

University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Piotr Oszer

Deformation theory and algebraic group actions

PhD dissertation

Supervisor
dr hab. Joachim Jelisiejew
Institute of Mathematics
University of Warsaw

Author's declaration:

I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

March 26, 2026
Piotr Oszer

Supervisors' declaration:

The dissertation is ready to be reviewed.

March 26, 2026
dr hab. Joachim Jelisiejew

Abstract

This dissertation is devoted to the study of several moduli spaces. We generalise the Białynicki-Birula decomposition to abstract deformation functors, providing a new method of studying deformation theory.

We provide an explicit description of the Białynicki-Birula cells on the Hilbert scheme of points on $\mathbb{A}^2$ with isolated fixed points, with the Hilbert-Burch matrices as a main tool; we prove a conjecture by Homs and Winz which we realise as a special case of our construction.

We prove that the Gorenstein locus of the Hilbert scheme of points on $\mathbb{A}^n$ is non-reduced for $n \geq 12$; we construct examples of non-reduced points that come from apolar algebras of the sum of general cubics. As a corollary, we get a non-reducedness result for the cactus scheme. Our construction gives us fractal structures on the nested Hilbert scheme.

We show that the stratification of the Territory scheme by the associated semigroup is an instance of the Białynicki-Birula decomposition. We show that the Territory scheme does not admit any reduced $\mathbb{k}$ -point; we introduce the Territory with the multiplicity bounded and get a structural comparison of it with the usual Territory.

Keywords: deformation theory, Hilbert scheme, Gorenstein algebra, curve singularities, inverse system, apolarity.

AMS MSC 2020 classification: 14B07, 14B10 14B12, 14C05, 14H20, 13D10.

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Acknowledgements

First and foremost I would like to pay my gratitude to my advisor Joachim, without whom I would not even begin to write this thesis, for his invaluable help both during writing this thesis and during the whole adventure in the doctoral school.

Second, I would like to thank all my mathematical colleagues, both those that pursue mathematical research to this day, and also to those that do not.

Third, I would like to thank my wife Laura and my whole family for continuing support.

Last but not least, I would also like to thank the Library of the Faculty of Physics, University of Warsaw, where most of this dissertation was written, for great hospitality.

Nikt nic nie czyta; jeśli czyta, nic nie rozumie; jeśli rozumie, natychmiast zapomina.

Stanisław Lem, *Prowokacja*

Chapter 1

Introduction

Let us take a smooth variety X and choose a point $x \in X$. Consider the ideal $\mathfrak{m}$ of (locally defined) functions on X that vanish on x . The quotient space $\mathfrak{m}/\mathfrak{m}^2$ is identified with the cotangent space of X at x . We denote the (locally defined around x) functions on X as R , the quotient $R/\mathfrak{m}^2$ is the ring of all first-order approximations around x . If we replace the number 2 by an arbitrary positive integer m , we get m -th order approximations, the other name for that is the jet space.

Assume that X admits an action of a one-parameter group $\mathbb{G}_m$, such that x is an isolated fixed point, then we get a flow. Some trajectories of the points in the neighbourhood of x , flow from x , other to it, and some do not meet x at all. This leads to the decomposition into stable and unstable varieties, for reference see [Per01, 2.7 The Stable Manifold Theorem]. The Białynicki-Birula decomposition is an algebraic geometry version of this construction. It was introduced in [BB73, BB76] and later extended in [Dri15, JS19]. If we take any m -th order approximation it decomposes into a sum of the positive and the negative part. Both parts are uniquely determined by their values on the stable and the unstable variety.

We forget about variety X altogether, but preserve the key features of a compatible system of higher order approximations and end up with a deformation functor. If we encode the one-parameter group action on a deformation functor appropriately, we get a variant of the Białynicki-Birula decomposition on deformation functors.

In this dissertation, we introduce this infinitesimal variant of the Białynicki-Birula decomposition, and employ it to study local properties of several moduli spaces. We extend the non-reduceness results of the nested Hilbert scheme [GGGL24], the Hilbert scheme of points [Jel20, Sza21, Jel24], and the Territory scheme [Ish80]. Our results are twofold.

1. On one hand we show new examples of non-reduced points on these moduli spaces, for example it was not known before that the Gorenstein locus of the Hilbert scheme is non-reduced.
2. On the other hand we provide refined descriptions of local structures by determining the singularity types in many cases.

We also study the Białynicki-Birula decomposition of the Hilbert scheme of points on $\mathbb{A}^2$, in particular we prove [HW21, Conjecture 5.14].

The studied moduli spaces admit plenty of applications outside of algebraic geometry

- To commutative algebra, where the Hilbert scheme is a moduli space of finite algebras, the Territory scheme is a moduli of subalgebras of a fixed algebra.
- To mathematical physics, where the Territory scheme is closely related to the moduli space of curves which in turn is used in String theory, see [Pol81, BM86].
- To combinatorics, where the Hilbert scheme was used in [Hai01] to prove $n!$ conjecture; we show that the Territory scheme is related to Wilf conjecture, see [MS25, Problem 30].

Structure of the dissertation

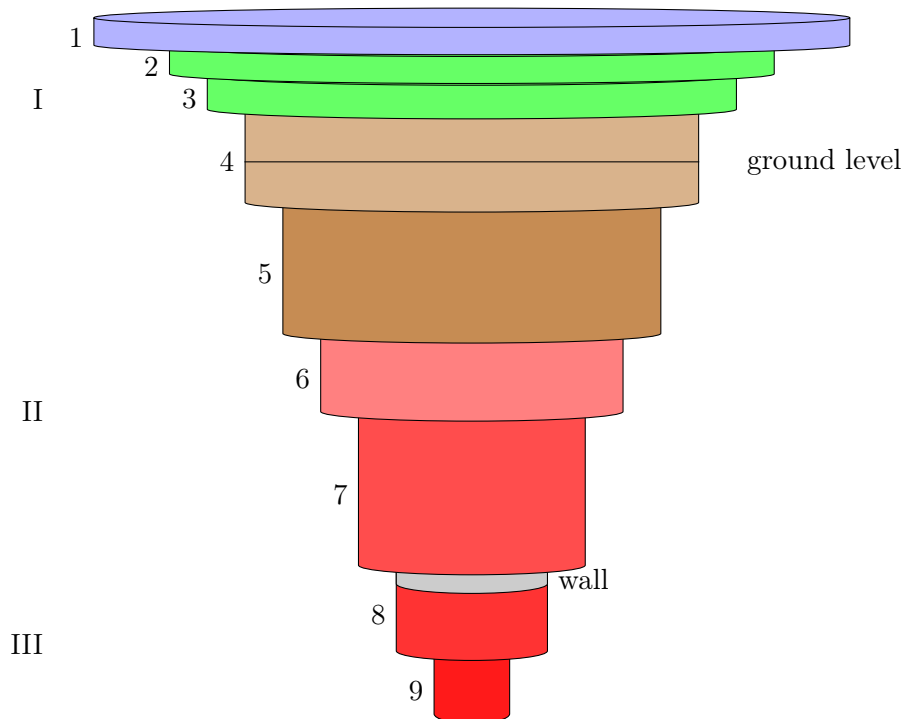


Figure ▼: Nine concentric circles of moduli ¹

The dissertation is divided into three parts and consists of nine chapters. The chapters form nine concentric circles as illustrated in Figure ▼. The deeper underground we go, the more obscure and possibly pathological objects we find. Everything that is above the ground level is part of state of the art. The parts under the ground contain new contributions to the field.

First four circles

The first part of the dissertation is an introduction of both classical and new techniques in study of the five moduli spaces that we are concerned with.

The first three circles are over the ground, the respective chapters contain preliminaries on commutative algebra and deformations theory. For the first item the main reference is [Eis95] and for the second [Ser06].

¹Actually only from the fifth downwards the circles represent moduli spaces. This alludes to the fact that there are plenty of moduli spaces that are not presented in this dissertation and that there is also many moduli spaces to dig up!

The forth circle is partially underground, that is due to the fact that up to Section 4.2, the chapter contains a recollection of known and useful facts. The Section 4.2 is based on [Osz26, Section 3]. There we introduce the functor of filtered deformations for affine $\mathbb{G}_m$ -schemes, and afterwards for equivariant maps between them, see Theorem 4.60.

Definition 1.1 (Definition 4.49). *Let X be an affine $\mathbb{G}_m$ -scheme, together with the filtration given by vanishing order on $\mathcal{O}_X$. Let $\text{Art}_{\mathbb{k}}$ denote the category of local Artinian $\mathbb{k}$ -algebras with residue field isomorphic to $\mathbb{k}$. We define a functor*

$$\text{Def}_X^{\geq 0} : \text{Art}_{\mathbb{k}} \rightarrow \text{Set}$$

to parametrise filtered deformations of X . Let $(A, \mathfrak{m}) \in \text{Art}_{\mathbb{k}}$. Then $\text{Def}_X^{\geq 0}(A)$ is a set of isomorphism classes of pairs consisting of a diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \text{Spec}(A), \end{array}$$

where $\mathcal{O}_{\mathcal{X}}$ is a flat filtered A -algebra; and $\mathcal{O}_{\mathcal{X}} \otimes A/\mathfrak{m}$ is a filtered algebra, together with a filtered algebra isomorphism to $\mathcal{O}_X$.

The filtered functor should be thought of as the infinitesimal Białynicki-Birula decomposition for algebraic stacks. The Białynicki-Birula decomposition for smooth schemes was introduced in [BB73]. Much later it was extended to schemes of finite type in [JS19]. The modern formalism of θ -stratification, see [HL22], can be thought as a vast generalisation of this framework.

Relying on the classical results on $\mathbb{G}_m$ -equivariant deformation given in [Pin74], we identify the key vector spaces controlling the filtered deformations. Let X be an affine $\mathbb{G}_m$ -scheme. The functor Def_X admits an associated triple

$$T_X^0, T_X^1, T_X^2,$$

consisting of the infinitesimal automorphism group, the tangent space, and an obstruction space. The vector spaces T_X^i are Lichtenbaum-Schlessinger functors, defined in [Har10, Section 1.3. The T^i Functors]. The $\mathbb{G}_m$ -action on X induces an action on the functor and yields graded structures on each of T_X^i vector spaces.

Theorem 1.2 (Theorem 4.55). *The functor $\text{Def}_X^{\geq 0}$ admits a associated triple*

$$(T_X^0)_{\geq 0}, (T_X^1)_{\geq 0}, (T_X^2)_{\geq 0},$$

consisting of the infinitesimal automorphism group, the tangent space, and an obstruction space.

Fifth circle

The fifth circle is underground, but very close to the surface. The respective Chapter 5 contains published results from a single authored paper [Osz25].

The main theme of this chapter is the Hilbert scheme of points on the affine plane.

We describe explicitly the Białynicki-Birula cells on $\text{Hilb}_\delta(\mathbb{A}^2)$ endowed with $\mathbb{G}_m$ -action with isolated fixed points, see Theorem 5.40 and Theorem 5.44. Our methods extend those introduced in a series of papers [ES87, ES88] and later developed in [CV08, Con11, HW21, HW24]. The main tool here are matrices derived from Hilbert-Burch Theorem, see Theorem 2.4. If the fixed point locus is positive dimensional we obtain an étale rational map to the cell, see Theorem 5.39 and Theorem 5.43.

In Corollary 5.45, we prove [HW21, Conjecture 5.14], which we realize as a special case of our construction. We also show examples when the construction provides a rational étale map to the Hilbert scheme which is not contained in any Białynicki-Birula cell. At the end, in Theorem 5.46, we give an explicit description of the local functor of any ideal in the Hilbert scheme of points on the plane.

Sixth circle

The next circle is full of self-similar objects. The corresponding Chapter 6 is based on [Osz26, Chapter 5], and it is concerned with the nested Hilbert scheme.

Definition 1.3 (Definition 3.14). *The nested Hilbert scheme $\text{Hilb}_{\delta_1, \delta_2}(Y)$ parametrises nested pairs of finite subschemes of length δ_1 and δ_2 of Y . If $\delta_1 = 1$ then we get a pointed Hilbert scheme, as a scheme of length one is a point.*

In recent years there has been growing interest in the nested Hilbert scheme, especially in the case where Y is a surface, see [BE16, Son16, RY20, GNR21, RS24]. Classically, in [Che98] it was shown that the only smooth nested Hilbert scheme of point on a smooth surface U is $\text{Hilb}_{\delta_1, \delta_1+1}(Y)$. In [Fog73] it was shown that the pointed Hilbert scheme $\text{Hilb}_{1, \delta_2}(Y)$ for a smooth surface Y is normal and Cohen-Macaulay, and in [Son16, RS24] that it admits rational singularities.

More recently the interest arisen in the local structure of the nested Hilbert scheme of points on varieties of higher dimension, see [GGGL24, GL26]. This is also the focus of this chapter.

In [GGGL24] it was proven that for $\delta_1 = 1$, $\delta_2 \geq 4$ and dimension of Y at least 4, the nested Hilbert scheme is non-reduced. In Theorem 6.20, We extend this result by describing the smooth type of the non-reduced points.

The description heavily relies on the notion of fractalness, which is inspired by [Sza21, Definition 3.18]. Intuitively, the fractalness of a pointed family of schemes means that the base of the family naturally contains a copy of the special fibre, yielding a non-trivial deformation of the fibre over itself. The fractal families are always non-reduced Lemma 6.16.

We also study the Hilbert scheme $\text{Hilb}_{\delta-1, \delta}(Y)$. In Theorem 6.29, we obtain another fractality result. The examples of fractal points come from pairs consisting of a Gorenstein algebra of length δ and its quotient obtained by cutting the socle. In [Sza21] it was shown that $\text{Hilb}_{2n+1}(\mathbb{A}^n)$ is non-reduced for $n = 6$ and $n \geq 7$, [Sza21, Theorem 1.4], and described the smooth type of this non-reduced points [Sza21, Theorem 1.5]. We present a new, more conceptual proof of these results, putting them into wider context.

Seventh circle

In the next circle the objects are not exactly self-similar, but at least cacti grow here. The respective Chapter 7 is based on [Osz26, Chapters 6-8].

It is classical to study the secant varieties. The question whether the secant varieties admit mild singularities is central to this topic, see [Ull16] [CS18], [ENP20], [CLPS25, Theorem A].

The special case of the secants on the Veronese embedding is of exceptional interest, since it relates secant varieties to tensors, for reference see [Lan12]. There is also a great deal of interest in equations of the secant varieties [BGL13, BBF24], as well as in the syzygies [CLPS25, Theorem B, Corollary C].

Since it is notoriously difficult to study equations of secant varieties, the cactus variety was introduced in a seminal paper [BB14]. Since its introduction, the cactus variety has been an object of intensive research, see [BR13, Gal17, BBG19, Gal23, BBF24, BK24, Buc26, BBG25]. In this chapter we obtain that the natural scheme structure on the cactus variety is non-reduced.

Corollary 1.4 (Corollary 7.48). *Let $n \geq 12$, but not $n = 13$, and $r \geq 4n + 4$. Let $\nu_r(\mathbb{P}^n)$ denote the r -th Veronese embedding. Then the cactus scheme $\mathfrak{K}_{2n+2}(\nu_r(\mathbb{P}^n))$ is non-reduced.*

To get this result we study the open subscheme of the Hilbert scheme that parametrises only Gorenstein schemes, it is called Gorenstein locus, see Definition 3.12. The Gorenstein locus and the scheme structure on the cactus are closely related, see for example [CLPS25]. The main result of this chapter is the following

Theorem 1.5 (Corollary 7.43). *Let $n \geq 12$, but not 13, then the Gorenstein locus $\text{Hilb}_{2n+2}^{\text{Gor}}(\mathbb{A}^n)$ is not reduced.*

Non-reduceness of the Hilbert scheme is known, see [Jel20, Sza21, Sza21, Jel24], but the methods used in these works do not extend to the Gorenstein locus, see [Osz26]. An explicit example can be obtained by taking a pair of sufficiently general cubics Φ and Ψ in n_1 and n_2 variables respectively, such that $n_1 + n_2 = n$, and both n_1 and n_2 belong to the set $\{6, 8, 9, 10, \dots\}$. The apolar algebra of $\Phi + \Psi$ provides a non-reduced point of the Gorenstein locus, for reference on apolarity see Section 2.2.

The construction of examples goes as follows. First, we consider a pair of Gorenstein pointed schemes X and Y , then glue them.

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \cup_* Y, \end{array}$$

see Definition 7.4. We study the deformations of such union along the point and get the following result.

Theorem 1.6 (Theorem 7.22). *Assume that X, Y are pointed finite $\mathbb{G}_m$ -schemes which are spectra of t -level algebras, see Definition 7.20, and assume that neither is a reduced point. Then the morphism*

$$\text{Def}_{* \rightarrow X}^{<0} \times \text{Def}_{* \rightarrow Y}^{<0} \rightarrow \text{Def}_{X \cup_* Y}^{<0}$$

is smooth and induces an isomorphism on the tangent spaces.

From that we construct a subscheme called a connected sum, the sum was introduced in [AAM12], see Definition 7.28. Then, in Theorem 7.38, we argue that the functors of deformations of connected sum and union along the point share some properties.

Eighth circle

The next circles are located behind the wall, which separates the circles corresponding to the study of the Hilbert scheme from those that are concerned with the Territory functor. The first of the chapters behind the wall is Chapter 8, where we discuss general properties and applications of the Territory scheme.

Definition 1.7 ([Ish80, Definition 2]). *Let X be an affine scheme and δ be a positive integer. We define a contravariant functor of the Territory of an algebra $\mathcal{O}_X$, $\text{Ter}_X^\delta: \text{Sch}^{op} \rightarrow \text{Set}$ on S -points as*

$$\text{Ter}_X^\delta(S) := \left\{ (0 \rightarrow \mathcal{A} \xrightarrow{\phi} \mathcal{O}_X \otimes \mathcal{O}_S \rightarrow \text{coker}(\phi) \rightarrow 0) \left| \begin{array}{l} \phi \text{ is an } \mathcal{O}_S\text{-algebra homomorphism} \\ \text{coker}(\phi) \text{ is rank } \delta \text{ locally free } \mathcal{O}_S\text{-module} \end{array} \right. \right\}$$

This functor was introduced quite a long time ago, but it has not been extensively studied since. The story goes back to the study of simultaneous normalization of families of curves, it was studied, first in the analytical setting [Tei77], then in the algebraic one [CHL06, GP22]. The main result in that area is a characterization of that flat families of curves over a normal base that admits a simultaneous normalization, in terms of numerical invariant delta, see [GLS07, Theorem 2.56 (Teisser, Raynaud)]. The Territory parametrises families with the normalization fixed, these are called equinormalizable, for reference see [GLS07, Section 2.6 Equinormalizable deformations].

Let us consider the cuspidal curve $Y = \text{Spec } \mathbb{k}[x^2, x^3]$, its normalization induces an injective $\mathbb{k}$ -algebra homomorphism

$$\mathbb{k}[x^2, x^3] \subset \mathbb{k}[x],$$

the cokernel is a $\mathbb{k}$ -linear space of dimension 1. Therefore we get a $\mathbb{k}$ -point of $\text{Ter}_{\mathbb{A}^1}^1$, this is the only $\mathbb{k}$ -point it admits, see [Ish80, Example I].

The main application so far of the Territory is to construct moduli spaces of singular curves, see [Ish82, Ham20, BGS24]. One of the crucial tools is to consider the stratification of Territory by the associated semigroup, see [Ish80, Definition 9]. In Theorem 8.28 we show that this stratification is an instance of the Białynicki-Birula decomposition. There is a standing question, [Ham20, Question 4.3], whether every stratum associated with a semigroup with three generators is smooth. We believe that the equivariant perspective is a useful tool in the pursuit of the answer to this question. The Territory is also related to the study of crimping spaces [vdW10], they are realised, in [BGS24, Theorem 3.19], as orbits of certain automorphism group action on the Territory.

We connect Territories to combinatorics. In the study of numerical semigroup the central play Wilf conjecture, it is the 30-th problem on the list [MS25]. We introduce a geometric variant of Wilf conjecture and in Theorem 8.44 we show that it is equivalent to the original question.

Our main motivation for study of the Territory is the pursuit of examples of curve singularities in positive characteristic that do not lift to characteristic 0. It is classical that every smooth curve

does lift, see [Har10, Corollary 22.2], but there are non-liftable smooth surfaces, see [Vak06]. It is expected that non-liftable curve singularities exist, but this remains open, see for example [Har10, page 154]. An extensive research has been pursued regarding the dimension zero scheme in place of curves, see [Lan18, Zda18, Jel20], in particular it is known that there are non-liftable finite $\mathbb{k}$ -algebras. We follow [Eke04] and sketch a strategy that can lead to discovery of examples of curve singularities that do not lift on the Territory. These would make good candidates for abstractly non-liftable curve singularities.

Ninth circle

The bottom circle is only dimly lit by our understanding, since the main object of interest in the respective Chapter 9 is new and have not appeared in the literature before

The Territory scheme is non-reduced, which is known since its construction [Ish80, Remark II]. We construct a closed subfunctor that parametrises subalgebras with bounded multiplicity.

Definition 1.8 (Definition 9.6). *Let $(\mathcal{O}_X, \mathfrak{m})$ be a complete local ring, free over the base field $\mathbb{k}$. The Territory with the multiplicity bounded from below by m is defined as*

$$\mathrm{Ter}_{X,m}^\delta = \mathrm{Ter}_{\mathbb{k} \oplus \mathfrak{m}^m}^{\delta-t},$$

where t is rank of the free $\mathbb{k}$ -vector space $\mathcal{O}_X/(\mathbb{k} \oplus \mathfrak{m}^m)$.

Since multiplicity of nontrivial numerical semigroup is always at least 2, we get a moduli space with isomorphic reduced structure, but larger tangent space. There are two conclusions. The first, the Territory is non-reduced.

Theorem 1.9 (Theorem 9.16). *Let δ be a positive integer. The scheme $\mathrm{Ter}_{\mathbb{k}[x]}^\delta$ is nowhere reduced.*

The second, we get a reduced component on a bounded Territory.

Theorem 1.10 (Theorem 9.8). *The scheme $\mathrm{Ter}_{\mathbb{k}[[x]],2}^\delta$ admits a generically reduced irreducible projective component.*

We go further and show that the local structure of at a subalgebra of multiplicity m is derived from the local structure on the bounded moduli. Unlike on the Hilbert scheme, the exhibited non-reduced structure does not have intrinsic meaning, it is produced from the first-order infinitesimal neighbourhood of the distinguished point of the spectrum of given subalgebra, as exhibited in Theorem 9.22. From that we get

Corollary 1.11 (Corollary 9.26). *Let $X = \mathrm{Spec} \mathbb{k}[[x]]$. Let $V \subset \mathrm{Ter}_{X,k}$ be an irreducible component, such that the general $\mathbb{k}$ -point has multiplicity at least m , for $k < m$. Then V is generically non-reduced.*

The main conclusion is that to understand the geometry of the Territory functor one needs to consider components of fixed multiplicity and then get rid of excessive non-reducedness by studying the bounded Territory.

1.1 Notations

Throughout the text we try to stick to the following notations.

- $\mathbb{k}$ - algebraically closed field of characteristic p or 0 .
- X, Y, Z, W, U, V, S schemes, with S being a test scheme
- $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}, \mathcal{U}, \mathcal{V}$ - deformations of above schemes
- $p = \text{char } \mathbb{k}$
- q is a prime number
- R, P -polynomial rings or general rings/algebras
- $\mathbf{P}$ - divided power series ring
- $\mathbb{S}$ - polynomial ring, not to be confused with our notation for a *test* scheme S and its thickening $\mathbf{S}$.
- E, I, J are ideals
- M, N are $\mathbb{k}$ -valued matrices or modules
- $\mathcal{M}, \mathcal{N}$ are ring-valued matrices or family of modules
- A, B, C, D - Artin algebras
- Λ - Noetherian local ring with $\mathbb{k}$ as residue field
- Art_Λ - category of local Artinian Λ -algebras with the residue field isomorphic to $\mathbb{k}$.
- F, G, H are functors
- T is forbidden not to be mistaken with T^i
- lowercase Greek letters are morphisms
- Γ is a semigroup
- small letters f, g, h, k, s are elements
- small letters r, t, a, b, i, k, l are indices or exponents
- $\mathcal{F}, \mathcal{G}$ filtration
- λ - one-parameter subgroup
- $\mathbf{V}()$ is a vanishing subscheme

Part I

Methods

Chapter 2

Commutative algebra

We fix a field $\mathbb{k} = \bar{\mathbb{k}}$ and a complete Noetherian local ring Λ with the residue field isomorphic with $\mathbb{k}$. Frequently we are in the special case $\Lambda = \mathbb{k}$ other than that in some cases it is the ring of Witt vectors of $\mathbb{k}$.

Definition/Notation 2.1. *The category of local Artinian Λ -algebras with the residue field $\mathbb{k}$ is denoted by Art_Λ .*

We start this chapter by a lemma very useful in proving non-reducedness.

Lemma 2.2. *Let us take Noetherian $\mathbb{k}$ -algebras R and P . Let $\phi: R \rightarrow P$ be a homomorphism. Assume that the induced morphism of schemes $\phi^\#: \text{Spec } P \rightarrow \text{Spec } R$ has all fibres irreducible and that $\phi^\#$ admits a section $\psi^\#$. Let us take a $\mathbb{k}$ -point of $\text{Spec } R$, then we get a Cartesian diagram*

$$\begin{array}{ccc} \text{Spec } P \otimes_R \mathbb{k} & \longrightarrow & \text{Spec } P \\ \downarrow & & \downarrow \phi^\# \uparrow \psi^\# \\ \text{Spec } \mathbb{k} & \longrightarrow & \text{Spec } R. \end{array}$$

Assume that the fibre $P \otimes_R \mathbb{k}$ is finite, then either ϕ is locally an isomorphism around the chosen point or P is non-reduced.

Proof. First, we prove that the homomorphism ϕ induces a bijection between the irreducible components. Let us take the generic point η of an irreducible component V of $\text{Spec}(R)$. The fibre of ϕ over η contains of at least one generic point of $\text{Spec}(P)$ since $\phi^\#$ is surjective. And at most one since the fibres are irreducible. As presented, the image $\psi^\#(\text{Spec } R)$ contains generic points of $\text{Spec } P$, since ψ is a surjection, the morphism $\psi^\#$ is a closed embedding. Hence, $\psi^\#(\text{Spec } R)$ contains all points of $\text{Spec}(R)$. Therefore, the kernel of $\psi: P \rightarrow R$ is contained in the nilradical of P .

As a consequence, the ring P is reduced then ψ is an isomorphism, and ϕ as an inverse is also an isomorphism. $\square$

Now, we replicate the proof of the Hedgehog point theorem [Sza21, Theorem 3.1].

Lemma 2.3. *Let X be a $\mathbb{G}_m$ -scheme. Assume that there is a $\mathbb{G}_m$ -equivariant affine morphism $\pi: X \rightarrow X^{\mathbb{G}_m}$. Let $x \in X^{\mathbb{G}_m}$ be a closed point, and assume that fibre of π over y is finite and non-reduced. Then X is non-reduced.*

Proof. Since π is $\mathbb{G}_m$ -equivariant, then the natural embedding $X^{\mathbb{G}_m} \subset X$ is a section of π . Since π is finite $\mathbb{G}_m$ -scheme, we get X as a relative spectrum of $\mathbb{N}$ -graded Artinian algebra R over $\mathcal{O}_{X^{\mathbb{G}_m}}$. We consider projectivization and get a projective morphism $\text{Proj}_{\mathcal{O}_{X^{\mathbb{G}_m}}}(R) \rightarrow X^{\mathbb{G}_m}$. Since the fibre over y is empty, by the semicontinuity of dimension of fibres [Sta22, Tag 0D4I] the fibres of π are finite in the open neighbourhood of x in $X^{\mathbb{G}_m}$. Any such fibre is finite $\mathbb{G}_m$ -scheme with a unique fixed point hence it is local, in particular irreducible. Hence, by Lemma 2.2, we get non-reducedness. $\square$

2.1 Ideals of depth 2

Our first interest lie in the study of finite subschemes of a smooth surface, the local structural results on such is the following classical theorem.

Theorem 2.4 (Hilbert-Burch theorem). *Let R be Noetherian local ring.*

(H-B-1) [Eis95, Theorem 20.15b] *Suppose that we are given a $(t+1) \times (t)$ matrix $\mathcal{M}$ with entries in R , such that depth of the ideal I generated by t -minors is equal to 2. The sequence*

$$0 \rightarrow R^{\oplus t} \xrightarrow{\mathcal{M}} R^{\oplus t+1} \xrightarrow{\eta} R \rightarrow \frac{R}{I} \rightarrow 0, \quad (2.5)$$

where the i -th entry of η is $(-1)^i$ times the i -th minor of $\mathcal{M}$, is a free resolution.

(H-B-2) [Har10, Theorem 8.3, Schaps] *Additionally, assume that R is a $\mathbb{k}$ -algebra and let $A \in \text{Art}_{\mathbb{k}}$. If $\mathcal{M}_A$ is any lifting of $\mathcal{M}$ to $R_A = R \otimes_{\mathbb{k}} A$ and η_A is obtained as in (2.5), then the sequence*

$$0 \rightarrow R_A^{\oplus t} \xrightarrow{\mathcal{M}} R_A^{\oplus t+1} \xrightarrow{\eta} R_A \rightarrow \frac{R_A}{I_A} \rightarrow 0,$$

where I_A is the ideal generated by the t -minors of $\mathcal{M}_A$, is again a free resolution, and R_A/I_A is flat over A .

(H-B-3) [Eis95, Theorem 20.15a] *Conversely to the first bullet, if a complex*

$$0 \rightarrow R^{\oplus t} \xrightarrow{\mathcal{M}} R^{\oplus t+1} \xrightarrow{\eta} R \rightarrow \frac{R}{I} \rightarrow 0$$

is exact, then there is a nonzero divisor $z \in R$, such that $I = zI_t(\mathcal{M})$, where $I_t(\mathcal{M})$ is the ideal generated by the maximal minors. Then the i -th entry of the matrix η is $(-1)^i z$ times the i -th minor of $\mathcal{M}$. The ideal has the depth equal to 2.

Since we are dealing with minors, we need a few prerequisites on resultants. For basic results on resultants, see [CLO15, Chapter 3].

Lemma 2.6 ([CLO15, Chapter 3, §6 Proposition 1, Proposition 3]). *Let R be a $\mathbb{k}$ -algebra. Let $f, g \in R[x]$. Then*

1. *The resultant $\text{Res}_x(f, g)$ of f, g with respect to x belongs to the ideal $(f, g) \cap R$.*
2. *$\text{Res}_x(f, g) = 0$ if and only if f and g admits a common factor which has a positive degree with respect to x .*

Lemma 2.7 ([CLO15, Chapter 3, §6 Proposition 3]). *Let $f, g \in R[x]$ and f be a monic polynomial of the variable x . The resultant is stable under base change of R ; for every $\mathbb{k}$ -algebra homomorphism $\psi: R \rightarrow P$, the image of resultant $\text{Res}_x(f, g)$ in $P[x]$ is equal to the resultant of images of f, g in $P[x]$.*

2.2 Apolarity

We are also interested in Gorenstein algebras, hence we formulate basic results on them. The main reference for finite Gorenstein algebras is [Eis95, 21.2 Zero-Dimensional Gorenstein Rings].

Definition 2.8. The socle of finite local algebra $(A, \mathfrak{m})$ is the annihilator of the maximal ideal $\mathfrak{m}$.

Definition 2.9 ([Eis95, Proposition 21.5]). *A finite local algebra $(A, \mathfrak{m})$ is Gorenstein if the socle of A is an A -module generated by single element. A finite (not necessarily local) scheme is Gorenstein if it is locally a spectrum of Gorenstein algebra.*

Definition 2.10. Let us take a polynomial algebra $\mathbb{S} = \mathbb{k}[\partial_1, \dots, \partial_n]$.

Denote by $\mathbf{P}$ a divided power series ring in variables $x_1, \dots, x_n$. There is an action of $\mathbb{S}$ on $\mathbf{P}$ by contraction:

$$\partial^{\mathbf{a}} \circ x^{\mathbf{b}} = \begin{cases} x^{\mathbf{b}-\mathbf{a}} & \text{if } \mathbf{b} \geq \mathbf{a}, \text{ that is } b_i \geq a_i \text{ for } i = 1, \dots, n; \\ 0 & \text{otherwise.} \end{cases}$$

Here $\mathbf{b} = (b_1 \dots b_n)$ and $x^{\mathbf{b}} = x_1^{b_1} \dots x_n^{b_n}$. This action is extended by $\mathbb{k}$ -linearity on $\mathbb{S}$ and on $\mathbf{P}$.

Definition 2.11. Let $\Phi \in \mathbf{P}$ be a polynomial, then we define the apolar algebra to be:

$$\text{Apolar}(\Phi) := \frac{\mathbb{S}}{\text{Ann}(\Phi)} = \frac{\mathbb{S}}{(s \in \mathbb{S} : s \circ \Phi = 0)}.$$

The following theorem characterizes local Gorenstein algebras in terms of apolarity.

Theorem 2.12 ([Mac94]). *Let $(A, \mathfrak{m})$ be a finite local Gorenstein algebra given as a quotient of the polynomial ring $\mathbb{S}/I$, such that $I \subset (x_1 \dots x_n)$. Then there exists a polynomial $\Phi \in \mathbf{P}$, such that $I = \text{Ann}(\Phi)$ and $\mathbb{S}/I = \text{Apolar}(\Phi)$. Conversely, if $\Phi \in \mathbf{P}$ then $\text{Apolar}(\Phi)$ is a finite local Gorenstein algebra.*

Let $(A, \mathfrak{m})$ be a finite Gorenstein algebra, then the socle is the ideal generated by any element f , such that $f \circ \Phi = 1$.

Chapter 3

Schemes

3.1 Functors and representability

In this section, we establish basic results and notation for general functors, schemes and the Hilbert scheme.

Definition/Notation 3.1. *In the category of Sch_Λ there is a terminal object $\text{Spec}(\mathbb{k})$, it is denoted by $*$.*

Definition 3.2. *Let $F: \text{Sch}_\Lambda^{\text{op}} \rightarrow \text{Set}$ be functor. Let us fix a $\mathbb{k}$ -point $f \in F(\mathbb{k})$, then a tangent vector to F at f is an element $f_\varepsilon \in F(\mathbb{k}[\varepsilon]/\varepsilon^2)$ over f .*

$$\begin{array}{ccc} & & \text{Spec}(\mathbb{k}[\varepsilon]/\varepsilon^2) \\ & \nearrow & \downarrow f_\varepsilon \\ * & \xrightarrow{f} & F. \end{array}$$

The set of tangent vectors at f is the tangent space to F at f .

Under mild conditions on the functor, the tangent space admits a natural $\mathbb{k}$ -vector space structure, see [FM98].

Definition 3.3. *Let $F: \text{Sch}_\Lambda^{\text{op}} \rightarrow \text{Set}$ be functor. We say that F is representable if there exists a scheme $Z \in \text{Sch}_\Lambda$, such that $F \cong \text{Hom}_{\text{Sch}_\Lambda}(-, Z)$.*

Example 3.4. *The tangent space to representable functors agrees with the tangent space to representing scheme.*

Definition 3.5. *A pointed scheme over S is an S -scheme X together with a section of the structural morphism $S \rightarrow X$. A map of pointed S -schemes is a morphism of S -schemes $\phi: X \rightarrow Y$ making the following diagram commutative*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ \uparrow & \nearrow & \\ S & & \end{array}$$

For $S = *$ we denote a pointed $\mathbb{k}$ -scheme $\{x\} \rightarrow X$ as a pair (X, x) .

Definition 3.6. We say that two pointed $\mathbb{k}$ schemes X, Y are smoothly equivalent if there exists a pointed scheme (M, m) together with smooth pointed maps to X and to Y .

$$\begin{array}{ccc} & (M, m) & \\ \swarrow \text{sm} & & \searrow \text{sm} \\ (X, x) & & (Y, y). \end{array}$$

Definition 3.7. Let Y be a Λ -scheme. The functor $\text{Hilb}_\delta(Y): \text{Sch}_\Lambda^{\text{op}} \rightarrow \text{Set}$ given by

$$\text{Hilb}_\delta(Y)(S) := \left\{ (0 \rightarrow \mathcal{I} \rightarrow \mathcal{O}_{Y \times_\Lambda S} \rightarrow \mathcal{O}_X \rightarrow 0) \left| \begin{array}{l} X \rightarrow S \text{ is finite,} \\ \mathcal{O}_X \text{ is finite locally free } \mathcal{O}_S\text{-module of rank } \delta \end{array} \right. \right\}$$

is called Hilbert functor of δ points on Y . In other words an S -point is closed subscheme of $X \subset Y \times S$ flat over S .

The Hilbert functor is representable.

Theorem 3.8 ([FGI⁺05, Theorem 5.14]). For Y a quasi-projective scheme, $\text{Hilb}_\delta(Y)$ is representable by quasi-projective scheme called the Hilbert scheme of δ -points.

Even though in general the singularities of the Hilbert scheme are arbitrary bad, see [Jel20], in the special case of smooth surface the Hilbert scheme is even smooth.

Theorem 3.9 ([Fog68]). The Hilbert scheme $\text{Hilb}_\delta(\mathbb{A}^2)$ is a smooth variety of dimension 2δ .

Proposition 3.10 ([Ser06, Proposition 3.2.1]). The tangent space to the Hilbert scheme at a $\mathbb{k}$ -point

$$0 \rightarrow I \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow 0$$

is given by $\text{Hom}_{\mathcal{O}_Y}(I, \mathcal{O}_X)$.

Definition 3.11. A component of the Hilbert scheme $\text{Hilb}_\delta(\mathbb{A}^n)$ is called elementary if it parametrises subschemes supported at a single point.

Recall what is a finite Gorenstein scheme, Definition 2.9. We are interested in the subscheme of the Hilbert scheme parameterising Gorenstein subschemes of the affine space.

Definition 3.12. Let $\text{Hilb}_\delta^{\text{Gor}}(\mathbb{A}^n)$ denote the Gorenstein locus of the Hilbert scheme of δ -points on $\mathbb{A}^n$.

Proposition 3.13 ([CN07, Proposition 2.5]). $\text{Hilb}_\delta^{\text{Gor}}(\mathbb{A}^n)$ is an open subscheme of $\text{Hilb}_\delta(\mathbb{A}^n)$.

We have also a variant of the Hilbert functor with nesting of subschemes.

Definition 3.14. Let Y be a Λ -scheme. The S -points of the functor $\text{Hilb}_{\delta_1, \delta_2}(Y): \text{Sch}_\Lambda^{\text{op}} \rightarrow \text{Set}$ is given by diagrams

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_2 & \longrightarrow & \mathcal{O}_{Y \times_\Lambda S} & \longrightarrow & \mathcal{O}_{X_2} \longrightarrow 0 \\ & & \downarrow & & \parallel & & \downarrow \\ 0 & \longrightarrow & I_1 & \longrightarrow & \mathcal{O}_{Y \times_\Lambda S} & \longrightarrow & \mathcal{O}_{X_1} \longrightarrow 0 \end{array}$$

such that $X_i \rightarrow S$ is finite, flat S scheme of length d_i , for $i = 1, 2$. is called the nested Hilbert functor for tuple δ_1, δ_2 of points on Y .

Theorem 3.15 ([Ser06, Theorem 4.5.1]). *For X a quasi-projective scheme, $\text{Hilb}_{\delta_1, \delta_2}(Y)$ is representable by quasi-projective scheme called the nested Hilbert scheme of (δ_1, δ_2) -points. If $\delta_1 = 1$ we call this scheme the pointed Hilbert scheme.*

3.2 Białynicki-Birula decomposition

In this section we state known and useful results on the Białynicki-Birula decomposition. The main aim here is to make Section 4.2 more comprehensible (by the comparison with the classical situation) and to fix the notation. Assume that X is a scheme of locally finite type over Λ .

Definition/Notation 3.16. *The group scheme $\Lambda[t, t^{-1}]$ is denoted by $\mathbb{G}_m$. The affine line $\mathbb{A}_\Lambda^1$ admits a natural action induced by an open embedding $\mathbb{G}_m \rightarrow \mathbb{A}^1$ given by*

$$\begin{aligned} \Lambda[t^{-1}] &\rightarrow \Lambda[t, t^{-1}] \\ t^{-1} &\mapsto t^{-1}. \end{aligned}$$

Note that the embedding in Definition 3.16 is non-standard, this is a deliberate choice.

Definition/Notation 3.17. *Let $\mathbf{G}$ be a group scheme over Λ , then any group homomorphism $\lambda: \mathbb{G}_m \rightarrow \mathbf{G}$ we call a one-parameter subgroup.*

Definition/Notation 3.18. *We say that X is a $\mathbb{G}_m$ -scheme if it admits a $\mathbb{G}_m$ -action.*

Definition/Notation 3.19. *The fixed-point locus of the scheme X is a closed subscheme of X and is denoted by $X^{\mathbb{G}_m}$.*

Proposition 3.20. *The tangent space to X at a $\mathbb{G}_m$ -fixed $\mathbb{k}$ -point $x \in X^{\mathbb{G}_m}$ admits a $\mathbb{Z}$ -grading induced by $\mathbb{G}_m$ -action. This grading defines a decomposition*

$$T_x X = (T_x X)_{<0} \oplus (T_x X)_0 \oplus (T_x X)_{>0}.$$

Definition 3.21 ([Dri15], [JS19, Theorem 1.1]). *Let X be a $\mathbb{G}_m$ -scheme. There is a Białynicki-Birula functor of X given by:*

$$X^{\leq 0}(S) = X(\mathbb{A}^1 \times S)^{\mathbb{G}_m}.$$

We have a natural affine $\mathbb{G}_m$ -equivariant map $\pi: X^{\leq 0} \rightarrow X^{\mathbb{G}_m}$ given by the embedding $\{0\} \times S \rightarrow \mathbb{A}^1 \times S$.

The image of π of a point $x \in X^{\leq 0}$ is the limit of x with t going to ∞ .

Definition 3.22. *Let X be a $\mathbb{G}_m$ -scheme, and $x \in X^{\mathbb{G}_m}(\mathbb{k})$. The Białynicki-Birula fibre of X over x is given as the fibre*

$$\begin{array}{ccc} P & \hookrightarrow & X^{\leq 0} \\ \downarrow & & \downarrow \pi \\ \{x\} & \hookrightarrow & X^{\mathbb{G}_m}. \end{array}$$

In the context where x is fixed, we denote the fibre P as $X^{<0}$.

The projection $\mathbb{A}^1 \times S \rightarrow S$ is $\mathbb{G}_m$ -equivariant and it induces a map

$$\sigma(S): X(S)^{\mathbb{G}_m} \rightarrow X(\mathbb{A}^1 \times S)^{\mathbb{G}_m} = X^{\leq 0}(S),$$

The morphism σ is a section of π .

The embedding $\{1\} \times S \rightarrow \mathbb{A}^1 \times S$ induces a map

$$\iota(S): X^{\leq 0}(S) = X^{\mathbb{G}_m}(\mathbb{A}^1 \times S) \rightarrow X(S).$$

Proposition 3.23 ([JS19, Lemma 5.8]). *The map ι is a monomorphism.*

Let us make a very short summary. For a given fixed point $x \in X^{\mathbb{G}_m}$ we get the following diagram, describing Białynicki-Birula decomposition of X

$$\begin{array}{ccc} X^{<0} & \hookrightarrow & X^{\leq 0} \xrightarrow{\iota} X \\ \downarrow & & \downarrow \pi \uparrow \sigma \\ \{x\} & \hookrightarrow & X^{\mathbb{G}_m}. \end{array}$$

Theorem 3.24 ([JS19, Theorem 1.1]). *Let X be a quasi-projective $\mathbb{G}_m$ -scheme and let $X^{\mathbb{G}_m} = \bigsqcup X_i^{\mathbb{G}_m}$ be the decomposition of the fixed point locus into connected components. Then $X^{\leq 0}$ is represented by a disjoint union $\bigsqcup X_i^{\leq 0}$ of locally closed subschemes of X , such that $\pi^{-1}(X_i^{\mathbb{G}_m}) = X_i^{\leq 0}$.*

Definition/Notation 3.25. *Elements of the decomposition $X_i^{\leq 0}$ are Białynicki-Birula cells.*

Definition/Notation 3.26. *Let $x \in X^{\mathbb{G}_m}$, the Białynicki-Birula cell centred at x is the cell of X that contains x .*

Proposition 3.27. *The tangent space to $X^{\leq 0}$ (respectively to $X^{<0}$) at a fixed point $x \in X^{\mathbb{G}_m}$ is $(T_x X)_{\leq 0}$ (respectively to $(T_x X)_{<0}$).*

Example 3.28. *Let us take $\mathbb{A}^1$ with the standard $\mathbb{G}_m$ -action, that is $\mathbb{A}^1 = \text{Spec } \mathbb{k}[x]$ with $\deg(x) = 1$. Take S with trivial $\mathbb{G}_m$ -action, then a $\mathbb{G}_m$ -equivariant morphism $\mathbb{A}^1 \times S \rightarrow \mathbb{A}^1$ comes from a graded homomorphism*

$$\mathbb{k}[x] \rightarrow \mathcal{O}_S[t^{-1}],$$

since $\deg(t^{-1}) = 1$, see Definition/Notation 3.16, such homomorphism factors through $\mathbb{k}$. As a consequence, the Białynicki-Birula decomposition consists of a single point, the origin.

The tangent space to the origin $o \in \mathbb{A}^1$ is a one-dimensional and concentrated in degree 1, hence the non-positive part $(T_o \mathbb{A}^1)_{\leq 0}$ vanishes.

The Białynicki-Birula decomposition is a great tool to understand local behaviour of a scheme, especially since under certain conditions the locally $X^{\leq 0}$ is an open subscheme of X , in particular they have the same singularity type.

Theorem 3.29 ([JS19, Proposition 1.6]). *Let $x \in X^{\mathbb{G}_m}$ be a $\mathbb{G}_m$ -fixed $\mathbb{k}$ -point, such that $(T_x X)_{>0}$ vanishes. Then $\iota: X^{\leq 0} \rightarrow X$ is an open immersion near x .*

Definition/Notation 3.30. *If we consider inverse $\mathbb{G}_m$ -action on X , that is in Definition 3.16 put different embedding of $\mathbb{G}_m$ into $\mathbb{A}^1$*

$$\begin{aligned} \Lambda[t^{-1}] &\rightarrow \Lambda[t, t^{-1}] \\ t^{-1} &\mapsto t, \end{aligned}$$

and then apply Białynicki-Birula machinery we get another decomposition and associate notations of $X^{\geq 0}$ and $X^{> 0}$.

Definition/Notation 3.31. The condition of vanishing of $(T_x X)_{< 0}$ is called trivial negative tangency of X at x .

Lemma 3.32. Let us consider two affine spaces X, Y of the same dimension, both with linear actions of $\mathbb{G}_m$. Let us assume that $X^{\mathbb{G}_m} = \{x\}$ and $Y^{\mathbb{G}_m} = \{y\}$, $X = X^{\leq 0}$ and $Y = Y^{\leq 0}$. Then any equivariant morphism $\phi: X \rightarrow Y$, which induces isomorphism of graded vector spaces

$$\mathfrak{t}_\phi: T_x X \rightarrow T_y Y$$

is an isomorphism of the schemes.

Proof. Since X has a contracting $\mathbb{G}_m$ -action, we get a natural equivariant isomorphism

$$\begin{aligned} X &\cong T_x X, \\ x &\mapsto \frac{\partial(g \cdot x)}{\partial g}(x). \end{aligned}$$

To be precise, since X is the Białynicki-Birula cell for $\mathbb{G}_m$, the functor

$$X^{\leq 0}(S) = \{\mathbb{A}^1 \times S \rightarrow X, \mathbb{G}_m\text{-equivariant}\}$$

is represented by X . There is a natural transformation of the functors from $X^{\leq 0}$ to

$$X_\varepsilon^{\leq 0}(S) = \{\mathrm{Spec} \mathbb{k}[\varepsilon]/(\varepsilon^2) \times S \rightarrow X, \mathbb{G}_m\text{-equivariant}\}$$

given by the closed embedding of the thicken point $\mathrm{Spec} \mathbb{k}[\varepsilon]/(\varepsilon^2)$ into the line $\mathbb{A}^1$ at the origin

$$\begin{array}{ccc} \mathbb{A}^1 \times S & \xrightarrow{\quad} & X \\ \uparrow & \searrow & \\ \mathrm{Spec} \mathbb{k}[\varepsilon]/(\varepsilon^2) \times S & & \end{array}$$

Since the fixed point locus of X is the point $\{x\}$, the functor $X_\varepsilon^{\leq 0}$ is represented by $T_x X$. The morphism $X \rightarrow T_x X$ coming from the natural transformation $X^{\leq 0} \rightarrow X_\varepsilon^{\leq 0}$ is an isomorphism because given an S -point of $X_\varepsilon^{\leq 0}$ there is only one way to complete the following diagram of $\mathbb{G}_m$ -schemes

$$\begin{array}{ccc} \mathbb{A}^1 \times S & \xrightarrow{\quad \dots \quad} & X \\ \uparrow & \searrow & \\ \mathrm{Spec} \mathbb{k}[\varepsilon]/(\varepsilon^2) \times S & & \end{array}$$

The same holds for Y , hence we obtain the following diagram

$$\begin{array}{ccc} T_x X & \xrightarrow{\mathfrak{t}_\phi} & T_y Y \\ \cong \uparrow & & \cong \uparrow \\ X & \xrightarrow{\phi} & Y. \end{array}$$

By functoriality, the diagram is commutative and since $d\phi$ is an isomorphism, the morphism ϕ is also an isomorphism. $\square$

Proposition 3.33. *A $\mathbb{G}_m$ -action on $\mathbb{A}^n$ yields an action on the Hilbert scheme $\text{Hilb}_\delta(\mathbb{A}^n)$. The fixed points of $\text{Hilb}_\delta(\mathbb{A}^n)$ are coming from homogeneous ideals of $H^0(\mathcal{O}_{\mathbb{A}^n}) = \mathbb{k}[x_1 \dots x_n]$, in particular every monomial ideal of colength δ yields a fixed point of the Hilbert scheme.*

3.2.1 Comparison with Gröbner cells

In this subsection we specialize to $X = \text{Hilb}_\delta(\mathbb{A}^2)$, where $\mathbb{A}^2 = \text{Spec}(\mathbb{k}[x, y])$. Since the motivation of Chapter 5 comes from commutative algebra, we discuss the relation between monomial orders and $\mathbb{Z}$ -graded structures on the polynomial ring $\mathbb{k}[x, y]$. For reference on monomial orders see [GLS07, Section 1.2].

