Compensated compactness and DiPerna-Majda measures

by

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Declaration

Author's declaration:

Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Through these six tough years I had several good and bad moments. The times of belief and of disbelief, of confidence and of anxiety, of awareness and of misguidance. In every of them, my beloved wife was standing right beside me. I dedicate this thesis to her.

Main topics of the dissertation

In the thesis we discuss several topics connected to compensated compactness and DiPerna-Majda measures. The thesis is divided into chapters, most of which are based on the articles written by the author exclusively or under co-authorship.

The first chapter briefly provides an overview of the discussed topics. We discuss the main motivations for taking on the chosen topics, as well as sketch the history of related research.

Chapter 2 is based on the joint paper written with Agnieszka Kałamajska [1]. We study geometric conditions for integrand f to define lower semicontinuous functional of the form $I_f(u) = \int_\Omega f(u)dx$, where u satisfiesthe conservation law $Pu = 0$, $P = (P_1, P_2, P_3)$ and

$$P_1 v = \frac{\partial v_1}{\partial y}, P_2 v = \frac{\partial v_2}{\partial x}, P_3 v = \frac{\partial v_3}{\partial x} - \frac{\partial v_3}{\partial y}.$$

Of our particular interest is tetrahedral convexity condition introduced Kałamajska in 2003, in connection of the study of the quasiconvexity condition in Calculus of Variation, which is the variant of maximum principle expressed on tetrahedrons, and the new condition which we call tetrahedral polyconvexity. We prove that second condition is sufficient but it is not necessary for lower semicontinuity of I_f, tetrahedral polyconvexity condition is non-local and both conditions are not equivalent. Problems we discuss are strongly connected with the rank–one conjecture of Morrey known in the multidimensional calculus of variations.

Chapter 3 is based on the author's own papers [4, 5]. In the first part, based on [5] we present a constructive proof of the fact, that for any subset $\mathscr{A} \subseteq \mathbb{R}^m$ and countable family $\mathscr{F}$ of bounded functions $f : \mathscr{A} \to \mathbb{R}$ there exists a compactification $\mathscr{A}' \subset \ell^2$ of $\mathscr{A}$ such that every function $f \in \mathscr{F}$ possesses a continuous extension to a function $\bar{f} : \mathscr{A}' \to \mathbb{R}$. However related to more classical theorems, our result is direct and hence applicable in Calculus of Variations. Our construction is then used to represent limits of weakly convergent sequences $\{f(u^\nu)\}$ via DiPerna-Majda measures methods. In particular, as our main application, we generalise the known Representation Theorem from the Calculus of Variations. In the second part of the chapter we focus our attention on the example of a non-supported measure given in [4], which explains the importance of metrizability of the compactification constructed in [5].

Chapter 4 is based on two joint works with Elvira Zappale. In the first part of the chapter, based on [2], we get in the realm of $3D - 2D$ dimensional reduction problems. We prove that, up to an extraction, it is possible to decompose a sequence (u_n), whose 'scaled gradients' $\left(\nabla_\alpha u_n, \frac{1}{\varepsilon_n}\nabla_3 u_n\right)$ are bounded in $L^\Phi(\omega \times (-1,1), \mathbb{R}^{3\times 3})$ for a suitable Orlicz function Φ, as $u_n = v_n + z_n$, such that v_n describes the oscillations, $\left(\Phi\left(\left|\nabla_\alpha v_n, \frac{1}{\varepsilon_n}\nabla_3 v_n\right|\right)\right)$, is equi-integrable and the 'remainder' z_n, accounting for concentration effects, converges to zero in measure. The second part, based on [3], is an application of the results from [2] to the optimal design problem. In particular, as the tickness of the film tends to zero, the $\Gamma - \lim$ of the sequence of optimal design functionals is computed.

References

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Contents

Notations

The following notations are used consequently throughout the dissertation.

The function f, with the domain D and target space T will be denoted by $f : D \to T$, while a multifunction (in other words – set-valued function) F will be denoted by $F : D \rightrightarrows T$.

For any Banach space X and an element $x \in X$, the norm of x in X is denoted by $||x||_X$.

For any subset $T \subseteq X$ the convex hull of T will be denoted by $CH\ T$. Analogously, the closed convex hull of T will be denoted by $\overline{CH}\ T$.

For any function $f : \mathbb{R}^n \to \mathbb{R}$ the convex envelope of f will be denoted by Cf. Similarly, for $f : \mathbb{R}^{n\times m} \to \mathbb{R}$ the quasiconvex envelope will be denoted by Qf.

For any topological space $\mathbb{X}$, by $C(\mathbb{X})$ we denote the space of all continuous, real-valued functions on $\mathbb{X}$. Analogously, by $C_0(\mathbb{X})$ denote the space of all compactly supported, continuous, real-valued functions on $\mathbb{X}$.

The space of all signed measures with finite variation on $\mathbb{X}$ will be denoted by $\mathcal{M}(\mathbb{X})$. The subspace of all probabilistic measures (i.e. positive and of variation equal to 1) will be denoted by $\mathcal{P}(\mathbb{X})$.

For any measured set M, the Lebesgue space of functions $u : M \to X$, integrable with power p, will be denoted by $L^p(M, X)$. The corresponding Sobolev space will be denoted by $W^{1,p}(M, X)$. We will omit X in the notation in case $X = \mathbb{R}$.

Similarly, the Orlicz space with the Orlicz function Φ will be denoted by $L^{\Phi}(M, X)$. The corresponding Sobolev-Orlicz space will be denoted by $W^{1,\Phi}(M, X)$. Again, we will omit X in the notation in case $X = \mathbb{R}$.

The set of functions of bounded variation $u : M \to T$ will be denoted by $BV(M, T)$. The perimeter of a subset $N \subseteq M$ will be denoted by $P(N, M)$.

Chapter 1

An overview of the thesis

1.1 Compensated compactness

1.1.1 The historical overview

One of the most significant problems in the calculus of variations is the lower semicontinuity of variational functionals. For open and bounded domain $\Omega \subset \mathbb{R}^n$ and the function $f : \mathbb{R}^{m\times n} \to \mathbb{R}$, we define the functional

$$I_f(u) \stackrel{\text{def}}{=\!=} \int_\Omega f(Du)dx, \tag{1.1}$$

on the proper space $X(\Omega)$ of mappings defined on Ω with values in $\mathbb{R}^m$. One investigates, under what conditions the functional is sequentially lower semicontinuous with respect to the given topology, i.e. when it satisfies the condition

$$u^\nu \rightharpoonup u \Rightarrow I_f(u) \le \liminf I_f(u^\nu), \tag{1.2}$$

where $u^\nu \rightharpoonup u$ denotes convergence in weak topology on $X(\Omega)$ (shortly lsc property). Application of Direct Methods of Calculus of Variations shows that lsc-property is one of the sufficient conditions for existence of minimisers of the functional I_f. Usually X is a Sobolev space, it is however also common to consider Orlicz-Sobolev or BV. Having in mind, that physical models often introduce a convex energy f, it seems that Orlicz-Sobolev spaces are a reasonable choice for the investigated space. This is a path we follow in Chapter 4.

In 1952 [127] Morrey proved that the lower semicontinuity of I_f, in case of $X(\Omega)$ consisting of Lipschitz functions, is equivalent to quasiconvexity of the function f, i.e. the following property

$$\forall \Phi \in C_0^\infty(\Omega, \mathbb{R}^m)\ \forall A \in M^{m\times n} \Longrightarrow f(A) \le \frac{1}{|\Omega|}\int_\Omega f(A + D\Phi)dx.$$

Unfortunately, the quasiconvexity condition is usually very hard, or even impossible, to verify. This was the reason to look for some other conditions, which would be more clear geometrically and easier to verify. Several of them appeared, however the equivalence between them and quasiconvexity is obtained only in case of u depending on one variable or having values in $\mathbb{R}$. What is more, they are equivalent to standard notion of convexity in that case.

One of the main is rank-one convexity of f: for any matrix $A \in M^{m\times n}$ and any matrix $B \in M^{m\times n}$ of rank one the following condition holds

$$\text{function } t \mapsto f(A+tB) \text{ is convex.}$$

In 1952 in [127] C. B. Morrey stated the conjecture on equivalence between rank-one convexity and quasiconvexity. The statement was disproved by Vladimír Šverák in 1992 in paper [146], but only in case when $m \geq 3, n \geq 3$. His counterexample doesn't work in case when $n = m = 2$.

A certain generalization of functional (1.1) reads as

$$I_f(u) \stackrel{\text{def}}{=\!=} \int_\Omega f(u)dx, \tag{1.3}$$

where u lies in the kernel of some differential operator P of constant coefficients. For example, in the classical case, one may consider gradients as functions lying in the kernel of curl operator. Similarly as in the case of the classical variational questions, one asks about conditions for lower semicontinuity of I_f, but functions u might not be only gradients.

Compensated compactness has found so far a wide spectrum of analytical and geometric applications, and furthermore seems a successful tool in investigation of conservation laws.

Dependently on satisfying the constant rank condition (the algebraic condition given on a constant rank of the characteristic matrix of the system $Pu = 0$), the functional I_f defined on $\ker P$ can posses various properties. The constant rank condition was introduced by Fonseca and Müller in 1999 in [63].

For P satisfying constant rank condition the geometric conditions equivalent to lower semicontinuity has been obtained [61].

There are no known equivalent conditions in the other cases, that is when the operator P disobeys the constant rank condition. Let us consider for example

$$P = (P_1, P_2, P_3) = (\frac{\partial u_1}{\partial x_2}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_3}{\partial x_1} - \frac{\partial u_3}{\partial x_2}). \tag{1.4}$$

We will refer to P as the operator of the type $(2,3)$. The operator acts on functions $u : \Omega \to \mathbb{R}^3, \Omega \subseteq \mathbb{R}^2$ and does not satisfy the constant rank condition. There are no known conditions equivalent to the lower semicontinuity of the functional I_f on $L^\infty(\Omega, \mathbb{R}^3) \cap \ker P$.

The work over quasiconvexity condition is considered one of the most important in functional analysis. It requires a variety of methods (like for example elliptic regularity and coercivity, as in [7]), benefiting often with some inequalities with optimal constants. These investigations were kept by such great mathematicians as Kari Astala [12, 13], John Ball [17, 18], Tadeusz Iwaniec [13, 77, 78], Jan Kristensen [109], Pablo Pedregal [135–138]. In the Polish group, however not directly but related research, is provided by Krzysztof Chełmiński, Agnieszka Kałamajska or Adam Osękowski. The notion of quasiconvexity has also many applications in non-linear elasticity theory.

Also the theory of compensated compactness was used in such important and hard branches of mathematics like geometric optics, conservation laws and elasticity. What is also vital, the theory seems to be a good tool to investigate the Morrey conjecture. Such approaches were already made by the likes of Irene Fonseca [61,63], Francois Murat [18,130], Jeffrey Rauch [79,80] or Luc Tartar [148–150].

It is worth mentioning, that related solved problem in the field of compensated compactness due to the young authors Guido De Philippis and Filip Rindler [46] was recently published in the most prestigious mathematical journal Annals of Mathematics.

1.1.2 A brief explanation of the contribution to the discipline

In Chapter 2, based on [90], we investigate the particular problem of sequential weak-$\star$ lower semicontinuity of functionals described in (1.3), defined not on the whole $L^\infty(\Omega)$, but only on the kernel of the $(2,3)$ operator. The main goal is to look for some new convexity-type conditions on f, which would explain the lower semicontinuity of I_f in some geometric way. In particular, basing on brilliant ideas coming from [79] and developing achievements raised in [37,84], we investigate functionals defined on the kernel of the operator P defined in (1.4), being the prototype of compensated compactness theory for operators which do not satisfy constant rank condition.

In Chapter 2, the condition of tetrahedral polyconvexity is proposed and its properties are considered. The condition is geometrically clear. It is proven to be sufficient for sequential lower semicontinuity of the functional described by (1.3), but defined on the kernel of P (see (1.4)). Also, certain Carathéodory type theorem for tetrahedral polyconvexity is proven. As its consequence, the locality of this condition is proven to fail, exactly as it happens for quasiconvexity in the classical case. Introducing this condition does not succeed however in closing the main research topic. An easy modification of a function proposed by Alibert and Dacorogna in [7] shows that the condition is not necessary.

1.2 Control of the discontinuous integrands

The work on the control of weak convergence of bounded sequences composed with discontinuous functions was inspired by the construction of measures of DiPerna and Majda, proposed in [49]. Agnieszka Kałamajska proved the Representation Theorem in [86] and her work was continued in [85, 89, 103].

1.2.1 The Representation Theorem

We briefly describe the Representation Theorem 3.2.11, originally proven in [86]. Let Ω be open and bounded domain in $\mathbb{R}^n$ and $u^\nu : \Omega \to \mathbb{R}^m$. Moreover, assume $\mathbb{R}^m$ is equipped in a certain compactification, i.e.

$$\mathbb{R}^m = A_1 \cup \ldots \cup A_k$$

and for every i the Borel set A_i is mapped homeomorphically and densely (by Φ_i) into the compact subset of $\mathbb{R}^{N_i}$, denoted by γA_i. Define also on $\mathbb{R}^m$ the density function g such that $g_i \overset{\text{def}}{=\!=} g_{|A_i} \in C(A_i)$ and $g_i(\lambda) \geq \alpha > 0$ for every $\lambda \in A_i \cap \partial A_i$. Let for every i

$$\tilde{f}_i \overset{\text{def}}{=\!=} (f/g_i) \circ \Phi^{-1} \in C(\gamma A_i), \tag{1.5}$$

(i.e. $(f/g_i) \circ \Phi^{-1}$, which is a continuous function on $\Phi(A_i)$, is extendable into a continuous function on γA_i). Then (up to some technical details) for any bounded sequence $\{u^\nu\}$ there exist a subsequence $\{u_j\}$, measures on Ω $\bar{m}^i, m^i$ (with some additional properties), families of probabilistic measures $\{\mu_x\}_{x\in\Omega}$ on $\mathbb{R}^m$, $\{\nu_x^i\}_{x\in\Omega}$ on the remainder $\gamma A_i \setminus \Phi(A_i)$ and $\{\bar{\nu}_x^i\}_{x\in\Omega}$ on $\partial A_i \cap A_i$ such that in the space of measures $\{f(u_j(x))dx\}$ converges weakly-$\star$ to

$$\sum_{i=1}^{k} \Bigg(\int_{\operatorname{int} A_i} f(\lambda)\mu_x(d\lambda)\mu(dx) + \int_{\partial A_i \cap A_i} f(\lambda)\bar{\nu}_x^i(d\lambda)\bar{m}^i(dx) + \int_{\gamma A_i \setminus \Phi(A_i)} \tilde{f}_i(\lambda)\nu_x^i(d\lambda)m^i(dx)\Bigg),$$

Furthermore, the obtained measures do not depend on a choice of a function f satisfying (1.5).

The above theorem gives us some sort of control of weak-$\star$ convergence of sequences composed with discontinuous functions. It is worth noting that whenever we assume $u^\nu(x) \to u(x)$ almost everywhere, then the formula for such limits is given by the Convergence Theorem 3.2.12 (see [14]). In that case the set of cluster points of $u^\nu(x)$ is precisely $\{u(x)\}$. We don't know about more subtle investigations involving precise analysis of the support of the involved measures.

The Young measures generated by gradients were investigated by David Kinderlehrer and Pablo Pedregal [95–97] and then Irene Fonseca, Stefan Müller and Pablo Pedregal in [64]. The full classification is obtained. The characterization of the measures of DiPerna and Majda controlled by the continuous functions and generated by gradients of functions from Sobolev space was obtained by Agnieszka Kałamajska and Martin Kružik in [92]. To my best knowledge, general results for control of gradients by discontinuous functions have not been obtained so far.

1.2.2 The problem of compatifications

The problem caused by the formulation of the Representation Theorem, which we will pay much attention to, reads as follows. Given a subset $A \subseteq \mathbb{R}^m$ and a continuous function $f : A \to \mathbb{R}$ we would like to find a metric compactification γA of A and a dense embedding $\varphi : A \hookrightarrow \gamma A$ such that the function $f \circ \varphi^{-1} : \varphi(A) \to \mathbb{R}$ possesses a continuous extension $\bar{f} : \gamma A \to \mathbb{R}$. The existence of such γA is indicated by the assumption in numerous papers in the field, for example [85–89, 92]. A verification of whether this assumption may be satisfied seems to be not entirely taken. To be more precise – the question of whether such compactification can be metrizable remains untaken.

Let us explain the troubles hidden deeper with a natural example. A very classic solution to the problem of compactification seems to be the well-recognized Čech-Stone compactification βA. Indeed, every continuous function $f : A \to \mathbb{R}$ possesses an extension to a continuous function $\bar{f} : \beta A \to \mathbb{R}$. Unfortunately, taking a very simple $A = \{1 - \frac{1}{n} : n \in \mathbb{N}\} \subset [0, 1]$ we are delivered a compactification $\beta \mathscr{A}$ non-metrizable, non-second countable and of cardinality $2^{2^{\mathbb{N}}}$ (see [53, Corollary 3.6.12] for details). This shows that a more specific construction is needed to obtain a compactification with metric and visible geometric structure. The original formulation of the Representation theorem by Kałamajska [86] requires γA to be a subset of an Euclidean space, but a careful analysis of the proof shows that this requirement can be relaxed. Nevertheless, metrizability of γA, as well as its embedding into a locally compact vector space (required for the Reschetnyak slicing argument [139]) are needed. There are several approaches towards this problem, due to Gelfand and Naimark [68,69], Engelking [53] or discussed by Keesling [93], which we will review in Section 3.5. Unfortunately, none of these delivers a rewarding answer.

1.2.3 A brief description of the own contribution

The main purpose of the Chapter 3 is to give a complete and positive answer to the compactification problem described above. We present a constructive proof of the fact, that for any subset $A \subseteq \mathbb{R}^m$ and a countable family $\mathscr{F}$ of bounded functions $f : A \to \mathbb{R}$ there exists a compactification $\kappa A \subset \ell^2$ of A such that every function $f \in \mathscr{F}$ possesses a continuous extension to a function $\bar{f} : \kappa A \to \mathbb{R}$. However related to a number of classical theorems, our result is direct and, in this way, new. By direct, we mean that the method of constructing the compactification is geometrically clear and gives us a straight formula on both the shape of κA and the dense embedding $\varphi : A \to \kappa A$. Furthermore, κA is naturally embedded into the Tychonoff's cube in ℓ^2. Let us remind, that the Tychonoff's cube in space $\ell^p, 1 \le p \le +\infty$ is the compact set

$$\mathscr{T} \stackrel{\text{def}}{=} \prod_{i=0}^{\infty} [0, 2^{-i}] = [0, 1] \times [0, \frac{1}{2}] \times [0, \frac{1}{4}] \times \dots .$$

Our construction is then used to represent limits of weakly convergent sequences $\{f(u^\nu)\}$ via methods related to DiPerna-Majda measures. In particular, as our main application, we generalise the aforementioned Representation Theorem from [86] to the case of the integrand f dependent on u and x. Without the careful analysis of the compactification problem, the study of the functionals $\int_\Omega f(x,u)dx$ was not possible for discontinuous f.

In the final part of the chapter we present arguments showing that the standard notion of the support of a probabilistic Borel measure is not well defined in every topological space. We stress that the notion of the support played an essential role in the proof of Theorem 3.4.3.

Our goal is to create a "very inseparable" space and to show the existence of a family of closed sets such that each of them is of full measure, but their intersection is empty. The presented classic construction is credited to Jean Dieudonné and dates back to 1939. We also propose certain, up to our best knowledge, new simplifications. The example is a good illustration of what may happen, if we abandon the assumptions on regularity of γA. The problems arise then not only in the proof of the Representation Theorem. In fact, some of the Theorem's statements, like inclusions of the supports, become meaningless, when the support of a probabilistic measure may not be well defined.

1.3 An application of Young and DiPerna-Majda theory

1.3.1 An overview on the commonly used methods

In the same spirit of seeking conditions related to sequential lower semicontinuity of (1.1) as above, one of the aims of my studies is that of determining the asymptotic behaviour of families of problems as in (1.1), arising from applications in Material Science, Elasticity, particularly related to Optimal Design, Modelling of Thin Structures. In fact, in the framework of Elasticity, given a family of functionals $\{I_{f_\nu}\}_\nu$, defined as

$$I_{f_\nu}(u) \stackrel{\text{def}}{=} \int_\Omega f_\nu(Du)dx, \tag{1.6}$$

where Ω is the reference configuration, u is the deformation (or displacement), and f_ν the stored energy density, under suitable boundary conditions and given loads, the minimal configurations u^ν, if any, represent the equilibrium states.

Clearly, if there is a lack of lower semicontinuity in (1.6), these equilibria may not exist but it is useful for applications to understand if the family of almost minimizers $\{u^\nu\}$ admits some cluster points $\overline{u}$, with respect to a suitable topology. Moreover, one wants to determine which minimum problem $\overline{u}$ solves, namely one wants to detect a suitable functional $\overline{I_{f_\nu}}$ such that

$$u^\nu \rightharpoonup u \Rightarrow \overline{I_{f_\nu}}(u) \leq \liminf I_{f_\nu}(u^\nu). \tag{1.7}$$

In principle, given certain convergences, the functional $\overline{I_{f_\nu}}$ may not even exist, or it may not be of integral type (see [28, Theorem 4.3.2]) and not even related to any pointwise limit f of f_ν (even in case f_ν is a constant sequence, as it is shown in [108]).

Here we would like to stress that the main role in the technical parts of the proofs of related relaxation results is played by variants of decomposition lemma, which seems to be the mostly applied tool when dealing with the notion of equiintegrability. In all the proofs of Decomposition Lemma the classical Young Theorem is exploited (see for example [23, 24, 26, 64] or [104] in the Orlicz setting). The lemma has been sharpened in the literature (see [89]) with the use of more general, DiPerna-Majda measures. Also, the control of concentration effects was proposed in [64]. On the other hand, the key role in the formulation of the result is played by the notion of quasiconvexity and quasiconvexifications, which were investigated in Chapter 2. As energies coming from elasticity (also in the context of compensated compactness) are very often convex, Orlicz spaces seem to be a natural habitat for such considerations.

1.3.2 The Optimal Design Problem

The model I will focus on is the following optimal design problem

$$\inf_{\substack{v\in W^{1,M}(\Omega(\varepsilon);\mathbb{R}^3)\\ \chi_{E(\varepsilon)}\in BV(\Omega(\varepsilon);\{0,1\})}} \Big\{ \tfrac{1}{\varepsilon}\Big(\int_{\Omega(\varepsilon)} \big(\chi_{E(\varepsilon)}W_1 + (1-\chi_{E(\varepsilon)})W_2\big)(\nabla v)dx - \int_{\Omega(\varepsilon)} \hat{f}\cdot v\, dx + \alpha P(E(\varepsilon);\Omega(\varepsilon))\Big) : \ v = 0 \text{ on } \partial\omega\times(-\varepsilon,\varepsilon), \tfrac{1}{\mathcal{L}^3(\Omega(\varepsilon))}\int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)}\, dx = \lambda \Big\}, \tag{1.8}$$

where

$$\beta'\left(M(|\xi|)-1\right) \le W_i(\xi) \le \beta(1+M(|\xi|)) \quad \forall \xi\in\mathbb{R}^{3\times 3}, \quad i=1,2, \text{ for suitable } \beta\ge\beta'>0. \tag{1.9}$$

where M is an Orlicz convex function, which, for some technical reasons, satisfies the ∇_2 and Δ_2 conditions (see (4.6) and (4.5) respectively). $E(\varepsilon) \subset \Omega(\varepsilon)$ is a measurable subset of $\Omega(\varepsilon)$ with finite perimeter. We assume that

$$P(E(\varepsilon);\Omega(\varepsilon)) \overset{\text{def}}{=\!=} \sup\Big\{ \int_{E(\varepsilon)} div\varphi\, dx : \varphi\in C^1_c(\Omega(\varepsilon);\mathbb{R}^3), \|\varphi\|_{L^\infty}\le 1\Big\} < +\infty, \tag{1.10}$$

and the load $\hat{f}\in L^{M^*}(\Omega(\varepsilon);\mathbb{R}^3)$, where M^* is the complementary (conjugate) Orlicz N-function of M.

In order to study the asymptotic behaviour we first rescale the problem in a fixed 3D domain and then we perform $\Gamma-$convergence with respect to the pair (deformation, design region). Thus one performs $\frac{1}{\varepsilon}$–dilation in the transverse direction x_3. Set

$$\Omega \stackrel{\text{def}}{=} \omega \times (-1,1), E_\varepsilon \stackrel{\text{def}}{=} \{(x_1,x_2,x_3) \in \Omega : (x_1,x_2,\varepsilon x_3) \in E(\varepsilon)\},$$
$$u(x_1,x_2,x_3) \stackrel{\text{def}}{=} v(x_1,x_2,\varepsilon x_3), f(x_1,x_2,x_3) \stackrel{\text{def}}{=} \hat{f}(x_1,x_2,\varepsilon x_3),$$
$$\chi_{E_\varepsilon}(x_1,x_2,x_3) \stackrel{\text{def}}{=} \chi_{E\ (\varepsilon)}(x_1,x_2,\varepsilon x_3), \tag{1.11}$$

where v is any admissible field for (1.8).

In the sequel we will denote $x_\alpha \stackrel{\text{def}}{=} (x_1,x_2)$, $dx_\alpha \stackrel{\text{def}}{=} dx_1dx_2$ and ∇_α and D_α will be identified with the pair (∇_1,∇_2), (D_1,D_2), respectively.

Note that by (1.10) and by the definition of total variation,

$$P(E(\varepsilon);\Omega(\varepsilon)) = \left|D\chi_{E(\varepsilon)}\right|(\Omega(\varepsilon)).$$

By the change of variables $y_3 \stackrel{\text{def}}{=} \varepsilon x_3$ and $y_\alpha \stackrel{\text{def}}{=} x_\alpha$ we have

$$\frac{1}{\varepsilon}\left|D\chi_{E(\varepsilon)}\right|(\Omega(\varepsilon)) = \left|\left(D_\alpha\chi_\varepsilon \left| \frac{1}{\varepsilon}D_3\chi_\varepsilon\right.\right)\right|(\Omega),$$

where χ_{E_ε} denotes the characteristic function of E_ε, that in the sequel we will denote simply by χ_ε. Hence we are lead to a rescaled minimum problem that, up to a dilation of $\frac{1}{\varepsilon}$ can be studied, introducing the functional in(1.12) below.

For every $\varepsilon > 0$, let $J_\varepsilon : L^1(\Omega;\{0,1\}) \times L^p(\Omega;\mathbb{R}^3) \to [0,+\infty]$ be the functional defined as follows

$$J_\varepsilon(\chi,u) \stackrel{\text{def}}{=} \begin{cases} \int_\Omega \left(\chi W_1\left(\nabla_\alpha u \left| \tfrac{1}{\varepsilon}\nabla_3 u\right.\right) + (1-\chi)W_2\left(\nabla_\alpha u \left| \tfrac{1}{\varepsilon}\nabla_3 u\right.\right)\right)dx & \\ -\displaystyle\int_\Omega f\cdot u dx + \alpha\left|\left(D_\alpha\chi\left|\frac{1}{\varepsilon}D_3\chi\right.\right)\right|(\Omega) & \text{in } BV(\Omega;\{0,1\}) \times W^{1,M}(\Omega;\mathbb{R}^3), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.12}$$

I emphasize that if the constant α is $=0$ (i.e. no penalization of the interfaces) then the asymptotic analysis will be very different, and the weak-$\star$ convergence in the sense of measures for $\chi's$ will be replaced by a the weak-$\star$ convergence in L^∞ with limit $\theta \in L^\infty(\Omega;[0,1])$ and not anymore a BV function.

Such models were considered in the contexts of conductivity, chemotaxis or elasticity by several marvellous mathematicians, including Irene Fonseca [59, 60], Robert Kohn [101, 119], Pablo Pedregal [60] or Elvira Zappale [34, 35, 105], to name a few.

The general Orlicz growth was considered in [114–116], where the non-power growth was considered, but still in case of a function M satisfying Δ_2 and ∇_2 conditions. They studied lower weak semicontinuity of variational functionals. In [104, 105] we managed to retrieve certain representation formula for Γ-limit from [34] in similar setting, that is for energy densities of the Orlicz growth. We believe however, that even this required conditions for growth (namely Δ_2) may be still weakened.

We stress that the notion of quasiconvexification, one of the main topics in Chapter 2, naturally rises in the definition of the functional J_0 (see (4.28)), which happens to be the $\Gamma - \lim$ of J_ε, as it is proven in Theorem 4.4.2.

Chapter 2

Tetrahedral polyconvexity

2.1 Introduction

One of the most challenging problems in the modern multidimensional calculus of variations is the so-called rank-one conjecture of Morrey which reads as follows. Let us consider the classical functional of the calculus of variations:

$$I_{\tilde{f}}(u) = \int_{\Omega} f(Du)dx,$$

where $\Omega \subseteq \mathbb{R}^n$, $u : \Omega \to \mathbb{R}^m$, $u = (u_1, \dots, u_m), u_i \in W^{1,\infty}(\Omega)$, $i = 1, \dots, m$, $Du = (\nabla u_1, \dots, \nabla u_m) \in \mathbb{R}^{n\times m}$. One asks about the characterization of the space of admitted functions f such that the functional I_f is sequentially lower semicontinuous with respect to the sequential weak-$\star$ convergence of its arguments (gradients) in $L^\infty(\Omega, \mathbb{R}^{n\times m})$ (to abbreviate let us call this property shortly $sw\star -lsc$). In the paper [127] Morrey proved that I_f is $sw^\star - lsc$ if and only if it satisfies the following condition called the quasiconvexity condition:

$$\frac{1}{|Q|}\int_Q f(A + D\phi)dx \geq f(A), \tag{2.1}$$

whenever $\phi \in C_0^\infty(\Omega, \mathbb{R}^m)$, $A \in \mathbb{R}^{n\times m}$ is an arbitrary matrix and Q is an arbitrary cube in $\mathbb{R}^n$. The quasiconvexity condition seems to be impossible to be verified in practice. Therefore it is natural to ask if there are some geometric conditions which are equivalent to the quasiconvexity condition (2.1). It was proven by Morrey in 1952 [127] that every quasiconvex function is convex in the directions of rank-one matrices and this property is called nowadays rank-one property. He conjectured (to be more precise he had expressed his doubts) that rank-one property is equivalent to the quasiconvexity condition. Since that time this conjecture is called rank one conjecture of Morrey. It required 40 years when this conjecture was disproved by Šverák [146] in cases $m \geq 3, n \geq 2$, while up to nowadays the conjecture is open in the remaining cases, which reduce to $m = n = 2$.

Many famous authors contributed further to that challenging question, like, among others, Alibert [7], Ball [17], Murat [18], Dacorogna [41–44], Kristensen [108, 109], Marcelini [120], Morrey [127, 128], Müller [131, 132], Pedregal [135, 136, 138] and Šverák [138, 144–146]. We also refer to e.g. [13,36,55,72,110,133,143,153,154]. Iwaniec in [77] has pointed out the strong relation between Morrey's conjecture and some important open problems in the theory of quasiconformal mappings (see also the paper by Astala [12]). Šverák has shown in [147] that quasiconvexity is strongly related to compactness properties of approximate solutions of the system $Du \in K$.

We are interested in geometric conditions which could be helpful for better understanding the quasiconvexity condition.
For this, we consider the case $m = n = 2$ and the special subset in the space of gradients, namely, denoting $z = (x, y)$

$$\begin{aligned} u(z) &= \xi_1 R(\xi_1 \cdot z) + \xi_2 S(\xi_2 \cdot z) + \xi_3 T(\xi_3 \cdot z) =: u_1(z) + u_2(z) + u_3(z), \text{ where} \\ \xi_1 &= (1, 0),\ \xi_2 = (0, 1),\ \xi_3 = (1, 1) \end{aligned}$$

and $R, S, T : \mathbb{R} \to \mathbb{R}$ are Lipschitz functions, $R^{'} = r, S^{'} = s, T^{'} = t$. We observe that

$$\begin{aligned} Du(z) &= \sum_{i=1}^{3} Du_i(z) = r(x) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + s(y) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + t(x+y) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &\sim (r(x), s(y), t(x+y)). \end{aligned}$$

This way the function $tildef$ defined on 2×2 symmetric matrices can be identified with the function f defined on $\mathbb{R}^3$ and our original functional reduces to the simpler one

$$I_f(v) = \int_Q \tilde{f}\big(v_1(x), v_2(y), v_3(x+y)\big) dxdy \tag{2.2}$$

(here $v_1 = r, v_2 = s, v_3 = t$). Note that the function $v(x, y) = (v_1(x), v_2(y), v_3(x+y))$ belongs to the kernel of differential operator $P = (P_1, P_2, P_3)$, i. e. $Pv = 0$, where

$$P_1 v = \frac{\partial v_1}{\partial y},\ P_2 v = \frac{\partial v_2}{\partial x},\ P_3 v = \frac{\partial v_3}{\partial x} - \frac{\partial v_3}{\partial y}. \tag{2.3}$$

In particular this reduction step links the problem of quasiconvexity with the problem in the compensated compactness theory (originated by the pioneering works by Murat [130] anrd Tatar [149], see also [150]), where one investigates the $sw\star{-}lsc$-property of functionals defined on functions which lie in the kernel of the given differential operator P. In the special case when P is the rotation operator one deals with gradients. The rather well understood case is the case when operator P satisfies the so-called constant rank condition (we refer to Braides, Fonseca and Leoni [25], Fonseca and Kinderlehrer [61], Fonseca and Müller [63]. For the cases when the constant rank condition might not be satisfied we refer to [79, 80] and their further extensions (involving many applications), like [82, 83] by Kałamajska. For recent works in this direction we refer also to [10, 140] and to references enclosed therein. We emphasize that our operator P given by (2.3) does not satisfy constant rank condition, i.e. it deals with the case which is less understood.

Let us skip this general approach and concentrate on the very special functional given by (2.2) which will be called the functional of the type $(2,3)$. Integrands which define $sw\star-lsc$ functional well be called $(2,3)$ quasiconvex.

As there are no constraints on the involved functions v_1, v_2, v_3 in (2.2), we thought that similarly as in the case of the classical unconstrained functional, one could expect that the $sw\star-lsc$ property of this functional can be expressed by the purely geometric constraints. The candidate for such geometric condition was found by first author in the paper [84] (Theorem 3.1, see also [37, 137] for the related issues). The condition has to be verified on three dimensional oriented simplex'es (oriented tetrahedrons) by the purely geometric means. To be more precise, in Theorem 3.1 in [84] it was shown that if $\tilde{f}$ defines the functional (2.2) with the $sw\star-lsc$ property then it necessarily must satisfy the two conditions and one of the conditions has purely geometric interpretation. We omit the formulation of the second one, which is not that directly geometric, and focus on first one only. It says the following. Having given an arbitrary tetrahedron $D \subseteq \mathbb{R}^3$ with three edges paralel to the axis and the polynomial P_f from seven dimensional space of polynomials $\mathcal{A} = \mathrm{span}\{1, x_1, x_2, x_3, x_1x_2, x_1x_3, x_2x_3\}$ such that $f = P_f$ in every corner of D and its three neighbours (we omit their definition, such P_f is defined uniquely), one has

$$f \leq P_f \text{ inside } D.$$

This property serves as the version of the maximum principle for f. We will call this condition weak tetrahedral convexity condition.

We address the following questions:

Question A Is the weak tetrahedral convexity condition equivalent to the $(2,3)$ quasiconvexity condition, i.e. lower semicontinuity of the related functional?

Question B Are there some other simple geometric conditions which guarantee $sw\star-lsc$ property of the related functional (2.2), i.e. $(2,3)$ quasiconvexity?

In this chapter we try to approach them. We did not succeed in answering Question A. However, when looking for some other simple geometric conditions, we have introduced another geometric condition called tetrahedral polyconvexity condition, similarly as one deals with polyconvexity condition in the calculus of variations [17]. Trying to approach both questions, we have shown that tetrahedral polyconvexity condition is not equivalent to $(2,3)$ quasiconvexity. For this we use the tool known in the calculus of variations, namely fourth order polynomial constructed by Alibert and Dacorogna in [7] and embedding of our special functions v in (2.2) into the space of gradients. Main statement in this direction, where we compared several convexity type conditions useful for understanding Question A, is formulated in Theorem 2.3.3. Moreover, we have shown that weak tetrahedral polyconvexity condition is the non-local one, i.e. it cannot be expressed by conditions which hold in an arbitrary small neighbourhoods of points. This is done by adapting to our setting the technique of Kristensen from [109], which is known in the calculus of variations, showing that polyconvexity is not the local condition. The adaptation required perhaps not so automatic modification of the Carathéodory Theorem (Theorem 2.4.4). Main statement about the non-locality is formulated as Theorem 2.5.2.

We hope that the presented issue will contribute to the discussion of the quasiconvexity condition in the calculus of variations, as well as will indicate on some new interesting questions in pure geometry.

The chapter is based on the joint work with Agnieszka Kałamajska [90]. The part delivered solely by the author contains proofs, as well as the whole section devoted to Carathéodory type theorem. The estimated contribution is 70%.

2.2 Preliminaries and basic notation

2.2.1 Functions of the type (2,3)

In this section we will be dealing with the following set of functions.

Definition 2.2.1 (Function of the type $(2,3)$)**.** Let Ω be an arbitrary open subset of $\mathbb{R}^n$. Function $u : \Omega \to \mathbb{R}^3$ having the form

$$u(z) = \Big(r\big(z \cdot \xi_1\big), s\big(z \cdot \xi_2\big), t\big(z \cdot \xi_3\big) \Big),$$

where $(\xi_i)_{i=1}^3$ is a triple of vectors from $\mathbb{R}^n$ which is pairwise independent, but dependent as a triple, $a \cdot b$ stays for a standard scalar product of the vectors a, b and r, s, t are scalar functions of one variable, will be called `function of the type (2,3)` defined on Ω.

Justification of this notion comes from the fact that we deal with two variables (z may be here a vector from $\mathbb{R}^n$, however a function is dependent only on its projection to a two dimensional plane: span $\{\xi_1, \xi_2, \xi_3\}$) and three functions. As an example of such function we may consider function

$$v(x, y) = \Big(r(x), s(y), t(x + y) \Big), \text{ where } (x, y) \in \Omega \subseteq \mathbb{R}^2.$$

Functions of the form $v(x, y) = (r(x), s(y), t(x + y))$ and $\Omega \subseteq \mathbb{R}^2$ will be called a `special (2,3) functions`.

However not defined so far, functions of that type appear in several papers in calculus of variations as a tool to investigate quasiconvexity condition ([37, 135]).

In our considerations we will use the fact that functions of the type $(2, 3)$ can be canonically embedded into the space of symmetric gradients. Let us explain how it is done.

For this we will use the convention:

$$Dv = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \cdots & \frac{\partial v_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial v_m}{\partial x_1} & \cdots & \frac{\partial v_m}{\partial x_n} \end{pmatrix}$$

where $v = (v_1, \dots, v_m) : \Omega \to \mathbb{R}^m$ is vector-valued function and $\Omega \subseteq \mathbb{R}^n$.

Let $r, s, t :\ \mathbb{R} \to \mathbb{R}$ be given scalar one-variable bounded functions and R, S, T be their absolutely continuous primitives (so Lipschitz), i.e. $R' = r, S' = s, T' = t$. Consider $u : \Omega \to \mathbb{R}^n$ such that for any $z \in \Omega$

$$u(z) = \xi_1 R(\xi_1 \cdot z) + \xi_2 S(\xi_2 \cdot z) + \xi_3 T(\xi_3 \cdot z) =: u_1(z) + u_2(z) + u_3(z).$$

Then we obtain

$$Du(z) = \xi_1 \otimes \xi_1 r(\xi_1 \cdot z) + \xi_2 \otimes \xi_2 s(\xi_2 \cdot z) + \xi_3 \otimes \xi_3 t(\xi_3 \cdot z) = \sum_{i=1}^{3} Du_i(z). \tag{2.4}$$

Taking $n = 2$ and

$$\xi_1 = (1, 0), \quad \xi_2 = (0, 1), \quad \xi_3 = (1, 1),$$

we may consider the special subsets in the space of gradients:

$$Du = Du_1 + Du_2 + Du_3, \text{ where}$$

$$\begin{aligned} u_1(x, y) &\stackrel{\text{def}}{=} (R(x), 0),\ u_2(x, y) := (0, S(y)),\ u_3(x, y) := (T(x + y), T(x + y)), && (2.5) \\ Du &= r(x) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + s(y) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + t(x + y) \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. && (2.6) \\ &\sim \big(r(x), s(y), t(x + y)\big). \end{aligned}$$

In particular, function of the type $(2, 3)$ is embedded into the space of symmetric gradients. We arrive at a following observation.

Proposition 2.2.2. *Any special function of the type* $(2, 3)$ *may be uniquely identified with a gradient of a certain function* $u : \Omega \to \mathbb{R}^3, u = u_1 + u_2 + u_3$ *defined by* (2.5) *via expression* (2.6).

We will consider $r, s, t \in L^\infty(\mathbb{R})$, so that R, S, T are Lipschitz. In fact, for the definition of a function of a type $(2, 3)$ we only need to know the values on projections of Ω into three lines along ξ_1, ξ_2, ξ_3 respectively. Note that any Lipschitz function defined on a closed subset may be extended to a Lipschitz function on whole $\mathbb{R}^N$ with no change of Lipschitz constant by the Kirszbraun theorem (see [2] or [98] in German).

2.2.2 Functionals of the type (2,3)

We start with recalling the definition of special functionals from [84].

Definition 2.2.3 (Functional of the type $(2,3)$)**.** Let Ω be an open subset of $\mathbb{R}^n$, $\xi = (\xi_i)_{i=1}^3$ be a triple of vectors belonging to $\mathbb{R}^n$ which are linearly dependent and pairwise independent. Let

$$v(z) = \Big(r\big(z\cdot\xi_1\big), s\big(z\cdot\xi_2\big), t\big(z\cdot\xi_3\big)\Big) \text{ be given function of the type } (2,3).$$

A functional of the form

$$I_f(v,\xi) = \int_\Omega f\big(r(z\cdot\xi_1), s(z\cdot\xi_2), t(z\cdot\xi_3)\big)dz$$

will be called a `functional of the type (2,3)`, while a functional of the form

$$I_f(v) = \int_\Omega f\big(r(x), s(y), t(x+y)\big)dxdy, \text{ where } (x,y)\in\Omega\subseteq\mathbb{R}^2$$

will be called the `special functional of the type (2,3)`.

A crucial problem for us is the investigation of lower semicontinuity property of such functionals with respect to weak-$\star$ convergence of $v's$ in $L^\infty(\Omega,\mathbb{R}^3)$. For this we use the following definition.

