem.

Theorem 55 *The unique feedback social optimum profile is given by*

$$C_i^{\text{SO}}(x) = c_i^{\text{SO}} := \frac{r}{n}, \tag{6.1.1}$$

while the value function for all players is given by

$$V^{\text{SO}}(x) := A^{\text{SO}} + B^{\text{SO}} \ln(x),\ \textit{for} \tag{6.1.2}$$

$$A^{\text{SO}} = \frac{1}{r}\left(n \ln\left(\frac{r}{n}\right) + \frac{n\xi}{r} - n\right),\ \textit{and}\ B^{\text{SO}} = \frac{n}{r}.$$

At this social optimum, the value function of player i is defined as

$$V_i^{\text{SO}}(x) := \frac{V^{\text{SO}}(x)}{n}. \tag{6.1.3}$$

Proof: It is enough to prove Eq. (6.1.1)–(6.1.2) on the restricted state set $(0, +\infty)$, since 0 cannot be reached from a positive initial state while $V^{\mathrm{SO}}(0) = -\infty$, whatever $S_i^{\mathrm{SO}}(0)$ is, it influences neither the current nor the total payoff. Use Theorem 7: for this model, the HJB Eq. (1.3.16) is

$$rV(x) = \max_{c\in[0,x]^n} \left(\sum_{i=1}^{n} \ln(c_i x)\right) + \left(\xi - \sum_{i=1}^{n} c_i\right) x \frac{\partial V(x)}{\partial x}. \tag{6.1.4}$$

Calculate the point of zero derivative of Eq. (6.1.4) with respect to c_i to get optimal c_i defined by

$$\frac{1}{\bar{c}_i} = \left(x \cdot \frac{\partial V(x)}{\partial x}\right), \; i = 1, 2 \cdots n. \tag{6.1.5}$$

This is identical for all i, therefore, all $\bar{c}_i$ are symmetric and given by Eq. (6.1.5) for M large enough. Substitute $c_i = \bar{c}_i$ into Eq. (6.1.4) and solve for $V(x)$ to get

$$rV(x) = n \ln(\bar{c}_i x) + (\xi - n\bar{c}_i)\, x \cdot \frac{\partial V(x)}{\partial x}. \tag{6.1.6}$$

The logarithmic structure of the optimization problem suggests that the value function is of logarithmic form. Therefore, assume the form of the value function $V(x)$ as

$$V(x) = A^{\mathrm{SO}} + B^{\mathrm{SO}} \ln x, \tag{6.1.7}$$

so that a logarithmic equation in the state x results. Since this equation has to hold for all x, the coefficients of $\ln x$ and the constant term on the left hand side and the right hand side have to be equal, which gives

$$A^{\mathrm{SO}} = \frac{1}{r}\left(n \ln\left(\frac{r}{n}\right) + \frac{n\xi}{r} - n\right), \text{ and } B^{\mathrm{SO}} = \frac{n}{r}.$$

Substitute B^{SO} in Eq. (6.1.5), to get the social optimum catch rate

$$C_i^{\mathrm{SO}}(x) = c_i^{\mathrm{SO}} = \frac{r}{n}. \tag{6.1.8}$$

To prove the terminal condition $\limsup_{t\to\infty} V(x)e^{-rt} = 0$, first note that $\xi X(t) \geq \dot{X}(t) \geq (\xi - nM)X(t)$ which implies that $x_0 e^{\xi t} \geq X(t) \geq x_0 e^{(\xi-nM)t}$. Since $V(x)$ is increasing, $V(x_0 e^{\xi t})e^{-rt} \geq V(x)e^{-rt} \geq V(x_0 e^{(\xi-nM)t})e^{-rt}$, i.e.,

$$\left(n \ln \frac{r x_0 e^{\xi t}}{n} + \frac{n\xi}{r} - n\right) \frac{e^{-rt}}{r} \geq V(x)e^{-rt} \geq \left(n \ln \frac{r x_0 e^{(\xi-nM)t}}{n} + \frac{n\xi}{r} - n\right) \frac{e^{-rt}}{r}.$$

The limits for $t \to \infty$, of both the first and last expression are equal to 0. So,

$0 \geq \lim_{t\to\infty} V(X(t))e^{-rt} \geq 0$. Therefore, $\limsup_{t\to\infty} V(X(t))e^{-rt} = 0$. Hence, the value function is given by Eq. (6.1.2) and the unique symmetric social optimum strategy is given by Eq. (6.1.1). ■

6.1.2 Nash Equilibrium

Next, we consider the Nash equilibrium problem.

At a Nash equilibrium for every i, C_i is the best response to $C_{\sim i}$.

Theorem 56 *The symmetric feedback Nash equilibrium profile is given by*

$$C_i^{\mathrm{NE}}(x) = c_i^{\mathrm{NE}} := r, \tag{6.1.9}$$

while the value function of player i is given by

$$V_i^{\mathrm{NE}}(x) := A_i^{\mathrm{NE}} + B_i^{\mathrm{NE}} \ln(x), \text{ for} \tag{6.1.10}$$

$$A_i^{\mathrm{NE}} = \frac{1}{r}\left(\ln r + \frac{\xi}{r} - n\right), B_i^{\mathrm{NE}} = \frac{1}{r}.$$

Proof: By analogous reasoning as in the social optimum problem, restrict the state space to $(0, +\infty)$.