Table 3.1: Dictionary between graded structures and monomial orders

$\mathbb{Z}$ -grading	Monomial ordering
a $\mathbb{Z}$ -grading structure $\deg$ on $k[x, y]$	a monomial order $\succeq$ on $\mathbb{k}[x, y]$
$D = \left\{ (\alpha - \alpha', \beta - \beta') \mid \deg(x^\alpha y^\beta) > \deg(x^{\alpha'} y^{\beta'}) \right\}$	$D_\succeq = \left\{ (\alpha - \alpha', \beta - \beta') \mid x^\alpha y^\beta \succeq x^{\alpha'} y^{\beta'} \right\}$
$\psi(\alpha, \beta) = \alpha \deg(x) + \beta \deg(y)$	a functional $\psi: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ such that $\psi(D_\succeq) \subset \mathbb{Z}_{\geq 0}$

In case of the Hilbert scheme of δ -points we consider only finitely many monomials. Thus, for a chosen monomial order $\succeq$, we can define ψ that maps the set $D_\succeq$ to strictly positive numbers, for reference see [GLS07, Lemma 1.2.11]. A graded structure on $\mathbb{k}[x, y]$ translates to a $\mathbb{G}_m$ -action on Hilbert scheme, and thus we obtain the following dictionary, Table 3.2.

Let $I \subset \mathbb{k}[x, y]$ be an ideal of colength δ , and let J be a fixed monomial ideal of colength δ .

Table 3.2: Dictionary between the limits of $\mathbb{G}_m$ -action and the initial term ideals

$\mathbb{G}_m$ -action	Monomial ordering
$\lim_{t \rightarrow 0} t \cdot [I] \in \text{Hilb}_\delta(\mathbb{A}^2)$	$\text{in}_\succeq(I) = (\text{in}_\succeq(f), f \in I)$
The Białynicki-Birula cell centred at $[J]$	The Gröbner cell $\{ I \subset \mathbb{k}[x, y], \text{in}_\succeq(I) = J \}$.

The limit $\lim_{t \rightarrow 0} t \cdot [I] \in \text{Hilb}_\delta(\mathbb{A}^2)$ is $\pi([I])$ in the notation of Definition 3.21 for an inverse $\mathbb{G}_m$ -action, see Notation 3.30, since it is the limit in 0 not in infinity.

Example 3.34. [GLS07, Example 1.2.8] *Let us consider the negative degree lexicographical ordering on $\mathbb{k}[x, y]$:*

$$x^\alpha y^\beta \succeq x^{\alpha'} y^{\beta'} \iff \alpha + \beta < \alpha' + \beta' \text{ or } \alpha + \beta = \alpha' + \beta' \text{ and } \alpha > \alpha'. \quad (3.35)$$

We follow the dictionary from Table 3.1, and look for an equivalent graded structure. From the second part of Condition (3.35) we get

$$\deg(x) < \deg(y) < \deg(1) = 0.$$

The first part of Condition (3.35) says that $\deg(x)$ and $\deg(y)$ should be almost equal. From that we see that, if we restrict ourselves to ideals of colength δ , the graded structure

$$\deg(y) = -\zeta,$$

$$\deg(x) = -\zeta(1 - \varepsilon),$$

for small enough $\varepsilon \in \mathbb{Q}_{>0}$, and $\zeta \in \mathbb{Z}_{>0}$ divisible enough, such that $\varepsilon\zeta \in \mathbb{Z}$, induces the same cell structure according to the dictionary from Table 3.2 as the chosen order.

Chapter 4

Local deformation functors

This chapter rhymes with the previous one, it starts with the functors and the classical theory, Section 4.1 and then switches to the local variant of Białynicki-Birula decomposition, Section 4.2.

4.1 Good deformation functors

This section is mainly an overview of the classical deformation theory, the main sources and references are [Sch68], [FM98], [Ser06] and [Har10].

To be able to check the homogeneity condition of a local functor, see (H4) in Definition 4.12, we use language of stacks, and get the group of infinitesimal automorphisms which yields computable criterion for prorepresentability, see Proposition 4.31. The main references to the part referring to stacks are [Sta22] and [Alp25]. The last part of the section describes various deformation functors of a morphism, the references for that are [Ill71],[Buc81] and [GLS07].

Recall, from Definition 2.1, that Art_Λ denotes the category of local Artinian Λ -algebras with the residue field isomorphic to $\mathbb{k}$, where Λ is Noetherian complete local algebra with the residue isomorphic to $\mathbb{k}$.

Definition 4.1 ([Sch68, Functors on C], [Ser06, 2.2 Functors of Artin rings]). A predeformation functor is a covariant functor $F: \text{Art}_\Lambda \rightarrow \text{Set}$, such that $F(\mathbb{k}) = *$.

Definition 4.2. Let us take a complete local ring R , then we can define a functor $h_R: \text{Art}_\Lambda \rightarrow \text{Set}$ by setting

$$h_R(A) = \text{Hom}(R, A).$$

We say that a predeformation functor F is prorepresentable if there exists a complete local ring such that

$$F \cong h_R.$$

Definition/Notation 4.3. In the category predeformation functors there is a terminal object

$$\begin{aligned} * &:= h_{\mathbb{k}}: \text{Art}_\Lambda \rightarrow \text{Set}, \\ A &\mapsto \text{Hom}_\Lambda(A, \mathbb{k}). \end{aligned}$$

Definition 4.4. Let us take a functor $\tilde{F}: \text{Sch}_\Lambda^{\text{op}} \rightarrow \text{Set}$, choose an element of $x \in \tilde{F}(\text{Spec}(\mathbb{k}))$.

Restrict $\tilde{F}$ to a covariant functor $\mathbf{Art} \rightarrow \mathbf{Set}$, and consider the pullback functor F_x

$$\begin{array}{ccc} F_x & \longrightarrow & \tilde{F}(A) \\ \downarrow & & \downarrow \\ * & \xrightarrow{x} & \tilde{F}(\mathrm{Spec}(\mathbb{k})) \end{array}$$

then F_x is a predeformation functor. We call F_x the local functor of F at x .

Definition 4.5. Let X be a Λ -scheme and $x \in X$ a $\mathbb{k}$ -point. We define a local functor of X at x X_x to be the local functor of $\mathrm{Hom}_{\mathrm{Sch}_\Lambda}(_, X)$ at x . Such predeformation functor is prorepresentable by the completed local ring of X at x , $\widehat{\mathcal{O}_{X_x}}$.

Definition 4.6. Let X be a Λ -scheme. We define a predeformation functor $\mathrm{Def}_X: \mathbf{Art}_\Lambda \rightarrow \mathbf{Set}$, let $A \in \mathbf{Art}_\Lambda$, then $\mathrm{Def}_X(A)$ is the set of isomorphism classes of Cartesian diagrams

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow \phi & & \downarrow \phi_A \\ * & \longrightarrow & \mathrm{Spec}(A). \end{array} \quad (4.7)$$

Definition 4.8. We say that a morphism of $F \rightarrow G$ of predeformation functors is smooth if for every surjection $B \rightarrow A$ in $\mathbf{Art}_\mathbb{k}$ the natural map:

$$F(B) \rightarrow F(A) \times_{G(A)} G(B)$$

induced by the commutative diagram

$$\begin{array}{ccc} F(B) & \longrightarrow & G(B) \\ \downarrow & & \downarrow \\ F(A) & \longrightarrow & G(A) \end{array}$$

is surjective.

Definition 4.9. We say that a predeformation functor F is smooth, if $F \rightarrow *$ is a smooth map.

Definition 4.10 ([Ser06, Definition 2.2.6]). Let F be a predeformation functors. We say it admits a semi-universal family if there is a smooth map $H \rightarrow F$ from a prorepresentable functor H , such that the induced map of the tangent spaces is isomorphic.

Definition 4.11. A small extension is a surjection in $\mathbf{Art}_\mathbb{k}$:

$$B \rightarrow A,$$

such that the kernel is annihilated by the maximal ideal of B , in particular it is a $\mathbb{k}$ -vector space.

Definition 4.12 ([Sch68, Schlessinger conditions, Theorem 2.11]). Let F be a predeformation functors. Let $B \rightarrow A$ and $C \rightarrow A$ be a pair of homomorphisms in $\mathbf{Art}_\mathbb{k}$, and consider the natural map

$$F(B \times_A C) \rightarrow F(B) \times_{F(A)} F(C). \quad (\diamond)$$

Then the following conditions are the Schlessinger conditions

(H1) if $C \rightarrow A$ is a small extension then the map $(\diamond)$ is surjective.

(H2) whenever $A = \mathbb{k}$ and $C = \mathbb{k}[\varepsilon]/\varepsilon^2$, then the map $(\diamond)$ is bijective.

(H3) The conditions (H1) and (H2) are satisfied, and $F(\mathbb{k}[\varepsilon]/\varepsilon^2)$ is a finitely dimensional vector space, see Remark 4.13.

(H4) if $C \rightarrow A$ is a small extension then the map $(\diamond)$ is bijective.

(L) whenever $A = \mathbb{k}$ then the map $(\diamond)$ is bijective.

Remark 4.13. By [FM98, Lemma 2.12] the conditions (H1)-(H2) imply that $F(\mathbb{k}[\varepsilon]/\varepsilon^2)$ is a $\mathbb{k}$ -vector space, called the tangent space to F (and denoted $\mathfrak{t}_F$).

Remark 4.14. The original condition (L) of linearity of obstruction theory from [FM98, Definition 2.9] is different but by [FM98, Lemma 2.11] our condition is stronger.

Theorem 4.15 ([Sch68, Theorem 2.11]). If F satisfies the conditions (H1),(H2),(H3), then it admits a semi-universal family. Furthermore, if it satisfies (H4) then F is prorepresentable.

Definition 4.16. Let $\mathbf{A}$ be a set, a pointed set is a pair $(\mathbf{A}, a)$ such that $a \in \mathbf{A}$. Let $(\mathbf{A}, a)$ be a pointed set and $\mathbf{B}, \mathbf{C}$ sets. Then an exact sequence of pointed sets is

$$\mathbf{C} \rightarrow \mathbf{B} \rightarrow \mathbf{A}$$

such that the image of $\mathbf{C}$ in $\mathbf{B}$ is the preimage of the point a .

For a given $\mathbb{k}$ -vector space V (or more generally a group), that it admits an obvious pointed set structure $(V, 0)$.

Definition 4.17. Let F be a predeformation functor, a (linear) obstruction space $\mathfrak{o}_F$ is a $\mathbb{k}$ -vector space, such that for any small extension $0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$ we get a functorial exact sequence of pointed sets

$$F(B) \rightarrow F(A) \rightarrow \mathfrak{o}_F \otimes I.$$

The image of $f \in F(A)$ in $\mathfrak{o}_F \otimes I$ is the obstruction of f to lifting to B .

Remark 4.18. Functoriality in Definition 4.17 means that if we take a morphism of small extensions

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & J & \longrightarrow & C & \longrightarrow & D \longrightarrow 0, \end{array}$$

we get a commutative diagram of pointed sets

$$\begin{array}{ccccc} F(B) & \longrightarrow & F(A) & \longrightarrow & \mathfrak{o}_F \otimes I \\ \downarrow & & \downarrow & & \downarrow \\ F(C) & \longrightarrow & F(D) & \longrightarrow & \mathfrak{o}_F \otimes J. \end{array}$$

Definition 4.19. For a given predeformation functor F , a tangent-obstruction theory is a pair of $\mathbb{k}$ -vector spaces

$$\mathfrak{t}_F, \mathfrak{o}_F,$$

where $\mathfrak{t}_F$ is the tangent space and $\mathfrak{o}_F$ is an (linear) obstruction space.

Remark 4.20. *The tangent-obstruction theories that we are going to consider are functorial in the following sense. Let us take a pair F and G of predeformation functors with tangent-obstruction theories $(\mathfrak{t}_F, \mathfrak{o}_F)$ and $(\mathfrak{t}_G, \mathfrak{o}_G)$. We say that these theories are functorial if for any map $F \rightarrow G$, we get induced maps of tangent spaces and of obstruction spaces.*

$$\begin{aligned} \mathfrak{t}_\phi: \mathfrak{t}_F &\rightarrow \mathfrak{t}_G, \\ \mathfrak{o}_\phi: \mathfrak{o}_F &\rightarrow \mathfrak{o}_G, \end{aligned}$$

such that for any small extension $0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0$, we get the following commutative diagram

$$\begin{array}{ccccccc} \mathfrak{t}_F \otimes I & \longrightarrow & F(B) & \longrightarrow & F(A) & \longrightarrow & \mathfrak{o}_F \otimes I \\ \downarrow \mathfrak{t}_\phi \otimes I & & \downarrow \phi & & \downarrow \phi & & \downarrow \mathfrak{o}_\phi \otimes I \\ \mathfrak{t}_G \otimes I & \longrightarrow & G(B) & \longrightarrow & G(A) & \longrightarrow & \mathfrak{o}_G \otimes I. \end{array}$$

We have defined two vector spaces controlling a predeformation functor. The functors coming from the geometric context admit a third one, which can be defined only by leaving the classical notions and diving into the domain of stacks. This is the proper setting to study the Schlessinger homogeneity condition (H4), as we see in Proposition 4.31. The objects of our interest are good deformation functors, which intuitively are functors *coming from the geometry*, by which we mean they are functors associated with stacks satisfying Rim-Schlessinger condition, see Definition 4.26.

Definition 4.21 ([Sta22, Tag 06GS]). A predeformation category $\mathfrak{F}$ is a category cofibred in groupoids over $\mathbf{Art}_\Lambda$ such that $\mathfrak{F}(\mathbb{k})$ is equivalent to category with one object and one morphism.

Definition 4.22. Let $\mathfrak{F}$ be a predeformation theory. We define a functor $\overline{\mathfrak{F}}: \mathbf{Art}_\Lambda \rightarrow \mathbf{Set}$ by setting $\overline{\mathfrak{F}}(A)$ to be the set of isomorphism classes of the category $\mathfrak{F}(A)$.

Definition 4.23 ([Alp25, Theorem 3.11 (Rim-Schlessinger's criteria)]). Let $\mathfrak{F}$ be a predeformation category. Let $B \rightarrow A$ and $C \rightarrow A$ be a pair of homomorphisms in $\mathbf{Art}_\mathbb{k}$, and consider the natural map of groupoids

$$\mathfrak{F}(B \times_A C) \rightarrow \mathfrak{F}(B) \times_{\mathfrak{F}(A)} \mathfrak{F}(C). \quad (\clubsuit)$$

Then the following conditions is the Rim-Schlessinger condition

(RS) whenever $C \rightarrow A$ is a small extension the map $(\clubsuit)$ is an equivalence of categories.

Remark 4.24. If $\mathfrak{F} = \overline{\mathfrak{F}}$ then the Rim-Schlessinger condition (RS) is exactly the Schlessinger homogeneity condition (H4).

Definition 4.25 ([Sta22, Tag 06J9]). A deformation category is a predeformation category $\mathfrak{F}$ satisfying (RS).

Definition 4.26. A predeformation functor F is a good deformation functor if there is a deformation category $\mathfrak{F}$ such that $\overline{\mathfrak{F}} = F$.

The notion of a good deformation functor is inspired by functors with good deformation and obstruction theory introduced in [FM98, Definition 2.10]. By Proposition 4.27, any good deformation functor is a functor with good deformation and obstruction theory. Recall that, from Remark 4.14, by Lemma [FM98, Lemma 2.11], the condition (L) is stronger than the condition of linearity of obstructions defined in [FM98, Definition 2.9].

Proposition 4.27 ([Sta22, Tag 06I0], [FM98, Theorem 6.11]). *A good deformation functor F satisfies the conditions (H1), (H2) and (L), consequently admits a tangent-obstruction theory.*

Definition 4.28. *The group of infinitesimal automorphisms $\mathbf{aut}_{\mathfrak{F}}$ is the group of automorphisms of any object in $\mathfrak{F}(\mathbb{k}[\varepsilon]/(\varepsilon^2))$. By [Sta22, Lemma 06JX], the group $\mathbf{aut}_{\mathfrak{F}}$ admits a natural $\mathbb{k}$ -vector space structure.*

Definition 4.29. *For a given good deformation functor F together with an associated deformation category $\mathfrak{F}$, such that $\overline{\mathfrak{F}} = F$, we define an associated triple*

$$\mathbf{aut}_{\mathfrak{F}}, \mathbf{t}_F, \mathbf{o}_F,$$

where $\mathbf{aut}_{\mathfrak{F}}$ is the infinitesimal automorphism group, see Definition 4.28, and the rest constitutes a tangent-obstruction theory of F , see Definition 4.19.

If context is clear we denote $\mathbf{aut}_{\mathfrak{F}}$ by $\mathbf{aut}_F$.

Proposition 4.30 ([Har10, Theorem 10.1]). *Let X be an affine $\mathbb{k}$ -scheme then the functor Def_X admits a natural associated triple induced by*

$$T_X^0, T_X^1, T_X^2,$$

where $T_X^i = T_{\mathbb{k}}^i(\mathcal{O}_X, \mathcal{O}_X)$, are the Schlessinger-Lichtenbaum functors, Definition A.6 in Appendix.

The group of infinitesimal automorphisms gives us a computable way to check the homogeneity condition (H4).

Proposition 4.31 ([Sta22, Lemma 06K0]). *Let $\mathfrak{F}$ be a deformation category. If $\mathbf{aut}_{\mathfrak{F}}$ vanishes then $\mathfrak{F} = F$. If additionally $\mathbf{t}_F$ is finite dimensional then F is prorepresentable.*

Lemma 4.32 ([Man99, Corollary 2.11]). *Assume that $\mathbf{aut}_G = 0$, then a smooth morphism $F \rightarrow G$ inducing an isomorphism on the tangent spaces is an isomorphism of good deformation functors.*

Corollary 4.33. *Let $\phi: F \rightarrow G$ be a smooth morphism of good deformation functors that induces an isomorphism on the tangent spaces. Let us take $H \rightarrow G$ a map from prorepresentable functor H , then the induced morphism $H \times_F G \rightarrow H$ is an isomorphism.*

Proof. As smoothness and property of being an isomorphism on the tangent spaces is preserved by the pullback, we get the statement by Lemma 4.32. $\square$

Definition 4.34. *Let $\phi: F \rightarrow G$ be a morphism of good deformation functors. We say that ϕ is unobstructed if there is a choice of obstruction spaces $\mathbf{o}_F$ and $\mathbf{o}_G$ such that the map induced by ϕ*

$$\mathbf{o}_F \rightarrow \mathbf{o}_G$$

is injective.

Theorem 4.35 ([Ser06, Proposition 2.3.6], [FM98, Lemma 6.1]). *Let $\phi: F \rightarrow G$ be an unobstructed morphism of good deformation functors. Assume that the induced map of the tangent spaces is surjective. Then ϕ is smooth.*

Definition 4.36. We say that functors G and F are smoothly equivalent if there exists a predeformation functor H with smooth maps as below.

$$\begin{array}{ccc} & H & \\ sm \swarrow & & \searrow sm \\ F & & G. \end{array}$$

A pair of pointed schemes (X, x) , (Y, y) are smoothly equivalent, see Definition 3.6, if the local functors at the chosen points are smoothly equivalent.

Definition 4.37. Let F be a predeformation functor. We say that F is non-reduced if F is smoothly equivalent to a functor h_R for R a non-reduced completed local ring.

Lemma 4.38. Let us take a diagram of deformation functors

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ * & \longrightarrow & H. \end{array}$$

Assume that the diagram above induces the following exact sequence of $\mathbb{k}$ -vector spaces,

$$\mathfrak{t}_F \rightarrow \mathfrak{t}_G \rightarrow \mathfrak{t}_H \xrightarrow{0} \mathfrak{o}_F \rightarrow \mathfrak{o}_G \rightarrow \mathfrak{o}_H.$$

Then the induced map $F \xrightarrow{\phi} P := G \times_H *$ is smooth.

Proof. The exactness of the sequence implies that ϕ is surjective on the tangent spaces. The map

$$\mathfrak{o}_F \rightarrow \mathfrak{o}_G$$

is injective and factors through $\mathfrak{o}_P$, thus we get that the map ϕ is unobstructed. By Theorem 4.35, we get the conclusion. $\square$

We have several local analogs of the nested Hilbert scheme, recall Definition 3.14.

Definition 4.39 ([Ser06, Section 3.4]). Let $\phi: X \rightarrow Y$ be a morphism of affine $\mathbb{k}$ -schemes. Let $A \in \text{Art}_\Lambda$. We define the following good deformation functors.

- [Ser06, Definition 3.4.1] Deformations of a morphism.

$\text{Def}_{X \rightarrow Y}(A)$ is a set of isomorphism classes of Cartesian diagrams

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow \phi & & \downarrow \phi_A \\ Y & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \psi \\ * & \longrightarrow & \text{Spec}(A), \end{array} \quad (\heartsuit)$$

where ψ and $\psi \circ \phi_A$ are flat morphisms.

- [Ser06, Subsection 3.4.2] Deformations of a morphism with target fixed.

$\text{Def}_{X/Y}(A)$ is the set of isomorphism classes of Cartesian diagrams as in Diagram (♡) but with chosen flat Λ -lift $\mathcal{Y}_\Lambda$ of Y and we put $\mathcal{Y} = \mathcal{Y}_\Lambda \times \text{Spec}(A)$.

- Deformations of a morphism with source fixed.

$\text{Def}_{X \setminus Y}(A)$ is the set of isomorphism classes of Cartesian diagrams as in Diagram (♡) but with chosen flat Λ -lift $\mathcal{X}_\Lambda$ of X and we put $\mathcal{X} = \mathcal{X}_\Lambda \times \text{Spec}(A)$.

- [Ser06, Subsection 3.4.1] Deformations of a morphism with target and source fixed.

$\text{Def}_{X \setminus X \rightarrow Y/Y}(A)$ is a set of isomorphism classes of Cartesian diagrams as in Diagram (♡) but $\mathcal{X} = \mathcal{X}_\Lambda \times \text{Spec}(A)$ and $\mathcal{Y} = \mathcal{Y}_\Lambda \times \text{Spec}(A)$.

Definition 4.40. Let us take a closed embedding $X \subset Y$, then $\text{Def}_{X \subset Y} := \text{Def}_{X \rightarrow Y}$ is called the local Hilbert functor denoted hilb_X^Y . In the context in which it is clear what is the codomain we omit the superscript in the notation

Proposition 4.41 ([Buc81, (2.4.4) Le théorème d'Illusie]). The following good deformation functors admit associated triples. The Lichtenbaum-Schlessinger functors are discussed in Appendix, Definition A.6.

Table 4.1: Associated triples	
A good deformation functor	An associated triple
Deformations of a morphism. see Appendix A.10.	$T_{X \rightarrow Y}^0, T_{X \rightarrow Y}^1, T_{X \rightarrow Y}^2$.
Deformations of a morphism with target fixed.	$T_{X/Y}^0, T_{X/Y}^1, T_{X/Y}^2$. $T_{X/Y}^i = T_{\mathcal{O}_Y}^i(\mathcal{O}_X, \mathcal{O}_X)$.
The local Hilbert functor hilb_X^Y $0 \rightarrow I \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow 0$	$0, \text{Hom}_{\mathcal{O}_X}(I/I^2, \mathcal{O}_X), T_{X/Y}^2$.
Deformations of a morphism with source fixed. see Appendix A.11.	$T_{X \setminus Y}^0, T_{X \setminus Y}^1, T_{X \setminus Y}^2$,
There are two special cases:	
$X \rightarrow Y$ is a closed embedding given by $0 \rightarrow I \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow 0$	$T_{X \setminus Y}^i = T_{\mathbb{k}}^i(\mathcal{O}_Y, I)$.
$\mathcal{O}_Y \rightarrow \mathcal{O}_X$ is injective $0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow C \rightarrow 0$	$T_{X \setminus Y}^i = T_{\mathbb{k}}^{i-1}(\mathcal{O}_X, C)$.
Deformations of a morphism with target and source fixed.	$0, T_{X \setminus X \rightarrow Y/Y}^1, T_{X \setminus X \rightarrow Y/Y}^2$. $T_{X \setminus X \rightarrow Y/Y}^i = T_{\mathbb{k}}^{i-1}(\mathcal{O}_Y, \mathcal{O}_X)$.

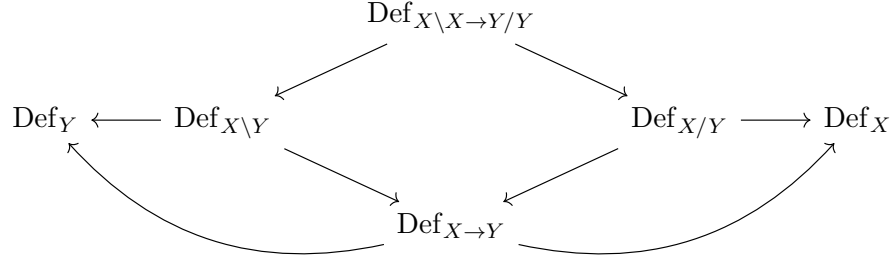
Definition 4.42. Let $\phi: F \rightarrow G$ be a morphism of good deformation functors lifting to a morphism of deformation categories $\mathcal{F} \rightarrow \mathcal{G}$. A relative triple $\text{aut}_r, \mathfrak{t}_r, \mathfrak{o}_r$ of ϕ is a triple of vector

spaces fitting into exact sequence of $\mathbb{k}$ -vector spaces

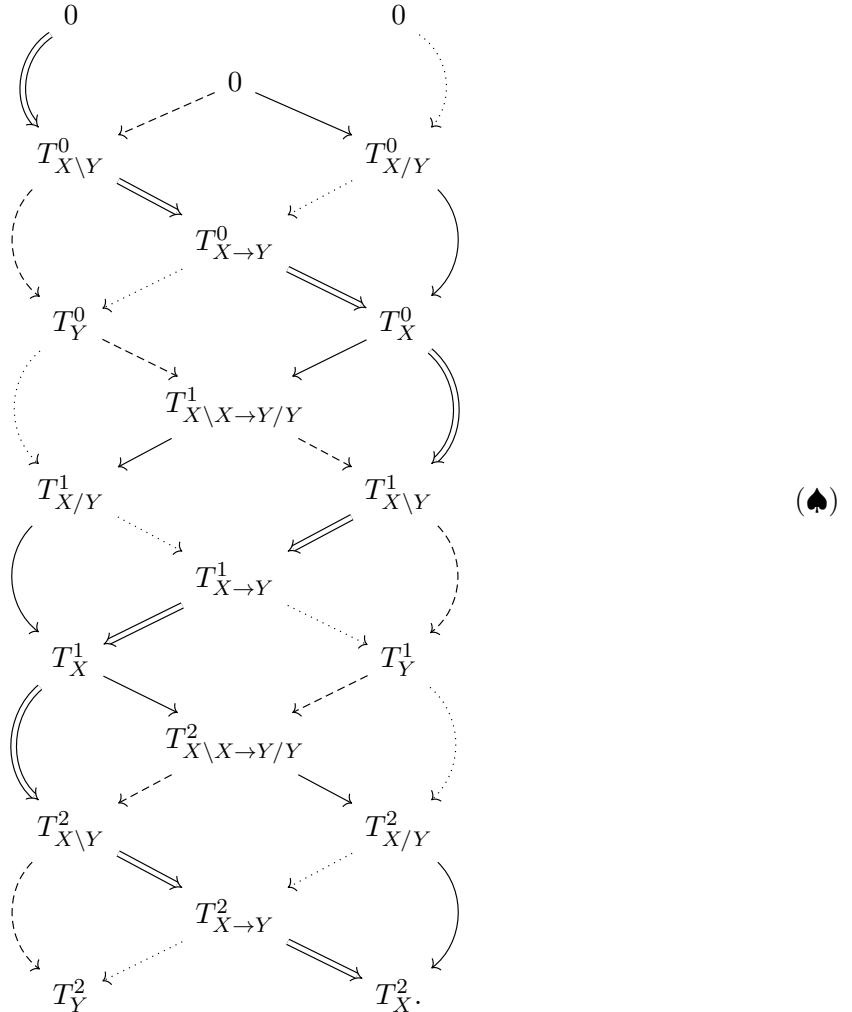
$$\mathbf{aut}_r \rightarrow \mathbf{aut}_F \rightarrow \mathbf{aut}_G \rightarrow \mathbf{t}_r \rightarrow \mathbf{t}_F \rightarrow \mathbf{t}_G \rightarrow \mathbf{o}_r \rightarrow \mathbf{o}_F \rightarrow \mathbf{o}_G.$$

Data of relative triples of functors coming from a map of schemes $F \rightarrow G$ is encoded into a commutative diagram called Braid ♠.

Theorem 4.43 ([Buc81, 2.4.3.2 page 69]). *Let $X \rightarrow Y$ be a morphism of $\mathbb{k}$ -schemes then the following diagram of good deformation functors*



induces four interdependent long exact sequences (each denoted by a differently styled arrow: usual, double, dashed and dotted), that form a commutative diagram as follows



Lemma 4.44. *Let $X \subset \mathbb{A}_{\mathbb{k}}^n$ be a closed embedding then the induced map $\mathrm{hilb}_X^{\mathbb{A}^n} \rightarrow \mathrm{Def}_X$ is smooth.*

Proof. The relative obstruction space is $T_{X \setminus X \rightarrow \mathbb{A}^n / \mathbb{A}^n}^2 = T_{\mathbb{k}}^1(\mathbb{S}, \mathcal{O}_X)$ where $\mathbb{S}$ is a polynomial ring over $\mathbb{k}$ in n -variables, by [Har10, Proposition 3.7] it vanishes. By Theorem 4.35 the statement follows. $\square$

4.2 The filtered deformations

This section is no longer just an overview of the literature, it contains actual contribution to the field. We introduce a new methods of study based on the functor of filtered deformation, see Definition 4.49, which is analogous to the Białynicki-Birula decomposition. The construction is based on the previous results on equivariant deformations, see for example [Pin74], [Kle79]. In case of affine schemes, the main advantage we get by working with abstract filtered deformations in place of the Białynicki-Birula decomposition is this formalism is better suited to deal with smooth equivalence, an instance of that is Lemma 4.69. Similarly to the previous section, at the end we discuss (this time filtered) deformations of a (this time $\mathbb{G}_m$ -equivariant) scheme morphism, see Corollary 4.70.

Definition 4.45. *Let A be a Λ -algebra. A filtered A -algebra $\mathcal{R}$ is a pair $(\mathcal{R}, \mathcal{F})$, where $\mathcal{R}$ is an A -algebra and $\mathcal{F}$ is a filtration by A -modules*

$$\mathcal{R} \supset \dots \supset \mathcal{F}_{-1} \supset \mathcal{F}_0 \supset \mathcal{F}_1 \supset \dots,$$

such that for every $i, j \in \mathbb{Z}$, we have $\mathcal{F}_i \mathcal{F}_j \subset \mathcal{F}_{i+j}$.

Definition 4.46. *A homomorphism of filtered algebras $(\mathcal{R}_1, \mathcal{F})$ and $(\mathcal{R}_2, \mathcal{F}')$ is a homomorphism of algebras $\phi: \mathcal{R}_1 \rightarrow \mathcal{R}_2$ such that $\phi(\mathcal{F}_i) \subset \mathcal{F}'_i$.*

Definition 4.47. *A filtered A -algebra $\mathcal{R}$ is flat if all $i \in \mathbb{Z}$ the module $\mathcal{F}_i / \mathcal{F}_{i+1}$ is a flat A -module.*

Definition 4.48. *Let X be an affine $\mathbb{G}_m$ -scheme, then $\mathcal{O}_X$ is a graded algebra. We can consider two $\mathbb{Z}$ -grading structures on $\mathcal{O}_X = \bigoplus (\mathcal{O}_X)_i$, the given one, and the opposite one $\mathcal{O}_X = \bigoplus (\mathcal{O}_X)_{-i}$.*

Then $\mathcal{O}_X$ admits the filtration by vanishing order

$$\mathcal{F}_i = \bigoplus_{j \geq i} (\mathcal{O}_X)_j$$

and the filtration by degree

$$\mathcal{G}_i = \bigoplus_{j \geq i} (\mathcal{O}_X)_{-j}.$$

This way we have two natural filtered algebras associated with any graded algebra $\mathcal{O}_X$.

Definition 4.49. *Let X be an affine $\mathbb{G}_m$ -scheme, together with the filtration given by vanishing order on $\mathcal{O}_X$. We define a predeformation functor*

$$\mathrm{Def}_X^{\geq 0}: \mathrm{Art}_{\Lambda} \rightarrow \mathrm{Set}$$

to parametrise filtered deformations of X . Let $(A, \mathfrak{m}) \in \text{Art}_\Lambda$. Then $\text{Def}_X^{\geq 0}(A)$ is a set of isomorphism classes of pairs consisting of a diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \text{Spec}(A), \end{array}$$

where $\mathcal{O}_\mathcal{X}$ is a flat filtered A -algebra; and $\mathcal{O}_\mathcal{X} \otimes A/\mathfrak{m}$ is a filtered algebra, together with a filtered algebra isomorphism to $\mathcal{O}_X$.

Definition 4.50. Let X be an affine $\mathbb{G}_m$ -scheme, together with the induced grading on $\mathcal{O}_X$. We define a predeformation functor

$$\text{Def}_X^0: \text{Art}_\Lambda \rightarrow \text{Set}$$

to parametrise equivariant deformations of X . Let $(A, \mathfrak{m}) \in \text{Art}_\Lambda$. Then $\text{Def}_X^0(A)$ is a set of isomorphism classes of the following diagrams

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \text{Spec}(A), \end{array}$$

where $\mathcal{O}_\mathcal{X}$ is a flat graded A -algebra; $\mathcal{O}_\mathcal{X} \otimes A/\mathfrak{m}$ is a graded algebra, together with a graded algebra isomorphism to $\mathcal{O}_X$.

Proposition 4.51. There is a morphism $\pi: \text{Def}_X^{\geq 0} \rightarrow \text{Def}_X^0$, on A -points defined by

$$(\mathcal{O}_\mathcal{X}, \mathcal{F}) \mapsto \text{gr}_\mathcal{F} \mathcal{O}_\mathcal{X}.$$

Proof. We check that π is a well-defined morphism. We choose Artin algebra $(A, \mathfrak{m}) \in \text{Art}_\Lambda$, and an A -point $(\mathcal{O}_\mathcal{X}, \mathcal{F}) \in \text{Def}_X^{\geq 0}(A)$. We take the associated graded algebra $\text{gr}_\mathcal{F} \mathcal{O}_\mathcal{X}$ since, by Definition 4.49, the quotients $\mathcal{F}_i/\mathcal{F}_{i+1}$ are flat over $\mathbb{A}$, we get a flat A -algebra. Since, by Definition 4.49, we have an isomorphism of filtered algebras

$$\mathcal{O}_\mathcal{X} \otimes A/\mathfrak{m} \cong \mathcal{O}_X$$

we get also an isomorphism of associated graded algebras

$$\text{gr}_\mathcal{F} \mathcal{O}_\mathcal{X} \otimes A/\mathfrak{m} \cong \mathcal{O}_X.$$

□

We define a local analogue of the Białynicki-Birula fibre, see Definition 3.22.

Definition 4.52. We define the functor of strictly positive deformations $\text{Def}_X^{\geq 0}$ as the following pullback:

$$\begin{array}{ccc} \text{Def}_X^{\geq 0} & \longrightarrow & \text{Def}_X^{\geq 0} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \text{Def}_X^0. \end{array}$$

Definition 4.53. Let X be a $\mathbb{G}_m$ -scheme and $x \in X^{\mathbb{G}_m}$ a fixed $\mathbb{k}$ -point. The filtered local deformation functor of X at x is $X_x^{\geq 0}$ (resp. > 0), see Definition 4.5. This functor is prorepresentable by the completed local ring of $X^{\geq 0}$ (resp. > 0) at x .

Lemma 4.54 ([Pin74, Propositions 2.2-2.4]). For $i = 0, 1, 2$ the module T_X^i is graded, in particular

$$T_X^i = (T_X^i)_{\geq 0} + (T_X^i)_{\leq 0} = (T_X^i)_{> 0} \oplus (T_X^i)_0 \oplus (T_X^i)_{< 0}.$$

Theorem 4.55. The functor $\text{Def}_X^{\geq 0}$ is a good deformation functor with an associated triple

$$(T_X^0)_{\geq 0}, (T_X^1)_{\geq 0}, (T_X^2)_{\geq 0}.$$

Proof. As too technical for good exposition, the proof is delegated to Appendix, see Corollary A.13. $\square$

Remark 4.56. The structure of filtered deformation functors can also be derived from a much more general construction of stack of filtered objects Filt , [HL22, Definition 1.1.11]. In case of finite schemes, one takes $\mathfrak{X}$ to be a stack of finite algebras, then apply [HL22, Proposition 1.1.2] to get that $\text{Filt}(\mathfrak{X})$ is an algebraic stack. Then we take a $\mathbb{k}$ -point x on $\text{Filt}(\mathfrak{X})$, that is a finite algebra together with a filtration. The local stack at x satisfies Rim-Schlessinger conditions, see [Sta22, Tag 06J1], hence the associated functor is a good deformation functor. Finally, the functor associated with such stack is the functor defined in Definition 4.49.

We introduce trivial tangent condition which is a local analogue of the condition from Definition 3.31

Definition 4.57. We say that a $\mathbb{G}_m$ -scheme X has trivial negative tangents (denoted as **TNT**) if $(T_X^1)_{< 0} = 0$.

Lemma 4.58. Let m be an integer. The conditions that $T^2(\mathcal{O}_X, \mathbb{k})_{< -m} = 0$, $(T_X^1)_{< m} = 0$, $(T_X^0)_{< m} = 0$ and the condition that X is TNT are open in flat $\mathbb{G}_m$ -equivariant families.

Proof. Fix $i \in \{0, 1, 2\}$. Let us take a flat family $\mathcal{X} \rightarrow S$ with the special fibre X . We can endow S with a trivial $\mathbb{G}_m$ -structure, for any $j \in \mathbb{Z}$ the module $T^i(\mathcal{X}/S)_j$ is coherent over S , hence its rank is upper-semicontinuous and vanishing is an open condition. The module $T^i(\mathcal{X}/S)$ is coherent over $\mathcal{O}_{\mathcal{X}}$, hence $T^i(\mathcal{X}/S)_{< m}$ does not vanish for only finitely many graded parts, consequently we get the claim. $\square$

Lemma 4.59 ([Kle79, Theorem 2.3]). Let $X \rightarrow Y$ be a morphism of affine $\mathbb{G}_m$ -schemes then for each variant $XY \in \{X \rightarrow Y, X/Y, X \setminus Y, X \setminus X \rightarrow Y/Y\}$ the following holds. For $i = 0, 1, 2$ the module T_{XY}^i is graded, in particular

$$T_{XY}^i = (T_{XY}^i)_{\geq 0} + (T_{XY}^i)_{\leq 0} = (T_{XY}^i)_{> 0} \oplus (T_{XY}^i)_0 \oplus (T_{XY}^i)_{< 0}.$$

Theorem 4.60. Let us take a morphism $X \rightarrow Y$ of affine $\mathbb{G}_m$ -schemes. Then for each variant $XY \in \{X \rightarrow Y, X/Y, X \setminus Y, X \setminus X \rightarrow Y/Y\}$ we define a functor $\text{Def}_{XY}^{\geq 0}$ of non-negative deformations. Those are good deformation functors and admit associated triples

$$(T_{XY}^0)_{\geq 0}, (T_{XY}^1)_{\geq 0}, (T_{XY}^2)_{\geq 0}.$$

Proof. As a too technical for good exposition, the proof is delegated to Appendix, see Corollary A.13. $\square$

Remark 4.61. *Braid ($\spadesuit$) is compatible with the graded structures, thus also with filtered ones. As a consequence, we obtain a useful method to study the (non)-positive deformations of maps from Braid ($\spadesuit$).*

4.2.1 Applications

In this subsection we apply the results from the previous section, to get several corollaries and lemmas to be used in further chapters. Lemma 4.69 is worth to be highlighted, since it shows that our machinery provides concrete results on the classical objects such as the Hilbert scheme of points, while not referring to the filtered deformations explicitly.

Lemma 4.62. *Let X be a an affine $\mathbb{G}_m$ -scheme, fix a closed $\mathbb{G}_m$ -equivariant embedding $X \subset \mathbb{A}^n$. There is a natural smooth transformation $\mathrm{hilb}_X^{\geq 0} \rightarrow \mathrm{Def}_X^{\geq 0}$, the induced morphism $\mathrm{hilb}_X^{\geq 0} \rightarrow \mathrm{Def}_X^{\geq 0}$ is also smooth.*

Proof. Since the map $X \subset \mathbb{A}^n$ is equivariant, we restrict Braid $\spadesuit$ to non-negative (resp. negative) parts and repeat the proof of Lemma 4.44. $\square$

Definition/Notation 4.63. *We define $\mathrm{Def}_X^{\leq 0}$ and $\mathrm{Def}_X^{< 0}$ as the filtered deformation functor of X , but with the opposite filtration on $\mathcal{O}_X$, see Definition 4.48.*

Lemma 4.64. *The following diagram is Cartesian*

$$\begin{array}{ccc} \mathrm{Def}_X^0 & \longrightarrow & \mathrm{Def}_X^{\leq 0} \\ \downarrow & & \downarrow \\ \mathrm{Def}_X^{\geq 0} & \longrightarrow & \mathrm{Def}_X. \end{array}$$

Proof. Let us take a small extension

$$0 \rightarrow I \rightarrow B \rightarrow A \rightarrow 0. \quad (4.65)$$

From the universal property there is a map

$$\mathrm{Def}_X^0 \rightarrow \mathrm{Def}_X^{\geq 0} \times_{\mathrm{Def}_X} \mathrm{Def}_X^{\leq 0}$$

By induction on small extensions we assume that the map

$$\mathrm{Def}_X^0(A) \rightarrow \mathrm{Def}_X^{\geq 0}(A) \times_{\mathrm{Def}_X(A)} \mathrm{Def}_X^{\leq 0}(A) \quad (4.66)$$

is bijective. We claim that if we replace A with B in (4.66), we still get a bijection. Let us take a B -point of the pull-back functor over a graded A -point $\mathcal{O}_X$, after tensoring with small extension (4.65) we get

$$0 \rightarrow I \otimes \mathcal{O}_{X_B} \rightarrow \mathcal{O}_{X_B} \rightarrow \mathcal{O}_X \rightarrow 0. \quad (4.67)$$

The algebra $\mathcal{O}_{X_B}$ is equipped with pair of filtrations $\mathcal{F}, \mathcal{G}$

$$\mathcal{F}_i \otimes A = F_i := \bigoplus_{j \geq i} (\mathcal{O}_X)_j,$$

$$\mathcal{G}_{-i} \otimes A = G_{-i} := \bigoplus_{j \leq i} (\mathcal{O}_{\mathcal{X}})_j,$$

by Definition 4.47 every element of both filtrations is flat over B . We prove that $\mathcal{O}_{X_B}$ comes with a natural graded structure, we claim that the following submodules form one

$$(\mathcal{O}_{X_B})_i = \mathcal{F}_i \cap \mathcal{G}_{-i}.$$

The subalgebra $\bigoplus_i (\mathcal{O}_{X_B})_i \subset \mathcal{O}_{X_B}$ is clearly graded. The question is whether the modules $(\mathcal{O}_{X_B})_i$ are flat and whether the constructed algebra is the whole algebra $\mathcal{O}_{X_B}$.

Let us take the subtraction map $\mathcal{F}_i \oplus \mathcal{G}_{-i} \rightarrow \mathcal{O}_{X_B}$ and tensor it with the small extension (4.65).

$$\begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ (\mathcal{F}_i \oplus \mathcal{G}_{-i}) \otimes I & \longrightarrow & \mathcal{O}_{X_B} \otimes I & \twoheadrightarrow & \text{coker} \otimes I \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_i \oplus \mathcal{G}_{-i} & \longrightarrow & \mathcal{O}_{X_B} & \twoheadrightarrow & \text{coker} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{F}_i \oplus G_{-i} & \longrightarrow & \mathcal{O}_{\mathcal{X}} & \longrightarrow & 0. \end{array}$$

The top row is a short exact sequence of $\mathbb{k}$ -vector spaces, by the fact that $\mathcal{O}_{X_B} \otimes \mathbb{k} = \mathcal{O}_{\mathcal{X}}$ is graded we get that $\text{coker} \otimes I$ vanishes. This implies that coker also vanishes. Since the module $(\mathcal{O}_{X_B})_i$ fits in the following exact sequence

$$0 \rightarrow (\mathcal{O}_{X_B})_i \rightarrow \mathcal{F}_i \oplus \mathcal{G}_{-i} \rightarrow \mathcal{O}_{X_B} \rightarrow 0,$$

it is flat. By a similar argument as above, but for $\bigoplus_i (\mathcal{O}_{X_B})_i \rightarrow \mathcal{O}_{X_B}$, in the place of $\mathcal{F}_i \oplus \mathcal{G}_{-i}$, we get that $\bigoplus_i (\mathcal{O}_{X_B})_i \rightarrow \mathcal{O}_{X_B}$ is an isomorphism of algebras, which finishes the proof. $\square$

Lemma 4.68. *Let us assume that the vector space $(T_X^1)_{<0}$ vanishes. Then the maps $\text{Def}_X^0 \rightarrow \text{Def}_X^{\leq 0}$ and $\text{Def}_X^{\geq 0} \rightarrow \text{Def}_X$ are smooth.*

Proof. In both cases we check condition of standard smoothness criterion, Theorem 4.35.

For the first map, since the negative tangent space vanishes, by Lemma 4.54, we get an isomorphism on tangent spaces and an injection on obstructions.

For the second map, consider the following pull-back diagram induced by Lemma 4.64.

$$\begin{array}{ccc} (T_X^1)_0 & \twoheadrightarrow & (T_X^1)_{\leq 0} \\ \downarrow & & \downarrow \\ (T_X^1)_{\geq 0} & \hookrightarrow & T_X^1. \end{array}$$

By Lemma 4.54, we conclude that there is a surjection on the tangent spaces and an injection on obstructions. $\square$

Lemma 4.69. *Assume that the vector space $(T_X^1)_{<0}$ vanishes, and fix $X \subset \mathbb{A}^n$, then $\text{hilb}_X^{\geq 0}$ and hilb_X are smoothly equivalent.*

Proof. By Lemma 4.64, we know that $\text{Def}_X^{\geq 0} \rightarrow \text{Def}_X$ is smooth. Subsequently, by Lemma 4.44 and Lemma 4.62, we get the following diagram

$$\begin{array}{ccc} \text{hilb}_X^{\geq 0} & \longrightarrow & \text{hilb}_X \\ \downarrow \text{sm} & & \downarrow \text{sm} \\ \text{Def}_X^{\geq 0} & \xrightarrow{\text{sm}} & \text{Def}_X. \end{array}$$

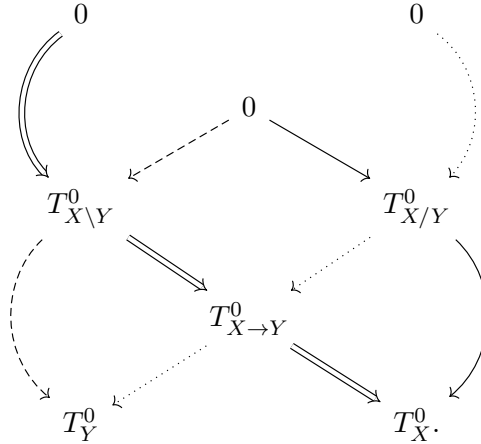
This way, we get that both $\text{hilb}_X^{\geq 0}$ and hilb_X are smooth over Def_X . $\square$

Corollary 4.70. *Let $X \rightarrow Y$ be a closed embedding of affine $\mathbb{G}_m$ -schemes. Assume that the negative tangent spaces $(T_X^1)_{<0}$, $(T_Y^1)_{<0}$ and $(T_{X/Y}^1)_{<0}$ are finitely dimensional and that the infinitesimal automorphisms $(T_X^0)_{<0}$, $(T_Y^0)_{<0}$ vanish. Then all three variants of deformations of the map $X \rightarrow Y$*

$$\begin{aligned} &\text{Def}_{X \rightarrow Y}^{<0}, \\ &\text{Def}_{X/Y}^{<0}, \\ &\text{Def}_{X \setminus Y}^{<0} \end{aligned}$$

are prorepresentable.

Proof. Recall the following part of Braid ($\spadesuit$)



We check conditions of Proposition 4.31.

By Proposition 4.41, the infinitesimal automorphisms of the local Hilbert functor vanish. By specialising Braid (4.2.1) above to the negative parts, simple diagram chasing shows that all negative T^0 -modules vanish.