Definition 2.2.4 (weak-$\star$ lower semicontinuity, weak-$\star$ continuity)**.**

(i) A functional of the type $(2,3)$ is lower semicontinuous with respect to the sequential weak-$\star$ convergence in $L^\infty(\Omega,\mathbb{R}^3)$ whenever for any sequence $v^\nu \overset{\star}{\rightharpoonup} v$ (i.e. v^ν weak-$\star$ converges to v in $L^\infty(\Omega,\mathbb{R}^3)$) of the functions of the type $(2,3)$ we have

$$\liminf_{\nu\to+\infty} I_f(v^\nu,\xi) \geq I_f(v,\xi).$$

To abbreviate we will call this condition the `(2,3) LSC` property. Those integrands which define functionals having the $(2,3)$ LSC property will be referred as `(2,3) quasiconvex`.

(ii) If $\liminf_{\nu\to+\infty} I_f(v^\nu,\xi) = I_f(v,\xi)$ whenever $v^\nu \overset{\star}{\rightharpoonup} v$, we say that I_f is weakly-$\star$ continuous with respect to the sequential weak-$\star$ convergence in $L^\infty(\Omega,\mathbb{R}^3)$ of the functions of the type $(2,3)$. Integrands defining such functionals will be referred as `(2,3) quasiaffine`.

The weak-$\star$ convergence of the v^ν's is equivalent to convergence of functions building their coordinates: r^ν, s^ν, t^ν with respect to weak-$\star$ convergence in L^∞ on the respective projections of set Ω onto the lines. Moreover, the limiting function v is also of the type $(2,3)$.

Let us note that the non-abstract description of set of $(2,3)$ quasiconvex functions has not been systematically investigated.

From the following fact stated below, it follows that the lower semicontinuity of any functional of the type $(2,3)$ reduces to the case of $\Omega = [0,1]^2$ and $\xi_1 = (1,0), \xi_2 = (0,1), \xi_3 = (1,1)$.

Fact 2.2.5 (Lemma 2.2 in [84]). *Let f be a continuous function $f : \mathbb{R}^3 \to \mathbb{R}$. The following conditions are equivalent.*

i) For $Q = [0,1]^2$ the special functional

$$I_f(v) = \int_Q f\big(r(x), s(y), t(x+y)\big)dxdy \tag{2.7}$$

is lower semicontinuous with respect to weak-$\star$ convergence of r, s, t in $L^\infty(\mathbb{R})$.

ii) For any domain $\Omega \subset \mathbb{R}^N$ and arbitrary triple of vectors $\xi = (\xi_i)_{i=1}^3$ which are pairwise independent, but linearly dependent as a triple, the functional $I_f(v,\xi)$ is lower semicontinuous with respect to weak-$\star$ convergence of r, s, t in $L^\infty(\mathbb{R})$.

In formulation of part (i) Lemma 2.2 in [84] one dealt with a fixed cube, however an easy translation and dilation argument shows that the statement in the lemma is equivalent to the one above.

According to Proposition 2.2.2 an arbitrary special functional of the type $(2,3)$ can be uniquely identified with certain functional defined on the subset of gradients given by (2.6):

$$I_f(v) = \int_\Omega f(r(x), s(y), t(x+y))dxdy = \int_\Omega \tilde{f}(Du_1 + Du_2 + Du_3),$$

where the u_i's were defined in (2.5) and

$$f(r,s,t) = \tilde{f}\left(r\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + s\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + t\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right) \overset{def}{=:} \tilde{f}\left(\begin{bmatrix}r+t & t\\ t & s+t\end{bmatrix}\right), \tag{2.8}$$

$$\tilde{f} \text{ is defined on span } \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}\right\} = M^{2\times 2}_{sym}$$

and we use the standard notation $M^{2\times 2} \cong \mathbb{R}^4$ to denote 2×2 matrices and $M^{2\times 2}_{sym}$ to denote symmetric 2×2 matrices.

2.2.3 Convexity-type conditions

It is not obvious whether there is a geometric interpretation of $(2,3)$ quasiconvex functions. To understand it better we discuss here several convexity-type conditions of geometric type for functions $f : \mathbb{R}^3 \to \mathbb{R}$.

For this we will start with introducing some geometric and algebraic objects we will deal with.

Definition 2.2.6 (Regular symplex)**.** Let p be a point in $\mathbb{R}^3$ and $\{t_i\}_{i=1}^3$ be nonzero real numbers and $(e_i)_{i=1}^3$ the standard $\mathbb{R}^3$ basis. We will call a symplex D `regular`, whenever D is the convex hull of four points: $p, \{p + t_i e_i\}_{i=1}^3$, for some $p, \{t_i\}_{i=1}^3$.

Such a D is obviously a tetrahedron with vertices $p, \{p + t_i e_i\}_{i=1}^3$, having three edges parallel to the axis.

Every regular symplex D defines a cuboid, which has eight vertices, i.e. $p, \{p + t_i e_i\}_{i=1}^3$, $\{p + t_i e_i + t_j e_j\}_{i \neq j}, p + \sum t_i e_i$. For any vertex q of that cuboid let us define the `neighbours` of q - that is those vertices which are linked with q by a single edge.

We introduce the seven dimensional subspace of polynomials

$$\mathcal{A} \stackrel{\text{def}}{=} \text{span}\ \{1, r, s, t, rs, rt, st\}.$$

Folowing [84], having given regular symplex D, we define the projection operator $P_D : C(\mathbb{R}^3) \to \mathcal{A}$ by choosing for any continuous function f such $P_D f \in \mathcal{A}$ that the equality

$$P_D f(r, s, t) = f(r, s, t),$$

holds in every vertex in D and all its neighbours.

Note that vertices $p, \{p + t_i e_i\}_{i=1}^3$ and their neighbours $\{p + t_i e_i + t_j e_j\}_{i \neq j}$ form a set of seven points - vertices of the cuboid defined by D apart from $p + \sum_{i=1}^3 t_i e_i$. As $\mathcal{A}$ is seven-dimensional and those seven points are affinely independent, Kronecker-Capelli theorem shows that $P_D f$ is uniquely defined.

We will deal with the following convexity-type conditions, which contribute to the understanding of (2,3) quasiconvexity condition.

Definition 2.2.7 (Convexity-type conditions)**.** The function $f : \mathbb{R}^3 \to \mathbb{R}$ will be called

(i) `tetraaffine`, whenever f is a polynomial belonging to the linear space $\mathcal{A}$;

(ii) `weakly tetrahedrally convex`, whenever the inequality

$$f(r, s, t) \leq P_D f(r, s, t)$$

holds for every point $(r, s, t) \in D$ and any regular symplex D.

(iii) `tetrahedrally polyconvex` if there exists convex function $g : \mathbb{R}^6 \to \mathbb{R}$ such that

$$f((x_1, x_2, x_3)) = g \circ e((x_1, x_2, x_3))$$

where $e : \mathbb{R}^3 \to \mathbb{R}^6$ is an embedding given by

$$e((x_1, x_2, x_3)) = (x_1, x_2, x_3, x_1 x_2, x_1 x_3, x_2 x_3);$$

(iv) `reduced polyconvex` if there exists convex function $g : \mathbb{R}^4 \to \mathbb{R}$ such that

$$f((x_1, x_2, x_3)) = g \circ i((x_1, x_2, x_3))$$

where $i : \mathbb{R}^3 \to \mathbb{R}^4$ is an embedding given by

$$i((x_1, x_2, x_3)) = \left(x_1, x_2, x_3, \det \begin{bmatrix} x_1 + x_3 & x_3 \\ x_3 & x_2 + x_3 \end{bmatrix}\right).$$

For any convexity type condition $\mathcal{C}$ a class of $\mathcal{C}-$affine functions is understood as functions f such that both f and $-f$ satisfy $\mathcal{C}$. This way we will deal with weakly tetrahedrally affine, tetrahedrally polyaffine and reduced polyaffine functions, respectively.

The following remark is in order.

Remark 2.2.8.

(i) Tetraaffine functions have appeared in the paper [84]. The following Proposition has been obtained there.

Proposition 2.2.9. *[84, Corollary 3.4] The function $f : \mathbb{R}^3 \to \mathbb{R}$ is tetraaffine if and only if it is $(2, 3)$ quasiaffine.*

(ii) The weak tetrahedral convexity was one of the two conditions established in [84], which together were called "tetrahedral convexity". That is the motivation of adding the word "weak" in the above definition. The following statement follows from Theorem 3.2 obtained in [84]. In the formulation we omit the second condition obtained there.

Proposition 2.2.10. *If f is $(2, 3)$ quasiconvex then f is weakly tetrahedrally convex.*

(iii) The notions of tetrahedral polyconvexity and reduced polyconvexity are related to the classical notion of polyconvexity condition due to Ball [17]. In case of functions defined on 2×2 matrices, function F is called `polyconvex`, if there exist the convex function $G : M^{2\times 2} \times \mathbb{R} \to \mathbb{R}$ such that $F = G \circ E$ and $E(X) = (X, \det X)$ for any matrix $X \in M^{2\times 2}$.

Define $I : \mathbb{R}^3 \to M^{2\times 2}_{sym}$ be the isomorphism given by

$$I(x_1, x_2, x_3) = \left(x_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + x_3 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_3 & x_3 \\ x_3 & x_2 + x_3 \end{bmatrix}.$$

Let us now define another isomorphism $J : M^{2\times 2}_{sym} \times \mathbb{R} \to \mathbb{R}^4$ by

$$J\left(\begin{bmatrix} r + t & t \\ t & s + t \end{bmatrix}, x\right) = (r, s, t, x).$$

For any symmetric matrix M and real number x we have

$$J(M, x) = (I^{-1}(M), x) \text{ and } J \circ E \circ I(r, s, t) = i(r, s, t).$$

Therefore, when f is reduced polyconvex, we have $f(r, s, t) =$

$$g \circ i(r, s, t) = (g \circ J) \circ E \circ \big(I(r, s, t)\big) = G \circ E\big(I(r, s, t)\big) = F \circ I(r, s, t),$$

involving convex function g, where $G = g \circ J$ is convex and $F = G \circ E$ is polyconvex in the classical sense. As I was a linear isomorphism between $\mathbb{R}^3$ and $M^{2\times 2}_{sym}$, f is identified through I with the classical polyconvex function reduced to the space of symmetric matrices.

(iv) The equality

$$\det \begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix} = rs + rt + st \tag{2.9}$$

shows that $\det \circ I$ is a tetraaffine function. It is also a `reduced polyaffine` function, i.e. such function f that both f and $-f$ are reduced polyconvex.

The following proposition characterises tetrahedrally polyconvex functions as supremas of tetraaffine polynomials. Note that this is a similar concept to convexity (resp. polyconvexity) - that is being a supremum of some family of affine (resp. quasiaffine) functions.

Proposition 2.2.11. *Function f is tetrahedrally polyconvex if and only if it is equal to supremum of some family of tetraaffine functions.*

Proof. Let $\mathcal{F}$ be a family of tetraaffine functions such that

$$\forall (r, s, t) \in \mathbb{R}^3,\ \sup_{p\in\mathcal{F}} p(r, s, t) < +\infty.$$

$$\text{Then } P(r, s, t) = \sup_{p\in\mathcal{F}} p(r, s, t)$$

is tetrahedrally polyconvex. It's easy to check, as the function

$$g(r, s, t, x, y, z) = \sup_{p\in\mathcal{F}}(a_0 + a_1 r + a_2 s + a_3 t + a_4 x + a_5 y + a_6 z)$$

(where coefficients $(a_i)_{i=0}^{6}$ are such that $(a_0 + a_1 r + a_2 s + a_3 t + a_4 rs + a_5 rt + a_6 st) \in \mathcal{A}$) is convex (because g is a supremum of the affine functions) and $P = g \circ e$ (obvious from the definition of g and e).

For the inverse notice that the function g from the tetrahedral polyconvexity definition, as a convex function, always needs to be supremum of affine functions (in 6 coordinates) and thus, after embedding e, any tetrahedrally polyconvex function needs to be a supremum of tetraaffine functions. □

Our next statement compares the introduced convexity conditions.

Lemma 2.2.12. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the given continuous function. Then the following implications hold:*

$$f \text{ is reduced polyconvex} \overset{(1)}{\Rightarrow} f \text{ is tetrahedrally polyconvex} \overset{(2)}{\Rightarrow} f \text{ is } (2,3) \text{ quasiconvex}$$
$$\overset{(3)}{\Rightarrow} f \text{ is weakly tetrahedrally convex} \overset{(4)}{\Rightarrow} f \text{ is convex along the axis.}$$

Moreover, the inverse implications to (1) *and* (4) *do not hold.*

In the following section we will show that the inverse implication to (2) also does not hold.

Proof.

"$\overset{(1)}{\Rightarrow}$" Assume there exists a convex function $G : M^{2\times 2}_{sym} \times \mathbb{R} \to \mathbb{R}$ such that, under the notation from Remark 2.2.8 we have

$$G(A, \det A) = G \circ E(A) = F(A) \text{ and } f = F \circ I.$$

We construct the projection $\pi : \mathbb{R}^3 \times \mathbb{R}^3 \to M^{2\times 2}_{sym} \times \mathbb{R}$ by

$$\pi([r,s,t],[x,y,z]) \overset{\text{def}}{=} \left(\begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix}, x+y+z \right).$$

Function π is affine and $\pi \circ e(r,s,t) =$

$$\pi(r,s,t,rs,rt,st) = \left(\begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix}, \det \begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix} \right) = E \circ I(r,s,t),$$

because of (2.9). Thus $g := G \circ \pi$ is a convex function and

$$g \circ e = G \circ \pi \circ e = G \circ E \circ I = F \circ I = f,$$

as required for tetrahedral polyconvexity.

"$\overset{(1)}{\not\Leftarrow}$" Let $f(r,s,t) = rs$. To verify tetrahedral polyconvexity condition for f it is sufficient to consider $g(r,s,t,x,y,z) := x$. Then g is convex and $g \circ e(r,s,t) = rs = f(r,s,t)$. However f is not reduced polyconvex. We will proceed with a contradiction. Assume there exists a convex function $G : \mathbb{R}^4 \to \mathbb{R}$ such that $G \circ E = f$. Therefore $G(r,s,t,rs+rt+st) = rs$. As G is convex, it is bounded from below by an affine function $h : \mathbb{R}^4 \to \mathbb{R}$,

$$h(r,s,t,x) = h_0 + h_1 r + h_2 s + h_3 t + h_4 x.$$

We have then for every r, s, t, that

$$h \circ E(r,s,t) = h(r,s,t,rs+rt+st) \leq rs = G \circ E(r,s,t) = f(r,s,t).$$

in particular we have that $h(r,0,t,rt) = h_0 + h_1 r + h_3 t + h_4 rt \leq rs$. Taking $r = 0$, from arbitrariness of t it follows that $h_0 \leq 0, h_3 = 0$. Furthermore, $h(r,s,0,rs) = h_0 + h_1 r + h_2 s + h_4 rs \leq rs$. Taking $s = 0$ shows that $h_1 = 0$. Analogously taking $r = 0$ shows that $h_2 = 0$. We obtain now that $h_0 + h_4 rs \leq rs$ for any r, s and thus $h_4 = 1$. So, if there exists such a function h, it is $h(r,s,t,x) = h_0 + x$ for some nonpositive h_0. Thus

$$h \circ E(r,s,t) = h_0 + rs + rt + st \leq rs$$

for any r, s, t. Taking now however $r = s = t$ we obtain $h_0 + 3r^2 \leq r^2$ which obviously doesn't hold for any h_0.

"$\overset{(2)}{\Rightarrow}$" It's easy to check that if $\{f_j\}_{j\in J}$ is a family $(2,3)$ quasiconvex functions, then $\sup\limits_{j\in J} f_j$ is also $(2,3)$ quasiconvex. From Proposition 2.2.11 any tetrahedrally polyconvex is a supremum of some family $\{p_j\}_{j\in J}$ of tetraaffine functions. As any tetraaffine function p_j is $(2,3)$ quasiconvex, the proof is done.

"$\overset{(3)}{\Rightarrow}$" This implication is just Proposition 2.2.10.

"$\overset{(4)}{\Rightarrow}$" For simplicity let us show the convexity of weakly tetrahedrally convex function f along the axis e_1. Let $p_1 = (r_1, s, t), p_2 = (r_2, s, t)$ be two point spanning the line parallel to e_1 axis. Let us prescribe any regular symplex D on the segment connecting points p_i. Now we obtain $f \leq P_D f$ in D, so also on the segment. As $P_D f$ is affine along every axis, we have shown that f is subaffine along e_1.

"$\overset{(4)}{\not\Leftarrow}$" Note that the function $f(r,s,t) = rst$ is indeed convex along every axis. It is not however weakly tetrahedrally convex. To prove that, take simplex D with vertices $(0,0,0), (1,0,0), (0,1,0)$ and $(0,0,1)$ and note that $P_D f \equiv 0$, however $p = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}) \in D$ and $f(p) = \frac{1}{64} > 0$. □

More precise statement holds for respective affinity conditions.

Theorem 2.2.13. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the given continuous function. Then the following implications hold:*

$$f \text{ is reduced polyaffine} \overset{(1)}{\Rightarrow} f \text{ is tetrahedrally polyaffine} \overset{(2)}{\Longleftrightarrow} f \text{ is } (2,3) \text{ quasiaffine}$$
$$\overset{(3)}{\Longleftrightarrow} f \text{ is weakly tetrahedrally affine} \overset{(4)}{\Rightarrow} f \text{ is affine along the axis.}$$

Moreover, the inverse implications to (1) and (4) do not hold.

Remark 2.2.14. Theorem 2.2.13 shows that the classes of tetrahedrally polyaffine, $(2,3)$ quasiaffine and weakly tetrahedrally affine functions are the same and equal to the class of tetraaffine functions.

Proof of Theorem 2.2.13. Implications to the right are already proven in Lemma 2.2.12. Also the inverse to (1) is contradicted. Let us here remind that the class of $(2,3)$ quasiaffine functions coincide with the class of tetraaffine functions (see Proposition 2.2.9). We will prove the following implications.

"$\overset{(2)}{\Leftarrow}$" Tetraaffine functions are tetrahedrally polyaffine because any $p \in \mathcal{A}$ satisfies $p = g \circ e$ for certain affine g.

"$\overset{(3)}{\Leftarrow}$" We will prove that a weakly tetrahedrally affine function f must be equal to certain tetraaffine p on every regular symplex D. This is the case indeed, because on such D we have $f \leq P_D f$ and $-f \leq -P_D f$. This however finishes the proof because two tetraaffine functions equal on any open set are equal in every point.

"$\overset{(4)}{\not\Leftarrow}$" Consider $f(r,s,t) = rst$. It is affine in the directions of the axis and not tetraaffine. □

2.3 Tetrahedral polyconvexity $\not\Leftarrow$ (2,3) quasiconvexity

In our analysis we are interested in functions defined on $\mathbb{R}^3$ and the respective integrands, which define functionals of the type $(2,3)$. We are now to prove that the inverse implication to (2) in Lemma 2.2.12 does not hold. For that we benefit from the well-known result in calculus of variations due to Alibert and Dacorogna [7] (see also [13] for related results). This is possible due to the canonical embedding of functions of the type $(2,3)$ into the special subspace of gradients - see Proposition 2.2.2.

The authors of [7] have introduced the following function $\tilde{f}_\gamma : M^{2\times 2} \to \mathbb{R}$, defined by

$$\tilde{f}_\gamma(A) = |A|^2(|A|^2 - 2\gamma \det A), \tag{2.10}$$

where $|A|$ stays for the Euclidean norm of A, i. e. $\left|\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\right|^2 = a^2 + b^2 + c^2 + d^2.$

The following theorem holds.

Theorem 2.3.1 (Alibert, Dacorogna [7])**.** *The following statements hold.*

a) *Function $\tilde{f}_\gamma$ defined in* (2.10) *is convex* $\iff$ $|\gamma| \leq \frac{2}{3}\sqrt{2}$.

b) *Function $\tilde{f}_\gamma$ defined in* (2.10) *is polyconvex* $\iff$ $|\gamma| \leq 1$ *(see Remark 2.2.8 point (iii)).*

c) *Function $\tilde{f}_\gamma$ defined in* (2.10) *is rank-one convex (i.e. the function $t \mapsto \tilde{f}_\gamma(A + tB)$ is convex for any matrix A and any rank-one matrix B)* $\iff$ $|\gamma| \leq \frac{2}{\sqrt{3}}$.

d) *There exists* $\varepsilon > 0$ *such that* $I_{\tilde{f}_\gamma}$ *is lower semicontinuous with respect to the sequential weak-* $\star$ *convergence of gradients of Lipschitz functions in* $L^\infty(\Omega, \mathbb{R}^2)$ $\Big($*i. e. it has the property: when* $\{u^\nu\} \subseteq W^{1,\infty}(\Omega, \mathbb{R}^2)$ *is a bounded sequence and* $Du^\nu \overset{\star}{\rightharpoonup} Du$ *in* $L^\infty(\Omega, M^{2\times 2})$, $u^\nu \to u$ *in* $L^1(\Omega, \mathbb{R}^2)$ *then* $\liminf_{\nu\to+\infty} I_{\tilde{f}_\gamma}(Du^\nu) \geq I_{\tilde{f}_\gamma}(Du)\Big)$ $\iff$ $|\gamma| \leq 1 + \varepsilon$.

Defining $A(r, s, t) = \begin{bmatrix} r+t & t \\ t & s+t \end{bmatrix}$ and using identification (2.8) and (2.10) we obtain

$$f_\gamma(r, s, t) := \tilde{f}_\gamma(A(r, s, t)) = \\ \big(r^2 + s^2 + 4t^2 + 2(r+s)t\big)\big(r^2 + s^2 + 4t^2 + 2(r+s)t - 2\gamma(rs + rt + st)\big).$$

We will investigate tetrahedral polyconvexity condition for function $f_\gamma : \mathbb{R}^3 \to \mathbb{R}$.

Theorem 2.3.2. *Function* $f_\gamma(r, s, t)$, *is tetrahedrally polyconvex if and only if* $|\gamma| \leq 1$.

Proof.

" $\Leftarrow$ " At first we note that f_γ is reduced polyconvex for $|\gamma| \leq 1$ as we have $\tilde{f}_\gamma = G \circ E$ under the notation of Remark 2.2.8 point (iii), where G is convex. We have then that

$$f_\gamma = G \circ E \circ I = G \circ J^{-1} \circ J \circ E \circ I = \big(G \circ J^{-1}\big) \circ i =: \tilde{G} \circ i,$$

where $\tilde{G}$ is convex. As reduced polyconvexity implies tetrahedral polyconvexity (see Lemma 2.2.12) the proof is done.

" $\Rightarrow$ " Assume on the contrary that there exists γ such that $|\gamma| > 1$ and f_γ is tetrahedrally polyconvex. Then there exists an affine function $p : \mathbb{R}^6 \to \mathbb{R}$,

$$p(r, s, t, x, y, z) := p_0 + v \cdot (r, s, t, x, y, z),$$

where $(v_1, v_2, v_3, v_4, v_5, v_6) \in \mathbb{R}^6$ is a constant vector and p is such that $f_\gamma(r, s, t) \geq p \circ e(r, s, t)$ for any r, s, t. We compute that

$$\begin{aligned} f_\gamma(r, cr, 0) &= r^4(1 + 2c^2 + c^4 - 2\gamma(c + c^3)), \\ p(r, cr, 0, cr^2, 0, 0) &= p_0 + v_1 r + c v_2 r + c v_4 r^2. \end{aligned}$$

This implies the inequality

$$r^4(1 + 2c^2 + c^4 - 2\gamma(c + c^3)) \geq p_0 + (v_1 + c v_2) r + c v_4 r^2,$$

holding for every $r, c \in \mathbb{R}$. We obviously need a coefficient $\kappa_\gamma(c) := (1 + 2c^2 + c^4 - 2\gamma(c + c^3))$ to be nonegative for every c. For $\gamma > 1$ we obtain that $\kappa_\gamma(1) = 4 - 4\gamma < 0$, for $\gamma < -1$ we get $\kappa_\gamma(-1) = 4 + 4\gamma < 0$. Therefore f_γ cannot be tetrahedrally polyconvex. $\square$

We end up with the following statement, which is the main result in this section.

Theorem 2.3.3. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the given continuous function. Then the following implications hold:*

$$f \text{ is reduced polyconvex} \overset{(1)}{\Rightarrow} f \text{ is tetrahedrally polyconvex} \overset{(2)}{\Rightarrow} f \text{ is } (2,3) \text{ quasiconvex}$$
$$\overset{(3)}{\Rightarrow} f \text{ is weakly tetrahedrally convex} \overset{(4)}{\Rightarrow} f \text{ is convex along the axis.}$$

Moreover, the inverse implications to $(1), (2), (4)$ *do not hold.*

Proof. Implications $\overset{(1)}{\Rightarrow}, \overset{(2)}{\Rightarrow}, \overset{(3)}{\Rightarrow}, \overset{(4)}{\Rightarrow}$, as well as $\overset{(1)}{\not\Leftarrow}$ and $\overset{(4)}{\not\Leftarrow}$ have been already established in Lemma 2.2.12. For the proof of the property $\overset{(2)}{\not\Leftarrow}$ we have to show that there exists a $(2,3)$ quasiconvex function that is not tetrahedrally polyconvex. Let $\gamma = 1 + \varepsilon$ as in point d) of Theorem 2.3.1. Now f_γ is $(2,3)$ quasiconvex due to embedding (2.6). Theorem 2.3.2 shows however, that f_γ is not tetrahedrally polyconvex. □

We address the following problem.

Open Problem 2.3.4. We do not know whether the inverse implication to (3) holds.

However the problem is open, let us note that the case is much simpler with bilinear forms.

Fact 2.3.5. *A bilinear form P is convex along each axis if and only if P is tetrahedrally polyconvex.*

Proof. Of course we only need to prove to prove the implication "$\Rightarrow$". Let $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $P(x) = \sum_{i=1}^{3} a_i x_i^2 + \sum_{i<j} a_{ij} x_i x_j =: Q + T$ be convex along the axis. As T is affine along the axis, it follows that Q must be convex along the axis. It shows that $a_i \geq 0$ for any $1 \leq i \leq 3$. Thus P is equal to the sum of tetrahedrally polyaffine form T and convex form Q. It is obvious that any convex function is tetrahedrally polyconvex and thus P is tetrahedrally polyconvex. □

2.4 Carathéodory type theorem

Our goal now is to obtain a variant of Carathéodory theorem for tetrahedrally polyconvex functions. This statement will be needed later to discus the locality properties of tetrahedral polyconvexity condition. For our analysis we have to define and investigate the tetrahedral polyconvex envelope of the given function f.

Let us start by recalling the definitions of convex hulls of sets, as well as the classical Carathéodory theorem.

Definition 2.4.1 (Convex hull, convex envelope)**.** For any subset X of a linear space $\mathcal{V}$ we define `convex hull` of X as

$$CH\ X \stackrel{\text{def}}{=} \bigcap\{C \mid C \text{ is convex and } X \subseteq C\}.$$

For any function $f : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ we define `convex envelope` of f as

$$Cf \stackrel{\text{def}}{=} \sup\{g(x) \mid g \text{ is convex and } g \leq f\}.$$

We use the convention that $\sup \emptyset = -\infty$.

Theorem 2.4.2 (Carathéodory Theorem, 1911, [33])**.** *Let X be a subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. Then*

i) $CH\ X = \{x \in \mathbb{R}^n \mid x = \sum_{i=1}^{n+1} \lambda_i x_i, x_i \in X, \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \in [0,1]\}$,

ii) $Cf(x) = \inf\left\{\sum_{i=1}^{n+1} \lambda_i f(x_i) \mid \sum_{i=1}^{n+1} \lambda_i x_i = x, \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \in [0,1]\right\}$.

To proceed further we require the following definition.

Definition 2.4.3 (Tetrahedral polyconvex envelope)**.** For the function $f : \mathbb{R}^3 \to \mathbb{R}$ we define `tetrahedral polyconvex envelope` via

$$TPEf(r,s,t) \stackrel{\text{def}}{=} \sup\{g(r,s,t) \mid g \text{ is tetrahedrally polyconvex and } g \leq f\}.$$

It is clear that if $TPEf \neq -\infty$, it is then a tetrahedrally polyconvex function. This is because at the same time $TPEf(\cdot,\cdot,\cdot)$ is a supremum of tetraaffine functions.

We are now to prove the variant of part ii) in Carathéodory theorem dealing with tetrahedral polyconvex envelopes of functions. Our arguments are based on variants of Carathéodory Theorem similar as presented in [41], Chapter 5. For readers convenience we present the proof in detail, as it contains not so trivial arguments.

Theorem 2.4.4 (Carathéodory Theorem with tetrahedrally polyconvex envelope)**.** *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a given function. Then the following statements hold.*

i) *The function $g^f : \mathbb{R}^6 \to \mathbb{R} \cup \{-\infty\}$ given by*

$$g^f(r,s,t,x,y,z) \stackrel{def}{=} \inf\left\{\sum_{i=1}^{7} \lambda_i f(r_i,s_i,t_i) \mid \sum_{i=1}^{7} \lambda_i e(r_i,s_i,t_i) = (r,s,t,x,y,z) \right. \\ \left. \text{where } \{\lambda_i\}_{i=1}^{7} : \text{ such that } \sum_{i=1}^{7} \lambda_i = 1, \lambda_i \in [0,1]\right\},$$

is well defined and convex.

ii) *If f is tetrahedrally polyconvex, then*

$$f(r,s,t) = g^f(e(r,s,t)) \text{ for any } (r,s,t) \in \mathbb{R}^3.$$

Moreover, for any f we have

$$g^f \circ e(r,s,t) = g^f(r,s,t,rs,rt,st) = TPEf.$$

Proof. *"Part i):"*
We begin with constructing desired convex function g^f. For integers $N \geq 7$ we define

$$g^f_N(r,s,t,x,y,z) \stackrel{\text{def}}{=} \inf S_N(r,s,t,x,y,z), \text{ where we set}$$

$$S_N = \left\{ \sum_{i=1}^{N} \lambda_i f(r_i,s_i,t_i) \mid \sum_{i=1}^{N} \lambda_i e(r_i,s_i,t_i) = (r,s,t,x,y,z), \lambda_i \in [0,1], \sum_{i=1}^{N} \lambda_i = 1 \right\}.$$

We divide the proof into steps.

Step 1. We show that

$$CH\ e(\mathbb{R}^3) = \mathbb{R}^6.$$

Step 2. We prove that g^f_7 is well defined (and thus also g^f_N whenever $N \geq 7$) and for any $N \geq 7$ we have $g^f_N = g^f_7 \stackrel{\text{def}}{=} g^f$.

Step 3. We prove that g^f is convex.

Proof of Step 1: Assume than $CH\ e(\mathbb{R}^3) \neq \mathbb{R}^6$. Thus $CH\ e(\mathbb{R}^3)$, as a convex set, lies in some halfspace of the form

$$\mathcal{H} = \{v \in \mathbb{R}^6 \mid \alpha \cdot v < \beta\}$$

for some nonzero $\alpha \in (\mathbb{R}^6)^* \cong \mathbb{R}^6$ and real β. Now $e(\mathbb{R}^3) \subseteq CH\ e(\mathbb{R}^3) \subseteq \mathcal{H}$. To show a contradiction we will find a triple (r,s,t) such that $\alpha \cdot (r,s,t,rs,rt,st)$ is not less then β. Let $\alpha = (\alpha_i)_{i=1}^6$ and i_0 be the smallest index i such that $\alpha_i \neq 0$. Assume first that $i_0 \leq 3$. Let then $(\bar{r},\bar{s},\bar{t})$ be equal to the i_0^{th} vector of the standard basis of $\mathbb{R}^3$. Now $e(\bar{r},\bar{s},\bar{t}) = (e_{i_0},0,0,0)$ (which is the i_0^{th} vector of the standard basis of $\mathbb{R}^6$) and for $(r,s,t) = (\frac{\beta}{\alpha_{i_0}}(\bar{r},\bar{s},\bar{t}))$ we arrive at

$$\alpha \cdot e(r,s,t) = \beta,$$

which contradicts the inclusion $e(\mathbb{R}^3) \subseteq \mathcal{H}$. For the case where $i_0 = 4$ take $(r,s,t) = \frac{\beta}{\alpha_4}(1,1,0)$ (so that α and $e(r,s,t)$ meet only in fourth place). Similar reasoning holds for $i_0 = 5, 6$, which finishes the proof of Step 1.

Proof of Step 2: To begin let us note that from Carathéodory Theorem (Theorem 2.4.2) and Step 1 we see that

$$CH\ e(\mathbb{R}^3) = \{\sum_{i=1}^{7} \lambda_i e(r_i, s_i, t_i), \sum_{i=1}^{7} \lambda_i = 1, \lambda_i \in [0,1]\} = \mathbb{R}^6$$

and thus g_7^f is well defined, that is $S_7 \neq \emptyset$.

Now let us introduce two substeps.

Substep 2A: We prove that for $N \geq 8$ we have $S_N = S_8$.

Let us recall the definition of the epigraph of a function f:

$$epif \overset{\text{def}}{=} \{(r,s,t,x) \in \mathbb{R}^4 \mid f(r,s,t) \leq x\}$$

and define

$$\hat{e}(epif) \overset{\text{def}}{=} \{(e(r,s,t),x) \mid f(r,s,t) \leq x\} \subseteq \mathbb{R}^7.$$

Note that $(e(r,s,t), f(r,s,t)) \in \hat{e}(epif)$, therefore any convex combination of such points belongs to $CH\ \hat{e}(epif)$.

As $\hat{e}(epif) \subseteq \mathbb{R}^7$, from Carathéodory Theorem (Theorem 2.4.2) it follows that

$$CH\ \hat{e}(epif) = \left\{ \sum_{i=1}^{8} \lambda_i (e(r_i,s_i,t_i), f(r_i,s_i,t_i) \mid \lambda_i \in [0,1], \sum_{i=1}^{8} \lambda_i = 1 \right\}.$$

It implies that $S_N \subseteq S_8$. Indeed, let $\bar{f} = \sum_{i=1}^{N} \lambda_i f(r_i,s_i,t_i) \in S_N(r,s,t,x,y,z)$ i.e.

$$\sum_{i=1}^{N} \lambda_i e(r_i,s_i,t_i) = (r,s,t,x,y,z), \lambda_i \in [0,1], \sum_{i=1}^{N} \lambda_i = 1.$$

Hence

$$\sum_{i=1}^{N} \lambda_i (e(r_i,s_i,t_i), f(r_i,s_i,t_i)) \in CH\ \hat{e}(epif).$$

Therefore there exist $\{\bar{\lambda}_i\}_{i=1}^{8}$ and $\{(\bar{r}_i,\bar{s}_i,\bar{t}_i)\}_{i=1}^{8}$ such that

$$\sum_{i=1}^{8} \bar{\lambda}_i (e(\bar{r}_i,\bar{s}_i,\bar{t}_i), f(\bar{r}_i,\bar{s}_i,\bar{t}_i)) = \sum_{i=1}^{N} \lambda_i (e(r_i,s_i,t_i), f(r_i,s_i,t_i)).$$

We obtain $\bar{f} \in S_8$. As sequence of sets S_N is nondecreasing, we see that for $N \geq 8$ we have $S_N = S_8$. As $g_N^f(r,s,t,x,y,z) = \inf S_N(r,s,t,x,y,z)$, we establish $g_N^f = g_8^f$ for any $N \geq 8$.

Substep 2B: We show that $g^f = g_7^f = g_8^f$. It suffices to prove $g_7^f \leq g_8^f$.

Take any $v \in \mathbb{R}^6 = CH\ e(\mathbb{R}^3)$, a sequence $\{\alpha_i\}_{i=1}^8$ satisfying $\alpha_i \in [0,1], \sum_{i=1}^8 \alpha_i = 1$ and points $\{(r_i, s_i, t_i)\}_{i=1}^8$ such that

$$\sum_{i=1}^{8} \alpha_i e(r_i, s_i, t_i) = v.$$

From Carathéodory Theorem (Theorem 2.4.2) applied to the set $\{(r_i, s_i, t_i)\}_{i=1}^8$, there exists a sequence $\{\beta_i\}_{i=1}^8$ satisfying $\beta_i \in [0,1], \sum_{i=1}^8 \beta_i = 1$ such that at least one of β_i vanishes and

$$\sum_{i=1}^{8} \beta_i e(r_i, s_i, t_i) = v.$$

To finish the proof of this substep it suffices to show that

$$\sum_{i=1}^{8} \alpha_i f(r_i, s_i, t_i) \geq \sum_{i=1}^{8} \beta_i f(r_i, s_i, t_i). \tag{2.11}$$

We may assume that all $\alpha_i s$ are positive, as otherwise we could take $\{\beta_i\}_{i=1}^8 = \{\alpha_i\}_{i=1}^8$. Assume that this inequality does not hold, i. e.

$$\sum_{i=1}^{8} \alpha_i f(r_i, s_i, t_i) < \sum_{i=1}^{8} \beta_i f(r_i, s_i, t_i). \tag{2.12}$$

In particular the set $C := \{i \in \{1, 2, \ldots, 8\} \mid \beta_i > \alpha_i\}$ is nonempty, because $\{\alpha_i\}$ and $\{\beta_i\}$ are two different sequences with equal sum of coefficients. Set now

$$\gamma \stackrel{\text{def}}{=} \min_{i \in C} \left\{ \frac{\alpha_i}{\beta_i - \alpha_i} \right\}.$$

Note that γ is positive from the definition of C. Moreover, let

$$\lambda_i \stackrel{\text{def}}{=} \alpha_i + \gamma(\alpha_i - \beta_i).$$

We have now

$$\sum_{i=1}^{8} \lambda_i = \sum_{i=1}^{8} \alpha_i + \gamma \sum_{i=1}^{8} (\alpha_i - \beta_i) = 1$$

and from the definition of γ we have $\lambda_i \geq 0$ for every i. It follows that every $\lambda_i \leq 1$ for every $i \in \{1, \ldots, 8\}$. From the definition of γ it also follows that there exists i such that $\lambda_i = 0$ - this is exactly index i on which we obtain a minimum in the definition of γ. Furthermore,

$$\sum_{i=1}^{8} \lambda_i e(r_i, s_i, t_i) = (1 + \gamma)v - \gamma v = v, \text{ and}$$

$$\sum_{i=1}^{8}\lambda_i f(r_i,s_i,t_i)=\sum_{i=1}^{8}\alpha_i f(r_i,s_i,t_i)+\gamma(\sum_{i=1}^{8}(\alpha_i-\beta_i)f(r_i,s_i,t_i))\overset{(2.12)}{<}\sum_{i=1}^{8}\alpha_i f(r_i,s_i,t_i),$$

because we assumed $\sum(\alpha_i-\beta_i)f(r_i,s_i,t_i)<0$ and dealt positive γ. We have shown that when the inequality (2.11) did not hold with coefficients $\{\beta_i\}$, it holds with coefficient $\{\lambda_i\}$. This finishes Step 2.

Proof of Step 3: From Step 2 we already know $g^f=g^f_7$. We are now to show that for $\lambda\in[0,1]$ and any vectors $v,w\in\mathbb{R}^6$ we have inequality

$$\lambda g^f(v)+(1-\lambda)g^f(w)\geq g^f(\lambda v+(1-\lambda)w).$$

From the definition of g^f we have that for any $\varepsilon>0$ there exist $(\mu_i)_{i=1}^7,(\nu_i)_{i=1}^7$ satisfying $\mu_i,\nu_i\in[0,1],\sum_{i=1}^7\mu_i=\sum_{i=1}^7\nu_i=1$ and $(r_i,s_i,t_i)_{i=1}^7,(\bar r_i,\bar s_i,\bar t_i)_{i=1}^7$ such that

$$\sum_{i=1}^{7}\mu_i e(r_i,s_i,t_i)=v,\sum_{i=1}^{7}\nu_i e(\bar r_i,\bar s_i,\bar t_i)=w,\text{ and}$$

$$\lambda g^f(v)+(1-\lambda)g^f(w)+\varepsilon\geq\lambda\sum_{i=1}^{7}\mu_i f(r_i,s_i,t_i)+(1-\lambda)\sum_{i=1}^{7}\nu_i f(\bar r_i,\bar s_i,\bar t_i).$$

Defining new sequence as

$$\lambda_i\overset{\text{def}}{=}\lambda\mu_i,\ \lambda_{7+i}\overset{\text{def}}{=}(1-\lambda)\nu_i,$$

and new points as

$$(r_i,s_i,t_i)\overset{\text{def}}{=}(r_i,s_i,t_i),\ (r_{7+i},s_{7+i},t_{7+i})\overset{\text{def}}{=}(\bar r_i,\bar s_i,\bar t_i),\text{ where }i=1,\dots,7,$$

we arrive at

$$\lambda g^f(v)+(1-\lambda)g^f(w)+\varepsilon\geq\sum_{i=1}^{14}\lambda_i f(r_i,s_i,t_i),\text{ where}$$

$$\sum_{i=1}^{14}\lambda_i e(r_i,s_i,t_i)=\lambda v+(1-\lambda)w.$$

Using the definition of g^f and the fact that $g^f=g^f_{14}$ we get

$$\lambda g^f(v)+(1-\lambda)g^f(w)+\varepsilon\geq g^f\left(\lambda v+(1-\lambda)w\right)$$

and arbitrariness of ε finishes the proof of Step 3 and of Part i).
"Part ii):"
At first we will show that if f is tetrahedrally polyconvex, then $f=g^f\circ e$.

For this let us consider a convex function $g: \mathbb{R}^6 \to \mathbb{R}$ such that $g \circ e = f$. As g is convex we have that for any choice of points $v_i \in \mathbb{R}^6$

$$\sum_{i=1}^{7} \lambda_i g(v_i) \geq g\left(\sum_{i=1}^{7} \lambda_i v_i\right),$$

where $\lambda_i \in [0,1]$ and $\sum_{i=1}^{7} \lambda_i = 1$. Taking $\{\lambda_i\}_{i=1}^{7}$ and $\{(r_i, s_i, t_i)\}_{i=1}^{7}$ such that

$$\sum_{i=1}^{7} \lambda_i e(r_i, s_i, t_i) = e\bigg(\sum_{i=1}^{7} \lambda_i (r_i, s_i, t_i)\bigg) = e(r, s, t) \tag{2.13}$$

shows that

$$\sum_{i=1}^{7} \lambda_i f(r_i, s_i, t_i) \geq f\left(\sum_{i=1}^{7} \lambda_i (r_i, s_i, t_i)\right) = f(r, s, t). \tag{2.14}$$

Taking infimum over all possible coefficients $\{\lambda_i\}_{i=1}^{7}$ and points $\{(r_i, s_i, t_i)\}_{i=1}^{7}$ satisfying (2.13), (2.14) and using the definition of g^f we obtain $g^f(e(r,s,t)) \geq f(r,s,t)$. As we obviously have $g^f(e(r,s,t)) \leq f(r,s,t)$, we obtain $g^f \circ e = f$.