Define an auxiliary function $V_i : \mathbb{R}_+ \times \mathcal{C}^{n-1} \to \mathbb{R}$ given by

$$V_i(x, C_{\sim i}) = \max_{C_i \in \mathcal{C}} J_i(x, C) \tag{6.1.11}$$

Note that whenever the remaining players use their Nash equilibrium strategies $C_{\sim i}^{\mathrm{NE}}$,

$$V_i^{\mathrm{NE}}(x) = V_i\left(x, C_{\sim i}^{\mathrm{NE}}\right). \tag{6.1.12}$$

Use Theorem 7. In this model, for given $C_{\sim i}$, the HJB Eq. (1.3.16) is

$$rV_i(x, C_{\sim i}) = \max_{c_i \in [0,M]} \left(\ln(c_i x) + \left(\xi - c_i - \sum_{j \neq i}^{n} C_j(x) \right) x \cdot \frac{\partial V_i(x, C_{\sim i})}{\partial x} \right). \tag{6.1.13}$$

Calculate the point of zero derivative of Eq. (6.1.13) with respect to c_i to get optimal c_i defined by the equation,

$$\frac{1}{c_i^*} = \left(x \cdot \frac{\partial V_i(x, C_{\sim i})}{\partial x} \right). \tag{6.1.14}$$

The logarithmic structure of the optimization problem suggests that the value function is of logarithmic form. Therefore, assume the form of the value function $V_i(x, C_{\sim i})$ as

$$V_i(x, C_{\sim i}) = A_i(C_{\sim i}) + B_i(C_{\sim i}) \ln x, \ i = 1, 2, \cdots, n; \tag{6.1.15}$$

so that a logarithmic equation in the state x results. Eq. (6.1.14) can be rewritten as

$$\frac{1}{c_i^*} = B_i(C_{\sim i}) \tag{6.1.16}$$

by comparing the coefficients in Eq. (6.1.13) and whatever $C_{\sim i}$ are, $B_i(C_{\sim i}) = \frac{1}{r}$. Denote it by B_i^{NE}. Consequently, all $C_i^*(x) = c_i^* = r$, which implies that also all $A_i(C_{\sim i})$ are equal. So the HJB equation for $V_i(x)$ simplifies to:

$$rV_i(x, C_{\sim i}) = \ln(c_i^* x) + (\xi - nc_i^*)\, x \cdot \frac{\partial V_i(x, C_{\sim i})}{\partial x}. \tag{6.1.17}$$

Since this equation has to hold for all x, the coefficients of $\ln x$ and the constant term on the left hand side and the right hand side will have to be equal, which gives

$$A_i(C_{\sim i}) = \frac{1}{r}\left(\ln r + \frac{\xi}{r} - n\right).$$

Denote it by A_i^{NE}. To prove the terminal condition $\limsup_{t\to\infty} V_i(x)e^{-rt} = 0$, first note that $\xi X(t) \geq \dot{X}(t) \geq (\xi - nM)X(t)$, so, $x_0 e^{\xi t} \geq X(t) \geq x_0 e^{(\xi - nM)t}$. Since $x > 0$, $V_i(x_0 e^{\xi t}, C_{\sim i}^*)e^{-rt} \geq V_i(x, C_{\sim i}^*)e^{-rt} \geq V_i(x_0 e^{(\xi - nM)t})e^{-rt}$, which implies that

$$\left(\ln(rx_0 e^{\xi t}) + \frac{\xi}{r} - n\right)\frac{e^{-rt}}{r} \geq V_i(x, C_{\sim i}^*)e^{-rt} \geq \left(\ln(rx_0 e^{(\xi - nM)t}) + \frac{\xi}{r} - n\right)\frac{e^{-rt}}{r}.$$

The limits for $t \to \infty$, of both the first and the last expression are equal to 0. So, $0 \geq \lim_{t\to\infty} V_i(x, C_{\sim i}^*)e^{-rt} \geq 0$. Therefore, $\limsup_{t\to\infty} V_i(x, C_{\sim i}^*)e^{-rt} = 0$. Hence, for given $C_{\sim i}^*$, the auxiliary function $V_i(\cdot, C_{\sim i}^*)$ is equal to the value function of the optimization problem of player i

Therefore, the symmetric feedback Nash equilibrium strategy is given by Eq. (6.1.9), while the value functions of player i at the Nash equilibrium is given by Eq. (6.1.10). ■

6.1.3 Comparison between the social optima and the Nash equilibria

Here, we compare, also graphically, the behaviour of various model variables at social optima and Nash equilibria for the different values of the parameters. Graphical comparison of the catch rates is skipped since it is obvious.