Arguing analogously on the T^1 -level, we obtain that $(T_{X \setminus Y}^1)_{<0}$, $(T_{X \rightarrow Y}^1)_{<0}$ are finitely dimensional. As a consequence, by Proposition 4.31, we get prorepresentability. $\square$

Corollary 4.71. *Let $X \rightarrow Y$ be a morphism of affine $\mathbb{G}_m$ -schemes. If the vector space $(T_{X \setminus Y}^2)_{<0}$ vanishes, then the natural map $\phi: \text{Def}_{X \rightarrow Y}^{<0} \rightarrow \text{Def}_X^{<0}$ is smooth. Moreover, if $(T_{X \setminus Y}^1)_{<0}$ vanishes, then it is isomorphic on the tangent spaces.*

Proof. By Braid ($\spadesuit$), the space $(T_{X \setminus Y}^1)_{<0}$ is the relative tangent space of ϕ , and $(T_{X \setminus Y}^2)_{<0}$ is a relative obstruction space. We apply Theorem 4.35 to get the statement. $\square$

Corollary 4.72. *Let $X \rightarrow Y$ be a morphism of affine $\mathbb{G}_m$ -schemes. If the vector space $(T_Y^1)_{<-1}$ vanishes then the natural map $\psi: \mathrm{Def}_{X/Y}^{<0} \rightarrow \mathrm{Def}_{X \rightarrow Y}^{<0}$ is unobstructed. Moreover, if $(T_Y^1)_{-1}$ vanishes, then the map is smooth.*

Proof. By Braid ($\spadesuit$), the space $(T_Y^1)_{<0}$ is a relative obstruction space of ψ . Since the obstructions lie in degrees < -1 , we get the first part of the statement. By Theorem 4.35, vanishing of the relative obstruction space implies smoothness. $\square$

Part II

The Hilbert scheme

Chapter 5

The Hilbert scheme of points on the plane

5.1 Introduction

The Hilbert scheme of δ -points on a surface is a moduli space of closed subschemes of finite length lying on the surface. It is a classical object in algebraic geometry, very well studied and admits a large number of applications, e.g. [ES87, Hai98, Eva04, EGL01, GN15, OR18, PT19, BOR20, KZ25]. In this chapter, we are interested in the Hilbert scheme of points on the plane, which is smooth, by classical result by Fogarty, Theorem 3.9.

Ellingsrud and Strømme in [ES87] have computed Betti numbers of the Hilbert scheme by constructing a particular basis of homology coming from the Białynicki-Birula cells. To do so, they have studied $\mathbb{G}_m^2$ -fixed points on $\text{Hilb}_\delta(\mathbb{A}^2)$, the fixed points on the level of algebra are the monomial ideals of colength δ . They described the tangent space in terms of first-order deformations of the Hilbert-Burch matrix. Such a matrix induces a free resolution of the monomial ideal. Subsequently in [ES88] they have exploited those techniques further and claimed to describe the Białynicki-Birula cells explicitly. Some issues with arguments in that paper were pointed out and addressed in [Hui05], as a consequence the description for one-dimensional tori with isolated fixed points was completed. Prior to that, Laurent Evain, coming from a different perspective, has fixed these issues in some cases, see [Eva04, Remark 16].

Since the foundational work of Ellingsrud and Strømme, many people have studied points of this Hilbert scheme explicitly, see [CV08, Con11, HW21, HW24]. The main method revolves around deforming the Hilbert-Burch matrix for a given monomial ideal. This way, for a given monomial ideal E , Conca and Valla described Gröbner cells centred at E for the lexicographic term order. Later Roser Homs and Anna-Lena Winz, motivated by the study of the punctual Hilbert scheme, studied Gröbner cells for the negative degree lexicographical monomial order. In [HW24] they have described explicitly those cells for lex monomial ideals, see [HW21, Theorem 5.7]. They have conjectured that the description extends to Gröbner cells for any monomial ideal E , see [HW24, Conjecture 4.2].

It remained open to construct all the Białynicki-Birula/Gröbner cells explicitly in terms of Hilbert-Burch matrices. Recall the correspondence between the Białynicki-Birula and Gröbner cells, see Table 3.2. There is also a question when such constructions provide us with explicit open subschemes of the Hilbert scheme. In this chapter, we answer these questions, after the

authors article [Osz25]. We create a unified setting for the previous results, making the cells specializations of a general construction. One of the advantages of our setting is that, even though we do not get a global description of the Hilbert scheme, we get global results yielding the Białynicki-Birula cells and local results yielding the infinitesimal deformations.

Choose a monomial ideal $E = (x^t, x^{t-1}y^{m_1}, \dots, xy^{m_{t-1}}, y^{m_t})$ and take $\delta_i = m_i - m_{i-1}$, with the convention that $m_0 = 0$. We consider the *spread-out* matrix:

$$\mathcal{M} := \begin{pmatrix} y^{\delta_1} + a_{1,1} & c_{1,2} & c_{1,3} & \dots & c_{1,t} \\ -x + c_{2,1} & y^{\delta_2} + c_{2,2} & c_{2,3} & \dots & c_{2,t} \\ c_{3,1} & -x + c_{3,2} & y^{\delta_3} + c_{3,3} & \dots & c_{3,t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{t,1} & \dots & \dots & -x + c_{t,t-1} & y^{\delta_t} + c_{t,t} \\ c_{t+1,1} & \dots & \dots & c_{t+1,t-1} & -x + c_{t+1,t} \end{pmatrix},$$

where $c_{i,j} \in \mathbb{k}[y]$ and

$$\begin{cases} \deg_y(c_{i,j}) < \delta_j, & \text{for } i > j, \\ \deg_y(c_{i,j}) < \delta_i, & \text{for } i \leq j, \end{cases}$$

where $c_{i,j} = \sum_k c_{i,j,k} y^k$, we introduce a notation $\mathbf{c} = (c_{i,j,k})$. The Hilbert-Burch matrix associated with E is the matrix with coefficients in $\mathbb{k}[x, y]$ obtained from $\mathcal{M}$ by putting all parameters $c_{i,j,k}$ to zero. The spread-out matrix defines a family π given by the ideal of maximal minors $I_t(\mathcal{M})$

$$\begin{array}{ccc} \mathbf{V}(I_t(\mathcal{M})) & \xrightarrow{cl} & \mathbb{A}^2 \times \text{Spec}(k[\mathbf{c}]) \\ & \searrow \pi & \downarrow \\ & Y := \text{Spec}(\mathbb{k}[\mathbf{c}]). & \end{array}$$

We show that this family controls all infinitesimal deformations of the ideal E . The origin of the affine space Y is denoted as o .

Theorem 5.1 (Theorem 5.28). *Let E be a monomial ideal of colength δ . The matrix $\mathcal{M}$ via the family π induces an isomorphism of local deformation functors*

$$Y_o \rightarrow \text{hilb}_{\mathbb{S}/E}.$$

The formal-local situation described above, in general does not extend to the global one. Even though the matrix $\mathcal{M}$ is defined over whole Y , the family π in general is not finite and flat.

Example (Example 5.30). *Let us take a monomial ideal $E = (y^3, x^2)$. The Hilbert-Burch matrix associated with E is*

$$M_E = \begin{pmatrix} y^3 & 0 \\ -x & 1 \\ 0 & -x \end{pmatrix}.$$

One of the restrictions of the spread-out matrix of E is

$$\mathcal{M} = \begin{pmatrix} y^3 & y^2 \\ -x + y^2 & 1 \\ 0 & -x \end{pmatrix}.$$

One can check that the ideals generated by 2×2 minors of M_E and $\mathcal{M}$ have different colengths, thus the family cannot induce a morphism to the Hilbert scheme of points.

We introduce a $\mathbb{G}_m^2$ -action on Y making the family π equivariant. Using equivariant techniques, we describe an explicit $\mathbb{G}_m^2$ -stable subscheme $Y^\sharp$ of Y such that the family π restricted to $Y^\sharp$ induces a morphism of the Hilbert scheme, see Corollary 5.35.

Let us choose a one-parameter subgroup $\lambda: \mathbb{G}_m \rightarrow \mathbb{G}_m^2$. We consider the $\mathbb{G}_m$ -action on the Hilbert scheme induced by λ

$$\begin{aligned} \mathbb{G}_m \times \text{Hilb}_\delta(\mathbb{A}^2) &\rightarrow \text{Hilb}_\delta(\mathbb{A}^2), \\ (t, [I]) &\mapsto (\lambda(t) \cdot [I]). \end{aligned}$$

Consider the fixed point locus $\text{Hilb}_\delta^{\mathbb{G}_m}(\mathbb{A}^2)$ together with its decomposition into connected components $\text{Hilb}_\delta^{\mathbb{G}_m}(\mathbb{A}^2) = \bigsqcup H_i$. The Białynicki Birula cell associated to ψ and H_i is a locally closed subscheme of $\text{Hilb}_\delta(\mathbb{A}^2)$ given by

$$\left\{ [I] \in \text{Hilb}_\delta(\mathbb{A}^2) \mid \lim_{t \rightarrow 0} t \cdot [I] \in H_i \right\}.$$

The cell centred at $[E]$, see Definition 3.26, is the one that contains the chosen monomial ideal E . In that case, this point is the only $\mathbb{G}_m^2$ -fixed point of the cell. The Białynicki-Birula cell of Y with respect to λ is denoted $Y^{+, \lambda}$.

The following two results give us an explicit rational $\mathbb{G}_m$ -equivariant étale map to the Białynicki-Birula cells of $\text{Hilb}_\delta(\mathbb{A}^2)$. We consider two cases according to the degree y with respect to λ . First, we describe the Białynicki-Birula cell for the y -axis being contracted to the origin by the chosen $\mathbb{G}_m$ -action.

Theorem 5.2 (Theorem 5.39). *Choose a one-parameter subgroup λ , such that y has a non-negative degree with respect to the induced grading. Let us restrict Y to $Y^{+, \lambda}$, then the morphism π induces a rational $\mathbb{G}_m^2$ -equivariant map $\phi^+: Y^{+, \lambda} \rightarrow \text{Hilb}_\delta(\mathbb{A}^2)$. The image of ϕ^+ lies in the Białynicki-Birula cell associated with ψ and centred at E . After restricting the codomain to the Białynicki-Birula cell, the map ϕ^+ is étale in a $\mathbb{G}_m^2$ -stable neighbourhood of the origin.*

When the degree of y being negative, the points outside of x -axis are diverging by the $\mathbb{G}_m$ -action to infinity. In such case the Białynicki-Birula cell on the Hilbert schemes parametrises only schemes supported on the x -axis. To get adequate situation on the Hilbert-Burch side, we need to modify our construction, by intersecting the family with a thickened x -axis. This forces the family over Y to parametrise only schemes supported on the x -axis.

$$\begin{array}{ccccc} \mathbf{V}(I_t(\mathcal{M}) + (y^{tm_t})) & \xrightarrow{cl} & \mathbf{V}(I_t(\mathcal{M})) & \xrightarrow{cl} & \mathbb{A}^2 \times Y \\ & & & \searrow \pi & \downarrow \\ & & & & Y \\ & \searrow \pi_- & & & \end{array}$$

For the detailed discussion of the morphism π_- we refer to the discussion of Figure 5.1.

Theorem 5.3 (Theorem 5.43). *Choose a one-parameter subgroup λ , such that y has a negative degree with respect to the induced grading. Let us restrict Y to $Y^{+, \lambda}$, then the morphism π_- induces a rational $\mathbb{G}_m^2$ -equivariant map $\phi^+: Y^{+, \psi} \rightarrow \text{Hilb}_\delta(\mathbb{A}^2)$. The image of ϕ^+ lies in the*

Białynicki-Birula cell associated with ψ and centred at E . After restricting the codomain to the Białynicki-Birula cell, the map ϕ^+ is étale in a $\mathbb{G}_m^2$ -stable neighbourhood of the origin.

We apply Theorem 5.3 and Theorem 5.3 to generalise Theorem 5.1 from only $\mathbb{G}_m^2$ -fixed points, to any point of the Hilbert scheme, that is to arbitrary ideals.

Theorem 5.4 (Theorem 5.46). *Let us take an ideal $[I] \in \text{Hilb}_\delta(\mathbb{A}^2)$. Every monomial ideal E belonging to the closure of $\mathbb{G}_m^2$ -orbit of $[I]$ induces an isomorphism of local functors*

$$\hat{\phi}: Y_{p_I} \rightarrow \text{hilb}_{\mathbb{S}/I},$$

where $p_I \in Y_E$ is a point such that the ideal of maximal minors of the matrix $\mathcal{M}_E$ over Y_E is equal to I .

Our construction puts all Białynicki-Birula cells in a coherent setting. Using the dictionary between Gröbner picture and $\mathbb{G}_m$ -equivariant picture Table 3.1, Table 3.2, we see that the construction generalizes the one in [CV08] and solves [HW24, Conjecture 4.2]. If we consider a dominating Białynicki-Birula cell centred at $[E]$, then Theorem 5.2 provides us with an explicit description of an open subset of the Hilbert scheme. However, the condition that $[E]$ is the centre of a dominating Białynicki-Birula cell is not necessary, as shown in the following example.

Example (Example 5.47). *Let us consider the monomial ideal $E = (y^2, x^3)$. There is no half-plane through the origin containing weights of all parameters $c_{i,j,k}$. This means that no open neighbourhood of the origin is fully contained in a Białynicki-Birula cell for any one-parameter subgroup λ . Thus the potential existence of the rational map to the Hilbert scheme cannot be predicted by Theorem 5.2. But one can check that the construction from Definition 5.8 provides us with an open subscheme of the Hilbert scheme centred at $[E]$.*

Motivated by the example above we pose the following question.

Question (Question 5.48). *What are the monomial ideals E such that the construction from Definition 5.8 induces a rational map to the Hilbert scheme, but there is no open Białynicki-Birula cell containing E . Can one distinguish specific infinite classes of such ideals?*

The answer for that question would provide a collection of explicit open subschemes of the Hilbert scheme, coming from the original strategy by Ellingsrud and Strømme. Such open subschemes should be, in principle, distinct from those constructed in [Hai98, Proposition 2.1].

The structure of the chapter is as follows. We begin in Section 5.2, by introducing the Hilbert-Burch matrices and the spread-out matrix $\mathcal{M}$. This matrix defines a family of ideals that is the central object of this chapter, we endow the construction with a $\mathbb{G}_m^2$ -action. We consider the local situation and prove Theorem 5.1. From that point, we consider the global question. In Section 5.3 we show that in general the family does not extend well. We develop a simple technique that allows us to control the well-definiteness of extended family. As a key technical idea, we consider a particular resultant of the outermost minors of the spread-out matrix. This resultant is a polynomial of one variable giving us an explicit criterion of well-definiteness of the morphism to the Hilbert scheme in terms of its coefficients, see Lemma 5.35. Finally, in Section 5.4, we apply our methods to describe the Białynicki-Birula cells. Afterwards we apply the obtained results to extend Theorem 5.1 from the case of monomial ideals to the case of arbitrary ones, see Theorem 5.4. In the last part Section 5.5 we open a further discussion on the topic.

5.2 Hilbert-Burch matrices

The cornerstone of this section is the Hilbert-Burch theorem (Theorem 2.4).

Setting 5.5. *In this chapter we fix $\Lambda = \mathbb{k}$ and a positive integer δ . We denote the polynomial ring $\mathbb{k}[x, y]$ as $\mathbb{S}$. It admits a bi-grading*

$$\deg(x^i y^j) = (i, j). \quad (5.6)$$

Moreover, we fix a monomial ideal $E \subset \mathbb{S}$ of colength δ . Since E is a monomial ideal, it can be uniquely presented as

$$(x^t, x^{t-1}y^{m_1}, \dots, xy^{m_{t-1}}, y^{m_t}), \quad (5.7)$$

where m_i is a sequence of non-decreasing positive integers.

Having such a presentation, we define the Hilbert-Burch matrix for E .

Definition 5.8. *Let $\delta_i = m_i - m_{i-1}$ with a convention that $m_0 = 0$. The Hilbert-Burch matrix of E is*

$$M_E = \begin{pmatrix} y^{\delta_1} & 0 & 0 & \dots & \dots & 0 \\ -x & y^{\delta_2} & 0 & \dots & \dots & 0 \\ 0 & -x & y^{\delta_3} & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & -x & y^{\delta_t} \\ 0 & \dots & \dots & \dots & 0 & -x \end{pmatrix}. \quad (5.9)$$

Proposition 5.10 ([ES87, Lemma 3.1]). *The matrix M_E satisfies the conditions of Theorem 2.4 and hence it gives us a resolution*

$$0 \rightarrow \mathbb{S}^{\oplus t} \xrightarrow{M_E} \mathbb{S}^{\oplus t+1} \xrightarrow{\eta} \frac{\mathbb{S}}{E} \rightarrow 0. \quad (5.11)$$

Moreover, the resolution (5.11) is $\mathbb{G}_m^2$ -equivariant with the following bi-graded structure

$$0 \rightarrow \bigoplus_{j=1}^t \mathbb{S}[-t+j-1, -m_j] \xrightarrow{M_E} \bigoplus_{i=1}^{t+1} \mathbb{S}[-t+i-1, -m_{i-1}] \rightarrow E \rightarrow 0, \quad (5.12)$$

Proof. The first part is a direct consequence of Theorem 2.4 (H-B-1).

For the second part, let us check that M_E is homogeneous. Subsequently, non-zero entries of M_E are either y^{d_i} on the (i, i) -th place or $-x$ on the $(i+1, i)$ -th. Let us recall that from the definition $m_i = \sum_{j \leq i} d_j$. In the first case

$$(-t+i-1, -m_i) + (0, d_i) = (-t+i-1, -m_i + d_i) = (-t+i-1, -\sum_{j < i} d_j) = (-t+i-1, -m_{i-1})$$

which is the degree of the i -th component of the target module.

In the second case

$$(-t+i-1, -m_i) + (1, 0) = (-t+(i+1)-1, -m_{(i+1)-1})$$

which is the degree of $(i + 1)$ -th component of the target module. Therefore the homomorphism of modules M_E is homogeneous. To complete the proof we notice that maximal minors of M_E are $(x^t, x^{t-1}y^{m_1}, \dots, xy^{m_{t-1}}, y^{m_t})$, in particular, they are homogeneous. $\square$

Definition 5.13. An admissible index for E is a triple (i, j, k) of non-negative integers such that

$$1 \leq i \leq t + 1, \quad (5.14)$$

$$1 \leq j \leq t, \quad (5.15)$$

$$\begin{cases} k < \delta_j & \text{if } i > j, \\ k < \delta_i & \text{if } i \leq j. \end{cases} \quad (5.16)$$

To see how the admissible indices appear see Example 5.21.

Lemma 5.17. *There is exactly 2δ admissible indices for E .*

Proof. To the monomial ideal E we have assigned a sequence of exponents $(\delta_1, \dots, \delta_t)$. First, note that the following equality holds

$$\delta = \sum_{1 \leq i \leq t} m_i.$$

Since $\delta_i = m_i - m_{i-1}$, we get $\sum_{1 \leq j \leq i} \delta_j = m_i$, hence

$$\sum_{1 \leq i \leq t} \left(\sum_{1 \leq j \leq i} \delta_j \right) = \delta. \quad (5.18)$$

To compute the number of admissible indices, we argue by a induction on t . Assume that $t = 1$, then (i, j, k) is an admissible index if and only if $1 \leq i \leq 2$, $j = 1$ and $0 \leq k < \delta_1$. Hence, there is $2\delta_1 = 2\delta$ admissible indices.

Now assume that statement holds for $t' < t$. The set of admissible indices for the sequence $(\delta_1, \dots, \delta_t)$ contains the set for $(\delta_1, \dots, \delta_{t-1})$. The complement consists of indices with $i = t + 1$ or $j = t$, there are exactly $2 \sum_{1 \leq i \leq t} \delta_i$ of them, consequently, by Equation (5.18), we get the conclusion. $\square$

Now, we are going to spread out the matrix M_E , by which we mean make M_E a central fibre of a certain family of matrices, as defined in Definition 5.8, in order to get a family of the Hilbert-Burch matrices. The spread-out matrix was introduced by the author in [Osz25] and it generalises previous constructions for example [ES87, Lemma 3.1], [CV08, Definition 3.2] and [HW21, Definition 2.2].

Definition 5.19. *Let us consider a $\mathbb{S}$ -algebra $\mathbb{S}[\mathbf{c}]$, where $\mathbf{c} = (c_{i,j,k})$ and the indices (i, j, k) run over the set of admissible indexes for E , see Definition 5.13. We define the following matrix over*

$\mathbb{S}[\mathbf{c}]$

$$\mathcal{M}_E := \begin{pmatrix} y^{\delta_1} + c_{1,1} & c_{1,2} & c_{1,3} & \dots & c_{1,t} \\ -x + c_{2,1} & y^{\delta_2} + c_{2,2} & c_{2,3} & \dots & c_{2,t} \\ c_{3,1} & -x + c_{3,2} & y^{\delta_3} + c_{3,3} & \dots & c_{3,t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_{t,1} & \dots & \dots & -x + c_{t,t-1} & y^{\delta_t} + c_{t,t} \\ c_{t+1,1} & \dots & \dots & c_{t+1,t-1} & -x + c_{t+1,t} \end{pmatrix}, \quad (5.20)$$

where $c_{i,j} = \sum_k c_{i,j,k} y^k$. If the context is clear we omit E in the subscript and write $\mathcal{M}$.

Example 5.21. Let E be the ideal generated by (y^3, xy, x^2y, x^3) , then the Hilbert-Burch matrix for E is:

$$M_E = \begin{pmatrix} y & 0 & 0 \\ -x & 1 & 0 \\ 0 & -x & y^2 \\ 0 & 0 & -x \end{pmatrix}$$

and our spread-out matrix is

$$\mathcal{M} = \begin{pmatrix} y + c_{1,1,0} & c_{1,2,0} & c_{1,3,0} \\ -x + c_{2,1,0} & 1 & 0 \\ c_{3,1,0} & -x & y^2 + c_{3,3,1}y + c_{3,3,0} \\ c_{4,1,0} & 0 & c_{4,3,1}y + c_{4,3,0} - x \end{pmatrix}.$$

Note that $c_{*,1} = c_{*,1,0}$ and the same goes for the first column. The admissible indices are

$$\{ (1, 1, 0), \dots, (1, 3, 0), (2, 1, 0), (3, 1, 0), (4, 1, 0), (3, 3, 1), (3, 3, 0), (4, 3, 1), (4, 3, 0) \}.$$

The length of the quotient algebra $\mathbb{S}/E$ is 5. Thus the ideal E induces a point in $\text{Hilb}_5(\mathbb{A}^2)$ which is of dimension 10, see Theorem 3.9, which the number of admissible indices.

Based on Example 5.21, one could expect that Y actually parametrizes certain resolutions coming from the Hilbert-Burch Theorem and hence controls the deformations of $\text{Spec}(\mathbb{S}/E)$ as a finite subscheme of $\mathbb{A}^2$. This speculation is investigated in Section 5.3. Infinitesimally the speculation holds, see Theorem 5.28. Usually the global case does not, see Examples 5.30 and 5.29. Before investigating these issues further, we endow the family with a $\mathbb{G}_m^2$ -action.

Definition 5.22. Let Y be a $\mathbb{G}_m^2$ -scheme given as spectrum of the polynomial ring $\mathbb{k}[\mathbf{c}]$ with the following bi-grading

$$\deg(c_{i,j,k}) = (i - j, m_j - m_{i-1} - k).$$

The origin $o \in Y$ is the unique $\mathbb{G}_m^2$ -fixed point.

Definition 5.23. We define the two matrices encoding the bi-degree of $\mathcal{M}_E$, $U_x(E)$ and $U_y(E)$ of degrees with respect to the x -grading and the y -grading, respectively. In [CV08, page 5] authors consider the degree matrix $U(E)$ which is exactly $U_x(E) + U_y(E)$.

Example 5.24. Let us see what are the degree encoding matrices in the case of Example 5.21. From Definition 5.22 we see that the degrees with respect to the x -grading can be read out straight

from the dimensions of the matrix M_E

$$U_x(E) = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}.$$

We can observe that matrix encoding the y -grading is determined by the diagonal (i, i) of the matrix M_E

$$U_y(E) = \begin{pmatrix} 1 & 1 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \\ -2 & -2 & 0 \end{pmatrix}.$$

Remark 5.25. To avoid confusion, we want to emphasise the degree of a monomial with respect to x is **not** the same as the degree with respect to the x -grading. For example, the monomial $a_{i,j,k}x^l$ has the degree l with respect to x , and $i - j + l$ with respect to the x -grading.

Lemma 5.26. The matrix $\mathcal{M}$ induces us the following map π :

$$\begin{array}{ccc} \mathbf{V}(I_t(\mathcal{M})) & \hookrightarrow & \mathbb{A}^2 \times Y \\ & \searrow \pi & \downarrow \\ & & Y, \end{array}$$

where $I_t(\mathcal{M})$ is the ideal generated by the maximal minors of the matrix $\mathcal{M}$. The map ϕ is $\mathbb{G}_m^2$ -equivariant.

Proof. We proceed as in Lemma 5.10. We need to check that the matrix $\mathcal{M}$ is homogeneous

$$(-t + j - 1, -m_j) + (i - j, m_j - m_{i-1}) = (-t + i - 1, -m_{i-1}),$$

which is the degree of i -th component of the target module.

This implies that the vanishing locus $\mathbf{V}(I_t(\mathcal{M}))$ is a $\mathbb{G}_m^2$ -stable subscheme of $\mathbb{A}^2 \times Y$. Thus the morphism π , being the projection onto the second factor restricted to the vanishing locus, is $\mathbb{G}_m^2$ -equivariant. $\square$

Theorem 5.27. Let us consider the first-order infinitesimal neighbourhood of $o \in Y$, that is $Y_1 = \text{Spec}(\mathcal{O}_Y/\mathfrak{m}_o^2)$, where $\mathfrak{m}_o$ is the maximal ideal of associated the the origin $o \in Y$. The family π restricted to Y_1 , is finite and flat.

Hence, $\pi|_{Y_1}$ induces a map to the Hilbert scheme $\text{Hilb}_\delta(\mathbb{A}^2)$, which in turn yields a map

$$\mathfrak{t}_\phi: T_o Y \rightarrow T_{[E]} \text{Hilb}_\delta(\mathbb{A}^2).$$

The map $\mathfrak{t}_\phi$ is a $\mathbb{G}_m^2$ -equivariant isomorphism.

Proof. By the Hilbert-Burch Theorem 2.4 (H-B-2) morphism $\pi|_{\pi^{-1}(Y_1)}$ is flat and by Nakayama's lemma [Sta22, Tag 051F] it is finite, thus it induces a morphism to $\text{Hilb}_\delta(\mathbb{A}^2)$. This map induces a linear homomorphism on tangent spaces which we denote by $\mathfrak{t}_\phi: T_o Y \rightarrow T_{[E]} \text{Hilb}_\delta(\mathbb{A}^2)$. Since, by Lemma 5.26, π is $\mathbb{G}_m^2$ -equivariant, the map $\mathfrak{t}_\phi$ is $\mathbb{G}_m^2$ -equivariant.

Recall that, by Lemma 5.17, the tangent spaces have the same dimension. Hence, it is enough to show that the kernel of $\mathfrak{t}_\phi$ is trivial. It is enough to show injectivity for the subspace of the weight w_x for the fixed weight w_x with respect to the x -grading. On the domain it is a linear subspace consisting of $(t+1) \times t$ matrices "with the third diagonal":

$$\mathcal{M} = \begin{pmatrix} y^{\delta_1} & 0 & 0 & \dots & e_1 & \dots & \dots & 0 \\ -x & y^{\delta_2} & 0 & \dots & \dots & e_2 & \dots & 0 \\ 0 & -x & y^{\delta_3} & \dots & \dots & \dots & e_3 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & \dots & e_{t-w_x} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & \dots & -x & y^{\delta_t} \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & -x \end{pmatrix},$$

where e_j is on the $(w_x + j, j)$ entry. Let us assume that $w_x < 0$, so that "the third diagonal" $\{e_i\}$ lies over the diagonal. In the other case, the argument is almost identical.

We show that the map of tangent spaces is injective when restricted to such matrices. Let us assume that the lowest index i such that $c_i \neq 0$ is i_0 , then the $(i_0 + 1)$ -th minor depends only on c_{i_0} . This minor is exactly:

$$y^{m_{i_0}} x^{t-i_0} - e_{i_0} (-x)^{t-i_0} \prod_{i < i_0} y^{\delta_i} = y^{m_{i_0}} x^{t-i_0} - e_{i_0} (-x)^{t-i_0} y^{m_{i_0}-1}.$$

Since $\deg(e_{i_0}) < i_0$, this minor does not belong to E , and the thus image of matrix M is not zero, which means that the kernel of $\mathfrak{t}_\phi$ is trivial. $\square$

We have seen that the map π controls all first-order deformations, now we extend this result to completed local rings.

Theorem 5.28. *The map $\mathfrak{t}_\phi$ extends to an isomorphism $\hat{\phi}$ of local deformation functors*

$$\hat{\phi}: Y_o \rightarrow \text{hilb}_{\mathbb{S}/E}.$$

Proof. Since both Y and the Hilbert scheme are smooth of the same dimension, both local functors are smooth. Any lift of the isomorphism of the tangent spaces $\mathfrak{t}_\phi$ is automatically isomorphic, by Lemma 4.32. We consider an infinitesimal thickening of the origin $o \in Y$, let us call it

$$\begin{array}{ccccc} (\text{Spec}(A), 0) & \rightarrow & (Y, o), \\ \mathbf{V}(I_t(M_E)) & \hookrightarrow & \mathbf{V}(I_t(\mathcal{M}))|_A & \hookrightarrow & \mathbf{V}(I_t(\mathcal{M})) \\ \downarrow \pi_0 & & \downarrow \pi_A & & \downarrow \pi \\ \{0\} & \hookrightarrow & (\text{Spec}(A), 0) & \hookrightarrow & (Y, o). \end{array}$$

The map π_A is coming from a lift of M_E to a matrix over the Artinian algebra A , thus by the Hilbert-Burch Theorem 2.4 (H-B-2) this map is flat. By the Nakayama's lemma [Sta22, Tag 051F], this map is also finite of degree equal δ .

By $\mathfrak{m} \subset \mathbb{k}[\mathbf{c}]$ we denote the maximal ideal generated by $(c_{i,j,k})_{i,j,k}$. Subsequently, we go to the inverse limit to consider the completed local ring. From the local criterion for flatness [Eis95, Corollary 6.9], we again get a flat family. We have a system of $k[[\mathbf{c}]]$ -modules:

$$\begin{array}{ccccc}
\mathbb{k}^d & \xrightarrow{\cong} & (\mathbb{k}[\mathbf{c}]/\mathfrak{m})^d & \longrightarrow & \mathbb{S}/E \\
& & \uparrow & & \uparrow \\
& & (\mathbb{k}[\mathbf{c}]/\mathfrak{m}^2)^d & \longrightarrow & \mathbb{S}[\mathbf{c}]/(E + \mathfrak{m}^2) \\
& & \uparrow & & \uparrow \\
& & \cdots & \longrightarrow & \cdots \\
& & \uparrow & & \uparrow \\
(\widehat{\mathcal{O}_{Y,o}})^d & \xrightarrow{\cong} & (\mathbb{k}[[\mathbf{c}]])^d & \longrightarrow & \mathbb{S}[[\mathbf{c}]]/\hat{E},
\end{array}$$

both vertical systems satisfy the Mittag-Leffler condition since all vertical arrows are surjective, thus the bottom horizontal arrow is also surjective. As a consequence, $\mathbb{S}[[\mathbf{c}]]/\hat{E}$ is finite over $\mathbb{k}[[\mathbf{c}]]$. This way we have obtained that the morphism π over the spectrum of the completed local ring induces a map $\hat{\phi}$ to the Hilbert scheme. $\square$

5.3 Failure of the dream scenario and the resultant fix

In the previous section, we have considered the infinitesimal behaviour of our family around the origin, in this section we consider the global question.

The dream scenario would be if the map π was finite and flat, thus induced a map $\phi: Y \rightarrow \text{Hilb}_d(\mathbb{A}^2)$. Then its induced local map would be $\hat{\phi}$ from Theorem 5.28. This way we would end up with a generically étale equivariant map to the Hilbert scheme constructed in a very explicit manner. The open subsets obtained that way would contain the Białynicki-Birula cells for any one-dimensional subgroup of $\mathbb{G}_m^2$ that has finitely many fixed points on the Hilbert scheme, see Corollary 5.40. As a consequence, we would construct an interesting collection of explicit $\mathbb{G}_m^2$ -stable open subsets giving more information than the Białynicki-Birula cells.

The construction would go in the following manner. We take the spread-out matrix $\mathcal{M}$, see Definition 5.8, it gives us a sequence

$$\mathbb{S}[\mathbf{c}]^{\oplus t} \xrightarrow{\mathcal{M}} \mathbb{S}[\mathbf{c}]^{\oplus t+1} \rightarrow I_t(\mathcal{M}) \rightarrow 0,$$

just as in the Hilbert-Burch Theorem, Theorem 2.4. If $\mathcal{M}$ satisfied the conditions of the Hilbert-Burch Theorem on every fibre of π and π was finite, from the local criterion for flatness, we would get that π is a flat morphism, thus inducing a map ϕ to the Hilbert scheme. The problem is that the dream scenario falls apart, $\mathcal{M}$ does not need to satisfy the condition of the Hilbert-Burch Theorem. The morphism π can admit positive dimensional fibres and even if we restrict it to the locus where it does have only zero-dimensional fibres it does not have to be finite.

Example 5.29. *The matrix $\mathcal{M}$ does not need to satisfy the conditions of the Hilbert-Burch Theorem and the morphism π does not need to have all fibres zero-dimensional.*

Let us take $E = (y^2, x^2) = (x^2, xy^2, y^2)$. In this case, the Hilbert-Burch matrix is

$$M_E = \begin{pmatrix} y^2 & 0 \\ -x & 1 \\ 0 & -x \end{pmatrix}.$$

Instead of working over the full base Y , let us restrict ourselves to the linear subspace spanned by $c_{1,2,1}$ and $c_{2,1,1}$ and we consider the family of matrices restricted to this subspace:

$$\mathcal{M} = \begin{pmatrix} y^2 & yc_{1,2,1} \\ -x + yc_{2,1,1} & 1 \\ 0 & -x \end{pmatrix}.$$

Let us consider the maximal minors of matrix $\mathcal{M}$

$$\begin{aligned} f_0 &= x^2 - xyc_{2,1,1}, \\ f_1 &= -xy^2, \\ f_2 &= y^2(1 - c_{2,1,1}c_{1,2,1}) + xyc_{1,2,1}. \end{aligned}$$

If $c_{2,1,1}c_{1,2,1} = 1$ then we get the ideal generated by (x^2, xy) . The vanishing locus of such an ideal contains the line $\mathbf{V}(x) \subset \mathbb{A}^2$. Thus not every fibre of π is zero-dimensional. This means, by Hilbert Burch Theorem 2.4 (H-B-3), that the sequence

$$0 \rightarrow \mathbb{S}[\mathbf{c}]^{\oplus t} \xrightarrow{\mathcal{M}} \mathbb{S}[\mathbf{c}]^{\oplus t+1} \rightarrow I_t(\mathcal{M}) \rightarrow 0$$

is not exact.

The existence of positive dimensional fibres is not a major issue by itself, there always is an open neighbourhood of the origin on which the morphism π is flat and has zero-dimensional fibres, see Lemma 5.33 below. The next example showcases the main problem.

Example 5.30. *The morphism π does not need to be finite over any open neighborhood of the origin. Let us take $E = (y^3, x^2) = (x^2, xy^3, y^3)$. The colength of E is 6, as the quotient algebra has a basis $\{1, x, xy, xy^2, y, y^2\}$.*

In this case, the Hilbert-Burch matrix is

$$M_E = \begin{pmatrix} y^3 & 0 \\ -x & 1 \\ 0 & -x \end{pmatrix}.$$

Instead of working over the full base Y , let us restrict ourselves to the linear subspace spanned by $c_{1,2,2}$ and $c_{2,1,2}$. We consider the family of matrices restricted to this subspace

$$\mathcal{M} = \begin{pmatrix} y^3 & y^2c_{1,2,2} \\ -x + y^2c_{2,1,2} & 1 \\ 0 & -x \end{pmatrix}.$$

Let us consider maximal minors of the matrix $\mathcal{M}$

$$\begin{aligned} f_0 &= x^2 - xy^2 c_{2,1,2}, \\ f_1 &= -xy^3, \\ f_2 &= -y^4 c_{2,1,2} c_{1,2,2} + y^3 + xy^2 c_{1,2,2}. \end{aligned}$$

If $c_{2,1,2}, c_{1,2,2} \neq 0$, then the set $\{1, x, xy, xy^2, y, y^2, y^3\}$ is linearly independent in the quotient algebra, thus the colength of the ideal is at least 7 which is more than it should be. The morphism π is flat on some open neighbourhood of the origin $o \in Y$, see Lemma 5.33, thus if it was finite, it would have a constant length of the fibres on some neighbourhood.

The dream scenario falls apart as exhibited in Examples 5.29, 5.30. The strategy to deal with those issues is to create manageable criteria for finiteness and flatness for algebras obtained from $\mathcal{M}$, see Definition 5.8. Since the minor that is obtained after removing the first row of $\mathcal{M}$ has a form

$$x^t + (\text{elements of lower degree with respect to } x),$$

to show finiteness of π it is enough to show that the element y in $\mathbb{S}[\mathbf{c}]/I_t(\mathcal{M})$ is integral over $\mathbb{k}[\mathbf{c}]$. That is why we adopt the technique coming from elimination theory.

Definition/Notation 5.31. *Let us consider the matrix $\mathcal{M}$ from Definition 5.8. The vanishing set of the ideal generated by its maximal minors defines the map π , see Definition 5.26. Let us take the first and last maximal minors of the matrix $\mathcal{M}$, that is, determinants of $\mathcal{M}$ after excluding the first and last row respectively. We treat these two minors as polynomials of variable x over $\mathbb{k}[\mathbf{c}, y]$. We consider their resultant with respect to the variable x , it is a polynomial in $\mathbb{k}[\mathbf{c}, y]$, and we denote it by Res_E .*

Remark 5.32. *Since the first minor is monic with respect to x , by Lemma 2.7, after changing the base Res_E continues to be the resultant of two minors of the restricted matrix.*

The following lemma resolves the issue exhibited in Example 5.29.

Lemma 5.33. *Let $Y^\flat$ be the subscheme of Y defined as the complement of vanishing locus of all coefficients of the resultant $\text{Res}_E \in \mathbb{k}[\mathbf{c}][y]$ as a polynomial of the variable y . Then $Y^\flat$ is a $\mathbb{G}_m^2$ -stable open subscheme of Y . The map π is flat over $Y^\flat$.*

Proof. For the first part of the statement, the resultant of two homogeneous elements with respect to a variable is homogeneous. This implies that the $\mathbb{G}_m^2$ -action restricts to $Y^\flat$.

For the second part of the statement, let $\mathfrak{p} \subset \mathbb{k}[\mathbf{c}]$ be a maximal ideal corresponding to a closed point of $Y^\flat \subset Y$, let $\kappa(\mathfrak{p})$ be a residual field of $\mathfrak{p}$. We check that the ideals $(I_t(\mathcal{M}))_{\mathfrak{p}}$ and $I_t((M) \otimes \kappa(\mathfrak{p}))$ have the depth equal to 2. Let us denote $I = I_t(\mathcal{M})$. Let $f_0, f_t \in \mathbb{S}[\mathbf{c}]$ be the first and the last maximal minor of $\mathcal{M}$. If they do not have a common factor, they constitute a regular sequence of length 2 so $\text{depth}(f_0, f_t) = 2$, and thus $\text{depth}(I) \geq 2$. Since π is a surjective map, $\text{codim}(I) \leq 2$. By [Eis95, Proposition 18.2.] $\text{codim}(I) \geq \text{depth}(I)$, implying that $\text{depth}(I) = 2$. The definition of the resultant says that the resultant Res_E vanishes if and only if f_0 and f_t admit a common factor that does not belong to $\mathbb{S}[\mathbf{c}, y]$, see Lemma 2.6. Since f_0 is monic with respect to x , it does not admit factors that do not depend on x . Therefore, the resultant vanishes if and only if they do admit a common factor. This implies that the depth of ideals $I_t(\mathcal{M})_{\mathfrak{p}}$ and $I_t((M) \otimes \kappa(\mathfrak{p}))$ over each point of $Y^\flat$ is constant and equal to 2.

Since the depth of $(I_t(\mathcal{M}))_{\mathfrak{p}}$ is equal to 2, the matrix $\mathcal{M}$, by the Hilbert-Burch Theorem 2.4 (H-B-1), induces the resolution:

$$0 \rightarrow \mathbb{S}[\mathbf{c}]_{\mathfrak{p}}^{\oplus t} \xrightarrow{\mathcal{M}} \mathbb{S}[\mathbf{c}]_{\mathfrak{p}}^{\oplus t+1} \rightarrow \mathbb{C}[\mathbf{c}]_{\mathfrak{p}} \rightarrow \frac{\mathbb{S}[\mathbf{c}]_{\mathfrak{p}}}{I_t(\mathcal{M})} \rightarrow 0.$$

By the local criterion for flatness [Eis95, Theorem 6.8] it is enough to check that

$$0 \rightarrow \left(\frac{\mathbb{S}[\mathbf{c}]}{\mathfrak{p}} \right)^{\oplus t} \xrightarrow{\mathcal{M}} \left(\frac{\mathbb{S}[\mathbf{c}]}{\mathfrak{p}} \right)^{\oplus t+1} \rightarrow \frac{\mathbb{S}[\mathbf{c}]}{\mathfrak{p}} \rightarrow \frac{\mathbb{S}[\mathbf{c}]/\mathfrak{p}}{I_t(\mathcal{M})} \rightarrow 0,$$

is an exact sequence of $\kappa(\mathfrak{p})$ -modules. Again since the depth of $I_t((\mathcal{M})) \otimes \kappa(\mathfrak{p})$ is 2, this sequence satisfies the assumptions of Hilbert-Burch Theorem 2.4 (H-B-1). $\square$

To understand finiteness of the map π we compute the resultant after restriction to the origin, this computation is useful in Theorem 5.35

Example 5.34. *After restricting to the origin o of Y , the first minor f_0 is x^t and the last one is y^{m_t} , the resultant $\text{Res}_x(x^t, y^{m_t})$ is equal to y^{tm_t} as it is a determinant of the $t \times t$ Sylvester matrix*

$$\begin{pmatrix} y^{m_t} & 0 & 0 & 0 & \dots & \dots \\ 0 & y^{m_t} & 0 & 0 & \dots & \dots \\ 0 & 0 & y^{m_t} & 0 & \dots & \dots \\ 0 & 0 & 0 & y^{m_t} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \dots & 0 & 0 & 0 & \dots & y^{m_t} \end{pmatrix}.$$

Now, we state the key technical result of this chapter.

Theorem 5.35 ([Osz25, Corollary 7.5]). *Let us consider the locally closed subscheme $Y^{\sharp} \subset Y^{\flat}$ containing the origin, on which the resultant is a polynomial of degree exactly tm_t . The map π restricted to $Y^{\sharp}$ is flat and finite.*

Proof. As the statement is local on the target, we substitute $Y^{\sharp}$ with an affine open subscheme. As we have seen in the computation above, Example 5.34, the resultant Res_E is equal to y^{tm_t} modulo the ideal generated by parameters $(c_{i,j})$. In particular, $Y^{\sharp}$ is non-empty. On $Y^{\sharp}$ the resultant is of the form

$$uy^{tm_t} + \text{terms with lower degrees with respect to } y$$

where u is invertible on $Y^{\sharp}$. Since $\text{Res}_E \in I_t(\mathcal{M})$, this relation makes $y \in H^0(Y^{\sharp}, \mathcal{O}_{Y^{\sharp}})[x, y]/I_t(\mathcal{M})$ integral over $H^0(Y^{\sharp}, \mathcal{O}_{Y^{\sharp}})$. As the maximal minor f_0 is monic with respect to x , it makes x integral over $H^0(Y^{\sharp}, \mathcal{O}_{Y^{\sharp}})[y]$, and thus we obtain that $H^0(Y^{\sharp}, \mathcal{O}_{Y^{\sharp}})[x, y]/I_t(\mathcal{M})$ is finite over $H^0(Y^{\sharp}, \mathcal{O}_{Y^{\sharp}})$. Since the resultant is monic of degree tm_t , it is not zero independently on the values of parameters $c_{i,j,k}$, consequently, by Lemma 5.33, we get flatness. $\square$

In general, the subscheme $Y^{\sharp}$ is not closed nor open in Y since it is given by:

- One open condition that the coefficient with y^{tm_t} does not vanish.
- Closed conditions that the coefficients with higher exponents of y are zero.

To see how the resultant strategy works in practice, let us look at Example 5.29 and Example 5.30 using this tool.

Example 5.36. *Let us continue the investigation started in Example 5.29, and apply Corollary 5.35. We take $E = (y^2, x^2)$, and we aim to compute the resultant of $x^2 - xyc_{2,1,1}$ and $xy_{1,2,1} + y^2(1 - c_{2,1,1}c_{1,2,1})$ with respect to the variable x . Let us denote $c_{1,2,1}$ by c and $c_{2,1,1}$ by e . To compute the resultant is take the Sylvester matrix*

$$\begin{pmatrix} 1 & -ye & 0 \\ ya & y^2(1 - ce) & 0 \\ 0 & yc & y^2(1 - ce) \end{pmatrix},$$

and compute that the determinant of whose is

$$y^4(1 - ce)^2 - y^4(-ce)(1 - ce) = y^4(1 - ce).$$

We see that the resultant is 0 if and only if $ce = 1$, which is exactly when $\mathcal{M}$ admits a non-trivial kernel in Example 5.29. Thus, we see that in the case of the ideal (x^2, y^2) , we got an induced rational map to the Hilbert scheme.

Example 5.37. *Let us see how this technique works out in the case presented in Example 5.30. We take $E = (y^3, x^2)$, and we want to compute the resultant of $x^2 - xy^2c_{2,1,2}$ and $xy^2c_{1,2,2} - y^4c_{2,1,2}c_{1,2,2} + y^3$ with respect to the variable x . Let us denote $c_{1,2,2}$ by c and $c_{2,1,2}$ by e . To compute the resultant is to take the Sylvester matrix*

$$\begin{pmatrix} 1 & -y^2e & 0 \\ y^2c & y^3(1 - yce) & 0 \\ 0 & y^2c & y^3(1 - yce) \end{pmatrix},$$

and compute that the determinant of whose is

$$y^7(ce) - y^6.$$

In Example 5.30, we have seen that over the open subset $ce \neq 0$ the length of the fibres is higher than over the origin. Corollary 5.35 tells us that on the closed locus $ce = 0$ we again get a map to the Hilbert scheme of points.

5.4 Applications

In this section we apply the resultant strategy to get a very explicit description of Białynicki-Birula cells and local deformation functors for an arbitrary ideal.

Definition/Notation 5.38. *Let $\lambda: \mathbb{G}_m \rightarrow \mathbb{G}_m^2$ be a one-parameter subgroup, and let $\mathbb{G}_m^2$ act on Y with only one fixed point. Then $Y^{+, \lambda}$ is the Białynicki-Birula cell $Y^{\geq 0}$, see Notation 3.30, for $\mathbb{G}_m$ -action provided by λ . Observe that, by Definition 3.21, the origin $o \in Y$ belongs to $Y^{+, \lambda}$. The $\mathbb{Z}$ -grading induced by λ is denoted by $\deg_\lambda$.*

Theorem 5.39 ([Osz25, Theorem 8.4]). *Choose a one-parameter subgroup λ , such that y has a non-negative degree with respect to the induced grading. Let us restrict Y to $Y^{+, \lambda}$, then the*

morphism π induces a rational $\mathbb{G}_m^2$ -equivariant map $\phi^+ : Y^{+, \lambda} \rightarrow \text{Hilb}_\delta(\mathbb{A}^2)$. The image of ϕ^+ lies in the Białynicki-Birula cell associated with ψ and centred at E . After restricting the codomain to the Białynicki-Birula cell, the map ϕ^+ is étale in a $\mathbb{G}_m^2$ -stable neighbourhood of the origin.

Proof. By Lemma 5.33, the resultant is homogeneous and, as observed in Example 5.34, the resultant Res_E is of degree $\deg_\lambda(y)tm_t$. We need to show that, when restricted to $Y^{+, \lambda}$, the resultant has no terms with degree higher than that with respect to y . If there is a non-zero term $\Psi(\mathbf{c})y^N$ for $N > tm_t$, then $\deg_\lambda(W(\mathbf{c})) = \deg_\lambda(y)(tm_t - N) < 0$. Since Y^+ is the Białynicki-Birula cell, $\Psi(\mathbf{c})$ depends only on parameters with non-negative degrees with respect to λ , hence $\deg_\lambda \Psi(\mathbf{c}) \geq 0$.

Now let us assume that $\deg_\lambda(y) = 0$. That means that ψ has a form:

$$t \rightarrow (t^\alpha, 1),$$

for $\alpha \neq 0$. Again we consider $\Psi(\mathbf{c})$ as before, since Res_E is homogeneous with respect to the full $\mathbb{Z}^2$ -grading, $\Psi(\mathbf{c})$ is also homogeneous and has the degree $(0, tm_t - N)$. Let us consider what parameters can $\Psi(\mathbf{c})$ depend on. Since we restrict ourselves to the Białynicki-Birula cell, it can only depend on the parameters $c_{i,j,k}$ with degree $(w_x, w_y) := (i - j, m_j - m_{i-1})$ satisfying the following inequality:

$$\alpha w_x + 0 \cdot w_y \geq 0.$$

The only way that one can obtain 0 as a sum of non-negative/non-positive numbers is by taking a sum of 0-s. Thereupon, the term $\Psi(\mathbf{c})$ depend only on parameters lying on the main diagonal (i, i) . As a consequence, we only can restrict ourselves to the parameters on the main diagonal, and put $c_{i,j,k} = 0$ for $i \neq j$. In that case, the first minor is just x^t , and the last one is a product of elements on the diagonal. In particular, the latter minor is a polynomial in y of degree m_t .

Therefore in both cases above, the resultant is the last minor to the power of t , so has the predicted degree. By Theorem 5.35, we get that the family π is finite and flat.

The last part is a direct corollary of Theorem 5.28. □

The only $\mathbb{G}_m^2$ -fixed points on the Hilbert scheme are the monomial ideals, thus the same is true for a general choice of the one-parameter subgroup. In this special case, we will get that our map is an isomorphism onto the cell.