To prove the second statement note that $g^f \circ e$ is tetrahedrally polyconvex, because g^f is convex. What is left is to establish that $g^f = TPEf$. From the definition of $TPEf$ and tetrahedral polyconvexity of $g^f \circ e$, we have that $g^f \circ e \leq TPEf$, because $g^f \circ e \leq f$. Observe that the following monotonicity property holds: when $h \leq f$ we have $g^h \leq g^f$. Moreover, from the already established first statement in this part, $h \mapsto g^h \circ e$ is a projection onto tetrahedrally polyconvex functions. Therefore we have $h = g^h \circ e \leq g^f \circ e$, whenever $h \leq f$ and h is tetrahedrally polyconvex. Taking $h = TPEf$ finishes the proof. □

We end this section with the following characterisation of tetrahedrally polyconvex functions, which is the consequence of the Theorem 2.4.4.

Corollary 2.4.5. *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the given function. The following conditions are equivalent:*

a) f is tetrahedrally polyconvex;

b) for any $(r,s,t) \in \mathbb{R}^3$, any coefficients $\{\lambda_i\}_{i=1}^{7}$ such that $\sum_{i=1}^{7} \lambda_i = 1$, $\lambda_i \in [0,1]$ and any triples of real numbers $\{(r_i, s_i, t_i)\}_{i=1}^{7}$ such that $\sum_{i=1}^{7} \lambda_i e(r_i, s_i, t_i) = e(r,s,t)$ the following Jensen-type inequality holds:

$$f(r,s,t) \leq \sum_{i=1}^{7} \lambda_i f(r_i, s_i, t_i).$$

Proof.

"$a) \Rightarrow b)$" We proceed exactly like in the proof of Part ii) of Theorem 2.4.4.

"$a) \Leftarrow b)$" Having a function f satisfying b), from the definition of the g^f in Theorem 2.4.4, Part i), we see that $g^f \circ e = f$ (while we always have $g^f \circ e \leq f$ and the converse inequality follows from b)). As a function g^f is convex (see Theorem 2.4.4, Part i), Step 3) the proof is done.

□

2.5 Non-locality of tetrahedral polyconvexity

In this section we are going to prove, that there exist no local condition for tetrahedral polyconvexity. We proceed in a similar way to Kristensen in [109]. We begin with some definitions, useful in the reasoning.

For f - a function of class $C^2(\mathbb{R}^3;\mathbb{R})$ recall the Taylor formula

$$f(z+w) = f(z) + Df(z)w + \frac{1}{2}D^2f(z)(w;w) + \rho(z,w),$$

where $\rho(z,w)$ is given by

$$\rho(z,w) = \int_0^1 (1-t)\left(D^2f(z+tw)(w;w) - D^2f(z)(w;w)\right)dt.$$

We also define function

$$\Lambda(r,s) = \sup\{|D^2f(z+w) - D^2f(z)| \ : \ |z| \leq r, |w| \leq s\}. \tag{2.15}$$

The function Λ is defined in such a way that we obtain an obvious estimate, for $|z| \leq r$ and any w we have

$$|\rho(z,w)| \leq \frac{1}{2}\Lambda(r,|w|)|w|^2. \tag{2.16}$$

We start with the following result.

Lemma 2.5.1. *Let f be any function of class $C^2(\mathbb{R}^3;\mathbb{R})$ such that $D^2f(z)(w;w) \geq 0$ for any z, w such that $|z| \leq r$ and w is parallel to one of the axis. Take any $\epsilon > 0$ and define $\delta \stackrel{def}{=} \frac{1}{2}\sup\{t \in (0,r) \ : \ \epsilon \geq \Lambda(r,t)\}$. Then there exists a tetrahedrally polyconvex function g such that*

$$g(z) = f(z) + \epsilon|z|^2 \text{ for } |z| < \delta.$$

Proof. Define

$$f_\epsilon(z) = f(z) + \epsilon|z|^2,$$

$$G(z) := \begin{cases} f_\epsilon(z) \text{ for } |z| \le \delta. \\ \sup_{|w|<\delta} \big(f_\epsilon(w) + Df_\epsilon(w)(z-w) + \frac{1}{2}D^2 f_\epsilon(w)(z-w; z-w)\big) \text{ for } |z| > \delta \end{cases}$$

and $g = TPE(G)$. It's now obvious that g is tetrahedrally polyconvex and that $g \le f_\epsilon$ for $|z| < \delta$. To check that $g = f_\epsilon$ for $|z| < \delta$ take any z such that $|z| \le \delta$. By Theorem 2.4.4 for any σ we may choose a convex combination $\{\lambda_j z_j\}_{i=1}^7$ of z such that

$$e(z) = \sum \lambda_j e(z_j) \tag{2.17}$$

and

$$g(z) + \sigma > \sum_{j=1}^{7} \lambda_j G(z_j).$$

From the definition of G it follows that

$$g(z) + \sigma > \sum_{|z_j| \le \delta} \lambda_j f_\epsilon(z_j) + \sum_{|z_j|>\delta} \lambda_j \bigg(f_\epsilon(z) + Df_\epsilon(z)(z_j - z) + \frac{1}{2}D^2 f_\epsilon(z)(z_j - z; z_j - z)\bigg)$$

$$= \sum_{|z_j| \le \delta} \ldots + \sum_{|z_j|>\delta} \ldots =: A + B.$$

Applying Taylor's formula to $f_\epsilon(z_j)$ in A yields

$$A = \sum_{|z_j| \le \delta} \lambda_j f_\epsilon(z) + \sum_{|z_j| \le \delta} \lambda_j Df_\epsilon(z)(z_j - z)$$

$$+\frac{1}{2} \sum_{|z_j| \le \delta} \lambda_j D^2 f_\epsilon(z)(z_j - z; z_j - z) + \sum_{|z_j| \le \delta} \lambda_j \rho(z, z_j - z),$$

hence

$$g(z) + \sigma > \sum_{|z_j| \le \delta} \lambda_j \rho(z, z_j - z)$$

$$+ \sum_j \lambda_j \bigg(f_\epsilon(z) + Df_\epsilon(z)(z_j - z) + \frac{1}{2}D^2 f_\epsilon(z)(z_j - z; z_j - z)\bigg).$$

From linearity of $Df(z)$ and the fact that $z = \sum \lambda_j z_j$ we obtain

$$g(z) + \sigma > \sum_{|z_j| \le \delta} \lambda_j \rho(z, z_j - z) + f_\epsilon(z) + \sum_j \lambda_j \left(\frac{1}{2}D^2 f_\epsilon(z)(z_j - z; z_j - z)\right)$$

$$= \sum_{|z_j| \le \delta} \ldots + f_\epsilon(z) + \sum_j \ldots =: C + f_\epsilon(z) + D. \tag{2.18}$$

Having in mind (2.16) we have

$$|\rho(z; z_j - z)| \leq \frac{1}{2}\Lambda(r, |z_j - z|)|z_j - z|^2.$$

Note that for $|z_j| \leq \delta$

$$|z_j - z| \leq |z| + |z_j| \overset{|z|<\delta}{<} 2\delta = \sup\{t \in (0, r) : \epsilon \geq \Lambda(r, t)\}$$

and therefore $|z_j - z| \in \{t \in (0, r) : \epsilon \geq \Lambda(r, t)\}$. Consequently $\Lambda(r, |z_j - z|) \leq \epsilon$ and $\rho(z; |z_j - z|) \geq -\frac{1}{2}\epsilon|z_j - z|^2$. It follows that

$$C \geq -\frac{\epsilon}{2} \sum_{|z_j|\leq\delta} \lambda_j |z_j - z|^2 \geq -\frac{\epsilon}{2} \sum_j \lambda_j |z_j - z|^2.$$

We also notice that $D^2 f_\epsilon(z) = D^2 f(z) + 2\epsilon Id$ and so

$$D = \sum_j \lambda_j \left(\frac{1}{2} D^2 f(z)(z_j - z; z_j - z)\right) + \epsilon \sum_j \lambda_j |z_j - z|^2.$$

Therefore

$$C + D \geq \sum_j \lambda_j \left(\frac{1}{2} D^2 f(z)(z_j - z; z_j - z)\right) + \frac{\epsilon}{2} \sum_j \lambda_j |z_j - z|^2.$$

What we need is to show that $C + D \geq 0$, so that from (2.18) we get

$$g(z) + \sigma > f_\epsilon(z).$$

Take now any bilinear symmetric form P and note that $P(x-y; x-y) = P(x; x) + P(y; y) - 2P(x; y)$. This shows however that

$$\begin{aligned}
\sum_j \lambda_j \left(\frac{1}{2} D^2 f(z)(z_j - z; z_j - z)\right) &= \sum_j \lambda_j \left(\frac{1}{2} D^2 f(z)(z_j; z_j)\right) \\
&\quad + \sum_j \lambda_j \left(\frac{1}{2} D^2 f(z)(z; z)\right) - 2\sum_j \lambda_j \left(\frac{1}{2} D^2 f(z)(z_j; z)\right) \\
&= \sum_j \lambda_j \left(\frac{1}{2} D^2 f(z)(z_j; z_j)\right) + \frac{1}{2} D^2 f(z)(z; z) - D^2 f(z)(z; z) \\
&= \sum_j \lambda_j \left(\frac{1}{2} D^2 f(z)(z_j; z_j)\right) - \frac{1}{2} D^2 f(z)(z; z).
\end{aligned}$$

From our assumptions $P(v) = D^2 f(z)(v;v)$ is the bilinear form convex along each axis. Therefore, from the Fact 2.3.5, it is tetrahedrally polyconvex. According to Corollary 2.4.5 and (2.17) we get the following Jensen-type inequality

$$\sum_j \lambda_j \left(D^2 f(z)(z_j;z_j)\right) = \sum_j \lambda_j P(z_j) \geq P(\sum_j \lambda_j z_j) = P(z) = D^2 f(z)(z;z),$$

which concludes the proof. □

We end this section with the following result.

Theorem 2.5.2. *There exists a function that is not tetrahedrally polyconvex such that its restriction to any ball of radius one may be extended to a tetrahedrally polyconvex function.*

Proof. Let $h(r,s,t) = -rst : \mathbb{R}^3 \to \mathbb{R}$. We note that h is not tetrahedrally polyconvex, but convex in the direction of each axis (see Lemma 2.2.12). Take now two functions $\alpha, \beta : [0,\infty) \to \mathbb{R}$, $\alpha, \beta \in C^1(0,\infty)$ such that

$$\begin{aligned} \alpha(t) &= \begin{cases} 1 & \text{for } t < 4, \\ \cos^2\left((t-4)\frac{\pi}{2}\right) & \text{for } t \in [4,5] \\ 0 & \text{for } t > 5, \end{cases} \\ \beta(t) &= \begin{cases} 0 & \text{for } t < 3,5 \\ (t-\frac{7}{2})^2 & \text{for } t \geq 3,5 \end{cases} \end{aligned}$$

We consider trunk function $\varphi_\delta(t) = \delta^{-1}\varphi(\frac{t}{\delta})$, where $\varphi \in C_0^\infty(\mathbb{R}), 0 \leq \varphi \leq 1, \int \varphi = 1, \varphi \equiv 1$ in some neighbourhood of 0 and supp $\varphi \subseteq [-1,1]$. Then we set $\alpha_\delta := \alpha * \varphi_\delta, \beta_\delta := \beta * \varphi_\delta$.

It is easy to check that there exist $k > 0, \delta \in (0, \frac{1}{2})$ such that the function g given by

$$g(z) \stackrel{\text{def}}{=} h(z)\alpha_\delta(|z|) + k\beta_\delta(|z|)$$

is smooth and convex in the direction of each axis. It is not tetrahedrally polyconvex. To confirm that, we use the argument from paper by Šverák [146] and substitute the sequence $u^\nu(x,y) = (\cos(2\pi x\nu), \cos(2\pi y\nu), \cos(2\pi(x+y)\nu))$, $\Omega = [0,1]^2$. Applying the Riemann-Lebesgue Lemma (see [41], Theorem 1.5) we see immediately that $u^\nu \rightharpoonup u = 0$ weakly-$\star$ in $L^\infty(\Omega,\mathbb{R}^3)$. However, a direct computation shows that

$$\liminf_{\nu\to\infty} I_g(u^\nu) = -\frac{1}{4} < I_g(u) = 0, \tag{2.19}$$

which shows that h is not $(2,3)$ quasiconvex and consequently it is not tetrahedrally polyconvex as well. Furthermore, we may find $\epsilon > 0$ such that

$$g_\epsilon(z) = g(z) + \epsilon|z|^2$$

is not tetrahedrally polyconvex because of minor modification of (2.19). Take now Λ defined in (2.15) for function $f = g_\epsilon$. As g_ϵ is smooth and its third derivative has a compact support, we get that

$$|\frac{\partial^2}{\partial z_i \partial z_j} g_\epsilon(z+w) - \frac{\partial^2}{\partial z_i \partial z_j} g_\epsilon(z)| = |\int_0^1 \nabla \frac{\partial^2}{\partial z_i \partial z_j} g_\epsilon(z+\theta w) \cdot w d\theta| \leq ||\nabla^3 g_\epsilon||_\infty |w|$$

and it follows that there exists a constant C such that $\Lambda(r,t) \leq Ct$, where C is independent on r. In particular, $\epsilon \geq \Lambda(r, \frac{\epsilon}{C})$ and so for any r

$$\frac{\epsilon}{2C} \leq \frac{1}{2} \sup\{t \in (0,r) \ : \ \epsilon \geq \Lambda(r,t)\}.$$

We claim that for fixed z_0 there exists a tetrahedrally polyconvex extension of g_ϵ from a ball with center in z_0 and of radius $\frac{\epsilon}{2C}$. Note that the radius does not depend on z_0. The existence of such extension follows from Lemma 2.5.1, when we substitute g_ϵ by a shifted function

$$g_\epsilon^{z_0}(z) \stackrel{\text{def}}{=} g_\epsilon(z_0 + z),$$

so that we extend the function $g_\epsilon^{z_0}$ from the ball centred at 0. Defining now

$$f(z) \stackrel{\text{def}}{=} g_\epsilon^{z_0}(\frac{2C}{\epsilon} z)$$

provides the radius 1 in the extension property and finishes the proof of existence of the function which is not tetrahedrally polyconvex, having however a tetrahedrally polyconvex extension from any ball of radius 1. □

Chapter 3

Compactifications in DiPerna-Majda measure Theory

3.1 Introduction

In modern Calculus of Variations a notion of compactification of an Euclidean spaces $\mathbb{R}^m$ gathered a certain weight. We recall that a compactification is a dense embedding into a compact space. One of the most natural examples is the use of so-called recession function in relaxing functionals of linear growth, see for example [20, 22, 35, 62]. Let us recall that for a function $f : \mathbb{R}^m \to \mathbb{R}$, by its recession function we mean

$$f^\infty : S^{m-1} \to \mathbb{R};\ f^\infty(\vartheta) \overset{\text{def}}{=\!=} \lim_{t \to +\infty} \frac{f(t\vartheta)}{t}.$$

The existence of the computed limits, as well as the continuity of f^∞, typically requires to be assumed. In other words, $\mathbb{R}^m$ – the domain of the integrand f, is compactified precisely with a unit ball by adding a sphere to $\mathbb{R}^m$. Furthermore, the compactification is designed in such a way, that $\dfrac{f(u)}{1+|u|}$ may be extended to a continuous function on a closed unit ball.

Compactifications appear also naturally in the field of DiPerna-Majda measures [86, 92, 111]. Papers dealing with DiPerna-Majda measures at most times assume directly, that some compactification exists and more – it is metric and separable, as these properties are often vital for further results. Of course, there also exist several papers in the field of DiPerna-Majda measures, dealing with the aforementioned compactification by a unit ball, for example [6, 49].

The following problem appears. Given a subset $\mathscr{A} \subseteq \mathbb{R}^m$ and a continuous function $f : \mathscr{A} \to \mathbb{R}$ we need to find a compactification $\mathscr{A}'$ of $\mathscr{A}$ and a dense embedding $\varphi : \mathscr{A} \hookrightarrow \mathscr{A}'$ such that the function $f \circ \varphi^{-1} : \varphi(\mathscr{A}) \to \mathbb{R}$ possesses a continuous extension $\bar{f} : \mathscr{A}' \to \mathbb{R}$. A very natural solution to that problem seems to be the classical Čech-Stone compactification $\beta\mathscr{A}$. Indeed, every continuous function $f : \mathscr{A} \to \mathbb{R}$ possesses an extension to a continuous function $\bar{f} : \beta\mathscr{A} \to \mathbb{R}$. Unfortunately, taking a very simple $\mathscr{A} = \{1 - \frac{1}{n} : n \in \mathbb{N}\} \subset [0, 1]$ we are delivered a compactification $\beta\mathscr{A}$ non-metrizable, non-second countable and of cardinality $2^{2^{\mathbb{N}}}$ (see [53, Corollary 3.6.12] for details). This shows that a more modest construction is needed to obtain a compactification with metric and visible geometric structure. There are several approaches towards this problem, due to Gelfand and Naimark [68,69], Engelking [53] or discussed by Keesling [93], which we will review in Section 3.5.

All the known earlier attitudes require only that $\mathscr{A}$ is a subset of an arbitrary Tychonoff's space $\mathbb{X}$ (see Definition 3.2.2 for the details), which is not necessarily an Euclidean space. They use abstract arguments, which do not benefit from any knowledge of special properties of $\mathbb{X}$. We present a variant of a solution to the problem stated above, as well as an easy and constructive proof in the particular case $\mathbb{X} = \mathbb{R}^m$, which is the first of our main results. The precise formulation is given in Theorem 3.3.2. What is here also some additional quality is a direct construction of the desired compact space, as well as of a homeomorphic embedding $\varphi : \mathscr{A} \to \mathscr{A}'$. In our case $\mathscr{A}'$ is proved to be a compact subset of $\mathbb{R}^{m+1}$ for any single function f.

In the next step we can consider a countable family $\mathscr{F}$ of functions $f : \mathscr{A} \to \mathbb{R}$ and construct a compactification $\mathscr{A}' \subset \ell^2$ such that $f \circ \varphi^{-1}$ possesses a continuous extension for every $f \in \mathscr{F}$. In particular, our compact space $\mathscr{A}'$ is metric and separable, as it inherits these properties from ℓ^2. Hence every measure on this set possesses a well-defined support, which is in general not always the case (see [102]). This important feature was not guaranteed by the classical methods, which we explain in Section 3.5. A precise and direct construction was needed for particular applications in Calculus of Variations we had in mind.

To motivate the second of our main results, let us discuss two theorems which may cause some interest in the field of Calculus of Variations. The first is a variant of Young (DiPerna-Majda) Theorem for discontinuous integrands – Theorem 3.2.11 due to Kałamajska [86], see also [85, 87–89] for related results. The theorem shows a representation formula for the weak-$\star$ limit for the sequences of compositions $\{f(u^\nu)d\mu\}$, where $f : \mathbb{R}^m \to \mathbb{R}$ is a continuous function on every set $A_i, i = 1, \ldots, k$. It is assumed that the sets $\{A_i\}_{i=1}^k$ form a partition of $\mathbb{R}^m$ and every Borel set A_i is compactified by some $\gamma A_i \subset \mathbb{R}^N$ for some N. The representation of the limit, given in (3.1), requires an integration of f with respect to some measure over the remainder $\gamma A_i \setminus \varphi_i(A_i)$, where $\varphi_i : A_i \to \gamma A_i$ is a homeomorphic embedding. In particular, the Theorem requires a knowledge about the shape of the set γA_i, as well as a construction of an embedding φ_i. Without that we are unable to compute the aforementioned limit of $\{f(u^\nu)d\mu\}$. The proof of the Representation Theorem 3.2.11 exploits a distance function on γA_i, while one of other assertions of the statement uses a support of the certain DiPerna-Majda measure defined on γA_i. For purposes of Representation Theorem 3.2.11, we need to know the precise shape of the compactification, construction of homeomorphic embeddings φ_i and warranty that γA_i are metric spaces. The classical methods are hence not helpful.

Several questions appear around the Representation Theorem from [86]. First doubt is whether the assumption $\gamma A_i \subseteq \mathbb{R}^N$ decreases the class of integrands f compatible with the

Representation Theorem. We remind that in Representation Theorem 3.2.11 it is required for f to be continuously extendable to a function on γA_i. In Lemma 3.3.1 we answer that this assumption may be satisfied by any integrand f and a proper γA_i and one may take $N = m + 1$ for $A_i \subset \mathbb{R}^m$.

Another question is whether these methods let us investigate a non-studied class of functionals, namely once of the type $\int_\Omega f(x, u(x))dx$, i.e. such that the integrand depends not only on the values of the function $u(x)$, but also on x. In this case we would look for a weak-$\star$ limit of $\{f(x, u^\nu)d\mu\}$. To proceed with such tasks we require a compactification $\gamma\mathbb{R}^m$ of $\mathbb{R}^m$ – a target space for functions u^ν – such that every function $i_x : p \mapsto f(x, p) \in \mathbb{R}$ is continuously extendable to function defined on $\gamma\mathbb{R}^m$ and the space $\gamma\mathbb{R}^m$ is independent of x. However, in further analysis it is required that $\gamma\mathbb{R}^m$ is metric, separable space (see [85, 87–89, 92]). Arranging the compactification for every function i_x separately it too naïve for that purposes.

In this chapter we present Theorem 3.4.3 – a generalisation of the Representation Theorem 3.2.11, dealing with integrands dependent on x, as well as some sufficient conditions for integrand $f = f(x, u)$ to admit a proper compactification. This is the second of our main results.

Let us note that Representation Theorem is related to the classical Convergence Theorem from Set-valued Analysis – Theorem 3.2.12 [14, Theorem 7.2.1]. The Convergence Theorem can be used to describe in terms of inclusions the limits of $f(u^\nu(x))$, where f can be possibly discontinuous. It assumes that u^ν is converging almost everywhere to u and $f(u^\nu)$ is weakly convergent in $L^1(\Omega)$. It is clear from the proof given in [14, p. 271] that some variants of Convergence Theorem may be deduced from the Representation Theorem from [86]. For more precise information see Remarks in [85, p. 4], [86, p. 2], [87, p. 4]. Letting $u^\nu(x)$ converge strongly to $u(x)$ in Representation Theorem gives us a variant of Convergence Theorem 3.2.12. This way, contrary to the formulation of Convergence Theorem, we obtain a precise integral formula instead of an inclusion. In Representation Theorem 3.2.11 only weak convergence is needed. In Theorem 3.2.11 however we assume some special properties on integrand f, while in Theorem 3.2.12 such assumption is not mandatory. It becomes natural to ask for a theorem working in possibly general setting, so that both Representatrion Theorem 3.2.11 and Convergence Theorem 3.2.12 become its special cases. Such a generalisation may contribute to both Calculus of Variations and Set-valued Analysis. This is our desired future application of the result.

3.2 Preliminaries

3.2.1 Notation

In the chapter we will use several notions and notations. For any subset D of a normed vector space by $\text{conv}(D)$ we will mean the convex hull of D and by $\overline{\text{conv}}(D)$ we will mean the closed convex hull, that is the closure of $\text{conv}(D)$. The function f with the domain D and values in

T will be denoted by $f : D \to T$. Following [14] and analogously to the previous notation the function F with the domain D and values in 2^T (that is – subsets of T) will be denoted by $F : D \rightrightarrows T$ and referred as a multifunction. The space of continuous, real-valued functions on D will be denoted by $C(D)$. By $C_0(D)$ we will mean the subspace of $C(D)$ consisting of compactly supported functions.

For any topological space T, by $\mathcal{M}(T)$ we will mean the space of finite, signed, Borel measures defined on T. We recall that the variation of a measure $m \in \mathcal{M}(T)$ is a measure on T defined for any subset $S \subset T$ via

$$|m|(S) = \sup_{\{v \in C(T), |v| \leq 1\}} \int_S v(x) m(dx).$$

Let us recall that the space $\mathcal{M}(T)$ is a normed space with the total variation norm, that is $||m|| = |m|(T)$. For any $m \in \mathcal{M}(T)$, by the support of a measure m we will mean the smallest closed set $C \subseteq T$ such that $|m|(T \setminus C) = 0$. The support of a measure m will be denoted by $\operatorname{supp} m$. The subspace of $\mathcal{M}(T)$ consisting of positive, probabilistic measures will be denoted by $\mathcal{P}(T)$.

Let us now recall the standard Lebesgue spaces. Let $m \in \mathcal{M}(T)$ and $p \in [1, +\infty)$. We will say that the function $f : T \to \mathbb{R}$ belongs to $L^p(T, m)$ whenever $\int_T \big(f(x)\big)^p |m|(dx) < +\infty$ and that f belongs to $L^\infty(T, m)$, whenever there exist a value v such that $|m|(\{x \in T : f(x) > v\}) = 0$. We stress that, even when m is a signed measure, the definitions of spaces and natural norms provided by them are dependent only on the variation measure $|m|$, which is a positive measure.

For every $m \in \mathcal{M}(D)$ and any subset of an Euclidean space D we will say that the mapping $\{\nu_x\} : D \to \mathcal{M}(T), x \mapsto \nu_x$ is weakly-$\star$ measureable with respect to m, whenever for every $v \in C_0(T)$ the mapping $D \to \mathbb{R}, x \mapsto \int_T v(\lambda)\nu_x(d\lambda)$ is measureable in the usual sense; we will write then $\{\nu_x\} \in \mathcal{M}(D, T, m)$. Whenever measures ν_x belong to $\mathcal{P}(D)$ for almost every x, we will write $\{\nu_x\} \in \mathcal{P}(D, T, m)$; For any sequence of subsets of an Euclidean space D_n, we recall the notions of set limits, that is

$$\limsup D_n \stackrel{\text{def}}{=\!=} \bigcap_{N \in \mathbb{N}} \bigcup_{n > N} D_n \text{ and } \liminf D_n \stackrel{\text{def}}{=\!=} \bigcup_{N \in \mathbb{N}} \bigcap_{n > N} D_n.$$

3.2.2 Basic properties of compactifications

In this section we will present basic notions needed to deal with compactifications.

Definition 3.2.1. Let $\mathbb{X}$ be a topological space. We say that a topological space $\gamma\mathbb{X}$ is a `compactification of` $\mathbb{X}$, whenever $\gamma\mathbb{X}$ is compact, there exists a homeomorphism $\varphi : \mathbb{X} \to \varphi(\mathbb{X}) \subseteq \gamma\mathbb{X}$ and $\varphi(\mathbb{X})$ is dense in $\gamma\mathbb{X}$.

The sets $\varphi(\mathbb{X})$ and $\mathbb{X}$ are often identified in the literature. In this thesis, however, we will directly distinguish between them to strengthen the role of the embedding φ.

Let us now recall the classic definitions of Hausdorff's and Tychonoff's spaces.

Definition 3.2.2. Let $\mathbb{X}$ be a topological space. We will say that $\mathbb{X}$ is a `Hausdorff's space`, whenever for any pair of points $x \neq x'$, there exist disjoint open sets U, U' such that $x \in U$ and $x' \in U'$.

We will say that $\mathbb{X}$ is `Tychonoff's space` whenever for any closed set $F \subseteq \mathbb{X}$ and a point $x \in \mathbb{X} \setminus F$ there exists a continuous function $f : \mathbb{X} \to \mathbb{R}$ such that $\forall_{y \in F} f(y) = 0$ and $f(x) = 1$.

It is clear that any Tychonoff's space is Hausdorff, while the converse does not hold (see [53, Example 1.5.6]). The classes are however equivalent in the category of compact spaces.

Fact 3.2.3 (Theorem 3.1.9 in [53])**.** *Let $\mathbb{X}$ be a compact, Hausdorff space. Then $\mathbb{X}$ is Tychonoff's.*

Proof. First let us notice that it is sufficient to prove that for every pair of disjoint, closed sets $A, B \subset \mathbb{X}$ there exist disjoint open sets U, V such that $A \subset U, B \subset V$. Indeed, from Tietz Theorem [53, Theorem 1.5.11] follows at once, that every space satisfying the mentioned property is Tychonoff's.

Let us now fix $a \in A, b \in B$ and from Hausdorff property let us find open, disjoint sets $U_{a,b}, V_{a,b}$ such that $a \in U_{a,b}$ and $b \in V_{a,b}$. Executing this procedure for every $a \in A$ gives us a cover of set A by family $\left\{U_{a,b}\right\}_{a\in A}$. As A is compact, we may choose a finite subcover and get $A \subset U_{a_1,b} \cup U_{a_2,b} \cup \ldots \cup U_{a_n,b}$. Define now $U_b \stackrel{\text{def}}{=\!=} U_{a_1,b} \cup U_{a_2,b} \cup \ldots \cup U_{a_n,b}, V_b \stackrel{\text{def}}{=\!=} V_{a_1,b} \cap V_{a_2,b} \cap \ldots \cap V_{a_n,b}$. This way we obtain disjoint open sets U_b, V_b such that $A \subset U_b, b \in V_b$. Similarly we may execute this procedure for every $b \in B$ and choose a finite subcover, getting $B \subset V_{b_1} \cup V_{b_2} \cup \ldots V_{b_k}$. Keeping in mind that $A \subset U_b$ for any $b \in B$, we obtain that $A \subset U_{b_1} \cap U_{b_2} \cap \ldots \cap U_{b_k}$. Taking now $U \stackrel{\text{def}}{=\!=} U_{b_1} \cap U_{b_2} \cap \ldots \cap U_{b_k}$ and $V \stackrel{\text{def}}{=\!=} U_{b_1} \cap U_{b_2} \cap \ldots \cap U_{b_k}$ finishes the proof. ☐

Definition 3.2.4. Let $\mathbb{X}, \mathbb{Y}$ be topological spaces and $f : \mathbb{X} \to \mathbb{Y}$ – a continuous function. Let $\gamma\mathbb{X}$ be certain compactification of $\mathbb{X}$ defined via homeomorphism $\varphi : \mathbb{X} \to \varphi(\mathbb{X}) \subseteq \gamma\mathbb{X}$. We will say that the function f is `admissible` for compactification $\gamma\mathbb{X}$, whenever there exists a continuous function $\bar{f} : \gamma\mathbb{X} \to \mathbb{Y}$ such that $\bar{f}(x) = \left(f \circ \varphi^{-1}\right)(x)$ for every $x \in \varphi(\mathbb{X})$.

From the definition of compactification it follows that, whenever f is admissible, the function $\bar{f}$ is uniquely determined by f. We will often refer to $\bar{f}$ as `extension of` f.

From now on, we will focus our interest in admissibility of real-valued functions, that is – we take $\mathbb{Y} = \mathbb{R}$ in the Definition 3.2.4. It is easily visible in that case, that a necessary condition for admissibility of such function is its boundedness. Keeping that in mind, let us recall the two useful notions dealing with real-valued functions.

Definition 3.2.5. Let $\mathscr{F}$ be a set of continuous real-valued functions on a topological space $\mathbb{X}$. We say that $\mathscr{F}$ forms a `ring of continuous functions` whenever the function $z \equiv 0$ belongs to $\mathscr{F}$ and for any $f, g \in \mathscr{F}$ we have $f \pm g, f \cdot g \in \mathscr{F}$.

We will often deal with rings consisting only of bounded functions. Such a ring will be referred as `ring of bounded continuous functions`.

Let us note, that the ring of continuous functions needs not to be not unital, that is – may not possess the '1' element. The examples of such a non-unital ring considered most often are continuous functions on $\mathbb{R}^n$ vanishing at infinity, or compactly supported. This circumstance changes in case of the following notion.

Definition 3.2.6. Let $\mathscr{F}$ be a ring of continuous functions on a Tychonoff's space $\mathbb{X}$. We will say that the ring $\mathscr{F}$ is `complete` whenever

(a) every constant function belongs to $\mathscr{F}$,

(b) for any closed set $F \subseteq \mathbb{X}$ and a point $x \in \mathbb{X} \setminus F$ there exists function $f \in \mathscr{F}$ such that $\forall_{y \in F} f(y) = 0$ and $f(x) = 1$ (in other words – $\mathscr{F}$ separates closed sets from points outside of them),

(c) $\mathscr{F}$ is closed with respect to uniform convergence.

Due to condition (a), every complete ring of functions contains a constant function equal to one – the '1' element in the ring. From that and Kuratowski-Zorn Lemma one may prove that there exist maximal ideals in the ring, which happens to be a crucial feature for purposes of the Engelking's statement, which we discus later. Let us focus on some properties of compacifications and functions, which are admissible for them.

Proposition 3.2.7 (Properties of the class of admissible functions)**.** *Let $\mathbb{X}$ be Tychonoff's space and $\gamma\mathbb{X}$ – its compactification. Let $\mathscr{F}$ be the set of all functions, which are admissible for this compactification. Then $\mathscr{F}$ is a ring of bounded continuous functions. Furthermore, it satisfies conditions (a), (c) from the Definition 3.2.6. If $\mathbb{X}$ is Hausdorff's, also (b) is satisfied.*

Proof. Conditions for the ring, as well as boundedness and (a) from Definition 3.2.6 do not reuire explanation. To prove (c), let us assume that the sequence f_i converges uniformly on $\mathbb{X}$ to f. We will prove that the extensions of f_i's – $\bar{f}_i$ – form a Cauchy sequence in the space of continuous functions on $\gamma\mathbb{X}$ with the supremum norm. Indeed, as $\varphi(\mathbb{X})$ is dense in $\gamma(\mathbb{X})$, we have

$$\sup_{y \in \gamma(\mathbb{X})} |\bar{f}_n(y) - \bar{f}_k(y)| = \sup_{y \in \varphi(\mathbb{X})} |\bar{f}_n(y) - \bar{f}_k(y)| = \sup_{x \in \mathbb{X}} |f_n(x) - f_k(x)|$$

and from the uniform convergence of f_i's on $\mathbb{X}$ we see that for n, k large enough the right-hand side of the above equality is bounded by ε. We have shown that $\bar{f}_i$ forms a Cauchy sequence, and, as $C(\gamma(\mathbb{X}))$ is complete, this sequence has a continuous limit $\bar{f}$. Checking that $\bar{f}$ is an extension of f is straightforward.

For the proof of (b) let us remind that from Fact 3.2.3 it follows that $\gamma\mathbb{X}$ is Tychonoff's, whenever it is Hausdorff. Let us consider now any closed $F \subset \mathbb{X}$ and $x \in \mathbb{X} \setminus F$ and assume that $\gamma\mathbb{X}$ is a compactification of $\mathbb{X}$ via homeomorphism $\varphi : \mathbb{X} \to \gamma\mathbb{X}$. As $\varphi(F)$ is closed in $\varphi(\mathbb{X})$, from the definition of inherited (so-called trace) topology it follows that there exists a set $K \subset \gamma\mathbb{X}$ which is closed in $\mathbb{X}$ and its intersection with $\varphi(\mathbb{X})$ is precisely $\varphi(F)$. Hence, $\varphi(x) \notin K$. From the definition of Tychonoff's space – there exist then a continuous function g on $\gamma\mathbb{X}$ such that $g \equiv 0$ on K and $g\big(\varphi(x)\big) = 1$. Now the desired admissible function is precisely $f = g_{|\varphi(\mathbb{X})} \circ \varphi$. □

3.2.3 Engelking's statement on compactifications

We are in position to state the Engelking's Theorem. However based on ideas from [69, Lemma 1], it was Engelking who formulated the statement this way.

Theorem 3.2.8 (Engelking in [53], p. 240)**.** *Let $\mathscr{F}$ be a complete ring of bounded continuous functions on Tychonoff's space $\mathbb{X}$. Then there exists a topological space $\Sigma\mathbb{X}$, which is a compacification of $\mathbb{X}$ and satisfies*

(a) every function $f \in \mathscr{F}$ is admissible for compactification $\Sigma\mathbb{X}$,

(b) every function, which is admissible for compactification $\Sigma\mathbb{X}$, belongs to $\mathscr{F}$.

Remark 3.2.9. Theorem 3.2.8 is commonly quoted and well-known. Up to our knowledge its proof has not been written so far. In [53] it is given as an exercise. The hinted proof of the entire statement requires introducing the topology on the set of ideals of the ring $\mathscr{F}$, it is hence technically complicated and unnecessarily demanding if in our mind only the case $\mathbb{X} \subseteq \mathbb{R}^m$ is needed. The stronger assumption on $\mathbb{X}$ allows a different proof, using essentially less advanced tools and geometrically clear, non-abstract ideas.

Some further information about the other, less direct methods of creating compactifications are given in Section 3.5.

3.2.4 The Representation Theorem from [86] and the Convergence Theorem

We begin with the Classical Young Theorem, highly inspired by [16]. We present a variant similar to the one in [75].

Theorem 3.2.10 (The Classical Young Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded and measureable set with respect to certain Borel measure μ. Assume that $K \subseteq \mathbb{R}^m$ is closed and $u^\nu : \Omega \to \mathbb{R}^m$ is a sequence of functions such that for every open $U \supseteq K$ we have*

$$\lim_{\nu\to+\infty} |\{x \in \Omega : u^\nu(x) \notin U\}| = 0 \text{ and } \lim_{M\to+\infty} \sup_\nu |\{x \in \Omega : |u^\nu(x)| > M\}| = 0.$$

Then there exists a subsequence (still denoted by $\{u^\nu\}$) and a mapping $\{\nu_x\} \in \mathcal{P}(\Omega, \mathbb{R}^m, \mu)$ such that $\operatorname{supp} \nu_x \subseteq K$ for almost every $x \in \Omega$ and for every Carathéodory function $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that the sequence $\{f(\cdot, u^\nu(\cdot)\}$ is uniformly integrable we have

$$f\big(\cdot, u^\nu(\cdot)\big) \rightharpoonup < \nu_x, f > \stackrel{def}{=\!=} \int_{\mathbb{R}^m} f(x,\lambda)\nu_x(d\lambda) \ \ in\ L^1(\Omega,\mu).$$

In order to recall the Representation Theorem from [86], let us form a set of assumptions and notations used in the sequel. We deal with the following assumptions.

(H1) Ω is a open and bounded domain in $\mathbb{R}^n$ equipped with measure μ.

(H2) Sets $A_1, A_2, \dots, A_k$ form a partition of $\mathbb{R}^m$.

(H3) Function $g : \mathbb{R}^m \to [0, +\infty)$ satisfies $g_i \stackrel{\text{def}}{=\!=} g_{|A_i} \in C(A_i)$ and $g_i(\lambda) \geq \alpha > 0$ for every $\lambda \in A_i \cap \partial A_i$ and some α.

(H4) For every $i = 1, 2, \dots, k$ and some N a space $\gamma A_i \subset \mathbb{R}^N$ is a compactification of A_i.

(H5) $\mathscr{F}$ is the class of functions $f : \mathbb{R}^m \to \mathbb{R}$ such that the function $f_i \stackrel{\text{def}}{=\!=} f_{|A_i}/g_i$ is admissible for compactification γA_i for every $i = 1, 2, \dots, k$.

We present the Representation Theorem due to Kałamajska [86], which will be generalised by Theorem 3.4.3 with the help of Theorem 3.3.2.

Theorem 3.2.11. *[86, Representation Theorem 3.1] Under assumptions (H1-5) let the sequence $\{u^\nu\}$ of μ-measureable functions $u^\nu : \Omega \to \mathbb{R}^m$ satisfy*

(T) $\limsup_\nu \mu\big(\{x \in \Omega : |u^\nu(x)| \geq r\}\big) \xrightarrow{r\to+\infty} 0,$

(D) $\sup_\nu \int_\Omega g(u^\nu)\mu(dx) < \infty.$

Then there exist

(a) a subsequence of $\{u^\nu\}$, denoted by the same expression,

(b) measures $\bar{m}^i, m^i$ on Ω, such that $\bar{m}^i$ is absolutely continuous with respect to μ and $\operatorname{supp} m^i \subseteq \operatorname{supp} \mu$ *for any $i = 1, 2, \dots, k$,*

(c) a family of probability measures $\{\mu_x\}_{x\in\Omega}$ defined on $\mathbb{R}^m$ and such that the function $x \mapsto \mu_x$ is weakly-$\star$ measureable with respect to μ (to abbreviate we just denote it by $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbb{R}^m, \mu)$),

(d) families $\{\bar{\nu}^i_x\} \in \mathcal{P}(\Omega, \partial A_i \cap A_i, \mu)$ and $\{\nu^i_x\} \in \mathcal{P}(\Omega, \gamma A_i \setminus \Phi_i(A_i), m^i)$

such that for every $f \in \mathscr{F}$ the subsequence $\{f(u^\nu(x))\mu(dx)\}$ converges weakly-$\star$ in the space of signed measures to the signed measure represented by

$$\sum_{i=1}^{k}\Bigg(\int_{\operatorname{int} A_i} f(\lambda)\mu_x^i(d\lambda)\mu(dx) + \int_{\partial A_i\cap A_i} f(\lambda)\bar{\nu}_x^i(d\lambda)\bar{m}^i(dx) + \int_{\gamma A_i\setminus\Phi_i(A_i)} \tilde{f}_i(\lambda)\nu_x^i(d\lambda)m^i(dx)\Bigg). \tag{3.1}$$

Moreover, $\{\mu_x\}_{x\in\Omega}$ is the classical Young Measure generated by the sequence $\{u^\nu\}$, as in Theorem 3.2.10.

Now we switch our attention to the classical Convergence Theorem from [14], which plays an important role in Set-Valued analysis.

Theorem 3.2.12. *[14, Theorem 7.2.1] Let $n \in \mathbb{N}$ and $F_n : \mathbb{R}^k \rightrightarrows \mathbb{R}^m$ be such multifunctions, that for every $x \in \mathbb{R}^k$ there exist an open neighbourhood V such that $\bigcup_{n\in\mathbb{N}} F_n(V)$ is bounded. We denote the graph of F_n as G_n. Assume further, that Ω is an open subset of $\mathbb{R}^n$ and $x_j : \Omega \to \mathbb{R}^k, y_j : \Omega \to \mathbb{R}^m$ – measureable functions such that*

a) x_j converges to x almost everywhere;

b) $y_j \in L^1(\Omega, \mathbb{R}^m)$ is weakly convergent in L^1 to y;

c) for almost every $w \in \Omega$ and every U – open neighbourhood of 0 in $\mathbb{R}^k \times \mathbb{R}^m$ there exist K such that $\forall_{k>K}(x_k(w), y_k(w)) \in G_k + U$.

Then for almost every $w \in \Omega$ we have $y(w) \in \overline{CH}F^{\#}(x(w))$, where $F^{\#}$ is such multifunction, that its graph is equal to $\limsup G_n$.

3.3 Compactification of an arbitrary subset of an Euclidean space

3.3.1 The proof in the chosen case

In this section we present our construction of compactification of an arbitrary subset of $\mathbb{R}^m$ and show some of its properties. In the sequel the construction will be exploited to generalise Representation Theorem 3.2.11.