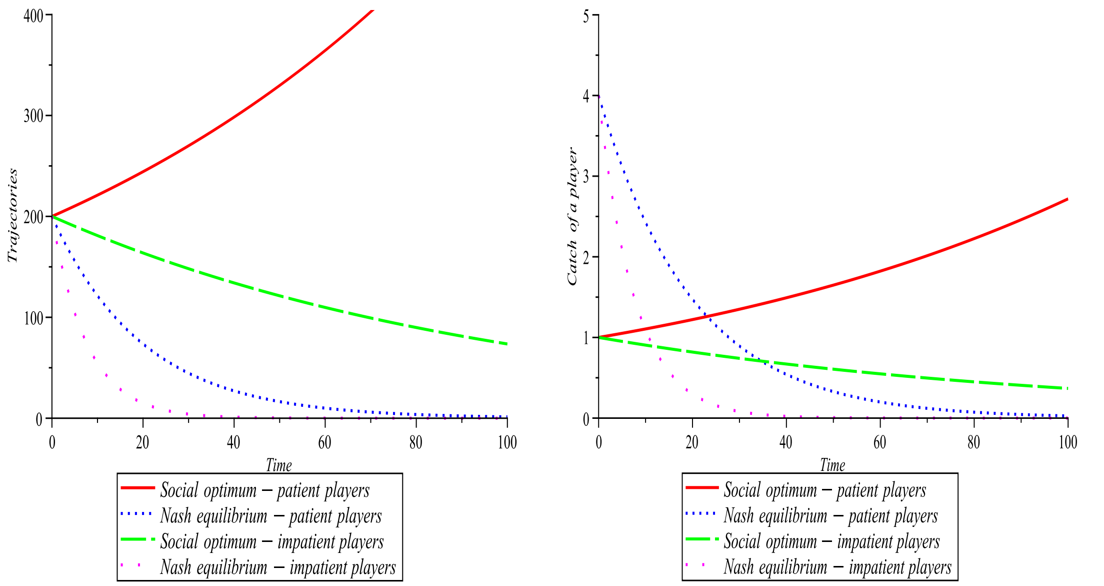
Fig. 6.1a–6.2b are for the values of the parameters as: $\xi = 0.03$, $n = 4$ and $x_0 = 200$. There are two different values of r: for patient players $r = 0.02$ while for impatient players $r = 0.04$.

Note that, generally, there can be three situations concerning sustainability at social optima: for $\xi > r$ (related to as *patient players*) the state trajectory then converges to $+\infty$ as $t \to \infty$, for $\xi = r$ it is constant, while for $\xi < r$ (*impatient players*) it converges to 0.

Usually, in the real world r and ξ are close to each other, so for Nash equilibria, especially for n players, the trajectory of the biomass always converges to 0.

Note that, while for impatient players, the resource is sustainable neither at the social optimum nor at the Nash equilibrium. In the remaining cases, there is sustainability at social optima while Nash equilibria always result in depletion of the resource, which we can see in Fig. 6.1a.

In the subsequent figures, we compare graphically relations between trajectories, catches, current payoffs and accumulated payoffs from a time instant on for social optima and Nash equilibria, for patient and impatient players.



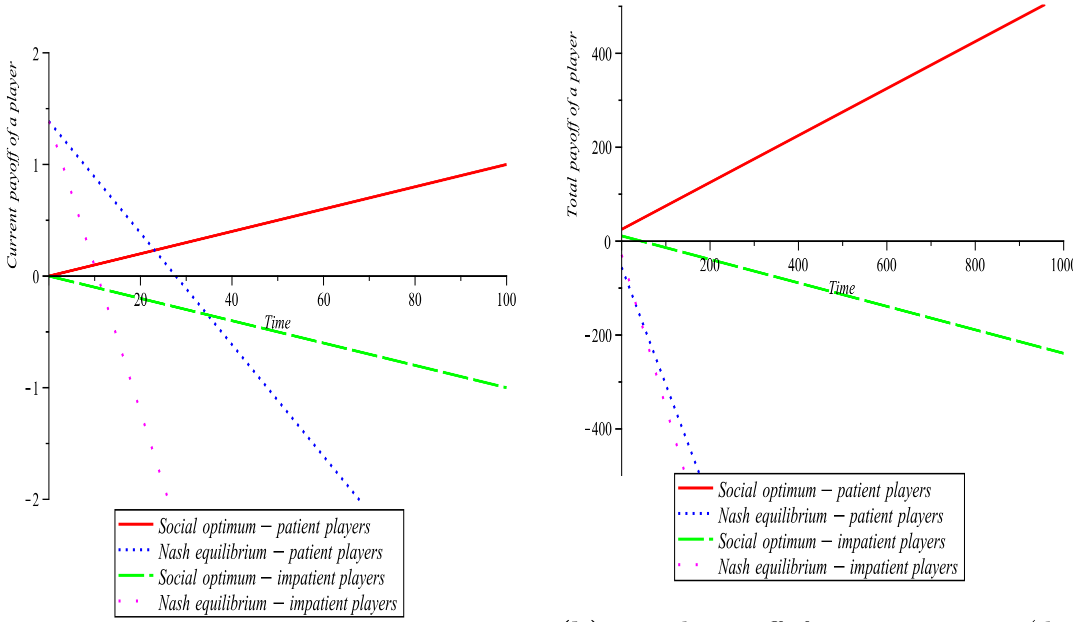
(a) Trajectories of the state for social optima and Nash equilibria, for patient and impatient players