Corollary 5.40. *Let us take the setup from Theorem 5.39, but also let us assume that the only $\mathbb{G}_m$ -fixed point on Y^+ is the origin o . Then the induced map ϕ^+ is an isomorphism onto the Białynicki-Birula cell.*

Proof. Let us consider the coefficient Ψ with y^{tm_t} in Res_E . Since the resultant is homogeneous, we have $\deg_\lambda \Psi = 0$. The only fixed point is the origin, therefore $\deg_\lambda(c_{i,j,k}) > 0$ for every parameter appearing as a function on Y^+ . As a consequence, $\Psi = 1$ and we get that π is flat over whole Y^+ . By the proof of Theorem 5.39, we get that globally the resultant has the degree at most tm_t , thus we obtain a map $\phi : Y^+ \rightarrow \text{Hilb}_\delta(\mathbb{A}^2)$ to the Hilbert scheme, globally.

The morphism ϕ is a map between the Białynicki-Birula cells (with single fixed points) that is an isomorphism on tangent spaces, Theorem 5.27. By Lemma 3.32, the map ϕ is an isomorphism. □

Remark 5.41. Lemma 5.40 has been essentially proven before in [Hui05, Lemma 1]. In that paper, the matrices considered were slightly different, and at least in principle, the proof there assumed that $\mathbb{k} = \mathbb{C}$, by considering Euclidean neighbourhoods.

For the case of $\deg_\lambda(y) < 0$ we need to restrict the family π , to get a family of closed subschemes set-theoretically supported at $y = 0$. In such case the Białyński-Birula cell on the Hilbert schemes parametrises only scheme supported on the x -axis.

Definition 5.42. The restriction π_- is defined by the following diagram

$$\begin{array}{ccccc} \mathbf{V}(I_t(\mathcal{M}) + (y^{tm_t})) & \xrightarrow{cl} & \mathbf{V}(I_t(\mathcal{M})) & \xrightarrow{cl} & \mathbb{A}^2 \times Y \\ & & \searrow \pi & & \downarrow \\ & & & & Y \\ & \searrow \pi_- & & & \\ & & & & \end{array}$$

Figure 5.1: Family supported on x -axis

The morphism π_- is finite, since y is forced to be integral, but there is no reason, at least a priori, that π_- is again flat.

Theorem 5.43 ([Osz25, Theorem 8.10]). Choose a one-parameter subgroup λ , such that y has a negative degree with respect to the induced grading. Let us restrict Y to $Y^{+, \lambda}$, then the morphism π_- induces a rational $\mathbb{G}_m^2$ -equivariant map $\phi^+ : Y^{+, \psi} \rightarrow \text{Hilb}_\delta(\mathbb{A}^2)$. The image of ϕ^+ lies in the Białyński-Birula cell associated with ψ and centred at E . After restricting the codomain to the Białyński-Birula cell, the map ϕ^+ is étale in a $\mathbb{G}_m^2$ -stable neighbourhood of the origin.

Proof. We imitate the proof of Theorem 5.39. Let us first consider the morphism π on $Y^{+, \lambda}$, and denote the ideal $I_t(\mathcal{M})$ by I . The resultant Res_E (on Y^+) is homogeneous of degree $\deg_\lambda(y)(tm_t)$, since $\deg(y) < 0$, it is a negative number. Let $\Psi \in \mathbb{k}[\mathbf{c}]$ be a coefficient with y^{tm_t} , then

$$\text{Res}_E = uy^{tm_t} + y^{tm_t+1}\Psi(\mathbf{a}, y) = y^{tm_t}(u + y\Psi(\mathbf{c}, y)).$$

Let us restrict ourselves to the open locus of Y^+ where u is invertible. We want to argue that the subscheme containing those points of $\mathbf{V}(I)$ that are not supported at $y = 0$ is closed. Let us observe that

$$(I + (y^{tm_t})) \cap (I + (u + y\Psi(\mathbf{c}, y))) = I,$$

that is because the principal ideals (y^{tm_t}) and $(u + y\Psi(\mathbf{c}, y))$ are co-prime. The ideal $I + (u + y\Psi(\mathbf{c}, y))$ cuts exactly part of $\mathbf{V}(I)$ that is not supported at $y = 0$. Thus its complement $\mathbf{V}(I + (y^{tm_t}))$ is open in $\mathbf{V}(I)$, therefore π_- is flat over open locus of Y^+ where u is invertible. By adding y^{tm_t} to the ideal we have enforced finiteness and as a consequence get the induced rational map ϕ^+ .

The last part follows from Theorem 5.28. $\square$

In the case of $\deg_\psi(y) < 0$, we have also an analogue of Corollary 5.40.

Corollary 5.44. Let us take the setup from Theorem 5.43 but also let us assume that the only $\mathbb{G}_m$ -fixed point on Y^+ is the origin o . Then the induced map ϕ^+ is an isomorphism onto the Białyński-Birula cell.

Proof. Analogous to the proof of Corollary 5.40. $\square$

We apply our results to answer the question asked in [HW24].

Let $E = (x^t, x^{t-1}y^{m_1}, \dots, xy^{m_{t-1}}, y^{m_t}) \subset \mathbb{k}[[x, y]]$ be a monomial ideal of finite colength, recall that $\delta_i = m_i - m_{i-1}$. Let $\mathcal{N}_{\leq \delta}$ denote the affine space consisting of matrices $N = (n_{i,j})$ of size $(t+1) \times t$ with entries in $\mathbb{k}[y]$, such that

$$\begin{cases} u_{i,j} < \text{ord}(n_{i,j}) < \delta_i, & i \leq j, \\ u_{i,j} \leq \text{ord}(n_{i,j}) < \delta_j, & i > j, \end{cases}$$

where the matrix $U = (u_{i,j})$ is the total degree matrix $U(E)$ from Definition 5.23. The matrix H is the Hilbert-Burch matrix M_E from Definition 5.9. The scheme $\mathbf{V}(E)$ is the Gröbner cell of E with respect to the negative degree lexicographical ordering, see Example 3.34.

Conjecture 1 ([?, ?, Conjecture 5.14] RoserWinz-1[HW24, Conjecture 4.2]). *Let $E \subset \mathbb{k}[[x, y]]$ be a monomial ideal of finite colength. Then the map*

$$\begin{aligned} \Phi_e: \mathcal{N}_{< \delta} &\rightarrow \mathbf{V}(E) \\ N &\mapsto I_t(H + N) \end{aligned}$$

is an isomorphism

Corollary 5.45 ([Osz25, Corollary 8.11]). *Conjecture 1 holds.*

Proof. We are following the dictionaries from Tables 3.1 and 3.2. According to Example 3.34, there is a grading with $\deg(x), \deg(y) < 0$, such that $\mathbf{V}(E)$ is the Białynicki-Birula cell. We can observe that $H + \mathcal{N}_\delta$ is the spread-out matrix $\mathcal{M}_E$ restricted to the Białynicki-Birula cell associated with the grading above. Thus the map Φ_E is the same map as ϕ^+ considered in Corollary 5.44, thus making the statement the special case of Corollary 5.44. $\square$

As we have, to some extent, exhausted the possibilities of our methods to the global study, we go back to the infinitesimal setting. We generalise Theorem 5.28 from monomial ideal to any ideal. The idea is the following. We know that the family controls higher-order infinitesimal deformations of the monomial ideal. Since every ideal can be deformed to a monomial one by going to the limit with $\mathbb{G}_m$ -action, we want to translate the isomorphism of tangent spaces by the group action. Subsequently, we repeat the argument from the proof Theorem 5.28 saying that by Hilbert-Burch Theorem, Theorem 2.4 (H-B-3) we get an isomorphism of local deformation functors.

Theorem 5.46 ([Osz25, Theorem 9.1]). *Let us take an ideal $[I] \in \text{Hilb}_\delta(\mathbb{A}^2)$. Every monomial ideal E belonging to the closure of $\mathbb{G}_m^2$ -orbit of $[I]$ induces an isomorphism of local functors*

$$\hat{\phi}: Y_{p_I} \rightarrow \text{hilb}_{\mathbb{S}/I},$$

where $p_I \in Y_E$ is a point such that the ideal of maximal minors of the matrix $\mathcal{M}_E$ over Y_E is equal to I .

Proof. Let us take a monomial ideal E such that $[E] \in \overline{\mathbb{G}_m^2 \cdot [I]}$. The claim is preserved under the field extension, so we may assume that $\mathbb{k}$ is algebraically closed, thus perfect. Now, we can use the classical result by Kempf [Kem78, Corollary 4.3] to say that if the monomial ideal $[E]$ is

inside the closure of the orbit $\mathbb{G}_m^2 \cdot [I]$, then there exists a one-parameter subgroup $\lambda: \mathbb{G}_m \rightarrow \mathbb{G}_m^2$ such that $E \in \overline{\mathbb{G}_m \cdot [I]}$. We choose such one-parameter subgroup λ additionally requiring that

$$\lim_{t \rightarrow 0} t \cdot [I] = [E].$$

As a consequence, $[I]$ lies in the Białynicki-Birula cell centred at E . There are two cases, either the $\mathbb{G}_m$ -action contracts y -axis or it expands it. In the first case we consider the family constructed in Theorem 5.39

$$\begin{array}{ccc} \mathbf{V}(I_t(\mathcal{M}_E))|_{Y^+} & \longrightarrow & \mathbf{V}(I_t(\mathcal{M}_E)) \\ \downarrow \pi|_{Y^+} & & \downarrow \pi \\ Y^+ & \longrightarrow & Y, \end{array}$$

where Y^+ is the Białynicki-Birula cell. For simplicity we will denote $\mathbf{V}(I_t(\mathcal{M}_E))$ as $\mathbf{V}(I)$. In the second case we follow the discussion of Diagram 5.1 and Theorem 5.43 and consider

$$\begin{array}{ccc} \mathbf{V}(I_t(\mathcal{M}_E) + y^{tm_t})|_{Y^+} & \longrightarrow & \mathbf{V}(I_t(\mathcal{M}_E) + y^{tm_t}) \\ \downarrow \pi_-|_{Y^+} & & \downarrow \pi_- \\ Y^+ & \longrightarrow & Y, \end{array}$$

again Y^+ is the Białynicki-Birula cell. We denote $\mathbf{V}(I_t(\mathcal{M}_E) + y^{tm_t})$ as $\mathbf{V}(I + y^{tm_t})$. We will consider both cases simultaneously.

Let us take $Y_1^+ \subset Y$, a first-order thickening of Y^+ in Y such that $T_{pt}Y_1^+ = T_{pt}Y$ for every point pt of Y^+ . Now, we argue that $\mathbf{V}(I)|_{Y_1^+}$ (resp. $\mathbf{V}(I + y^{tm_t})$) induces a morphism to the Hilbert scheme. By Nakayama Lemma [Sta22, Tag 051F], the fact that the morphism $\pi|_{Y^+}$ (resp. $\pi_-|_{Y^+}$) is finite implies that the morphism $\pi|_{Y_1^+}$ (resp. $\pi_-|_{Y_1^+}$) is also finite. By Hilbert-Burch Theorem, Theorem 2.4 (H-B-2), we get also flatness of $\pi|_{Y_1^+}$. In the second case we are considering π_- instead of π . But since $\mathbf{V}(I + y^{tm_t})|_{Y^+}$ is open and closed in $\mathbf{V}(I)|_{Y^+}$, as shown in the proof of Theorem 5.43, its thickening is also open and closed, thus flatness of $\pi|_{\mathbf{S}}$ implies flatness of $\pi_-|_{Y_1^+}$. As a result, we get a morphism $\phi_{Y_1^+}: Y_1^+ \rightarrow \text{Hilb}_\delta(\mathbb{A}^2)$.

Let us denote the embedding of Y^+ into Y by $\iota: Y^+ \rightarrow Y$. Subsequently, the morphism $\phi_{Y_1^+}$ obtained above induces the following commutative diagram:

$$\begin{array}{ccc} i^*TY & \longrightarrow & T \text{Hilb}_\delta(\mathbb{A}^2) \\ \downarrow & & \downarrow \\ Y^+ & \xrightarrow{\phi^+} & \text{Hilb}_\delta(\mathbb{A}^2), \end{array}$$

where ϕ^+ is the morphism described in Theorem 5.39 (resp. Theorem 5.43). Thus we get a morphism of sheaves:

$$\tilde{\mathbf{t}}_\phi: \iota^*TY \rightarrow (\phi^+)^*T \text{Hilb}_\delta(\mathbb{A}^2).$$

The morphism $\tilde{\mathbf{t}}_\phi$ restricted to $0 \in Y$ is the isomorphism of tangent spaces $\mathbf{t}_\phi$ described in Theorem 5.27. As a consequence $\tilde{\mathbf{t}}_\phi$ is an isomorphism on some open subscheme U of Y^+ . Since π (resp. π_-) is $\mathbb{G}_m^2$ -equivariant, so is the morphism $\tilde{\mathbf{t}}_\phi$. And thus U is a $\mathbb{G}_m^2$ -stable subscheme of

Y^+ . Consequently it contains the preimage of $[I]$ under the map ϕ^+ .

This way we have obtained a map of tangent spaces at any point $pt \in Y$ such as in the statement of the theorem. The rest is to extend this morphism to a morphism of local deformation functors. The argument follows identically as in the special case of Theorem 5.46. $\square$

5.5 Further discussions

So far we have shown that our strategy shows that our family of matrices induces maps between the Białynicki-Birula cells. In this section, we see examples when the dream scenario from Section 5.3 occurs and it is not a direct consequence of Theorem 5.39.

Example 5.47. *Let us consider the monomial ideal $E = (y^2, x^3)$. There is no half-plane through the origin containing the weights of all parameters $c_{i,j,k}$. This means that no open neighbourhood of the origin is fully contained in a Białynicki-Birula cell for some one-parameter subgroup. Thus the potential existence of the rational map to the Hilbert scheme cannot be predicted by Theorem 5.39.*

Let us consider the spread-out matrix from Definition 5.19

$$\mathcal{M} = \begin{pmatrix} y^2 + c_{1,1} & c_{1,2} & ca_{1,3} \\ -x + c_{2,1} & 1 & 0 \\ c_{3,1} & -x & 1 \\ c_{4,1} & 0 & -x \end{pmatrix}.$$

Since we are interested only in the degree of the resultant of the first and last minor, we can consider only the variables $e_{i,j} := c_{i,j,1}$, since for every term in that the resultant that does contain $c_{i,j,0}$, there is always one with it replaced by $c_{i,j,1}$ and of degree one higher with respect to y . Thus, we can assume that our matrix is

$$\mathcal{M} = \begin{pmatrix} y^2 + y e_{1,1} & y e_{1,2} & y e_{1,3} \\ y e_{2,1} - x & 1 & 0 \\ y e_{3,1} & -x & 1 \\ y e_{4,1} & 0 & -x \end{pmatrix}.$$

The last minor is

$$f_3 = x^2 y e_{1,3} - x y^2 e_{2,1} e_{1,3} + x y e_{1,2} - y^2 e_{2,1} e_{1,2} - y^2 e_{3,1} e_{1,3} + y^2 + y e_{1,1}$$

and the first one

$$f_0 = -x^3 + x^2 y e_{2,1} + x y e_{3,1} + y e_{4,1}.$$

Their Sylvester matrix with respect to the variable x can be treated as a matrix of polynomials in y . Let us look at the matrix of the degrees of those polynomials.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & -1 & -1 \\ -1 & 1 & 2 & 2 & -1 \\ -1 & -1 & 1 & 2 & 2 \end{pmatrix},$$

where we denote degree of 0 as -1 . From this, by considering the Leibniz formula for determinants, we see that the degree of the resultant with respect to y is lower or equal to 7. Let us consider the 2×2 minors as highlighted below

$$\begin{pmatrix} 0 & 1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & -1 & -1 \\ -1 & 1 & 2 & 2 & -1 \\ -1 & -1 & 1 & 2 & 2 \end{pmatrix}.$$

The minor is

$$\begin{pmatrix} -1 & y e_{2,1} \\ y e_{1,3} & -y^2 e_{2,1} e_{1,3} + y e_{1,2} \end{pmatrix}.$$

The determinant of that minor has the degree equal to 1 with respect to y . This means that the contribution of the first two columns to the degree of the resultant is at most 1. The only way to get the degree 6 out of the last three columns is by multiplying the following entries:

$$\begin{pmatrix} 0 & 1 & 1 & 1 & -1 \\ -1 & 0 & 1 & 1 & 1 \\ 1 & 2 & 2 & -1 & -1 \\ -1 & 1 & 2 & 2 & -1 \\ -1 & -1 & 1 & 2 & 2 \end{pmatrix}.$$

This way we obtain the full degree of resultant to be 6. We have shown that the degree of resultant is not greater than 6, and thus by Lemma 5.33 and Lemma 5.35 we obtain a rational map to the Hilbert scheme.

An almost identical argument can be made to show that we get a rational map for the following monomial ideals:

$$(y^5, x y^4, x^2 y^3, x^3)$$

$$(y^8, x y^6, x^2 y^4, x^3)$$

$$(y^{11}, x y^8, x^2 y^5, x^3)$$

$$(y^{14}, x y^{10}, x^2 y^6, x^3)$$

We leave the following question open for further research.

Question 5.48 ([Osz25, Question 10.2]). *What are the monomial ideals E such that the construction from Definition 5.8 induces a rational map to the Hilbert scheme, but there is no open Białynicki-Birula cell containing E . Can one distinguish specific infinite classes of such ideals?*

Chapter 6

The nested Hilbert scheme

6.1 Introduction

The pointed Hilbert scheme $\mathrm{Hilb}_{1,\delta}(U)$ parametrizes finite schemes of length δ together with chosen $\mathbb{k}$ -point on a smooth quasi-projective variety. We generalise [GGGL24, Theorem 2.8] where the authors have shown that the pointed Hilbert scheme $\mathrm{Hilb}_{1,\delta}(U)$ is non-reduced when $\dim U \geq 4$ and $\delta \geq 7$. The generalisation is twofold, first we exhibit larger locus of non-reduceness, second we describe the structure of the completed local ring at the described non-reduced points.

For the first part, we require that the parametrised finite scheme cannot be deformed into a reducible scheme. This is a weaker condition than requiring that the scheme lies on a unique elementary component. The technical term for this is that the finite schemes are *uncleavable*, this notion was first introduced in [BBKT15, Definition 5.15], see Definition 6.17.

For the second part, the heart of the description lies in the notion of a *fractal family*, which we introduce in Definition 6.15. The *fractalness* is heavily influenced by fractal-like families introduced in [Sza21, Section 3.3]. A pointed family of schemes is called a *fractal family* if a local ring at the distinguished point is smoothly equivalent to a deformation of the special fibre. Intuitively, this means that the base of the pointed family naturally contains a copy of the special fibre, yielding a non-trivial deformation of the fibre over itself.

Lemma 6.1 (Lemma 6.16). *Let $\mathcal{U} \rightarrow \mathcal{M}$ be a fractal family at a $\mathbb{k}$ -point $x \in \mathcal{M}$. Then the $\mathbb{k}$ -point x is a non-reduced point of $\mathcal{M}$.*

Theorem 6.2 (Theorem 6.20). *Let U be a smooth quasi-projective variety and let $\delta \geq 2$ be a positive integer. Then the nested Hilbert scheme $\mathrm{Hilb}_{1,\delta}(U)$ admits a fractal structure at any point $[\ast, X] \in \mathrm{Hilb}_{1,\delta}(U)$, such that $[X] \in \mathrm{Hilb}_{\delta}(U)$ and X is uncleavable.*

In Theorem 6.29 we show that the nested Hilbert scheme $\mathrm{Hilb}_{\delta-1,\delta}(U)$ admits a fractal structure at pairs $(X \# \emptyset, X)$, where X is a finite uncleavable Gorenstein scheme. The subscheme $X \# \emptyset$ is given as the vanishing locus of the socle of $\mathcal{O}_X$, see Notation 6.5. This way, in Theorem 6.37, we reprove the result [Sza21, Theorem 1.5] describing the deformations of $X \# \emptyset$.

6.2 Setting

Setting 6.3. *For this chapter, we fix a positive integer δ and work over an algebraically closed field $\mathbb{k}$ of characteristic either 0 or co-prime with δ .*

This particular assumption on the characteristic of the field will be recalled in the statements that rely on it, the sole reason for this assumptions is Lemma 6.18.

Definition 6.4. *Let Y be a scheme, then a pointed scheme is a morphism $*$ $\rightarrow Y$. If Y admits $\mathbb{G}_m$ -action and the morphism $*$ $\rightarrow Y$ is equivariant, then it is a $\mathbb{G}_m$ -pointed scheme.*

Setting 6.5. *In this chapter we fix Y to be a pointed affine $\mathbb{G}_m$ -scheme, such that the structural sheaf $\mathcal{O}_Y$ is positively graded with zero component equal to the constant sheaf $\mathbb{k}$.*

6.3 Prorepresentability

This section has an auxiliary character and it is focused on the prorepresentability of various good deformation functors. In Proposition 6.6 we describe the structure of the Hilbert scheme of one point in terms of filtered deformations, introduced in Section 4.2. Subsequently, we comment on prorepresentability of negative deformations of finite schemes in Lemma 6.9. The main part of the section is a comparison of the Hilbert scheme of a point on a finite scheme Y and the functor of negative pointed deformations of Y in Corollary 6.10. At the end we check the necessity of the assumptions in Lemma 6.12.

Let us describe the functor of embedded deformations of a distinguished point in a scheme, it is denoted $\text{Def}_{*/Y}$, this functor is introduced in Definition 4.39. This functor is a source of all "fractal" results known to the author, for example *fractal-like* families introduced in [Sza21] and Theorem 6.2. Let us sketch idea of fractalness. We study a pointed families of schemes, we say that a point of the base is fractal if its local ring naturally contains a copy of the special fibre, yielding a non-trivial deformation of the fibre over itself.

Proposition 6.6. *Let $*$ $\rightarrow Y$ be a pointed affine $\mathbb{G}_m$ -scheme. There is a natural isomorphism $\text{Def}_{*/Y}^{<0} \cong \text{Def}_{*/Y}$, both are prorepresented by the completed local ring at the distinguished point of Y . The tangent spaces of these functors are concentrated in degrees equal to (-1) times the degrees of the minimal generators of the maximal ideal of $\mathcal{O}_Y$.*

Proof. Let us take the exact sequence describing the pointed structure of Y

$$0 \rightarrow \mathfrak{m} \rightarrow \mathcal{O}_Y \rightarrow \mathbb{k}.$$

The tangent space of the functor $\text{Def}_{*/Y}$ is the tangent to the Hilbert scheme at the same point

$$\text{Hom}_{\mathcal{O}_Y}(\mathfrak{m}, \mathbb{k}).$$

Hence, we get the last part of the statement, since $\mathfrak{m}$ is generated in positive degrees, the tangent space is strictly negative. The map of obstruction spaces is the top horizontal map in

the commutative diagram

$$\begin{array}{ccc} (T_{*/Y}^2)_{<0} & \longrightarrow & T_{*/Y}^2 \\ \downarrow & & \downarrow \\ \mathrm{Ext}_{\mathcal{O}_Y}^1(\mathfrak{m}, \mathbb{k})_{<0} & \longrightarrow & \mathrm{Ext}_{\mathcal{O}_Y}^1(\mathfrak{m}, \mathbb{k}). \end{array}$$

Hence, the map is unobstructed, see Definition 4.34 and surjective on the tangent spaces. Therefore, by Theorem 4.35 we get a smooth map $\mathrm{Def}_{*/Y}^{<0} \rightarrow \mathrm{Def}_{*/Y}$. As it is a map between prorepresentable functors, which is isomorphic on the tangent spaces, by Lemma 4.32 we get the first claim of the statement. The Hilbert scheme of one point on Y is isomorphic to Y . Consequently, the local functor is as claimed in the second part of the statement. $\square$

It is worth mentioning the vanishing

$$(T_Y^1)_{<-1} = 0$$

plays a remarkable role in all the results in next chapter. The reason for that is Corollary 4.72 applied for $(X \rightarrow Y) = (* \rightarrow Y)$.

Corollary 6.7. *Assume that $(T_Y^1)_{<-1}$ vanishes and that $\mathcal{O}_Y$ is generated in degree 1. Then the vector space $(T_{*\rightarrow Y}^1)_{<-1}$ also vanishes. Moreover, the diagram*

$$\begin{array}{ccc} \mathrm{Def}_{*/Y} & \longrightarrow & \mathrm{Def}_{*\rightarrow Y} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Def}_Y \end{array} \tag{6.8}$$

is Cartesian up to smooth equivalence and the map $\mathrm{Def}_{/Y} \rightarrow \mathrm{Def}_{*\rightarrow Y}$ is unobstructed.*

Proof. The first part is a direct consequence of the long exact sequence of T^i -functors, see Braid ($\spadesuit$)

$$T_{*/Y}^1 \rightarrow T_{*\rightarrow Y}^1 \rightarrow T_Y^1,$$

the space $(T_Y^1)_{<-1}$ vanishes by assumptions, and $(T_{*/Y}^1)_{<-1}$ by Lemma 6.6. By Lemma 6.6, the tangent space $T_{*/Y}^1$ is generated in negative degrees. Henceforth, by Corollary 4.72, we get that the map $\mathrm{Def}_{*/Y} \rightarrow \mathrm{Def}_{*\rightarrow Y}$ is unobstructed, thus by Lemma 4.38 we get that Diagram (6.8) is Cartesian up to smooth equivalence. $\square$

Since we are predominantly interested in finite local schemes, one could wonder if, in that case, the condition for prorepresentability in Corollary 4.31 for strictly negative deformations $\mathrm{Def}_Y^{<0}$, vanishing of $(T_Y^0)_{<0}$ is always satisfied. In characteristic 0, if the structural sheaf $\mathcal{O}_Y$ is generated in a fixed positive degree, it is true.

Lemma 6.9. *Assume that Y is finite and the structural sheaf $\mathcal{O}_Y$ is generated in a fixed positive degree. In characteristic zero or larger than the length of Y , the group of infinitesimal automorphisms $(T_Y^0)_{<0}$ vanishes. And thus $\mathrm{Def}_Y^{<0}$ is prorepresentable.*

Proof. An element of T_Y^0 is a derivation $\mathrm{Der}_{\mathbb{k}}(\mathcal{O}_Y, \mathcal{O}_Y)$. Let us take a non-zero negative derivation $\partial \in \mathrm{Der}_{\mathbb{k}}(\mathcal{O}_Y, \mathcal{O}_Y)_{<0}$. All generators of $\mathcal{O}_Y$ have fixed degree $l > 0$, thus for a generator $e \in \mathcal{O}_Y$,

the value $\partial(e)$ has the degree 0, hence, it is either 0 or invertible, recall Setting 6.5. Since $\partial \neq 0$, there exists e such that $\partial(e)$ is invertible. For the minimal positive integer $k \in \mathbb{Z}$ such that $e^k = 0$ we get

$$0 = \partial(e^k) = k!e^{k-1}d(e),$$

and since $\text{char } \mathbb{k} > k$, we get a contradiction. Since the negative tangent space is finitely dimensional, the last part is an immediate application of Proposition 4.31. $\square$

We show that if Y is a **TNT** scheme, recall Definition 4.57, then all negative deformations of the pointed maps come from the Hilbert scheme of the distinguished point.

Corollary 6.10. *If the vector spaces $(T_Y^0)_{<0}$ and $(T_Y^1)_{<0}$ vanish, then*

$$\text{Def}_{*/Y} \rightarrow \text{Def}_{*\rightarrow Y}^{<0} \quad (6.11)$$

is an isomorphism.

Proof. We argue that the morphism (6.11) is smooth, isomorphic on the tangent spaces and domain and the codomain are prorepresentable. Then, by Lemma 4.32, we get an isomorphism. By Corollary 4.72, the morphism (6.11) is smooth. The kernel of the induced homomorphism of tangent spaces $T_{*/Y}^1 \rightarrow T_{*\rightarrow Y}^1$ is the image of T_Y^0 . By our assumption, the injectivity of the tangent map follows. The functor $\text{Def}_{*/Y}^{<0}$ is prorepresentable. The only thing left is to argue that $\text{Def}_{*\rightarrow Y}^{<0}$ is prorepresentable, see Proposition 6.6. The vector space $(T_{*/Y}^1)_{<0}$ is finitely dimensional is satisfied, since Y has a finitely dimensional tangent space at the distinguished point. Hence, the assumptions of Corollary 4.70 are satisfied and we get the claim. $\square$

In the following lemma, we show that the condition on the infinitesimal deformations is necessary for Corollary 6.6.

Lemma 6.12. *Let Y be a pointed affine $\mathbb{G}_m$ -scheme, with the structural sheaf $\mathcal{O}_Y$ generated in a fixed positive degree. Then the map $(T_Y^0)_{<0} \rightarrow (T_{*/Y}^1)_{<0}$ is injective. If $(T_Y^0)_{<0}$ does not vanish, the map of the tangent spaces*

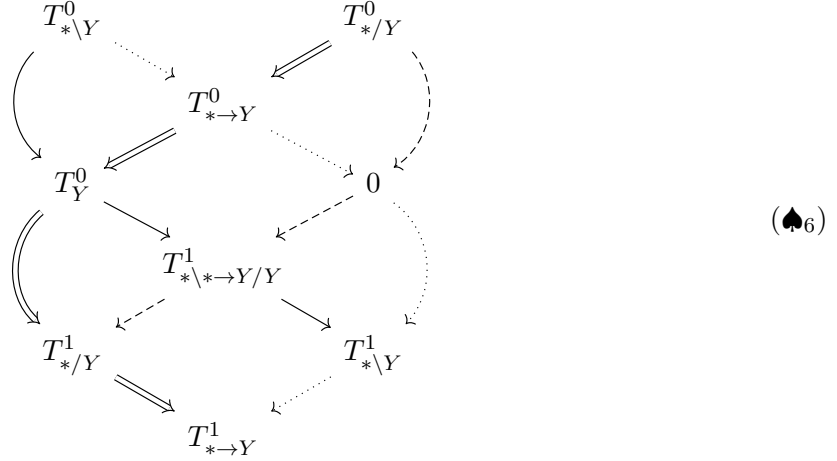
$$T_{*/Y}^1 \rightarrow T_{(*\rightarrow Y)}^1$$

is not injective.

Proof. We look at the exact sequence of T^i functors induced by the following exact sequence of $\mathcal{O}_Y$ -modules

$$\begin{aligned} 0 \rightarrow \mathfrak{m} \rightarrow \mathcal{O}_Y \rightarrow \mathbb{k} \rightarrow 0, \\ 0 \rightarrow T_{\mathbb{k}}^0(\mathcal{O}_Y, \mathfrak{m}) \rightarrow T_{\mathbb{k}}^0(\mathcal{O}_Y, \mathcal{O}_Y) \rightarrow T_{\mathbb{k}}^0(\mathcal{O}_Y, \mathbb{k}) = T_{*\setminus *\rightarrow Y/Y}^1. \end{aligned} \quad (6.13)$$

This is part of Braid $(\spadesuit)$ for the pointed map $(* \rightarrow Y)$ given by the single arrow.



By the definition [Har10, Proposition 3.6], the sequence (6.13) is equal to

$$0 \rightarrow \mathrm{Der}_{\mathbb{k}}(\mathcal{O}_Y, \mathfrak{m}) \rightarrow \mathrm{Der}_{\mathbb{k}}(\mathcal{O}_Y, \mathcal{O}_Y) \rightarrow \mathrm{Der}_{\mathbb{k}}(\mathcal{O}_Y, \mathbb{k}).$$

Since the module $\mathrm{Der}_{\mathbb{k}}(\mathcal{O}_Y, \mathfrak{m})$ is zero in negative degrees, the map

$$(T_Y^0)_{<0} = T_{\mathbb{k}}^0(\mathcal{O}_Y, \mathcal{O}_Y)_{<0} \rightarrow (T_{*\setminus *\rightarrow Y/Y}^1)_{<0}$$

is injective. Hence, when composed with the map $T_{*\setminus *\rightarrow Y/Y}^1 \rightarrow T_{*/Y}^1$, which is injective by Braid $(\spadesuit_6)$, we get the statement. $\square$

6.4 Pointed Hilbert scheme

In this subsection we define a notion of a fractal family, inspired by [Sza21, Section 3.3]. Subsequently, we show the existence of such a family on the pointed Hilbert scheme $\mathrm{Hilb}_{1,\delta}(\mathbb{A}^n)$, this is the main outcome of this subsection, Theorem 6.20, which is a partial generalisation of [GGGL24, Theorem 2.8].

We start with an example, which showcases the difference between functor of pointed and the usual deformations of a local scheme. We show our take on results shown in [GGGL24, 2.2. Some elementary generically non-reduced components], Example 6.14 showcases the phenomenon which can be deduced from the proof of [GGGL24, Theorem 2.8]. The technique is to take a finite pointed scheme and translate it affinely, while *fixing* the distinguished point. This provided us with nontrivial deformations, all first-order deformations used in [GGGL24, Theorem 2.8] are constructed this way. In that paper, the authors deduced from that construction the non-reducedness of the pointed Hilbert scheme, recall Definition 3.14. We focus on the singularity type, which is a more refined invariant.

Example 6.14. *Let $\mathrm{char} \mathbb{k} = p$ and take a finite scheme $Y = \mathrm{Spec} \mathbb{k}[x]/(x^q)$ with $\deg(x) = 1$. Then let us consider the unique pointed structure on Y given by the exact sequence*

$$0 \rightarrow (x) \rightarrow \mathcal{O}_Y \rightarrow \mathbb{k} \rightarrow 0.$$

The functor $\text{Def}_{*/Y}$ is the Hilbert scheme of a point on Y it is isomorphic to Y itself, see Lemma 6.6. The tangent space at the unique closed point is

$$\text{Hom}_{\mathcal{O}_Y}((x), \mathbb{k}),$$

it is one-dimensional and concentrated in degree -1 . Therefore, all embedded deformations are negative. The image of the maximal deformation $\text{Def}_{*/Y}(\mathcal{O}_Y)$ in $\text{Def}_{*\rightarrow Y}$ is

$$\mathbb{k}[\varepsilon]/(\varepsilon^q) \subset \mathbb{k}[x, \varepsilon]/((x + \varepsilon)^q, \varepsilon^q) \xrightarrow{x \mapsto 0} \mathbb{k}[\varepsilon]/(\varepsilon^q).$$

if $p|q$ then this is a trivial pointed deformation, since $(x + \varepsilon)^q = x^q + \varepsilon^q$. In the case p divides q , we show that this pointed deformation is not trivial.

The argument is by a contradiction, we assume that there is an isomorphism ϕ between the pointed family described above and the trivial one. We have a pair of pointed deformations, over $\mathbb{k}[\varepsilon]/(\varepsilon^q)$, since they are pointed, each of them can be seen as a deformation with a section.

$$\begin{array}{ccc} \mathbb{k}[x, \varepsilon]/(\varepsilon^q, (x + \varepsilon)^q) & \xrightarrow{\phi} & \mathbb{k}[x, \varepsilon]/(\varepsilon^q, x^q) \\ & \searrow \pi_1 & \nearrow \pi_2 \\ & \mathbb{k}[\varepsilon]/(\varepsilon^q) & \end{array}$$

First, by the commutativity of the diagram of $\mathbb{k}[\varepsilon]/(\varepsilon^q)$ -algebras above

$$\phi(\varepsilon) = \varepsilon.$$

Second, since ϕ lies over identity on $\mathbb{k}[x]/(x^q)$

$$\phi(x) = x \pmod{(\varepsilon)},$$

equivalently

$$\phi(x) - x \in (\varepsilon) \subset \mathbb{k}[x, \varepsilon]/(\varepsilon^q, x^q).$$

Hence, there exists $h \in \mathbb{k}[x, \varepsilon]/(\varepsilon^q, x^q)$ such that

$$\varepsilon h = \phi(x) - x,$$

but since

$$\varepsilon \pi_2(h) = \pi_2(\varepsilon h) = \pi_2(\phi(x) - x) = \pi_1(x) - \pi_2(x) = 0 - 0 = 0,$$

we get $h \in (\varepsilon^{q-1}, x) \subset \mathbb{k}[x, \varepsilon]/(x^q, \varepsilon^q)$. This implies that there are elements $g, g' \in \mathbb{k}[x, \varepsilon]/(x^q, \varepsilon^q)$ such that

$$h = xg + \varepsilon^{q-1}g'.$$

Thus $\phi(x) = x(1 + g\varepsilon)$, therefore $\phi(x)$ is nilpotent of order q , but $x^q \in \mathbb{k}[x, \varepsilon]/(\varepsilon^q, (x + \varepsilon)^q)$ is not zero, a contradiction with ϕ being an isomorphism.

Recall, from Definition 3.6, that two pointed schemes (X, x) and (Y, y) are smoothly equivalent if there exists a pointed scheme (Z, z) together with pair of smooth morphisms preserving

the pointed structures

$$\begin{array}{ccc} & (Z, z) & \\ \swarrow \text{sm} & & \searrow \text{sm} \\ (X, x) & & (Y, y). \end{array}$$

Definition 6.15. Let $\mathcal{U} \rightarrow \mathcal{M}$ be a flat morphism with irreducible fibres and with a section $\mathcal{M} \rightarrow \mathcal{U}$. Let $y \in \mathcal{M}$ be $\mathbb{k}$ -point such that $\mathcal{U}_{|\{y\}} = \mathcal{U} \times_{\mathcal{M}} \{y\}$ is a finite non-reduced scheme. We say that $\mathcal{M}$ admits a fractal structure at a point y if there is a morphism $\{y\} \rightarrow Z \rightarrow \mathcal{M}$ such that the pointed schemes $(\mathcal{U}_{|Z}, \{y\})$ and $(\mathcal{M}, \{y\})$ are smoothly equivalent.

$$\begin{array}{ccccc} & & & \nearrow \text{sm-eq} & \mathcal{M} \\ \mathcal{U}_{|\{y\}} & \longrightarrow & \mathcal{U}_{|Z} & \longrightarrow & \mathcal{U} \\ \uparrow \downarrow & & \uparrow \downarrow & & \uparrow \downarrow \\ \{y\} & \longrightarrow & Z & \longrightarrow & \mathcal{M}. \end{array}$$

We apply the definition for $\mathcal{M}$ being a moduli space of pointed objects, for example, the pointed Hilbert scheme of points $\text{Hilb}_{1,\delta}(\mathbb{A}^n)$, see Theorem 6.20. Definition 6.15 says that the completed local ring of the moduli space at $\{y\}$ looks like a deformation of $\mathcal{U}_{|\{y\}}$ over a scheme Z .

Lemma 6.16. Let $\mathcal{U} \rightarrow \mathcal{M}$ be a fractal family at a $\mathbb{k}$ -point $y \in \mathcal{M}$. Then the $\mathbb{k}$ -point y is a non-reduced point of $\mathcal{M}$.

Proof. Let us consider the following diagram

$$\begin{array}{ccc} \mathcal{U}_{|\{y\}} & \longrightarrow & \mathcal{U}_{|Z} \\ \uparrow \downarrow & & \uparrow \downarrow \\ \{y\} & \longrightarrow & Z. \end{array}$$

Since the morphism $\mathcal{U} \rightarrow \mathcal{M}$ satisfies the conditions of Lemma 2.2 and the fibre $\mathcal{U}_{|\{y\}}$ is not a point, $\mathcal{U}_Z$ is non-reduced. By Definition 6.16, $\mathcal{M}$ is also non-reduced at y . $\square$

We introduce a crucial notion of uncleavable scheme, its says that the scheme cannot be deformed into a reducible scheme. It is needed to get a fractal family, see Definition 6.15. Even more importantly it is needed to get a non-reduced structure by using Lemma 2.2.

Definition 6.17 ([BBKT15, Definition 5.15]). We say that a finite local scheme Y is uncleavable if for every flat finite family over irreducible scheme S

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \pi \\ * & \longrightarrow & S \end{array}$$

with Y as the fibre over special point, the every fibre of π is irreducible.

Before stating the main result of this section, we show a key property of families that do not decompose the fibres.

Lemma 6.18. *Let $V \subset \text{Hilb}_\delta \mathbb{A}^n$ be an elementary component. Assume that $\text{char } \mathbb{k}$ is either 0 or co-prime with δ . Then the universal family $\pi: \mathcal{U}_V \rightarrow V$ admits a section σ .*

Proof. By [Sza21, Proposition 2.27], the component $V \cong \mathbb{A}^n \times V_0$, where V_0 parametrizes schemes supported at the origin $0 \in \mathbb{A}^n$. This way we also obtain a decomposition of the universal family $\mathcal{U}_V = \mathbb{A}^n \times \mathcal{U}_{V_0}$. Hence, it is enough to construct a section of the restricted projection $\pi_0: \mathcal{U}_{V_0} \rightarrow V_0$. By the definition of the universal family, $\mathcal{U}_{V_0}$ is a closed subscheme of $\mathbb{A}^n \times V_0$, since V_0 parametrises schemes supported at the origin, $\mathcal{U}_{V_0}$ contains the subscheme $\{0\} \times V_0 \cong V_0$. $\square$

Remark 6.19. *The Lemma 6.18 is the sole reason why we assume anything about the characteristic of the field in further results, for example in Theorem 6.20 and Theorem 6.29. The assumption is needed to use [Sza21, Proposition 2.27], the argument goes through constructing the average morphism $\mathbb{A}^n / \Sigma_\delta \rightarrow \mathbb{A}^n$ in [Sza21, Theorem 2.26], hence the usage of the characteristic is non-trivial.*

Theorem 6.20. *Let $* \rightarrow Y$ be a finite uncleavable scheme of length δ . Assume that $\text{char } \mathbb{k}$ is either 0 or co-prime with δ . Then for a given embedding $Y \subset \mathbb{A}^n$, the induced point $[*, Y]$ in the pointed Hilbert scheme $\text{Hilb}_{1,\delta}(\mathbb{A}^n)$ is a fractal point.*

Remark 6.21. *Since the property of being uncleavable is weaker than lying on unique elementary component, we get a twofold generalisation of [GGGL24, Theorem 2.8]. First, we consider a larger locus of points. Second, not only do we show non-reducedness, but also describe structure of the local rings.*

Proof. By definition, an S -point of the universal family $\mathcal{U}$ of $\text{Hilb}_\delta(\mathbb{A}^n)$ is a pair consisting of a closed subscheme Y_S of $\mathbb{A}^n \times S$, finite and flat over S , and an S -point of Y_S , hence a section of $Y_S \rightarrow S$. The image of this section is a closed subscheme of Y_S , hence we obtain a map $\mathcal{U} \rightarrow \text{Hilb}_{1,\delta}(\mathbb{A}^n)$. Conversely, an S -point of $\text{Hilb}_{1,\delta}(\mathbb{A}^n)$ is a pair $Z_S \subset Y_S \subset \mathbb{A}^n \times S$, where $Z_S \rightarrow S$ is finite flat over S with fibres of degree one, hence $Z_S \rightarrow S$ is an isomorphism and Z_S is a section of $Y_S \rightarrow S$. This shows that $\mathcal{U}$ and $\text{Hilb}_{1,\delta}(\mathbb{A}^n)$ represent the same functor, hence are isomorphic.

Let V be an open neighbourhood of $[Y] \in \text{Hilb}_\delta(\mathbb{A}^n)$, such that V parametrises only uncleavable points. By Lemma 6.18, the universal family $\pi: \mathcal{U}_V \rightarrow V$ admits a section σ , hence we get the induced map to the nested Hilbert scheme $\text{Hilb}_{1,\delta}(\mathbb{A}^n)$. This yields the Cartesian diagram

$$\begin{array}{ccc} \mathcal{U}_V & \longrightarrow & \mathcal{U}_{1,\delta} \\ \pi \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) \sigma & & \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) \\ V & \longrightarrow & \text{Hilb}_{1,\delta}(\mathbb{A}^n), \end{array}$$

where $\mathcal{U}_{1,\delta}$ is the universal family of $\text{Hilb}_{1,\delta}(\mathbb{A}^n)$.

Let us choose a $\mathbb{k}$ -point $[Y]$ in V , by the universal property of $\mathcal{U}_V$ we get the Cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{U}_V \\ \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) & & \left(\begin{array}{c} \nearrow \\ \searrow \end{array} \right) \sigma \\ * & \longrightarrow & V. \end{array}$$

Since V is open in $\text{Hilb}_\delta(\mathbb{A}^n)$, $\mathcal{U}_V$ is open in $\mathcal{U} \cong \text{Hilb}_{1,\delta}(\mathbb{A}^n)$, in particular the pointed schemes $(\mathcal{U}_V, *)$ and $(\text{Hilb}_{1,\delta}(\mathbb{A}^n), [*, Y])$ are smoothly equivalent. Since by choice of V , the fibres of π are finite irreducible and non-reduced, we get the fractal structure. $\square$

By Lemma 6.16, we get an immediate non-reduceness result.

Corollary 6.22. *Let $* \rightarrow Y$ be a finite uncleavable scheme of length δ . Assume that $\text{char } \mathbb{k}$ is either 0 or co-prime with δ . Then $\text{Def}_{* \rightarrow Y}$ (respectively $\text{Def}_{* \rightarrow Y}^{<0}$) is non-reduced.*

Proof. Let us take an embedding $Y \subset \mathbb{A}^n$. By Theorem 6.20, we get that the Hilbert scheme has a fractal structure and the induced point, by Lemma 6.16 this point is non-reduced. By Lemma 4.44, we get the claim.

For the second variant of the statement, let V be a $\mathbb{G}_m$ -equivariant neighbourhood of $[Y] \in \text{Hilb}_\delta(\mathbb{A}^n)$. To prove that $\text{Def}_{* \rightarrow Y}^{<0}$ is non-reduced it is enough to show non-reduceness of $[*, Y] \in \text{Hilb}_{1,\delta}^{<0}(\mathbb{A}^n)$, see Lemma 4.62. We repeat the argument of Theorem 6.20 and get the following Cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{U}_V \\ \left(\begin{array}{c} \nearrow \\ \downarrow \end{array} \right) & & \left(\begin{array}{c} \nearrow \pi \\ \downarrow \end{array} \right) \sigma \\ * & \longrightarrow & V. \end{array} \quad (6.23)$$

The morphisms are $\mathbb{G}_m$ -equivariant. We restrict Diagram 6.23 to the Białynicki-Birula fibre of $[Y]$ and get

$$\begin{array}{ccc} Y^{<0} & \longrightarrow & (\mathcal{U}_V)^{<0} \\ \left(\begin{array}{c} \nearrow \\ \downarrow \end{array} \right) & & \pi^{<0} \left(\begin{array}{c} \nearrow \\ \downarrow \end{array} \right) \\ * & \longrightarrow & V^{<0}. \end{array}$$

To show non-reduceness we check the conditions of Lemma 2.2 for the family $\pi^{<0}$.

- The fibres of $\pi^{<0}$ are contained in fibres of π , and the latter are local, hence the former are local. In particular, the fibres of $\pi^{<0}$ are irreducible.
- By Lemma 6.6, we have that Y in Diagram 6.23 has a strictly negative $\mathbb{G}_m$ -action, hence the isomorphism $Y \cong Y^{<0}$. The special fibre is a finite non-reduced scheme.

Since the special fibre is non-reduced, Lemma 2.2 implies the claim. $\square$

6.5 Deformation of colength one subalgebra of a Gorenstein algebra

In this subsection, we show a second example of fractal structure, recall Definition 6.15. We show that the nested Hilbert scheme $\text{Hilb}_{\delta-1,\delta}(\mathbb{A}^n)$ admits a fractal structure. The main outcome is Theorem 6.37, in which we reprove [Sza21, Theorem 1.5]. This result describes *the fractal-like* structure of the Hilbert scheme on the component with the Hilbert function equal to $(1, n, n)$.

For this subsection, we specialise the setting of the previous subsection to the Gorenstein case.

Setting 6.24. *We take a scheme $Y = \text{Spec } B \subset \mathbb{A}^n$ of length δ , supported at the origin. The local ring $(B, \mathfrak{m})$ is a graded local Artinian Gorenstein algebra and $(T_Y^1)_{<-1} = 0$. Let $g \in B$ be a homogeneous generator of the socle. Let us denote the degree of g by t .*

Now, we study deformations of embedding $\text{Spec}(B/(g)) \rightarrow Y$, for clarity of exposition we introduce notation for this subscheme.

Notation. *The closed subscheme $\text{Spec}(B/(g)) \rightarrow Y$ is denoted by $Y \# \emptyset$.*

The notation 6.5 is to be compared with definition of connected sum, Definition 7.28. Note that $Y \# \emptyset$ is not Gorenstein. Let us apply a special case of so called *linkage* procedure [Har10, Exercise 9.3].

Lemma 6.25. *There is an isomorphism of the good deformation functors*

$$\text{Def}_{* \rightarrow Y} \cong \text{Def}_{Y \# \emptyset \rightarrow Y}.$$

and an analogous isomorphism for variant with the fixed codomain.

$$\text{Def}_{Y \# \emptyset / Y} \cong \text{Def}_{* / Y}.$$

Proof. Let us fix $(A, \mathfrak{n}) \in \text{Art}_{\mathbb{k}}$. Let us take an A -point on the left hand-side of the alleged isomorphism

$$0 \rightarrow I \rightarrow \mathcal{O}_{Y_A} \rightarrow A \rightarrow 0 \in \text{Def}_{* \rightarrow Y}(A). \quad (6.26)$$

We construct an element in $\text{Def}_{Y \# \emptyset / Y}(A)$. First, we dualise Sequence (6.26) and we get

$$0 \rightarrow A^\vee \rightarrow (\mathcal{O}_{Y_A})^\vee \rightarrow I^\vee \rightarrow 0.$$

Since $\mathcal{O}_{Y_A}$ is Gorenstein $(\mathcal{O}_{Y_A})^\vee \cong (\mathcal{O}_{Y_A})$, see [Har10, Exercise 7.5]. We obtain

$$0 \rightarrow A \rightarrow \mathcal{O}_{Y_A} \rightarrow I^\vee \rightarrow 0, \quad (6.27)$$

this way A is realised as a principal ideal in $\mathcal{O}_{Y_A}$. After tensoring with $A/\mathfrak{n}$, we get

$$0 \rightarrow \mathbb{k} \rightarrow \mathcal{O}_Y \rightarrow \text{coker} \rightarrow 0, \quad (6.28)$$

since the socle ideal of $\mathcal{O}_Y$ is principal, Sequence (6.28) is the unique element in $\text{Def}_{Y \# \emptyset \rightarrow Y}(\mathbb{k})$. Hence, Sequence (6.27) is an element in $\text{Def}_{Y \# \emptyset \rightarrow Y}(A)$. Since the order of the argument is reversible, we get an isomorphism.