We will use the letter $\mathscr{A}$ for an arbitrary subset of $\mathbb{R}^m$.

The key role in our construction is the following, simpler version of Theorem 3.3.2.

Lemma 3.3.1 (Finitely many functions case)**.** *Let $\mathscr{A}$ be a subset of $\mathbb{R}^m$ and $f_1, f_2, \dots, f_k : \mathscr{A} \to \mathbb{R}$ be continuous and bounded functions. Then there exists a compactification $\kappa\mathscr{A} \subseteq \mathbb{R}^{m+k}$ such that every f_i can be continuously extended to $\bar{f}_i : \kappa\mathscr{A} \to \mathbb{R}$.*

Proof. We will divide it into steps.

Step 1 We observe that without a loss of generality we may assume, that $\mathscr{A}$ is bounded. Indeed, we may apply diffeomorphism $\phi : \mathscr{A} \to \mathbb{R}^m, \phi(x) \stackrel{\text{def}}{=\!=} \frac{x}{1+|x|}$, so that a homeomorphic copy of $\mathscr{A}$ is a subset of the open unit ball and $\phi^{-1}(y) = \dfrac{y}{1-|y|}$. It is now sufficient to show the thesis for functions $\tilde{f} \stackrel{\text{def}}{=\!=} f \circ \phi^{-1}$ defined on the bounded copy of $\mathscr{A}$.

Step 2 Since $\mathscr{A}$ is assumed to be a bounded subset of $\mathbb{R}^m$, the closure of $\mathscr{A}$ is compact. We use now the classical homeomorphism between a domain and a graph of a continuous function. Namely, we set $\varphi : \mathscr{A} \to \mathscr{A} \times \mathbb{R}^k; \varphi(x) \stackrel{\text{def}}{=\!=} \big(x, f_1(x), f_2(x), \dots f_k(x)\big)$ and observe that the image $\varphi(\mathscr{A})$ is homeomorphic to $\mathscr{A}$.

Step 3 We define $\kappa\mathscr{A} \stackrel{\text{def}}{=\!=} \overline{\varphi(\mathscr{A})}$. Note that $\kappa\mathscr{A}$ is indeed compact, as $\mathscr{A}$ was a bounded set in $\mathbb{R}^m$ and f_i were assumed to be bounded functions. Thus the set $\varphi(\mathscr{A})$ is bounded in $\mathbb{R}^{m+k}$ and its closure is compact. Obviously, $\varphi(\mathscr{A})$ is a homeomorphic copy of $\mathscr{A}$ and it is dense in $\kappa\mathscr{A}$. We get that $\kappa\mathscr{A}$ is a compactification of $\mathscr{A}$

Step 4 Now, for any i we set $\bar{f}_i : \kappa(\mathscr{A}) \to \mathbb{R}, \bar{f}(x, y_1, y_2, \dots, y_k) \stackrel{\text{def}}{=\!=} y_i$. Such defined $\bar{f}_i$ is obviously continuous. Also it satisfies $\bar{f}_{i|\varphi(\mathscr{A})} = f \circ \varphi^{-1}$, which finishes the proof. □

The situation becomes more involved, when the given family of admissible functions is not finite, like in Lemma 3.3.1, but countable. The following statement is one of our main results in this chapter.

Theorem 3.3.2 (Countably many functions case)**.** *Let $\mathscr{A}$ be a bounded subset of $\mathbb{R}^m$ and $f_1, f_2 \dots : \mathscr{A} \to \mathbb{R}$ be continuous and bounded, functions. Then there exists a compactification $\kappa\mathscr{A} \subseteq \ell^2$ such that for any index i the function f_i can be continuously extended to $\bar{f}_i : \kappa\mathscr{A} \to \mathbb{R}$.*

Proof. Let us assume that $f_i \not\equiv 0$ for any i. This way we obtain, that for any i,

$$0 < \sup_{x \in \mathscr{A}} |f_i(x)| < +\infty.$$

For simplicity, $\sup_{x\in\mathscr{A}} |f_i(x)|$ we will denote by $\sup |f_i|$.

As previously, we may assume that $\mathscr{A}$ is bounded and thus $||x||_2 \le M$ for every $x \in \mathscr{A}$, where $||x||_2$ stays for the standard Euclidean norm. Let us now define the embedding $\varphi : \mathscr{A} \to \ell^2$ via

$$\varphi(x) \stackrel{\text{def}}{=\!=} \big(x, 2^{-1}(\sup|f_1|)^{-1} f_1(x), 2^{-2}(\sup|f_2|)^{-1} f_2(x), \dots, 2^{-j}(\sup|f_j|)^{-1} f_j(x), \dots\big). \quad (3.2)$$

Note that the image of φ is a subset of the set

$$\{(x_i) \in \ell_2 : ||(x_1, x_2, \ldots, x_m)||_2 \leq M, |x_{m+j}| \leq 2^{-j}\},$$

which is compact by the standard argument used for Tychonoff's cube. Of course φ^{-1} is continuous, as it is a projection. Obviously φ is bijective. What is non-trivial is the continuity of φ itself.

To that end let us first note that it is enough to check continuity in ℓ^1. Indeed, if for any $x_k \to x$ in $\mathscr{A}$ we will show that $\varphi(x_k) \to \varphi(x)$ in ℓ^1, then such a convergence in ℓ^2 also follows. Indeed, let us take $||x||_1$ as a Manhattan norm, that is $||(x_1, x_2, \ldots)||_1 = |x_1| + |x_2| + \ldots$. From Hölder inequality, $||\varphi(x_k) - \varphi(x)||_2 \leq ||\varphi(x_k) - \varphi(x)||_1 ||(1, 1, \ldots)||_\infty = ||\varphi(x_k) - \varphi(x)||_1$. It follows, that it is sufficient to check convergence in ℓ^1, where the calculations are less involving. We remind that the assumption $||x||_2 \leq M$ implies by Hölder inequality in $\mathbb{R}^m$, that $||x||_1 \leq M\sqrt{m}$, so we may still consider a bounded $\mathscr{A}$.

Our aim now is to show that for arbitrary $x \in \mathscr{A}$ and a sequence $x_k \to x$ we have

$$\sum_{i=1}^{\infty} \frac{|f_i(x_k) - f_i(x)|}{\sup|f_i| 2^i} \to 0 \text{ with } k \to +\infty.$$

For simplicity of notation let us take functions $v_i \stackrel{\text{def}}{=\!=} f_i / \sup|f_i|$. Such defined functions satisfy the condition $|v_i| \leq 1$ and we need to check that

$$\Delta_k \stackrel{\text{def}}{=\!=} \sum_{i=1}^{\infty} |v_i(x) - v_i(x_k)| 2^{-i} \to 0 \text{ with } k \to +\infty.$$

Note that $|v_i(x) - v_i(x_k)| \leq 2$ and thus the investigated sum is no bigger than 2.

For the proof that $\Delta_k \to 0$ we assume on the contrary that

$$\limsup_{k \to +\infty} \Delta_k = \delta > 0.$$

As δ is finite, let us take N such that

$$\sum_{i > N} 2 \cdot 2^{-i} < \delta/4$$

and then k_0 such big that for $i \leq N$ and $k > k_0$ we have

$$|v_i(x) - v_i(x_k)| < \delta/4.$$

Take now any $k > k_0$. We have that

$$\Delta_k = \sum_{i=1}^{N} |v_i(x) - v_i(x_k)| 2^{-i} + \sum_{i > N} |v_i(x) - v_i(x_k)| 2^{-i} < \delta/4 + \delta/4 = \delta/2.$$

This contradicts however the assumption on the $\limsup \Delta_k$, as it holds for every k big enough.

Regarding continuity of φ and its aforementioned properties, we obtain that φ is a homeomorphism $\mathscr{A} \hookrightarrow \ell^2$. We may thus take for $\kappa\mathscr{A}$ the closure of the image of φ and define

$$\bar{f}_i(x, y_1, y_2, \ldots) \stackrel{\text{def}}{=\!=} \sup |f_i| 2^i y_i,$$

which completes the proof. $\square$

Remark 3.3.3. In the theorem above we chose the compactification to be a subset of the Banach space ℓ^2, because we wanted this target space to be an infinite dimensional variant of an Euclidean space. As the Euclidean metric is analogous to the ℓ^2 norm, the choice seems to be natural. The proof shows however, that one may in fact take ℓ^1 instead of ℓ^2. Furthermore, having in mind that the canonical embedding $\ell^1 \hookrightarrow \ell^p$ is a contraction on the Tychonoff's cube for every $p \in [1, +\infty]$, the statement would be correct for every space $\ell^p, p \in [1, +\infty]$.

In fact, in the spirit of the classical Banach-Mazur Theorem [19], the choice of the space ℓ^1 is very natural as well. Let us briefly remind that the Theorem proves, that any separable Banach space is a continuous image of the space ℓ^1.

3.3.2 Properties of the compactification κ

We will follow by some observations regarding the compactification $\kappa\mathscr{A}$ described in Lemma 3.3.1 and Theorem 3.3.2.

Remark 3.3.4 (Non-minimality of the class of admissible functions)**.** Compactification $\kappa\mathscr{A}$ does not satisfy any natural condition similar to point $b)$ of Theorem 3.2.8. Indeed, let us consider $\mathscr{A} = (0, 2\pi)$ and functions $f_1(x) = \sin x, f_2(x) = \cos x$. Such a set of functions separates points of the space $\mathscr{A}$. Application of the construction described in Lemma 3.3.1 gives a compact space homeomorphic to the closed interval $[0, 2\pi]$. The function $f : \mathscr{A} \to \mathbb{R}, f(x) = x$ is admissible for $\kappa\mathscr{A}$. This function however does not belong to the complete ring of functions generated by f_1, f_2, as it is immediate to see that any function in that ring will posses equal limits in $x = 0$ and $x = 2\pi$.

Remark 3.3.5 (Uniformly continuous functions)**.** An easy generalisation of the previous reasoning shows the following. Let us take any at most countable set of real-valued, bounded functions f_i on $\mathscr{A}$ – a bounded subset of $\mathbb{R}^m$. Let $\kappa\mathscr{A}$ be the compactification generated by these functions via method from Theorem 3.3.2. Let $f : \mathscr{A} \to \mathbb{R}$ be any uniformly continuous and bounded function. Then f is admissible for $\kappa\mathscr{A}$.

Indeed, as f is uniformly continuous on a bounded set $\mathscr{A}$, it possesses limits $\lim\limits_{x \to x_0} f(x)$ for every $x_0 \in \overline{\mathscr{A}}$. Thus we may define $\bar{f}(x_0, y_1, y_2, \ldots) \stackrel{\text{def}}{=\!=} \lim\limits_{x \to x_0} f(x)$.

It is worth noticing that this is not the case for unbounded $\mathscr{A}$. To check this, let us take $\mathscr{A} = [1, +\infty)$ and compacitfication generated by $f(x) = e^{-x}$ via method from Lemma 3.3.1.

Applying Step 1. from the proof gives us the new space $\phi(\mathscr{A}) = [\frac{1}{2}, 1)$ and $(f \circ \phi^{-1})(y) = e^{-y/(1-|y|)}$. Again we see that when x approaches infinity (and thus y approaches 1), function f vanishes. Thus, $\kappa\mathscr{A}$ is homeomorphic $[\frac{1}{2}, 1]$. Let us now investigate the function $\sin : \mathscr{A} \to \mathbb{R}$. Obviously $\sin \circ \phi^{-1}$ does not possess a limit at 1, so it is not admissible for $\kappa\mathscr{A}$. The condition of uniform continuity is then not sufficient for admissibility.

3.4 Generalisation of the Representation Theorem

3.4.1 Representation Theorem for discontinuous integrand

We will construct here a certain application of Lemma 3.3.1 and Theorem 3.3.2 to the theory of measures of DiPerna-Majda. We begin with generalising Representation Theorem established in [86].

It is possibly worth noting that we have proven what was supposed to be an assumption in Representation Theorem 3.2.11. The assumption stated that every brick A_i possesses a compactification $\gamma A_i \subset \mathbb{R}^N$ such that a continuous and bounded function f_i/g_i is admissible. Our elementary reasoning presented in Lemma 3.3.1 shows that for any $A_i \subseteq \mathbb{R}^m$ and continuous and bounded $f : A_i \to \mathbb{R}$ we have the compactification $\kappa A_i \subseteq \mathbb{R}^{m+1}$, for which the function f is admissible. In particular, from the embedding of κA_i into an Euclidean space it follows that the space κA_i is separable and metric. Thus the support of an arbitrary measure on κA_i exists, which is not the case in the general setting: see for example [21, page 68, Example 7.1.3 and page 73, Proposition 7.2.5 points (i) and (iii)] or [102]. Let us stress here that a certain part of the further analysis is based on the behaviour of the support of certain measures. Its existence is essential for the theory.

Our aim is to generalise Representation Theorem 3.2.11 to a certain class of the integrands of the type $f(x, u)$, in other words such one that is dependent not only of a value of a function u, but also on particular $x \in \Omega$, where the value $u(x)$ is taken. We begin with the following remark.

Remark 3.4.1. Let us assume that for open and bounded $\Omega \subset \mathbb{R}^n$ we have $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ defined by $f(x, u) = f_1(x)f_2(u)$ and f is bounded. We further assume that f_1 is continuous and bounded, while f_2 – continuous and bounded on certain bricks A_i. In this situation we may choose any $x_0 \in \Omega$ such that $f_1(x_0) \neq 0$ and apply the compactification procedure shown in Lemma 3.3.1 for function $v \mapsto f(x_0, u)$, obtaining the compactifications κA_i. Let us now note that the for any x the function $u \mapsto f(x, u)$ is admissible for this compactification. Indeed, $f(x, u) = f(x_0, u)\dfrac{f_1(x)}{f_1(x_0)}$, and the statement follows from the fact that the admissible functions form an algebra and $f(x_0, u)$ is admissible.

In fact, the above remark can be generalised. Let us then assume that for an open and bounded $\Omega \subset \mathbb{R}^n$ we have $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$. We further assume that f is bounded, continuous with respect to x and continuous with respect to v on given bricks A_i. Our aim is to construct a compactification of A_i on which the function $f(x, v)$ is continuous for every $x \in \Omega$. For that purpose let us first fix countable, dense subset $\mathcal{O} = \{x_i\}$ in Ω. Let us then take $f_i(u) \stackrel{\text{def}}{=\!=} f(x_i, u)$ and apply Theorem 3.3.2 for created sequence f_i, obtaining compact space κA_i on which every function f_i is admissible. Let us define then function $\bar{f} : \Omega \times \kappa A_i$ in the following way. For fixed $x \in \Omega$ let us choose a sequence $x_i \in \mathcal{O}$ converging to x. We define

$$\bar{f}(x, u) \stackrel{\text{def}}{=\!=} \lim_{i \to \infty} \bar{f}_i(u).$$

Of course, such definition is in general not proper, but from the assumption of the continuity of f with respect to x it follows that indeed $\bar{f}(x, u)$ is independent of the choice of the sequence x_i. It is however not enough to establish continuity of $\bar{f}$ on bricks A_i. This will be satisfied under the following assumption:

$$\sup_{u \in A_i} |f(x_n, u) - f(x, u)| \to 0, \text{ as } x_n \to x. \tag{3.3}$$

The assumption reads as: $f(x, \cdot)$ is a uniform limit of functions $f(x_n, \cdot)$ whenever $x_n \to x$. In particular it gives us continuity of $f(x, \cdot)$ for every $x \in \Omega$. As a consequence, we obtain continuity of $\tilde{f}$. Indeed, let us fix (x, u) and take $(x_n, u_n) \to (x, u)$. We have

$$|\bar{f}(x, u) - \tilde{f}(x_n, u_n)| \leq |\bar{f}(x, u) - \bar{f}(x, u_n)| + |\bar{f}(x, u_n) - \bar{f}(x_n, u_n)|.$$

Now first term goes to 0 from the continuity of $\bar{f}(x, \cdot)$, while the second vanishes thanks to the property (3.3). Altogether, the following lemma is established.

Lemma 3.4.2. *Let Ω be an open and bounded set in $\mathbb{R}^n$. Set $A_1, A_2, \ldots, A_k$ to be a partition of $\mathbb{R}^m$ and assume that $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ is a bounded function, continuous on $\Omega \times A_i, i = 1, 2, \ldots, k$ and satisfying* (3.3)*. Then, for every i there exists a compctification κA_i of the set A_i such that the function $f(x, \cdot)$ is admissible for every $x \in \Omega$. Furthermore, the extension $\bar{f} : \Omega \times \kappa A_i$ is continuous.*

Let us also notice that the class of functions $f = f(x, u)$ satisfying (3.3) covers all functions of the type $f(x, u) = f_1(x) f_2(u)$ described by Remark 3.4.1. This is exactly why the following, generalised version of the Theorem 3.2.11 uses only observations covered by the more general Lemma 3.4.2.

Before stating the theorem, let us introduce one new assumption, that is

(H5') $\mathscr{F}$ is the class of functions $f : \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that the function $f_i(x, \cdot) \stackrel{\text{def}}{=\!=} f(x, \cdot)_{|\Omega \times A_i} / g_i(\cdot)$ is continuous and bounded on A_i for every $i = 1, 2, \ldots, k$, every $x \in \Omega$ and satisfy (3.3), that is

$$\sup_{u \in A_i} |f(x_n, u) - f(x, u)| \to 0, \text{ as } x_n \to x. \tag{3.4}$$

Theorem 3.4.3. *Assume (H1-3), (H5') and let $f \in \mathscr{F}$. Then, for every $i = 1, 2, \ldots, k$ there exist κA_i – compact subsets of ℓ_2, which are compactifications of A_i (with embeddings φ_i) and such that $f_i(x, u)/g_i(u)$ is extendable to a continuous function $\bar{f} : \Omega \times \kappa A_i \to \mathbb{R}$.*

Take any sequence $\{u^\nu\}$ of μ-measureable functions $u^\nu : \Omega \to \mathbb{R}^m$ satisfying

(T) $\limsup_\nu \mu\big(\{x \in \Omega : |u^\nu(x)| \geq r\}\big) \xrightarrow{r\to+\infty} 0,$

(D) $\sup_\nu \int_\Omega g(u^\nu)\mu(dx) < \infty.$

Then there exist

(a) a subsequence of $\{u^\nu\}$, denoted by the same expression,

(b) measures $\bar{m}^i, m^i$ on Ω, such that $\bar{m}^i$ is absolutely continuous with respect to μ and $\operatorname{supp} m^i \subseteq \operatorname{supp} \mu$ for any $i = 1, 2, \ldots, k$.

(c) a family of probability measures $\{\mu_x\}_{x\in\Omega}$ defined on $\mathbb{R}^m$ and such that the function $x \mapsto \mu_x$ is weakly-$\star$ measureable with respect to μ (to abbreviate we just denote it by $\{\mu_x\}_{x\in\Omega} \in \mathcal{P}(\Omega, \mathbb{R}^m, \mu)$).

(d) families $\{\bar{\nu}^i_x\} \in \mathcal{P}(\Omega, \partial A_i \cap A_i, \mu)$ and $\{\nu^i_x\} \in \mathcal{P}(\Omega, \kappa A_i \setminus \varphi_i(A_i), m^i)$.

such that for every function $f \in \mathscr{F}$, which is admissible for compactification $\kappa\mathscr{A}_i$, there exists a subsequence $\{f(x, u^\nu(x))\mu(dx)\}$ converging weakly-$\star$ in the space of signed measures to the signed measure represented by

$$\sum_{i=1}^{k} \Big(\int_{intA_i} f(x,\lambda)\mu^i_x(d\lambda)\mu(dx) + \int_{\partial A_i\cap A_i} f(x,\lambda)\bar{\nu}^i_x(d\lambda)\bar{m}^i(dx) + \int_{\kappa A_i\setminus\varphi_i(A_i)} \bar{f}_i(x,\lambda)\nu^i_x(d\lambda)m^i(dx)\Big).$$

Moreover, $\{\mu_x\}_{x\in\Omega}$ is a classical Young Measure generated by the sequence $\{u^\nu\}$, as in Theorem 3.2.10.

We stress here, that, due to Lemma 3.3.1, Theorem 3.4.3 recovers Representation Theorem 3.2.11, whenever the integrand f does not depend on x.

Remark 3.4.4. Let us mention, that the condition (D) is trivial when $\mu(\Omega) < +\infty$ and g is bounded. Indeed, in this case

$$\sup_\nu \int_\Omega g(u^\nu)\mu(dx) \leq (\sup g)\mu(\Omega).$$

On the other hand, whenever $g(v) \to +\infty$, as $|v| \to +\infty$, the condition (D) implies (T). To see that, take $M = \sup_\nu \int_\Omega g(u^\nu)\mu(dx)$ and for an arbitrary ε take such L, that $g(v) > M\varepsilon^{-1}$ whenever $|v| > L$. We see now that

$$\mu\big(\{x \in \Omega : |u^\nu(x)| \geq L\}\big) \leq \mu\big(\{x \in \Omega : g(u^\nu(x)) \geq M\varepsilon^{-1}\}\big).$$

As $\mu\big(\{x \in \Omega : g(u^\nu(x)) \geq M\varepsilon^{-1}\}\big)(M\varepsilon^{-1}) \leq \int_\Omega g(u^\nu)\mu(dx)$, we see that

$$\mu\big(\{x \in \Omega : g(u^\nu(x)) \geq M\varepsilon^{-1}\}\big) \leq \varepsilon M^{-1} \int_\Omega g(u^\nu)\mu(dx) \leq \varepsilon.$$

Before the proof, we will formulate and prove the following Lemma. It is just a very minor modification of [86, Lemma 3.3], but we will present the proof for reader's convenience.

Lemma 3.4.5. *Let $\Omega \subset \mathbb{R}^n$ be the compact set equipped with a Radon measure μ and $A \subset \mathbb{R}^m$ be Borel, compactified by metrizable γA with an embedding φ. Moreover, let us assume that $g \in C(A)$ is non-negative and a sequence $\{u^\nu\} : \Omega \to \mathbb{R}^m$ satisfies*

$$\sup_\nu \int_{\{x \in \Omega : u^\nu(x) \in A\}} g(u^\nu)\mu(dx) < +\infty$$

and generates Young measure $\{\mu_x\}_{x\in\Omega}$.

Let us define a sequence of measures $\{L^\nu\}$ on $\Omega \times \gamma A$ via the condition, that for any $F \in C(\Omega \times \gamma A)$ we have

$$(F, L^\nu) \overset{def}{=\!=} \int_{\{x \in \Omega : u^\nu(x) \in A\}} F(x, \varphi(u^\nu(x))g(u^\nu(x))\mu(dx). \tag{3.5}$$

Then, up to a choice of certain subsequence (without any change in notation), there exist measures L on $\Omega \times \gamma A$, $\tilde{m}$ on Ω and $\{\tilde{\nu}_x\} \in \mathcal{P}(\Omega, \gamma A, \tilde{m})$ such that

$$\begin{aligned} L^\nu &\overset{\star}{\rightharpoonup} L \textit{ in } \mathcal{M}(\Omega \times \gamma A), \\ (F, L) &= \int_\Omega \int_{\gamma A} F(x, \lambda)\tilde{\nu}_x(d\lambda)\tilde{m}(dx), \\ \operatorname{supp} \tilde{m} &\subseteq \operatorname{supp} \mu. \end{aligned} \tag{3.6}$$

Moreover, let $\tilde{m} = p(x)\mu + \tilde{m}_s$ be the Radon-Nikodym decomposition of $\tilde{m}$ with respect to μ. If $\gamma A \setminus \varphi(A) \neq \emptyset$, then $\tilde{\nu}_x(\gamma A \setminus \varphi(A)) = 1$ for $\tilde{m}_s$-almost every x, while for $\gamma A \setminus \varphi(A) = \emptyset$ we have $\tilde{m}_s \equiv 0$.

If we further take $U \overset{def}{=\!=} \operatorname{int} A$ and $U^0 \overset{def}{=\!=} \varphi(U)$ and assume that $f \in C(\Omega \times \gamma A)$ is such that the function $F(x, \lambda) \overset{def}{=\!=} f(x, \varphi(\lambda))g(\lambda)$ is continuous on $\Omega \times A$, as well as satisfies (3.3) *on A, vanishes on $\Omega \times \partial A$ and have 0 limits when the second coordinate tends to infinity, then*

$$int_U f(x, \varphi(\lambda))g(\lambda)\mu_x(d\lambda) = p(x) \int_{U^0} f(x, \lambda)\tilde{\nu}_x(d\lambda) \tag{3.7}$$

for μ-almost every $x \in \Omega$, where μ_x is a standard Young measure generated by u^ν.

Proof. As we assumed γA to be metrizable, we see that $C(\Omega \times \gamma A)$ is separable and hence from Banach-Alaouglu Theorem, measures on $(\Omega \times \gamma A)$ are weakly-$\star$ compact. This proves the existence of measure L.

Let us now define $\tilde{m}$ via the condition, that

$$(h, \tilde{m}) = \int_{\Omega \times \gamma A} h(x) L(dx, d\lambda) \text{ for every } h \in C(\Omega).$$

Applying the classical slicing measure argument from [139], we obtain the existence of the family of positive measures $\{\tilde{\nu}_x\}_{x\in\Omega} \in L^\infty(\Omega, \mathcal{M}(\gamma A), \tilde{m})$ such that (3.6) holds.

We will check that for $\tilde{m}$-almost every x, measures $\tilde{\nu}_x$ are probabilistic. Take any $l \in C(\Omega)$. Taking $F = l$ in (3.6), we get

$$(l, \tilde{m}) = \int_\Omega l(x) \big(\int_{\gamma A} \tilde{\nu}_x(d\lambda) \big) \tilde{m}(dx).$$

We see that $\tilde{\nu}_x(\gamma A)\tilde{m}(dx) = \tilde{m}(dx)$ as measures on Ω, hence $\tilde{\nu}_x$ are probabilistic $\tilde{m}$-almost everywhere.

We will split the remaining part of the reasoning into two cases. At first we will deal with the situation when $\gamma A \setminus \varphi(A) \neq \emptyset$. The alternative case will be considered separately.

Let us now assume that $\gamma A \setminus \varphi(A) \neq \emptyset$. We consider the function $D(\lambda) = \text{dist}(\lambda, \gamma(A) \setminus \varphi(A))$, i.e. the distance from the point $\lambda \in \gamma A$ to the remainder of the compactification. As γA was assumed to be metrizable, the function is well-defined, bounded and continuous. We define $h^\nu(x) \stackrel{\text{def}}{=} D(\varphi(u^\nu))g(u^\nu)\chi_{\{z\in\Omega: u^\nu(z)\in A\}}(x)$. We will show, that the sequence h^ν is uniformly integrable in $L^1(\Omega, \mu)$. To that end, let us define $A_\epsilon \stackrel{\text{def}}{=} \{\lambda \in A : \text{dist}(\varphi(\lambda), \gamma A \setminus \varphi(A)) < \epsilon\}$, set $M > 0$ and observe that

$$\int_{\{x\in\Omega: h^\nu(x) > M\}} h^\nu(x)\mu(dx) =$$

$$\int_{\{x: h^\nu(x)>M, u^\nu(x)\in A_\epsilon\}} h^\nu(x)\mu(dx) + \int_{\{x: h^\nu(x)>M, u^\nu(x)\in A\setminus A_\epsilon\}} h^\nu(x)\mu(dx).$$

The first term of the line below is bounded by $\epsilon \int_{\{x: u^\nu(x)\in A\}} g(u^\nu(x))\mu(dx)$. To deal with the second, let us observe that $\varphi(A \setminus A_\epsilon)$ is compact. We will show that its complement is open. The complement of $\varphi(A \setminus A_\epsilon)$ equals $\{\lambda \in \gamma A : \text{dist}(\lambda, \gamma A \setminus A) < \epsilon\}$ end hence it is open. The compactness of $\varphi(A \setminus A_\epsilon)$, together with the continuity of $g \circ \varphi^{-1}$ on $\varphi(A \setminus A_\epsilon)$ shows that the second term vanishes, when M is big enough.

From uniform integrability we get that there exist $h \in L^1(\Omega, \mu)$ such that $h^\nu \mu \stackrel{\star}{\rightharpoonup} h\mu$ in measures. We have, however, that for arbitrary $\psi \in C(\Omega)$

$$\int_\Omega \psi h^\nu \mu(dx) = (\psi D, L^\nu) \to (\psi D, L) = \int_\Omega \psi h \mu(dx) = \int_\Omega \psi(x) \int_{\gamma A} D(\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}(dx).$$

Using Radon-Nikodym decomposition $h = p(x)\mu + \tilde{m}_s$ and setting $\mathcal{F}(x) \stackrel{\text{def}}{=\!=} \int_{\gamma A} D(\lambda)\tilde{\nu}_x(d\lambda)$ we finally get

$$\int_\Omega \psi h\mu(dx) = \int_\Omega \psi \mathcal{F} p\mu(dx) + \int_\Omega \psi \mathcal{F}\tilde{m}_s(dx).$$

Having in mind, that $h^\nu\mu \stackrel{\star}{\rightharpoonup} h\mu$, we see that the second term vanishes, so $\mathcal{F}(x) = 0$ for $\tilde{m}_s$-almost every $x \in \Omega$. As D is strictly positive on $\varphi(A)$, we get that $\tilde{\nu}_x(\varphi(A)) = 0$ $\tilde{m}_s$-almost everywhere.

In the second case, that is when $\gamma A \setminus A = \emptyset$, we observe that A is compact and for any $F \in C(\gamma A)$ the sequence $h^\nu(x) \stackrel{\text{def}}{=\!=} F(\varphi(u^\nu(x))g(u^\nu(x))\chi_{\{x:u^\nu\in A\}}$ is uniformly bounded, hence uniformly integrable. Knowing that $h^\nu\mu \stackrel{\star}{\rightharpoonup} h\mu$ we calculate again that

$$\int_\Omega \psi h^\nu\mu(dx) \to \int_\Omega \psi\mathcal{F}p\mu(dx) + \int_\Omega \psi\mathcal{F}\tilde{m}_s(dx) = \int_\Omega \psi h\mu(dx).$$

We see that the second term vanishes. Plugging $F \equiv 1$, having in mind that ν_x are probabilistic we obtain that $\tilde{m}_s = 0$.

For the last part of the Lemma assume that $f \in C(\Omega \times \gamma A)$ is such that $F(x,\lambda) \stackrel{\text{def}}{=\!=} f(x,\varphi(\lambda))g(\lambda)$ belongs to $C(\Omega \times A)$ and satisfies (3.3) on A, vanishes on $\Omega \times \partial A$ and has 0 limits when the second coordinate tends to infinity. In particular, the function F can be extended to a function defined on the whole $\mathbb{R}^m$ and vanishing in the infinity. This lets us apply the classical Young Theorem 3.2.10 to see that

$$F(x,u^\nu(x)) \rightharpoonup \mathscr{F}(x) \stackrel{\text{def}}{=\!=} \int_{\mathbb{R}^m} F(x,\lambda)\mu_x(d\lambda) = \int_{\operatorname{int} A} f(x,\varphi(\lambda))g(\lambda)\mu_x(d\lambda) \text{ in } L^1(\Omega,\mu).$$

We have then $\mathscr{F}\mu = (f,\tilde{\nu}_x)\tilde{m} = (f,\tilde{\nu}_x)p(x)\mu + (f,\tilde{\nu}_x)\tilde{m}_s$. As f vanishes on $\gamma A \setminus \varphi(A)$, from the already proved parts of the lemma it follows that $(f,\tilde{\nu}_x)\tilde{m}_s \equiv 0$. □

Proof of the Theorem 3.4.3. The existence of appropriate spaces κA_i follows readily from Theorem 3.3.2 and Lemma 3.4.2. The remaining part of the proof is a slight modification of the proof of [86, Theorem 3.1], but we will present it for completeness.

Using additivity of the integral, we may assume that f vanishes on every brick except for one A_i, which will be referred as A. For $F(x,\lambda) \stackrel{\text{def}}{=\!=} f(x,\varphi^{-1}(\lambda))/g(\varphi^{-1}(\lambda))$ and $u^\nu(x) \in A$ we see that

$$f(x,u^\nu(x)) = (f/g)g = F(x,\varphi(u^\nu(x))g(u^\nu(x)).$$

Hence, from Lemma 3.4.5, we get

$$f(x,u^\nu(x))\mu(dx) \stackrel{\star}{\rightharpoonup} \int_{\kappa A} F(x,\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}(dx) =$$

$$\int_{\kappa A} F(x,\lambda)\tilde{\nu}_x(d\lambda)p(x)\mu(dx) + \int_{\kappa A} F(x,\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}_s(dx).$$

By Lemma 3.4.5 we know, that the second term is in fact an integral over $\kappa A \setminus \varphi(A)$. Splitting the first integral into two pieces and applying (3.7), we get

$$f(x, u^\nu(x))\mu(dx) \overset{\star}{\rightharpoonup} \int_{\operatorname{int} A} f(x,\lambda)\mu_x(d\lambda)\mu(dx) +$$

$$\int_{\kappa A\setminus\varphi(\operatorname{int} A)} F(x,\lambda)\tilde{\nu}_x(d\lambda)p(x)\mu(dx) + \int_{\kappa A\setminus\varphi(A)} F(x,\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}_s(dx) = a + b + c.$$

Noting that $\kappa A \setminus \varphi(\operatorname{int} A) = (\kappa A \setminus \varphi(A)) \cup \varphi(\partial A \cap A)$ lets us write

$$b + c = \int_{\varphi(\partial A\cap A)} F(x,\lambda)\tilde{\nu}_x(d\lambda)p(x)\mu(dx) + \int_{\kappa A\setminus\varphi(A)} F(x,\lambda)\tilde{\nu}_x(d\lambda)\tilde{m}(dx) = d + e.$$

Consider a function $h(x) \overset{\text{def}}{=\!=} \tilde{\nu}_x(\kappa A \setminus \varphi(A))$ and set $\Omega' \overset{\text{def}}{=\!=} \{x \in \Omega : h(x) \neq 0\}$. Choose any $y \in \kappa A \setminus \varphi(A)$. Let us define measures $m \overset{\text{def}}{=\!=} h\tilde{m}$ and ν_x by the condition, that for any $G \in C(\kappa A \setminus \varphi(A))$ we have

$$(G, \nu_x) = \begin{cases} 1/h(x)\int_{\kappa A\setminus\varphi(A)} G(\lambda)\tilde{\nu}_x(d\lambda) & \text{for } x \in \Omega', \\ G(y) & \text{for } x \notin \Omega'. \end{cases}$$

Notice that

$$e = \int_{\kappa A\setminus\varphi(A)} F(x,\lambda)\nu_x(d\lambda)m(dx).$$

To deal with d, we introduce a function

$$w(x) \overset{\text{def}}{=\!=} \int_{\varphi(\partial A\cap A)} 1/g(\varphi^{-1}(\lambda))\tilde{\nu}_x(d\lambda).$$

Choose an arbitrary $a \in \partial A \cap A$, set $\Omega'' \overset{\text{def}}{=\!=} \{x \in \Omega : \tilde{\nu}_x(\varphi(\partial A \cap A)) > 0\}$ and define $\bar{\nu}_x$ by the condition, that for any $G \in C(\partial A \cap A)$ we have

$$(G, \bar{\nu}_x) = \begin{cases} 1/w(x)\int_{\varphi(\partial A\cap A)} G/g(\varphi^{-1}(\lambda))\tilde{\nu}_x(d\lambda) & \text{for } x \in \Omega'', \\ G(a) & \text{for } x \notin \Omega'' \end{cases}$$

and see that now

$$d = \int_{\partial A\cap A} f(x,\lambda)\bar{\nu}_x(d\lambda)w(x)p(x)\mu(dx).$$

Setting $\bar{m} \overset{\text{def}}{=\!=} w(x)p(x)\mu$ finishes the proof. □

3.4.2 Examples

For the first application we deal with the following, very simple situation.

Example 3.4.6 (Discontinuity on a hyperplane)**.** Let us take an open and bounded set $\Omega \subset \mathbb{R}^n$ equipped with Lebesgue measure $\mathscr{L}^n$, functions $u = (u_1, u_2, \ldots, u_m) : \Omega \to \mathbb{R}^m$ and $f(x,u) = a_1(x)\chi_{(-\infty,0)}(u_1) + a_2(x)\chi_{\{0\}}(u_1) + a_3(x)\chi_{(0,+\infty)}(u_1)$, where a_i are arbitrary continuous and bounded functions on Ω.

Let us set $H \stackrel{\text{def}}{=} \{u \in \mathbb{R}^m : u_1 = 0\}$ and analogously $H^+ \stackrel{\text{def}}{=} \{u \in \mathbb{R}^m : u_1 > 0\}$, $H^- \stackrel{\text{def}}{=} \{u \in \mathbb{R}^m : u_1 < 0\}$.

We take bricks $A_1 = H^-, A_2 = H, A_3 = H^+$, and set functions $g_i \equiv 1, i = 1,2,3$. We will explain that the function f satisfies the assumptions of Theorem 3.4.3. Indeed, condition (H5') is satisfied. Let us take $f_i \stackrel{\text{def}}{=} f_{|A_i}$ for $i = 1,2,3$. Every function $f_i/g_i = a_i$ is continuous on $\Omega \times A_i$. Furthermore, we easily see that for any sequence $x_n \in \Omega, x_n \to x$ we have $f_i(x_n, u) \equiv a_i(x_n) \to a_i(x) \equiv f_i(x,u)$. As functions a_i are independent of u, the above convergence is the uniform convergence of functions dependent on u, which is exactly what was required in condition (3.3).

Let us know explain the shape of κA_i's in this situation. For brick A_1 we start by a homeomorphism $\phi : u \mapsto \dfrac{u}{1+|u|} = v$, which maps H^- into an open semiball, i.e. $B^- \stackrel{\text{def}}{=} \{v \in \mathbb{R}^m : |v| < 1, v_1 < 0\}$. Our function $f_1 : \Omega \times H^-, f_1(x,u) = a_1(x)$ is now formally transformed to $f_1^b : \Omega \times B^-, f_1^b(x,v) \stackrel{\text{def}}{=} f_1(x, \phi^{-1}(v)) = a_1(x)$.

We take any countable and dense subset $\mathcal{O} = \{x_1, x_2, \ldots, \}$ of Ω and use procedure described in Lemma 3.4.2. Since then we obtain a sequence of functions $f_i(v) \stackrel{\text{def}}{=} f_1(x_i, v) = a_1(x_i)$ and use Theorem 3.3.2 to construct κA_1. The embedding $\varphi : B^- \to \ell_2$ defined in (3.2) for such $f_i's$ reads as

$$\varphi(v) = (v, 2^{-1}, 2^{-2}, 2^{-3}, \ldots).$$

As we see, the image of φ is actually a semiball $B^- \subset \mathbb{R}^m$ naturally embedded in ℓ_2 via $x \mapsto (x, 0, 0, \ldots)$ and then shifted by a vector $(\underbrace{0, \ldots, 0}_{m}, 2^{-1}, 2^{-2}, 2^{-3}, \ldots)$. Its closure – κA_1 – is then a closure of B^- shifted in the same way. It is homeomorphic (and, up to an equivalent disturbance of metric in ℓ_2, isometric) with $\overline{B^-} = \{v \in \mathbb{R}^m : |v| \leq 1, v_1 \leq 0\}$. For simplicity we may thus take $\kappa A_1 = \overline{B^-}$.

The construction of κA_3 is perfectly analogous. We leave to the reader to check that $\kappa A_3 = \overline{B^+} = \{v \in \mathbb{R}^m : |v| \leq 1, v_1 \geq 0\}$. A very similar reasoning also shows that $\kappa A_2 = \overline{B_{m-1}} = \{v \in \mathbb{R}^m : |v| \leq 1, v_1 = 0\}$, i.e. it is a closed unit ball of dimension $(m-1)$.

Having κA_i's constructed, we may take any tight (i.e. satisfying condition (T) in Theorem 3.4.3) sequence $u^\nu : \Omega \to \mathbb{R}^m$ and from the Theorem 3.4.3 it follows that the measure

$f(x, u^\nu)\mathscr{L}^n$ converges weakly-$\star$ to a measure described by

$$a_1(x)\int_{H^-}\mu_x^1(dv)dx + a_1(x)\int_{\partial B^-\setminus B^-}\nu_x^1(dv)m^1(dx) +$$
$$a_2(x)\int_H \bar{\nu}_x^2(dv)\bar{m}^2(dx) + a_2(x)\int_{\{v\in\mathbb{R}^m: v_1=0, |v|=1\}}\nu_x^2(dv)m^2(dx) +$$
$$a_3(x)\int_{H^+}\mu_x^3(dv)dx + a_3(x)\int_{\partial B^+\setminus B^+}\nu_x^3(dv)m^3(dx),$$

where measures $\mu_x^1, \mu_x^3, \bar{\nu}_x^2, \bar{m}^2, \nu_x^1, m^1, \nu_x^2, m^2, \nu_x^3, m^3$ are like in Theorem 3.4.3.

Example 3.4.7 (Single brick case)**.** Let us assume that $f : \Omega \times \mathbb{R}^m$ is continuous and bounded, satisfying (3.3). In this situation we deal with one brick $A_1 = \mathbb{R}^m$ and obtain that, under assumptions of Theorem 3.4.3, there exist a subsequence of the sequence $f(x, u^\nu)dx$ converging weakly-$\star$ to

$$\int_{\mathbb{R}^m} f(x,\lambda)\mu_x(d\lambda)dx + \int_{\kappa\mathbb{R}^m\setminus\mathbb{R}^m} f(x,\lambda)\nu_x(d\lambda)m(dx),$$

retrieving the classic DiPerna-Majda Theorem from [49, Theorem 1].

Now we move to the more involving reasoning, which generalises [86, Theorem 4.2] to the situation of the integrand dependent on x, which was not considered so far.

Example 3.4.8 (Finitely many points of discontinuity)**.** Let us take open and bounded $\Omega \subset \mathbb{R}^n$, equipped with an arbitrary Borel measure μ and an arbitrary bounded function $\hat{f} : \Omega \times \mathbb{R}^m \to \mathbb{R}$ such that $\hat{f}$ is continuous with respect to x. We further require that

$$\hat{f} \in C\big(\Omega \times (\mathbb{R}^m \setminus \{P_1, P_2, \dots, P_k\})\big)$$

and satisfies there (3.3). Our aim is to derive a representation formulae for the weak-$\star$ limit of $\hat{f}(x, u^\nu)$.

First let us note that, as the number of points P_i is finite, we may find such a radius $r > 0$ that balls centred in P_i's and of radius r are disjoint. For such a fixed r, let us define the set $A_0 \overset{\text{def}}{=\!=} \{u \in \mathbb{R}^m : \text{dist}(u, P_i) > r/2 \text{ for every } i\}$. Now the stets $A_0, B(P_i, r), i = 1, 2, \dots, k$ form an open covering of $\mathbb{R}^m$, to which we may find a subordinate covering of unity – ψ_i. Now, in spite of deriving representation formulae for an arbitrary $\hat{f}$, we will work with $\psi_i\hat{f} = f$, – a function supported on $\Omega \times A_0$ or $\Omega \times B(P_i, r)$ for one particular $i = 1, 2, \dots, k$.