(b) The catch over time for social optima and Nash equilibria, for patient and impatient players

Figure 6.1

In Fig. 6.1b and 6.2a, it can be easily seen that the catch at a Nash equilibrium tends to 0 and the current payoff tends to $-\infty$ as time tends to infinity, while for patient players the social optimum catch and current payoff tends to $+\infty$. Note that the Nash equilibrium catch and payoff are initially higher than the social optimum catch and payoff, but from some time instant on, the negative results of over-exploitation start to dominate and the inequality becomes reverse.

For the accumulated payoff from time t on, discounted to time instant t, presented in Fig. 6.2b, the social optimum payoff is always higher than the Nash equilibrium payoff, and the difference between them increases over time.



(a) Instantaneous payoffs over time for social optima and Nash equilibria, for patient and impatient players

(b) Total payoff from time t on (discounted to time instant t) for social optima and Nash equilibria, for patient and impatient players

Figure 6.2

6.2 Enforcing social optimality by a tax-subsidy system

Here we consider the problem of enforcement of a social optimum profile by a *tax-subsidy system*, linear in the surplus over the social optimum level:

$$\mathrm{tax}(c_i, x) = \tau(x) \left(c_i - C_i^{\mathrm{SO}}(x)\right) x. \tag{6.2.1}$$

So, the current payoff in the modified game is

$$J_i^\tau(x, C) = \int_{t=0}^{\infty} e^{-rt} \left(\ln\left(C_i(X(t)) \cdot X(t)\right) - \tau(x) \left(C_i(X(t)) - \frac{r}{n} \right) X(t) \right) dt. \tag{6.2.2}$$

Theorem 57 *Consider the enforcement problem. Rate of linear tax-subsidy system τ which enforces the social optimal profile C^{SO} is given by*

$$\tau(x) = \frac{(n-1)}{rx}. \tag{6.2.3}$$

Fig. 6.3 represents the *rate of tax* of the *tax-subsidy system* enforcing the socially optimal profile. It can be easily observed that for low biomass of fish, large value of τ are required, moreover, τ decreases with x.

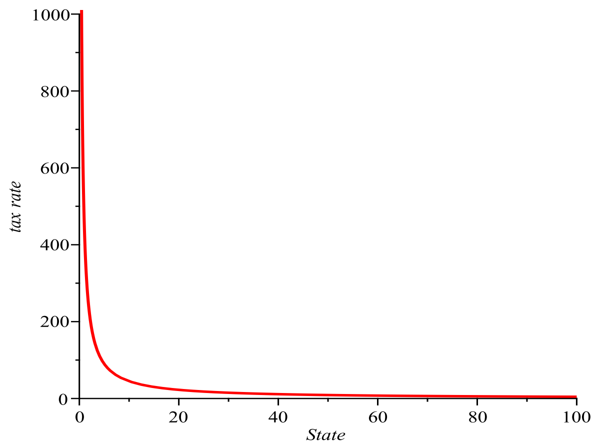


Figure 6.3: Tax rate $\tau(x)$ enforcing the socially optimal profile for the values of the parameters are $\xi = 0.03$, $n = 10$ and $r = 0.02$

Proof: Consider a game with enforcing the social optimum profile. If a player plays $c_i^{\text{SO}} = \frac{r}{n}$ then there is no tax to be paid or subsidy to be obtained. So, if every player plays c_i^{SO}, each of them obtains the payoff $\frac{V^{\text{SO}}(x)}{n}$ and this is the optimal payoff for such an appropriate $\tau(x)$, if it exists. So, the HJB Eq. (1.3.16) for the modified game is

$$\frac{r}{n}V^{\text{SO}}(x) = \max_{c_i \in [0,M]} \left(\ln(c_i x) - \tau(x)\left(c_i - \frac{r}{n}\right) x + \left(\xi - c_i - \sum_{j \neq i}^{n} c_j\right) x \cdot \frac{\partial V^{\text{SO}}(x)}{n \cdot \partial x} \right) \tag{6.2.4}$$

Calculate the zero derivative point of the right hand side of Eq. (6.2.4) to get the optimal value of c_i as

$$\tilde{c}_i = \frac{r}{(1 + \tau(x) r x)} \tag{6.2.5}$$

and the social optimum should be attained at c_i^{SO}. So, substitute $\tilde{c}_i = c_i^{\text{SO}}$ to get τ as given in Eq. (6.2.3). It can be easily checked that for this τ, $\tilde{c}_i$ fulfils the HJB Eq. (6.2.4). ■

6.2.1 Enforcement by a financial incentive — the general algorithm

The technique used in the proof of Theorem 57 can be extended to a quite general algorithm of calculating a successful enforcement of the socially optimal profile S_i^{SO} for games with arbitrary current payoffs P_i and state transition function ϕ in a certain class of financial incentives or proving that such a tool does not exist.