The argument for the second variant is the same but with $\mathcal{O}_{Y_A}$ fixed to be $\mathcal{O}_Y \otimes A$. $\square$

Similarly to Section 6.4 we get a fractal structure, recall Definition 6.15.

Theorem 6.29. *Assume that $\text{char } \mathbb{k}$ is either 0 or co-prime with δ . The nested Hilbert scheme $\text{Hilb}_{\delta-1, \delta}(\mathbb{A}^n)$ admits a fractal structure at any point $([Y \# \emptyset, Y])$ such that Y is an uncleavable scheme satisfying given in Setting 6.24.*

Proof. Let V be an open neighbourhood of $[X] \in \text{Hilb}_\delta(\mathbb{A}^n)$, such that V parametrises only uncleavable points. By Theorem 6.20, the universal family $\pi: \mathcal{U}_V \rightarrow V$ is isomorphic to a component $\tilde{V} \subset \text{Hilb}_{1, \delta}(\mathbb{A}^n)$ over V . The local functor of $\tilde{V}$ at $[*, Y]$ is smoothly equivalent $\text{Def}_{* \rightarrow Y}$. By Lemma 6.25, we have an isomorphism $\text{Def}_{* \rightarrow Y} \cong \text{Def}_{Y \# \emptyset \rightarrow Y}$. The last functor is smoothly equivalent to the local functor of $\text{Hilb}_{\delta-1, \delta}(\mathbb{A}^n)$ at $[Y \# \emptyset, Y]$. $\square$

We switch subjects a little, and focus on the deformations of $Y \# \emptyset$, more specifically, we describe *the fractal-like* at such points on $\text{Hilb}_{\delta-1}(\mathbb{A}^n)$, that is the content of Theorem 6.37. Even though the result itself is not stronger than [Sza21, Theorem 1.5], the argument is much more conceptual, and provides a good example of application of our techniques.

We prove Theorem 6.37 in a few steps. The first step is to show that the forgetful functor

$$\text{Def}_{Y \# \emptyset \rightarrow Y}^{<0} \rightarrow \text{Def}_{Y \# \emptyset}^{<0},$$

with appropriate assumptions is smooth.

Lemma 6.30. *Assume that the vector space $T^2(\mathcal{O}_Y, \mathbb{k})_{<-t}$ vanishes. Then the forgetful map*

$$\text{Def}_{Y \# \emptyset \rightarrow Y}^{<0} \rightarrow \text{Def}_{Y \# \emptyset}^{<0}$$

is smooth and isomorphic on the tangent spaces. Moreover, if $\mathcal{O}_Y$ is generated in degree 1, then $(T_{Y \# \emptyset}^1)_{<-1}$ vanishes.

Proof. By Corollary 4.71 it is enough to check the vanishing of

$$(T_{Y \# \emptyset \setminus Y}^i)_{<0} = 0,$$

for $i = 1, 2$.

$$T_{Y \# \emptyset \setminus Y}^i = T_{\mathbb{k}}^i(\mathcal{O}_Y, (g)) = T_{\mathbb{k}}^i(\mathcal{O}_Y, \mathbb{k})[\deg(g)].$$

For $i = 1$, the vanishing in negative degrees follows from the fact that $\deg(g) = t$ is the regularity of Y . For $i = 2$, this is the assumption.

The second part of the statement follows from the fact that $(T_{Y \# \emptyset \rightarrow Y}^1)_{<-1}$ vanishes. To prove that, we take a part of the long exact sequence of T^i functors

$$T_{Y \# \emptyset / Y}^1 \rightarrow T_{Y \# \emptyset \rightarrow Y}^1 \rightarrow T_Y^1.$$

The module $T_{Y \# \emptyset / Y}^1$ is concentrated in degree -1 as by Lemma 6.25, it is isomorphic to $T_{*/Y}^1$, which in turn by Lemma 6.6 is concentrated in degree -1 , as $\mathcal{O}_Y$ is generated in degree 1. The vanishing of $(T_Y^1)_{<-1}$ follows from the assumptions. $\square$

Recall that local Gorenstein algebras are characterised by apolarity, see Theorem 2.12. To get the desired description we need to specialise Setting 6.24 to algebras apolar to general enough cubics.

Definition 6.31 ([Sza21, Definition 3.2]). *Let $\Phi \in P$, be a cubic in n variables. We call it very general if the following conditions are satisfied:*

- (c₁) $\text{Apolar}(\Phi)$ has Hilbert function equal to $(1, n, n, 1)$.
- (c₂) The minimal graded free resolution of $\text{Apolar}(\Phi)$ has generators in degree 2 and syzygies in degree 3.
- (c₃) $\text{Apolar}(\Phi)$ admits **TNT** property.

Lemma 6.32. *Let Φ be a cubic, such that $\text{Apolar}(\Phi)$ has the Hilbert function $(1, n, n, 1)$, then the positive tangent space $(T_{\text{Apolar}(\Phi)}^1)_{>0}$ vanishes.*

Proof. From [Sza23, Proposition 2.16.] we know that the tangent space in the embedded case vanishes in degrees > 1 . We take the usual presentation of an apolar algebra, coming from Definition 2.11

$$0 \rightarrow I \rightarrow \mathbb{S} \rightarrow \text{Apolar}(\Phi) \rightarrow 0.$$

The embedding above presents $\text{Apolar}(\Phi)$ as a point on the Hilbert scheme. The positive tangent space to the Hilbert scheme at $\text{Apolar}(\Phi)$ is concentrated in degree 1 and of dimension $\binom{n+1}{2} - n$. Let us denote $\text{Apolar}(\Phi)$ by B . By [Har10, Proposition 3.10], we can compute the dimension of T^1 from the following sequence:

$$0 \rightarrow \text{Der}(B, B) \rightarrow \text{Der}(\mathbb{S}, B) \rightarrow \text{Hom}_B(I/I^2, B) \rightarrow T^1(B, B) \rightarrow 0.$$

Since the degree 1 part $\text{Der}(B, B)_1$ has dimension $n^2 - \binom{n+1}{2} - n$ and $\text{Der}(\mathbb{S}, B)_1$ has dimension n^2 . This concludes the statement. $\square$

The local Gorenstein algebras with the Hilbert function $(1, n, n, 1)$ have been extensively studied. Recall, from Definition 3.22, that the Białynicki-Birula fibre is the fibre of the Białynicki-Birula decomposition over chosen $\mathbb{G}_m$ -fixed point. It is known that the Białynicki-Birula fibre over such an algebra with the Hilbert function $(1, n, n, 1)$ is smooth, see [Jel17, Proposition 1.1], [ER12, Theorem 3.3].

Recall, Definition 3.12, that $\text{Hilb}_\delta^{\text{Gor}}(\mathbb{A}^n)$ denotes the Gorenstein locus of the Hilbert scheme of δ points on $\mathbb{A}^n$.

Proposition 6.33 ([Sza23, Theorem 1]). *Let $n \geq 6$ be a positive integer not equal to 7. The locus of algebras with the Hilbert function $(1, n, n, 1)$ is open and dense in a generically smooth irreducible component of $\text{Hilb}_{2n+2}^{\text{Gor}}(\mathbb{A}^n)$.*

Lemma 6.34 ([Sza21, Proposition 3.4]). *Let Φ be a very general cubic. Let $Y = \text{Spec } \text{Apolar}(\Phi)$. Then the positive tangent space $(T_{Y \# \emptyset}^1)_{>0}$ vanishes.*

Proof. By [Har10, Proposition 3.10], we get that a surjection

$$\text{Hom}(I, \mathbb{S}/I) \rightarrow (T_{Y \# \emptyset}^1) \rightarrow 0,$$

where $I = \text{Ann}(\Phi) + (g)$. Recall that (g) is the socle of $\text{Apolar}(\Phi)$. We prove that already the module $\text{Hom}(I, \mathbb{S}/I)_{>0}$ vanishes. Suppose that $\partial \in \text{Hom}(I, \mathbb{S}/I)_k$, for $k > 0$. As $Y \# \emptyset$ has its Hilbert function $(1, n, n)$, the image $\partial(I_2) \in (\mathbb{S}/I)_{2+k} = 0$. Similarly for I_3 , we get $\partial(I_3) = 0$. By (c_2) of Definition 6.31, the ideal $\text{Ann}(\Phi)$ is generated in degree 2, and $\deg(g) = 3$. $\square$

Lemma 6.35. *Let Φ be a very general cubic. Let $Y = \text{Spec } \text{Apolar}(\Phi)$. Then the space $(T^2(\mathcal{O}_Y, \mathbb{k}))_{<-3}$ vanishes.*

Proof. Let us take a minimal graded free resolution of $\text{Apolar}(\Phi)$

$$\dots \rightarrow F_2 \rightarrow F_1 \rightarrow \mathbb{S} \rightarrow 0.$$

By [Ser06, Lemma 3.1.11] and Definition A.6, the module $T^2(\mathcal{O}_Y, \mathbb{k})$ is a quotient of submodule of $\text{Hom}(F_2, \mathbb{k})$. By $(\mathfrak{c}_2)$ in Definition 6.31, the minimal second syzygies F_2 are in degree 3, hence $T^2(\mathcal{O}_Y, \mathbb{k})$ vanishes in degrees < -3 . $\square$

Definition 6.36. For a fixed $\mathbb{G}_m$ -equivariant embedding $Y \# \emptyset \subset \mathbb{A}^n$. The Białynicki-Birula fibre of $Y \# \emptyset$ is $\text{hilb}_{[Y \# \emptyset]}^{<0}$.

At last, we proof the main result of this section.

Theorem 6.37 ([Sza21, Theorem 1.5]). *Let us take $B = \text{Apolar}(\Phi)$ such that Φ is a very general cubic, such that $Y = \text{Spec } B$. Then the Białynicki-Birula fibre of $Y \# \emptyset$ has the smooth type of Y . Moreover, $\text{hilb}_{[Y]}$ is non-reduced.*

Proof. Since Φ is a very general cubic, by $(\mathfrak{c}_3)$ of Definition 6.31, the vector space $(T_Y^1)_{<0}$ vanishes. This implies that

$$Y \cong \text{Def}_{*/Y} \rightarrow \text{Def}_{*\rightarrow Y}^{<0}$$

and

$$\text{Def}_{Y \# \emptyset / Y}^{<0} \rightarrow \text{Def}_{Y \# \emptyset \rightarrow Y}^{<0}$$

are smooth by Corollary 4.72. The morphism $\text{Def}_{Y \# \emptyset \rightarrow Y}^{<0} \rightarrow \text{Def}_{Y \# \emptyset}^{<0}$ is smooth by Lemma 6.30, as by Lemma 6.35 the relative obstruction space $T^2(\mathcal{O}_Y, \mathbb{k})_{<-3}$ vanishes.

So far, we have concluded that there is a smooth map $Y \rightarrow \text{Def}_{Y \# \emptyset}^{<0}$. Thereupon, we get the following Cartesian diagram up to a smooth equivalence with a section.

$$\begin{array}{ccc} Y & \longrightarrow & \text{Def}_{Y \# \emptyset}^{<0} \\ \downarrow & & \downarrow \uparrow \\ * & \longrightarrow & \text{Def}_{Y \# \emptyset}^0 \end{array} \quad (6.38)$$

This implies the first part of the statement.

For the last part, by Lemma 2.3, we get that $\text{Def}_{Y \# \emptyset}^{<0}$ is non-reduced. The positive tangent space $(T_{Y \# \emptyset}^1)_{>0}$ vanishes since Y is apolar to a very general cubic, see Lemma 6.34, hence, by Lemma 4.68, we get that $\text{Def}_{Y \# \emptyset}$ is also non-reduced. As a consequence, the Hilbert scheme is also non-reduced at the point $[Y]$. $\square$

Remark 6.39. *If we assume that $\text{Def}_{Y \# \emptyset}^0$ is reduced, then we get flatness of $\text{Def}_{Y \# \emptyset} \rightarrow \text{Def}_{Y \# \emptyset}^0$. Since in Diagram (6.38), the fibre is Y not $Y \# \emptyset$, we do not get a fractal family according to Definition 6.15. The Białynicki-Birula fibre there naturally contains the deformed scheme $Y \# \emptyset$, but it is clearly **not** smoothly equivalent to it. The phenomenon showcased above has more to do with the connected sum of Gorenstein algebras (modulo the fact that it is not Gorenstein) than with this pointed case, for comparison, see Section 7.5.*

Chapter 7

The Gorenstein locus of the Hilbert scheme

7.1 Introduction

The Gorenstein locus $\text{Hilb}_\delta^{\text{Gor}}(\mathbb{A}^n)$ is the open locus of the Hilbert scheme $\text{Hilb}_{1,\delta}(\mathbb{A}^n)$ parameterising the Gorenstein finite subschemes, see [CN07, Proposition 2.5]. The global structure of $\text{Hilb}_\delta^{\text{Gor}}(\mathbb{A}^n)$ has been an object of intensive study. Irreducibility and smoothness for $n \leq 3$ is part of the folklore, see [KMR07]. It is known that the Gorenstein locus is irreducible for $\delta \leq 13$ and for $\delta = 14$ and $n \leq 5$, see [CJN15, Theorem A], which completes a series of papers on this topic [CN07, CN09, CN14, CN11, CN16]. In particular, the Gorenstein locus is not dense, since the Hilbert scheme is reducible for $n = 3$ and $\delta \geq 78$, as well as for $n \geq 4$ and $\delta \geq 8$. This shows that the Gorenstein locus is better behaved than the whole Hilbert scheme.

There has also been an interest in the question of the existence of nonsmoothable components. It has been proven, [IK99, Lemma 6.21], that $\text{Hilb}_\delta^{\text{Gor}}(\mathbb{A}^n)$ is reducible for $n \geq 6$ and $\delta \geq 14$. By a parameter count, it is known that the reducibility also holds for $n = 4$, $\delta \geq 140$ and $n = 5$, $\delta \geq 42$, see [BB14, Proposition 6.2].

In [IK99, Lemma 6.21] new smooth components of $\text{Hilb}_{2n+2}^{\text{Gor}}(\mathbb{A}^n)$ have been constructed for $6 \leq n \leq 13$ and not 7. A recent study [Sza23, Theorem 1] completed that work by constructing explicit new smooth components of $\text{Hilb}_{2n+2}^{\text{Gor}}(\mathbb{A}^n)$ for $n \geq 6$ except for $n = 7$. This particular construction fails for $n \leq 5$ and for $n = 7$, see [CN11] and [BCR22].

In this article, we focus on the local structure, especially on the singularity type. Our major outcome is that the Gorenstein locus is not reduced.

Theorem 7.1 (Corollary 7.43). *Let $n \geq 12$, but not 13, then the Gorenstein locus $\text{Hilb}_{2n+2}^{\text{Gor}}(\mathbb{A}^n)$ is not reduced.*

It is useful to look at the Gorenstein locus through the lenses of Macaulay duality, hence focusing on apolar algebras, see Definition 2.11. Our construction is explicit, and the exhibited examples of non-reduced points arise as apolar algebras of the sums of very general cubics of n_1 and n_2 variables with $n_1 + n_2 = n$, see Definition 6.31. The examples provide us embedded points rather than generically non-reduced component, since the locus of algebras with the Hilbert function $(1, 2n, 2n, 1)$ is generically smooth, see Proposition 6.33. An immediate corollary is that $\text{Hilb}_\delta^{\text{Gor}}(\mathbb{A}^n)$ is non-reduced for $n > 11$ for δ large enough. The case of $4 \leq n \leq 11$ remains open.

The constructed Gorenstein schemes are examples of connected sum. The connected sum of Gorenstein algebras was first introduced in [AAM12, Section 2. Connected sums]. Subsequently, in [IMS22, Theorem 1], the structure of the sum was described in the language of apolarity.

7.1.1 Application to the study of tensors

The local structure of the Gorenstein locus is related to the theory of tensors. The main goal of the theory of tensors is to understand the ranks and border ranks of tensors, and the analogous notions for symmetric tensors. The findings in this area have a large variety of applications outside of algebraic geometry, for example, to algebraic statistics [PS05] and complexity theory, see [Lan17], where it was famously applied by Strassen to study bounds on the exponent of the matrix multiplication [Str87]. The notion of border rank is closely related to the study of secant varieties. Let $U \subset \mathbb{P}^N$ be a smooth projective scheme and let δ be a fixed positive integer. We define the secant variety as

$$\sigma_\delta(U) := \overline{\bigcup_{p_1, \dots, p_\delta \in U} \{\langle p_1 \dots p_\delta \rangle\}} \subset \mathbb{P}^N,$$

where $\langle p_1, \dots, p_\delta \rangle$ is the projective span. In particular, if the scheme U is the Veronese embedding of $\mathbb{P}^n$, the $\mathbb{k}$ -points of Y represent powers of linear forms. Then the $\mathbb{k}$ -points of $\sigma_\delta(Y)$ represent sums of δ -powers of such linear forms and their limits. Since it is notoriously difficult to study equations of secant varieties, the cactus varieties $\mathfrak{K}_\delta(Y)$ were introduced as a remedy in a seminal work [BB14, Section 2]. They are defined as

$$\mathfrak{K}_\delta(U) := \overline{\bigcup \{\langle R \rangle \mid R \subset U, R \text{ is a subscheme of degree at most } \delta\}} \subset \mathbb{P}^N,$$

where $\langle R \rangle$ is the projective span, the minimal linear subspace $\mathbb{P}^k$ containing the scheme R . The cactus variety carries a natural scheme structure given by minors of catalecticant matrices, see [BB14, Theorem 1.5]. This structure was used in [CLPS25] and in [BK24] to show that singularities of the Gorenstein locus of the Hilbert scheme all yield singularities of the cactus scheme. In particular, our main outcome, Corollary 7.43, implies the following.

Corollary 7.2 (Corollary 7.48). *Let ν_r be the r -th Veronese embedding, let $n \geq 12$, but not $n = 13$, and $r \geq 4n + 4$. The cactus scheme $\mathfrak{K}_{2n+2}(\nu_r(\mathbb{P}^n))$ is non-reduced.*

7.1.2 Non-reducedness of the Hilbert scheme

The non-reducedness of the Hilbert scheme was exhibited for the Hilbert scheme of a large number of points on $\mathbb{A}^{16}$ in [Jel20, Corollary 5.1], for the Hilbert scheme of 13 points on $\mathbb{A}^6$ in [Sza21, Theorem 1.4], for at least 21 points on $\mathbb{A}^4$ in [Jel24, Theorem 1.1] and for the pointed Hilbert scheme $\text{Hilb}_{1,d}(\mathbb{A}^n)$, for $\delta \geq 8$ and $n \geq 4$ in [GGGL24, Theorem B].

The main method yielding non-reducedness results was based on **TNT** property, see Definition 4.57. However, in the Gorenstein case, the vanishing of the negative tangents rarely happens. For example it is open if there is a local Artinian Gorenstein algebra in four variables with only trivial negative tangents, see [SS23, Question 1.7]. In the known cases, cubics with the Hilbert function $(1, n, n, 1)$ and quartics with the Hilbert function $(1, n, \binom{n+1}{2}, n, 1)$, this condition implies smoothness of the positive spike [IK99, Lemma 6.21], [ER15, Theorem 3.1], [ER12, Theorem 3.1], [Jel17, Proposition 1.1]. Hence, the methods cannot be applied to prove non-reducedness.

To overcome this problem, we use methods from Section 4.2 in Preliminaries.

Apart from proving non-reducedness, we describe the structure of the completed local ring at the described non-reduced points of the Gorenstein locus. To do that we refer to the methods and results obtained in Chapter 6.

In Section 4.1, we glue the schemes along the distinguished points

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \cup_* Y, \end{array}$$

and obtain an intermediate result, of independent interest, on the negative deformations of the union along the point.

Theorem 7.3 (Theorem 7.22). *Assume that X, Y are pointed finite $\mathbb{G}_m$ -schemes which are spectra of t -level algebras, for reference on level algebras see Definition 7.20, and assume that neither is a reduced point. Then the morphism*

$$\mathrm{Def}_{*\rightarrow X}^{<0} \times \mathrm{Def}_{*\rightarrow Y}^{<0} \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0}$$

is smooth and induces an isomorphism on the tangent spaces.

Subsequently, we assume that both X and Y are Gorenstein. We take the union of X and Y along the point, then identify the socle of X with the socle of Y . This way, we obtain the connected sum $X \# Y$, see Definition 7.28. Such a scheme is Gorenstein and leads up to the main outcome, Theorem 7.1.

7.2 Setting

For this chapter, we fix a positive integer δ and work over an algebraically closed field $\mathbb{k}$ of characteristic either 0 or co-prime with δ . The only reason for that is that we are going to apply Corollary 6.22 in non-reducedness results, hence the setting has to be compatible with Setting 6.2.

7.3 Sum along the point

7.3.1 Generalities on the sum along the point

In this section we take a pair of pointed $\mathbb{G}_m$ -schemes and we glue them along the distinguished points, see Definition 7.4. The section is devoted to the study of deformations of the obtained scheme, this is an intermediate step from deformations of pointed schemes, which where the topic of Chapter 6, to deformations of the connected sum, which are the topic of Section 7.5.

Definition 7.4. *Let X, Y be a pair of pointed $\mathbb{G}_m$ -schemes. The sum along the point is the pushforward $X \cup_* Y$*

$$\begin{array}{ccc} * & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \cup_* Y \end{array}$$

To be precise, the projections from the product $X \times Y$

$$\begin{array}{ccc} & X \times Y & \\ \swarrow & & \searrow \\ X & & Y, \end{array}$$

admit sections given by the pointed structures, $(X, *) \subset X \times Y$ and $(*, Y) \subset X \times Y$. Then the scheme theoretic union of X and Y as subschemes of $X \times Y$ is $X \cup_* Y$.

By definition, the scheme $X \cup_* Y$ is embedded into the product $X \times Y$, and hence admits a $\mathbb{G}_m^2$ -action. We assume that $\mathcal{O}_X$ and $\mathcal{O}_Y$ are generated in degree 1, both can be realised as subalgebras of $\mathcal{O}_{X \cup_* Y} = \mathcal{O}_X \times_{\mathbb{k}} \mathcal{O}_Y$ generated in bi-degree $(1, 0)$ and $(0, 1)$, respectively.

We introduce several notations related to the bi-graded structure.

Definition 7.5. Let us fix a bi-graded sheaf $\mathcal{F}$ on $X \cup_* Y$. The elements of bi-degree $(0, k)$ in $\mathcal{F}$, for any positive integer k , we call the elements of pure y -degree, and analogously with x . Any bi-homogeneous element not of pure degree is of mixed degree.

Definition 7.6. The $\mathbb{k}$ -vector space spanned by the pure y -degree elements of $\mathcal{F}$ is called the pure y -degree component, and analogously with x . The y -degree component admits a module structure over $\mathcal{O}_X$, but not over $\mathcal{O}_{X \cup_* Y}$. The $\mathbb{k}$ -vector space spanned by the mixed degree elements is called the mixed degree component.

We want to stress that according to this notation, the elements of bi-degree $(0, 0)$ are **not** of pure degree.

Definition 7.7. A bi-graded element of $\mathcal{F}$ of degree (k, l) is called negative if $k + l < 0$.

We take negative elements to be as in Definition 7.7, not with the condition that both weights are negative. This is because we choose the diagonal torus action on $X \cup_* Y$, as the one inducing grading of the structural sheaf in view of considering filtered deformations, see Definition 4.49.

We finish this subsection with a useful lemma.

Lemma 7.8. Fix $A \in \text{Art}_{\mathbb{k}}$. Let us take a pair of flat pointed affine A -schemes, corresponding to the algebras $(B_1, \mathfrak{n}_2)$ and $(B_2, \mathfrak{n}_1)$, where $\mathfrak{n}_i = \ker(B_i \rightarrow A)$. Let us consider the fibre product of algebras $B := B_1 \times_A B_2$. Then we get the following exact sequences of B -modules

$$0 \rightarrow \mathfrak{n}_x \rightarrow B \rightarrow B_y \rightarrow 0,$$

$$0 \rightarrow \mathfrak{n}_y \rightarrow B \rightarrow B_x \rightarrow 0.$$

In particular the ideal $\mathfrak{n} = \ker(B \rightarrow A)$ is equal to $\mathfrak{n}_1 + \mathfrak{n}_2$ and the product of B -ideals $\mathfrak{n}_1 \mathfrak{n}_2$ is 0.

Proof. Let us take the pointed structure of B_2

$$0 \rightarrow \mathfrak{n}_2 \rightarrow B_2 \rightarrow A \rightarrow 0,$$

the structural homomorphism gives us a section of $B_2 \rightarrow A$, this way we get a section of the projection $B \rightarrow B_1$, in particular B has a B_1 -module structure.

In the sequence

$$0 \rightarrow J \rightarrow B \rightarrow B_1 \rightarrow 0,$$

the kernel J is actually a B_2 -module (and B by the forgetful functor), as it is annihilated by the maximal ideal $\mathfrak{n}_1 \subset B_1$. This way, we get

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J & \longrightarrow & B & \longrightarrow & B_1 & \longrightarrow & 0 \\ \downarrow & & \parallel & & \downarrow & & \downarrow & & \downarrow \\ & & Id & & \otimes B_1/\mathfrak{n}_1 & & \otimes B_1/\mathfrak{n}_1 & & \\ 0 & \longrightarrow & \mathfrak{n}_2 & \longrightarrow & B_2 & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

□

7.4 Deformations of sum along the point

We are going to compare deformations of sum along the point and pointed deformations of the schemes X and Y . We focus on the negative deformations (assuming $\mathbb{G}_m$ -action on both schemes), and show that under certain conditions every (negative) infinitesimal deformation of sum along the point preserves to be a sum along the point, which is a property of independent interest.

Proposition 7.9. *There is a morphism of good deformation functors*

$$\mathrm{Def}_{*\rightarrow X}^{<0} \times \mathrm{Def}_{*\rightarrow Y}^{<0} \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0} \quad (\diamond)$$

for given $A \in \mathrm{Art}_{\mathbb{k}}$ it is given by

$$\begin{array}{ccc} \begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Spec}(A) \end{array} & \begin{array}{ccc} Y & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Spec}(A) \end{array} & \mapsto \begin{array}{ccc} X \cup_* Y & \longrightarrow & \mathcal{X} \cup_A \mathcal{Y} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Spec}(A) \end{array} \end{array}$$

Proof. The only thing to check is that $\mathcal{X} \cup_A \mathcal{Y}$ is flat over A . Since the question is local, we assume that X and Y are affine. Let us consider the exact sequence of A -modules given by the pointed structure of $\mathcal{X}$

$$0 \rightarrow \mathfrak{m}_{\mathcal{X}} \rightarrow \mathcal{O}_{\mathcal{X}} \rightarrow A \rightarrow 0.$$

The structural morphism of $\mathcal{X}$ gives us a section $A \rightarrow \mathcal{O}_{\mathcal{X}}$, which implies:

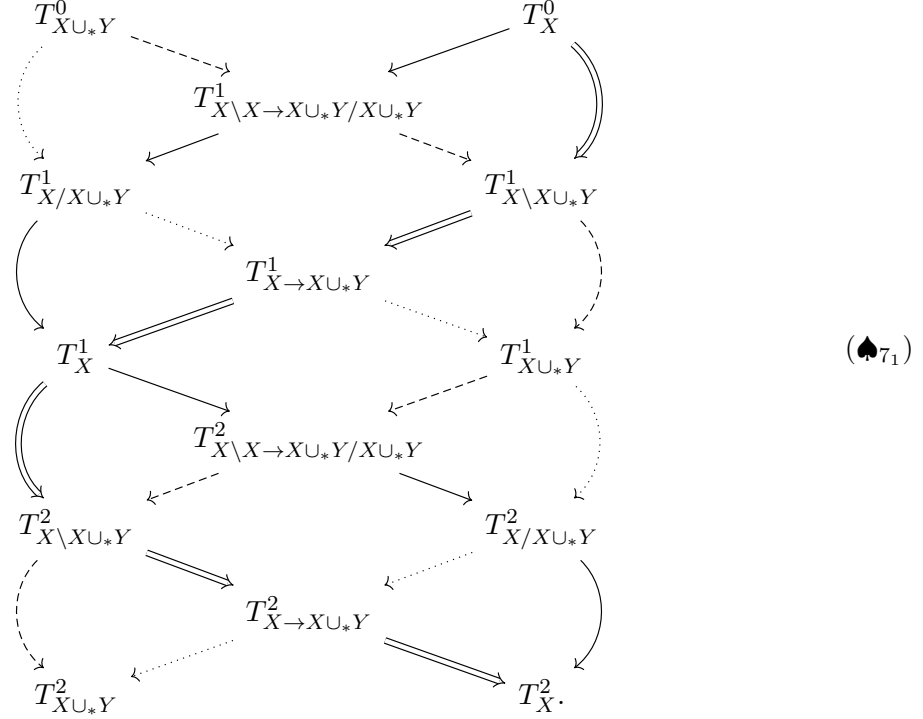
$$A \oplus \mathfrak{m}_{\mathcal{X}} \cong \mathcal{O}_{\mathcal{X}}$$

as A -modules. Since $\mathcal{O}_{\mathcal{X}}$ is flat over A , the direct summand $\mathfrak{m}_{\mathcal{X}}$ is also flat. Since $\mathcal{O}_{\mathcal{X} \cup \mathcal{Y}} \cong \mathcal{O}_{\mathcal{X}} \times_A \mathcal{O}_{\mathcal{Y}}$, the kernel of $\mathcal{O}_{\mathcal{X} \cup \mathcal{Y}} \rightarrow \mathcal{O}_{\mathcal{Y}}$ is isomorphic to $\mathfrak{m}_{\mathcal{X}}$, see Lemma 7.8. We see that $\mathcal{O}_{\mathcal{X} \cup \mathcal{Y}}$ is an extension of flat modules, therefore it is flat. □

The aim of this section is to prove that the morphism $\diamond$ is smooth. In various parts of the argument we require that neither X nor Y is a reduced point, see Remark 7.14, Remark 7.17 and Remark 7.19.

The key technical tool in this sections is Braid ($\spadesuit$), here applied for the natural maps

$X \rightarrow X \cup_* Y$ and for $Y \rightarrow X \cup_* Y$.



The line of the arguments in this section is the following.

First, we define a map $\text{Def}_{* \rightarrow Y} \rightarrow \text{Def}_{X \setminus X \cup_* Y}$ and show that the former is smoothly equivalent to pure y -degree elements of the latter. This is the content of Lemma 7.10. Since the tangent-obstruction theory of the map $\text{Def}_{X \setminus X \cup_* Y} \rightarrow \text{Def}_{X \rightarrow X \cup_* Y}$ is given by T^1_X, T^2_X , see Braid ($\spadesuit_{71}$), we immediately get a similar result for the composition

$$\text{Def}_{* \rightarrow Y} \rightarrow \text{Def}_{X \setminus X \cup_* Y} \rightarrow \text{Def}_{X \rightarrow X \cup_* Y}.$$

Now, for the final composition $\text{Def}_{* \rightarrow Y} \rightarrow \text{Def}_{X \cup_* Y}$, the relevant tangent-obstruction theory is $T^i_{X/X \cup_* Y}$, vanishing of pure y -degrees of those functors is the content of Lemma 7.15. This way, we get that $\text{Def}_{* \rightarrow Y}$ is smoothly equivalent to the pure y -degree part of the deformations of the sum along the distinguished point, Lemma 7.18. We get a symmetrical result for $\text{Def}_{* \rightarrow X}$. Finally, to get the desired description of all the negative deformations this way, we need to make sure that there are no negative tangents of mixed degree; this is the content of Lemma 7.21. To prove that statement, we take X and Y to be t -level algebras, see Definition 7.20. At last, Theorem 7.22 gives us the result announced at the beginning of the section.

Lemma 7.10. *There is a natural morphism*

$$\text{Def}_{* \rightarrow Y} \rightarrow \text{Def}_{X \setminus X \cup_* Y}$$

The morphism induces an injection on tangent spaces and obstruction spaces. The image of the tangent space is the whole pure y -degree component.

Proof. The morphism

$$\text{Def}_{* \rightarrow Y} \rightarrow \text{Def}_{X \setminus X \cup_* Y} \tag{7.11}$$

is given by taking a local Artinian algebra A and a deformation $\mathcal{Y}$ of Y over A , then

$$(X \times \operatorname{Spec} A) \cup_{\operatorname{Spec} A} \mathcal{Y}$$

is a deformation over A fixing the scheme X .

We ask about properties of the maps of the tangent and obstruction spaces induced by (7.11)

$$T_{\mathbb{k}}^i(\mathcal{O}_Y, \mathfrak{m}_Y) \rightarrow T_{\mathbb{k}}^i(\mathcal{O}_{X \cup_* Y}, \mathfrak{m}_Y), \quad (7.12)$$

for $i = 1, 2$. First, let us take pointed structure of Y

$$0 \rightarrow \mathfrak{m}_Y \rightarrow \mathcal{O}_Y \rightarrow \mathbb{k} \rightarrow 0.$$

By Lemma 7.8, the kernel J in the sequence

$$0 \rightarrow J \rightarrow \mathcal{O}_{X \cup_* Y} \rightarrow \mathcal{O}_X \rightarrow 0$$

is isomorphic to $\mathfrak{m}_X$. The map (7.12) is a part of a long exact sequence of T^i -functors, [Har10, Theorem 3.5]

$$T_{X \cup_* Y}^i(\mathcal{O}_Y, \mathfrak{m}_Y) \rightarrow T_{\mathbb{k}}^i(\mathcal{O}_Y, \mathfrak{m}_Y) \rightarrow T_{\mathbb{k}}^i(\mathcal{O}_{X \cup_* Y}, \mathfrak{m}_Y). \quad (7.13)$$

We prove that the pure y -degree part and the degree 0 part of the group $T_{X \cup_* Y}^i(\mathcal{O}_Y, \mathfrak{m}_Y)$ vanish (for $i = 1, 2$). As the tangent and obstruction spaces of $\operatorname{Def}_{* \rightarrow Y}$ the pair $T_{\mathbb{k}}^1(\mathcal{O}_Y, \mathfrak{m}_Y)$, $T_{\mathbb{k}}^2(\mathcal{O}_Y, \mathfrak{m}_Y)$ is spanned by the pure y -degree and the degree 0 part, the vanishing yields the statement. Consider the following exact sequence:

$$0 \rightarrow \mathfrak{m}_X \rightarrow \mathcal{O}_{X \cup_* Y} \rightarrow \mathcal{O}_Y \rightarrow 0.$$

By [Har10, Proposition 3.8],

$$T_{X \cup_* Y}^1(\mathcal{O}_Y, \mathfrak{m}_Y) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathfrak{m}_X / \mathfrak{m}_X^2, \mathfrak{m}_Y),$$

this satisfies the conditions, as the group is concentrated in degree of $(-1, l)$, for $l \geq 0$.

We also prove that pure y -degree component and the degree 0 component of $T_{X \cup_* Y}^2(\mathcal{O}_Y, \mathfrak{m}_Y)$ vanishes. The module $T_{X \cup_* Y}^2(\mathcal{O}_Y, \mathfrak{m}_Y)$ is a sub-quotient of $\operatorname{Hom}_{X \cup_* Y}(F_2, \mathfrak{m}_Y)$, where F_2 are the second minimal $\mathcal{O}_{X \cup_* Y}$ -syzygies of $\mathcal{O}_Y$.

$$F_2 \rightarrow F_1 \rightarrow \mathfrak{m}_X \rightarrow 0.$$

Each minimal syzygy of $\mathfrak{m}_X$ as $\mathcal{O}_{X \cup_* Y}$ -module is either of the form

$$y \cdot x = 0,$$

where y is a generator of $\mathfrak{m}_Y$, and x is a generator of $\mathfrak{m}_X$, such a syzygy is of degree $(1, 1)$. Or it comes from relations of $\mathfrak{m}_X$ in $\mathcal{O}_X$, then it is of pure x -degree, $(l, 0)$, $l > 0$. A homomorphism $\operatorname{Hom}_{X \cup_* Y}(F_2, \mathfrak{m}_Y)$ of degree $(0, k)$ would send the first kind of generators to the degree $(1, 1+k)$, and the second to the degree (l, k) . But all the elements of $\mathfrak{m}_Y$ have only pure y -degrees, which leads to a contradiction. $\square$

Remark 7.14. *If in Lemma 7.10 we assume that $X = \{x\}$, then we get that*

$$\mathrm{Def}_{*\rightarrow Y} \rightarrow \mathrm{Def}_{*\setminus Y}$$

is smooth and an isomorphism on the tangent spaces. It provides us with a section of the natural map

$$\mathrm{Def}_{*\setminus Y} \rightarrow \mathrm{Def}_{*\rightarrow Y},$$

proving an isomorphism of the functors.

The next lemma, says that the functor $\mathrm{Def}_{X/X \cup_* Y}$ admits a trivial $\mathbb{G}_m$ -action with respect to the one-parameter group $\mathbb{G}_m$ coming from action on Y . We want to stress that this time we exclude the case $X = \{x\}$, see Remark 7.17.

Lemma 7.15. *Let $X \neq \{x\}$. $\mathrm{Def}_{X/X \cup_* Y}$ has no tangents nor obstructions lying in the pure y -degree.*

Proof. The tangent-obstruction theory of $\mathrm{Def}_{X/X \cup_* Y}$ is $T_{X \cup_* Y}^i(\mathcal{O}_X, \mathcal{O}_X)$, for $i = 1, 2$, see Proposition 4.41. Let us recall, from Lemma 7.8, that we have the following exact sequence of $\mathcal{O}_{X \cup_* Y}$ -modules

$$0 \rightarrow \mathfrak{m}_Y \rightarrow \mathcal{O}_{X \cup_* Y} \rightarrow \mathcal{O}_X \rightarrow 0,$$

$$0 \rightarrow \mathfrak{m}_X \rightarrow \mathcal{O}_{X \cup_* Y} \rightarrow \mathcal{O}_Y \rightarrow 0.$$

By the usual argument on the Hilbert schemes, it is enough to show the vanishing of pure y -degree part for:

$$\mathrm{Hom}_{\mathcal{O}_X \times_{\mathbb{k}} \mathcal{O}_Y}(\mathfrak{m}_Y, \mathcal{O}_X).$$

This is an $\mathcal{O}_Y$ -module, since it is annihilated by $\mathfrak{m}_X$. The module $\mathfrak{m}_Y$ is generated in degree $(0, 1)$. Let us assume that there is a homomorphism $\phi \in \mathrm{Hom}_{\mathcal{O}_X \times_{\mathbb{k}} \mathcal{O}_Y}(\mathfrak{m}_Y, \mathcal{O}_X)$ of pure y -degree. The generators should be mapped to degree zero in $\mathcal{O}_X$, which is $\mathbb{k} \subset \mathcal{O}_X$. If ϕ is non-zero, then there are invertible elements in the image, but this contradicts the $\mathcal{O}_Y$ -structure of $\mathfrak{m}_Y$.

We also need to consider the obstruction module

$$T_{X \cup_* Y}^2(\mathcal{O}_X, \mathcal{O}_X) \subset \mathrm{Ext}_{\mathcal{O}_{X \cup_* Y}}^1(\mathfrak{m}_Y, \mathcal{O}_X).$$

We consider the resolution of $\mathfrak{m}_Y$ as $\mathcal{O}_{X \cup_* Y}$ -module

$$F_3 \xrightarrow{d_2} F_2 \xrightarrow{d_1} F_1 \rightarrow \mathfrak{m}_Y \rightarrow 0.$$

As in Lemma 7.10 (but for $\mathfrak{m}_Y$ instead of $\mathfrak{m}_X$), the minimal second syzygies are in degrees $(1, 1)$ or $(0, l)$, for $k, l > 0$.

Assume that we have a homomorphism $\psi \in \mathrm{Hom}(F_2, \mathcal{O}_X)$ of pure y -degree that provides us a class in $\mathrm{Ext}_{\mathcal{O}_X \times_{\mathbb{k}} \mathcal{O}_Y}^1(\mathfrak{m}_Y, \mathcal{O}_X)$.

The second kind of syzygies goes with ψ to the degree $(0, 0)$, since $\mathcal{O}_X$ is generated as an algebra by pure x -degrees. If the image of those syzygies is non-zero then ψ sends one of these syzygies to the unit. If $\psi: F_2 \rightarrow \mathcal{O}_X$ lifts to an element in $\mathrm{Ext}_{\mathcal{O}_X \times_{\mathbb{k}} \mathcal{O}_Y}^1(\mathfrak{m}_Y, \mathcal{O}_X)$ group, then the

following diagram commutes

$$\begin{array}{ccc} F_3 & & \\ \downarrow d_2 & \searrow 0 & \\ F_2 & \xrightarrow{\psi} & \mathcal{O}_X. \end{array} \quad (7.16)$$

Let us say that there is a syzygy $s \in F_2$ of degree $(0, l)$ such that $\psi(s) = 1$. Let $x \in \mathcal{O}_X \subset \mathcal{O}_{X \cup_* Y}$ be a non-zero element of pure x -degree, then $\psi(xs) \neq 0$.

As $\deg(s) = (0, l)$, we get $d_1(s) \in \mathfrak{m}_Y F_1$ and consequently $d_1(xs) = x d_1(s) = 0$ (note that we are in the category of $\mathcal{O}_{X \cup_* Y}$ -modules). Hence, there is an element $z \in F_3$, such that $d_2(z) = xs$. The 3-syzygy $z \in F_3$ contradicts the commutativity of Diagram (7.16). Consequently, the second kind of syzygies are annihilated by ψ .

Let us consider the submodule of F_2 generated by the syzygies of the first kind, those of degree $(1, 1)$. The quotient of this submodule by (the restriction of) the image of d_2 is isomorphic to $\bigoplus \mathfrak{m}_X[0, -1]$, where the sum is indexed by the homogeneous generators of $\mathfrak{m}_Y$. The set of elements in

$$\mathrm{Hom}_{\mathcal{O}_X \times_{\mathbb{k}} \mathcal{O}_Y}(\mathfrak{m}_X[0, -1], \mathcal{O}_X)$$

of pure y -degree is the set of degree zero elements of

$$\mathrm{Hom}_{\mathcal{O}_X \times_{\mathbb{k}} \mathcal{O}_Y}(\mathfrak{m}_X, \mathcal{O}_X).$$

There is only one such homogeneous element, coming from the embedding. But such a homomorphism factors through the map d_1 . Thereupon, its class in the extensions group is trivial. $\square$

Remark 7.17. *We comment on the necessity of the assumption that $X \neq \{x\}$. Assume that X is a reduced point, then we get a functor $\mathrm{Def}_{*/Y}$, this is the Hilbert scheme of one point on Y , then the tangent space lies in the pure y -degree, see Proposition 6.6. This contradicts the statement as long as Y is not a reduced point as well.*

We compose Lemma 7.10 and Lemma 7.15 to get an intermediate result which already gives some properties of the morphism

$$\mathrm{Def}_{* \rightarrow X}^{<0} \times \mathrm{Def}_{* \rightarrow Y}^{<0} \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0}. \quad (\diamond)$$

Lemma 7.18. *Assume that $X \neq \{x\}$. Then the map $\mathrm{Def}_{* \rightarrow Y}^{<0} \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0}$ is injective on tangents and obstructions, and surjects onto y -pure degree tangents.*

Proof. Lemma 7.10 implies the statement for the map

$$\mathrm{Def}_{* \rightarrow Y}^{<0} \rightarrow \mathrm{Def}_{X \setminus X \cup_* Y}^{<0}.$$

A tangent-obstruction theory of the map

$$\mathrm{Def}_{X \setminus X \cup_* Y} \rightarrow \mathrm{Def}_{X \rightarrow X \cup_* Y}$$

is given by (T_X^0, T_X^1) , see Braid $(\spadesuit_{71})$. Therefore, the tangent-obstruction theory is concentrated in pure x -degrees, we get the statement for the composition

$$\mathrm{Def}_{* \rightarrow Y}^{<0} \rightarrow \mathrm{Def}_{X \rightarrow X \cup_* Y}^{<0}.$$

By Braid ($\spadesuit_{7_1}$) the tangent-obstruction theory of the map

$$\mathrm{Def}_{X \rightarrow X \cup_* Y} \rightarrow \mathrm{Def}_{X \cup_* Y}$$

is given by the tangent-obstruction theory of $\mathrm{Def}_{X/X \cup_* Y}$, by Lemma 7.15 it is concentrated in pure x -degrees. Thus the composition

$$\mathrm{Def}_{* \rightarrow Y}^{<0} \rightarrow \mathrm{Def}_{X \rightarrow X \cup_* Y}^{<0} \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0},$$

again satisfies the conditions, the fact that this composition is the map from the statement follows from the commutativity of Braid ($\spadesuit_{7_1}$). $\square$

Remark 7.19. *We want to stress the necessity of X not being a reduced point. If in Lemma 7.18 we assume that $X = \{x\}$, then the statement says that $\mathrm{Def}_{* \rightarrow Y}^{<0} \rightarrow \mathrm{Def}_Y^{<0}$ is smooth. In general, this is not true. Corollary 6.10 states that under certain open conditions $\mathrm{Def}_{*/Y} \rightarrow \mathrm{Def}_{* \rightarrow Y}^{<0}$ is smooth. Since the composition of these maps factors through a point, as presented in the diagram below*

$$\begin{array}{ccc} \mathrm{Def}_{*/Y} & \longrightarrow & \mathrm{Def}_{* \rightarrow Y}^{<0} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Def}_Y^{<0}. \end{array}$$

We get that $\mathrm{Def}_{/Y}$ is smooth. But, by Proposition 6.6, this functor is prorepresented by the complete local ring of the distinguished point of Y , that implies that Y is smooth at the distinguished point.*

To get the main result, Theorem 7.22, we need both schemes have certain bounding properties on the socle, recall its Definition 2.8.

Definition 7.20. *A graded Artinian local algebra $(B, \mathfrak{m})$ is a t -level algebra if the socle B is equal to the t -th graded part B_t . A finite local $\mathbb{G}_m$ -scheme Y is t -level if $\mathcal{O}_Y$ is a t -level algebra.*

Recall definition of mixed elements of bi-graded module, Definition 7.6.

Lemma 7.21. *Let us assume that both X and Y are finite t -level $\mathbb{G}_m$ -schemes. There are no negative mixed tangents in $T_{X \cup_* Y}^1$.*

Proof. Let us choose a resolution

$$0 \rightarrow J \rightarrow \mathbb{S} \rightarrow \mathcal{O}_{X \cup_* Y} \rightarrow 0,$$

where $\mathbb{S} = \mathbb{k}[x_1, \dots, x_n, y_1, \dots, y_m]$ is a polynomial ring formally generated by minimal homogeneous generators of $\mathcal{O}_X$ and $\mathcal{O}_Y$. By [Har10, Proposition 3.10], there is the following surjection:

$$\mathrm{Hom}(J/J^2, \mathcal{O}_{X \cup_* Y}) \rightarrow T_{X \cup_* Y}^1 \rightarrow 0.$$

We compute the Hilbert function of the former module. Let us take a bi-homogeneous element $\phi \in \mathrm{Hom}(J/J^2, \mathcal{O}_{X \cup_* Y})$. The ideal J is generated by the forms $x_i y_j$ and by some pure x -degree and some pure y -degree polynomials. Since there is no elements of mixed degree in the ring $\mathcal{O}_{X \cup_* Y}$, the degree of ϕ is either $(-1, -1)$ or one of the weights is positive. First, assume that the bi-degree of ϕ is $(-1, -1)$. In such a case, only generators of the form $x_i y_j$ can survive. But

then $\phi(x_i y_j^N) = 0$ for a minimal positive integer N such that $y_j^N \in J$, hence $\phi(x_i y_j) y_j^{N-1} = 0$, but since $\phi(x_i y_j)$ is of degree $(0, 0)$ it either vanishes or it is invertible, since N is minimal, we get the vanishing.

Now, let us assume that ϕ is of degree $(k, -l)$ for integers satisfying $k, l > 1$ and $k - l < 0$. Let us take a minimal generator $g \in J$ of pure y -degree, if the y -degree of g is smaller than l then $\phi(g)$ vanishes; thus its y -degree is greater or equal to l . Since $x_i g \in (x_i y_j)_{1 \leq i \leq n, 1 \leq j \leq m}$, and $\phi(x_i y_j) = 0$, the form $x_i \phi(g) = \phi(x_i g)$ vanishes, hence $\phi(g)$ is of degree $(k, 0)$ and is an element of socle of $\mathcal{O}_X$. Since $\mathcal{O}_X$ is a t -level algebra we get $k = t$. Since $\mathcal{O}_Y$ is a t -level algebra, the y -degree of g is at most t , in particular $l < t$ and as a consequence, $k - l$ cannot be negative. $\square$

Having all necessary prerequisites, we prove the result on the negative deformations of the sum along the point, which is announced at the beginning of the section. We also show that the sum along point inherits the vanishing of tangent space in degrees ≤ -1 , which is useful in the arguments of the next section, see Theorem 7.38.