Let us begin with the case where f is supported on $\Omega \times A_0$. In this situation it is enough to deal with the previous example, as we may extend function f by 0 to the whole domain $\Omega \setminus \mathbb{R}^m$.

In the latter case we may decompose $\mathbb{R}^m$ into three bricks, that is $A_1 = \mathbb{R}^m \setminus B(P_i, r)$, $A_2 = B(P_i, r) \setminus \{P_i\}$, $A_3 = \{P_i\}$, where radius r is precisely the same as before. Note that in this case, from the construction of the unit partition ψ_i, we have $f \equiv 0$ on $\Omega \times A_1$, as well as $f \equiv 0$ on the sufficiently small neighbourhood of $\Omega \times \partial B(P_i, r)$.

This observations show that the representation formulae will meet a null ingredient when dealing with the brick A_1. Also, if we use hemoeomorphism $\beta : A_2 \to R \stackrel{\text{def}}{=} B(0, r+1) \setminus B(0,1)$ given by $\beta : u \mapsto u - P_i + \dfrac{u - P_i}{|u - P_i|}$ we see that the function $\check{f} : \Omega \times R \to \mathbb{R}, \check{f}(x, v) \stackrel{\text{def}}{=} f(x, \beta^{-1}(v))$ vanishes on $\partial B(r+1, 0)$.

Since then, the limit measure given by Theorem 3.4.3 will have a form $M = M_1 + M_2 + M_3$, where $M_1 = 0$, $M_3 = f(x, P_i)\bar{m}dx$ and

$$M_2 = \int_{A_2} f(x,v)\mu_x(dv)\mu(dx) + \int_{\kappa R \setminus \varphi(R)} f(x,v)\nu_x(dv)m(dx),$$

and if we divide the remainder set $\kappa R \setminus \varphi(R)$ into parts

$$R_1 \stackrel{\text{def}}{=} \{(v_1, v_2, \ldots) \in \ell_2 : (v_1, \ldots, v_m) \in \partial B(0,1)\},$$
$$R_2 \stackrel{\text{def}}{=} \{(v_1, v_2, \ldots) \in \ell_2 : (v_1, \ldots, v_m) \in \partial B(0, r+1)\}$$

we may write

$$M_2 = \int_{A_2} f(x,v)\mu_x(dv)\mu(dx) + \int_{R_1} f(x,v)\nu_x(dv)m(dx).$$

Altogether, we get that

$$M = \int_{A_2} f(x,v)\mu_x(dv)\mu(dx) + \int_{R_1} f(x,v)\nu_x(dv)m(dx) + f(x, P_i)\bar{m}dx.$$

Moreover, as the sequence $\{f(x, u^\nu)\}$ is bounded, we see that the measures $m, \bar{m}$ are absolutely continuous with respect to μ. This results in

$$M = \int_{A_2} f(x,v)\mu_x(dv)\mu(dx) + \int_{R_1} f(x,v)\nu_x(dv)p(x)\mu(dx) + f(x, P_i)q(x)\mu(dx).$$

If we take now any function $h = h(v)$ continuous on $\mathbb{R}^m$, supported in $B(P_i, r)$ and $f \equiv 1$ in the neighbourhood of P_i, we see that it is admissible for compactifications κA_i and its extension is constantly equal to 1 on R_1. Hence we have

$$h(u^\nu)\mu \stackrel{\star}{\rightharpoonup} \int_{B(P_i,r)\setminus\{P_i\}} (h(v)\mu_x(dv) + p(x) + q(x))\mu(dx).$$

On the other hand, the Young Theorem 3.2.10 yields

$$h(u^\nu)\mu(dx) \stackrel{\star}{\rightharpoonup} \int_{B(P_i,r)} h(v)\mu_x(dv)\mu(dx).$$

Since then, we see that $p(x) + q(x) = \int_{P_i} h(v)\mu_x(dv)$ and, in general,

$$p(x) + q(x) = \int_{P_i} f(x,v)\mu_x(dv).$$

3.5 The information about the existing methods

Engelking Theorem

In the book by Engleking [53], chapter 3.12.22(e), page 240, there is given an exercise leading to the formulation of Theorem [53]. The exercise shows that for every complete ring of continuous, bounded functions $\mathscr{F}$ on a Tychonoff's space $\mathbb{X}$ there exists a compactification $\Sigma\mathbb{X}$ such that the class of admissible functions is precisely $\mathscr{F}$. The solution of the exercise is however not given. The statement is well-known and broadly quoted in papers dealing with DiPerna-Majda measures theory. The proof hinted by Engelking requires introducing topology on the set of ideals of the ring of continuous real valued functions defined on a space $\mathbb{X}$, which is meant to be compactified. From the proof it does not follow whether the resulting compact set can be embedded into any well-understood Banach space. On the other hand, the set $\mathbb{X}$ is only assumed to be Tychonoff regular, which, in some cases, is unnecessarily general for applications.

The idea of the hinted proof is to define space $\Sigma\mathbb{X}$ as the set of all maximal ideals in the ring $\mathscr{F}$. We introduce there the topology by defining its basis. For that purpose for any $f \in \mathscr{F}$, we define U_f – the set of all ideals m in the ring $\mathscr{F}$ such that $f \notin m$. The basis for topology of $\Sigma\mathbb{X}$ is now precisely the family $\{U_f\}_{f\in\mathscr{F}}$.

Such construction gives us a valuable information – whenever a ring $\mathscr{F}$ is separable (as a space of continuous functions with topology of uniform convergence), we may extract from the family $\{U_f\}_{f\in\mathscr{F}}$ a countable basis of topology. Precisely, we take a dense set $\{f_i\}_{i\in\mathbb{N}} \subset \mathscr{F}$ and notice that the family $\{U_{f_i}\}_{i\in\mathbb{N}}$ forms a countable basis of topology in $\Sigma\mathbb{X}$.

Existence of a countable basis of topology, the so-called second-countablity of the space, is equivalent to metrizability in the class of Hausdorff compact spaces. To see that second-countability implies metrizability we need first to recall [53, Theorem 2.3.23]. The Theorem says that any second-countable Tychonoff's space (so, by fact 3.2.3, in particular any compact Hausdorff space) may be homeomorphically embedded into Tychonoff's cube of countable weight, that is $[0,1]\times[0,1]\times\ldots$. This space is however homeomorphic to $[-1,1]\times[\frac{-1}{2},\frac{1}{2}]\times[\frac{-1}{2^2},\frac{1}{2^2}]\times\ldots\subset\ell^2$, which is metric. After all, any second-countable Hausdorff compact space is homeomorophic to a subspace of a metric space, hence it is metric.

To see that any metric compact space K is second-countable, it is enough to take, for fixed n, a particular cover of K – $\{B(x,1/n)\}_{x\in K}$ – and choose a finite subcover $\mathcal{U}_n$. Now the the family of open sets chosen in at least one of the subcovers $\mathcal{U}_n$, that is $\bigcup_{n\in\mathbb{N}}\mathcal{U}_n$ forms a countable basis of topology.

Therefore we see that whenever the ring $\mathscr{F}$ is separable, the space $\Sigma\mathbb{X}$ is metrizable and may be described as a compact subset of Banach space ℓ^2. This fact gives us an information on the topological structure of that space. Nevertheless, the sketched reasoning is not constructive, as we cannot precisely determine the image of the embedding of $\Sigma\mathbb{X}$ into ℓ^2.

The Gelfand-Naimark Theorem

Another source of knowledge about compactifications, which seems very natural for specialists in analysis, but is not mentioned in the literature around Calculus of Variations, is the classical Gelfand-Naimark Theorem, see [69, Theorem 1]. The most useful for our purposes variant of the theorem reads as follows [11, Theorem 1.1.1]: *Every commutative $\mathbb{C}^\star$ algebra A with "1" is isometric to the $\mathbb{C}^\star$ algebra of continuous, complex-valued functions on some compact space Φ_A.*

The compact set Φ_A is precisely identified as a subset of the dual space to A consisting of such non-zero linear functionals $\phi : A \to \mathbb{C}$ that are also multiplicative (the so-called characters). The topology of Φ_A is an inherited weak-$\star$ topology from the dual space to A.

Let us explain how the statement of the Gelfand-Naimark Theorem contributes to understanding of the problem of compactifications. Take any locally compact and Hausdorff space $\mathbb{X}$. Let us choose any A – a complete ring (see Definition 3.2.6) of bounded, real-valued functions on $\mathbb{X}$. Obviously, A forms a $\mathbb{C}^\star$ algebra with the identity $\star$ operation. The space Φ_A is then a compactification of $\mathbb{X}$ and the space of continuous functions $C(\Phi_A) = A$. The last equality can be understood in the following manner. The set of continuous functions on $\mathbb{X}$, which can be continuously extended to continuous functions on Φ_A, is precisely A. Surprisingly, a careful analysis, see the proof of [69, Lemma 1 and III on p. 1] and [68, Satz 2], shows that the space Φ_A is homeomorphic to the Engelking's compactification of $\mathbb{X}$. Indeed, assigning to every character its kernel is a homeomorphism between Φ_A and $\Sigma\mathbb{X}$. Let us however note that the set Φ_A is identified with a certain subset of the dual space to A, which is a Banach space.

In general case, the compactness of Φ_A holds only for weak-$\star$ topology of the dual of A. Nevertheless, Φ_A happens to be compact in strong, and hence metric, topology whenever A is a countably generated algebra. Furthermore, every continuous function on Φ_A with weak-$\star$ topology is automatically continuous in strong topology (the converse is false). It follows that in case when A is countably generated, Φ_A with metric topology is a compactification of $\mathbb{X}$ such that every function from A possesses a continuous extension to a function on Φ_A. The existence of such metric compactification was not visible from the Engelking's construction. Unfortunately, the shape of Φ_A is hard to determine.

The embedding into a long product due to Keesling

The last idea we would like to consider, and seems to be a little noticed, is presented by Keesling in [93]. The author explains there a construction of the compactification analogous to the one by Engelking in more geometric fashion. Let us brief this construction here. Take the set $\mathscr{F}$ of functions on $\mathbb{X}$ (an arbitrary Tychonoff's space), which are expected to be extendable to continuous functions on compactification. We then use an embedding $i : \Omega \to \prod_{\{f\in\mathscr{F}\}} \mathbb{R}$, $i(x) \stackrel{\text{def}}{=\!=} \big(f(x)\big)_{\{f\in\mathscr{F}\}}$. The compactification is then the closure of the image of i. It is worth explaining here that injectivity of i is guaranteed by the structure of the set $\mathscr{F}$, while compactness follows from boundedness of every single function $f \in \mathscr{F}$ and Tychonoff's Theorem. However the construction is essentially less demanding for non-specialists, the problem of metrizability remains untaken.

3.6 An example of a non-supported measure

Apart from its applications, the construction presented in the Theorem 3.3.2 shows several regularity properties of the abstract compactification $\kappa\mathscr{A}$ of an arbitrary $\mathscr{A}$ used in Theorem 3.4.3. Much effort is taken to guarantee the metrizability of $\kappa\mathscr{A}$. One may wonder whether the result wouldn't hold unbothered if we relaxed our expectations in this direction. Of course the first answer is that the proof of Lemma 3.4.5 exploits the advantages of metrizability. This could be however overridden by some other, more abstract assumption on the regularity of $\kappa\mathscr{A}$, which could possibly be more general than metrizability. This would possibly let us use compactifications constructed in the shape described by Keesling (see section 3.5 in the previous chapter).

The purpose of this section is to present a supercalifragilisticexpialidocious example due to Dieudonne [48], showing that measures defined on non-metrizable topological spaces, especially non-second countable ones, happen to deny some essential properties of the Borel measures known from the Euclidean space.

It is worth stressing that the richness of the Čech-Stone compactification forces us to look very carefully on spaces much less intuitive than subsets of an Euclidean space. In fact, even the set of natural numbers possesses a Čech-Stone compactification of cardinality $2^{\mathfrak{c}}$ and weight (in other words – the minimal cardinality of the basis of topology) $\mathfrak{c}$ [53, Corollary 3.6.12], which gives a brief of how involved is the space. Furthermore, the Parovičenko Theorem [134] states that every compact Hausdorff space of weight no bigger than $\aleph_1$ is a continuous image of the remainder $\beta\mathbb{N}\setminus\mathbb{N}$. Analogous statement to the Parovičenko Theorem holds for the real half-line $[0, +\infty)$ as well [51, Theorem 1]. This shows that even for easy subsets of Euclidean space, there exists a compact subset of their Čech-Stone compactification, which can be mapped onto ω_1. This, in the spirit of the example presented below, may be found disturbing.

3.6.1 Introduction

Let us consider an arbitrary Borel measure defined on a topological space. Intuitively, one would expect such a measure to possess a support, where by the support it is meant the smallest closed set of full measure. For example, let μ be a Lebesgue measure on a cube $[0, 1]^2$ in a plane. Then the support of μ is the entire cube, because otherwise one would need some non-empty open set of a zero measure. However, it appears that measures not possessing a support do exist.

The purpose of this section is to present the construction of the so-called Dieudonné measure – a Borel measure possessing no support, as well as giving a simplified, up to our best knowledge unknown so far, example. The measure possessing no support was found by Jean Dieudonné in 1939 (see [48]). It appears that his ingenious and enlightening example is not difficult to explain to a non-specialist. For his construction, Dieudonné used tools that were already known in the very beginning of the XXth century, introduced by Gerhard Hessenberg in 1906 (see [73]). By now the notions given by Hessenberg have become classical tools in set theory. The Dieudonné discovery ignited several deep investigation topics that have resulted in completely new discoveries, not only in measure theory, but also in set theory. This example has contributed to the understanding of the concept of a measureable cardinal number given by Ulam in [151]. A great development of measure theory on abstract topological spaces was made since then by, among others, Alexandroff [3–5], Rohlin [142] and Marczewski [121–123]. The most notable development was the notion of τ-additivity of measures and its detailed study in series of papers by Alexandroff [3–5], which are being considered as a milestone in the field of abstract measure theory. This seems to be widely inspired by the construction given by Dieudonné.

However, the construction may be found in a variety of places after Dieudonné's paper, as for example in [21, Example 7.1.3], the existing presentations are hard to follow for non-specialists in set theory or abstract analysis. Perhaps this is the reason why this beautiful construction is not as commonly known as it deserves to be. It was the author's aim to present a possibly self-contained proof, which would be widely accessible for non-specialists. We also present another example of a non-supported measure, which is based on Dieudonné's ideas, but the new construction allows to simplify it. Contrary to the classical examples, the measure we present is not finite. On the other hand, the construction is essentially simpler. To our best knowledge, infinite measures constructed on ordinal numbers have not been considered to far. The analysis of supports of Borel measures is visible in recent papers dealing with DiPerna-Majda measures, which are modern tools in the Calculus of Variations [85, 92, 113], seem influential in PDEs [6, 40] and applied mathematics, like the analysis of microstructures [112] or fluid mechanics [49].

Especially in [85], dealing with DiPerna-Majda measures, one may observe that the problem of the existence of the support of a measure seems to be an important topic for formulating essential assumptions on the main theorem. It was puzzling to the author if a violation of this assumption could lead to a construction experiencing similar phenomena. Such consideration could possibly weaken the standard assumptions in the field. This led to the surprising idea of connecting Calculus of Variations – in fact very applicable for engineering – with very abstract measure theory. It appears that these theories meet when looking for conditions guaranteeing the existence of the support of a Borel measure.

The section begins with recalling some introductory information about the notions we deal with. Throughout the process, we introduce measure-theoretic and set-theoretic notions and prove needed properties. In the last part, the desired example comes easily as a consequence of presented considerations, as well as the new simplification is given. Although the section is intended to be self-contained, some technical details are left to the reader to figure out. Even then, we refer to some outer sources where the facts are directly established.

3.6.2 Auxiliary measure- and set-theoretic definitions

Classical measure-theoretic notions

We start by recalling some measure-theoretic definitions and begin studying measures possessing support in a wide class of spaces, containing most classical examples.

Definition 3.6.1 (Measure, see Definition 1.3.2 in [21])**.** Let Ω be any set and $\mathcal{F}$ – any σ-field of subsets of Ω. By a measure on Ω (measuring elements of $\mathcal{F}$) we mean a function

$$\mu : \mathcal{F} \to [0, +\infty]$$

which vanishes on $\emptyset$ and for any countable family of subsets $\{A_i\}_{i\in\mathbb{N}}$ of $\mathcal{F}$ such that $i \neq j \Rightarrow A_i \cap A_j = \emptyset$, we have

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i).$$

Definition 3.6.2 (Support of a measure, just above Proposition 7.2.9 in [21])**.** By a support of measure μ we mean a closed set $C \stackrel{\text{def}}{=\!=} \operatorname{supp}\mu$ such that

i) $\mu(\Omega \setminus C) = 0$;

ii) if C_1 is closed and $\mu(\Omega \setminus C_1) = 0$, then $C \subseteq C_1$.

We may thus interpret support as the smallest closed set which has a μ-"almost empty" complement. In particular, if a support exists, it must be equal to the intersection of all closed sets whose complements have the measure 0.

The question on the existence of a support of measure μ is very simple in typical situations - for example every Borel measure defined on a metric, separable space possesses a support. Even more generally, the following proposition holds.

Proposition 3.6.3. *Let Ω be a topological space with a countable basis of topology and μ be a Borel measure. Then the support of the measure μ is well-defined.*

Proof. Let us take a family $\mathscr{C}$ of all closed subsets C of Ω such that $\mu(\Omega \setminus C) = 0$. We show that the following equality holds

$$\mathfrak{S} \stackrel{\text{def}}{=\!=} \bigcap_{C\in\mathscr{C}} C = \operatorname{supp}\mu.$$

It is clear that $\mathfrak{S}$ is closed. It is also convenient, that if C is closed and $\mu(\Omega \setminus C) = 0$, then $\mathfrak{S} \subseteq C$, because $C \in \mathscr{C}$. From that point we know that it is the smallest closed set in the family $\mathscr{C} \cup \{\mathfrak{S}\}$.

It remains to check, however, that $\mu(\Omega \setminus \mathfrak{S}) = 0$, i.e. $\mathfrak{S} \in \mathscr{C}$. To that end, let us denote the countable basis of topology on Ω as $\mathscr{U} \stackrel{\text{def}}{=\!=} \{U_i\}_{i\in\mathbb{N}}$. Notice then, that for any closed $C \subseteq \Omega$ the set $\Omega \setminus C$ is open and thus it may be written as a sum of some sets $U_i \in \mathscr{U}$. For every closed $C \subseteq \Omega$ let us define

$$I(C) \stackrel{\text{def}}{=\!=} \{i \in \mathbb{N} : \; U_i \subseteq \Omega \setminus C\}.$$

In this way, for every closed $C \subseteq \Omega$ we have

$$\Omega \setminus C = \bigcup_{i \in I(C)} U_i.$$

Notice that for every $C \in \mathscr{C}$ and every $i \in I(C)$ we have $\mu(U_i) = 0$. Hence we have

$$\Omega \setminus \mathfrak{S} = \bigcup_{C\in\mathscr{C}} \Omega \setminus C = \bigcup_{C\in\mathscr{C}} \bigcup_{i\in I(C)} U_i = \bigcup_{\{i\in\mathbb{N}:\; \exists_{C\in\mathscr{C}} \; i\in I(C)\}} U_i.$$

The set $\{i \in \mathbb{N} : \; \exists_{C\in\mathscr{C}} \; i \in I(C)\}$ is of course countable, because it is a subset of $\mathbb{N}$. Thus $\Omega \setminus \mathfrak{S}$ is a countable union of sets of measure 0 and thus it is of measure 0. □

Remark 3.6.4. Note that any metric separable space has a countable basis of topology. For the proof it is enough to take balls centred in the dense countable subset with rational radii. On the other hand, any space with a countable basis of topology is separable – for the proof one takes a single point from every open set of the countable basis.

Remark 3.6.5. The proposition above, shown for example in [21, Proposition 7.2.2 (iv)], is a special case of some deeper theorems, such as [21, Proposition 7.2.9].

Ordinal numbers

We move to recalling the definition of ordinal numbers and some basic facts about them. We will apply them to create a certain "very inseparable" topological space where our desired non-supported measure will lie. Let us also recall that by a well-ordered set we mean a set with such a relation of ordering, where every subset possesses its smallest element.

We begin with defining ordinal numbers in a non-standard, but very compact manner. The interested reader is welcome to compare this notion with [129, Excersices 12.2-12.4].

Definition 3.6.6 (Ordinal numbers)**.** We define ordinal numbers by the procedure of transfinite induction. Take

(i) $0 \stackrel{\text{def}}{=\!=} \emptyset$;

(ii) for any ordinal number α we define $\alpha + 1 \overset{\text{def}}{=\!=} \alpha \cup \{\alpha\}$;

(iii) for any set of ordinals $\{\alpha_t\}_{t\in T}$, where T is any set of indices, we define the ordinal number $\sup\{\alpha_t\}_{t\in T} \overset{\text{def}}{=\!=} \bigcup_{t\in T} \alpha_t$.

Remark 3.6.7. It is worth noting that simultaneously we have $\alpha \in \alpha + 1$ and $\alpha \subset \alpha + 1$.

Example 3.6.8. Having $0 = \emptyset$ already defined, note that step (ii) gives us

$$1 = \emptyset \cup \{\emptyset\} = \{\emptyset\} = \{0\}.$$

We then have

$$2 = 1 \cup \{1\} = \{\emptyset\} \cup \big\{\{\emptyset\}\big\} = \big\{\emptyset, \{\emptyset\}\big\} = \{0, 1\}.$$

Following this procedure we obtain the set of natural numbers (understood as the set of all finite ordinal numbers), having in mind that for any natural number n we have

$$n + 1 = \{0, 1, 2, \ldots, n\}.$$

Note now that this is the moment when, for every already defined ordinal number n, the ordinal number $n + 1$ is also defined and thus we may not use step (ii) for defining new ordinal numbers. Applying step (iii) to the set of natural numbers defined in this way we get the first infinite ordinal number

$$\omega_0 = \bigcup_{n\in\mathbb{N}} n = \bigcup_{n\in\mathbb{N}} \{0, 1, 2, \ldots, n-1\} = \{0, 1, 2, \ldots\},$$

which is a supremum of all natural numbers. After that, step (ii) defines the ordinal number $\omega_0 + 1 = \{0, 1, 2, \ldots, \omega_0\}$. Continuing analogously, we will get the ordinal number $\omega_0 + n = \{0, 1, \ldots, \omega_0, \omega_0 + 1, \ldots\ \omega_0 + (n-1)\}$ and then applying step (iii) one gets $\omega_0 + \omega_0 = \{0, 1, \ldots, \omega_0, \omega_0 + 1, \ldots\ \}$. We will denote this number as $2 \times \omega_0$.

Proceeding further, we will then obtain $2 \times \omega_0 + n = \{0, 1, \ldots \omega_0, \omega_0 + 1, \ldots, 2 \times \omega_0, 2 \times \omega_0 + 1, \ldots, \omega_0 + (n-1)\}$ and applying step (iii) again, $3 \times \omega_0$. Similarly, we will obtain ordinal numbers $4 \times \omega_0, 5 \times \omega_0$ and, as the supremum of numbers $n \times \omega_0$, arrive at $\omega_0 \times \omega_0$.

From the definition it follows that every ordinal number α is defined as a set consisting of all ordinal numbers previously defined. Furthermore, for any two ordinal numbers $\alpha \neq \beta$ we confirm that exactly one of the relations $\alpha \subsetneq \beta, \beta \subsetneq \alpha$ holds. Note also that $\alpha \in \beta \iff \alpha \subsetneq \beta$. We may thus define ordering $\alpha \prec \beta \overset{\text{def}}{\iff} \alpha \in \beta$. Any ordinal number γ is a well-ordered set with such ordering, which seems visible, but is quite demanding from the technical side. Also, for any well-ordered set T (with order $<$) there exists precisely one ordinal number ϑ (with order $\prec$) and bijection $\varphi : T \to \vartheta$ such that $t_1 < t_2 \iff \varphi(t_1) \prec \varphi(t_2)$. The interested reader could find many details for example in [129, Chapter 12]. In particular, a precise proof of the well-ordering of ordinal numbers is given in [129, Theorem 12.15].

Definition 3.6.9 (Successor, compare to 12.2 in [129]). The number α is called a successor (of β), if there exists an ordinal number β such that $\beta + 1 = \alpha$.

Successors are precisely the numbers which (as sets) possess the largest element. Other numbers, i. e. those not possessing its biggest element, are called `limit numbers`.

Example 3.6.10. Every non-zero natural number n is, of course, a successor because $n = (n-1)+1$. On the other hand, ω_0 is a limit number. For every $n \neq 0$, however, $\omega_0 + n$ is a successor and $2 \times \omega_0$ is again a limit number.

We continue with defining certain symbols. By ω_0 we will mean the first non-empty limit ordinal number, as in Example 3.6.8.
By ω_1 we will mean the first ordinal number which is non-bijective (as a set) with ω_0.
Then by the transfinite induction let us set ω_α to be the first ordinal number which is non-bijective with ω_β for every $\beta < \alpha$.

It is probably worth explaining how the transfinite induction works in the particular situation of defining numbers ω_α. The induction is made with respect to the indices. The first point is then to define number ω_0 and this is done with natural numbers, i.e. the number ω_0 coincides with the first infinite ordinal number. Then, having defined ω_β we may define a number $\omega_{\beta+1}$ as the first ordinal number non-bijective with ω_β. What is left to explain is the limit step. Having defined ω_β for every number $\beta < \alpha$ we may define number ω_α as the first ordinal number which is non-bijective with any of numbers ω_β.

It is not visible at the first glance that the procedure of defining ordinal numbers ω_α does not stop, i.e. one may wonder if there exists some 'untouchable' ordinal number δ such that there exists no ω_δ. The following observation shows that this is not the case.

Proposition 3.6.11 (Existence of ordinal numbers, 12.29 in [129]). *For any ordinal number α there exists the ordinal number ω_α.*

Proof. Zermelo's Theorem [129, Theorem 8.9] states that any set may be well-ordered. Let us also recall Cantor's Theorem [129, Theorem 2.21], stating that the set $P(T)$ of all subsets of a set T has more elements than the set T itself (in the sense that there exists no bijection between these sets, while obviously there exists a one-to-one function from T to $P(T)$ – namely $t \mapsto \{t\}$).

Suppose then that there exists an ordinal number α such that ω_α does not exist. From the procedure of the transfinite induction it follows that for any $\beta > \alpha$ the number ω_β also does not exist.

We may assume then that every ordinal number γ, such that there exists ω_γ, is bounded from above by α, hence these numbers γ form an ordinal number which is a subset of α. Let us call this number Γ. Note that $\Gamma \notin \Gamma$, so specifically there exists no ω_Γ. Take $\Gamma' = \sup_{\gamma\in\Gamma} \omega_\gamma$. Then the set $P(\Gamma')$ has more elements than any of the sets ω_γ. Via Zermelo's Theorem this set may be well-ordered, so there exists an ordinal number bijective to $P(\Gamma')$. It follows that there exists an ordinal number which has more elements than any of the numbers ω_γ for $\gamma \in \Gamma$.

We will show that there exists a number ω_Γ, which provides a contradiction. Indeed, we have shown that there exists an ordinal number (let's call it ϑ) which has more elements than ω_γ for any $\gamma \in \Gamma$. Note now that $\vartheta \in \vartheta + 1$. The set of those elements of $\vartheta + 1$, which have more elements than ω_γ for any $\gamma \in \Gamma$ is thus non-empty and hence possesses its smallest element. From the very definition, this element is precisely ω_Γ, thus ω_Γ exists – a contradiction. □

Definition 3.6.12 (Cardinal numbers, 12.26 in [129])**.** The class of such ordinals α that there exists an ordinal number β such that $\alpha = \omega_\beta$ will be called the class of cardinal numbers. We will say that a set T is of cardinality κ whenever κ is such a cardinal number that T is bijective with κ and denote $|T| = \kappa$.

Note that any cardinal number is a limit ordinal number. This is because of the bijection between numbers α and $\alpha + 1$, which holds whenever α is infinite.

Having the notion of the cardinality of a set, it is necessary to mention the classical and deep Hessenberg Theorem, which is attributed to the German mathematician Gerhard Hessenberg and dates back to 1906. The first proof of the result was presented by Hessenberg in [73]. To find a proof in English in any outer source, the author recommends [129, Lemma 9.15]. We will present the proof only in the case of a countable set T, because this is the only case we will use and the proof is significantly simpler.

Theorem 3.6.13 (Hessenberg in [73], 1906)**.** *Let T be an infinite set. Then $|T| = |T \times T|$.*

Proof – only in case of T countable. First we prove that whenever sets T, T' satisfy $|T| = |T'|$, then also $|T \times T| = |T' \times T'|$. For that it is enough to check that whenever $\phi : T \to T'$ is a bijection, so is $\tilde{\phi} : T \times T \to T' \times T'$ defined via $\tilde{\phi}[(t_1, t_2)] = \Big(\phi(t_1), \phi(t_2)\Big)$, which is standard.

It follows that it is sufficient to check that $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$, as by the assumption there exists a bijection $\phi : \mathbb{N} \to T$. This is done by a standard diagonal procedure, which is presented below for completeness.

Let us define the ordering $\prec$ on the set $\mathbb{N} \times \mathbb{N}$ as follows. We say $(n, m) \prec (n', m')$ whenever either $n + m < n' + m'$ or $n + m = n' + m'$ and $n < n'$. It is obvious that $\prec$ is a well-ordering of $\mathbb{N} \times \mathbb{N}$ and thus, there exists an ordinal number α and bijection $\psi : \mathbb{N} \times \mathbb{N} \to \alpha$ such that $(n, m) \prec (n', m') \iff \psi[(n, m)] < \psi[(n', m')]$. Of course the set $\mathbb{N} \times \mathbb{N}$ is infinite, nevertheless for any pair (n, m) there exist less than $\frac{1}{2}(n + m + 1)^2$ (so finitely many) pairs (n', m') such that $(n', m') \prec (n, m)$. Note that if $\omega_0 \in \alpha$, then there exist infinitely many $\beta \in \alpha = \psi[\mathbb{N} \times \mathbb{N}]$ such that $\beta < \omega_0$. Hence $\omega_0 \notin \psi[\mathbb{N} \times \mathbb{N}]$ and α is either finite or equal to ω_0. As the first option is not the case, we see that $\mathbb{N} \times \mathbb{N}$ is bijective with ω_0, which completes the proof of $|\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$.

□

Remark 3.6.14. In particular from the Hessenberg Theorem it is simple to deduce, that whenever sets T_i are countable, then $\bigcup_{i\in\mathbb{N}} T_i$ also is. Hence, ω_1 is not a supremum of any countable family of its elements. We will use this fact in the sequel.

Let us skip to the ordinal topology, that is to the way we define the topology on ordinal sets. For that purpose we reformulate [21, Example 6.1.21] to a form more accurate for our purposes.

Definition 3.6.15 (Ordinal topology, compare to 12.24 in [129])**.** We say that subset C of the ordinal α is closed in α, if for any set T and the choice of elements $c_t \in C$, where $t \in T$, the ordinal number c defined via

$$c \stackrel{\text{def}}{=\!=} \bigcup_{t\in T} c_t$$

belongs to C, whenever it belongs to α.

Example 3.6.16. It follows straightforwardly from the definition of ordinal topology, that for any $\gamma < \alpha$ the set $C_\gamma = \{\beta \in \alpha : \ \beta > \gamma\}$ is closed in α. It is also unbounded in the following sense: there is no such $\delta < \alpha$ (or equivalently $\delta \in \alpha$) such that $\beta \in C_\gamma \Rightarrow \beta \leq \delta$. In the sequel we will often refer to these two properties, namely to closeness and unboundedness. In the literature, closed and unbounded sets are often called `clubs`.

Definition 3.6.17 (Bounded and unbounded sets)**.** Let α be an ordinal number. We say that the set $T \subset \alpha$ is bounded in α, whenever there exists $\tau \in \alpha$ such that for each $t \in T$ the inequality $t < \tau$ holds.

Remark 3.6.18. Notice that for any limit number α and its unbounded subset C (for example $C = \alpha$) the sum of the elements of C is precisely α and thus does not belong to α as its element. This is the reason for the "whenever it belongs to α" condition in the definition of ordinal topology and actually the only case when it is used.

Remark 3.6.19. Such defined topology is somehow comparable to the typical "trace topology" of a subspace of a certain topological space. Note, however, that the interval $[0, 1)$ is not closed in $\mathbb{R}$ (here we consider $\mathbb{R}$ with standard metric topology), but closed in the trace topology inherited from the interval $(-1, 1)$. The same way the set $\mathscr{M}$ of all countable ordinals is closed and unbounded in ω_1 (because it actually coincides with ω_1) but in $\omega_1 + 1$ it is neither closed nor unbounded (because its supremum – ω_1 – belongs to $\omega_1 + 1$ and does not belong to $\mathscr{M}$).

The ordinal ω_0 can be homeomorphically embedded into $[0, 1]$ via $0 \mapsto 0$ and for $n \geq 1$ $n \mapsto 1 - \frac{1}{n+1}$. This homeomorphism (onto the image) preserves order. It is worth observing that every point in ω_0 is open, so we experience no problems with continuity. Similarly, one defines a map $\omega_0 + 1 = \omega_0 \cup \{\omega_0\} \to [0, 1]$ via $n \mapsto 1 - \frac{1}{n+1}$, $\omega_0 \mapsto 1$. Intuitively, such defined function is continuous because the supremum of n's is mapped into the supremum of $(1 - \frac{1}{n+1})$'s.

Further set-theoretical tools

The next proposition establishes an important feature of clubs, that is closed and unbounded sets. It will be crucial for the construction of the desired measure.

Proposition 3.6.20 (Compare to Lemma 3.4 in [76])**.** *Let $\mathscr{C}$ be a countable family whose elements are closed and unbounded sets in ω_1. Then the set $\bigcap_{C\in\mathscr{C}} C$ is closed and unbounded in ω_1.*

Proof. As the intersection of closed sets is always closed, it is sufficient to prove the unboundedness. Take then any $\gamma < \omega_1$. Let us also enumerate sets C by natural numbers, so that $\mathscr{C} = \{C_i\}_{i\in\mathbb{N}}$. We will construct by induction a particular sequence $\{\xi_i\}_{i\in\mathbb{N}}$ of ordinals smaller then ω_1. Let $\xi_1 = \gamma$. Assuming now that for a certain i that ξ_i is well defined, let us define ordinal numbers $\vartheta_{i,j}$ for any $j \in \mathbb{N}$. For that purpose we take any $\vartheta_{i,j} \in C_j$ such that $\vartheta_{i,j} > \xi_i$. It is possible for every j, as every C_j is unbounded in ω_1. Let

$$\xi_{i+1} = \sup_{j\in\mathbb{N}} \vartheta_{i,j}.$$

Now, by Remark 3.6.14, $\xi_{i+1} \in \omega_1$ because it is a supremum of a countable family of elements of ω_1. Therefore, the sequence $\{\xi_i\}_{i\in\mathbb{N}}$ is properly defined.

Let us take $\delta = \sup \xi_i$ and note again that $\delta \in \omega_1$. Observe that for any fixed j one has $\xi_i \prec \vartheta_{i,j} \prec \xi_{i+1}$, and hence δ is also a supremum of elements of C_j. As a result, and from the closeness of C_j, we obtain that $\delta \in C_j$ for every j. Thus δ belongs to the intersection of C_j. In particular, the intersection of $C_j's$ possesses an element bigger than γ. As γ was arbitrary, the unboundedness is established. □

Now we are ready to define the σ-field of measurable sets for the Dieudonné measure. Let us take

$$\mathscr{B} \overset{\text{def}}{=} \{A \subseteq \omega_1 \mid \text{there exists a closed and unbounded set } C \text{ in } \omega_1 \text{ such that either } C \subseteq A \text{ or } C \subseteq A^{\complement}\}.$$

We will now make some observations regarding the family $\mathscr{B}$, using Remark 3.6.14 and Proposition 3.6.20.

Corollary 3.6.21. *The following conditions hold*

i) *$\mathscr{B}$ is a $\sigma - field$. Furthermore, every Borel set belongs to $\mathscr{B}$.*

ii) *Sets*

$$\mathscr{B}_1 \overset{def}{=} \{A \subseteq \omega_1 | \textit{there exists a closed and unbounded set } C \textit{ in } \omega_1 \textit{ st. } C \subseteq A\},$$
$$\mathscr{B}_2 \overset{def}{=} \{A \subseteq \omega_1 | \textit{there exists a closed and unbounded set } C \textit{ in } \omega_1 \textit{ st. } C \subseteq A^{\complement}\}$$

are disjoint.

iii) If for every $i \in \mathbb{N}$ the set $A_i \in \mathscr{B}_2$, then $\bigcup_{i\in\mathbb{N}} A_i \in \mathscr{B}_2$.

Proof. First, let us note that whenever C_1, C_2 are closed and unbounded, the set $C_1 \cap C_2$ is also closed and unbounded, so in particular is non-empty. On the other hand, if $C_1 \subseteq A, C_2 \subseteq A^{\complement}$, then $C_1 \cap C_2$ is empty. Thus, A is an element of at most one of sets $\mathscr{B}_1, \mathscr{B}_2$. This finishes the proof of part ii).

Obviously $\emptyset \subseteq \omega_1$ and ω_1 is closed and unbounded, so taking $C = \omega_1$ in the definition of $\mathscr{B}$ shows that both $\emptyset$ and ω_1 of them belong to $\mathscr{B}$.

From Proposition 3.6.20 it readily follows, that whenever for every $i \in \mathbb{N}$ one has $C_i \subseteq A_i$, then $\bigcap C_i$ is closed and unbounded and $\bigcap C_i \subseteq \bigcap A_i$. In addition, whenever A_1 is such that $C_1 \subseteq A_1^{\complement}$ and $A_i \in \mathscr{B}$ for any i, then $C_1 \subseteq A_1^{\complement} \subseteq \left(\bigcap A_i\right)^{\complement}$. We have shown then that a countable intersection of sets from $\mathscr{B}$ belongs to $\mathscr{B}$.

Finally – whenever A is an element of particular $\mathscr{B}_i$, the set $A^{\complement}$ is an element of the other one. In fact this is straightforward because $\left(A^{\complement}\right)^{\complement} = A$. We have established that the complements of sets from $\mathscr{B}$ belong to $\mathscr{B}$ and thus, in view of the two previous paragraphs, proved that $\mathscr{B}$ is a σ-field.

It is left for us to observe that any closed set K is either unbounded (and thus belong to $\mathscr{B}$, because $K \subseteq K$) or bounded by some $\gamma < \omega_1$. In the latter case note that the set $C_\gamma = \{\beta \in \alpha : \beta > \gamma\}$ is closed and unbounded, so there exists a closed and unbounded set C such that $C \subseteq K^{\complement}$. Since then every closed set belongs to $\mathscr{B}$. Knowing that $\mathscr{B}$ is a σ-field, we conclude that any Borel set belongs to $\mathscr{B}$ and establish point i).

Point iii) is already established throughout the proof of ii), but it seems necessary for the sequel and thus we will formulate and prove it separately. Take closed and unbounded sets C_i such that $C_i \subseteq A_i^{\complement}$. From Proposition 3.6.20 it follows that $\bigcap C_i$ is closed and unbounded. Thus one gets $\bigcap C_i \subseteq \bigcap A_i^{\complement} = \left(\bigcup A_i\right)^{\complement}$. □

Remark 3.6.22. It is worth to stress, that whenever C is a closed and unbounded set, it obviously belongs to $\mathscr{B}_1$.

3.6.3 An example of a Borel measure with no support

Let us now construct the non-supported measure on the topological space ω_1. For any $S \subseteq \omega_1$ such that $S \in \mathscr{B}$ we define measure

$$\mathscr{D}(S) = \begin{cases} 1 & \text{whenever } S \in \mathscr{B}_1 \\ 0 & \text{whenever } S \in \mathscr{B}_2. \end{cases}$$

Proposition 3.6.23. *With such defined $\mathscr{D}$ we have*

i) $\mathscr{D}$ is a Borel measure on Ω;

ii) for every element $\beta \in \Omega$ there exists a closed set C_β of measure 1 such that $\beta \notin C_\beta$. In particular, the intersection of all closed sets with complements of measure 0 is empty.

Proof.

i) First let us check that the measure is properly defined. As sets $\mathscr{B}_1$ and $\mathscr{B}_2$ are disjoint, every set has been assigned with precisely one measure. The measure of empty set is 0, so we need to check additivity on at most countable families of disjoint sets. As Corollary 3.6.21 part iii) shows, a countable sum of sets with a measure 0 has a measure 0. On the other hand, from Proposition 3.6.20 we see that any at most countable family of sets of measure 1 has a non-empty intersection (even of measure 1), hence the sets are not disjoint. It easily follows then, that $\mathscr{D}$ is a well-defined measure. From 3.6.21 part i) it follows that this measure is Borel.

ii) Indeed, for any β take $C_\beta = \{\gamma \in \Omega \mid \gamma > \beta\}$ as in Example 3.6.16. Every set C_β is closed and unbounded, thus of measure 1. Its complement is hence of measure 0. As $\beta \notin C_\beta$, in particular $\beta \notin \bigcap_{\alpha \in \Omega_1} C_\alpha$. Arbitrariness of β shows that $\bigcap_{\alpha \in \Omega_1} C_\alpha = \emptyset$.

□

The Dieudonné example is in fact an inspiration for plenty of other, just slightly more involving, examples. Let us mention that a technical modification of Proposition 3.6.20 reads as follows.

Proposition 3.6.24. *Let α be any non-zero, non-limit ordinal number and $\mathscr{C}$ – a countable family whose elements are closed and unbounded sets in ω_α. Then the set $\bigcap_{C \in \mathscr{C}} C$ is closed and unbounded in ω_α.*

This statement may still be improved by the use of the notion of cofinality, however we do not intend to introduce it. Proposition 3.6.24 shows in fact, that whenever α is a non-limit ordinal number, a variant of Dieudonné measure may by found on ω_α. Namely, we define

$$\mathscr{B}_1 \stackrel{\text{def}}{=} \{A \subseteq \omega_\alpha \mid \text{there exists a closed and unbounded set } C \text{ in } \omega_\alpha \text{ such that } C \subseteq A\},$$
$$\mathscr{B}_2 \stackrel{\text{def}}{=} \{A \subseteq \omega_\alpha \mid \text{there exists a closed and unbounded set } C \text{ in } \omega_\alpha \text{ such that } C \subseteq A^{\complement}\}$$

and set

$$\mathscr{D}_\alpha(S) = \begin{cases} 1 & \text{whenever } S \in \mathscr{B}_1; \\ 0 & \text{whenever } S \in \mathscr{B}_2. \end{cases}$$

In our closing remarks we intend to present one more, slightly easier example. To be more precise – one without any of the tools described in subsection 3.6.2. The notion of a measureable number, which was introduced by Ulam in [151], dealt with probabilistic measures. This is probably the reason why such examples as the one below were hardly ever considered. Despite making an effort, the author has not found such a simplification of Dieudonné construction in the existing literature. Let us define a simplified Dieudonné measure on ω_1 via

$$\mathscr{E}\mathscr{D}(S) = \begin{cases} +\infty & \text{if it unbounded in } \omega_1 \\ 0 & \text{otherwise.} \end{cases}$$

To check that $\mathscr{E}\mathscr{D}$ is a measure it is sufficient to note that set S is bounded in ω_1 if and only if it is at most countable. As we have proven, the countable union of countable sets stays countable, hence a countable sum of sets of $\mathscr{E}\mathscr{D}$-measure 0 is still of measure 0. On the other hand, a support of this measure does not exist, as the identical reasoning has shown in case of $\mathscr{D}$-measure.