This financial incentive of a general form $\mathcal{T}(x, s_i) = \varphi(x, s_i, p)$, dependent on a vector p of m parameters. The tax should be purely regulatory and such that $\mathcal{T}(x, S_i^{\mathrm{SO}}(x)) = 0$ for all x.

Further extension of this method to finding a financial incentive enforcing a previously specified profile $\bar{S}$ (or proving that such an incentive does not exist in a certain class of incentives) if only we are able to calculate the players' payoffs for this profile, $J_i(x, S)$, is also immediate.

For the given financial incentive φ defined by a vector of parameters p, the general algorithm is as follows.

1. Write the HJB equation from Theorem 6 for the optimization problem of player i given $\bar{S}_{\sim i}$ and $V_i = J_i(x, \bar{S}_i^{\mathrm{SO}})$, for $i = 1, \ldots, n$.

2. Calculate the strategy $S_i^p(t, x)$ (for finite time horizon, the functions of the model may be directly dependent on time, or $\bar{S}$ may be dependent also on time, so we usually cannot avoid direct dependence of time) such that $S_i^p(t, x)$ maximizes the right-hand side of HJB equation of player i for x and t.

3. Find p for which $S^p \equiv \bar{S}_i^{\mathrm{SO}}$.

By Theorem 5 for a finite horizon case and by Theorem 6 for the infinite time horizon case, $\bar{S}$ is the Nash equilibrium in the game modified by this financial incentive.

6.3 Self-enforcing environmental agreement

Here, we solve a model of a self-enforcing environmental agreement, considering a unilateral deviation of a player.

In this binding agreement model, all the players commit themselves to play their social optimum strategies until the deviation is not observed. However, we assume that the observation is delayed — it takes ϵ to notice a deviation of a deviating player. So, if player i deviates at time 0, then the remaining players observe it at time ϵ, and the agreement is broken and from time ϵ on, all the players play their Nash equilibrium strategies.

Fix a time instant $\bar{t}$. We compare the payoff of player i for not deviating (and playing the social optimum strategy) and $V_i^{\mathrm{D}}(x)$ being the optimal payoff if they deviate at $\bar{t}$ assuming that the other players play their agreement strategies, $C_{\sim i}^{\mathrm{SO}}(x)$ before they notice the deviation, and $C_{\sim i}^{\mathrm{NE}}(x)$ afterwards.

Definition 14 *An agreement is* self-enforcing *if there is no incentive to unilaterally deviate from it, whatever the state of the system x is. So, for every x, $V_i^{\mathrm{D}}(x) \leq \frac{V^{\mathrm{SO}}(x)}{n}$.*

Obviously, if $V_i^{\mathrm{D}}(x) > V_i^{\mathrm{SO}} = \frac{V^{\mathrm{SO}}(x)}{n}$, then player i will deviate and the agreement will be broken whereas if $V_i^{\mathrm{D}}(x) < \frac{V^{\mathrm{SO}}(x)}{n}$, then player i will not deviate at time $\bar{t}$ at which the state is x. Generally, if $V_i^{\mathrm{D}}(x) = \frac{V^{\mathrm{SO}}(x)}{n}$ in real life, we cannot be sure whether the players will break the agreement or not, but such an option is possible.

Obviously, the feedback form of strategies is not applicable any more, and strategies in this modification become dependent also on the time instant.

For such a binding agreement a different information structure has to be considered. The fact that deviation has been observed has to taken into account by players who want to be in agreement as long as it is not violated by the others. To describe strategies of a player who abides by the agreement as long as they do not observe defection of some other player, an additional binary variable a is needed with $a = 1$ denoting that any deviation has not been observed, as an additional argument of strategy. The trajectory of this a is denoted by A.

$$\hat{C}_i^{\mathrm{A}}(a, x) = \begin{cases} c_i^{\mathrm{SO}} & \text{if } a = 1, \\ c_i^{\mathrm{NE}} & \text{if } a = 0. \end{cases} \tag{6.3.1}$$

Theorem 58 *The agreement is self enforcing if and only if*

$$\epsilon \leq \epsilon^{\mathrm{crit}} = \frac{1}{r} \ln \left(\frac{(n-1)^2}{1 - n + n \ln(n)} \right). \tag{6.3.2}$$