Theorem 7.22. *Assume that X, Y are pointed finite $\mathbb{G}_m$ -schemes which are spectra of t -level algebras, neither is a reduced point. Then the morphism:*

$$\mathrm{Def}_{*\rightarrow X}^{<0} \times \mathrm{Def}_{*\rightarrow Y}^{<0} \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0} \quad (\diamond)$$

is smooth and induces isomorphism on the tangent spaces. Moreover, if the groups $(T_X^1)_{<-1}$, $(T_Y^1)_{<-1}$ vanish, then

$$(T_{X \cup_* Y}^1)_{<-1} = 0.$$

Proof. By Lemma 7.18, the morphisms

$$\mathrm{Def}_{*\rightarrow Y}^{<0} \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0}$$

and

$$\mathrm{Def}_{*\rightarrow X}^{<0} \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0}$$

are injective on tangents and obstructions. Since the tangent-obstruction theory of $\mathrm{Def}_{*\rightarrow Y}^{<0}$ (respectively $\mathrm{Def}_{*\rightarrow X}^{<0}$) is equal to its pure y -degree parts (resp. x -degree), we get that the morphism $(\diamond)$ is injective on tangents and obstructions. Moreover, it surjects onto pure x -degree and pure y -degree tangents. The only thing left is to show that there are no mixed negative tangents. This is exactly Lemma 7.21.

For the second part of the statement, recall that we assume that $\mathcal{O}_X$ and $\mathcal{O}_Y$ are generated in degree 1, hence by Corollary 6.7, the modules $T_{*\rightarrow Y}^1$ or $T_{*\rightarrow X}^1$ vanish in degrees < -1 . Since by the first part of the statement, the negative tangents $(T_{X \cup_* Y}^1)_{<-1}$ are coming either from $T_{*\rightarrow Y}^1$ or $T_{*\rightarrow X}^1$, we conclude the statement. $\square$

Corollary 7.23. *Let X, Y as in the assumptions of Theorem 7.22, assume additionally that the groups $(T_X^1)_{<-1}$, $(T_Y^1)_{<-1}$ vanish, then we get the following diagram*

$$\begin{array}{ccc} X \times Y & \longrightarrow & \mathrm{Def}_{X \cup_* Y}^{<0} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Def}_X^{<0} \times \mathrm{Def}_Y^{<0} \end{array} \quad (7.24)$$

to be Cartesian up to smooth equivalence.

Proof. By Corollary 6.7 we get a diagram, Cartesian up to smooth equivalence

$$\begin{array}{ccc} \mathrm{Def}_{*/X}^{<0} \times \mathrm{Def}_{*/Y}^{<0} & \longrightarrow & \mathrm{Def}_{*\rightarrow X}^{<0} \times \mathrm{Def}_{*\rightarrow Y}^{<0} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Def}_X^{<0} \times \mathrm{Def}_Y^{<0}. \end{array}$$

By Theorem 7.22, the functor $\mathrm{Def}_{X \cup_* Y}^{<0}$ is smoothly equivalent to $\mathrm{Def}_{*\rightarrow X}^{<0} \times \mathrm{Def}_{*\rightarrow Y}^{<0}$. $\square$

Let us state another interesting corollary on non-reducedness. The point is that smoothness of the morphism $(\blacklozenge)$ implies that the functor of deformations of the union comes with the fractal structures of $\mathrm{Def}_{*\rightarrow X}$ and $\mathrm{Def}_{*\rightarrow Y}$.

Corollary 7.25. *Let X, Y be a pair of uncleavable scheme satisfying the assumptions of Theorem 7.22. Assume that lengths of X and Y are co-prime with characteristic of $\mathbb{k}$, or $\mathrm{char} \mathbb{k} = 0$. Then $\mathrm{Def}_{X \cup_* Y}^{<0}$ is non-reduced.*

Proof. The functors $\mathrm{Def}_{*\rightarrow X}^{<0}$ $\mathrm{Def}_{*\rightarrow Y}^{<0}$ are non-reduced by Corollary 6.22. By Theorem 7.22, the functor $\mathrm{Def}_{X \cup_* Y}^{<0}$ is smoothly equivalent to the product $\mathrm{Def}_{*\rightarrow X}^{<0} \times \mathrm{Def}_{*\rightarrow Y}^{<0}$, hence the claim. $\square$

7.5 Connected sum of Gorenstein algebras

In this section, at last, we focus on the Gorenstein case. We move from the union along the point $X \cup_* Y$ to the connected sum $X \# Y$, first introduced in [AAM12, Section 2. Connected sums]. We use its description in terms of apolar algebras developed in [IMS22], see Definition 7.28. Our main goal is to transform the fact (Theorem 7.22) that the map

$$\mathrm{Def}_{*\rightarrow X}^{<0} \times \mathrm{Def}_{*\rightarrow Y}^{<0} \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0}$$

is smooth and isomorphic on the tangent spaces, to a description of negative deformations of the connected sum

$$\mathrm{Def}_{X \# Y}^{<0}.$$

This functor does not admit a smooth map from the functor $\mathrm{Def}_{X \cup_* Y}^{<0}$, but it is equivalent to a hypersurface cut out by a one equation coming from the socle; this we prove in Theorem 7.38.

Recall the Macaulay duality, Theorem 2.12. It is worth to look at Gorenstein schemes through lenses of apolarity, see Definition 2.10. Let $\mathbf{P}_x$ (respectively $\mathbf{P}_y$) be a divided power series ring in variables $x_1, \dots, x_{n_1}$ (resp. $y_1, \dots, y_{n_2}$). Let $\mathbb{S}_x$ (respectively $\mathbb{S}_y$) be a polynomial ring $\mathbb{k}[\partial_{x_1}, \dots, \partial_{x_{n_1}}]$ (resp. $\mathbb{k}[\partial_{y_1}, \dots, \partial_{y_{n_2}}]$).

Setting 7.26. *Let*

$$X = \mathrm{Spec} \mathrm{Apolar}(\Phi),$$

$$Y = \mathrm{Spec} \mathrm{Apolar}(\Psi),$$

where $\Phi \in \mathbf{P}_x$ and $\Psi \in \mathbf{P}_y$, the polynomials Φ, Ψ are homogeneous of degree $t \geq 3$. such that

$$(T_X^1)_{<-1} = (T_Y^1)_{<-1} = 0$$

is satisfied. We add two conditions.

- The following second cotangent space vanish

$$T^2(\mathcal{O}_X, \mathbb{k})_{<-t} = T^2(\mathcal{O}_Y, \mathbb{k})_{<-t} = 0.$$

- The ideals $\text{Ann}(\Phi) \subset \mathbb{S}_x$ and $\text{Ann}(\Psi) \subset \mathbb{S}_y$ are generated in the degrees $\leq t - 1$.

Remark 7.27. The conditions of being t -level from Theorem 7.22 is satisfied by the condition of being an apolar algebra of a polynomial of degree t .

Since $\mathcal{O}_{X \cup_* Y}$ admits a socle of length 2, spanned by the socles of $\mathcal{O}_X$ and $\mathcal{O}_Y$, the union along the point is not Gorenstein. To get a Gorenstein algebra, we identify the socles.

Definition 7.28 ([IMS22, Theorem 4.6]). The connected sum of Gorenstein finite schemes $X = \text{Spec Apolar}(\Phi)$ and $Y = \text{Spec Apolar}(\Psi)$ is given by the formula

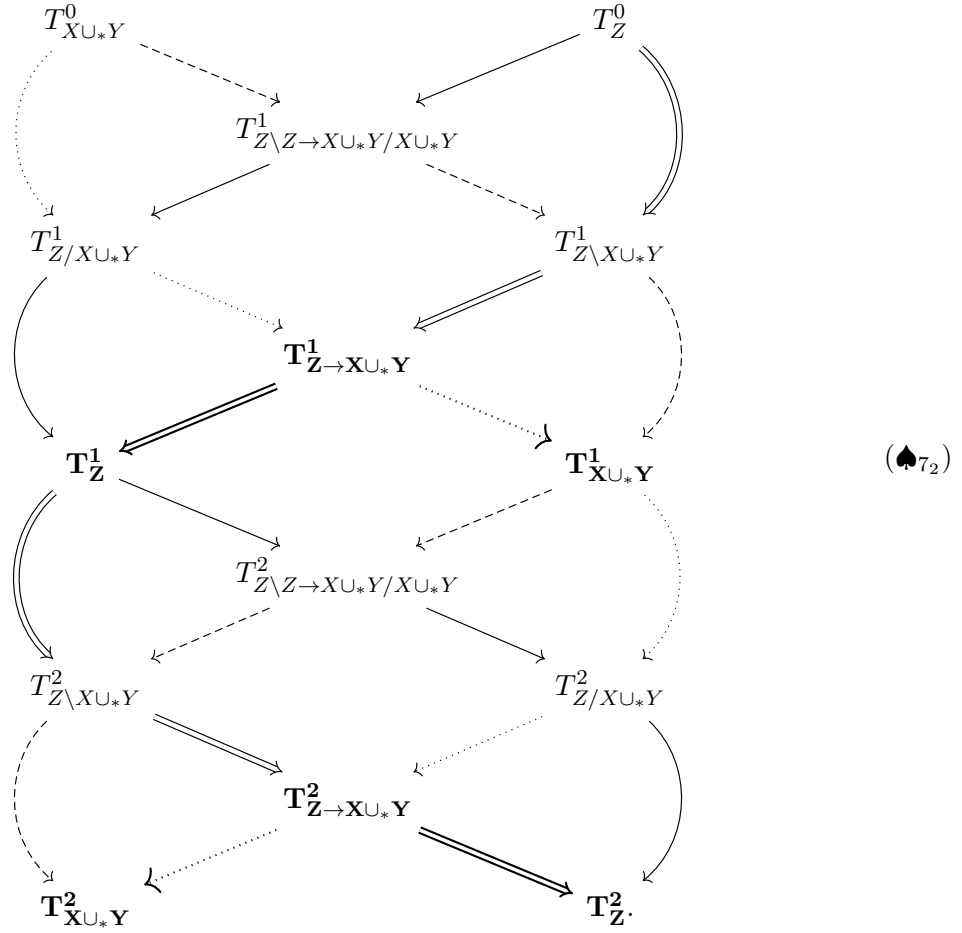
$$X \# Y = \text{Spec Apolar}(\Phi + \Psi) = \text{Spec } \frac{\mathcal{O}_{X \cup_* Y}}{f - g}$$

where $f \circ \Phi = 1$ and $g \circ \Psi = 1$, are homogeneous elements generating respective socles of $\mathcal{O}_X$ and $\mathcal{O}_Y$.

Let us take a closed subscheme $Z \subset X \cup_* Y$ given by a homogeneous (but not necessarily bi-homogeneous) equation h in the socle, in particular $\deg(h) = t$.

$$Z := \text{Spec}(\mathcal{O}_{X \cup_* Y}/(h)) \subset X \cup_* Y. \quad (7.29)$$

The socle of $\mathcal{O}_{X \cup_* Y}$ is the vector space $\langle f, g \rangle$, hence there are three possible variants of h . Recall, from Definition 6.5, that $X \# \emptyset$ is a hypersurface of X cut by the socle. The first case is when h is a multiplicity of f , then $Z = X \# \emptyset \cup_* Y$, analogously we get the second case if h is a multiplicity of g , then $Z = X \cup_* Y \# \emptyset$. The third case is if $h = af + bg$ with both a and b non-zero, then Z is isomorphic with $X \# Y$. The main example of Z is $X \# Y$, but since the connected sum does not admit $\mathbb{G}_m^2$ -action, we need more flexibility and we also consider $X \# \emptyset \cup_* Y$, which in turn has a $\mathbb{G}_m^2$ -action. Once again, our main technical tool will be Braid ($\spadesuit$), in this section, we apply it for the embedding $Z \rightarrow X \cup_* Y$.



We proceed as follows. We compare $\text{Def}_Z^{\leq 0}$ and $\text{Def}_{X \cup_* Y}^{\leq 0}$, we do so via the functor of deformation of the given embedding.

$$\begin{array}{ccc}
 & \text{Def}_{Z \rightarrow X \cup_* Y}^{\leq 0} & \\
 \swarrow \text{dotted} & & \searrow \text{double-lined} \\
 \text{Def}_Z^{\leq 0} & & \text{Def}_{X \cup_* Y}^{\leq 0}
 \end{array} \tag{7.30}$$

Thereupon, we focus on the maps between those T^i functors that are written in Braid $(\spadesuit_{7_2})$ with the bold font. The left arrow of Diagram (7.30) is smooth; this is the content of Lemma 7.31. The right arrow is obstructed and its obstructness is described in Lemma 7.32. Theorem 7.33 and Theorem 7.38 conclude the argument by showing the result announced at the beginning of the section.

Lemma 7.31. *The morphism $\text{Def}_{Z \rightarrow X \cup_* Y}^{\leq 0} \rightarrow \text{Def}_Z^{\leq 0}$ is smooth and isomorphic on the tangent spaces.*

Proof. Corollary 4.71 states that it is enough to show that a relative tangent-obstruction theory vanishes, as we see in Braid $(\spadesuit_{7_2})$. That is

$$(T_{Z \setminus X \cup_* Y}^i)_{<0} = 0,$$

for $i = 1, 2$. Recall the definition of Z , see Formula (7.29). By Proposition 4.41, we get the following identities

$$T_{Z \setminus X \cup_* Y}^i = T_{\mathbb{k}}^i(\mathcal{O}_{X \cup_* Y}, (h)) = T_{\mathbb{k}}^i(\mathcal{O}_{X \cup_* Y}, \mathbb{k})[t].$$

Hence, the negative part is isomorphic to

$$T_{\mathbb{k}}^i(\mathcal{O}_{X \cup_* Y}, \mathbb{k})_{< -t}.$$

Since X and Y are spectra of t -level algebras, by Theorem 7.22, the tangent part vanishes. It remains to prove that $T_{\mathbb{k}}^2(\mathcal{O}_{X \cup_* Y}, \mathbb{k})_{< -t}$ vanishes.

Let us consider the minimal free bi-graded resolution of $\mathcal{O}_{X \cup_* Y}$ over a polynomial ring $\mathbb{S}$ generated by the minimal generators of $\mathcal{O}_X$ and $\mathcal{O}_Y$.

$$F_3 \xrightarrow{d_2} F_2 \xrightarrow{d_1} F_1 \rightarrow \mathbb{S} \rightarrow \mathcal{O}_{X \cup_* Y} \rightarrow 0.$$

Recall that

$$T_{\mathbb{k}}^2(\mathcal{O}_{X \cup_* Y}, \mathbb{k}) \subset \text{Ext}_{\mathbb{S}}^2(\mathcal{O}_{X \cup_* Y}, \mathbb{k}).$$

We first consider the pure degree components of $T_{\mathbb{k}}^2(\mathcal{O}_{X \cup_* Y}, \mathbb{k})$, we claim that they either come from $T_{\mathbb{k}}^2(\mathcal{O}_X, \mathbb{k})$ and $T_{\mathbb{k}}^2(\mathcal{O}_Y, \mathbb{k})$. We lift a pure x -degree element to a homomorphism $\phi \in \text{Hom}_{\mathbb{S}}(F_2, \mathbb{k})$ of the same degree. Such a homomorphism annihilates all syzygies that are not pure x -degree. Hence, we obtain a candidate for a class in $T^2(\mathcal{O}_X, \mathbb{k})$. By Setting 7.26, the space $T_{\mathbb{k}}^2(\mathcal{O}_X, \mathbb{k})_{< -t}$ vanishes. Hence, we get a homomorphism from x -degree component of F_1 to $\mathbb{k}$, we extend it by 0 on the pure y -degree and mixed-degree generators and get a homomorphism $\psi \in \text{Hom}_{\mathbb{S}}(F_1, \mathbb{k})$ such that precomposed with d_1 gives ϕ . Therefore, the pure degree components vanish in degrees $< -t$.

Subsequently, we consider the mixed degree component of $T_{\mathbb{k}}^2(\mathcal{O}_{X \cup_* Y}, \mathbb{k})$. We choose a homomorphism $\phi \in \text{Hom}_{\mathbb{S}}(F_2, \mathbb{k})$ as before. Since the elements of the codomain $\mathbb{k}$ are of bi-degree $(0, 0)$, all the elements of pure degree are mapped to 0 by ϕ . The minimal homogeneous mixed degree syzygies are of one of two kinds. The first kind consists of syzygies of the ideal $\mathbf{xy}$ (the product of maximal ideals of $\mathcal{O}_X$ and $\mathcal{O}_Y$). Those syzygies are linear, hence of degree 3, a homomorphism of degree strictly smaller than $-t \leq -3$ has to annihilate them.

The second kind consists of syzygies expressing the fact that a pure x -degree (respectively y) generator times any element of pure y -degree (resp. x) is in the ideal $\mathbf{xy}$. Since, by Setting 7.26, the degrees of the generators are smaller than t , their bi-degree can be maximally $(1, t-1)$ or $(t-1, 1)$. As a consequence, a homomorphism of degree strictly lower than $-t$ has to annihilate them all. $\square$

The next step is to study a relative tangent-obstruction theory of map $\text{Def}_{Z \rightarrow X \cup_* Y} \rightarrow \text{Def}_{X \cup_* Y}$, see Braid ($\spadesuit_{72}$). Recall that Z is defined as the vanishing locus of equation h , see Formula (7.29).

Lemma 7.32. *The following cotangent space vanishes*

$$(T_{Z/X \cup_* Y}^1)_{< 0} = 0.$$

Moreover, if Z is Gorenstein, then the second cotangent module

$$(T_{Z/X \cup_* Y}^2)_{<0}$$

is contained in a one-dimensional $\mathbb{k}$ -vector space generated spanned by a homogeneous element in degree $-t$.

Proof. The cotangent module is a one-dimensional $\mathbb{k}$ -linear space generated by an element of degree zero.

$$T_{Z/X \cup_* Y}^1 = \text{Hom}_{\mathcal{O}_{X \cup_* Y}}((h), \mathcal{O}_{X \# Y}) = \text{Hom}_{\mathcal{O}_{X \cup_* Y}}(\mathbb{k}, \mathcal{O}_Z)[t] = \mathbb{k}[-t][t] = \mathbb{k}.$$

Now, we assume that Z is Gorenstein and compute the obstruction space

$$T_{Z/X \cup_* Y}^2 \subset \text{Ext}_{\mathcal{O}_{X \cup_* Y}}^1((h), \mathcal{O}_Z) = \text{Ext}_{\mathcal{O}_{X \cup_* Y}}^1(\mathbb{k}, \mathcal{O}_Z)[t],$$

the latter module is concentrated in degree $-t$. Since $\mathcal{O}_Z$ is Gorenstein we get $\mathcal{O}_Z[t] = \mathcal{O}_Z^\vee$, and

$$\text{Ext}_{\mathcal{O}_{X \cup_* Y}}^1(\mathbb{k}, \mathcal{O}_Z) = \text{Ext}_{\mathcal{O}_{X \cup_* Y}}^1(\mathbb{k}, \mathcal{O}_Z^\vee[-t]) = \text{Ext}_{\mathcal{O}_{X \cup_* Y}}^1(\mathbb{k}, \mathcal{O}_Z^\vee)[-t].$$

Consequently

$$T_{Z/X \cup_* Y}^2 \subset \text{Ext}_{\mathcal{O}_{X \cup_* Y}}^1((h), \mathcal{O}_Z) = \text{Ext}_{\mathcal{O}_{X \cup_* Y}}^1(\mathbb{k}, \mathcal{O}_Z^\vee).$$

By [Rot09, Corollary 10.63], we get

$$\text{Ext}_{\mathcal{O}_{X \cup_* Y}}^1(\mathbb{k}, \mathcal{O}_Z^\vee) = \text{Tor}_1^{\mathcal{O}_{X \cup Y}}(\mathbb{k}, \mathcal{O}_Z)^\vee.$$

By the symmetry of the Tor functor, we can change the order of the arguments:

$$\text{Tor}_1^{\mathcal{O}_{X \cup Y}}(\mathbb{k}, \mathcal{O}_Z) = \text{Tor}_1^{\mathcal{O}_{X \cup Y}}(\mathcal{O}_Z, \mathbb{k})$$

and look at the projective resolution of $\mathcal{O}_Z$ over $\mathcal{O}_{X \cup_* Y}$:

$$\begin{array}{ccccccccc} F_2 & \longrightarrow & \mathcal{O}_{X \cup_* Y} & \xrightarrow{h} & \mathcal{O}_{X \cup_* Y} & \xrightarrow{1} & \mathcal{O}_Z & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \otimes_{\mathcal{O}_{X \cup_* Y}} \mathbb{k} & & \\ F_2 \otimes \mathbb{k} & \longrightarrow & \mathbb{k} & \xrightarrow{0} & \mathbb{k} & \xrightarrow{Id} & \mathbb{k} & \longrightarrow & 0. \end{array}$$

Hence, we conclude that $\text{Tor}_1^{\mathcal{O}_{X \cup Y}}(\mathcal{O}_Z, \mathbb{k}) = \mathbb{k}[-t]$. That proves this part of the argument. $\square$

If we take h , see Formula (7.29), to be of degree $(t, 0)$, then h arises as a lift of a generator of the socle of $\mathcal{O}_X$, in particular $\mathcal{O}_Z = \mathcal{O}_{X \cup_* Y}/(h)$ is isomorphic to $\mathcal{O}_{X \# \emptyset} \times_{\mathbb{k}} \mathcal{O}_Y$, which in turn is isomorphic to the structural sheaf of $X \# \emptyset \cup_* Y$. Recall from Definition 7.28 that f is a generator of the socle of $\mathcal{O}_X$. Any element $h \in \mathcal{O}_{X \cup_* Y}$ of degree $(t, 0)$ is f rescaled by a scalar.

Theorem 7.33. *The morphism $\text{Def}_{X \# \emptyset \cup_* Y \rightarrow X \cup_* Y}^{<0} \rightarrow \text{Def}_{X \cup_* Y}^{<0}$ is not smooth.*

Proof. Assume that $\text{Def}_{X \# \emptyset \cup_* Y \rightarrow X \cup_* Y}^{<0} \rightarrow \text{Def}_{X \cup_* Y}^{<0}$ is smooth. By Lemma 7.31, we obtain the

following diagram

$$\begin{array}{ccc}
 & \text{Def}_{X\#\emptyset\cup_*Y\rightarrow X\cup_*Y}^{<0} & \\
 \swarrow \text{sm} & & \searrow \text{sm} \\
 \text{Def}_{X\#\emptyset\cup_*Y}^{<0} & & \text{Def}_{X\cup_*Y}^{<0},
 \end{array}$$

by Lemma 7.32, both maps are isomorphisms on the tangent spaces. By Theorem 7.22, the map

$$\text{Def}_{*\rightarrow X}^{<0} \times \text{Def}_{*\rightarrow Y}^{<0} \rightarrow \text{Def}_{X\cup_*Y}^{<0} \quad (7.34)$$

is smooth and an isomorphism on the tangent spaces. By Lemma 7.18, the map

$$\text{Def}_{X\#\emptyset}^{<0} \rightarrow \text{Def}_{X\#\emptyset\cup_*Y}^{<0} \quad (7.35)$$

is unobstructed and injective on the tangent spaces. By Lemma 6.30, we get the vanishing $(T_{X\#\emptyset}^1)_{<-1} = 0$, and hence, by Lemma 6.7, we get that

$$\text{Def}_{*/X\#\emptyset}^{<0} \rightarrow \text{Def}_{X\#\emptyset}^{<0} \quad (7.36)$$

is unobstructed.

By composing the maps (7.35) and (7.36) and use Corollary 4.33 twice, the first time for the morphism $\text{Def}_{X\#\emptyset\cup_*Y\rightarrow X\cup_*Y}^{<0} \rightarrow \text{Def}_{X\#\emptyset\cup_*Y}^{<0}$, and the second time for (7.34); we get a morphism

$$\text{Def}_{*/X\#\emptyset}^{<0} \rightarrow \text{Def}_{*\rightarrow X}^{<0} \times \text{Def}_{*\rightarrow Y}^{<0},$$

which is unobstructed. By $\mathbb{G}_m^2$ -equivariance, the composed morphism $\text{Def}_{*/X\#\emptyset}^{<0} \rightarrow \text{Def}_{*\rightarrow X}^{<0}$ is still unobstructed. We compose this morphism even further and get a map $\text{Def}_{*/X\#\emptyset}^{<0} \rightarrow \text{Def}_X^{<0}$, it is trivial, since, by Lemma 6.30, the functor $\text{Def}_{X\#\emptyset\setminus X}^{<0}$ is trivial, as its tangent and obstruction spaces vanish. Therefore, by the universal property of pullback, we get an unobstructed map

$$X\#\emptyset = \text{Def}_{*/X\#\emptyset}^{<0} \rightarrow \text{Def}_{*\rightarrow X}^{<0} \times_{\text{Def}_X^{<0}} \{*\} =: F. \quad (7.37)$$

By Setting 7.26, the space $(T_X^1)_{<-1}$ vanishes, consequently, by Lemma 6.7, there is a smooth map $X \cong \text{Def}_{*/X}^{<0} \rightarrow F$. Since the tangent space to $X\#\emptyset$ at the unique closed point is naturally isomorphic to the tangent space to X , the map (7.37) is surjective on the tangent spaces, and hence smooth. We get that $X\#\emptyset$ is smoothly equivalent to X , which is false. $\square$

The final argument is derived from the previous one.

Theorem 7.38. *The functor $\text{Def}_{X\#Y}^{<0}$ is smoothly equivalent to a closed hypersurface of $\text{Def}_{X\cup_*Y}^{<0}$ given by an equation of degree $-t$. We get a diagram, Cartesian up to smooth equivalence*

$$\begin{array}{ccc}
 \text{Spec}(\mathcal{O}_X \otimes \mathcal{O}_Y/(f-g)) & \longrightarrow & \text{Def}_{X\#Y}^{<0} \\
 \downarrow & & \downarrow \\
 * & \longrightarrow & \text{Def}_X^{<0} \times \text{Def}_Y^{<0},
 \end{array}$$

where $\mathcal{O}_X \otimes \mathcal{O}_Y$ in the diagram are generated in degree -1 . In particular, if X and Y are TNT, algebras then we get a smooth map

$$\mathrm{Spec} \frac{\mathcal{O}_X \otimes \mathcal{O}_Y}{f - g} \rightarrow \mathrm{Def}_{X \# Y}^{<0}.$$

Proof. We take the following diagram

$$\begin{array}{ccc} P & \longrightarrow & \mathrm{Def}_{X \# Y}^{<0} \\ \downarrow & & \downarrow \\ X \times Y & \longrightarrow & \mathrm{Def}_{X \cup_* Y}^{<0} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Def}_X^{<0} \times \mathrm{Def}_Y^{<0}, \end{array} \quad (7.39)$$

where P makes the top square Cartesian. In Diagram 7.39, we abuse the notation and write $\mathrm{Def}_{X \# Y}^{<0}$ in place of $\mathrm{Def}_{X \# Y \rightarrow X \cup_* Y}^{<0}$, as, by Lemma 7.31, there is a smooth map between those functors. The pullback map $P \rightarrow X \times Y$ is an isomorphism on the tangent spaces and, by Lemma 7.32, has one obstruction in degree $-t$. If we prove that this map is not smooth, then it is a closed embedding given by one non bi-homogeneous equation of degree $-t$. Note that the tangent space is generated in degree -1 , see Lemma 6.6. As a consequence, it is isomorphic to the spectrum of $\mathcal{O}_X \otimes \mathcal{O}_Y / (f - g)$. That is exactly the statement of the theorem.

Let us assume that the pullback map $P \rightarrow X \times Y$ is smooth.

Let us take the element in $\mathrm{Def}_{X \cup_* Y}^{<0}(X \times Y)$ induced by the map $X \times Y \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0}$. This is a local algebra $(B, \mathfrak{n})$ over $(A, \mathfrak{m}) = (\mathcal{O}_X \otimes \mathcal{O}_Y, \mathfrak{m})$, which over the closed point specializes to $B/(\mathfrak{m}B) \cong \mathcal{O}_{X \cup_* Y}$.

By Theorem 7.22, the morphism $X \times Y \rightarrow \mathrm{Def}_{X \cup_* Y}^{<0}$ factors through the product $\mathrm{Def}_{* \rightarrow X}^{<0} \times \mathrm{Def}_{* \rightarrow Y}^{<0}$, hence the deformation decomposes

$$(B, \mathfrak{n}) = (B_x \times_A B_y, \mathfrak{n}_x + \mathfrak{n}_y),$$

where $(B_x, \mathfrak{n}_x)$ (respectively $(B_y, \mathfrak{n}_y)$) is a trivial deformation of $\mathcal{O}_X$ (resp. $\mathcal{O}_Y$) with a possibly non-trivial pointed structure. Recall, from Lemma 7.8, that we get the following exact sequences

$$0 \rightarrow \mathfrak{n}_x \rightarrow B \rightarrow B_y,$$

$$0 \rightarrow \mathfrak{n}_y \rightarrow B \rightarrow B_x,$$

in particular $\mathfrak{n}_x \mathfrak{n}_y = 0$.

Since $\mathcal{O}_{X \# Y}$ is given as a quotient $\mathcal{O}_{X \cup_* Y} / (f - g)$, a lift of B to $\mathrm{Def}_{X \# Y}^{<0}$ is given by a lift $\tilde{f} - \tilde{g}$ of the equation $f - g$ (we assume that $\tilde{f}$ lift of f belonging to $\mathfrak{n}_x$ and analogously for $\tilde{g}$ and g). By flatness, we obtain the following map of exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ann}(\tilde{f} - \tilde{g}) & \longrightarrow & B & \xrightarrow{\tilde{f} - \tilde{g}} & B & \longrightarrow & B/(\tilde{f} - \tilde{g}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ann}(f - g) & \longrightarrow & \mathcal{O}_{X \cup_* Y} & \longrightarrow & \mathcal{O}_{X \cup_* Y} & \longrightarrow & \mathcal{O}_{X \# Y} & \longrightarrow & 0. \end{array}$$

The following identities hold

$$\text{Ann}(f - g) = \text{Ann}(f) = \text{Ann}(g) = \frac{\mathfrak{n}}{\mathfrak{m}B},$$

since these are equations from the socle. Let us take $b \in \text{Ann}(\tilde{f} - \tilde{g}) \subset \mathfrak{n}$ it can be decomposed into $b = b_x + b_y$ with $b_x \in \mathfrak{n}_x$ and $b_y \in \mathfrak{n}_y$.

$$0 = (b_x + b_y)(\tilde{f} - \tilde{g}) = b_x\tilde{f} + b_y\tilde{f} - b_x\tilde{g} - b_y\tilde{g} = b_x\tilde{f} - b_y\tilde{g},$$

consequently we get that $b_x\tilde{f} \in \mathfrak{n}_x\mathfrak{n}_y$, but this by definition vanishes. We get $(b_x + b_y)(\tilde{f}) = 0$, as a consequence

$$\text{Ann}(\tilde{f} - \tilde{g}) \subset \text{Ann}(\tilde{f}).$$

Now, we consider the problem of A -flatness of $B/\tilde{f}$. By the local criterion for flatness [Eis95, Theorem 6.8], we need to compute if $\text{Tor}_A^1(A/\mathfrak{m}, B/(\tilde{f}))$ vanishes.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \text{Ann}(\tilde{f}) & \longrightarrow & B & \xrightarrow{\tilde{f}} & B & \longrightarrow & B/(\tilde{f}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ & & \text{Ann}(\tilde{f}) \otimes A/\mathfrak{m} & \longrightarrow & \mathcal{O}_{X \cup_* Y} & \longrightarrow & \mathcal{O}_{X \cup_* Y} & \longrightarrow & \mathcal{O}_{X \# \emptyset \cup_* Y} & \longrightarrow & 0. \end{array}$$

Since $\text{Ann}(\tilde{f} - \tilde{g}) \subset \text{Ann}(\tilde{f})$, we get that $\text{Ann}(\tilde{f}) \otimes A/\mathfrak{m}$ contains lifts of $\mathfrak{n}/\mathfrak{m}B$. That implies the desired vanishing. Consequently, we have shown that assuming smoothness of $P \rightarrow X \times Y$ implies that

$$\text{Def}_{X \# \emptyset \cup_* Y}^{<0} \rightarrow \text{Def}_{X \cup_* Y}^{<0}$$

is smooth. But this contradicts Theorem 7.33. $\square$

Corollary 7.40. *Let X and Y be a pair of uncleavable schemes satisfying Setting 7.26, and such that lengths of X and Y are co-prime with $\text{char } \mathbb{k}$, or $\text{char } \mathbb{k} = 0$. Then $\text{Def}_{X \# Y}^{<0}$ is non-reduced.*

Proof. Theorem 7.38 shows that $\text{Def}_{X \# Y}^{<0}$ is a closed subfunctor of $\text{Def}_{X \cup_* Y}^{<0}$ given by a single equation in degree $-t$. By Setting 7.26, the tangent space to $\text{Def}_X^{<0} \times \text{Def}_Y^{<0}$ lies in degree -1 , together with Theorem 7.38, it implies that the tangent space to $\text{Def}_{X \cup_* Y}^{<0}$ also lies in degree -1 .

Since, by Corollary 7.25, the functor $\text{Def}_{X \cup_* Y}^{<0}$ is non-reduced, we get a hypersurface of degree $-t < -1$ of a non-reduced functor with the tangent space lying in degree -1 . By pulling back to the semi-universal families, we see that this hypersurface is non-reduced. $\square$

7.6 Applications

In this section, we return to the investigation of the Gorenstein locus of the Hilbert scheme of points to show applications of the obtained results. In the second part of the section we apply our results to the study of cactus schemes.

Theorem 7.41. *Let us take a pair of uncleavable Gorenstein finite schemes*

$$X = \text{Spec Apolar}(\Phi),$$

$$Y = \text{Spec Apolar}(\Psi),$$

where $\Phi \in \mathbf{P}_x$ and $\Psi \in \mathbf{P}_y$ is a pair of homogeneous polynomials of degree $t \geq 3$, such that

$$(T_X^1)_{<-1} = (T_Y^1)_{<-1} = 0$$

is satisfied. We assume two additional conditions.

- The following second cotangent space vanish

$$T^2(\mathcal{O}_X, \mathbb{k})_{<-t} = T^2(\mathcal{O}_Y, \mathbb{k})_{<-t} = 0.$$

- The ideals $\text{Ann}(F) \subset S_x$ and $\text{Ann}(G) \subset S_y$ are generated in degrees $\leq t - 1$.

Then the negative spike $\text{Hilb}_{[X\#Y]}^{\leq 0}$ has the smooth type of a family over the product of Białynicki-Birula fibres of X and Y , such that the special fibre is

$$\frac{\mathcal{O}_X \otimes \mathcal{O}_Y}{f - g},$$

where $f \circ \Phi = 1$ and $g \circ \Psi = 1$.

Proof. By Theorem 7.38, the smooth type of $\text{Def}_{X\#Y}^{\leq 0}$ is as claimed in the statement. By Lemma 4.62, the map $\text{Hilb}_{[X\#Y]}^{\leq 0} \rightarrow \text{Def}_{X\#Y}^{\leq 0}$ is smooth, and therefore the functors are smoothly equivalent. $\square$

Corollary 7.42. Take X, Y as in Theorem 7.41, such that lengths of X and Y are co-prime with $\text{char } \mathbb{k}$, or $\text{char } \mathbb{k} = 0$. and assume that the positive tangent space $(T_{X\#Y}^1)_{>0}$ vanishes. Then $\text{Def}_{X\#Y}$ is non-reduced and consequently $X\#Y$ gives us a non-reduced point on the Hilbert scheme.

Proof. By Corollary 7.40, we obtain non-reducedness of $\text{Def}_{X\#Y}^{\leq 0}$. We get a Cartesian diagram with a section as presented below:

$$\begin{array}{ccc} \text{Def}_{X\#Y}^{\leq 0} & \longrightarrow & \text{Def}_{X\#Y}^{\leq 0} \\ \downarrow & & \downarrow \uparrow \\ * & \longrightarrow & \text{Def}_{X\#Y}^0. \end{array}$$

Lemma 2.3 implies that $\text{Def}_{X\#Y}^{\leq 0}$ is non-reduced. By Lemma 4.62, we get that $\text{Def}_{X\#Y}^{\leq 0}$ is smoothly equivalent to $\text{Hilb}_{[X\#Y]}^{\leq 0}$. Since the positive tangent space vanishes, by Lemma 4.69, we get $\text{Hilb}_{[X\#Y]}^{\leq 0}$ is smoothly equivalent to $\text{Hilb}_{[X\#Y]}$. $\square$

Corollary 7.43. For every $n_1, n_2 \geq 6$, but neither is equal to 7, then $\text{Hilb}_{2n_1+2n_2+2}^{\text{Gor}}(\mathbb{A}^{n_1+n_2})$ is not reduced, providing that $\text{char } \mathbb{k}$ is either 0 or co-prime with $2n_1 + 2n_2 + 2$.

Proof. By Lemma 6.32, for a cubic in m variables with the Hilbert function equal $(1, m, m, 1)$, the positive tangent space vanishes. If we take two such cubics Φ and Ψ in n_1 and n_2 variables respectively, then the connected sum of affine spectra of their apolar algebras $X\#Y$ has length $2n_1 + 2n_2 + 2$.

By Corollary 7.42, if such cubics satisfy Setting 7.26, we get the statement.

Since the moduli space of cubics in m variables is irreducible and all conditions are open, we only need to prove that there is a cubic for every condition and every integer $m \geq 6$ which is not 7.

Recall from Definition 6.31, what are the very general cubics. The condition of vanishing of $T^2(\mathcal{O}_X, \mathbb{k})_{<-3}$ is satisfied by them, by Lemma 6.35, and open by Lemma 4.58. Similarly, the condition that generators of the ideal are in degree 2 is open, by semicontinuity of Betti numbers, and satisfied by very general cubics.

The condition on vanishing of T_X^1 in degrees < -1 is satisfied by very general cubics, which do exist for $n > 7$ and $n = 6$, see [IK99, Lemma 6.21], [Sza23, Theorem 2]. This concludes the proof. $\square$

7.6.1 Application to Cactus variety

Let $U \subset \mathbb{P}^N$ be a smooth projective scheme. We fix a non-negative integer δ and recall the classical notion of secant variety.

Definition 7.44. *Let $X \subset \mathbb{P}^n$ be a projective scheme. The projective span $\langle X \rangle$ is the minimal linear subspace $\mathbb{P}^k \subset \mathbb{P}^n$, that contains X .*

Definition 7.45. *The δ -th secant variety of U is the subscheme*

$$\sigma_\delta(U) := \overline{\bigcup_{p_1, \dots, p_\delta \in U} \{\langle p_1 \dots p_\delta \rangle\}} \subset \mathbb{P}^n,$$

with the reduced scheme structure.

The study of higher secant varieties enjoys large interest, the question whether they admit mild singularities is central, see [Ull16] [CS18], [ENP20], [CLPS25, Theorem A]. The special case $U = \nu_r(\mathbb{P}^m)$, where ν_r is the r -th Veronese embedding, is of special interest, since it relates secant varieties to tensors, for reference see [Lan12]. There is also a great deal of interest in equations of the secant varieties [BGL13, BBF24] as well as in the syzygies [CLPS25, Theorem B, Corollary C].

One of the approaches to the study of equations is to study a cactus variety, introduced in a seminal work [BB14].

Definition 7.46 ([BB14]). *The δ -th cactus variety of U is given set-theoretically as*

$$\mathfrak{K}_\delta(U) := \overline{\bigcup \{ \langle X \rangle \mid X \subset U, X \text{ is a subscheme of degree at most } \delta \}} \subset \mathbb{P}^N.$$

Clearly $\sigma_\delta(U) \subset \mathfrak{K}_\delta(U)$, in some special cases the equality holds, but in general the secants are thought as an irreducible component of the cacti. The cactus variety plays an important role in new findings on singularities of the secant variety [CLPS25, Corollary E].

In the special case of $\nu_r(\mathbb{P}^n)$, the cactus variety admits a (possibly non-reduced) natural scheme structure coming from minors of catalecticant matrices, see [BB14, Theorem 1.5], the cactus variety with this structure is called *cactus scheme*. It can be also defined in other way [BK24, Theorem 1.1, Definition 4.3] by employing notion of familiars, see [BK24, Definition 3.10]. In that case the singularities of the Gorenstein locus of the Hilbert scheme yield all singularities of the cactus scheme.

Theorem 7.47 ([BK24, Theorem 1.2],[CLPS25]). *Let $\delta \geq 1$ and $r \geq 2\delta$. There exists a diagram of smooth maps*

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \text{sm} \\ \mathfrak{K}_\delta(\nu_r(\mathbb{P}^m)) & & \text{Hilb}_\delta^{\text{Gor}}(\mathbb{A}^n), \end{array}$$

where $Z \rightarrow \mathfrak{K}_\delta(\nu_r(\mathbb{P}^m))$ is an open embedding, $\mathfrak{K}_{\delta-1}(\nu_r(\mathbb{P}^m))$ the complement of the image, and the morphism $Z \rightarrow \text{Hilb}_\delta^{\text{Gor}}(\mathbb{A}^n)$ is smooth and surjective.

The relationship between the cactus scheme and the Gorenstein locus rise far beyond the case of Veronese embedding, for details see [CLPS25, Theorem D].

We apply the main result of this chapter, Theorem 7.43, to Theorem 7.47 and get non-reduceness of the cactus scheme.

Corollary 7.48. *Let $n_1, n_2 \geq 6$ but neither equal to 7, and $r \geq 4n_1 + 4n_2 + 4$. The cactus scheme $\mathfrak{K}_{2n_1+2n_2+2}(\nu_r(\mathbb{P}^{n_1+n_2}))$ is non-reduced.*

Proof. By Theorem 7.47, it is enough to show the non-reducedness for $\text{Hilb}_{2n_1+2n_2+2}^{\text{Gor}}(\mathbb{A}^{n_1+n_2})$, but this is the statement of Corollary 7.43. $\square$

Part III

The Territory scheme

Chapter 8

The Territory scheme

8.1 Introduction

This chapter is devoted to the study of the Territory scheme Ter , a moduli space that thus far did not attract as much attention as the Hilbert scheme. The Territory scheme was introduced in [Ish80, Definition 2] and it parametrises co-finite subrings of $\mathcal{O}_X$, for a given ring $\mathcal{O}_X$, see Definition 8.2. This construction is originally motivated by the question how many isolated singularities can be normalized to a fixed normal singularity X .

If we take $\mathcal{O}_X$ to be the ring of section of the affine line, then a co-finite subring $\mathcal{O}_Y \subset \mathcal{O}_X$ yields a surjective map of affine spectra $\mathbb{A}^1 \rightarrow Y$, which is the normalization of Y . One can think about Y as an affine model of a unibranched singularity, this view is explained in Section 8.3. The findings on Ter are applied to study of several variants of moduli of curves, for example to construct the Moduli space of equisingular complete curves, see [Ish82], to construct the stack of equinormalizable curves [BGS24, Theorem I], and to prove connectedness of the Moduli space of reduced curves, see [Boz25, Theorem 1.1].

We take a subalgebra of a polynomial ring $\mathbb{k}[x]$, we associate with it a semigroup of achieved valuations, see Definition 8.12. One of the key ideas used to study the geometry of Territory scheme, used for example in [Ish80] and in [Ham20], is to stratify the scheme by locally-closed subschemes indexed by the associated semigroups. In Theorem 8.28 we show that this stratification is an instance of the Białynicki-Birula decomposition. The Białynicki-Birula cells are indexed by the connected components of $\mathbb{G}_m$ -fixed point locus, in this case the fixed points are isolated. The fixed points correspond to semigroup algebras. Given a semigroup Γ , the semigroup algebra $\mathbb{k}[\Gamma]$ is a subring of $\mathbb{k}[x]$ spanned by monomials x^γ with γ running over the semigroup Γ .

8.1.1 New applications

A co-finite submonoid of natural numbers is called a numerical semigroup. These combinatorial objects admit various interesting connections to several areas of mathematics such as number theory or coding theory, see [ADGS20]. The central problem in the study of numerical semigroup is the Wilf conjecture. It is listed as problem 30 on list of open problems concerning numerical semigroups [MS25]. It was observed in [MS15] that the Wilf conjecture can be phrased in algebraic terms, from that the following question arisen.

Question ([MS15, Question 9]). *Let R be a one-dimensional Cohen-Macaulay local ring such*

that the integral closure of R in its total ring of fractions is a finite R -module. Under what assumptions do we have

$$\text{length}(\overline{R}/\mathfrak{C}) \leq \text{edim}(R)\text{length}(R/\mathfrak{C}), \quad (8.1)$$

where $\mathfrak{C}$ is the largest maximal ideal of R and its integral closure $\overline{R}$.

We partially answer this question, if R is an integral $\mathbb{k}$ -algebra, then by completion, we get equivalent question with the integral closure given as the ring of formal series $\mathbb{k}[[x]]$. Then the question can be phrased in terms of sheaves on the Territory scheme. From that we formulate Conjecture 3, that all such algebras satisfy (8.1). In Theorem 8.44 we show that whether Conjecture 3 holds is equivalent to the Wilf conjecture. We do that by introducing a Wilf locus consisting of algebras satisfying (8.1), and showing that it is a locally closed $\mathbb{G}_m$ -equivariant subscheme of Ter . This way we show, as far as we know, unknown to this point, connection between the study of Territory scheme and purely combinatorial problems.

In the last section of this chapter, we consider the problem of lifting curve singularities from positive characteristic to 0. It is classical that the smooth curves lift, see [Har10, Corollary 22.2], but whether curve singularities lift remains open, see [Har10, Page 155]. Following [Eke04, Lemma 5.7], we sketch a strategy aiming to find non-liftable singularities using the Territory as the source of examples. As presented in the next chapter the Territory scheme admits a very non-reduced structure, hence it is plausible that it fits into the pattern of *pathological moduli spaces*. The examples of components lying purely in characteristic p would put in the line with various other moduli spaces, such as the Hilbert scheme of points and several others as exhibited in [Vak06, Jel20, Sch23].

8.2 Generalities on Territory scheme

Recall that $\mathbb{k}$ is an algebraically closed field of characteristic p and that Λ is a Noetherian completed local ring with the residue field isomorphic to $\mathbb{k}$.

Definition 8.2 ([Ish80, Definition 2]). *Let X be a flat affine Λ -scheme and δ be a positive integer. We define a contravariant functor of the Territory of Λ -scheme X , $\text{Ter}_X^\delta : \text{Sch}_\Lambda^{op} \rightarrow \text{Set}$ on S -points as*

$$\text{Ter}_X^\delta(S) := \left\{ (0 \rightarrow \mathcal{A} \xrightarrow{\phi} \mathcal{O}_X \otimes_\Lambda \mathcal{O}_S \rightarrow \text{coker}(\phi) \rightarrow 0) \left| \begin{array}{l} \phi \text{ is an } \mathcal{O}_S\text{-algebra homomorphism} \\ \text{coker}(\phi) \text{ is rank } \delta \text{ locally free } \mathcal{O}_S\text{-module.} \end{array} \right. \right\}$$

Since the homomorphism ϕ is an $\mathcal{O}_S$ -algebra homomorphism, the module $\mathbb{A}$ yields an S -scheme $Y := \text{Spec } \mathcal{A}$.

The set $\text{Ter}_X^\delta(\text{Spec}(\mathbb{k}))$ consists of subalgebras of $\mathcal{O}_X \otimes_\Lambda \mathbb{k}$, with cokernel having dimension δ over $\mathbb{k}$.

Notation. We denote the points of $\text{Ter}_X^\delta(S)$ either on the level of the structural sheaves, omitting the pushforward in the first term,

$$(0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \otimes_\Lambda \mathcal{O}_S \rightarrow \text{coker}(\phi) \rightarrow 0),$$

sometimes we also omit the cokernel; or as a morphism of schemes

$$(X \times_{\Lambda} S \rightarrow Y).$$

We denote $X \times_{\mathbb{k}} S$ by X_0 , and a $\mathbb{k}$ -point of Ter_X^{δ} either as

$$\mathcal{O}_{Y_0} \rightarrow \mathcal{O}_{X_0}$$

or

$$(X_0 \rightarrow Y_0).$$

Lemma 8.3 ([BGS24, Lemma 2.3]). *Let us take pair of flat Λ -algebras $\mathcal{O}_{X_1} \subset \mathcal{O}_{X_2}$, such that $\mathcal{O}_{X_2}/\mathcal{O}_{X_1}$ is a locally free Λ -module of rank t . There is a closed immersion of functors $\mathrm{Ter}_{X_1}^{\delta} \rightarrow \mathrm{Ter}_{X_2}^{\delta+t}$ induced by the inclusion.*

Proof. We define the map on S -points. We abuse the notation and omit $\mathcal{O}_S$. and take an element

$$(0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{X_1} \rightarrow \mathrm{coker}_{X_1} \rightarrow 0) \in \mathrm{Ter}_{X_1}^{\delta}$$

and construct the following diagram

$$\begin{array}{ccccccc}
& & 0 & \longrightarrow & 0 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_Y & \xrightarrow{Id} & \mathcal{O}_Y & \longrightarrow & 0 \longrightarrow \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_{X_1} & \longrightarrow & \mathcal{O}_{X_2} & \longrightarrow & \mathcal{O}_{X_2}/\mathcal{O}_{X_1} \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \longrightarrow & \mathrm{coker}_{X_1} & \longrightarrow & \mathrm{coker}_{X_2} & \longrightarrow \mathcal{O}_{X_2}/\mathcal{O}_{X_1} \longrightarrow 0 \\
& & & \downarrow & & \downarrow & \downarrow \\
& & & 0 & & 0 & 0.
\end{array}$$

From Snake's lemma we see that coker_X is in fact of rank $\delta + t$. All objects above are locally free over $\mathcal{O}_S$, so it is a transformation of functors. As presented, it is injective on objects. It is left to show that it is closed. Let us assume that we have an element of $\mathrm{Ter}_{X_2}^{\delta+t}(S)$ an algebra $\mathcal{O}_Y$, to say that is coming from $\mathrm{Ter}_{X_1}^{\delta}$ is to say that $\mathcal{O}_Y \rightarrow \mathcal{O}_{X_2}/\mathcal{O}_{X_1}$ is 0, this is a closed condition. $\square$

Theorem 8.4 ([Ish80, Theorem 1], [BGS24, Theorem 2.5]). *Let $\mathcal{O}_X$ be locally free over Λ of rank n , then Ter_X^{δ} is represented by a closed subscheme of Grassmannian $\mathrm{Gr}_{\Lambda}(\delta, \mathcal{O}_X)$.*

Let us recall that to any contravariant functor from schemes to sets and any $\mathbb{k}$ -point of such scheme we associate a predeformation functor, Definition 4.4.