We have presented this example to show that the Dieudonné construction may be simplified. It is, however, vital to observe, that the $+\infty$ substitution in the definition is necessary. To see it, let us take two unbounded sets. Let A be the set of all limit ordinal numbers in ω_1 and B be defined via $B \stackrel{\text{def}}{=\!=} \{\alpha+1 : \alpha \in A\}$. Of course A consists only of limit numbers, while B – only of successors. In particular, they are disjoint. It shows that having any real number R instead of $+\infty$ in the definition of $\mathscr{E}\mathscr{D}$ would lead to an improper definition. Realizing this allows us once again appreciate the elegance of the construction given in [48].

Chapter 4

Thin structures in Orlicz-Sobolev spaces

4.1 Introduction

In the study of thin structures, i.e. when the structure's size along one or more dimensions is much smaller than along the others, say of order $\varepsilon << 1$, rigorous analysis via dimensional reduction proves to be an useful tool to deduce properties of thin domains starting from thicker models. In this analysis, one deals with sequences of functions defined on cylindrical sets, which are 'thin' (ε sized) in some dimensions. In the $3D$ setting, thin films are modelled as $\omega \times (-\varepsilon, \varepsilon)$ with $\omega \subset \mathbb{R}^2$ being a bounded open set. In order to perform an asymptotic analysis as $\varepsilon \to 0$, with the aim of deducing a theory settled in ω, functions are usually rescaled to an ε-independent reference configuration, so that a new sequence (u_ε) is constructed, satisfying, in the standard Sobolev setting, some 'degenerate' bounds of the form

$$\int_{\omega\times(-1,1)} \left(|\nabla_\alpha u_\varepsilon|^p + \frac{1}{\varepsilon^p}|\nabla_3 u_\varepsilon|^p\right) dx \leq C < +\infty \tag{4.1}$$

if the sequence of unscaled gradients (∇w_ε) satisfied some corresponding L^p bound on the unscaled domain $\omega \times (-\varepsilon, \varepsilon)$.

Above and in the sequel ∇_α represents the gradient with respect to the unscaled coordinates (denoted by x_α) and ∇_3 represents the gradient with respect to the 'thin' coordinate direction denoted by x_3. In particular, $\Omega := \omega \times (-1, 1) = \{(x_\alpha, x_3) : (x_\alpha, \varepsilon x_3) \in \omega \times (-\varepsilon, \varepsilon)\}$ and $w_\varepsilon(x_\alpha, \varepsilon x_3) \stackrel{\text{def}}{=} u_\varepsilon(x_\alpha, x_3)$.

Bocea and Fonseca in [23] (see also Braides and Zeppieri in [26] for any dimension) proved an equi-integrability Lemma for scaled gradients satisfying a bound as (4.1). Indeed they generalized the Fonseca, Mueller and Pedregal's result (see [64, Lemma 1.2]) which allows to substitute a sequence (u_n), whose gradients (∇u_n) are bounded in L^p, by a sequence (v_n) with $(|\nabla v_n|^p)$ equi-integrable, such that the two sequences are equal except on a set of vanishing measure. The purpose of such a result is due to the fact that when applying the direct methods of the Calculus of Variations, or some Γ-convergence argument, it is very convenient to replace a given sequence with one having better regularity and integrability properties.

In this chapter of the thesis we extend Braides and Fonseca results, namely [23, Theorem 1.1, Corollary 1.2], to the Orlicz-Sobolev setting. See Section 4.2 for details and properties about Orlicz spaces L^Φ and Orlicz-Sobolev ones $W^{1,\Phi}$. Indeed, via Young measures techniques, we prove the following Theorem.

Theorem 4.1.1. *Let $\omega \subset \mathbb{R}^2$ be a bounded open set with Lipschitz boundary and $\Omega \stackrel{def}{=} \omega \times (-1,1)$. Let $\Phi : [0,+\infty) \to [0,+\infty)$ be an Orlicz function satisfying conditions Δ_2 and ∇_2 (see (4.5) and (4.6) respectively). Let $(u_n) \subset W^{1,\Phi}(\Omega;\mathbb{R}^3)$. Assume that (ε_n) is a sequence of numbers converging to 0, such that*

$$\sup_n \int_\Omega (\Phi(|\nabla_\alpha u_n, \tfrac{1}{\varepsilon_n}\nabla_3 u_n|))dx = C < +\infty. \tag{4.2}$$

Then there exists a (non-relabelled) subsequence (u_n) and a sequence $(v_n) \subset W^{1,\Phi}(\Omega;\mathbb{R}^3)$ such that

(i) sequence $(\Phi(|\nabla_\alpha v_n, \frac{1}{\varepsilon_n}\nabla_3 v_n|))$ is equi-integrable,

(ii) $v_n \rightharpoonup u_0$ in $W^{1,\Phi}(\Omega;\mathbb{R}^3)$, where u_0 is the weak limit of (u_n) in $W^{1,\Phi}(\Omega;\mathbb{R}^3)$,

(iii) $|\{x \in \Omega : u_n \neq v_n \text{ or } \nabla u_n \neq \nabla v_n\}| \to 0$, as $n \to +\infty$,

(iv) $v_{n|\partial\omega\times(-1,1)} = u_0$.

We stress that the above result holds for any sequence of scaled gradients appearing in any dimensional reduction problem, besides the proof is presented for the total number of dimensions $N = 3$ and the number of fixed-size dimensions $K = 2$.

Having in mind the equilibrium problems related to membranes, where the total energy of the thin film under a deformation $w_\varepsilon : \omega \times (-\varepsilon,\varepsilon) \to \mathbb{R}^3$ is given by

$$E_\varepsilon(w_\varepsilon) \stackrel{\text{def}}{=} \int_{\omega\times(-\varepsilon,\varepsilon)} W(\nabla w_\varepsilon(y))dy - \int_{\omega\times(-\varepsilon,\varepsilon)} f^\varepsilon(y)\cdot w_\varepsilon(y)dy,$$

with $f^\varepsilon \in L^\Psi(\omega\times(-\varepsilon,\varepsilon),\mathbb{R}^3)$ – an appropriate dead loading body force density (we refer to [114] for the asymptotic analysis of the above energy), it is important to prove the existence of an 'attaining' sequence for the limit density, which is Φ-equi-integrable. Indeed, the following result holds. We stress that it is a natural generalisation of [24, Remark 3.3], but formulated for Orlicz spaces instead of classical Lebesgue. It is worth observing that the result deals with the notion of quasiconvexification, which was one of the investigated points in Chapter 2.

Theorem 4.1.2. *Let Ω and Φ be as in Theorem 4.1.1. Let $u_0 \in W^{1,\Phi}(\omega, \mathbb{R}^3)$ be an affine mapping with gradient $\xi_0 \in \mathbb{R}^{3\times 2}$ and let $W : \mathbb{R}^{3\times 3} \to \mathbb{R}$ be a continuous function satisfying*

$$\beta\Phi(|\xi|) - c \leq W(\xi) \leq \beta'\Phi(|\xi|) + C \quad \textit{for every } \xi \in \mathbb{R}^{3\times 3}, \tag{4.3}$$

for suitable constant $0 < \beta \leq \beta'$, $c, C > 0$.

For any matrix $\xi \in \mathbb{R}^{3\times 2}$ and a vector $z \in \mathbb{R}^3$ let $(\xi|z)$ be a 3×3 matrix, whose first and second column ale columns from the matrix ξ, while the third column of $(\xi|z)$ coincides with z. Given any sequence (ε_n) of positive real numbers converging to zero, there exist a subsequence (not relabelled) of (ε_n), and a sequence of functions $(u_{\varepsilon_n}) \subset W^{1,\Phi}(\Omega, \mathbb{R}^3)$ such that

(i) $\lim_{n\to+\infty} \frac{1}{|\Omega|} \int_\Omega W\left(\nabla_\alpha u_{\varepsilon_n}, \frac{1}{\varepsilon_n}\nabla_3 u_{\varepsilon_n}\right) dx = Q\overline{W}(\xi_0)$, *where* $\overline{W}(\xi_0) = \min_{z\in\mathbb{R}^3} W(\xi_0|z)$ *and* $Q\overline{W}$ *denotes the quasiconvex envelope of* $\overline{W}$, *namely*

$$Q\overline{W}(\xi_0) = \inf_{\varphi\in W_0^{1,\infty}(Q_b,\mathbb{R}^3)} \left\{ |Q_b|^{-1} \int_{Q_b} \overline{W}(\xi_0 + \nabla_\alpha\varphi(x_\alpha))dx_\alpha \right\} \tag{4.4}$$

for any cube $Q_b \subseteq \omega$,

(ii) $\lim_{n\to+\infty} \|u_{\varepsilon_n} - u_0\|_{L^\Phi(\Omega;\mathbb{R}^3)} = 0$,

(iii) $u_{\varepsilon_n|\partial\omega\times(-1,1)} = u_0$.

(iv) $\Phi\left(\left|\nabla_\alpha u_{\varepsilon_n}, \frac{1}{\varepsilon_n}\nabla_3 u_{\varepsilon_n}\right|\right)$ *is equi-integrable.*

It is worth to observe that such a result can be seen as a counterpart of the characterization of the Young measures generated by scaled gradients in the Orlicz-Sobolev setting. Indeed, formula (i) is entirely analogous to one proposed in [96] (the formula just before (1.16)).

The proof of Theorem 4.1.1 develops first by proving a Decomposition Lemma for standard gradients (see Theorem 4.3.2) which relies on properties of maximal functions, and exploits the Fundamental Theorem of Young measures (see Theorem 4.2.6). Then the proof of Theorem 4.1.1 follows as a consequence making use of the fine homogenization technique introduced in [26]. These are the subject of Section 4.3, while all the preliminary results, together with properties of Hardy maximal operator are contained in Section 4.2.

4.2 Notations and Preliminaries

We will use the following notations:

- ⋆ $|A|$ denotes the Lebesgue measure of a set $A \subseteq \mathbb{R}^N$ for $N \geq 2$, or an Euclidean norm of the vector or matrix A, it will be clear from the context;

- ⋆ the symbol dx will also be used to denote integration with respect to the Lebesgue measure $\mathcal{L}^N$, while the symbol dx_α will be used to denote integration with respect to the Lebesgue measure $\mathcal{L}^2$;
- ⋆ the symbol $\nabla_\alpha u$ denotes the derivatives with respect to $x_\alpha \stackrel{\text{def}}{=\!=} (x_1, x_2)$ of a given field u;
- ⋆ a matrix $\xi \in \mathbb{R}^{3\times 3}$, will be often written as (ξ_α, ξ_3) where ξ_α stands for the first two columns and ξ_3 represents the third;
- ⋆ a sequence (f_n) is said to be Φ-equi-integrable whenever the sequence $(\Phi(|f_n|))$ is equi-integrable.
- ⋆ C represents a generic positive constant that may change from line to line;

We say that $\Phi : [0, +\infty) \to [0, +\infty)$ is an Orlicz function whenever it is continuous, strictly increasing, convex, vanishes only at 0 and $\lim_{t\to 0^+} \Phi(t)/t = 0; \lim_{t\to+\infty} \Phi(t)/t = +\infty$. This statement is equivalent to demanding that $\Phi(t) = \int_0^t \phi(s)ds$ for some right-continuous, non-decreasing ϕ s.t. $\phi(t) = 0 \iff t = 0$ and $\lim_{t\to+\infty} \phi(t) = +\infty$.

We say that Φ satisfies Δ_2 (denoted by $\Phi \in \Delta_2$) condition whenever

$$\text{there exists } C > 0 \text{ and } t \geq t_0 \text{ such that } \Phi(2t) < C\Phi(t) \text{ for all } t \geq t_0. \tag{4.5}$$

Orlicz functions Φ possess the complementary Orlicz function $\Psi(s) \stackrel{\text{def}}{=\!=} \Phi^\star(s)$, where the latter denotes the standard Fenchel's conjugate of Φ, i.e.

$$\Psi(s) \stackrel{\text{def}}{=\!=} \sup_{t\geq 0}\{st - \Phi(t)\}, \quad s \geq 0,$$

and, it results that $\Psi(s) = \int_0^s \phi^{-1}(\tau)d\tau$, where ϕ^{-1} stands for right inverse function of ϕ.
Clearly $\Psi^\star = (\Phi^\star)^\star = \Phi$.

If $\Psi \in \Delta_2$ then (see [106, Theorem 4.2]) its conjugate Φ satisfies

$$\text{there exists } C > 0 \text{ and } t_0 \geq 0 \text{ such that } \Phi(t) \leq 1/(2C)\Phi(Ct) \text{ for any } t > t_0. \tag{4.6}$$

The condition (4.6) is often referred to as ∇_2 condition, i.e. $\Phi \in \nabla_2$.
Given two Orlicz functions Φ and Φ_1, we say that Φ dominates Φ_1 near infinity ($\Phi_1 \prec \Phi$ or $\Phi \succ \Phi_1$ in symbols) if there exists $C > 1$ and $t_0 > 0$ such that $\Phi_1(t) \leq \Phi(Ct)$ for all $t > t_0$. We say that Orlicz functions Φ, Φ_1 are equivalent whenever $\Phi \prec \Phi_1 \prec \Phi$.

Remark 4.2.1. In the literature, often a different formulation of the definition of dominance is considered. Namely, it is being stated that Φ dominates Φ_1, whenever there exist $C > 1$, $D > 0$ and $t_0 > 0$ such that $\Phi_1(t) \leq D\Phi(Ct)$ for all $t > t_0$. This notion is however equivalent to the notion of dominance introduced by us. To see that, first observe that one of the implications is trivial and it is enogh to take $D = 1$. For the other implication it is sufficient to prove that for every Orlicz function Φ and any constants C, D there exists a constant C' such that

$$L(t) \stackrel{\text{def}}{=} D\Phi(Ct) \le R(t) \stackrel{\text{def}}{=} \Phi(C't).$$

Indeed, let us observe that $L(0) = R(0) = 0$ and, as both functions are absolutely continuous,

$$\begin{aligned} L(t) &= \int_0^t DC\phi(Ct)ds \\ R(t) &= \int_0^t C'\phi(C't)ds, \end{aligned}$$

where by ϕ we mean the pointwise derivative, defined almost everywhere. Taking now

$$C' \ge C \max(1, D)$$

ensures that $C' \ge DC$, while the fact that ϕ is increasing shows that $\phi(C't) \ge \phi(Ct)$. Putting that together, we obtain that $R(t) \ge L(t)$.

For an arbitrary set of positive Lebesgue measure $E \subset \mathbb{R}^N$ we define the Orlicz class $L_\Phi(E)$ of functions u on E as functions satisfying inequality

$$\int_E \Phi(|u|)dx < +\infty$$

In general the class $L_\Phi(E)$ is not a linear space, and the Orlicz space $L^\Phi(E)$ is defined as the linear hull of $L_\Phi(E)$. It is easy to check that (see [106, Theorem 8.2]) Orlicz class $L_\Phi(E)$ coincides with its Orlicz space $L^\Phi(E)$ if and only if $\Phi \in \Delta_2$.

Orlicz spaces are equipped with Luxemburg norm, namely

$$||u||_{L^\Phi(E)} = \inf_{k>0} \int_E \Phi(|u|/k) \le 1 \tag{4.7}$$

and are complete (see [106, Theorems 9.2 and 9.5]).

The following properties hold.

Lemma 4.2.2. *Let Φ be an Orlicz function satisfying Δ_2 condition* (4.5) *and let E be a bounded open set in $\mathbb{R}^N$. Then*

(i) $C_c^\infty(E)$ is dense in $L^\Phi(E)$;

(ii) $L^\Phi(E)$ is separable and it is reflexive when Φ satisfies (4.6)*;*

(iii) *the dual of* $L^{\Phi}(E)$ *is identified with* $L^{\Psi}(E)$, $(\Psi = \Phi^{\star})$ *and the dual norm on* $L^{\Psi}(E)$ *is equivalent to* $\|\cdot\|_{L^{\Psi}}$;

(iv) *given* $u \in L^{\Phi}(E)$ *and* $v \in L^{\Psi}(E)$, *then* $u \cdot v \in L^{1}(E)$ *and the following generalized Hölder inequality holds*

$$\left|\int_E u \cdot v dx\right| \leq 4\|u\|_{L^{\phi}} \cdot \|v\|_{L^{\Psi}};$$

(v) *for every* $v \in L^{\Phi}(E)$ *the linear functional* L_v *on* $L^{\Psi}(E)$ *defined as*

$$L_v(u) \overset{def}{=\!=} \int_E u(x)v(x)dx$$

belongs to the dual of $L^{\Psi}(E)$ *with* $\|v\|_{L^{\Phi}} \leq \|L_v\|_{[L^{\Psi}(E)]'} \leq 2\|v\|_{L^{\Phi}}$

(vi) *given* Φ *and* $\tilde{\Phi}$, *the continuous embedding* $L^{\Phi}(E) \hookrightarrow L^{\tilde{\Phi}}(E)$ *holds iff* $\Phi \succ \tilde{\Phi}$ *near infinity;*

(vii) *in particular, in view of (vi), we have* $L^{\Phi}(E) \hookrightarrow L^{1}(E) \hookrightarrow L^{1}_{\text{loc}}(E) \hookrightarrow \mathcal{D}'(E)$;

(viii) *the product of* d *identical copies of* $L^{\Phi}(E)$, $(L^{\Phi}(E))^{d} \overset{def}{=\!=} L^{\Phi}(E) \times \cdots \times L^{\Phi}(E)$ *endowed with the norm* $\|v\|_{(L^{\Phi}(E))^{d}} \overset{def}{=\!=} \sum_{i=1}^{d} \|v_i\|_{L^{\Phi}(E)}$ *is an Orlicz space, i.e. the norm is equivalent to the* $L^{\Phi}(\sqcup_1^d E)$ *norm, where* $\sqcup$ *stays for sum of disjoint copies of the set.*

Proof. The point (i) is Theorem 1 in [70]. The separability stated in point (ii), is proven in [106, point 4 at page 85], while the reflexivity under the condition ∇_2 and point (iii) are stated in [106, Theorem 14.2]. The point (iv) follows from [106, Theorem 9.3] and formula (9.24) therein. The same formula, together with [106, Theorem 9.5] gives (v). The point (vi) coincides with [106, Theorem 8.1] and (vii) immediately follows from (vi). The proof of the point (viii) is standard. □

The proofs of chosen facts stated above are presented in Section 4.6.

Sobolev-Orlicz spaces $W^{1,\Phi}(E)$ are defined via

$$W^{1,\Phi}(E) \overset{\text{def}}{=\!=} \{u \in \mathcal{D}'(E) : u \in L^{\Phi}(E), \nabla u \in (L^{\Phi}(E))^{N}\}$$

and endowed with the norm

$$||u||_{W^{1,\Phi}(E)} \overset{\text{def}}{=\!=} ||u||_{L^{\Phi}(E)} + ||\nabla u||_{(L^{\Phi}(E))^{N}}.$$

Identifying the elements of $W^{1,\Phi}(E)$ with the couples $(u, \nabla u)$, we see them as a closed subspace of $L^{\Phi}(E)^{N+1}$, thus $W^{1,\Phi}(E)$ is a Banach space.

The Sobolev-Orlicz space $W^{1,\Phi}(E;\mathbb{R}^d)$, $d \in \mathbb{N}$ is defined as the Banach space of $\mathbb{R}^d$ valued functions $u \in L^\Phi(E;\mathbb{R}^d)$ with distributional derivative $\nabla u \in L^\Phi(E;\mathbb{R}^{N\times d})$, equipped with the norm

$$\|u\|_{W^{1,\Phi}(E;\mathbb{R}^d)} \overset{\text{def}}{=\!=} \|u\|_{L^\Phi(E;\mathbb{R}^d)} + \|\nabla u\|_{L^\Phi(E;\mathbb{R}^{N\times d})},$$

where the meaning of the norm $\|\cdot\|_{L^\Phi(E;\mathbb{R}^\cdot)}$ is easily understood from (viii) in Lemma 4.2.2. On the other hand, all the other properties in Lemma 4.2.2 extend with obvious meaning to the vectorial setting.

Remark 4.2.3. As shown in [70, Theorem 1], whenever $\Phi \in \Delta_2$, smooth functions are dense in the space $W^{1,\Phi}(E)$ and thus $W^{1,\Phi}(E)$ is separable. Without the assumption of $\Phi \in \Delta_2$ separability is not valid.

If E has Lipschitz boundary, then the embedding

$$W^{1,\Phi}(E;\mathbb{R}^d) \hookrightarrow L^\Phi(E;\mathbb{R}^d) \tag{4.8}$$

is compact (see [1] and [67, Theorems 2.2 and Proposition 2.1]).

For Sobolev-Orlicz space $W_0^{1,\Phi}(E)$, where E has a Lipschitz boundary, we have a Poicaré inequality (see [50, Theorem 3.4 (a)])

$$||u||_{L^\Phi(E)} \leq C||\nabla u||_{L^\Phi(E)} \text{ for some constant } C = C(E,\Phi) > 0$$

and if $\Phi \in \Delta_2$, there exists a linear continuous trace operator $\text{Tr} : W^{1,\Phi}(E) \to L^\Phi(\partial E)$ [91, Theorem 3.13].

Let $\mathcal{M}$ be a (centred) Hardy maximal operator, i.e. for any $f \in L^1_{loc}(E) \cap L^\Phi(E)$ let

$$\mathcal{M}f(x) \overset{\text{def}}{=\!=} \sup_r |B(x,r)|^{-1} \int_{B(x,r)\cap E} |f(y)|dy. \tag{4.9}$$

The following result, an easy corollary from [94, Theorem 1], will be exploited in the sequel.

Proposition 4.2.4 (Weak estimate for Hardy maximal operator)**.** *Let Φ be an Orlicz function satisfying (4.5) and (4.6). For any $f \in L^\Phi(E)$ there exists a constant $C = C\big(E,\Phi\big)$ such that*

$$|\{\mathcal{M}f > t\}| \leq \frac{C}{\Phi(t)} \int_E \Phi(|f|)dx, \tag{4.10}$$

for every $t > 0$.

The proof of the Proposition is presented with Theorem 4.6.5

We quote the Fundamental Theorem on Young measures, which will be invoked in the proof of our main results. We refer to the classical presentation in [16]. Our formulation is however kept in the spirit of the one given in [118]. For details regarding Young measures generated by gradients we refer to [95, 97].

We start by recalling a classical definition, which can be found for example in [141, 14(8) and Definition 14.27].

Definition 4.2.5. Let $f : E\times\mathbb{R}^N \to \mathbb{R}\cup\{+\infty\}$ be a mapping. By the epigraphical mapping of f we will mean the multifunction $S_f : E \rightrightarrows \mathbb{R}^N \times \mathbb{R}$ defined by

$$S_f(x) \stackrel{\text{def}}{=\!=} \{(\xi,\alpha) \in \mathbb{R}^N \times \mathbb{R} : f(x,\xi) \le \alpha\}.$$

We will say, that an integrand f is normal, whenever the mapping S_f is closed-valued and measureable.

Theorem 4.2.6. *Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure and let (z_n) be a sequence of measurable functions, $z_n : E \to \mathbb{R}^m$. Then there exists a subsequence (z_{n_k}) and a weak-$\star$ measurable map $\nu : E \to \mathcal{M}(\mathbb{R}^m)$ such that the following hold:*

(i) $\nu_x \ge 0, \|\nu_x\|_{\mathcal{M}(R^d)} = \int_{\mathbb{R}^d} d\nu_x \le 1$ *for a.e.* $x \in E$*;*

(ii) one has (i') $\|\nu_x\|_{\mathcal{M}} = 1$ for a.e. $x \in E$ if and only if the so-called tightness condition is satisfied, i.e.

$$\lim_{R\to+\infty} \sup_k |\{|z_{n_k}| \ge R\}| = 0$$

(iii) if $K \subset \mathbb{R}^m$ is a compact subset and $\operatorname{dist}(z_{n_k}, K) \to 0$ *in measure, then* $\operatorname{supp}\nu_x \subset K$ *for a.e. $x \in E$;*

(iv) if (i') holds, then in (iii) one may replace 'if' with 'if and only if';

(v) if $f : E \times \mathbb{R}^d \to \mathbb{R}$ is a normal integrand, bounded from below, then

$$\liminf_{n\to+\infty} \int_E f(x, z_{n_k}(x))dx \ge \int_E \int_{\mathbb{R}^d} f(x,y)d\nu_x(y)dx$$

(vi) if (i') holds and if $f : E \times \mathbb{R}^m \to \mathbb{R}$ is Carathéodory and bounded from below, then

$$\lim_{n\to+\infty} \int_E f(x, z_{n_k}(x))dx = \int_E \int_{\mathbb{R}^d} f(x,y)d\nu_x(y)dx$$

if and only if $(f(x, z_{n_k}(x)))$ is equi-integrable. In this case

$$f(x, z_{n_k}(x)) \rightharpoonup \int_{\mathbb{R}^d} f(x,y)d\nu_x(y) \text{ in } L^1(E).$$

The map $\nu : E \to \mathcal{M}(\mathbb{R}^m)$ is called the Young measure generated by (z_{n_k}).

4.3 Proofs of the Theorems 4.1.1 and 4.1.2

This section is devoted to the proof of our main result.

We start by proving a Lemma which generalizes the result of Fonseca and Leoni [118, Lemma 8.13] to the Orlicz setting. The Lemma is needed to achieve a good control of the behaviour of truncations of the functions from the Orlicz space.

Lemma 4.3.1. *Let Φ be an Orlicz function satisfying (4.5) and (4.6). Let $E \subset \mathbb{R}^N$ be a Lebesgue measurable set of finite measure and let (u_n) be a uniformly bounded sequence in $L^\Phi(E;\mathbb{R}^m)$. For any r define the standard truncature operators $\tau_r : \mathbb{R} \to \mathbb{R}$ as*

$$\tau_r(t) \stackrel{def}{=\!=} \begin{cases} t, & \text{whenever } |t| \leq r, \\ r\frac{t}{|t|} & \text{otherwise.} \end{cases} \tag{4.11}$$

Then there exist a (non-relabeled) subsequence (u_n) and an increasing sequence of numbers $r_n \to +\infty$ such that $\tau_{r_n} \circ u_n$ are $\Phi-$equi-integrable and the set $|\{x \in E : \tau_{r_n} \circ u_n \neq u_n\}| \to 0$.

Proof. By (i) in Theorem 4.2.6, we may assume that (u_n) generates the Young measure ν_x. The uniform boundedness of the sequence (u_n), together with (iii) therein, guarantees that

$$\int_E \int_{\mathbb{R}^m} \Phi(|z|) d\nu_x(z) dx < +\infty.$$

So we have

$$\lim_{r\to+\infty} \lim_{n\to\infty} \int_E \Phi(|\tau_r \circ u_n|)dx = \lim_{r\to+\infty} \int_E \int_{\mathbb{R}^m} \Phi\big(|\tau_r(z)|\big) d\nu_x(z) dx = \int_E \int_{\mathbb{R}^m} \Phi(|z|) d\nu_x(z) dx.$$

where the first equality relies on (vi) of Theorem 4.2.6, and the second one on Lebesgue Monotone Convergence theorem. Take r_n such that

$$\lim_{n\to+\infty} \int_E \Phi(|\tau_{r_n} \circ u_n|)dx = \int_E \int_{\mathbb{R}} \Phi(|z|) d\nu_x(z) dx.$$

As $r_n \to +\infty$ and (u_n) is bounded, one has

$$|\{x \in E : \tau_{r_n} \circ u_n \neq u_n\}| \to 0.$$

Thus, we can conclude that $(\tau_{r_n} \circ u_n)$ generates the same Young measure as (u_n) (see [118, Corollary 8.7]).

Finally (vi) in Theorem 4.2.6 ensures $\Phi-$equi-integrability of $(\tau_{r_n} \circ u_n)$. □

Now we prove a Decomposition Lemma for gradients. In the sequel we will extend this result to the scaled ones.

Theorem 4.3.2. *Let $E \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let Φ be an Orlicz function satisfying Δ_2 and ∇_2 conditions (see (4.5) and (4.6) respectively), and let $(u_n) \subset W^{1,\Phi}(E;\mathbb{R}^d)$ be a sequence of functions converging to u_0 weakly in $W^{1,\Phi}(E;\mathbb{R}^d)$. Then there exists a subsequence (u_{n_k}) and a sequence $(v_k) \subset W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that (v_k) converges to u_0 weakly in $W^{1,\Phi}(E;\mathbb{R}^d)$,*

$$|\{x \in E : v_k(x) \neq u_{n_k}(x) \text{ or } \nabla u_{n_k}(x) \neq \nabla v_k(x)\}| \to 0 \text{ as } k \to +\infty$$

and $(\Phi(|\nabla v_k|))$ is equi-integrable.

Proof. Since

$$\sup_n \|u_n\|_{W^{1,\Phi}(E;\mathbb{R}^d)} \leq C$$

and $\Phi \in \Delta_2$, we have

$$\sup_n \left\{ \int_E (\Phi(|u_n|) + \Phi(|\nabla u_n|))dx \right\} \leq C.$$

It follows from the continuity of the maximal operator (see [66, Theorem 2.1] for the original source or Theorem 4.6.8 in the Section 4.6), that

$$\sup_n \left\{ \int_{\mathbb{R}^N} \Phi(\mathcal{M}(|u_n| + |\nabla u_n|)\chi_E)dx \right\} \leq C,$$

where $\mathcal{M}((|u_n| + |\nabla u_n|)\chi_E)$ is the maximal function of $(|u_n| + |\nabla u_n|)\chi_E$. By Lemma 4.3.1, there exists an increasing sequence $t_n \to +\infty$ such that $(\Phi(|\tau_{t_n} \circ (\mathcal{M}((|u_n| + |\nabla u_n|)\chi_E))|))$ is equi-integrable, where τ_{t_n} is defined in (4.11).

Define

$$A_n \overset{\text{def}}{=\!=} \{x \in E : |\mathcal{M}\big((|u_n| + |\nabla u_n|)\chi_E\big)| > t_n\}. \tag{4.12}$$

By [118, Theorem 4.32], there exists $(v_n) \subset W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$ such that

$$\|v_n\|_{W^{1,\infty}} \leq Ct_n, \tag{4.13}$$

where C depends on E and N, and such that $v_n = u_n$ almost everywhere in the sense of $\mathcal{L}^N$ on $E \setminus A_n$ and by (4.10)

$$|A_n| \leq \frac{C}{\Phi(t_n)} \int_{\mathbb{R}^N} \Phi(|u_n| + |\nabla u_n|)dx.$$

In order to show that $(\Phi(|\nabla v_n|))$ is equi-integrable we observe that for $\mathcal{L}^N$ a.e. x in $E \setminus A_n$

$$|\nabla v_n| = |\nabla u_n| \leq \mathcal{M}((|u_n| + |\nabla u_n|)\chi_E) = |\tau_{t_n} \circ \mathcal{M}((|u_n| + |\nabla u_n|)\chi_E)|$$

while if $x \in A_n$ then

$$|\nabla v_n| \leq Ct_n \leq C|\tau_{t_n} \circ \mathcal{M}((|u_n| + |\nabla u_n|)\chi_E)|.$$

It remains to prove the weak convergence of (v_n) to u_0 in $W^{1,\Phi}(E;\mathbb{R}^d)$. To this end, first we observe that (4.13) and (4.10) ensure

$$\begin{aligned}\int_E \Phi(|v_n|+|\nabla v_n|)dx &= \int_{E\setminus A_n} \Phi(|u_n|+|\nabla u_n|)dx + \int_{A_n} \Phi(|v_n|+|\nabla v_n|)dx \\ &\le \int_{E\setminus A_n} \Phi(|u_n|+|\nabla u_n|)dx + \Phi(Ct_n)|A_n| \\ &\le C\int_E \Phi(|u_n|+|\nabla u_n|)dx.\end{aligned}$$

Next the reflexivity of $W^{1,\Phi}(E;\mathbb{R}^d)$ under (4.5),(4.6) (see Lemma 4.2.2) and the Banach-Alaoglu-Bourbaki theorem ensure that, up to the choice of a non-relabelled subsequence, $v_n \rightharpoonup v_0$ in $W^{1,\Phi}(E;\mathbb{R}^d)$. Thus, since $|\{x\in E : v_n \ne u_n \text{ or } \nabla u_n \ne v_n\}| \to 0$ as $n\to+\infty$ we can conclude, via the compact imbedding (see (4.8)) that $v_0 = u_0$ $\mathcal{L}^N$- a.e. in E. □

Proof of Theorem 4.1.1. The proof of the claims (i) and (iii) follows line by line as in [26, Theorem 3.1]. Namely, we define $\bar{u}_n \overset{\text{def}}{=\!=} u_n(x_1,x_2,\frac{2x_3-\varepsilon_n}{\varepsilon_n})$ (so it is a shifted and scaled version of u_n, and it is defined on $\omega\times(0,\varepsilon_n)$) and observe that

$$\sup_n \int_{\omega\times(0,\varepsilon_n)} \Phi(|\nabla \bar{u}_n|)dx = C, \text{ where } C \text{ is exactly like in (4.2).}$$

We now extend $\bar{u}_n$ by reflection to $\omega\times(-\varepsilon_n,\varepsilon_n)$ and then produce its periodic extension, $\hat{u}$, to $\omega\times(-1,1)$.

For such constructed sequence $\hat{u}_n$ one can obtain the uniform bound of its norm in $W^{1,\Phi}\big(\omega\times(-1,1)\big)$ as in [26, formula (3.6)]. Thus, we apply Theorem 4.3.2 and obtain a sequence $(\hat{v}_n)$ with $(\nabla\hat{v}_n)$ Φ-equi-integrable. The use of de la Vallée Poussin Criterion (see [118, Theorem 2.29]) and the ingenious computation due to Braides and Zeppieri (see [26, formula (3.7)]) gives us the sequence $(\bar{v}_n)$ satisfying claim (i) and (iii).

The presentation of the aforementioned computation held in [26] is less clear then in our case. That is because we deal with the fixed dimension $x=(x_1,x_2,x_3)$ and only the third dimension is scaled. Braides and Zeppieri worked with $x=(x_\alpha,x_\beta)$ and $x_\alpha\in\mathbb{R}^{n-k}, x_\beta\in\mathbb{R}^k$. Taking the number of scaled dimensions $k=1$, as in our case, doesn't change the proof, but significantly simplifies the notation and makes the presentation simpler to read. For this reason we present it with details. A reader familiar with Braides' and Zeppieri's paper is invited to skip this part of the proof and move instantly to the proof of the claim (iv) below.

We are at the point, when the use of Theorem 4.3.2 gave us the sequence $(\hat{v}_n)$ such that $\Phi(|\nabla\hat{v}_n|)$ (or equivalently $\Phi(|\nabla_\alpha\hat{v}_n|)+\Phi(|\nabla_3\hat{v}_n|)$) is equi-integrable on $\omega\times(-1,1)$ and

$$\mathscr{L}^3\big((\{\hat{v}_n\ne\hat{u}_n \text{ or } \nabla\hat{v}_n\ne\nabla\hat{u}_n\})\cap(\omega\times(-1,1))\big)\to 0.$$

De la Vallée Poussin Criterion guarantees, that there exists a positive Borel function $f:\mathbb{R}\to\mathbb{R}$ such that

$$\lim_{t\to+\infty}\frac{f(t)}{t}=+\infty \text{ and } \sup_n\int_{\omega\times(-1,1)} f\big(\Phi(|\nabla_\alpha\hat{v}_n|)+\Phi(|\nabla_3\hat{v}_n|)\big)<+\infty.$$

Let $\lfloor r \rfloor$ be the integer part of the real number r, i.e. the biggest integer not exceeding r. Obviously, $\lfloor 1/\varepsilon_n \rfloor \varepsilon_n \leq 1$ and hence positivity of f and monotonicity of $\mathscr{L}^3$ yield

$$\int_{\omega \times (-\lfloor 1/\varepsilon_n \rfloor \varepsilon_n, \lfloor 1/\varepsilon_n \rfloor \varepsilon_n)} f\big(\Phi(|\nabla_\alpha \hat{v}_n|) + \Phi(|\nabla_3 \hat{v}_n|)\big) \leq \int_{\omega \times (-1,1)} f\big(\Phi(|\nabla_\alpha \hat{v}_n|) + \Phi(|\nabla_3 \hat{v}_n|)\big).$$

and

$$\mathscr{L}^3\big((\{\hat{v}_n \neq \hat{u}_n \text{ or } \nabla \hat{v}_n \neq \nabla \hat{u}_n\}) \cap (\omega \times (-\lfloor 1/\varepsilon_n \rfloor \varepsilon_n, \lfloor 1/\varepsilon_n \rfloor \varepsilon_n))\big) \leq \mathscr{L}^3\big((\{\hat{v}_n \neq \hat{u}_n \text{ or } \nabla \hat{v}_n \neq \nabla \hat{u}_n\}) \cap (\omega \times (-1,1))\big).$$

Let us set the notation

$$M_n \overset{\text{def}}{=\!=} \int_{\omega \times (-1,1)} f\big(\Phi(|\nabla_\alpha \hat{v}_n|) + \Phi(|\nabla_3 \hat{v}_n|)\big);$$
$$m_n \overset{\text{def}}{=\!=} \mathscr{L}^3\big((\{\hat{v}_n \neq \hat{u}_n \text{ or } \nabla \hat{v}_n \neq \nabla \hat{u}_n\}) \cap (\omega \times (-1,1))\big)$$

and remind that

$$\sup_n M_n < +\infty; \ m_n \to 0. \tag{4.14}$$

Having in mind, that $(-\lfloor 1/\varepsilon_n \rfloor \varepsilon_n, \lfloor 1/\varepsilon_n \rfloor \varepsilon_n) = \bigcup_{i=-\lfloor 1/\varepsilon_n \rfloor}^{\lfloor 1/\varepsilon_n \rfloor - 1} (i\varepsilon_n + (0, \varepsilon_n))$ we see that

$$\sum_{i=-\lfloor 1/\varepsilon_n \rfloor}^{\lfloor 1/\varepsilon_n \rfloor - 1} \int_{\omega \times (i\varepsilon_n + (0,\varepsilon_n))} f\big(\Phi(|\nabla_\alpha \hat{v}_n|) + \Phi(|\nabla_3 \hat{v}_n|)\big) < M_n \tag{4.15}$$

$$\sum_{i=-\lfloor 1/\varepsilon_n \rfloor}^{\lfloor 1/\varepsilon_n \rfloor - 1} \mathscr{L}^3\big((\{\hat{v}_n \neq \hat{u}_n \text{ or } \nabla \hat{v}_n \neq \nabla \hat{u}_n\}) \cap (i\varepsilon_n + (0, \varepsilon_n))\big) < m_n. \tag{4.16}$$

Now, for fixed n,, let us consider only intervals $(i\varepsilon_n + (0, \varepsilon_n))$ with i being an even integer, i.e. $i = 2h$ for some $h \in \mathscr{I}_n \overset{\text{def}}{=\!=} \mathbb{Z} \cap [-1/2\lfloor 1/\varepsilon_n \rfloor, 1/2(\lfloor 1/\varepsilon_n \rfloor - 1)]$. What is worth mentioning now is that on such intervals the extended function $\hat{u}_n$ coincides with its 'mother function' $\bar{u}_n$ shifted by the vector $(0, 0, 2h\varepsilon_n)$.