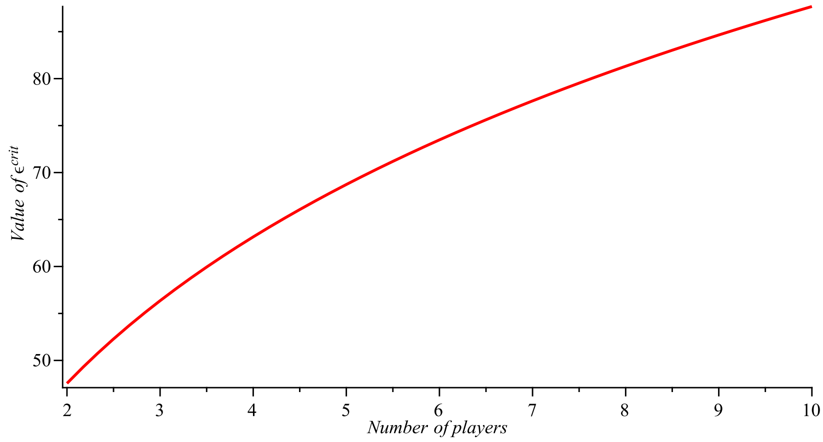


Figure 6.4: The critical value of the delay ϵ^{crit} with which the deviation is being observed as a function of the number of players for $\xi = 0.03$, and $r = 0.02$

Fig. 6.4 shows how the critical ϵ^{crit} behaves as the number of player increases — it is increasing in n because of the fact that the agreement is such that the larger the number of players in the agreement, the more severe is the punishment for the deviating player.

This seems counter-intuitive — it is generally well known that it is easier to obtain cooperation in smaller society. However, we have to take into account a specific form of agreement. First, it is more costly for a deviating player to deviate for a larger number of players because the punishment by many players is more severe. Besides, the punishers punish because, if the agreement is broken, the others play Nash equilibrium, so the best response is to play Nash equilibrium too.

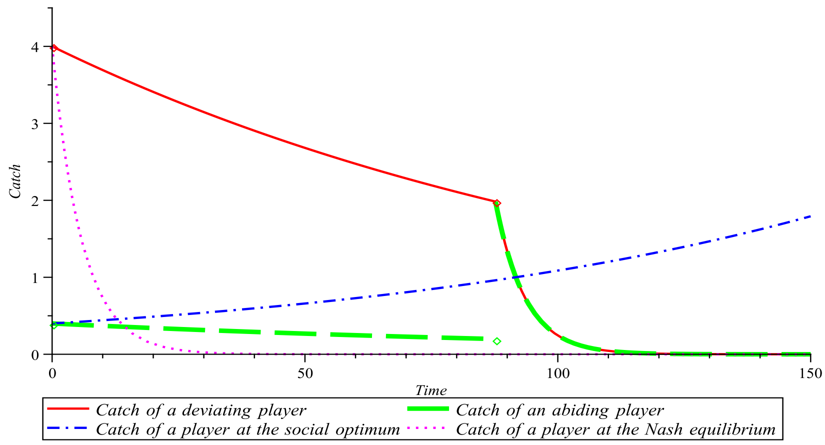


Figure 6.5: Catches of a deviating player and of an abiding player, compared to the catch of a player at the social optimum and Nash equilibrium for $n = 10$, $\xi = 0.03$, $r = 0.02$ and $\epsilon = \epsilon^{\text{crit}}(n)$

From Fig. 6.5, it can be easily seen that the catch of the deviating player is initially higher even than at the Nash equilibrium while the catch of each of the abiding players is initially even lower than at the social optimum, then after a discontinuity of the catch of the abiding player at the time of observing the deviation, they are both equal and decreasing to zero (because of depletion of the resource).

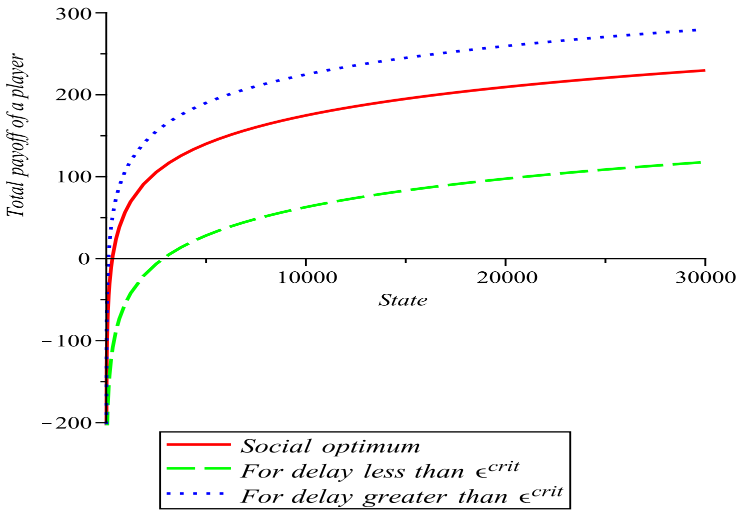


Figure 6.6: Total payoff of a deviating player compared to the agreement payoff depending on the delay for $n = 10$, $\xi = 0.03$, $r = 0.02$ and two different values of delay: $\epsilon = 40$ (below critical) and $\epsilon = 150$ (above critical) depending on the initial state