Definition 8.5. *For a $\mathbb{k}$ -point $\{X_0 \rightarrow Y_0\} \in \mathrm{Ter}_X^{\delta}(\mathbb{k})$, we define a predeformation deformation functor $\mathrm{ter}_{Y_0}^X: \mathrm{Art}_{\Lambda} \rightarrow \mathrm{Set}$, to be the local functor of Ter at the point $\{X_0 \rightarrow Y_0\}$.*

In the context where X is fixed, we omit the superscript in the notation and write ter_{Y_0} .

Theorem 8.6. *There is an isomorphism of predeformation functors*

$$\mathrm{Def}_{X \setminus Y_0} \cong \mathrm{ter}_{Y_0}^X.$$

Proof. Both functor parametrise infinitesimal deformations of an injective algebra homomorphism $\mathcal{O}_{Y_0} \rightarrow \mathcal{O}_{X_0}$. Let us denote the cokernel by C_0 . We choose an exact sequence in Art_Λ

$$0 \rightarrow K \rightarrow A \rightarrow \mathbb{k} \rightarrow 0. \quad (8.7)$$

An element of $\text{Def}_{X \setminus Y}(A)$ is a homomorphism $\mathcal{O}_{Y_A} \rightarrow \mathcal{O}_X \otimes A$ of flat A -algebras with cokernel denoted by C_A , such that after tensoring with (8.7) we get

$$\begin{array}{ccccc} & & & & 0 \\ & & & & \downarrow \\ \mathcal{O}_{Y_A} \otimes_\Lambda K & \longrightarrow & \mathcal{O}_{Y_A} & \longrightarrow & \mathcal{O}_{Y_0} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}_X \otimes_\Lambda K & \longrightarrow & \mathcal{O}_X \otimes_\Lambda A & \longrightarrow & \mathcal{O}_{X_0} \\ \downarrow & & \downarrow & & \downarrow \\ C_A \otimes_\Lambda K & \longrightarrow & C_A & \longrightarrow & C_0 \\ \downarrow & & \downarrow & & \downarrow \\ 0 & & 0 & & 0. \end{array}$$

Since the condition that homomorphism is injective is open, by [Sta22, Tag 00ME], the homomorphism $\mathcal{O}_{Y_A} \rightarrow \mathcal{O}_X \otimes_\Lambda A$ is injective. The only thing left to prove is that C_A is flat over A . By local criterion for flatness [Eis95, Theorem 6.8], it is enough to prove that $\text{Tor}_A^1(C_A, \mathbb{k})$ vanishes. By the long exact sequence of Tor-functors we get

$$\text{Tor}^1(\mathcal{O}_X \otimes_\Lambda A, \mathbb{k}) \rightarrow \text{Tor}^1(C_A, \mathbb{k}) \rightarrow \mathcal{O}_{Y_0} \rightarrow \mathcal{O}_{X_0} \rightarrow C_0 \rightarrow 0.$$

By the assumption $\text{Tor}^1(\mathcal{O}_X \otimes_\Lambda A, \mathbb{k})$ vanishes, and map $\mathcal{O}_{Y_0} \rightarrow \mathcal{O}_{X_0}$ is injective, hence $\text{Tor}_A^1(C_A, \mathbb{k}) = 0$. $\square$

Corollary 8.8. *Let us take a $\mathbb{k}$ -point of Ter_X^δ*

$$0 \rightarrow \mathcal{O}_{Y_0} \rightarrow \mathcal{O}_{X_0} \rightarrow C_0 \rightarrow 0.$$

The functor $\text{ter}_{Y_0}^X$ is a good deformation functor and admits an associated triple

$$0, T_{\mathbb{k}}^0(\mathcal{O}_{Y_0}, C_0), T_{\mathbb{k}}^1(\mathcal{O}_{Y_0}, C_0)$$

in particular the functor is prorepresentable.

Proof. This is an immediate corollary to Theorem 8.6 and Theorem 4.41. $\square$

8.2.1 Group actions

Proposition 8.9. *Let us take $\mathbb{G}$ to be group scheme over Λ . Then a $\mathbb{G}$ -action on X induces a $\mathbb{G}$ -action on Ter_X^δ .*

Proof. We construct a morphism

$$\mathbb{G}(S) \times \text{Ter}_X^\delta(S) \rightarrow \text{Ter}_X^\delta(S). \quad (8.10)$$

An element of $\mathbb{G}(S)$ is a morphism $g: S \rightarrow \mathbb{G}$, it induces an automorphism

$$\begin{aligned}\tilde{g}: S \times X &\rightarrow S \times X, \\ (s, x) &\mapsto (s, g(s)x).\end{aligned}$$

An element of $\text{Ter}_X(S)$ is a commutative diagram

$$\begin{array}{ccc} S \times X & \longrightarrow & Y_S \\ & \searrow & \downarrow \\ & & S. \end{array}$$

where the map $S \times X \rightarrow Y_S$ induces injection $\mathcal{O}_{Y_S} \rightarrow \mathcal{O}_X \otimes \mathcal{O}_S$, with locally free cokernel. Now we consider the following diagram

$$\begin{array}{ccc} S \times X & & \\ \tilde{g} \downarrow & \searrow & \\ S \times X & \longrightarrow & Y_S \\ & \searrow & \downarrow \\ & & S. \end{array}$$

We need to check that induced map $\mathcal{O}_{Y_S} \rightarrow \mathcal{O}_X \otimes \mathcal{O}_S$ has cokernel locally free over $\mathcal{O}_S$. We perform a diagram chasing on

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{Y_S} & \longrightarrow & \mathcal{O}_X \otimes \mathcal{O}_S & \longrightarrow & C_S \longrightarrow 0 \\ & & \downarrow & & \downarrow \cong & & \downarrow \\ 0 & \longrightarrow & \mathcal{O}_{Y_S} & \longrightarrow & \mathcal{O}_X \otimes \mathcal{O}_S & \longrightarrow & C'_S \longrightarrow 0, \end{array}$$

and get $C_S \cong C'_S$ as $\mathcal{O}_S$ -modules. □

Corollary 8.11. *Let X admit a $\mathbb{G}_m$ -action, then $\mathcal{O}_{Y_0} \subset \mathcal{O}_{X_0}$ induces a natural filtration on $\mathcal{O}_{Y_0}$. Then the local functor $\text{ter}_{Y_0}^{\geq 0}$ is a good deformation functor and admits an associated triple*

$$0, T^0(\mathcal{O}_{Y_0}, C_0)_{\geq 0}, T^1(\mathcal{O}_{Y_0}, C_0)_{\geq 0}$$

in particular the functor is prorepresentable.

Proof. By Theorem 8.6 we get

$$\text{Def}_{X \setminus Y_0} \cong \text{ter}_{Y_0}^X.$$

Hence the statement follows from Theorem 4.60. □

8.2.2 Semigroups

Given a curve singularity, the main invariant is an associated semigroup. In this subsection we define basic notions related to numerical semigroups and show the relation with the Territory scheme, see Lemma 8.19, Theorem 8.20.

Definition 8.12. A numerical semigroup is a submonoid Γ of natural numbers $(\mathbb{N}, +)$ such that $\mathbb{N} \setminus \Gamma$ is a finite set.

For the rest of this subsection we fix a numerical semigroup Γ .

Definition 8.13. The cardinality of $\mathbb{N} \setminus \Gamma$ is called the genus of Γ .

Definition 8.14. The minimal positive integer $\mathbf{c}$, such that all elements of $\mathbb{N} \setminus \Gamma$ are less than $\mathbf{c}$ is called the conductor.

Lemma 8.15. Let Γ be a numerical semigroup with genus δ , then the conductor is at most 2δ .

Proof. Let us denote $\mathbb{N} \setminus \Gamma$ as Γ^c . By definition of genus $|\Gamma^c| = \delta$. For an element $x \in \Gamma$ the set $(\Gamma^c - x)$ intersects Γ emptily. Assume that the conductor is larger than 2δ , then

$$|\Gamma \cap \{1 \dots 2\delta\}| > \delta. \quad (8.16)$$

We take an element $e \in \Gamma^c$ such that $e > 2\delta$, then $e - (\Gamma \cap \{1 \dots 2\delta\}) \subset \Gamma^c$, but

$$|e - (\Gamma \cap \{1 \dots 2\delta\})| = |\Gamma \cap \{1 \dots 2\delta\}| > \delta,$$

which contradicts the cardinality of Γ^c . □

Definition 8.17. Let us take a local affine scheme $(\mathcal{O}_X, \mathfrak{m})$. For an S -point $(\mathcal{O}_Y \subset \mathcal{O}_X \otimes_{\Lambda} \mathcal{O}_S) \in \text{Ter}_X$, the submonoid

$$\Gamma_Y = \left\{ n \in \mathbb{N}, \frac{\mathfrak{m}^n}{\mathfrak{m}^{n+1}} = \frac{\mathcal{O}_Y \cap \mathfrak{m}^n}{\mathcal{O}_Y \cap \mathfrak{m}^{n+1}} \right\}$$

is the associated semigroup to Y_0 .

Definition 8.18. Let $(\mathcal{O}_X, \mathfrak{m})$ be a local Λ -algebra. The conductor number $\mathbf{c}_Y$ of a $\mathbb{k}$ -point $(\mathcal{O}_{Y_0} \subset \mathcal{O}_{X_0}) \in \text{Ter}_X$ is the conductor of the associated semigroup Γ_{Y_0} .

Note that, the conductor $\mathbf{c}_Y$ in Definition 8.18 is the minimal $c \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{m}^c$ belongs to $\mathcal{O}_{Y_0}$

Lemma 8.19. For a $\mathbb{k}$ -point

$$(0 \rightarrow \mathcal{O}_{Y_0} \subset \mathcal{O}_{X_0} \rightarrow C_0 \rightarrow 0) \in \text{Ter}_X^\delta(\mathbb{k}),$$

the associated semigroup Γ_{Y_0} is numerical and its genus is less or equal to δ . If for every $n \in \mathbb{N}$, the module $\mathfrak{m}^n / \mathfrak{m}^{n+1}$ is a one-dimensional $\mathbb{k}$ -linear space, then equality holds, in particular for $\mathcal{O}_X = \mathbb{k}[[x]]$.

Proof. For every element of the complement $n \in \mathbb{N} \setminus \Gamma_{Y_0}$, there is an element $f_n \in \mathcal{O}_{X_0}$ such that $[f_n] \in C_0$ is non-zero. For $n \neq n'$, $[f_n]$ and $[f_{n'}]$ are linearly independent in C_0 . Consequently, $\delta = \dim_{\mathbb{k}} C_0 \geq |\mathbb{N} \setminus \Gamma_{Y_0}|$.

If for every $n \in \mathbb{N}$ $\mathfrak{m}^n / \mathfrak{m}^{n+1}$ has length 1, then the complement of Γ_{Y_0} is equal to the set

$$\left\{ n \in \mathbb{N}, \frac{\mathcal{O}_Y \cap \mathfrak{m}^n}{\mathcal{O}_Y \cap \mathfrak{m}^{n+1}} = 0 \right\},$$

and $[f_n]$'s form a basis of C_0 . □

We prove a generalisation of [Ish80, Proposition 2].

Theorem 8.20. *Let $(\mathcal{O}_X, \mathfrak{m})$ be a local Λ -algebra, then $\mathrm{Ter}_X^{\delta,r} = \mathrm{Ter}_{X_r}^\delta$, where $X_r = \mathrm{Spec}(\mathcal{O}_X/\mathfrak{m}^r)$. Then we get a sequence of closed immersions*

$$\mathrm{Ter}_X^{\delta,1} \subset \mathrm{Ter}_X^{\delta,2} \subset \dots \subset \mathrm{Ter}_X^{\delta,r} \subset \dots$$

Let $A \in \mathrm{Art}_\Lambda$ be a local Artinian Λ -algebra of length l . Then the sequence

$$\mathrm{Ter}_X^{\delta,1}(A) \subset \mathrm{Ter}_X^{\delta,2}(A) \subset \dots \subset \mathrm{Ter}_X^{\delta,r}(A) \subset \dots$$

stabilizes for $r \geq 2\delta l$.

Proof. Consider a point $\{\mathcal{O}_{Y_A} \rightarrow \mathcal{O}_X \otimes A\} \in \mathrm{Ter}_X^\delta(A)$. We argue that the conductor of Y_A is at most $2\delta l$, which is equivalent to the claim.

The cokernel C_A is a free A -module of rank δ , hence $\mathbb{k}$ -vector space of dimension δl . By abuse of notation, we denote the ideal $\mathfrak{m} \otimes A \subset \mathcal{O}_X \otimes A$ as $\mathfrak{m}$. We consider a filtration of C_A given by images of powers of $\mathfrak{m} \subset \mathcal{O}_X \otimes A$.

$$C_A \supset [\mathfrak{m}] \supset [\mathfrak{m}^2] \supset \dots$$

Since C_A has the length equal to δl , only that many of elements of filtration can be distinct. The set of minimal exponents i_j such that

$$C_A \supset [\mathfrak{m}^{i_1}] \supsetneq [\mathfrak{m}^{i_2}] \supsetneq \dots$$

is a filtration by distinct submodules is by definition equal to the set $\mathbb{N} \setminus \Gamma_{Y_A}$. Consequently, the set $\mathbb{N} \setminus \Gamma_{Y_A}$ has the cardinality at most δl . By Lemma 8.15, this implies that the conductor is at most $2\delta l$. $\square$

Corollary 8.21. *For $\mathcal{O}_X$ a completed local ring, Ter_X^δ is an ind-scheme given as closed subscheme of infinite Grassmannian $\mathrm{Gr}(\mathcal{O}_X, \delta)$.*

8.3 Deformations of curve singularities

The aim of this section is to show that the study of curve singularities motivates the study of the Territory scheme. The perspective of curve singularities was pursued in [Ish82],[Ham20],[BGS24] and [Boz25]. One of the key tools was the stratification by the associated semigroup, see Definition 8.24, first considered in [Ish80, Definition 10]. In this section, we show that such stratification is an example of the Białynicki-Birula decomposition, Theorem 8.28. Thanks to that, we obtain a description of local deformations tangent to stratum, Corollary 8.32.

We fix a numerical semigroup Γ of genus δ .

Definition 8.22 ([GLS07, Definition 2.44]). *Let us fix a Λ -scheme S . A flat family $\mathcal{Y}$ of reduced curves over S is equinormalizable if the normalization morphism $\eta: \overline{\mathcal{Y}} \rightarrow \mathcal{Y}$ induces the*

normalization η_s of the fibre $\mathcal{Y}_s$ over any point $s \in S$:

$$\begin{array}{ccc} \overline{\mathcal{Y}}_s & \longrightarrow & \overline{\mathcal{Y}} \\ \downarrow \eta_s & & \downarrow \eta \\ \mathcal{Y}_s & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ s & \longrightarrow & S. \end{array}$$

Definition 8.23. Let $\mathcal{Y}$ be an equinormalizable family over S . The delta invariant over a point $s \in S$ is

$$\delta(s) = \dim_{\kappa(s)}(\eta_* \mathcal{O}_{\overline{\mathcal{Y}}_s} / \mathcal{O}_{\mathcal{Y}_s}).$$

The cornerstone result in the study of equinormalizable deformations is [GLS07, Theorem 2.56], it states that equinormalizable families over a normal base is characterised by a locally constant delta invariant.

Let us consider a pointed equinormalizable family, $(\mathcal{Y}, y) \rightarrow (S, s)$, inducing $\kappa(y) \cong \kappa(s) \cong \mathbb{k}$. Assume also that $y \in \mathcal{Y}_s$ is a unibranched singular point of fibre $\mathcal{Y}_s$, in such case the preimage of y under normalization is one point and we denote it by $\bar{y}$. After an étale base change on S , we can assume that we get the affine line over S as étale neighbourhood of $\bar{y} \in \overline{\mathcal{Y}}$.

$$\begin{array}{ccccc} (\overline{\mathcal{Y}}, \bar{y}) & \longleftarrow & (U, u) & \xrightarrow{\text{ét}} & (\mathbb{A}_S^1, \bar{y}) \\ \downarrow & & & \swarrow & \\ (S, s) & & & & \end{array}$$

Since the residue field of y is $\mathbb{k}$, so is residue field of $\bar{y}$, and thus this étale neighborhood induces an isomorphism of completed local rings:

$$\widehat{\mathcal{O}_{\overline{\mathcal{Y}}, \bar{y}}} \cong \widehat{\mathcal{O}_S[x]}.$$

Since the completed local ring $\widehat{\mathcal{O}_{\overline{\mathcal{Y}}, \bar{y}}}$ injects as a subring into $\widehat{\mathcal{O}_S[x]}$ and cokernel is an finite locally free $\mathcal{O}_S$ -module, we can replace $\mathcal{Y}$ by a pointed affine S -scheme $(\mathcal{Y}', y')$, such that at completed stalks at y , $\widehat{\mathcal{O}_{\mathcal{Y}, y}} = \widehat{\mathcal{O}_{\mathcal{Y}', y'}}$ and as subalgebras of $\widehat{\mathcal{O}_S[x]}$, but $\mathcal{O}_{\mathcal{Y}', y'} \subset \mathcal{O}_S[x] = \mathcal{O}_S \otimes \Lambda[x]$ is an S -point of $\text{Ter}_{\text{Spec } \Lambda[x]}^\delta$. This way we get that the study of equinormalizable deformations of unibranched singularities is equivalent to the study of Territory of formal series of one variable. A similar approach has been pursued in [BGS24], but they consider families with fixed conductance, see [BGS24, Definition 5.4]. This is due to irregular behaviour of the Territory scheme.

Definition 8.24 ([Ish80, Definition 9]). Let us take $\Gamma \subset \mathbb{N}$ a numerical semigroup with genus δ .

$$\text{Ter}^{\delta, r, \Gamma}(S) = \left\{ (\mathcal{O}_Y \subset \mathcal{O}_X \otimes_\Lambda \mathcal{O}_S) \in \text{Ter}^{\delta, r}(S) \mid \text{the associated semigroup } \Gamma_Y = \Gamma \right\}.$$

Proposition 8.25. Let $X = \text{Spec } \mathbb{k}[[x]]$, the sequence of the closed immersions

$$\text{Ter}_X^{\delta, 1, \Gamma} \subset \text{Ter}_X^{\delta, 2, \Gamma} \subset \dots \subset \text{Ter}_X^{\delta, r, \Gamma} \subset \dots$$

stabilizes for $r \geq 2\delta$.

Proof. Let us take a point $\mathcal{O}_Y \subset \mathcal{O}_X \otimes \mathcal{O}_S$. Since, by definition, $\Gamma_Y = \Gamma$, then the conductor of $\Gamma_Y = \Gamma$ is at most 2δ , see Lemma 8.15. $\square$

Motivated by Proposition 8.25 and the fact that, by Theorem 8.20, the reduced structure of Ter_X^δ is the same as on $\text{Ter}_X^{\delta, 2\delta}$, we introduce the following notation.

Notation. The scheme $\text{Ter}_X^{\delta, 2\delta}$ is denoted by Ter^δ . We are going to fix δ , and denote it by Ter . Similarly $\text{Ter}_X^{\delta, 2\delta, \Gamma}$ is denoted by Ter^Γ .

The advantage of working with Ter^δ is that by Theorem 8.4 is represented by a projective scheme. Moreover, since $\Lambda[x]/(x^{2\delta})$ is naturally graded, by Proposition 8.9, we get a $\mathbb{G}_m$ -action on Ter^δ .

Proposition 8.26. *Let Γ be a numerical semigroup with genus δ . Then the graded algebra*

$$\mathcal{O}_S[\Gamma] := \mathcal{O}_S[x^e]_{e \in \Gamma} \subset \mathcal{O}_S[x]$$

induces a $\mathbb{G}_m$ -fixed S -point of Ter . The $\mathbb{G}_m$ -fixed point locus $\text{Ter}^{\mathbb{G}_m} = \bigsqcup_H \text{Spec}(\Lambda)_H$ is indexed by semigroups with genus δ .

Proof. The first part of the statement is immediate. For the second part, let us take S to be a local scheme and consider an S -point of Ter .

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_S[x] \rightarrow C_S \rightarrow 0,$$

the grading is given by x , since C_S is a free $\mathcal{O}_S$ -module, and hence $\mathcal{O}_Y$ is generated by monomials of x , which means that $\mathcal{O}_Y = \mathcal{O}_S[\Gamma]$ as a subalgebra of $\mathcal{O}_S[x]$. This means that $\mathcal{O}_Y$ factors through $\text{Spec}(\Lambda)$ -point given by

$$\begin{aligned} \text{Spec } \Lambda &\rightarrow \text{Ter} \\ \{ \Lambda[\Gamma] \subset \Lambda[x] \}. \end{aligned}$$

$\square$

Having described the fixed point locus, we proceed with the Białynicki-Birula decomposition. Recall that, from Definition 3.21, the decomposition is given by the formula

$$\text{Ter}^+(S) = \left\{ \mathcal{O}_Y^h \rightarrow \frac{\mathcal{O}_S[x][t^{-1}]}{(x^{2\delta})} \in \text{Ter}(\mathbb{A}_S^1) \mid \mathcal{O}_Y^h \text{ is a graded subalgebra of } \frac{\mathcal{O}_S[x][t^{-1}]}{(x^{2\delta})} \right\}$$

The projection $\pi: \text{Ter}^+ \rightarrow \text{Ter}^{\mathbb{G}_m}$ is given by restricting $\mathbb{A}^1$ to the origin. On the level of algebras the image is given by taking the tensor product with $\Lambda[t^{-1}]/(t)$.

The forgetful map $\iota: \text{Ter}^+ \rightarrow \text{Ter}$ is given by restricting $\mathbb{A}^1$ to 1. On the level of algebras it is given by tensoring with $\Lambda[t^{-1}]/(t-1)$.

Definition 8.27. *The Białynicki-Birula cell centred at $\Lambda[\Gamma]$, see Definition 3.26, is denoted by $\text{Ter}^{+, \Gamma}$. By definition, it is a pullback*

$$\begin{array}{ccc} \text{Ter}^{+, \Gamma} & \longrightarrow & \text{Ter}^+ \\ \downarrow & & \downarrow \pi \\ \text{Spec}(\Lambda)_\Gamma & \longrightarrow & \bigsqcup_H \text{Spec}(\Lambda)_H. \end{array}$$

Conceptually, the elements of $\text{Ter}^{+, \Gamma}(S)$ are filtered subalgebras with the associated graded equal to $\mathcal{O}_S[\Gamma]$.

Theorem 8.28. *There is an isomorphism of functors*

$$\text{Ter}^{\Gamma} \cong \text{Ter}^{+, \Gamma}$$

Proof. Let us take an S -point of Ter^{Γ} .

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_S[x]/x^{2\delta} \rightarrow C_S \rightarrow 0 \quad (8.29)$$

The algebra $\mathcal{O}_Y$ admits a filtration $\mathcal{F}$ by flat $\mathcal{O}_S$ -modules induced by vanishing order on $\Lambda[x]/x^{2\delta}$, see Definition 4.48. The say that $\Gamma_Y = \Gamma$ is the same as to say the semigroup of the associated graded algebra $\text{gr}_{\mathcal{F}} \mathcal{O}_Y$ is Γ . But since $\text{gr}_{\mathcal{F}} \mathcal{O}_Y$ is graded, by Proposition 8.26, it is isomorphic to $\mathcal{O}_S[\Gamma]$. We construct a Rees algebra associated of $\mathcal{O}_Y$ to $\mathcal{F}$

$$\mathcal{R}(\mathcal{O}_Y, \mathcal{F}) = \bigoplus t^{-i} \mathcal{F}_i \subset \frac{\mathcal{O}_S[x][t^{-1}]}{x^{2\delta}}.$$

By Lemma A.1, there is an equivalence of categories

$$\begin{aligned} \text{Mod}_{(\mathcal{O}_Y, \mathcal{F})}^{filt} &\rightarrow \text{Mod}_{\mathcal{R}(\mathcal{O}_Y, \mathcal{F})}^{gr}, \\ (M, \mathcal{F}_i) &\mapsto \bigoplus_{i \in \mathbb{Z}} t^{-i} \mathcal{F}_i. \end{aligned}$$

Hence, flatness of the cokernel of Rees algebra over $\mathcal{O}_S[t^{-1}]$ follows from flatness of C_S over $\mathcal{O}_S$. We get a point in Ter^+ , since the associated graded algebra is $\mathcal{O}_S[\Gamma]$, this point lies in $\text{Ter}^{+, \Gamma}$.

Conversely, if we take an S -point of $\text{Ter}^{+, \Gamma}$

$$0 \rightarrow \mathcal{O}_Y^h \rightarrow \mathcal{O}_S[x][t^{-1}]/x^{2\delta} \rightarrow C_S^h \rightarrow 0, \quad (8.30)$$

taking the image by ι , recovers the point in Ter^{Γ} , of which $\mathcal{O}_Y^h$ is the Rees algebra. $\square$

Corollary 8.31 ([Ish80, Theorem 3, Theorem 4]). *The functor Ter^{Γ} is represented by an affine locally closed subscheme of Ter . If the point associated with the subalgebra $\mathbb{k}[\Gamma] \subset \mathbb{k}[x]$ is smooth on Ter^{Γ} , then Ter^{Γ} is an affine space.*

Proof. All the claims follow from standard properties of the Białyński-Birula decomposition, Theorem 3.24. The second part follows from the classical Białyński-Birula decomposition [JS19, Theorem 1.5]. $\square$

Corollary 8.32. *Let Γ be a numerical semigroup of genus δ , we construct an exact sequence of $\mathbb{k}$ -vector spaces*

$$0 \rightarrow \mathbb{k}[\Gamma] \rightarrow \mathbb{k}[x] \rightarrow C_{\Gamma} \rightarrow 0.$$

This sequence is a $\mathbb{k}$ -point in $\text{Ter}_{\text{Spec}(\mathbb{k}[x])}^{\Gamma}$. The local functor at this $\mathbb{k}$ -point admits an associated triple

$$0, T^0(\mathbb{k}[\Gamma], C_{\Gamma})_{>0}, (T^1(\mathbb{k}[\Gamma], C_{\Gamma})_{>0}$$

in particular the functor is prorepresentable.

Proof. This is a consequence of Corollary 8.11. There is a triple

$$0, T^0(\mathbb{k}[\Gamma], C_\Gamma)_{\geq 0}, (T^1(\mathbb{k}[\Gamma], C_\Gamma)_{\geq 0}$$

by Proposition 8.26, the fixed point locus consists of isolated reduced points, the 0-th degree tangent space vanishes. $\square$

8.4 Wilf conjecture

In the previous sections we have developed fundamentals of the theory of Territory functors, now we switch to applications.

This section is devoted to explore the link between the study of the Territory functor and combinatorics, mainly to connect it to the Wilf conjecture. The conjecture was posed in 1978 by Wilf in [Wil78]. Since then a lot of effort was made to solve it, see for example [Sam12], [MS15] [Eli18], [EMA21], [DEF25], [SS25], to name a few, but the general problem remains open.

Notation. Fix $\Lambda = \mathbb{k}$, fix a positive integer δ and X to be $\text{Spec } \mathbb{k}[x]$. In this section, we omit the subscripts and superscripts in Ter_X^δ and just write Ter .

We begin by setting up the notation for standard notions in the study of numerical semi-groups.

Definition 8.33. The Frobenius number F_Γ of a numerical semigroup Γ is equal to its conductor number minus one.

Definition 8.34. Let $(\mathcal{O}_Y \subset \mathcal{O}_X \otimes \mathcal{O}_S) \in \text{Ter}(S)$. Then the Frobenius number of the point $\mathcal{O}_Y$ is the rank of the $\mathcal{O}_S$ -module

$$\mathcal{O}_X \otimes \mathcal{O}_S / (\mathfrak{m}^{\mathbf{c}_Y}),$$

where $\mathbf{c}_Y$ is the conductor number of Y .

Definition 8.35. The conductor $\mathfrak{C}_Y$ of an S -point $(0 \rightarrow \mathcal{O}_Y \subset \mathcal{O}_X \otimes \mathcal{O}_S \rightarrow C_S \rightarrow 0)$ is the kernel

$$0 \rightarrow \mathfrak{C}_Y \rightarrow \mathcal{O}_Y \rightarrow \text{Hom}_{\mathcal{O}_S}(C_S, C_S).$$

In general $\mathfrak{m}^{\mathbf{c}_Y}$ and $\mathfrak{C}_Y$ are not the same, see [BGS24, Example 2.15], this motivates the notion of Territory with fixed conductance see Definition 8.38.

Conjecture 2 ([Wil78]). *[Wilf conjecture] Let $\eta(\Gamma)$ be number of integers in Γ smaller than Frobenius number F_Γ . Let $\text{edim}(\Gamma)$ denote the minimal number of generators of Γ . Then the following inequality holds*

$$F_\Gamma \leq \text{edim}(\Gamma) \cdot \eta(\Gamma). \quad (8.36)$$

Let $S = \text{Spec}(\mathbb{k})$, $\mathcal{O}_X = \mathbb{k}[x]$ and $\mathcal{O}_Y = \mathbb{k}[\Gamma]$. Then the Wilf inequality (8.36) is equivalent to

$$\text{length}(\mathcal{O}_X \otimes \mathcal{O}_S / \mathfrak{C}) \leq \text{rk}(\Omega_Y) \text{length}(\mathcal{O}_Y / \mathfrak{C}), \quad (8.37)$$

over $\mathbb{k}$ the rank $\text{rk}(\Omega_Y)$ is the dimension of tangent space to Y at the closed point, hence it is the embedding dimension of Y . Let us take $\mathcal{O}_Y$ to be an arbitrary local Cohen-Macaulay ring of

dimension one, and $\mathcal{O}_X$ be the integral closure of $\mathcal{O}_Y$ in the total ring of fractions. It was asked [MS15, Question 9], what are the assumptions on Y making the inequality (8.37) to hold.

In Theorem 8.44, we partially answer the question, we show that whether the inequality holds for all integral $\mathbb{k}$ -algebras is equivalent to the Wilf conjecture.

Definition 8.38 ([BGS24, Definition 2.15]). The Territory with fixed conductance $\mathbf{c}$ is defined on S -points as

$$\mathrm{Ter}^{\mathbf{c}}(S) := \left\{ 0 \rightarrow \mathcal{O}_Y \subset \mathcal{O}_X \otimes \mathcal{O}_S \rightarrow C_S \rightarrow 0 \in \mathrm{Ter}(S) \mid \begin{array}{l} \text{coker}(\mathcal{O}_Y \rightarrow \mathrm{Hom}_{\mathcal{O}_S}(C_S, C_S)) \\ \text{is locally free of rank } \delta^2 - \mathbf{c} + \delta \end{array} \right\}.$$

Lemma 8.39. A S -point Y of Ter is a point of $\mathrm{Ter}^{\mathbf{c}}(S)$ if and only if $\mathfrak{C}_Y = \mathbf{m}^{\mathbf{c}}$.

Proof. Since the inclusion $\mathbf{m}^{\mathbf{c}_Y} \subset \mathfrak{C}_Y$ is always satisfied, we get the factorization

$$\begin{array}{ccc} & \mathcal{O}_Y / (\mathbf{m}^{\mathbf{c}_Y}) & \\ \nearrow & & \searrow \\ \mathcal{O}_Y & \xrightarrow{\quad\quad\quad} & \mathrm{Hom}_{\mathcal{O}_S}(C_S, C_S), \end{array}$$

hence, the equality $\mathfrak{C}_Y = \mathbf{m}^{\mathbf{c}_Y}$ holds on the $\mathrm{Ter}^{\mathbf{c}}(S)$ as the ranks agree.

Let us assume that $\mathfrak{C}_Y = \mathbf{m}^{\mathbf{c}}$ holds. On any $\mathbb{k}$ -point Y_0 the equality $\mathfrak{C}_{Y_0} = \mathbf{m}^{\mathbf{c}_{Y_0}}$ holds, hence, by local criterion for flatness [Eis95, Theorem 6.8], the cokernel is locally free. Since the rank of $\mathcal{O}_Y / \mathbf{m}^{\mathbf{c}}$ over $\mathcal{O}_S$ is $\mathbf{c} - \delta$, we get the rank of the cokernel as in Definition 8.38. $\square$

Lemma 8.40. The stratification by the semigroup $\bigsqcup \mathrm{Ter}^{\Gamma} \rightarrow \mathrm{Ter}$ is finer than the stratification by the conductance $\bigsqcup \mathrm{Ter}^{\mathbf{c}} \rightarrow \mathrm{Ter}$.

Proof. By Lemma 8.39, it is enough to check that $\mathfrak{C} = \mathbf{m}^{\mathbf{c}}$. Recall that $\mathbf{m}^{\mathbf{c}} \subset \mathfrak{C}_Y$.

Let us take an S -point on Ter^{Γ}

$$0 \rightarrow \mathcal{O}_Y \subset \mathcal{O}_S[x] / (x^{2\delta}) \rightarrow C_S \rightarrow 0.$$

This is an exact sequence of free filtered $\mathcal{O}_S$ -modules, the filtration $\mathcal{F}$ comes from vanishing order on $\mathcal{O}_X[x] / (x^{2\delta})$, see Definition 4.48. By Theorem 8.28, when we take the associated graded modules we get

$$0 \rightarrow \mathcal{O}_S[\Gamma] \rightarrow \mathcal{O}_S[x] / (x^{2\delta}) \rightarrow \mathrm{gr}_{\mathcal{F}} C_S \rightarrow 0.$$

By definition $\mathfrak{C}$ is the kernel of

$$\mathcal{O}_Y \rightarrow \mathrm{Hom}_{\mathcal{O}_S}(C_S, C_S).$$

Let us take an element $f \in \mathcal{O}_Y$ such that the multiplication map $C_S \xrightarrow{f} C_S$ is zero. Then on the associated graded modules this multiplication still vanishes

$$\mathrm{gr}_{\mathcal{F}} C_S \xrightarrow{[f]} \mathrm{gr}_{\mathcal{F}} C_S.$$

But the module $\mathrm{gr}_F C_S$ is the cokernel of

$$0 \rightarrow \mathcal{O}_S[\Gamma] \rightarrow \mathcal{O}_S[x]/(x^{2\delta}) \rightarrow \mathrm{gr}_F C_S \rightarrow 0,$$

and the conductor of $\mathcal{O}_S[\Gamma]$ is just $\mathfrak{m}^c$. $\square$

Definition 8.41. *We define the Wilf locus of genus δ .*

$$\mathrm{Wilf}^\delta(S) = \left\{ (\mathcal{O}_Y \subset \mathcal{O}_X \otimes \mathcal{O}_S) \in \bigsqcup_{\Gamma} \mathrm{Ter}^\Gamma \mid \mathrm{length}(\mathcal{O}_X \otimes \mathcal{O}_S / \mathfrak{C}) \leq \mathrm{rk}(\Omega_Y) \mathrm{length}(\mathcal{O}_Y / \mathfrak{C}) \right\}$$

The elements of Wilf locus we call Wilf subalgebras.

Theorem 8.42. *The Wilf locus is a locally closed $\mathbb{G}_m$ -invariant subscheme of Ter .*

Proof. By Theorem 8.28, we have an equality $\mathrm{Ter}^\Gamma = \mathrm{Ter}^{+, \Gamma}$. By the general theory of Białynicki Birula cells, Theorem 3.24, $\mathrm{Ter}^{+, \Gamma}$ is a locally closed subscheme of Ter . The only thing to check is that the inequality (8.37)

$$\mathrm{length}(\mathcal{O}_X \otimes \mathcal{O}_S / \mathfrak{C}) \leq \mathrm{edim}(\mathcal{O}_Y) \mathrm{length}(\mathcal{O}_Y / \mathfrak{C})$$

is a locally closed condition. By Lemma 8.40, the equality $\mathfrak{C} = \mathfrak{m}^c$ holds, hence $\mathrm{length}(\mathcal{O}_X \otimes \mathcal{O}_S / \mathfrak{C}) = F_\Gamma$, and $\mathrm{length}(\mathcal{O}_Y / \mathfrak{C}) = F_\Gamma - \delta$, and the inequality is equivalent to

$$F_\Gamma \leq \mathrm{rk}(\Omega_Y)(F_\Gamma - \delta). \quad (8.43)$$

The inequality (8.43) gives a closed condition, by the upper-semicontinuity of the rank. $\square$

Conjecture 3 (The geometric Wilf conjecture). *The Wilf locus agrees with Ter on the reduced structure level. In particular every complete local $\mathbb{k}$ -algebra with normalization $\mathbb{k}[[x]]$ is a Wilf subalgebra.*

Theorem 8.44. *The geometric Wilf conjecture is equivalent to the Wilf conjecture.*

Proof. We prove that if the Wilf conjecture holds then the geometric Wilf conjecture holds also. The Wilf conjecture states that the $\mathbb{G}_m$ -fixed points are Wilf. We consider the inequality (8.43), if we prove that this condition is open inside Ter^Γ , then the condition is satisfied in some open neighborhoods of the fixed points. By Theorem 8.42, the Wilf locus is $\mathbb{G}_m$ -equivariant, hence the whole Ter^Γ is contained in Wilf.

To show openness of the condition (8.43) it is enough to prove that the minimal embedding dimension of $\mathbb{k}$ -points on Ter^Γ is achieved at $\mathbb{k}[\Gamma]$. The embedding dimension of $\mathbb{k}[\Gamma]$ is the minimal number of generators of Γ , but since every other $\mathbb{k}$ -point has the same associated semigroup, it has to be generated by at least that many elements. $\square$

8.5 Lifting to characteristic 0

We propose a strategy to find a curve singularity that does not lift to characteristic 0, which is based on the one used for Calabi-Yau threefolds in [Eke04, Lemma 5.7]. The strategy is, in the first step, to find examples of curve singularities that do not lift on Territory scheme modulo p^2 .

Having methods to produce such examples, the next step would be to get singularities that do not lift abstractly modulo p^2 . Their classes would be a good source of potential candidates for non-liftability to characteristic 0. The reasoning for that is presented on this diagram, which also summarizes this section.

$$\begin{array}{ccc}
Y \text{ liftable with a smooth base, Definition 8.48} & \implies & Y \text{ liftable, Definition 8.47} \\
\updownarrow \text{Proposition 8.49} & & \\
Y \text{ liftable over } W(\mathbb{k}) & \xleftrightarrow{\text{Definition 8.46}} & Y \text{ liftable over } W_k(\mathbb{k}) \ \forall k \\
& & \downarrow \\
Y \text{ liftable mod } p^2 \text{ on Ter, Definition 8.51} & \implies & Y \text{ liftable mod } p^2, \text{ Definition 8.50.}
\end{array} \tag{8.45}$$

In this section we fix $\mathbb{k}$ to be an algebraically close field of characteristic p .

Definition 8.46. The ring of Witt vectors of length n $W_n(\mathbb{k})$ is the ring characterised by the following properties

- $W_n(\mathbb{k})$ is flat over $\mathbb{Z}/p^n$,
- $W_n(\mathbb{k})/p \cdot W_n(\mathbb{k}) \cong \mathbb{k}$.

The ring of Witt vectors $W(\mathbb{k})$ of $\mathbb{k}$ is the colimit

$$W(\mathbb{k}) := \varprojlim W_n(\mathbb{k}).$$

Definition 8.47. Let us take a scheme Y over $\mathbb{k}$. A lifting to characteristic zero is a flat morphism $\mathcal{Y} \rightarrow S$ such that (S, s) is a pointed integral $\mathbb{Z}$ -scheme of characteristic 0. We assume that residue field of S at s isomorphic to $\mathbb{k}$ and that

$$\begin{array}{ccc}
Y & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\{s\} & \longrightarrow & S
\end{array}$$

is a Cartesian diagram. A scheme Y is liftable if there is a lifting $\mathcal{Y}$ of Y to characteristic zero.

Definition 8.48. Let Y be a scheme over $\mathbb{k}$, we say that Y is liftable with a smooth base if there is a lifting of Y

$$\begin{array}{ccc}
Y & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\{s\} & \longrightarrow & S
\end{array}$$

with the base S being a separated scheme, formally smooth over $\mathbb{Z}$.

Proposition 8.49. Let us assume that Y is liftable with a smooth base, then there is a lifting with the spectrum of ring of Witt vectors $W(\mathbb{k})$ as the base.

Proof. Let us take a lifting of Y with a smooth base S

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ \{s\} & \longrightarrow & S, \end{array}$$

by Definition 8.48, $S \rightarrow \operatorname{Spec}(\mathbb{Z})$ is formally smooth. We specialise $\mathcal{Y}$ to the completed local ring of S at s , $(\widehat{\mathcal{O}}_{S,s}, \mathfrak{m})$. Since, by Definition 8.46, $W_1(\mathbb{k}) \cong \mathbb{k}$, we get a homomorphism $\widehat{\mathcal{O}}_{S,s} \rightarrow W_1(\mathbb{k})$. The ring $R := (\widehat{\mathcal{O}}_{S,s}, \mathfrak{m})$ admits a formally smooth homomorphism $\mathbb{Z} \rightarrow \widehat{\mathcal{O}}_{S,s}$. We proceed inductively on n . We construct the diagram

$$\begin{array}{ccc} \widehat{\mathcal{O}}_{S,s} & \longrightarrow & W_{n-1}(\mathbb{k}) \\ \uparrow & \searrow & \uparrow \\ \mathbb{Z} & \longrightarrow & W_n(\mathbb{k}), \end{array}$$

and by formal smoothness get that the dotted arrow exists. Thus, we get a homomorphism $\widehat{\mathcal{O}}_{S,s} \rightarrow W(\mathbb{k})$. We pullback the family $\mathcal{Y}$ by the induced morphism of affine schemes to $W(\mathbb{k})$, and get the claim. $\square$

Definition 8.50. We say that a scheme Y is liftable modulo p^2 , if there is a flat $W_2(\mathbb{k})$ family $\mathcal{Y} \rightarrow \operatorname{Spec} W_2(\mathbb{k})$ such that there is a Cartesian diagram

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{Y} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \operatorname{Spec} W_2(\mathbb{k}). \end{array}$$

Definition 8.51. We say that a $\mathbb{k}$ -point $(\mathbb{A}^1 \rightarrow Y) \in \operatorname{Ter}_{\mathbb{A}^1}^\delta$ is liftable modulo p^2 if it factors through a $W_2(\mathbb{k})$ -point of $\operatorname{Ter}_{\mathbb{A}^1 W_2(\mathbb{k})}^\delta$.

Proposition 8.52. Let $(\mathcal{O}_Y \subset \mathcal{O}_X) \in \operatorname{Ter}_{\mathbb{A}^1}(\mathbb{k})$. There is an element $\mathfrak{o}_{p^2}$ in the obstruction space $T^1(\mathcal{O}_Y, C)$ such that vanishing is equivalent to existence of the lift modulo p^2 . Moreover, the image of $\mathfrak{o}_{p^2}$ in T_Y^2 is the obstruction to lifting the scheme Y modulo p^2 .

Proof. This is a direct consequence of functoriality of Lichtenbaum-Schlessinger functors. Let $\Lambda = W_2(\mathbb{k})$, the sequence

$$0 \rightarrow pW_2(\mathbb{k}) \rightarrow W_2(\mathbb{k}) \rightarrow \mathbb{k} \rightarrow 0$$

is a small extension in $\operatorname{Art}_\Lambda$. There is a forgetful morphism $\phi: \operatorname{ter}_Y \cong \operatorname{Def}_{\mathbb{A}^1 \setminus Y} \rightarrow \operatorname{Def}_Y$, see Theorem 4.43. By Definition 4.17, this morphism induces the following commutative diagram

$$\begin{array}{ccccc} \operatorname{ter}_Y(W_2(\mathbb{k})) & \longrightarrow & \operatorname{ter}_Y(\mathbb{k}) & \xrightarrow{\circ} & T^1(\mathcal{O}_Y, C) \\ \downarrow \phi & & \downarrow \phi & & \downarrow \\ \operatorname{Def}_Y(W_2(\mathbb{k})) & \longrightarrow & \operatorname{Def}_Y(\mathbb{k}) & \longrightarrow & T_Y^2. \end{array} \tag{8.53}$$

If Y is liftable on Ter then we take $\mathfrak{o}_{p^2}$ to be 0, the image of the lifting under ϕ gives us an

abstract lifting. If Y is non-liftable, then we put $\mathfrak{o}(Y) = \mathfrak{o}_p^2$, by commutativity of Diagram 8.53, we get that its image in T_Y^2 is the obstruction to lifting Y modulo p^2 . $\square$

In Appendix B we present a Macaulay2 code checking the vanishing of the obstruction $\mathfrak{o}_{p^2}$.

Chapter 9

Territory with the multiplicity bounded from below

9.1 Introduction

One of the major problems in studying the Territory scheme is that it is notoriously non-reduced, even if we fix the genus $\delta = 1$ we get a non-reduced structure, see [Ish80, Remark II]. In [Ish80] this difficulty is avoided exactly by restricting to the reduced structure. In [BGS24], in order not cut the part of the non-reduced structure that admits some geometric meaning, they stratify the Territory by locally closed subschemes parametrised by certain combinatorial data, see [BGS24, Definition 2.15].

Our approach is different. Instead of trying to replace the moduli for one that is easier to control, we try to understand the full local structure. It is feasible since the local functor is prorepresentable, see Corollary 8.8.

Given a subalgebra $\mathcal{O}_Y \subset \mathbb{k}[x]$, its multiplicity is the minimal non-zero valuation obtained in $\mathcal{O}_Y$. We define the functor of the Territory with the multiplicity bounded from below by m , to parametrise the subalgebras with multiplicity at least m , see Definition 9.6. Since every non-trivial subalgebra of $\mathbb{k}[x]$ has multiplicity greater equal than 2, the usual Territory functor and bounded by 2 are isomorphic on $\mathbb{k}$ -points. We show that the bounded variant at least partially omits the problem with non-reducedness.

Theorem 9.1 (Theorem 9.8). *Let δ be a positive integer. The functor $\mathrm{Ter}_{\mathbb{k}[[x]],2}^\delta$ admits a generically reduced irreducible projective component.*

This is important since unlike on the Hilbert scheme of points, where one can study the smoothable component, there is no obvious generically reduced component. No matter what genus δ we choose there is no reduced $\mathbb{k}$ -point.

Theorem 9.2 (Theorem 9.16). *Let δ be a positive integer. The functor $\mathrm{Ter}_{\mathbb{k}[x]}^\delta$ is nowhere reduced.*

Theorem 9.2 was expected since in the simplest example $\delta = 1$ the Territory is a single non-reduced point, but as far as author is aware general argument was not presented before.

In the second section of this chapter we study local structures of bounded Territory scheme. The local functor with the multiplicity bounded by m at a point $\mathcal{O}_Y \subset \mathbb{k}[x]$ is denoted $\mathrm{ter}_{Y,m}$. The main result in this chapter is the structural comparison.

Theorem 9.3 (Theorem 9.22). *Let Y be a spectrum of a co-finite subalgebra $\mathcal{O}_Y \subset \mathbb{k}[x]$ with multiplicity at least m . We get the following Cartesian diagram of good deformation functors*

$$\begin{array}{ccc} \mathrm{ter}_{Y,m} & \longrightarrow & \mathrm{ter}_{Y,m-1} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \mathrm{Der}(\mathcal{O}_{Y,-}), \end{array} \quad (9.4)$$

where $\mathrm{Der}(\mathcal{O}_{Y,-})$ is a functor that does depend only on the ring $\mathcal{O}_Y$, precise description of this functor is given in Definition 9.17. As a corollary, we obtain a refinement of Theorem 9.2.

Corollary 9.5 (Corollary 9.26). *Let $X = \mathrm{Spec} \mathbb{k}[[x]]$. Let $V \subset \mathrm{Ter}_{X,k}$ be an irreducible component, such that the general $\mathbb{k}$ -point has multiplicity at least m , for $k < m$. Then V is generically non-reduced.*

9.2 Generalities on the bounded Territory

Definition 9.6. *Let $(\mathcal{O}_X, \mathfrak{m})$ be a complete local ring, free over Λ . The Territory with the multiplicity bounded from below by m is defined as*

$$\mathrm{Ter}_{X,m}^\delta = \mathrm{Ter}_{\Lambda \oplus \mathfrak{m}^m}^{\delta-t},$$

where t is rank of the free Λ -module $\mathcal{O}_X/(\Lambda \oplus \mathfrak{m}^m)$.

By Lemma 8.3, the functor $\mathrm{Ter}_{X,m}^\delta$ is a closed subfunctor of Ter_X^δ .

Definition 9.7. *A point in Ter_X^δ that belongs to $\mathrm{Ter}_{X,m}^\delta$ is called a subalgebra with multiplicity at least m .*

A $\mathbb{k}$ -point $\mathcal{O}_{Y_0} \subset \mathcal{O}_{X_0}$ is a subalgebra with multiplicity at least m , if the maximal ideal of $\mathcal{O}_{Y_0}$, $\mathfrak{m}_{Y_0}$ is contained in $\mathfrak{m}^m$. If $X = \mathrm{Spec} \mathbb{k}[[x]]$ then it means that $\mathcal{O}_{Y_0}$ is generated by elements of valuation greater or equal to m .

We prove, in Theorem 9.16, that the usual Territory $\mathrm{Ter}_{\mathbb{k}[[x]]}^\delta$ is nowhere reduced, by the fact that there are negative tangents with values of valuation 1. If we consider $\mathrm{Ter}_{\mathbb{k}[[x]],2}^\delta$ we get a "tamer" moduli space, for example it contains generically reduced components.