Of course, positivity of f, together with monotonicity of $\mathscr{L}^3$, let us immediately conclude from (4.15) and (4.16) that

$$\sum_{h \in \mathscr{I}_n} \int_{\omega \times (2h\varepsilon_n + (0,\varepsilon_n))} f\big(\Phi(|\nabla_\alpha \hat{v}_n|) + \Phi(|\nabla_3 \hat{v}_n|)\big) \leq M_n \tag{4.17}$$

$$\sum_{h \in \mathscr{I}_n} \mathscr{L}^3\big((\{\hat{v}_n \neq \hat{u}_n \text{ or } \nabla \hat{v}_n \neq \nabla \hat{u}_n\}) \cap (2h\varepsilon_n + (0, \varepsilon_n))\big) \leq m_n. \tag{4.18}$$

We claim that for at least $\lfloor 1/2\#\mathscr{I}_n\rfloor$ indices $h \in \mathscr{I}_n$, we have

$$\int_{\omega\times(2h\varepsilon_n+(0,\varepsilon_n))} f\big(\Phi(|\nabla_\alpha \hat{v}_n|)+\Phi(|\nabla_3 \hat{v}_n|)\big) \le (\#\mathscr{I}_n - \lfloor 1/2(\#\mathscr{I}_n)\rfloor + 1)^{-1} M_n. \tag{4.19}$$

To check the claim, let us set $\mathscr{I}'_n \stackrel{\text{def}}{=\!=} \{h \in \mathscr{I}_n : (4.19) \text{ does not hold }\}$ and assume that $\#\mathscr{I}'_n \ge (\#\mathscr{I}_n - \lfloor 1/2(\#\mathscr{I}_n)\rfloor + 1)$. We easily compute, that

$$\begin{aligned}\sum_{h\in\mathscr{I}_n} \int_{\omega\times(2h\varepsilon_n+(0,\varepsilon_n))} f\big(\Phi(|\nabla_\alpha \hat{v}_n|)+\Phi(|\nabla_3 \hat{v}_n|)\big) &\ge \\ \sum_{h\in\mathscr{I}'_n} \int_{\omega\times(2h\varepsilon_n+(0,\varepsilon_n))} f\big(\Phi(|\nabla_\alpha \hat{v}_n|)+\Phi(|\nabla_3 \hat{v}_n|)\big) &\ge \\ \#\mathscr{I}'_n(\#\mathscr{I}_n - \lfloor 1/2(\#\mathscr{I}_n)\rfloor + 1)M_n&,\end{aligned}$$

which contradicts (4.17). An easy computation shows, that $\#\mathscr{I}_n = 2\lfloor 1/2(1/\varepsilon_n - 1)\rfloor + 2$, hence for sufficiently large n one has

$$\#\mathscr{I}_n - \lfloor 1/2\#\mathscr{I}_n\rfloor + 1 > 1/(4\varepsilon_n).$$

That, together with (4.19) gives us that for at least $\lfloor 1/2\#\mathscr{I}_n\rfloor$ indices $h \in \mathscr{I}_n$, we have

$$\int_{\omega\times(2h\varepsilon_n+(0,\varepsilon_n))} f\big(\Phi(|\nabla_\alpha \hat{v}_n|)+\Phi(|\nabla_3 \hat{v}_n|)\big) \le 4\varepsilon_n M_n \tag{4.20}$$

for n large enough. An averaging procedure shows that, among the indices satisfying (4.20) we can find one that satisfies

$$\begin{aligned}\mathscr{L}^3\big((\{\hat{v}_n \ne \hat{u}_n \text{ or } \nabla\hat{v}_n \ne \nabla\hat{u}_n\}) \cap (2h\varepsilon_n + (0,\varepsilon_n))\big) &\le \\ \lfloor 1/2\#\mathscr{I}_n\rfloor^{-1} m_n < 4\varepsilon_n m_n&. \end{aligned}\tag{4.21}$$

Let us take an index satisfying (4.20) and (4.21) simultaneously and name it $h^\star$. To simplify the notation, with no loss of generality, we assume that $h^\star = 0$. We focus our attention on the new function z_n – the restriction of the function $\hat{v}_n$ to the chosen piece of Ω, that is $\omega \times 2h^\star\varepsilon_n + (0,\varepsilon_n)$. We will construct the function $\bar{v}_n$ from z_n by unscaling. Let us set

$$\bar{v}_n(x) \stackrel{\text{def}}{=\!=} z_n(x_1, x_2, (\varepsilon_n(x_3+1)/2).$$

The function $\bar{v}_n \in W^{1,\Phi}(\omega \times (-1,1);\mathbb{R}^m)$ By virtue of (4.20) we have

$$\int_{\omega\times(-1,1)} f\big(\Phi(|\nabla_\alpha \bar{v}_n|)+\Phi(1/\varepsilon_n|\nabla_3 \bar{v}_n|)\big) \le 4M_n$$

and de la Vallée Poussin Criterion yields equi-integrability of $\Phi(|\nabla_\alpha \bar{v}_n|) + \Phi(1/\varepsilon_n |\nabla_3 \bar{v}_n|)$. This shows that the sequence $\bar{v}_n$ satisfies claim (i) of the Theorem 4.1.1. By (4.21) we deduce

$$\mathscr{L}^3\big((\{\bar{v}_n \neq \hat{u}_n \text{ or } \nabla\bar{v}_n \neq \nabla\hat{u}_n\}) \cap (2h\varepsilon_n + (0,\varepsilon_n))\big) \leq 4m_j.$$

Having in mind (4.14), we get claim (iii). Up to an extraction of a subsequence one may immediately deduce claim (ii).

To get (iv) we argue as in [23, Corollary 1.2]. We define sets

$$\omega_j \stackrel{\text{def}}{=} \{x \in \omega : \operatorname{dist}(x, \partial\omega) < 1/j\} \tag{4.22}$$

and cut-off functions $\theta_j \in C_0^\infty(\omega, [0,1])$, equal to 1 on $\omega \setminus \omega_j$, vanishing in a neighborhood of $\partial\omega$, and such that $|\nabla\theta_j| < Cj$ for some constant C. We set then $v_{n,j} \stackrel{\text{def}}{=} u_0 + \theta_j \bar{v}_n$. Via compact imbedding (see (4.8)) and the diagonal argument we may find such a function $n(j)$ that

$$||v_{n(j),j} - u_0||_{L^\Phi(\Omega;\mathbb{R}^3)} \to 0 \text{ and } ||v_{n(j),j}||_{L^\Phi(\Omega;\mathbb{R}^3)} < \frac{1}{j^2}.$$

To obtain (iv), it suffices to define $v_j \stackrel{\text{def}}{=} v_{n(j),j}$. It remains to deduce (i)-(iii) for this latter sequence. To prove (iii) we just observe that

$$\begin{aligned} &|\{x \in \Omega : u_j \neq v_j \text{ or } \nabla u_j \neq \nabla v_j\}| \\ &\leq |\{x \in \Omega : u_j \neq \bar{v}_j \text{ or } \nabla u_j \neq \nabla\bar{v}_j\}| + |\{x \in \Omega : \bar{v}_j \neq v_j \text{ or } \nabla u_j \neq \nabla\bar{v}_j\}|, \end{aligned}$$

and the claim follows from the control of the latter two sets. For (i), it suffices to exploit the definition of u_j and the Φ-equi-integrability of $\overline{v}_j$, (see also [23, formula (4.8)]). Up to the extraction of the subsequence we deduce (ii). □

Proof of Theorem 4.1.2. It can be deduced from [23, Corollary 1.2]. We sketch the main points for the reader's convenience. First let us observe that from density of smooth functions and properties of quasiconvex envelope and definition of $\overline{W}$ it can be easily proven that

$$\inf_{\varepsilon, u|_{\partial\omega\times(-1,1)} \equiv u_0} \frac{1}{|\Omega|}\int_\Omega W(\nabla_\alpha u, \frac{1}{\varepsilon}\nabla_3 u)dx = Q\overline{W}(\nabla u_0). \tag{4.23}$$

Now let us assume that ω is a square $(-c/2, c/2)^2$. Let (w_n, L_n) be the infimizing sequence of the left-hand side in (4.23). We may thus assume that, up to a reflection and then a periodic extension, functions $(w_n - u_0)$ are already defined on $\mathbb{R}^2 \times (-1,1)$. We define $w_{n,j}(x) \stackrel{\text{def}}{=} \varepsilon_j L_n(w_n - u_0)\big((\varepsilon_j L_n)^{-1}x_\alpha, x_3\big)$ and observe that

$$\lim_{n\to\infty}\lim_{j\to\infty} w_{n,j} = 0 \text{ weakly in } W^{1,\Phi}(\Omega;\mathbb{R}^3)$$

and

$$\lim_{n\to\infty}\lim_{j\to\infty}\frac{1}{|\Omega|}\int_\Omega W\big(\nabla_\alpha u_0 + \nabla_\alpha w_{n,j}, \frac{1}{\varepsilon_j}\nabla_3 w_{n,j}\big) = Q\overline{W}(\nabla u_0).$$

By a diagonal procedure and (4.8) we may choose $j(n)$ such that (denoting $w_{n,j(n)}$ as $\tilde{w}_n$ and $\varepsilon_{j(n)}$ as $\tilde{\varepsilon}_n$), $\lim \tilde{w}_n = 0$ strongly in $L^\Phi(\Omega)$, and

$$\lim_{n\to\infty} \frac{1}{|\Omega|}\int_\Omega W(\nabla_\alpha u_0 + \nabla_\alpha \tilde{w}_n, \frac{1}{\tilde{\varepsilon}}\nabla_3 \tilde{w}_n)dx = Q\overline{W}(\nabla u_0).$$

The latter equality, together with (4.2) gives us bound on the norm of $\tilde{w}_n$ in $W^{1,\Phi}(\Omega;\mathbb{R}^3)$. Up to an extraction of the subsequence (not relabelled) we may still assume that $\tilde{w}_n \rightharpoonup 0$ in $W^{1,\Phi}(\Omega;\mathbb{R}^3)$.

Applying Theorem 4.1.1 we obtain a sequence (v_n) satisfying (ii), (iii) and (iv). The point (i) follows from triangle inequality, Φ-equi-integrability of (v_n), point (iii) of the Theorem 4.1.1 and the fact that $|\omega_j| \to 0$ (see (4.22)).

To generalize the result to ω with Lipschitz boundary, the standard, but technically involving glueing procedure is used. For the detailed presentation we refer to the second step of the proof of [23, Corollary 1.2]. □

4.4 The Γ-convergence of energies in thin structures setting

Optimal design problems, devoted to find the minimal energy configurations of a mixture of two conductive (or elastic) materials, have requested much attention in the past years starting with the pioneering papers [101]. It is well known that, given a container Ω and prescribing only the volume fraction of the material where it is expected to have a certain conductivity, an optimal configuration might not exist. To overcome this difficulty, Ambrosio and Buttazzo in [8] imposed a perimeter penalization on the interface of the two materials. In [34, formula (2)] the same perimeter term has been added in order to deal with the model proposed in [59] and [24] in the framework of thin structures in the non-linear elasticity setting.

Here we are considering an analogous problem, where the continuous energy densities $W_i : \mathbb{R}^{3\times 3} \to \mathbb{R}$, $i = 1, 2$, do not satisfy growth conditions of order p but are of the type

$$\beta(\Phi(|F|) - 1) \leq W_i(F) \leq \beta'(1 + \Phi(|F|)) \tag{4.24}$$

for every $F \in \mathbb{R}^{3\times 3}$ and Φ – an Orlicz function (see section 4.2) with $0 < \beta \leq \beta'$. We refer to [114, 115] for related results in the framework of dimensional reduction problems casted in the Orlicz-Sobolev setting.

Let $\varepsilon > 0$ and consider $\Omega(\varepsilon) := \omega \times (-\varepsilon, \varepsilon)$, where ω is a bounded domain of $\mathbb{R}^2$, with Lipschitz boundary. Assume that $\Omega(\varepsilon)$ is clamped on its lateral boundary, and suppose that $\Omega(\varepsilon)$ is filled with two materials of respective energy densities W_i, $i = 1, 2$ as above, satisfying (4.24).

We study the following problem of minimization with respect to the couple $(v, E(\varepsilon))$

$$\inf_{\substack{v \in W^{1,\Phi}(\Omega(\varepsilon);\mathbb{R}^3) \\ \chi_{E(\varepsilon)} \in BV(\Omega(\varepsilon);\{0,1\})}} \left\{ \frac{1}{\varepsilon}\int_{\Omega(\varepsilon)} [(\chi_{E(\varepsilon)}W_1 + (1-\chi_{E(\varepsilon)})W_2)(\nabla v) - \overline{f}\cdot v]dx \right. \\ \left. + \frac{1}{\varepsilon}P(E(\varepsilon);\Omega(\varepsilon)) : v\lfloor_{\partial\omega\times(-\varepsilon,\varepsilon)} = 0, \frac{1}{|\Omega(\varepsilon)|}\int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)}\, dx = \lambda \right\}, \tag{4.25}$$

where $E(\varepsilon) \subset \Omega(\varepsilon)$ is a measurable set with finite perimeter (see Section 4.6 for more details) and $\bar{f} \in L^{\Phi^*}(\Omega(\varepsilon);\mathbb{R}^3)$, where Φ^* is the Legandre conjugate of Φ (see section 4.2) and $\lambda \in (0,1)$ is the volume fraction.

In order to study the asymptotic behaviour of (4.25) we first rescale the problem in a fixed 3D domain and then we perform Γ-convergence (see Definition 4.6.21) with respect to the pair (deformation, design region) as in [34]. We introduce a curious reader to see Section4.6 for more details. The definition of this notion reads as follows.

Definition 4.4.1 (Γ-convergence)**.** Let $\mathbb{X}$ be a topological space and $F_n : \mathbb{X} \to \overline{\mathbb{R}}$ be a sequence of functionals. We will say that $F = \Gamma - \lim\limits_{n\to+\infty} F_n$, whenever for every $x \in \mathbb{X}$ the following two conditions hold.

(LB) for any $x_n \to x$ we have $\liminf F_n(x_n) \geq F(x)$;

(UB) there exists such $x_n \to x$ that $\limsup F_n(x_n) \leq F(x)$.

In the literature, the conditions (LB) and (UB) are often referred as 'the lower bound' and 'the upper bound' respectively. The natural generalisation of the Definition to the families of functionals is given in Definition 4.6.22.

We refer to [28–31,45] for Γ-convergence theory. A very brief view of the essential theorems is given in Section 4.6. We consider a $\frac{1}{\varepsilon}-$dilation in the transverse direction x_3. Set $\Omega := \omega \times (-1,1)$,

$$E_\varepsilon \stackrel{\text{def}}{=\!=} \{(x_\alpha, x_3) \in \Omega : (x_\alpha, \varepsilon x_3) \in E(\varepsilon)\}, \; u(x_\alpha, x_3) \stackrel{\text{def}}{=\!=} v(x_\alpha, \varepsilon x_3), \\ f(x_\alpha, x_3) \stackrel{\text{def}}{=\!=} \overline{f}(x_\alpha, \varepsilon x_3), \quad \chi_{E_\varepsilon}(x_\alpha, x_3) \stackrel{\text{def}}{=\!=} \chi_{E(\varepsilon)}(x_\alpha, \varepsilon x_3), \tag{4.26}$$

where v is any admissible field for (4.25).

In the sequel we will denote $dx_\alpha \stackrel{\text{def}}{=\!=} dx_1 dx_2$ and ∇_α and D_α will be identified with the pair (∇_1, ∇_2) (D_1, D_2), respectively.
By the definition of total variation, $P(E(\varepsilon);\Omega(\varepsilon)) = |D\chi_{E(\varepsilon)}|(\Omega(\varepsilon))$, and the change of variables in (4.26) ensures that $\frac{1}{\varepsilon}|D\chi_{E(\varepsilon)}|(\Omega(\varepsilon)) = |(D_\alpha\chi_\varepsilon, \frac{1}{\varepsilon}D_3\chi_\varepsilon)|(\Omega)$, where χ_{E_ε} denotes the characteristic function of E_ε, that in the sequel we will indicate simply by χ_ε. We refer to [9] for sets of finite perimeter and BV functions.

For every $\varepsilon > 0$, let $J_\varepsilon : L^1(\Omega;\{0,1\}) \times L^\Phi(\Omega;\mathbb{R}^3) \to [0,+\infty]$ be the functional defined as follows

$$J_\varepsilon(\chi,u) \stackrel{\text{def}}{=\!=} \begin{cases} \int_\Omega (\chi W_1 + (1-\chi)W_2)\left(\nabla_\alpha u \,\middle|\, \frac{1}{\varepsilon}\nabla_3 u\right) dx - \int_\Omega f\cdot u dx \\ +\left|\left(D_\alpha \chi \,\middle|\, \frac{1}{\varepsilon} D_3\chi\right)\right|(\Omega) \quad \text{in } BV(\Omega;\{0,1\}) \times W^{1,\Phi}(\Omega;\mathbb{R}^3), \\ \\ +\infty \qquad\qquad \text{otherwise.} \end{cases} \tag{4.27}$$

Analogously, consider the functional $J_0 : L^1(\Omega;\{0,1\}) \times L^\Phi(\Omega;\mathbb{R}^3) \to [0,+\infty]$ as

$$J_0(\chi,u) \stackrel{\text{def}}{=\!=} \begin{cases} 2\int_\omega Q\overline{V}(\chi,\nabla_\alpha u)dx_\alpha - \int_{-1}^1\int_\omega f\cdot u dx_\alpha dx_3 \\ +2|D\chi|(\omega), \quad \text{in } BV(\omega;\{0,1\}) \times W^{1,\Phi}(\omega;\mathbb{R}^3), \\ \\ +\infty \qquad \text{otherwise,} \end{cases} \tag{4.28}$$

where $V : \{0,1\} \times \mathbb{R}^{3\times 3} \to [0,+\infty)$ is given by

$$V(d,F) \stackrel{\text{def}}{=\!=} dW_1(F) + (1-d)W_2(F), \tag{4.29}$$

with W_1 and W_2 satisfying (4.24), $\overline{V} : \{0,1\} \times \mathbb{R}^{3\times 2} \to [0,+\infty)$ is given by

$$\overline{V}\left(d,\overline{F}\right) \stackrel{\text{def}}{=\!=} d\overline{W}_1\left(\overline{F}\right) + (1-d)\,\overline{W}_2\left(\overline{F}\right), \tag{4.30}$$

with $\overline{W}_i(\overline{F}) \stackrel{\text{def}}{=\!=} \inf_{c\in\mathbb{R}^3} W_i\left(\overline{F}|c\right), \overline{F} \in \mathbb{R}^{3\times 2},\ i=1,2$, and $Q\overline{V}$ stands for the quasiconvexification of $\overline{V}$ in the second variable (compare with (2.1)). Namely, for every $(d,\overline{F}) \in \{0,1\}\times\mathbb{R}^{3\times 2}$

$$Q\overline{V}\left(d,\overline{F}\right) \stackrel{\text{def}}{=\!=} \inf\left\{\int_{Q'} \overline{V}\left(d,\overline{F} + \nabla_\alpha\varphi(x_\alpha)\right)dx_\alpha : \varphi \in C_0^\infty\left(Q';\mathbb{R}^3\right)\right\}, \tag{4.31}$$

where $Q' \subset \mathbb{R}^2$ denotes the unit cube.

We will prove that problems (4.25) Γ-converge, as $\varepsilon \to 0^+$, to the problem

$$\inf_{\substack{u\in W_0^{1,\Phi}(\omega;\mathbb{R}^3) \\ \chi\in BV(\omega;\{0,1\}) \\ \frac{1}{|\omega|}\int_\omega \chi dx_\alpha = \frac{1}{2}\lambda}} \left\{2\int_\omega Q\overline{V}(\chi,\nabla_\alpha u)dx_\alpha - \int_{-1}^1\int_\omega f\cdot u dx + 2|D\chi|(\omega)\right\}.$$

In fact, the above convergence relies on the following theorem that will be proven in Section 4.5. We underline that the strategy of the proof is similar to the analogous result in [34], but it requires to introduce ad hoc tools in the Orlicz-Sobolev setting. It is worth observing that the result deals with the notion of quasiconvexification, which was one of the investigated points in Chapter 2.

Theorem 4.4.2 (The Γ-convergence result)**.** *Let $\Omega = \omega \times (-1, 1)$ be a bounded open set, $\omega \subseteq \mathbb{R}^2$ open and bounded with Lipschitz boundary and let $W_i : \Omega \to [0, +\infty)$, $i = 1, 2$, be continuous functions satisfying* (4.24)*. Let (J_ε) be the family of functionals defined in* (4.27)*. Then (J_ε) Γ-converges, with respect to the strong topology of $L^1(\Omega; \{0, 1\}) \times L^\Phi(\Omega; \mathbb{R}^3)$, to J_0 in* (4.28)*, as $\varepsilon \to 0^+$.*

Indeed, for what concerns the volume constraint and the boundary conditions, it is enough to observe that the strong convergence in $L^1(\Omega; \{0, 1\}) \times L^\Phi(\Omega; \mathbb{R}^3)$ (and, by compactness arguments, weak-$\star$-$BV(\Omega; \{0, 1\})\times$ weak-$W^{1,\Phi}(\Omega; \mathbb{R}^3)$), of the rescaled sequence $(\chi_\varepsilon, u_\varepsilon)$ of almost minimizers of (4.25) to $(\chi, u) \in BV(\omega; \{0, 1\}) \times W_0^{1,\Phi}(\omega; \mathbb{R}^3)$, guarantees that the volume fraction

$$\frac{1}{|(\Omega(\varepsilon)|} \int_{\Omega(\varepsilon)} \chi_{E(\varepsilon)} dx = \frac{1}{|\Omega|} \int_\Omega \chi_\varepsilon dx = \lambda$$

is kept in the limit. The continuity of the trace operator [91, Theorem] entails that $u \in W_0^{1,\Phi}(\omega; \mathbb{R}^3)$.

4.5 Proof of the Γ-convergence result

The following result, whose proof is immediate, will be exploited in the sequel.

Proposition 4.5.1. *Let $\overline{V}$ be as in* (4.30)*. Then $\overline{V}$ is continuous and satisfies*

$$\beta' \left(\Phi(|\overline{F}|) - 1\right) \leq \overline{V}\left(\chi, \overline{F}\right) \leq \beta\left(1 + \Phi(|\overline{F}|)\right), \tag{4.32}$$

where β' and β are the constants in (4.24)*. Moreover,*

$$\left|\overline{V}\left(\chi, \overline{F}\right) - \overline{V}(\chi', \overline{F})\right| \leq 2\beta\,|\chi - \chi'|\,(1 + \Phi(|\overline{F}|)).$$

Furthermore, the function $Q\overline{V}$ in (4.31) *is continuous and satisfies* (4.32)*, and*

$$|Q\overline{V}(\chi, \overline{F}) - Q\overline{V}(\chi', \overline{F})| \leq C|\chi' - \chi|(1 + \Phi(|\overline{F}|)). \tag{4.33}$$

We will also use the following, classical integral representation theorem due to Buttazzo and Dal Maso ([28, Theorem 4.3.2] or [45, Theorem 20.1]).

Theorem 4.5.2 (The Integral Representation Theorem)**.** *Let $\mathcal{A}$ be a set of all open subsets of $\omega \subset \mathbb{R}^n$ and Φ be an Orlicz function satysfying the Δ_2 condition* (4.5)*. Let $G : L^\Phi(\omega) \times \mathcal{A} \to [0, +\infty]$ be an increasing functional satisfying the following properties:*

(i) $G(u, A) = G(v, A)$ whenever $u = v$, a.e. on ω,

(ii) $G(u, \cdot)$ is the restriction of a finite non-negative Radon measure on $\mathcal{A}(\omega)$,

(iii) there exist $b \in \mathbb{R}$ and $a \in L^1_{loc}(\omega)$ such that

$$0 < G(u, A) < \int_A \big(a(x) + b\Phi(|Du(x)|)\big)dx$$

for every $u \in W^{1,\Phi}(\omega)$ and every $A \in \mathcal{A}$,

(iv) $G(u + c, A) = G(u, A)$ for any $c \in \mathbb{R}$,

(v) G is lower semicontinuous.

Then there exists a Borel function $W : \omega \times \mathbb{R}^n \to [0, +\infty)$ such that

(i) for every $u \in L^\Phi(\omega)$ and for every $A \in \mathcal{A}$ such that $u_{|A} \in W^{1,\Phi}_{loc}$ we have

$$G(u, A) = \int_A W(x, Du(x))dx,$$

(ii) for almost every $x \in \omega$, the function $W(x, \cdot)$ is convex on $\mathbb{R}^n$,

(iii) for almost every $x \in \omega$ we have

$$0 \leq W(x, \xi) \leq a(x) + b\Phi(\xi)$$

for every $\xi \in \mathbb{R}^n$.

However the Theorem is stated in the original sources for Sobolev spaces, result holds without any substantial modifications in our Sobolev-Orlicz setting. In particular, all the crucial steps (like 'Zig-Zag' lemma or the passage through affine and piecewise affine functions) can be repeated word by word. Also, wee emphasize that the approximation of functions in Sobolev-Orlicz spaces (with Orlicz function $\Phi \in \Delta_2$) by piecewise constant functions holds as originally stated in [52, Proposition 2.8].

We start by motivating the choice of the topology in Theorem 4.4.2. We claim that energy bounded sequences $(\chi_\varepsilon, u_\varepsilon) \in BV(\Omega; \{0, 1\}) \times W^{1,\Phi}(\Omega; \mathbb{R}^3)$, admissible for the rescaled version of (4.25), i.e. such that there exists $C > 0$:

$$\Big| \int_\Omega [(\chi_\varepsilon W_1 + (1 - \chi_\varepsilon)W_2) \left(\nabla_\alpha u_\varepsilon, \tfrac{1}{\varepsilon}\nabla_3 u_\varepsilon\right) - f \cdot u_\varepsilon]dx + \left|\left(D_\alpha \chi_\varepsilon, \tfrac{1}{\varepsilon} D_3 \chi_\varepsilon\right)\right|(\Omega)\Big| \leq C, \quad (4.34)$$

with u_ε clamped on $\partial\omega \times (-1, 1)$ and $\frac{1}{|\Omega|}\int_\Omega \chi_\varepsilon dx = \lambda$, are compact in space $L^1(\Omega; \{0, 1\}) \times L^\Phi(\Omega; \mathbb{R}^3)$ and with limit in $BV(\omega; \{0, 1\}) \times W^{1,\Phi}(\omega; \mathbb{R}^3)$. Indeed, let $(\chi_\varepsilon, u_\varepsilon)$ be a sequence such that (4.34) holds, then there exists $C' \in \mathbb{R}^+$ such that

$$\|u_\varepsilon\|_{W^{1,\Phi}} \leq C', \ \left\|\frac{1}{\varepsilon}\nabla_3 u_\varepsilon\right\|_{L^\Phi} \leq C', \ \left|\left(D_\alpha \chi_\varepsilon, \frac{1}{\varepsilon} D_3 \chi_\varepsilon\right)\right|(\Omega) \leq C'. \quad (4.35)$$

Then, standard arguments in dimensional reduction (i.e. the application of [117, Lemma 3]), entail that every cluster point $u \in W^{1,\Phi}(\Omega;\mathbb{R}^3)$ of the sequence (u_ε) is such that $\nabla_3 u \equiv 0$, and so u can be identified with a function (still denoted in the same way, cf. [124, Theorem 1 in Section 1.1.3]) $u \in W^{1,\Phi}(\omega;\mathbb{R}^3)$. Analogous considerations hold for the limit of χ_ε. Thus there exists a subsequence, not relabelled, $(\chi_\varepsilon, u_\varepsilon)$ such that $u_\varepsilon \rightharpoonup u$ in $W^{1,\Phi}(\Omega;\mathbb{R}^3)$, and a measurable set $E \subset \Omega$ such that $\chi_\varepsilon \overset{\star}{\rightharpoonup} \chi_E$ and $D_3\chi_E \equiv 0$. Hence, there exists $E' \subset \omega$, with

$$|D\chi_E|(\Omega) = 2|D\chi_{E'}|(\omega), \tag{4.36}$$

where $E = E' \times (-1,1)$. In the sequel we will identify the set E with the set E' and denote $\chi_{E'}$ by χ. We stress that the doubling coefficient in (4.36) comes from the fact that $\Omega = \omega \times I$, where the length of the interval I equals 2.

We observe that Theorem 4.4.2 still holds without the coercivity assumption (4.24), provided the admissible sequences satisfy (4.35).

Proof of Theorem 4.4.2. For every $\varepsilon > 0$, let J_ε be the functional in (4.27). Let us remind that, as Φ satisfies the Δ_2 condition (4.5), Proposition 4.2.2 point (ii) shows that $L^\Phi(\Omega,\mathbb{R}^3)$ is separable. Consequently, the metric space $L^1(\Omega;\{0,1\}) \times L^\Phi(\Omega;\mathbb{R}^3)$ is separable. That ensures that for each sequence (ε) there exists a subsequence, still denoted by (ε), such that $\Gamma - \lim_{\varepsilon\to 0^+} J_\varepsilon$ with respect to the strong topology of $L^1(\Omega;\{0,1\}) \times L^\Phi(\Omega;\mathbb{R}^3)$ exists.
For every $(\chi,u) \in L^1(\Omega;\{0,1\}) \times L^\Phi(\Omega;\mathbb{R}^3)$, let $J(\chi,u)$ be this Γ-limit. By Urysohn property, it suffices to prove that any sequence (J_ε) admits a further subsequence whose Γ-limit, $J(\chi,u)$, coincides with $J_0(\chi,u)$ in (4.28).

It is easily seen that

$$J(\chi,u) = +\infty \text{ for every } (\chi,u) \in (L^1(\Omega;\{0,1\}) \times L^\Phi(\Omega;\mathbb{R}^3)) \setminus (BV(\omega;\{0,1\}) \times W^{1,\Phi}(\omega;\mathbb{R}^3)).$$

Indeed, if this is not the case, from the condition $J(\chi,u) < +\infty$, we would get the existence of a sequence $(\chi_\varepsilon, u_\varepsilon)$ converging to (χ,u) such that $J_\varepsilon(\chi_\varepsilon,u_\varepsilon) < +\infty$ and this would imply $(\chi,u) \in BV(\omega;\{0,1\}) \times W^{1,\Phi}(\omega;\mathbb{R}^3)$ – a contradiction.

The remaining part of the proof is divided into two steps. First we show the lower bound, then we prove the upper bound (compare with Definition 4.6.21).

Lower bound: We claim that for every $(\chi,u) \in BV(\omega;\{0,1\}) \times W^{1,\Phi}(\omega;\mathbb{R}^3)$

$$\begin{aligned} J(\chi,u) \geq J_0(\chi,u) = \; & 2\int_\omega Q\overline{V}\left(\chi\left(x_\alpha\right), \nabla_\alpha u\left(x_\alpha\right)\right) dx_\alpha \\ & - \int_{-1}^{1}\int_\omega f\left(x_\alpha, x_3\right) u\left(x_\alpha\right) dx_\alpha dx_3 \\ & + 2\left|D\chi\right|(\omega), \end{aligned} \tag{4.37}$$

where the terms of the right-hand side of the inequality can be referred as bulk term, forces and perimeter respectively.

To prove the claim, let $(\chi_\varepsilon, u_\varepsilon) \subset L^1(\Omega;\{0,1\}) \times L^\Phi(\Omega;\mathbb{R}^3)$ be a sequence converging to $(\chi,u) \in BV(\omega;\{0,1\}) \times W^{1,\Phi}(\omega;\mathbb{R}^3)$. For the forces, the strong convergence in L^Φ, together with continuity of the linear term with respect to strong topology, show the claim, i.e. show that $\liminf_{\varepsilon\to 0^+} -\int_\Omega f u_\varepsilon dx \geq -\int_{-1}^1\int_\omega f(x_\alpha,x_3)\,u(x_\alpha)\,dx_\alpha dx_3$, so that the forces term of the original family of functionals is bounded from below by the forces term of our limit candidate.

For the perimeter, the lower bound for the proper terms follows by the lower semicontinuity of the total variation (see Proposition 4.6.25 or [9, Remark 3.5]), i.e. $\chi_\varepsilon \to \chi$ strongly in $L^1 \Rightarrow \liminf ||D\chi_\varepsilon|| \geq ||D\chi||$, and an analogous reasoning to (4.36). Namely,

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} |(D_\alpha\chi_\varepsilon|\frac{1}{\varepsilon}D_3\chi_\varepsilon)|(\Omega) &\overset{(1)}{\geq} \liminf_{\varepsilon\to 0^+} |(D_\alpha\chi_\varepsilon|D_3\chi_\varepsilon)|(\Omega) \overset{(2)}{\geq} \\ |(D_\alpha\chi|D_3\chi)|(\Omega) &\overset{(3)}{=\!=} |(D_\alpha\chi|0)|(\Omega) \overset{(4)}{=} 2|D\chi|(\omega),\end{aligned}$$

where (1) follows by $\varepsilon<1$, (2) by the lower semicontinuity of the total variation, (3) by the fact that $\chi \in BV(\omega,;\{0,1\})$ (so that $D_3\chi=0$) and (4) by $\Omega=\omega\times(-1,1)$.

For what concerns the bulk energy, by Theorem 4.1.1, there exist (w_ε) and (A_ε), such that $A_\varepsilon \subset \Omega$, w_ε converges in $L^\Phi(\Omega;\mathbb{R}^3)$ to $u \in W^{1,\Phi}(\omega;\mathbb{R}^3)$, the scaled gradients $(\nabla_\alpha w_\varepsilon, \frac{1}{\varepsilon}\nabla_3 w_\varepsilon)$ are Φ-equi-integrable, $A_\varepsilon \subseteq \Omega$ and $u_\varepsilon \equiv w_\varepsilon$ in A_ε and $|\Omega\setminus A_\varepsilon|\to 0$ as $\varepsilon\to 0^+$. Denoting the bulk energy density of J_ε by V as in (4.29), one obtains

$$\begin{aligned}\liminf_{\varepsilon\to 0^+}\int_\Omega V\left(\chi_\varepsilon,\left(\nabla_\alpha u_\varepsilon \frac{1}{\varepsilon}\nabla_3 u_\varepsilon\right)\right)dx &\overset{(1)}{\geq} \\ \liminf_{\varepsilon\to 0^+}\int_{A_\varepsilon} V\left(\chi_\varepsilon,\left(\nabla_\alpha w_\varepsilon, \frac{1}{\varepsilon}\nabla_3 w_\varepsilon\right)\right)dx & \\ -\beta\limsup_{\varepsilon\to 0^+}\int_{\Omega\setminus A_\varepsilon}(1+\Phi(\left|(\nabla_\alpha w_\varepsilon, \frac{1}{\varepsilon}\nabla_3 w_\varepsilon)\right|))dx &\overset{(2)}{\geq} \\ \liminf_{\varepsilon\to 0^+}\int_\Omega V\left(\chi_\varepsilon,\left(\nabla_\alpha w_\varepsilon, \frac{1}{\varepsilon}\nabla_3 w_\varepsilon\right)\right)dx &\overset{(3)}{\geq} \\ \liminf_{\varepsilon\to 0^+}\int_\Omega \overline{V}\left(\chi_\varepsilon,\nabla_\alpha w_\varepsilon\right)dx &\overset{(4)}{\geq} \\ \liminf_{\varepsilon\to 0^+}\int_\Omega Q\overline{V}\left(\chi_\varepsilon,\nabla_\alpha w_\varepsilon\right)dx, & \end{aligned} \tag{4.38}$$

where the inequality (1) follows by (4.24), (2) by Φ-equiintegrability of w_ε and $|\Omega\setminus A_\varepsilon|\to 0$, while (3) and (4) are straightforward consequences of the definitions of the densities $\overline{V}$ and $Q\overline{V}$, given in (4.30) and (4.31) respectively.

Observe that, by (4.33), for a.e. $x\in\Omega$.

$$\int_\Omega |Q\overline{V}(\chi_\varepsilon,\nabla_\alpha w_\varepsilon) - Q\overline{V}(\chi,\nabla_\alpha w_\varepsilon)|\,dx \leq C\int_\Omega |\chi_\varepsilon-\chi|(1+\Phi(|\nabla_\alpha w_\varepsilon|))\,dx.$$

Thus, the Φ-equi-integrability of $(\nabla_\alpha w_\varepsilon \,|\, \frac{1}{\varepsilon}\nabla_3 w_\varepsilon)$ ensures that, as $\varepsilon \to 0^+$, χ_ε can be replaced by χ in the right-hand side of (4.38).

From [70, Theorem 1] it follows that smooth functions are dense in $W^{1,\Phi}(E)$ providing Φ satisfies (4.6) and the argument exploited in [117, Proposition 6] ensures that $Q\overline{V}(\chi(x_\alpha),\cdot)$ is quasiconvex also in $\mathbb{R}^{3\times 3}$. Thus, by the growth condition of $Q\overline{V}$,(4.32), and [58, Theorem 3.1] the functional $v \in W^{1,\Phi}(\Omega;\mathbb{R}^3) \longmapsto \int_\Omega Q\overline{V}(\chi(x_\alpha), \nabla_\alpha v(x))dx$ is sequentially weakly lower semicontinuous with respect to $W^{1,\Phi}$-weak topology (and, by (4.8), strongly in L^Φ). Hence,

$$\liminf_{\varepsilon\to 0^+} \int_\Omega Q\overline{V}(\chi, \nabla_\alpha w_\varepsilon)dx \geq 2\int_\omega Q\overline{V}(\chi, \nabla_\alpha u)dx_\alpha.$$

By the superadditivity of the lim inf and an arbitrary choice of $(\chi_\varepsilon, u_\varepsilon) \to (\chi, u)$ we achieve the claim.

Upper bound: To prove the claim, that is that for every $(\chi, u) \in BV(\omega;\{0,1\}) \times W^{1,\Phi}(\omega;\mathbb{R}^3)$, we have $J(\chi,u) \leq J_0(\chi,u)$, let us start by observing that for every $\chi \in BV(\omega;\{0,1\})$,

$$J(\chi,u) \leq \liminf_{\varepsilon\to 0^+} J_\varepsilon(\chi, u_\varepsilon) \text{ for every } u_\varepsilon \to u \text{ in } L^\Phi(\Omega;\mathbb{R}^3).$$

Having χ fixed, we observe that the perimeter term (see (4.37)) in J_ε coincides with the perimeter term of J_0. Thus, we can reduce to study the asymptotic behaviour with respect to the $W^{1,\Phi}$-weak convergence of

$$\int_\Omega \left(\chi W_1\left(\nabla_\alpha u_\varepsilon, \tfrac{1}{\varepsilon}\nabla_3 u_\varepsilon\right) + (1-\chi)W_2\left(\nabla_\alpha u_\varepsilon, \tfrac{1}{\varepsilon}\nabla_3 u_\varepsilon\right)\right)dx - \int_\Omega f\cdot u_\varepsilon dx. \tag{4.39}$$

Since χ is fixed, we can rewrite $\chi W_1(\cdot) + (1-\chi)W_2(\cdot)$ as a new function with explicit dependence on x_α.

Denoting

$$W(x_\alpha, F) \stackrel{\text{def}}{=\!=} V(\chi(x_\alpha), F),$$

it results that W is a Carathéodory function satisfying a growth condition of the type (4.24), i.e. $\frac{1}{C}\Phi(|F|) - C \leq W(x_\alpha, F) \leq C(1+\Phi(|F|))$ for a suitable constant $C \in \mathbb{R}^+$, for a.e. $x_\alpha \in \omega$ and for all $F \in \mathbb{R}^{3\times 3}$.

Next, we argue as in [24, Theorem 2.5] and [15, Lemma 2.5]. Let $\mathcal{A}(\omega)$ be the set of all open subsets $A \subseteq \omega$. We define the sequence of functionals (G_ε), where $G_\varepsilon : L^\Phi(\Omega;\mathbb{R}^3)\times\mathcal{A}(\omega) \to [0,+\infty)$ is given by

$$G_\varepsilon(u,A) = \begin{cases} \int_{A\times(-1,1)} W\left(x_\alpha, \left(\nabla_\alpha u, \frac{1}{\varepsilon}\nabla_3 u\right)\right)dx - \int_{A\times(-1,1)} f\cdot u dx & \text{if } u \in W^{1,\Phi}(\Omega;\mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

and we claim that

$$\limsup_{\varepsilon\to 0^+} G_\varepsilon(u,\omega) \leq \int_\omega \underline{W}(x_\alpha, \nabla_\alpha u)dx_\alpha - \int_{-1}^1\int_\omega f\cdot u dx_\alpha dx_3,$$

where $\underline{W} : \omega \times \mathbb{R}^{3\times 2}$ is defined by

$$\begin{aligned}&\underline{W}(x_\alpha, F) \stackrel{\text{def}}{=\!=}\\ &\inf\Big\{\tfrac{1}{2}\int_{Q\times(-1,1)} W(x_\alpha, F + \nabla_\alpha\varphi(y_\alpha, y_3), \lambda\nabla_3\varphi(y_\alpha, y_3))dy_\alpha dy_3 :\\ &\varphi \in W_0^{1,\Phi}(Q' \times (-1,1)), \varphi = 0 \text{ on } \partial Q' \times (-1,1), \lambda > 0\Big\}.\end{aligned} \tag{4.40}$$

To this end, we observe that there exists a subsequence of (G_ε) which Γ-converges to some lower semicontinuous functional (for example take one converging to $\Gamma - \liminf$), which will be referred to as $\overline{G}$, i.e.

$$\overline{G} \stackrel{\text{def}}{=\!=} \Gamma - \lim_{\varepsilon\to 0^+} G_\varepsilon.$$

With straightforward modifications to the long and technical argument of [24, Step 3. and Step 4. Theorem 2.5] (which can be adapted to $W^{1,\Phi}$ setting) one proves that for every $u \in W^{1,\Phi}(\omega, \mathbb{R}^3)$, the set function $\overline{G}(u, \cdot) : \mathcal{A}(\omega) \to \mathbb{R}$ satisfies

(i) $\overline{G}(u, A) = \overline{G}(v; A)$ whenever $u = v$, a.e. on $\mathbb{R}^2$,

(ii) $\overline{G}(u, \cdot)$ is the restriction of a finite non-negative Radon measure on $\mathcal{A}(\omega)$,

(iii) $\overline{G}(u, A) \leq 2\beta \int_A (1 + \Phi(|D_\alpha u|))dx$,

(iv) $\overline{G}(u + c; A) = \overline{G}(u, A) - \int_{-1}^{1}\int_A f \cdot c dx_\alpha$ for any $c \in \mathbb{R}^3$.

This proves that the bulk term of the functional $\overline{G}$ staisfies the assumptions of The Integral Representation Theorem 4.5.2. The existence of an energy density $\widetilde{W} : \omega \times \mathbb{R}^{3\times 2} \to \mathbb{R}$ such that $\overline{G}(u, A) = \int_A \widetilde{W}(x, \nabla_\alpha u)dx_\alpha - \int_{-1}^{1}\int_A f \cdot u dx_\alpha$ is thus guaranteed. Finally, applying [15, Lemma 2.5], whose proof can be repeated word by word in Sobolev-Orlicz setting instead of the classical Sobolev case, entails that $\widetilde{W}(x_\alpha, \overline{F}) \leq \underline{W}(x_\alpha, \overline{F})$, for a.e. $x_\alpha \in \omega$ and every $\overline{F} \in \mathbb{R}^{3\times 2}$, where $\underline{W}$ is the density in (4.40).

Next, we introduce for every $\overline{F} \in \mathbb{R}^{3\times 2}$, $\overline{W}(x_\alpha, \overline{F}) \stackrel{\text{def}}{=\!=} \inf_{c\in\mathbb{R}^3} W(x_\alpha, (\overline{F}|c))$, and denote by $Q\overline{W}$ the quasiconvexification of $\overline{W}$ with respect to the second variable, according to (4.31).

Finally, applying Theorem 4.1.2 point (i), we get that for a.e. $x_\alpha \in \omega$ and every $\overline{F} \in \mathbb{R}^{3\times 2}$ we have $\underline{W}(x_\alpha, \overline{F}) \leq Q\overline{W}(x_\alpha, \overline{F})$, while the inverse inequality follows straight from the definitions of these densities. As a result, we see that for a.e. $x_\alpha \in \omega$ and every $\overline{F} \in \mathbb{R}^{3\times 2}$ we get $\underline{W}(x_\alpha, \overline{F}) = Q\overline{W}(x_\alpha, \overline{F})$

The proof is concluded with observing that by (4.30) and the definitions of $\overline{W}_i$ below, we have

$$\overline{W}(x_\alpha, \overline{F}) = \chi(x_\alpha)\overline{W_1}(\overline{F}) + (1 - \chi(x_\alpha))\overline{W_2}(\overline{F}) = \overline{V}(\chi(x_\alpha), \overline{F})$$

and $Q\overline{W}(x_\alpha, \overline{F}) = Q\overline{V}(\chi(x_\alpha), \overline{F})$ for every $(x_\alpha, \overline{F}) \in \omega \times \mathbb{R}^{3\times 2}$.

□

4.6 Some additional information on the tools used

In this section we give a brief remainder of the tools used in the chapter. We also sketch the chosen proofs of some used facts from the Orlicz spaces theory. The tools described in the first subsection, dealing with Orlicz spaces, played an essential role in the proof of Theorem 4.1.1. The second subsection is devoted to the Theory of the Γ-convergence, which is the main notion used in Theorem 4.4.2. The third section is a brief presentation of the theory of BV spaces and sets of finite perimeter. These theories were exploited in the definition of the functionals J_ε (4.27) and J_0 (4.28), as well as in describing the topology hidden behind the notion of the Γ-convergence presented in Theorem 4.4.2.