Fig. 6.6 shows the total payoff of a deviating player depending on the initial state, compared to their non-deviating strategy assuming the others abide. **Proof:** The payoff of the deviating player i is given by

$$\hat{J}_i^{\mathrm{D}}(x_0, C_i) := \left(\int_{t=0}^{\infty} e^{-rt} \ln\left(C_i(X(t)) \cdot X(t)\right) dt \right), \text{ where} \tag{6.3.3a}$$

$$\dot{X}(t) = \left(\xi - C_i(X(t)) - \sum_{j \neq i}^{n} \hat{C}_j^{\mathrm{A}}(t, X(t), A(t)) \right) X(t), \tag{6.3.3b}$$

$$X(0) = x_0. \tag{6.3.3c}$$

Consider a situation when only a single player i deviates. If the deviation is at time $\bar{t}$ then from (6.3.1), for simplicity, at time t, the strategies of the abiding players can be equivalently written in form of *feedback strategies* dependent also on time as

$$C_i^{\mathrm{A}}(t, x) = \begin{cases} c_i^{\mathrm{SO}} & \text{if } t < \bar{t} + \epsilon, \\ c_i^{\mathrm{NE}} & \text{if } t \geq \bar{t} + \epsilon. \end{cases} \tag{6.3.4}$$

Consider a strategy in which player i cheats at time $\bar{t}$, then after $\bar{t}+\epsilon$, the best response to the strategies of the others is to play their Nash equilibrium strategies. So, with this additional assumption on the strategy of player i and changing the form of strategies of the others to (6.3.4), Eq. (6.3.3) can be rewritten as,

$$J_i^{\mathrm{D}}(x_0, C_i) = \left(\int_{t=0}^{\bar{t}+\epsilon} e^{-rt} \ln\left(C_i(X(t)) \cdot X(t)\right) dt \right) + e^{-r\epsilon} G_i(X(\bar{t}+\epsilon)) \tag{6.3.5a}$$

$$\dot{X}(t) = \left(\xi - C_i(X(t)) - \sum_{j \neq i}^{n} \bar{C}_j^{\mathrm{A}}(t, X(t)) \right) X(t), \tag{6.3.5b}$$

$$X(0) = x_0, \tag{6.3.5c}$$

for the terminal payoff $G_i(x) = V_i^{\mathrm{NE}}(x)$.

Note that, if for every $\bar{x} > 0$, $J_i^{\mathrm{D}}\left(\bar{x}, C_i, C_{\sim i}^{\mathrm{A}}\right) \leq V_i^{\mathrm{SO}}$, then it is enough to check the deviation only at $\bar{t} = 0$, since deviating at any time $\bar{t} > 0$ results in an analogous analysis whether the agreement is self-enforcing or not.

So, the total payoff of player i for deviation at time 0 is,

$$V_i^{\mathrm{D}}(0, x) = \max_{C_i \in \mathfrak{C}} J_i^{\mathrm{D}}(x, C_i). \tag{6.3.6}$$

Since the optimal strategy of the deviating player from time ϵ on is $\bar{S}_i^{\mathrm{NE}}$, to solve the optimization problem of the deviating player for $t \leq \epsilon$, it is enough to solve the optimization problem for finite time with horizon ϵ and the terminal payoff $\bar{V}_i^{\mathrm{NE}}$ by using Theorem 5. By solving Eq. (1.3.12)–(1.3.13), similarly to the proofs of Theorems 55 and 56, after assuming the logarithmic form $V_i^{\mathrm{D}}(t,x) = A_i^{\mathrm{D}}(t) + B_i^{\mathrm{D}}(t) \ln x$, to get $C_i^{\mathrm{D}}(x,t) = \frac{1}{B_i^{\mathrm{D}}}$ and B_i^{D} solves the differential equation $rB_i^{\mathrm{D}}(t) - \frac{d}{dt} B_i^{\mathrm{D}}(t) = 1$, with the terminal condition $B_i^{\mathrm{D}}(\epsilon) = B_i^{\mathrm{NE}} = \frac{1}{r}$, which results in $B_i^{\mathrm{D}}(t) \equiv \frac{1}{r}$. Consequently, the optimal strategy of the deviating player is $C_i^{\mathrm{D}}(t,x) = r = C_i^{\mathrm{NE}}$.

For the accurate value of A_i^{D} for comparison with the abiding strategy, solve the differential equation for it to get, $rA_i^{\mathrm{D}}(t) - \frac{d}{dt} A_i^{\mathrm{D}}(t) = \ln(r) + \frac{1}{r}\left(\xi - r - \frac{(n-1)r}{n}\right)$ with the terminal condition $A_i^{\mathrm{D}}(\epsilon) = A_i^{\mathrm{NE}} = \frac{1}{r}\left(\ln r + \frac{\xi}{r} - n\right)$, which gives,

$A_i^{\mathrm{D}}(t) = \frac{(\ln(r) n r + (-2\,n+1) r + n\xi) \exp^{r\epsilon} - \exp^{rt} r (n-1)^2}{r^2 \exp^{r\epsilon} n}$.