Theorem 9.8. *The functor $\mathrm{Ter}_{\mathbb{k}[[x]],2}^\delta$ admits a generically reduced irreducible projective component.*

Proof. Let us consider the semigroup $\Gamma := \langle 2, 2\delta + 1 \rangle \subset \mathbb{N}$. The algebra $\Lambda[[\Gamma]] \subset \Lambda[[x]]$ is associated with $\mathbb{G}_m$ -fixed point on $\mathrm{Ter}_{\mathbb{k}[x]}^\delta$. Since the inclusion $\Lambda[x^2, x^3] \subset \Lambda[x]$ is homogeneous, we get a $\mathbb{G}_m$ -fixed point on $\mathrm{Ter}_{\mathbb{k}[x],2}^\delta$. Let us consider the Białynicki-Birula cell Ter^Γ . We show that the point $\mathbb{k}[\Gamma] \subset \mathbb{k}[x]$ is smooth on Ter^Γ . The obstruction space is, by Corollary 8.11, equal to $T^1(\mathbb{k}[\Gamma], \mathbb{k}[x^2, x^3]/\mathbb{k}[\Gamma])_{>0}$. The algebra is generated by x^2 and $x^{2\delta+1}$, this gives us a presentation

$$0 \rightarrow I \rightarrow \mathbb{k}[e_1, e_2] \rightarrow \mathbb{k}[\Gamma] \rightarrow 0,$$

the degrees of syzygies in I are greater than $2\delta - 1$, which is the maximal degree of $\mathbb{k}[x^2, x^3]/\mathbb{k}[\Gamma]$. By [Har10, Proposition 3.10], we get a homogeneous surjection

$$\mathrm{Hom}_{\mathbb{k}[\Gamma]}(I/I^2, \mathbb{k}[x^2, x^3]/\mathbb{k}[\Gamma]) \rightarrow T^1(\mathbb{k}[\Gamma], \mathbb{k}[x^2, x^3]/\mathbb{k}[\Gamma]),$$

hence the positive obstructions vanish. Therefore the point is smooth and, by Corollary 8.31, we get that Ter^Γ is isomorphic to affine space $\mathbb{A}^\delta$. We see that on $\mathbb{k}$ -points in Ter^Γ are described as:

$$\mathbb{k}[[x^2 + \sum_{\substack{i < 2\delta \\ 2 \nmid i}} a_i x^i]] \oplus (x^{2\delta}) \subset \mathbb{k}[[x]].$$

We show that $\mathrm{Ter}^\Gamma \subset \mathrm{Ter}$ is an open immersion. By Corollary 8.8, the tangent space to Ter at $[\Gamma]$ is isomorphic to subspace of $\mathrm{Der}(\mathbb{k}[\Gamma], \mathbb{k}[x^2, x^3]/\mathbb{k}[\Gamma])$ consisting of those derivations that send the ideal $(x^{2\delta})$ to 0. A derivation from $\mathbb{k}[\Gamma]$ is determined by the image of x^2 , since in $\mathbb{k}[x^2, x^3]/\mathbb{k}[\Gamma]$ every non-zero element has valuation greater than 2, $\mathrm{ter}_{\mathbb{k}[\Gamma]}^{<0} = 0$. Thus from the general theory of Białynicki-Birula cells, Theorem 3.29, we get an open immersion. $\square$

In Theorem 9.8 we have used example from [Ish80, Corollary 5], where it was already shown that the considered cell is an affine space.

Let us consider a curve Y with a unibranched singularity. Assume that the normalization is the affine line $\mathbb{A}^1$. We can also assume that the origin of $\mathbb{A}^1$ is mapped to the singularity. Then the normalization morphism is an isomorphism outside of the origin, and contracts an infinitesimal neighbourhood of $\mathbb{A}^1$, such a neighbourhood contains the tangent vector. Consequently, the normalization factors through cuspidal curve $\mathbb{A}_{2,3}^1 := \mathrm{Spec} \Lambda[x^2, x^3]$, as on the diagram below. This is the same as to say that $\mathrm{Ter}_{\mathbb{k}[x]}^\delta$ agrees on $\mathbb{k}$ -points with $\mathrm{Ter}_{2, \mathbb{k}[x]}^\delta$.

We get the following diagram

$$\begin{array}{ccc} & \mathfrak{t} & \\ & \downarrow & \\ & \mathbb{A}^1 & \\ \swarrow & & \downarrow \\ \mathbb{A}_{2,3}^1 & & Y, \end{array} \tag{9.9}$$

where $\mathfrak{t}$ is $\mathrm{Spec} \mathbb{k}[\varepsilon]/(\varepsilon^2)$. There is a chain of morphism of good deformation functors induced by Diagram (9.9)

$$\mathrm{Def}_{\mathbb{A}_{2,3}^1 \setminus Y} \rightarrow \mathrm{Def}_{\mathbb{A}^1 \setminus Y} \rightarrow \mathrm{Def}_{\mathfrak{t} \setminus Y},$$

where $\mathrm{Def}_{\mathfrak{t} \setminus Y}$ is induced by the trivial map $\mathfrak{t} \rightarrow Y$. By Theorem 8.6, $\mathrm{Def}_{\mathbb{A}_{2,3}^1 \setminus Y}$ and $\mathrm{Def}_{\mathbb{A}^1 \setminus Y}$ are the local functors of the respective Territory functors. Our aim here is to understand how these local structures compare. This situation described above, when the bound on the multiplicity is 2, is distinguished since every $\mathbb{k}$ -point of $\mathrm{Ter}_{\mathbb{k}[x]}^\delta$ has multiplicity at least 2. From now on, we deal with general case and specialise back to $m = 2$ again in Theorem 9.16.

Definition 9.10. *Let $(Y, *)$ be a pointed $\mathbb{k}$ -scheme. The functor of deformations of a trivial tangent vector, denoted $\mathrm{Def}_{\mathfrak{t} \rightarrow Y}$, is the functor of deformations of the trivial map $\mathfrak{t} \rightarrow Y$. We*

can also consider the functor of deformations of fixed trivial tangent vector $\text{Def}_{\mathfrak{t} \setminus Y}$.

Definition 9.11. The local functor of $\text{Ter}_{\mathbb{k}[[x]],m}^\delta$ at the point given by Y is denoted by $\text{ter}_{Y,m}$.

There is a natural map $\text{Def}_{* \rightarrow Y} \rightarrow \text{Def}_{\mathfrak{t} \setminus Y}$ induced by the structural map $\mathfrak{t} \rightarrow *$.

Lemma 9.12. Let Y be a $\mathbb{k}$ -point of Ter_X^δ of multiplicity at least m . We get the following Cartesian diagram of good deformation functors

$$\begin{array}{ccc} \text{ter}_{Y,m} & \longrightarrow & \text{ter}_{Y,m-1} \\ \downarrow & & \downarrow \\ \text{Def}_{* \rightarrow Y} & \longrightarrow & \text{Def}_{\mathfrak{t} \setminus Y}. \end{array} \quad (9.13)$$

Proof. First, we describe the maps in Diagram (9.13).

Recall definition of the top horizontal map from Lemma 8.3.

We take the map

$$\text{Spec}(\Lambda \oplus \mathfrak{m}^m) \rightarrow Y$$

and compose it with the pointed structure $* \rightarrow \text{Spec}(\Lambda \oplus \mathfrak{m}^m)$ to get the left vertical map. Similarly, we take the map

$$\text{Spec}(\Lambda \oplus \mathfrak{m}^{m-1}) \rightarrow Y$$

and compose it with the trivial map $\text{Spec} \mathbb{k}[\varepsilon]/(\varepsilon^2) \rightarrow \text{Spec}(\Lambda \oplus \mathfrak{m}^{m-1})$ to get the other vertical map.

Second, let us check that the diagram is commutative. Let us take $B \in \text{Art}_\Lambda$ and B -point of $\text{ter}_{Y,m}$.

$$0 \rightarrow \mathcal{O}_{Y_B} \rightarrow B \oplus (x^m) \rightarrow C_m \rightarrow 0.$$

The image of this point by the composition of morphism through $\text{ter}_{Y,m-1}^\delta$ in $\text{Def}_{\mathfrak{t} \setminus Y}$ is

$$\mathcal{O}_{Y_B} \rightarrow B \oplus (x^{m-1})/(x^m) \cong B[\varepsilon]/(\varepsilon^2).$$

Since the algebra $\mathcal{O}_{Y_B}$ is contained in $B \oplus (x^m)$, this image lies in $B \subset B \oplus (x^{m-1})/(x^m)$ which shows commutativity.

We take $B \in \text{Art}_\Lambda$, a B -point of $\text{ter}_{m-1,Y}$ and its image in $\text{Def}_{\mathfrak{t} \setminus Y}$

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_{Y_B} & \longrightarrow & B \oplus (x^{m-1}) & \longrightarrow & C_B \longrightarrow 0 \\ & & \searrow & & \downarrow & & \downarrow \\ & & & & B \oplus (x^{m-1})/(x^m) & \longrightarrow & (x^{m-1})/(x^m) \longrightarrow 0. \end{array}$$

Assume that the image comes from $\text{Def}_{* \rightarrow Y}$, this implies that the image of B in $B \oplus (x^{m-1})/(x^m)$ is spanned by 1 as a B -module. This implies that the image of $\mathcal{O}_{Y_B} \rightarrow B \oplus (x^{m-1})$ lies in $B \oplus (x^m)$. $\square$

Corollary 9.14. Let Y be a $\mathbb{k}$ -point of Ter_X^δ of multiplicity. There is a morphism

$$\text{ter}_{Y,m-1} \rightarrow \text{ter}_{Y,m}. \quad (9.15)$$

such that the map $\text{ter}_{Y,m} \rightarrow \text{ter}_{Y,m-1}$ is its section.

Proof. There is a morphism $\phi: \text{Def}_{\mathfrak{t} \setminus Y} \rightarrow \text{Def}_{* \rightarrow Y}$ given by the embedding $* \rightarrow \mathfrak{t}$. Let us take a B -point

$$\mathcal{O}_{Y_B} \rightarrow B[\varepsilon]/(\varepsilon^2)$$

the morphism ϕ sends it to the composition of this map with $B[\varepsilon]/(\varepsilon^2) \rightarrow B$. The morphism $\text{Def}_{* \rightarrow Y} \rightarrow \text{Def}_{\mathfrak{t} \setminus Y}$ from Diagram (9.13) is a section of ϕ . By Lemma 9.12, the diagram is Cartesian, hence we obtain the claimed morphism. $\square$

Theorem 9.16. *The scheme $\text{Ter}_{\mathbb{k}[x]}^\delta$ is nowhere reduced.*

Proof. Let us take a $\mathbb{k}$ -point of $\text{Ter}_{\mathbb{k}[x]}^\delta$

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_X \rightarrow C \rightarrow 0.$$

We consider the long exact sequence of $\mathcal{O}_Y$ -modules induced by the short exact sequence:

$$0 \rightarrow C_2 \rightarrow C \rightarrow (x)/(x^2) \rightarrow 0,$$

$$0 \rightarrow T^0(\mathcal{O}_Y, C_2) \rightarrow T^0(\mathcal{O}_Y, C) \rightarrow T^0(\mathcal{O}_Y, (x)/(x^2)) \rightarrow T^1(\mathcal{O}_Y, C_2) \rightarrow T^1(\mathcal{O}_Y, C) \rightarrow T^1(\mathcal{O}_Y, (x)/(x^2)) \rightarrow \dots$$

By Corollary 9.14, the map $\text{ter}_{2,Y} \rightarrow \text{ter}_Y$ is a section, and hence we get exactness and splitting of

$$0 \rightarrow T^0(\mathcal{O}_Y, C_2) \rightarrow T^0(\mathcal{O}_Y, C) \rightarrow T^0(\mathcal{O}_Y, (x)/(x^2)) \rightarrow 0.$$

But $T^0(\mathcal{O}_Y, (x)/(x^2)) = \text{Der}(\mathcal{O}_Y, \mathbb{k})[-1]$; this is the tangent space to the distinguished point of Y at the distinguished point. Since the map $T^0(\mathcal{O}_Y, C_2) \rightarrow T^0(\mathcal{O}_Y, C)$ is the tangent map of $\text{Ter}_{k[x^2, x^3]} \subset \text{Ter}_{\mathbb{k}[x]}$, we get that $\text{Ter}_{k[x^2, x^3]} \subset \text{Ter}_{\mathbb{k}[x]}$ agrees on $\mathbb{k}$ -points and the cokernel of the induced map of tangent spaces is never trivial. $\square$

9.3 Comparison of structures of Ter_m and Ter

The aim of this section is to refine Theorem 9.16 to a structural comparison of the functor bounded from below with the usual one. The main result is Theorem 9.22, it presents the local bounded Territory functors $\text{ter}_m \subset \text{ter}$ as the fibre of a morphism from the usual local Territory functor.

Definition 9.17. *We define the functor of derivations $\text{Der}_{(\mathcal{O}_Y, -)}$ by the following Cartesian diagram.*

$$\begin{array}{ccc} \text{Der}_{(\mathcal{O}_Y, -)} & \longrightarrow & \text{Def}_{\mathfrak{t} \setminus Y} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \text{Def}_{* \setminus Y}. \end{array} \tag{9.18}$$

Lemma 9.19. *Take $(B, \mathfrak{n}) \in \text{Art}_{\mathbb{k}}$. The B -points of $\text{Def}_{\mathfrak{t} \setminus Y}$ are given by a pairs consisting of: a pointed deformation $\mathcal{O}_{Y_B} \rightarrow B$ and a derivation*

$$\text{Der}_B(\mathcal{O}_{Y_B}, (\varepsilon)/(\varepsilon^2) \otimes \mathfrak{n}).$$

Proof. Let $B \in \text{Art}_{\mathbb{k}}$, by definition a B -point of $\text{Def}_{\mathfrak{t} \setminus Y}$ is given by a map

$$\mathcal{O}_{Y_B} \rightarrow B[\varepsilon]/(\varepsilon^2)$$

making the following diagram commutative

$$\begin{array}{ccc} \mathcal{O}_{Y_B} & \longrightarrow & B[\varepsilon]/(\varepsilon^2) \\ \downarrow & & \downarrow \\ \mathcal{O}_Y & \longrightarrow & \mathbb{k}[\varepsilon]/(\varepsilon^2) \\ & \searrow & \uparrow \\ & & \mathbb{k}. \end{array}$$

Which yields the following diagram

$$\begin{array}{ccccccc} & & & & (\varepsilon)/(\varepsilon^2) \otimes B & & \\ & & & & \downarrow & \searrow 0 & \\ 0 & \longrightarrow & \mathfrak{m}_{Y_B} & \longrightarrow & \mathcal{O}_{Y_B} & \longrightarrow & B[\varepsilon]/(\varepsilon^2) \longrightarrow B \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathfrak{m}_Y & \longrightarrow & \mathcal{O}_Y & \longrightarrow & \mathbb{k}[\varepsilon]/(\varepsilon^2) \longrightarrow \mathbb{k}. \\ & & \searrow 0 & & \uparrow & \nearrow 0 & \\ & & & & (\varepsilon)/(\varepsilon)^2. & & \end{array}$$

Hence, we see that the point is given by a map $\mathfrak{m}_{Y_B} \rightarrow (\varepsilon)/(\varepsilon^2) \otimes B$ over $\mathfrak{m}_Y \xrightarrow{0} (\varepsilon)/(\varepsilon^2)$. Let $\mathfrak{n}$ be the maximal ideal of B , then the B -points are

$$\text{Hom}_{\mathcal{O}_{Y_B}}(\mathfrak{m}_{Y_B}, (\varepsilon)/(\varepsilon^2) \otimes \mathfrak{n}) = \text{Der}_B(\mathcal{O}_{Y_B}, (\varepsilon)/(\varepsilon^2) \otimes \mathfrak{n}).$$

□

Lemma 9.20. For $(B, \mathfrak{n}) \in \text{Art}_{\mathbb{k}}$, the B -points of $\text{Der}_{(\mathcal{O}_Y, _)}^{\mathfrak{t}}$ are given as

$$\text{Der}_B(\mathcal{O}_Y \otimes B, (x)/(x^2) \otimes \mathfrak{n}).$$

The tangent space is isomorphic to $\text{Der}(\mathcal{O}_Y, (\varepsilon)/(\varepsilon^2))$.

Proof. The first part is immediate from Lemma 9.19.

To find a tangent space we take $B = \mathbb{k}[\xi]/(\xi^2)$.

$$\text{Hom}_{\mathcal{O}_Y[\xi]/(\xi^2)}(\mathfrak{m}_Y \otimes \mathbb{k}[\xi]/(\xi^2), (\varepsilon)/(\varepsilon^2) \otimes (\xi)/(\xi^2))$$

Since $\mathcal{O}_Y \otimes (\xi)/(\xi^2)$ annihilates $(\varepsilon)/(\varepsilon^2) \otimes (\xi)/(\xi^2)$, we get $(\varepsilon)/(\varepsilon^2) \otimes (\xi)/(\xi^2) \cong (\varepsilon)/(\varepsilon^2)$. Hence, the surjection $\mathcal{O}_Y \otimes B \rightarrow \mathcal{O}_Y$ induces an isomorphism with

$$\text{Hom}_{\mathcal{O}_Y}(\mathfrak{m}_Y, (\varepsilon)/(\varepsilon^2)),$$

that in turn is isomorphic to $\text{Der}_{\mathbb{k}}(\mathcal{O}_Y, (\varepsilon)/(\varepsilon^2))$. $\square$

Since Diagram (9.18) is Cartesian, it induces a section $\text{Def}_{\mathfrak{t} \setminus Y} \rightarrow \text{Der}(\mathcal{O}_Y, -)$, and we can define the next diagram.

Lemma 9.21. *Let us assume that $\Lambda = \mathbb{k}$, hence any Artinian algebra $B \in \text{Art}_{\mathbb{k}}$ admits an injective map $\mathbb{k} \rightarrow B$. We get the following Cartesian diagram of good deformation functors*

$$\begin{array}{ccc} \text{Def}_{*\setminus Y} & \longrightarrow & \text{Def}_{\mathfrak{t} \setminus Y} \\ \downarrow & & \downarrow \\ * & \longrightarrow & \text{Der}_{(\mathcal{O}_Y, -)}. \end{array}$$

Proof. Let us consider a B -point of $\text{Def}_{\mathfrak{t} \setminus Y}$, a derivation $\partial \in \text{Der}_B(\mathcal{O}_{Y_B}, B \otimes \mathfrak{n})$, we are going to associate with it

$$\tilde{\partial} \in \text{Der}_B(\mathcal{O}_Y \otimes B, (\varepsilon)/(\varepsilon^2) \otimes \mathfrak{n}).$$

Let us consider the pointed structure on Y_B

$$0 \rightarrow \mathfrak{m}_{Y_B} \rightarrow \mathcal{O}_{Y_B} \rightarrow B[\varepsilon]/(\varepsilon) = B,$$

$$\phi: \mathfrak{m}_{Y_B} \rightarrow (\varepsilon)/(\varepsilon^2) \otimes \mathfrak{n}.$$

The derivation ∂ is uniquely determined by the map ϕ . By a base change

$$\text{Hom}_{\mathcal{O}_{Y_B}}(\mathfrak{m}_{Y_B}, (\varepsilon)/(\varepsilon^2) \otimes \mathfrak{n}) = \text{Hom}_B(\mathfrak{m}_{Y_B}/\mathfrak{m}_{Y_B}^2, (\varepsilon)/(\varepsilon^2) \otimes \mathfrak{n}).$$

Since we have a map $\mathbb{k} \rightarrow B$, we get a section of the natural surjection $\mathfrak{m}_{Y_B} \rightarrow \mathfrak{m}_Y$, this section $\mathfrak{m}_Y \rightarrow \mathfrak{m}_{Y_B}$ is a map of $\mathbb{k}$ -vector spaces. We perform a base change

$$\text{Hom}_{\mathbb{k}}(\mathfrak{m}_Y, \mathfrak{m}_{Y_B}) = \text{Hom}_B(\mathfrak{m}_Y \otimes B, \mathfrak{m}_{Y_B}).$$

Hence, we obtain $\tilde{\phi} \in \text{Hom}_B(\mathfrak{m}_Y/\mathfrak{m}_Y^2 \otimes B, (\varepsilon)/(\varepsilon^2) \otimes \mathfrak{n}) = \text{Hom}_{\mathcal{O}_Y \otimes B}(\mathfrak{m}_Y \otimes B, (\varepsilon)/(\varepsilon^2) \otimes \mathfrak{n})$. Note that $\tilde{\phi}$ vanishes if and only if ϕ vanishes.

The diagram is commutative and Cartesian, since trivial element of $\text{Der}_{(\mathcal{O}_Y, -)}$ comes from the zero map ϕ , which is equivalent to $\mathcal{O}_{Y_B} \rightarrow B[\varepsilon]/(\varepsilon^2)$ factoring through B . $\square$

Thanks to that, we show a structural theorem elegantly comparing local behaviour of the bounded Territory functor and the usual one. It shows that the local functor at point of Ter is constructed inductively on multiplicity from local bounded local functors, at each step adding copy of the derivation functor. In particular it shows that if the multiplicity is bounded by m at some open neighbourhood of a closed point, then all meaningful local properties are already present in Ter_m .

Theorem 9.22. *Let us assume that $\Lambda = \mathbb{k}$, in particular any $A \in \text{Art}_{\mathbb{k}}$ admits an injective map $\mathbb{k} \rightarrow A$. Let Y be a $\mathbb{k}$ -point of Ter with multiplicity at least m . We get the following Cartesian*

diagram of good deformation functors

$$\begin{array}{ccc}
 \mathrm{ter}_{Y,m} & \longrightarrow & \mathrm{ter}_{Y,m-1} \\
 \downarrow & & \downarrow \\
 * & \longrightarrow & \mathrm{Der}(\mathcal{O}_Y, -) .
 \end{array} \tag{9.23}$$

Proof. By Lemma 9.12 we get a Cartesian diagram

$$\begin{array}{ccc}
 \mathrm{ter}_{Y,m} & \longrightarrow & \mathrm{ter}_{Y,m-1} \\
 \downarrow & & \downarrow \\
 \mathrm{Def}_{* \rightarrow Y} & \longrightarrow & \mathrm{Def}_{t \setminus Y} .
 \end{array} \tag{9.24}$$

Utilizing Lemma 9.21, we get two Cartesian squares

$$\begin{array}{ccc}
 \mathrm{ter}_{Y,m} & \longrightarrow & \mathrm{ter}_{Y,m-1} \\
 \downarrow & & \downarrow \\
 \mathrm{Def}_{* \setminus Y} & \longrightarrow & \mathrm{Def}_{t \setminus Y} \\
 \downarrow & & \downarrow \\
 * & \longrightarrow & \mathrm{Der}(\mathcal{O}_Y, -) ,
 \end{array}$$

hence the claim. $\square$

Theorem 9.22 shows that the local structure of the Territory at a subalgebra with multiplicity m consists of a lot of redundant information encoding the tangent space to the subalgebra at the closed point. This redundancy is controlled as it comes from the functor $\mathrm{Der}(\mathcal{O}_Y, -)$, see Definition 9.17.

Corollary 9.25. *Let $X = \mathrm{Spec} \mathbb{k}[[x]]$. Let $V \subset \mathrm{Ter}_X$ be an irreducible component, such that general $\mathbb{k}$ -point has multiplicity at least m . For a general point $y \in V$ and $\mathcal{O}_{Y_0} \subset \mathcal{O}_{X_0}$ associated subalgebra*

$$\dim V \leq \dim(T_y \mathrm{Ter}_X) - (m - 1) \mathrm{rk}(\Omega_{\mathcal{O}_{Y_0}}).$$

Proof. Let $0 < k \leq m$.

We start by repeating the argument for Theorem 9.16, for multiplicity k in place of 2. Let us consider

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathbb{k} \oplus (x^{k-1}) \rightarrow C_{k-1} \rightarrow 0.$$

We consider the long exact sequence of $\mathcal{O}_Y$ -modules induced by the short exact sequence:

$$0 \rightarrow C_k \rightarrow C_{k-1} \rightarrow (x^{k-1})/(x^k) \rightarrow 0,$$

$$0 \rightarrow T^0(\mathcal{O}_Y, C_k) \rightarrow T^0(\mathcal{O}_Y, C_{k-1}) \rightarrow T^0(\mathcal{O}_Y, (x^{k-1})/(x^k)) \rightarrow T^1(\mathcal{O}_Y, C_k) \rightarrow \dots$$

By Corollary 9.14, the map $\mathrm{ter}_{k,Y} \rightarrow \mathrm{ter}_{k-1,Y}$ is a section, and hence we get exactness and splitting of the sequence

$$0 \rightarrow T^0(\mathcal{O}_Y, C_k) \rightarrow T^0(\mathcal{O}_Y, C_{k-1}) \rightarrow T^0(\mathcal{O}_Y, (x^{k-1})/(x^k)) \rightarrow 0,$$

as this is the sequence of the tangent spaces induced by Diagram (9.23).

Hence, by Theorem 9.22 we get

$$\dim \mathfrak{t}_{\text{ter}_{Y_0,k}} = \dim \mathfrak{t}_{\text{ter}_{Y_0,k-1}} + \dim \mathfrak{t}_{\text{Der}(\mathcal{O}_{Y_0,-})}.$$

By Lemma 9.20, we get

$$\dim \mathfrak{t}_{\text{Der}(\mathcal{O}_{Y_0,-})} = \text{rk}(\Omega_{\mathcal{O}_{Y_0}}).$$

Since $y \in V$ is a general point, it has multiplicity at least m , therefore

$$\dim T_y V = \dim \mathfrak{t}_{\text{ter}_{Y_0,m}} + (m-1) \text{rk}(\Omega_{\mathcal{O}_{Y_0}}).$$

Since V the general point has multiplicity at least m ,

$$\dim \mathfrak{t}_{\text{ter}_{Y_0,m}} \geq \dim V.$$

□

Corollary 9.26. *Let $X = \text{Spec } \mathbb{k}[[x]]$. Let $V \subset \text{Ter}_{X,k}$ be an irreducible component, such that the general $\mathbb{k}$ -point has multiplicity at least m , for $k < m$. Then V is generically non-reduced.*

Proof. Since the general point has multiplicity at least m , the closed subscheme $V \cap \text{Ter}_{X,m} \subset V$, gives an isomorphism of the reduced structures. But, by Theorem 9.22, the induced map of the tangent spaces is not an isomorphism. □

In the light of Corollary 9.26 and Theorem 9.8, for a given component with the multiplicity m , one should consider the Territory scheme with an appropriate bound, and study geometry of such.

Conjecture 4. *Let $V \subset \text{Ter}_{X,m}$ be an irreducible component, such that the general $\mathbb{k}$ -point has multiplicity exactly m . Then V is generically reduced.*

9.4 Computations of Ter for small genus

The reduced structures of the following Territories were previously described, for $\delta = 1, 2, 3$ in [Ish80, Example 1, Example 2] and for $\delta = 3, 4$ in [Gue23, 1.2.3 Some Examples]. We slightly extend those descriptions giving explicit description for the scheme-structure $\text{Ter}_{\mathbb{k}[[x]]}^{\delta, 2\delta}$ for the same numbers.

9.4.1 Genus 1

We describe $\text{Ter}_{\mathbb{k}[[x]]}^{1,2}$. There is only one $\mathbb{k}$ -point, coming from the cuspidal singularity $\mathbb{k}[[x^2, x^3]]$. To describe the scheme structure, we go back to the representability result, Theorem 8.4. We take an infinite Grassmannian $\text{Gr}(\mathbb{k}[x]/x^2, 1)$, since $1 \in \mathcal{O}_Y$ we can look at $\text{Gr}(x\mathbb{k}[x]/(x)^2, 1)$. Let us take an S point

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_S[x]/x^2 \rightarrow C \rightarrow 0.$$

The structural sheaf $\mathcal{O}_Y$ is an algebra hence contains the unit 1, then

$$0 \rightarrow \mathfrak{m}_Y \rightarrow x\mathcal{O}_S[x]/x^2 \rightarrow C \rightarrow 0.$$

The argument reduces to Grassmannian $\text{Gr}(1, 1)$ which is a reduced point. Note that reduced structure is provided by forcing the conductor to be (x^2) . For comparison, let us look at the full Territory $\text{Ter}_{\mathbb{k}[[x]]}^1$ the tangent space at the unique $\mathbb{k}$ -point is given by $\text{Der}(\mathbb{k}[x^2, x^3], \langle x \rangle)$ This space is two dimensional

$$\begin{aligned}\partial: \mathbb{k}[x^2, x^3] &\rightarrow \langle x \rangle, \\ x^2 &\mapsto ax, \\ x^3 &\mapsto bx.\end{aligned}$$

As showcased in [Ish80, Remark II], $\text{Ter}_{\mathbb{k}[[x]]}^1$ in general does not have scheme structure, since $\text{Ter}_{\mathbb{k}[[x]]}^{1,r}$ do not stabilize.

9.4.2 Genus 2

We compute $\text{Ter}_{\mathbb{k}[[x]]}^{2,4}$. We take an S -point

$$0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_S[x]/(x^4) \rightarrow C \rightarrow 0,$$

where C is locally free of rank 2 over S . The Territory scheme is a closed subscheme of $\text{Gr}(\mathcal{O}_X, 2)$. There are two $\mathbb{G}_m$ -fixed points on this Territory, associated with numerical semigroups:

$$\Gamma_1 = (2, 5) \subset \mathbb{N}, \tag{9.27}$$

$$\Gamma_2 = (3, 4, 5) \subset \mathbb{N}. \tag{9.28}$$

A generic point $\mathbb{k}$ -point of Ter^{Γ_1} is $\mathbb{k}[x^2 + ax^3] + (x^4)$, thus on the reduced structure we get $\mathbb{A}^1$. From a direct computation of positive tangent space we get that Ter^{Γ_2} is a point. This way we get that the reduced structure of Ter^2 is $\mathbb{P}^1$. Now, we compute the tangent space at $\mathbb{k}[x^2 + ax^3] + (x^4)$ for given $a \in \mathbb{k}$.

$$\partial \in \text{Der}(\mathbb{k}[x^2 + ax^3] + (x^4), \langle x, x^3 \rangle),$$

$$\partial: \mathbb{k}[x^2 + ax^3] + (x^4) \rightarrow \langle x, x^3 \rangle, \tag{9.29}$$

$$x^2 + ax^3 \mapsto bx + cx^3, \tag{9.30}$$

$0 = \partial(0) = \partial((x^2 + ax^3)^2) = 2(x^2 + ax^3)(bx + cx^3) = 2bx^3 + 2abx^4 + 2cx^5 + 2acx^6 = 2bx^3 \in C$ consequently we get $b = 0$. We get that the tangent space is one-dimensional, which agrees with the reduced structure.

On the other hand, we compute the tangent space at $\mathbb{k}[\Gamma_2]$.

$$\partial \in \text{Der}(\mathbb{k}[x^3] + (x^4), \langle x, x^2 \rangle),$$

$$\partial(x^3) = cx + dx^2, \tag{9.31}$$

since $(x^3)^2 = 0 \in \mathbb{k}[x]/(x^4)$, we get

$$0 = \partial(0) = \partial(x^6) = 2x^3\partial(x^3) = 2cx^4 + 2dx^5 = 0.$$

As a consequence, we get an additional tangent and $\text{Ter}_{\mathbb{k}[[x]]}^{2,4}$ is non-reduced, again a non-reduced structure comes from negative tangent such that the image of derivation has valuation equal to 1. In Appendix B, we attach a Macaulay2 code that gives equations of $\text{Ter}^{2,4}$ it shows that it is $\mathbb{P}^1$ with a non-reduced structure of first-order on the point associated with the semigroup $(3, 4)$. That is, all the structure is already described above.

9.4.3 Genus 3

We compute $\text{Ter}_{2, \mathbb{k}[[x]]}^{3,6}$. Let us say we take a subalgebra $\mathcal{O}_Y \subset \mathbb{k}[x^2, x^3]/(x^6)$, since $\delta = 3$, $\mathcal{O}_Y$ is a 3-dimensional, closed on multiplication, $\mathbb{k}$ -vector space spanned by 1 and two polynomials. We put those polynomials in a general form

$$F = a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5,$$

$$G = b_2x^2 + b_3x^3 + b_4x^4 + b_5x^5.$$

There is a reduced component coming from semigroup containing 2, see Theorem 9.8. On this component we can assume $a_2 = 1$, then we can assume that $b_2 = 0$, then also $b_3 = 0$

$$F = x^2 + a_3x^3 + a_4x^4 + a_5x^5,$$

$$G = b_4x^4 + b_5x^5.$$

We get that $G = \lambda F^2$, which give us linear equations on b 's, as a consequence we see that the cell is $\mathbb{A}^3$ as expected.

Now, we consider the semigroup $\Gamma = \langle 3, 4 \rangle$.

$$F = a_2x^2 + x^3 + a_4x^4 + a_5x^5,$$

$$G = b_2x^2 + b_4x^4 + b_5x^5.$$

To keep the genus, we have $G^2 = 0$. This imposes the following conditions

$$b_2^2 = 0. \tag{9.32}$$

We have that $F^2 = \lambda G$, for some (possibly zero) parameter λ .

$$2a_2x^5 + a_2^2x^4 = \lambda(b_5x^5 + b_4^2),$$

$$2b_4a_2 - b_5a_2^2 = 0. \tag{9.33}$$

Similarly $FG = \lambda G$, for some (possibly zero) $\lambda \in \mathbb{k}$.

$$(b_2)x^5 + a_2b_2x^4 = \lambda(b_2x^2 + b_5x^5 + b_4x^4).$$

We apply Equation (9.33)

$$\begin{aligned}
x^5(b_2 - \lambda b_5) + x^4(a_2 b_2 - \lambda b_4) - \lambda(b_2 x^2) &= 0, \\
b_2 &= \lambda b_5, \\
b_2 a_2^2 &= \lambda b_5 a_2^2 = 2\lambda b_4 a_2 = 2a_2^2 b_2, \\
a_2 b_2 &= 0,
\end{aligned} \tag{9.34}$$

$$b_2 b_4 = \lambda b_5 b_4 = a_2 b_2 b_4 = 0. \tag{9.35}$$

Consequently, Equations (9.32), (9.33), (9.34), (9.35) give us equations around the point given by Γ ,

$$\begin{aligned}
b_2^2 &= 0, \\
2b_4 a_2 - b_5 a_2^2 &= 0, \\
a_2 b_2 &= 0, \\
b_2 b_4 &= 0.
\end{aligned}$$

The reduced structure is given by

$$\begin{aligned}
b_2 &= 0, \\
(a_2)(2b_4 - b_5 a_2) &= 0.
\end{aligned}$$

This yields a decomposition into component with $a_2 = 0$ one with $\mathbb{k}[\Gamma]$ and second $(2b_4 - b_5 a_2) = 0$ a quadratic cone, as observed in [Ish80, Example 2] and [Gue23]. The non-reduced structure is expected by Theorem 9.16. We are left with two semigroups $\langle 3, 5, 7 \rangle$ and $\langle 4, 5, 6, 7 \rangle$ we omit direct computations, but we attach Macaulay2 code that computes the structure there.

Part IV

Appendix

Appendix A

Proof of Theorem 4.55

We split Theorem 4.55 and Theorem 4.60 into smaller parts.

Lemma A.1. *Let (R, F_i) be a filtered algebra, and $\tilde{R} = \bigoplus t^{-i} F_i$ is the graded Rees algebra associated with the filtration. The following natural transformation is an equivalence of categories:*

$$\begin{aligned} \text{Mod}_{(R, F_i)}^{filt} &\rightarrow \text{Mod}_{\tilde{R}}^{gr}, \\ (M, \mathcal{F}_i) &\mapsto \bigoplus_{i \in \mathbb{Z}} t^{-i} \mathcal{F}_i. \end{aligned}$$

Proof. We start by checking that this transformation is well-defined on morphisms. We take a homomorphism $f: (M, \mathcal{F}) \rightarrow (N, \mathcal{G})$ of filtered R -modules. We associate with it a map

$$\tilde{f}: \bigoplus_{i \in \mathbb{Z}} t^{-i} \mathcal{F}_i \rightarrow \bigoplus_{i \in \mathbb{Z}} t^{-i} \mathcal{G}_i,$$

in the usual way.

Now, we define the inverse transformation:

$$\begin{aligned} \text{Mod}_{\tilde{R}}^{gr} &\rightarrow \text{Mod}_{(R, F_i)}^{filt}, \\ \tilde{M} &\mapsto \left(\frac{M}{(t-1)}, \frac{M_i}{(t-1)} \right). \end{aligned}$$

In this manner, we get an inverse. Thereupon, the categories are isomorphic. $\square$

Theorem A.2. *Let X be an affine $\mathbb{G}_m$ -scheme. $\text{Def}_X^{\geq 0}$ is a good deformation functor.*

Proof. First, we build from $\mathcal{O}_X$ a Rees algebra $\widetilde{\mathcal{O}_X} = \bigoplus_{i \in \mathbb{Z}} t^{-i} \mathcal{F}_i \subset \mathcal{O}_X[t, t^{-1}]$, where $\mathcal{F}_i = \bigoplus_{j \geq i} (\mathcal{O}_X)_j$, it is a graded algebra flat over $\mathbb{k}[t^{-1}]$. Note that the degree of t^{-1} is -1 , and $\mathbb{k}[t^{-1}] \rightarrow \widetilde{\mathcal{O}_X}$ is a morphism of graded algebras. We consider the following deformation functor

$$\text{Def}_{\widetilde{\mathcal{O}_X}/\mathbb{k}[t^{-1}]}^0,$$

the functor of abstract equivariant deformations over $\mathbb{k}[t^{-1}]$.

By Lemma A.1 we get isomorphism of good deformation functors

$$\mathrm{Def}_X^{\geq 0}(A) \rightarrow \mathrm{Def}_{\widetilde{\mathcal{O}_X/\mathbb{k}[t^{-1}]}}^0(A), \quad (\text{A.3})$$

$$(\mathcal{O}_X, \mathcal{F}) \mapsto \widetilde{\mathcal{O}_X} = (\oplus_{i \in \mathbb{Z}} t^{-i} \mathcal{F}_{-i}). \quad (\text{A.4})$$

□

Theorem A.5. *Let $X \rightarrow Y$ be a $\mathbb{G}_m$ -equivariant morphism of affine $\mathbb{G}_m$ -schemes. $\mathrm{Def}_{XY}^{\geq 0}$ is a good deformation functor.*

Proof. The proof is analogous to one of Theorem A.2. □

Definition A.6 ([Har10, Construction 3.1]). *The Lichtenbaum-Schlessinger complex. Let R be a P -algebra. We consider $\mathbb{S}_P$ a polynomial algebra over P (possibly of infinitely many variables) which covers R . Then we consider the kernel I , and its cover η by a free $\mathbb{S}_P$ -module N . We also consider the kernel of this covering Q , its submodules N_0 generated by Koszul relations*

$$\eta(a)b - \eta(b)a,$$

for $a, b \in N$.

$$\begin{array}{ccccccc} & & 0 & & & & \\ & & \uparrow & & & & \\ 0 & \longrightarrow & \tilde{I} & \longrightarrow & \mathbb{S}_P & \longrightarrow & R \longrightarrow 0 \\ & & \uparrow & & & & \\ & & N & & & & \\ & & \uparrow & & & & \\ & & Q & \longleftarrow & N_0 & & \\ & & \uparrow & & & & \\ & & 0. & & & & \end{array} \quad (\text{A.7})$$

A Lichtenbaum-Schlessinger complex is

$$L_{R/P} = Q/N_0 \rightarrow N \otimes_{\mathbb{S}_P} R \rightarrow \Omega_{\mathbb{S}_P/P} \otimes_{\mathbb{S}_P} R. \quad (\text{L-S})$$

The Lichtenbaum-Schlessinger functor value on R -module M is

$$T_P^i(R, M) = \mathrm{Ext}^i(L_{R/P}, M),$$

defined only for $i = 0, 1, 2$.

Corollary A.8. *Let P be a graded ring, R be a graded P -algebra and M a graded R -module. Then the Lichtenbaum-Schlessinger functor*

$$T_P^i(R, M)$$

is naturally graded.

Proof. In Definition A.6 we can choose all components to be graded and end up with graded (L-S) complex. By [Har10, Lemma 3.2, Lemma 3.3], the modules T^i are independent on choice of covering polynomial algebra and covering module. $\square$

Definition A.9 ([Ill71, III 4.1.1]). Let $\phi: X \rightarrow Y$ be a $\mathbb{G}_m$ -equivariant morphism of affine $\mathbb{G}_m$ -schemes, then $\text{Mod}(X \rightarrow Y)$ is the category of triples

$$(M, N, u),$$

where that $M_X \in \text{Mod}_{\mathcal{O}_X}$, $M_Y \in \text{Mod}_{\mathcal{O}_Y}$ and $u: \phi^* M_Y \rightarrow M_X$. The structural sheaf is $\mathcal{O}_{XY} = (\mathcal{O}_X, \mathcal{O}_Y, \phi)$.

Definition A.10 ([Buc81, 2.4.1]). A (L-S) complex of ϕ is given by a triple (L_X, L_Y, ϕ_L) , where L_X, L_Y are (L-S) complexes and ϕ_L is a map which extends ϕ . The Lichtenbaum-Schlessinger functor of ϕ is for a given (L-S) complex L_{XY} given by the following formula

$$T_{X \rightarrow Y}^i = \text{Ext}_{\mathcal{O}_{XY}}^i(L_{XY}, \mathcal{O}_{XY}).$$

Note that Lemma A.1 still holds in the category $\text{Mod}(X \rightarrow Y)$.

Definition A.11. We define Lichtenbaum-Schlessinger functor for fixed source as

$$T_{X \setminus Y}^i = \text{Ext}_{\mathcal{O}_{XY}}^i((0, L_Y, 0), \mathcal{O}_{XY}).$$

Proposition A.12. Let P be a ring with trivial grading. Let R be a graded P -algebra, with induced filtration $\mathcal{F}_i$. Let L_R be a (L-S) graded complex of R , with induced filtration, then $\widetilde{L}_R$ is a (L-S) complex of $\widetilde{R}$.

Proof. Follows from construction in Definition A.6 and Lemma A.1. $\square$

This way we get Theorem 4.55 and Theorem 4.60.

Corollary A.13. For each variant $XY \in \{X, Y, X \rightarrow Y, X/Y, X \setminus Y, X \setminus X \rightarrow Y/Y\}$, the functor $\text{Def}_{XY}^{\geq 0}$ admits an associated triple

$$(T_{XY}^0)_{\geq 0}, (T_{XY}^1)_{\geq 0}, (T_{XY}^2)_{\geq 0}.$$

Proof. We choose a graded (L-S) complex L_{XY} with the induced filtration, as $\widetilde{L}_{XY}$ is a graded complex for $\widetilde{XY}$ we get that

$$\begin{aligned} (T_{XY}^i)_{\geq 0} &= H^i(\text{Hom}(L_{XY}, \mathcal{O}_{XY}))_{\geq 0} = H^i(\text{Hom}^{filt}(L_{XY}, \mathcal{O}_{XY})) = H^i(\text{Hom}(\widetilde{L}_{XY}, \widetilde{\mathcal{O}_{XY}})_0) = \\ &= H^i(\text{Hom}(\widetilde{L}_{XY}, \widetilde{\mathcal{O}_{XY}}))_0 = (T_{\widetilde{XY}}^i)_0. \end{aligned}$$

$\square$

Appendix B

Supplementary Macaulay2 code for Territories

For a given subalgebra of $\mathbb{k}[e]$ this function computes the tangent space and obstruction space.

```
ord = series -> (
  if series == 0 then return 510000; -- Grothendieck's infinity
  return min for el in terms series list first degree el;
);

tanob = method();
tanob(List):= subalg -> (
  R := ring subalg_0;
  T':= (coefficientRing R)[y_1 .. y_(#subalg), Degrees=>(for el in subalg list ord el)];
  subalgMap' := map(R, T', subalg);
  -- push forward the algebra
  -- no prune means no minimalization.
  --This is useful (only) for making the C' work as expected
  M' := pushFwd(subalgMap', module R, NoPrune=>true);
  C' := cokernel(presentation M' | matrix{(gens M')_0});
  -- get the generators of the pushforward of the algebra as a module
  -- the result is a matrix
  generatorsOfM := (pushAuxHgs subalgMap')_0;
  assert (generatorsOfM_(0,0) == 1_R);
  degs := for el in degrees source generatorsOfM list first el;
  -- construct the "Rees" modules of M' and C'
  Maug := augmentedModule(M', degs);
  Caug := augmentedModule(C', degs);
  (ARA, proj, inj, u) := getAugmentationData ring M';
  -- check that their Hilbert functions look reasonable
  HSM=hilbertSeries(proj ** Maug, Order=>30);
  HSC=hilbertSeries(proj ** Caug, Order=>30);
  -- proceed to construct the same for the kernel of subalgMap'
  (Raug, projRaug, injRaug, uRaug) := getAugmentationData R;
```

```

subalgMap = map(Raug, ARA, (for el in subalg list injRaug(el) // uRaug^(ord el)) | {uRaug});
assert(isHomogeneous subalgMap);
rels := ideal mingens saturate(kernel subalgMap, ideal{u});
-- this line make sense only for this fixed example with 10.13
ideal(apply(lista, n->xx_(n-1)));
relsred:=rels+ideal(apply(lista, n->xx_(n-1)));
--describe ARA
assert(isHomogeneous relsred);
HSAred:= hilbertSeries(proj(relsred),Order=>30);
HSA:=hilbertSeries(proj(rels), Order=>30);
assert(HSM-HSC-HSA==0);
coh0C := hilbertSeries(proj ** Hom(pseudoCotangentModule rels, Caug), Order=>20);
T1C:=cotangentCohomology1ModNew(rels, Caug);
coh1C := hilbertSeries(proj ** T1C, Order=>20);
print(coh0C,coh1C);
);
p=3;
kk=ZZ/p;
R := (kk)[e];
lista = ();
subalg :=join( {e^3,e^4},apply(lista , n-> e^n));
tanob(subalg)

```

This function checks if a subalgebra of $k[e]$ with generators given by "subalg" is liftable modulo p^2 .

```

isLiftableonTer = method();
isLiftableonTer(List):= subalg -> (
  S:= ring subalg_0;
  T := (coefficientRing S)[y_1 .. y_(#subalg),Degrees=>(for el in subalg list ord el)];
  --covering polynomial algebra
  subalgMap := map(S, T, subalg); --covering map
  S' := pushFwd(subalgMap, module S, NoPrune=>true); -- kk[e] as module over T'
  unit:=transpose matrix {join({1},(rank ambient S'-1:0))}; --unit vector
  M:=map(S',T^1, unit**T); -- map from subalgmap' on the level of T'-modules
  C:=coker M; -- the cokernel values of the obstructions are in it
  F:= mingens ker M; -- next module in Hartshorne's T^i construction
  assert(M**F==0);
  TZ := ZZ [ gens T,Degrees=>(for el in subalg list ord el)]; --lift of T' to ZZ
  presSZ:=sub(presentation S', TZ);
  SZ:= ZZ[gens S];
  FZ:=sub(F,TZ); --lift of F
  subalgZ:=for el in subalg list sub(el,SZ);
  subalgMapZ := map(SZ, TZ, subalgZ); -- we lift the ring map
  pre:= transpose matrix for i in 0..numColumns(FZ)-1 list {subalgMapZ((FZ_i)_0)};

```

```

if(pre==0) then {use S; return true;};
preobS:=sub(promote(pre,frac(SZ))/p,S); -- divide by p
obt:=pushFwd(subalgMap, preobS,NoPrune=>true); --change to T-modules
assert(numColumns obt== rank source F* rank ambient S');
obtt:=submatrix(obt,0..rank source pre-1);
--only images of generators are of interest
I:=module ideal image F;
ob:=map(C,I,obtt); --map yielding an obstruction
assert(target ob==C);
if(ob==0) then {use S; return true;};
use S;
return(((image(homomorphism' ob))/ (image(Hom(jacobian ideal F,C)))) ==0);
);

```

```

p=3;
S=ZZ/p[e];
lista = (8,15,21);
subalg = join( {e^5+e^7},apply(lista , n-> e^n));
print isLiftableonTer(subalg);

```

This code computes $\text{Ter}_{\mathbb{k}[[x]]}^{2,4}$

```

kk := QQ;
S := kk[a_1,a_2,a_3][x];
polya := a_1*x^1+a_2*x^2 + a_3*x^3
polya2 := polya^2 % x^4
polya3 := polya^3 % x^4
coef := p -> (coefficients (p,Monomials=>{x^1,x^2,x^3}))_1
mat := coef(polya)|coef(polya2)|coef(polya3)
I := minors(2, mat) : minors(1, mat)
print ideal mingens ideal mingens sub(I,{a_2=>1})
-- if a_2=1 then it is A^1 with parameter a_3
print ideal mingens ideal mingens sub(I,{a_2=>0,a_3=>1})
-- on the complement we get additional tangent

```

This code computes $\text{Ter}_{2,\mathbb{k}[[x]]}^{3,6}$

```

kk := QQ;
S := kk[a_2, a_3, a_4, a_5, b_2, b_3, b_4, b_5][x];
polya := a_2*x^2 + a_3*x^3 + a_4*x^4 + a_5*x^5;
polyb := b_2*x^2 + b_3*x^3 + b_4*x^4 + b_5*x^5;
mat := matrix({a_2, a_3, a_4, a_5}, {b_2, b_3, b_4, b_5}, {0, 0, a_2^2, 2*a_2*a_3},
{0, 0, b_2^2, 2*b_2*b_3}, {0, 0, a_2*b_2, a_2*b_3+a_3*b_2});
I := minors(3, mat) : minors(2, mat) -- general ideal
rI := radical I;

```

```

primaryDecomposition rI
pdI := primaryDecomposition I
for el in pdI list radical el -- third one is the line we seek
print ideal mingens ideal mingens sub(I,{a_2=>1,b_2=>0}) -- (2)
ideal mingens ideal mingens sub(I,{a_3=>1,b_3=>0}) -- (3,4)
sub(polyb^2% x^6,{a_3=>1,b_3=>0})
sub((polya * polyb) % x^6, {a_3=>1,b_3=>0})

```

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