Solve $V_i^{\mathrm{D}}(x,0) = V_i^{\mathrm{SO}}(x)$ for ϵ, to get Eq. (6.3.2) for the critical value of ϵ. ■

Chapter 7

Conclusions and future directions

7.1 Conclusions

In this dissertation, as it is stated in the title, we consider various models of renewable resource extraction. In those models, we study various aspects of the calculation of optima and equilibria. Firstly, we have considered a constrained linear-quadratic dynamic game, modeling the problem of exploitation of a common renewable resource in discrete time with the infinite time horizon and with increasing number of players. We consider the value of the constraint such that it makes depletion possible. To make the model realistic, we have imposed the constraints on strategies. As a consequence, calculation of a feedback Nash equilibrium has become complicated. We have calculated the social optima both for n-players and for the continuum of players. We have also calculated the Nash equilibria for the continuum of players, and in spite of very simple equilibria, the value function has turned out to be very complicated and irregular. For the n-player case, we have not been able to calculate the Nash equilibria for $n \geq 2$, and we have proved that solutions in some class of functions cannot be obtained. We return to this problem in a truncation of the game in Chapter 3 to show the reason that even in a 2-stage truncation such a continuous solution does not exist. Our results may be treated as a counterexample to the correctness of the *undetermined coefficient method*, used for solving the Nash equilibrium and/or optimal control problems. We have also investigated and found different kinds of enforcement of an optimal social profile by a Pigovioun tax-subsidy system.

In Chapter 3, we have calculated the symmetric feedback Nash equilibria in two-stage truncations of the considered constrained linear-quadratic dynamic game, but only with 2 players. In spite of the concave instantaneous payoffs and convex sets of available decisions, we have proven the non-existence of continuous symmetric feedback Nash equilibria (discontinuous with respect to the state variable). However, we have found a continuum of discontinuous symmetric feedback Nash equilibria with respect to the state variable. So, the result is a counterexample to the common belief in the continuity of equilibria for linear-quadratic dynamic games with concave payoffs.

Next, we have analysed a dynamic optimisation problem, closely related to the above two dynamic games — a generalisation of a linear-quadratic problem with state-dependent constraints on control. We have obtained a complicated form of the solution with a piecewise linear solution with infinitely many pieces in the infinite time horizon. However, we were able to prove that the Bellman equation holds and solve it by using concavity. Equivalently, it may be regarded as a linear-quadratic problem with non-negativity constraints, both on control and state. We have considered both the infinite time horizon problem and its finite horizon truncations. The problem is very important from the theoretical point of view — although it looks simple in its linear-quadratic form, calculation of the optimal control is nontrivial because of the more general discount factor β instead of the *golden rule* β.

In the next Chapter, we have started the work from computing and comparing the numerical and analytic methods to find the social optima and the Nash equilibria for the well known Levhari and Mirman Fish Wars model of a dynamic game for logarithmic current and terminal payoffs. In spite of singularities in payoffs, we have obtained a very good approximation of social optima and Nash equilibria along the corresponding optimal trajectory, although the value function was substantially overestimated on some sets and underestimated on some other sets.

This has been a starting point to the main achievement of the paper that is the formulation of general rules when such over-estimation or under-estimation of the value function does not result in wrong optimal trajectory and the optimal strategy along with it for the dynamic optimisation problems.

We have not restricted that the over-estimation or under-estimation is resulting from using numerical methods only, but it may also be a result of replacing the value function which is not known precisely by a constraint for it on some intervals a priori in order to simplify further computation or calculation.

Our results also prove that in some dynamic optimisation problems, solving the Bellman equation and finding the maximum of its right-hand side as the candidate for optimal control for the steady state of the state variable only, may lead to a correct result. It also justifies the procedure of calculating the optimal control only along the optimal trajectory.

Finally, in the thesis, we have considered a differential game — a continuous time version of the modified Fish Wars game with linear dynamics.

We have calculated the social optimum and a Nash equilibrium, and we have compared the results for different parameter depending on whether the players are patient or impatient. We have proved that over-exploitation of the resource always takes place and it may even lead to its depletion.

To have sustainability of the resource, we have solved two different problems of enforcing social optimality: external enforcement by a tax-subsidy system and a self-enforcing binding agreement with the assumption that there is a delay in observation of a default. As a consequence of solving the tax-subsidy problem, we have also proposed a general algorithm for finding financial incentives, defined by some parameters, enforcing a social optimum in a large class of differential games.

7.2 Scope for future study

There are several potential continuations of the linear-quadratic dynamic game model.

- Numerical computation of a belief distorted Nash equilibrium.
- Introducing a more complex spatial distribution of fish in the model, so that the current decisions of each of the players have more influence on the future level of biomass in their zone than decisions of any other player. Obtaining Nash equilibria in such a model, however, may turn out to be possible only in finite time horizon problems.
- An attempt to calculate all the symmetric feedback Nash equilibria for more than two stages (for the game considered in Chapter 3) can also be taken, however, because of discontinuity and non-uniqueness of the two-stage equilibria, its extension to more than two stages poses several technical challenges.
- Formulation of general rules when such over-estimation or under-estimation of the value function does not result in wrong optimal trajectory and the optimal strategy along with it for the *Nash equilibrium problems in dynamic games* are an obvious extension on which we already worked, and the results are available in Singh and Wiszniewska-Matyszkiel [94].

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