4.6.1 Selected facts about Orlicz spaces

We begin with the following, easy fact, connecting the inequalities on Orlicz functions with inequalities on their conjugates.

Theorem 4.6.1. *[106, Theorem 2.1] Let Φ, Φ_1 be Orlicz functions and Ψ, Ψ_1 – their conjugate functions respectively. Suppose that, for $t > t_0$, we have $\Phi(t) < \Phi_1(t)$. Then, tor $s > s_0$, we have $\Psi_1(s) < \Psi(s)$.*

Proof. Assume now that ϕ_1, ψ_1 are the right derivatives of Φ_1 and Ψ_1 respectively. Take $s_0 = \phi_1(t_0)$. From monotonicity, it follows that $\psi_1(s) \geq t_0$ whenever $\phi_1(t_0) \leq s$. Having in mind that

$$\psi_1(s)s = \Phi_1(\psi_1(s)) + \Psi_1(s),$$

we also see that by the Young inequality

$$\psi_1(s)s \leq \Phi(\psi_1(s)) + \Psi(s).$$

In the result,

$$\Phi_1(\psi_1(s)) + \Psi_1(s) \leq \Phi(\psi_1(s)) + \Psi(s)$$

and since $\Phi(\psi_1(s)) < \Phi_1(\psi_1(s))$, $\Psi_1(s) < \Psi(s)$ follows. □

Now we present another fact, which lets us rescale the Orlicz function and compute the conjugate after the rescaling.

Lemma 4.6.2. *Let Φ, Ψ be conjugate Orlicz functions and ϕ, ψ – their densities respectively. Set $\Phi_1(t) \overset{def}{=} A\Phi(bt)$. Then, the conjugate function to Φ_1 is*

$$\Psi_1(s) \overset{def}{=} A\Psi\Big(\frac{s}{Ab}\Big). \tag{4.41}$$

Proof. Let us assume that Ψ_1 is the conjugate to Φ_1 and prove the equality (4.41). The right derivative of Φ_1 is

$$\phi_1(t) = Ab\phi(bt)$$

and hence the right derivative of Ψ_1 is its right inverse, i.e.

$$\psi_1 = \frac{1}{b}\psi(\frac{s}{Ab}).$$

Since then, we compute

$$\Psi_1(s) = \int_0^s \psi_1(\sigma)d\sigma = \frac{1}{b}\int_0^s \psi(\frac{\sigma}{Ab})d\sigma = A\int_0^{\frac{s}{Ab}} \psi(\sigma)d\sigma,$$

which shows (4.41). □

The two above statements were required to prove the relation between the classical Δ_2 and ∇_2 conditions ((4.5) and (4.6) respectively).

Theorem 4.6.3. *[106, Theorem 4.2] Let Φ and Ψ be the conjugate Orlicz functions. Then Φ satisfies* (4.5) *if and only if Ψ satisfies* (4.6)*.*

Proof. First, let as assume that Ψ satisfies (4.6), i.e.

there exists $C > 0$ and $s_0 \geq 0$ such that $\Psi(s) \leq 1/(2C)\Psi(Cs)$ for any $s > s_0$.

Let us take then

$$\Psi_1(s) \overset{\text{def}}{=\!=} 1/(2C)\Psi(Cs),$$

so that the (4.6) condition is rewritten in the form $\Psi(s) < \Psi_1(s)$ for $s > s_0$. In the virtue of Theorem 4.6.1, $\Phi_1(t) < \Phi(t)$ follows for sufficiently big t, where Φ_1 is the conjugate function to Ψ_1. By Lemma 4.6.2 we may calculate that $\Phi_1(t) = 1/(2C)\Phi(2t)$. Hence, for sufficiently big t, we get $\Phi(2t) < 2C\Phi(t)$ and condition (4.5) is established.

The inverse implication is proven in the analogous way. □

Another variant of the Δ_2 condition (4.5) is given in the following result.

Theorem 4.6.4. *[106, Theorem 4.1] Let Φ be an Orlicz function and ϕ be its density. Then Φ satisfies* (4.5) *if and only if the inequality*

$$\frac{t\phi(t)}{\Phi(t)} < \alpha$$

holds for sufficiently big t and some finite α.

Proof. Assume first, that $\frac{t\phi(t)}{\Phi(t)} < \alpha$ for $t > t_0$ and some α. Then

$$\int_t^{2t} \frac{\phi(\tau)}{\Phi(\tau)} d\tau < \int_t^{2t} \frac{\alpha}{\tau} d\tau = \alpha \ln 2.$$

Hence $\ln \Phi(2t) - \ln \Phi(t) < \alpha \ln 2$ and consequently $\Phi(2t) < 2^\alpha \Phi(t)$, which shows that (4.5) holds.

On the other hand, under (4.5), we have

$$C\Phi(t) > \Phi(2t) = \int_0^{2t} \phi(\tau) d\tau > \int_t^{2t} \phi(\tau) d\tau > t\phi(t),$$

where the last inequality follows from the fact that ϕ is non-decreasing. This shows that

$$\frac{t\phi(t)}{\Phi(t)} < C$$

and finishes the proof.

□

Next, we will focus on the properties of the Hardy maximal operator. As it was already seen, the Hardy maximal operator is an essential tool in the proof of the Decomposition Lemma. For that reason, we want to present some of its properties more carefully and in a self-contained way.

The definition of the Hardy maximal operator, which was already given in (4.9), reads as follows. Let $f : E \to \mathbb{R}$ be a scalar function, $E \subset \mathbb{R}^n$ and $B(x,r)$ be a ball in $\mathbb{R}^n$, centred at x and of the radius r.

$$\mathcal{M}f(x) \overset{\text{def}}{=\!=} \sup_r |B(x,r)|^{-1} \int_{B(x,r)\cap E} |f(y)| dy.$$

Theorem 4.6.5. *Let Φ be an Orlicz function satisfying* (4.5). *For any $f \in L^\Phi(E)$ there exists a constant $C = C\big(E, \Phi\big)$ such that*

$$|\{\mathcal{M}f > t\}| \le \frac{C}{\Phi(t)} \int_E \Phi(|f|) dx,$$

for every $t > 0$.

We present a very simplified and well-known proof. In the original [105] paper the proof based on the reasoning held in [94]. It is worth mentioning that Kerman and Torchinsky dealt with weighted setting and for that reason, the requirement of (4.6) appeared. In isotropic setting, however, the result is well-known and straightforward.

Before the proof, we will recall the classical Five Covering Lemma, which will be soon used. The result was obtained by Giuseppe Vitali in [152] and hence is often referred in literature as Vitali Covering Lemma. For the proof in English, we recommend [54, Theorem 1 in Section 1.5].

Theorem 4.6.6 (The Five Covering Lemma)**.** *Let $\mathbb{X}$ be a separable metric space and $\mathcal{F}$ be a family of open balls with bounded diameter. Then $\mathcal{F}$ has a countable subfamily $\mathcal{F}'$ consisting of disjoint balls such that*

$$\bigcup_{B\in\mathcal{F}} B \subseteq \bigcup_{B\in\mathcal{F}'} 5B,$$

where 5B is B with 5 times radius.

We are ready to prove Theorem 4.6.5

Proof of Theorem 4.6.5. Consider the set of points x such that $\mathcal{M}f > t$. For every such point x there exists a ball B_x centred at x, such that

$$|B_x|^{-1} \int_{B_x} |f| dx > t.$$

From monotonicity and convexity of Φ, with the help of Jensen inequality, we obtain that

$$|B_x|^{-1} \int_{B_x} \Phi(|f|) dx \geq \Phi\Big(|B_x|^{-1} \int_{B_x} |f| dx\Big) > \Phi(t).$$

In particular, for any ball B_x like above, we have

$$\int_{B_x} \frac{\Phi(|f|)}{\Phi(t)} dx > |B_x|.$$

By the Five Covering Lemma, from the family of all such balls we exclude a countable subfamily of disjoint balls B_j, such that

$$\bigcup_j 5B_j \supseteq \bigcup B_x.$$

Since then,

$$|\{x : \mathcal{M}f > t\}| \leq \sum_j |5B_j| = 5^n \sum_j |B_j| \leq 5^n \sum_j \int_{B_j} \frac{\Phi(|f|)}{\Phi(t)} dx.$$

□

It is worth to observe that the result holds with the same proof in the vectorial case.

Our aim now is to characterise Orlicz spaces, for which the Hardy maximal operator is continuous. For that result we will exploit the following, straightforward fact.

Fact 4.6.7. *Let Φ be the Orlicz function and let Ψ be its conjugate Orlicz function. Let ϕ, ψ be their densities respectively. Then Ψ satifies* (4.5) *if and only if*

$$\beta \stackrel{def}{=\!=} \inf_{t>0} \frac{t\phi(t)}{\Phi(t)} > 1.$$

The above fact is well known. The proof of this result is straightforward, but quite computational and technical. We refer to [106, Theorem 4.3] or [66, Proposition 1.4]. The interpretation of the constant β is very instructing. We invite the curious reader to see [57] for details.

We are ready to prove the classical result due to Diego Gallardo [66]. A similar result, but dealing with different domain and target Orlicz spaces for the operator $\mathcal{M}$, was obtained by Kita [99, 100]. It seems hat Kita didn't know the Gallardo's paper. The statement of the main Theorems is essentially broader and the technique of the proof is significantly different.

We follow the idea of the proof due to Gallardo. The proof given in [66] is quite hard to follow. Several non-obvious computational steps are left to the reader and the whole structure of the proof is not clear. Our presentation was intended to require less concentration from the reader. For a clear presentation of the result in Lebesgue spaces we recommend [47, Theorem 7.19].

Theorem 4.6.8. *[66, Theorem 2.1] Let E be a measurable subset of $\mathbb{R}^n$. The Hardy maximal operatror $\mathcal{M} : L^\Phi(E) \to L^\Phi(\mathbb{R}^n)$ satisfies*

$$||\mathcal{M}f||_{L^\Phi(\mathbb{R}^n)} \leq D||f||_{L^\Phi(E)}$$

if and only if Φ satisfies (4.6).

Proof. In order to prove the claim, we will use the following, technical condition.

($\star$) For every f in $L^\Phi(E)$ there exist positive constants A and B such that

$$\int_{\mathbb{R}^n} \Phi(B\mathcal{M}f)dx \leq A \int_E \Phi(|f|)dx.$$

The proof will be divided into three steps. In **Step 1**, we prove that the ($\star$) condition implies the continuity of the Maximal Operator. In **Step 2**, we show that the continuity of the Maximal Operator implies that Φ satisfies the ∇_2 condition (4.6). In **Step 3**, it is proven that the condition (4.6) implies that the ($\star$) condition holds.

Step 1. We prove that the ($\star$) condition implies that for every $f \in L^\Phi(E)$ we have $||\mathcal{M}f||_{L^\Phi(\mathbb{R}^n)} \leq C||f||_{L^\Phi(E)}$. Indeed, as $f \in L^\Phi(E)$, for some λ we have $\lambda f \in L_\Phi(E)$ (let us remind that L_Φ stays for the Orlicz class, which is not a linear space under violation of (4.5)). Since then, from ($\star$) we get that $\lambda B\mathcal{M}f \in L_\Phi(\mathbb{R}^n)$, and hence $B\mathcal{M}f \in L^\Phi(\mathbb{R}^n)$. Furthermore, from ($\star$) applied to $f/||f||_{L^\Phi(E)}$ and the definition of the Luxemburg norm

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{B\mathcal{M}f}{\max(1, A)||f||_{L^\Phi(E)}}\Big)dx < A\max(1, A)^{-1} \int_E \Phi\Big(\frac{|f|}{||f||_{L^\Phi(E)}}\Big)dx \leq 1.$$

This shows that

$$||\mathcal{M}f||_{L^\Phi(\mathbb{R}^n)} \leq B^{-1}\max(1, A)||f||_{L^\Phi(E)}.$$

Step 2. We show that whenever

$$||\mathcal{M}f||_{L^\Phi(\mathbb{R}^n)} \le D||f||_{L^\Phi(E)}$$

holds, the function Φ needs to satisfy (4.6). Let us then take Ψ – the conjugate function to Φ and remind that Φ satisfying (4.6) is equivalent to Ψ satisfying (4.5).

We use the standard notation $B(x, r)$ for a ball centred at x and of radius r. Let α_n be the measure of the unit ball. For any pair (v, s), $v > 0, s > 1$ we consider balls $S(v, s) = B(0, (\alpha_n vs)^{-\frac{1}{n}}), L(v) = B(0, (\alpha_n v)^{-\frac{1}{n}})$.

For any point x outside $S(v, s)$ we see that $S(v, s) \subseteq B(x, 2|x|)$ and hence

$$\mathcal{M}\chi_{S(v,s)}(x) \ge \frac{|S(v, s)|}{|B(x, 2|x|)|} = (2^n \alpha_n vs|x|)^{-1}.$$

Let us take

$$g \stackrel{\text{def}}{=\!=} \Psi^{-1}(v)\chi_{L(v)} \in L^\Psi(\mathbb{R}^n),$$

where Ψ^{-1} stays for the inverse function to Ψ. We observe that

$$\int_{\mathbb{R}^n} \Psi(|g|)dx = v|L(v)| = 1.$$

This shows, that $||g||_{L^\Phi(R^n)} \le 1$. Hence, we compute

$$||\mathcal{M}\chi_{S(v,s)}||_{L^\Phi_\star(\mathbb{R}^n)} \ge \Psi^{-1}(v) \int_{L(v)} \mathcal{M}\chi_{S(v,s)}dx$$

$$\ge (2^n\alpha_n vs)^{-1}\Psi^{-1}(v) \int_{L(v)\setminus S(v,s)} |x|^{-n}dx = (2^n\alpha_n vs)^{-1}\Psi^{-1}(v) \ln s,$$

where $|| \cdot ||_{L^\Phi_\star}$ is the operator norm for Orlicz space, i.e.

$$||f||_{L^\Phi_\star} \stackrel{\text{def}}{=\!=} \sup_{||g||_{L^\Psi} \le 1} \int fg.$$

We remind that the operator norm is equivalent to the Luxemburg norm.

Having in mind, that

$$||\chi_{S(v,s)}||_{L^\Phi_\star(\mathbb{R}^n)} = (vs)^{-1}\Psi^{-1}(vs),$$

from the continuity of Hardy operator it follows that

$$2^{-n}\Psi^{-1}(v) \ln s \le D\Psi^{-1}(vs).$$

Taking $\ln s = 2^{n+1}D$ gives

$$2\Psi^{-1}(v) \le \Psi^{-1}(ve^{2^{n+1}D}).$$

Taking now $t = \Psi^{-1}(v)$ and putting Ψ on both sides shows that

$$\Psi(2t) \leq ve^{2^{n+1}D} = \Psi(t)e^{2^{n+1}D},$$

which proves that Ψ satisfies (4.5).

Step 3. What is left to do is to show that whenever Ψ satisfies (4.6), the $(\star)$ condition is satisfied. For that, let us first notice that the weak estimate established previously holds also for $\Phi(t) = t$. Having that in mind, let us observe that the following properties hold.

(i) $|\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda\}| \leq C\lambda^{-1} \int_E |f|dx$;

(ii) $||\mathcal{M}f||_{L^\infty(\mathbb{R}^n)} \leq ||f||_{L^\infty(E)}$;

(iii) $|\mathcal{M}(f+g)| \leq (|\mathcal{M}f| + |\mathcal{M}g|)$,

where the constant C comes from Theorem 4.6.5.

Let us now set $f^\lambda \overset{\text{def}}{=} f\chi_{\{x\in E:|f(x)|>\lambda\}}$. We claim that

$$|\{x : \mathcal{M}f(x) > 2\lambda\}| < |\{x : \mathcal{M}f^\lambda(x) > \lambda\}|. \tag{4.42}$$

To prove the claim, let $y \in \{x : \mathcal{M}f(x) > 2\lambda\}$. Hence,

$$2\lambda < \mathcal{M}f(y) \overset{\text{(iii)}}{\leq} \mathcal{M}f^\lambda(y) + \mathcal{M}(f - f^\lambda)(y),$$

but $(f - f^\lambda)(x) \leq \lambda$ for every $x \in E$ and, from (ii), also $\mathcal{M}(f - f^\lambda)(x) < \lambda$ for a.e. $x \in E$. This shows that $\mathcal{M}f^\lambda(y) > s$ and hence $y \in \{x : \mathcal{M}f^\lambda(x) > \lambda\}$. All in all, $\{x : \mathcal{M}f(x) > 2\lambda\} \subseteq \{x : \mathcal{M}f^\lambda(x) > \lambda\}$, which proves the claim.

We have then

$$\begin{aligned}\int_{\mathbb{R}^n} \Phi(\mathcal{M}f) &= \int_0^{+\infty} \phi(\lambda)|\{x : \mathcal{M}f(x) > \lambda\}|d\lambda \overset{(4.42)}{\leq} \int_0^{+\infty} \phi(\lambda)|\{x : \mathcal{M}f^\lambda(x) > \frac{\lambda}{2}\}|d\lambda \\ &\overset{(i)}{\leq} 2C \int_0^{+\infty} \phi(\lambda)\lambda^{-1} \int_E |f^\lambda(x)|dx\, d\lambda = 2C \int_E |f(x)|(\int_0^{|f(x)|} \lambda^{-1}\phi(\lambda)d\lambda)dx,\end{aligned} \tag{4.43}$$

where ϕ is the density of Φ.

With the integration by parts, we discover that for every positive s

$$\int_0^s \lambda^{-1}\phi(\lambda)d\lambda = s^{-1}\Phi(s) + \int_0^s \lambda^{-2}\Phi(\lambda)d\lambda. \tag{4.44}$$

Under the assumption (4.6) on Φ, we remind that from the Fact 4.6.7 it follows that there exists $\beta > 1$ such that for any positive s

$$\beta\Phi(s) < s\phi(s).$$

After rewriting it to the form

$$\frac{\beta}{s} < \frac{\phi(s)}{\Phi(s)},$$

we may integrate both sides from λ to 1, getting

$$-\beta \ln \lambda < \ln\big(\Phi(1)\big) - \ln\big(\Phi(\lambda)\big)$$

and consequently

$$\lambda^{\beta} > \frac{\Phi(\lambda)}{\Phi(s)}$$

for every $\lambda \in (0,1)$. Since, then

$$\Phi(\lambda) < \Phi(1)\lambda^{\beta}.$$

Multiplying the above by λ^{-2}, we get that the second ingredient of the right hand side in (4.44) is finite and

$$\int_0^s \lambda^{-1}\phi(\lambda)d\lambda < \frac{\beta}{\beta-1}s^{-1}\Phi(s).$$

Plugging it to (4.43) with $s = f(x)$ we show that

$$\int_{\mathbb{R}^n} \Phi(\mathcal{M}f)dx < 2C\frac{\beta}{\beta-1}\int_E \Phi(|f|)dx$$

what finishes the proof. □

4.6.2 Γ-convergence and its application to the thin structures setting

In the study of thin structures, i.e. when the studied domain is much smaller in one or some directions than in the others, say of order $\varepsilon << 1$, rigorous analysis via dimensional reduction proves to be a useful tool to deduce properties of thin domains starting from thicker models. In this analysis one deals with sequences of functions defined on cylindrical sets with some "thin" (ε sized) dimension. In the $3D$ setting, thin films are modelled as $\omega \times (-\varepsilon, \varepsilon)$ with $\omega \subset \mathbb{R}^2$ a bounded open set.

The standard way to deal with such problems is to search for Γ-limit with respect to the scaling factor. The essential feature of Γ-limits is that every functional, which is the $\Gamma - \lim$ of any sequence of functionals, is always lower semicontinuous. In particular, if F is not lower semicontinuous, then $\Gamma - \lim F \neq F$.

The purpose of this section is to briefly present the notion of Γ-convergence and explain the way we use it to deal with our optimal design problem.

To abbreviate, the compact topological space $\mathbb{R} \cup \{-\infty, +\infty\}$ (the real line with both ends, a two-point compactification of $\mathbb{R}$) will be referred as $\overline{\mathbb{R}}$. The classic definition, as in [45, Definition 4.1], reads as follows.

Definition 4.6.9. [Γ-convergence] Let $\mathbb{X}$ be a topological space and $F_n : \mathbb{X} \to \overline{\mathbb{R}}$ be functionals. Let $\mathscr{U}(x)$ be the family of all open neighbourhood of a point $x \in \mathbb{X}$. The Γ-lower limit and the Γ-upper limit are defined respectively as

(LL) $(\Gamma - \liminf F_n)(x) \overset{\text{def}}{=\!=} \sup_{U \in \mathscr{U}(x)} \liminf_{n \to +\infty} \inf_{y \in U} F_n(y);$

(UL) $(\Gamma - \limsup F_n)(x) \overset{\text{def}}{=\!=} \sup_{U \in \mathscr{U}(x)} \limsup_{n \to +\infty} \inf_{y \in U} F_n(y).$

The function $F : \mathbb{X} \to \overline{\mathbb{R}}$ is referred as the $\Gamma - \lim F_n$, whenever it coincides with both $\Gamma - \liminf F_n$ and $\Gamma - \limsup F_n$.

The notion of the Γ-convergence gives us a very good control on the behaviour of the minimizers of the functionals. We will present a few of essential facts showing how the information about the minimizers of F_n is transferred to $\Gamma - \lim F_n$ and its own minimizers. Some of the proofs are quite technical and so we omit them. We invite the curious reader to look for more details in [45, Chapter 7].

From now on we assume that $\mathbb{X}$ is a topological space and $K \subseteq \mathbb{X}$.

Definition 4.6.10. The set K is called countably compact, whenever every sequence $x_n \in K$ possesses a cluster point in K.

Proposition 4.6.11. *[45, Proposition 7.2] Let K be a countably compact subset of $\mathbb{X}$. Then*

$$\min_{x \in K} (\Gamma - \liminf F_n)(x) \leq \liminf_{n \to +\infty} \inf_{x \in K} F_n(x).$$

Proof. Since $\Gamma - \liminf F_n$ is lower semicontinuous and K is countably compact, $\Gamma - \liminf F_n$ attains its minimum on K. Let us take a sequence (n_k) such that

$$\liminf_{n \to +\infty} \inf_{x \in K} F_n(x) = \lim_{n \to +\infty} \inf_{x \in K} F_{n_k}(x)$$

and a sequence $(y_k) \in K$ such that

$$\lim_{n \to +\infty} \inf_{x \in K} F_{n_k}(x) = \lim_{n \to +\infty} F_{n_k}(y_k).$$

Let $y \in K$ be a cluster point of the sequence y_k. For every $U \in \mathscr{U}(y)$ and every m there exists $k > m$ such that $y_k \in U$ and hence

$$\inf_{x \in U} F_{n_k}(x) \leq F_{n_k}(y_k).$$

Since then,

$$\liminf_{n \to +\infty} \inf_{x \in U} F_n(x) \leq \liminf_{n \to +\infty} \inf_{x \in U} F_{n_k}(x) \leq \lim_{n \to +\infty} F_{n_k}(y_k) = \lim_{n \to +\infty} \inf_{x \in K} F_{n_k}(x) = \liminf_{n \to +\infty} \inf_{x \in K} F_n(x).$$

Taking supremum over $U \in \mathscr{U}(y)$ in the above inequality, we arrive with

$$(\Gamma - \liminf F_n)(y) \leq \liminf_{n\to+\infty} \inf_{x\in K} F_n(x),$$

while $\min_{x\in K}(\Gamma - \liminf F_n)(x) \leq (\Gamma - \liminf F_n)(y)$, because $y \in K$. □

Proposition 4.6.12. *[45, Theorem 7.4] Assume there exist a countably compact set $K \subseteq \mathbb{X}$ such that for every $n \in \mathbb{N}$*

$$\inf_{x\in K} F_n(x) = \inf_{x\in\mathbb{X}} F_n(x).$$

Then $\Gamma - \liminf F_n$ attains its minimum on $\mathbb{X}$ and

$$\min_{x\in\mathbb{X}}(\Gamma - \liminf F_n)(x) = \liminf_{n\to+\infty} \inf_{x\in\mathbb{X}} F_n(x).$$

If, in addition, F_n Γ-converges to F, then

$$\min_{x\in\mathbb{X}} F(x) = \lim_{n\to+\infty} \inf_{x\in\mathbb{X}} F_n(x).$$

Proof. From the Definition 4.6.9 point (LL) it follows, that

$$\inf_{x\in\mathbb{X}}(\Gamma - \liminf F_n)(x) \geq \liminf_{n\to+\infty} \inf_{x\in\mathbb{X}} F_n(x).$$

Having in mind the previous Proposition, we see that

$$\inf_{x\in\mathbb{X}}(\Gamma - \liminf F_n)(x) \leq \min_{x\in K}(\Gamma - \liminf F_n)(x) \leq \liminf_{n\to+\infty} \inf_{x\in\mathbb{X}} F_n(x),$$

hence

$$\inf_{x\in\mathbb{X}}(\Gamma - \liminf F_n)(x) = \min_{x\in K}(\Gamma - \liminf F_n)(x) = \liminf_{n\to+\infty} \inf_{x\in\mathbb{X}} F_n(x).$$

In case when the sequence F_n Γ-converges, we see that the above equality, together with

$$\inf_{x\in\mathbb{X}}(\Gamma - \liminf F_n)(x) \geq \limsup_{n\to+\infty} \inf_{x\in\mathbb{X}} F_n(x),$$

finish the proof. □

Definition 4.6.13 (Coercivity & Equicoercivity)**.** We say that the functional $F : \mathbb{X} \to \overline{\mathbb{R}}$ is coercive, whenever for every $t \in \mathbb{R}$ there exist closed and countably compact set K_t such that $\{x \in \mathbb{X} : F(x) \leq t\} \subseteq K_t$.

We say that the sequence of functionals $F_n : \mathbb{X} \to \overline{\mathbb{R}}$ is equicoercive, whenever for every $t \in \mathbb{R}$ there exist closed and countably compact set K_t such that for every n, $\{x \in \mathbb{X} : F_n(x) \leq t\} \subseteq K_t$.

Proposition 4.6.14. *[45, Theorem 7.8] Assume that the sequence of functionals $F_n : \mathbb{X} \to \overline{\mathbb{R}}$ is equicoercive. Then $\Gamma - \liminf F_n$ and $\Gamma - \limsup F_n$ are coercive and*

$$\min_{x \in \mathbb{X}} (\Gamma - \liminf F_n)(x) = \liminf_{n \to +\infty} \inf_{x \in \mathbb{X}} F_n(x).$$

Moreover, in F_n Γ-converges to F, then

$$\min_{x \in \mathbb{X}} F(x) = \lim_{n \to +\infty} \inf_{x \in \mathbb{X}} F_n(x).$$

Definition 4.6.15 (ε-minimizer)**.** Let $F : \mathbb{X} \to \overline{\mathbb{R}}$ be a functional and let $\varepsilon > 0$. We say that a point $x_\varepsilon \in \mathbb{X}$ is an ε-minimizer of F, whenever

$$F(x_\varepsilon) \leq \max\big(\inf_{x \in \mathbb{X}} F(x) + \varepsilon, \frac{-1}{\varepsilon} \big).$$

In opposition to a minimizer, an ε-minimizer always exists. Let us also note that if $\inf_{x \in \mathbb{X}} F(x) > -\infty$ and ε is sufficiently small, we can use just $\inf_{x \in \mathbb{X}} F(x) + \varepsilon$ instead of $\max\big(\inf_{x \in \mathbb{X}} F(x) + \varepsilon, \frac{-1}{\varepsilon}\big)$.

Proposition 4.6.16. *[45, Corollary 7.17] Let $F_n : \mathbb{X} \to \overline{\mathbb{R}}$ be a sequence of functionals. For every $n \in \mathbb{N}$ let x_n be an ε_n-minimizer of F_n, where ε_n is a sequence of non-negative real numbers converging to 0. If $x_n \to x$ in $\mathbb{X}$, then x is a minimizer of both $\Gamma - \liminf F_n$ and $\Gamma - \limsup F_n$ and*

$$(\Gamma - \liminf F_n)(x) = \liminf_{n \to +\infty} F_n(x_n); \ (\Gamma - \limsup F_n)(x) = \limsup_{n \to +\infty} F_n(x_n).$$

Proposition 4.6.17. *[45, Corollary 7.20] Let $F_n : \mathbb{X} \to \overline{\mathbb{R}}$ be a sequence of functionals and F – its $\Gamma - \lim$. For every $n \in \mathbb{N}$ let x_n be an ε_n-minimizer of F_n, where ε_n is a sequence of non-negative real numbers converging to 0. If x is a cluster point of x_n, then x is a minimizer of F and*

$$F(x) = \limsup_{n \to +\infty} F_n(x_n).$$

If furthermore $x_n \to x$ in $\mathbb{X}$, then

$$F(x) = \lim_{n \to +\infty} F_n(x_n).$$

Proposition 4.6.18. *[45, Corollary 7.24] Let $F_n : \mathbb{X} \to \overline{\mathbb{R}}$ be an equicoercive sequence of functionals and F – its $\Gamma - \lim$. Assume F possesses an unique minimizer x_0. For every $n \in \mathbb{N}$ let x_n be an ε_n-minimizer of F_n, where ε_n is a sequence of non-negative real numbers converging to 0. Then $x_n \to x_0$ and $F_n(x_n) \to F(x_0)$.*

Now we focus our attention on translating the topologically involved definition of Γ-convergence into the language of converging sequences. For that purpose we introduce the so-called first axiom of countability of a space $\mathbb{X}$. The definition reads as follows.

Definition 4.6.19 (The first axiom of countability)**.** Let $\mathbb{X}$ be a topological space. Let $\mathscr{U}(x)$ be the set of all open neighbourhoods of x. We will say that $\mathbb{X}$ satisfies the first axiom of countability, whenever for every $x \in \mathbb{X}$ there exist a sequence of neighbourhoods $U_n \in \mathscr{U}(x)$ such that for every $U \in \mathscr{U}(x)$ there exists $i \in \mathbb{N}$ such that $U_i \subseteq U$.

Proposition 4.6.20. *[45, Proposition 8.1] Assume that $\mathbb{X}$ satisfies the first axiom of countability. Assume that $F_n : \mathbb{X} \to \overline{\mathbb{R}}$ is a sequence of functionals. Then $\underline{F} = (\Gamma - \liminf F_n)$ if and only if the following two conditions hold.*

(a) For every $x \in \mathbb{X}$ and every $x_n \to x$ we have

$$\underline{F}(x) \leq \liminf_{n\to+\infty} F_n(x_n).$$

(b) For every $x \in \mathbb{X}$ there exists $x_n \to x$ such that

$$\underline{F}(x) \geq \liminf_{n\to+\infty} F_n(x_n).$$

Analogously, $\overline{F} = (\Gamma - \limsup F_n)$ if and only if the following two conditions hold.

(c) For every $x \in \mathbb{X}$ and every $x_n \to x$ we have

$$\overline{F}(x) \leq \limsup_{n\to+\infty} F_n(x_n).$$

(d) For every $x \in \mathbb{X}$ there exists $x_n \to x$ such that

$$\overline{F}(x) \geq \limsup_{n\to+\infty} F_n(x_n).$$

Under the assumption of the first axiom of countability, having in mind the above Proposition, we see that the definition given below is equivalent to the classic one, i.e. Definition 4.6.9.

Definition 4.6.21 (Γ-convergence in spaces satisfying the first axiom of countability)**.** Let $\mathbb{X}$ be a topological space satisfying the first axiom of countability and $F_n : \mathbb{X} \to \overline{\mathbb{R}}$ be a sequence of functionals. We will say that $F = \Gamma - \lim_{n\to+\infty} F_n$, whenever for every $x \in \mathbb{X}$ the following two conditions hold.

(LB) for any $x_n \to x$ we have $\liminf F_n(x_n) \geq F(x)$;

(UB) there exists such $x_n \to x$ that $\limsup F_n(x_n) \le F(x)$.

In the literature, the conditions (LB) and (UB) are often referred as 'the lower bound' and 'the upper bound' respectively. We naturally extend this definition to the families of functionals.

Definition 4.6.22. Given a family of functionals $F_r : \mathbb{X} \to \overline{\mathbb{R}}$, we say that (F_r) Γ-converges to the functional F, as $r \to \infty$, if for every $(r_n) \to +\infty$ the sequence (F_{r_n}) Γ-converges to F.

Let us close this part of the section with a very useful fact, describing the possibility to use the sequential variant of Γ-convergence in the case of Banach spaces with weak topology.

Proposition 4.6.23. *[45, Proposition 8.7] Assume that $\mathbb{X}$ is a Banach space and its dual $\mathbb{X}^\star$ is separable. Then there exists a metric d on $\mathbb{X}$ such that the topology induced by d coincides with the weak topology on every norm-bounded subset of $\mathbb{X}$.*

The above Proposition shows that every norm-bounded set in $\mathbb{X}$ with the weak topology is in fact a metric space. Every metric space satisfies the first axiom of countability (for any point x it is sufficient to take balls with positive, rational radia, centred at x). Since then, the Definitions 4.6.9 and 4.6.21 are equivalent on them. Having in mind, that every weakly convergent sequence in Banach space is norm-bounded, the aforementioned equivalence in fact holds on the whole Banach spaces with weak topology, because the sequences $x_n \rightharpoonup x$ from the Definition 4.6.21 belong to a fixed norm-bounded set in $\mathbb{X}$.

We remind that an easy variant of Banach-Alaoglu Theorem guarantees that the weak compactness of closed balls in $\mathbb{X}$ is provided by separability of $\mathbb{X}^\star$. As an easy example of the space $L^1([0,1])$ shows, the assumption of separability of $\mathbb{X}^\star$ is not negligible.

What is left to explain is how the notion of Γ-convergence is used in the thin structures setting. This is in fact non-trivial because performing the Γ-convergence directly is impossible. The visible obstacle is that $\mathbb{X}$ in the definition of the $\Gamma - \lim$ is a fixed topological space, while we want to perform it with respect to the topology of $L^\Psi(\Omega(\varepsilon))$. Our topological space depends on the parameter of convergence (ε), which is not permitted in the Γ-convergence. To fix this problem, a certain scaling operation is required.

Instead of working with functions $v(x_\alpha, x_3)$ defined $\Omega(\varepsilon) = \omega \times (-\varepsilon, +\varepsilon)$, we will use scaling $u(x_\alpha, x_3) \stackrel{\text{def}}{=\!=} v(x_\alpha, \varepsilon x_3)$, where u is now defined on $\Omega(1) = \Omega$.

4.6.3 The BV spaces and sets of finite perimeter

We present the chosen definitions and facts about BV spaces. We follow the classical presentation given in [9, Chapter 3]. The presented theory is classical and hence we omit the proofs. Each of them can be found in [9].

In what below, $\Omega \subset \mathbb{R}^n$ is open and of finite measure, that is $\mathscr{L}^n(\Omega) < +\infty$. The real-valued function ϕ belongs to $C_c^1(\Omega)$, which means that it is at least once differentiable and supported in a compact subset of Ω.

We begin with the definition of the space $BV(\Omega)$.

Definition 4.6.24. Let $u \in L^1(\Omega)$. We say that u is a `function of a bounded variation`, whenever there exist finite, $\mathbb{R}^n$-valued Radon measures $D_1u, D_2u, \ldots, D_nu$ such that

$$\int_\Omega u\frac{\partial \phi}{\partial x_i}dx = -\int_\Omega \phi dD_iu\ \forall_{\phi\in C_c^1(\Omega)}.$$

The set of all functions of bounded variations will be called the `BV space` and denoted $BV(\Omega)$.

As it is easily seen, it contains $W^{1,1}(\Omega)$ and the inclusion is strict. To see that, it is sufficient to take $\Omega = (-1,2)$ and $u = \chi_{(0,1)}$. We introduce the norm on $BV(\Omega)$ as $||u||_{BV(\Omega)} \stackrel{\text{def}}{=\!=} ||u||_{L^1(\Omega)} + |Du|(\Omega)$, where by $|\mu|(A)$ we mean the variation of the measure, i.e.

$$\sup\{\int_\Omega \text{div}\ \phi d\mu : \phi \in C_c^1(\Omega), \phi \leq 1\}.$$

Such defined $BV(\Omega)$ is a Banach space.

We introduce an essential property of the variation of the measure $|Du|$.

Proposition 4.6.25 (Remark 3.5 in [9])**.** *. The mapping $L^1(\Omega) \ni u \mapsto |Du|(\Omega)$ is lower semicontinuous with respect to the strong topology of $L^1(\Omega)$.*

In other words, the above Proposition shows that whenever $u_j \to u$ in $L^1(\Omega)$, then $\liminf_j |Du_j|(\Omega) \geq |Du|(\Omega)$.

The fact below shows however that the norm topology in $BV(\Omega)$ is too strict for some statements to hold.

Theorem 4.6.26 (Theorem 3.9 in [9])**.** *Let $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if and only if there exists a sequence $\phi_j \in C_c^1(\Omega)$ such that*

$$\phi_j \xrightarrow{L^1(\Omega)} u \text{ and } V \stackrel{def}{=\!=} \sup_j \int_\Omega |\nabla \phi_j| dx < +\infty.$$

Furthermore, the least possible V in the formula above is $|Du|(\Omega)$.

Knowing that the smooth functions are dense in Sobolev spaces, it is visible that the above Theorem may not be naturally sharpened. To be more precise, one cannot expect that the gradients of ϕ_j will converge strongly in $L^1(\Omega)$ to Du because Du needs not to be an element of $L^1(\Omega)$. However, the following Proposition explains that some weaker notion of convergence appears.

Proposition 4.6.27 (Proposition 3.13 in [9])**.** *Assume that the sequence $u_j \in BV(\Omega)$ is norm-bounded in $BV(\Omega)$ and $u_j \xrightarrow{L^1(\Omega)} u \in BV(\Omega)$ Then $Du_j \stackrel{\star}{\rightharpoonup} Du$ in the space $\mathcal{M}(\Omega)$.*

The two facts above show that another notion of convergence is worth considering.

Definition 4.6.28. Let $u_j, u \in BV(\Omega)$. We say that u_j `converges` BV`-weakly-`$\star$ to u, whenever $u_j \overset{L^1(\Omega)}{\longrightarrow} u$ and $Du_j \overset{\star}{\rightharpoonup} Du$ in $\mathcal{M}(\Omega)$.

Now we would like to distinguish a certain class of domains Ω.

Definition 4.6.29. We say that Ω is an `extension domain`, whenever there exists an operator $T : BV(\Omega) \to BV(\mathbb{R}^n)$ such that

(i) $Tu \equiv 0$ on $\mathbb{R}^n \setminus \Omega$ for every $u \in BV(\Omega)$;

(ii) $|DTu|(\partial\Omega) = 0$ for every $u \in BV(\Omega)$;

(iii) for every $p \in [1, +\infty]$ the operator T, restricted to $W^{1,p}(\Omega)$, induces a linear map

$$\tilde{T} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n).$$

A typical example of an extension domain is Ω with Lipschitz boundary (see [9, Proposition 3.21]). For extension domains, several natural facts hold.

Theorem 4.6.30 (Theorem 3.23 in [9])**.** *Every norm-bounded sequence u_j in $BV(\Omega)$ admits a subsequence (still denoted by u_j) such that $u_j \overset{L^1(\Omega)}{\longrightarrow} u$ for some $u \in L^1(\Omega)$.*
If Ω is an extension domain, we may take $u \in BV(\Omega)$ and u_j converges BV-weakly-$\star$.

We introduce the notation $u_\Omega \overset{\text{def}}{=\!=} \dfrac{1}{|\Omega|} \int_\Omega u(x)dx$ and present a typical Poincaré inequality for BV spaces.

Theorem 4.6.31 (Theorem 3.44 in [9])**.** *Let Ω be a connected extension domain. Then, there exists a constant C such that*

$$\int_\Omega |u - u_\Omega| < C|Du|(\Omega).$$

As usual, from the Poicaré inequality, one may deduce an embedding Theorem.

Theorem 4.6.32 (Corollary 3.49 in [9])**.** *Let $p \in [1, \frac{n}{n-1}]$. Let Ω be a connected extension domain. Then every norm-bounded subset of the space $BV(\Omega)$ is a bounded subset of $L^p(\Omega)$. Furthermore, if $p < \frac{n}{n-1}$, the closed and norm-bounded subsets of $BV(\Omega)$ are compact in $L^p(\Omega)$.*

Let us now focus our attention on the particular case of characteristic functions. We assume that E is a measureable subset of $\mathbb{R}^n$. We present a natural definition of a set of finite perimeter.

Definition 4.6.33. For any open domain $\Omega \subset \mathbb{R}^n$ we define the perimeter of E in Ω as

$$P(E,\Omega) \stackrel{\text{def}}{=\!=} |D\chi_E|(\Omega).$$

We say that the set E is of `finite perimeter in` Ω, whenever $P(E,\Omega)$ is finite.

When dealing with the sets of finite perimeter, we will use the notion of convergence in measure. Namely, for any two sets A, B we define $A\Delta B \stackrel{\text{def}}{=\!=} (A \setminus B) \cup (B \setminus A)$ and say that $E_j \to E$ in measure if and only if $|E_j \Delta E| \to 0$. Of course, such a convergence is equivalent to the strong convergence of χ_{E_j} in $L^1(\mathbb{R}^n)$.

We close the Chapter with the essential result due to Federer [56, Theorem 4.5.11]. Before we state the Theorem, let us introduce few definitions dealing with a concept of a boundary of a measureable set E. As usual, the closure of E will be denoted by $\overline{E}$ and a ball centred at x and of radius r will be denoted by $B(x,r)$.

Definition 4.6.34. Let E be a set of finite perimeter in Ω. The `reduced boundary of` E, denoted by $\mathcal{F}E$, is defined as the collection of all $x \in \Omega \cap \overline{E}$ such that

$$\nu_E(x) \stackrel{\text{def}}{=\!=} \lim_{\varepsilon \to 0^+} \frac{D\chi_E(B(x,\varepsilon))}{|D\chi_E|(B(x,\varepsilon))}$$

exists in $\mathbb{R}^n$ and satisfies $|\nu_E(x)| = 1$.

The function $\nu_E : \mathcal{F}E \to \partial B(0,1)$ will be called `the generalised inner normal to` E.

For $t \in [0,1]$ and $x \in \mathbb{R}^n$ we say that the point x is of `density` t `in` E, whenever

$$\lim_{\varepsilon \to 0^+} \frac{|B(x,\varepsilon) \cap E|}{|B(x,\varepsilon)|} = t.$$

The set of such points will be denoted by E^t.

We define the `essential boundary of` E as $\partial^\star E \stackrel{\text{def}}{=\!=} \mathbb{R}^n \setminus (E^0 \cup E^1)$.

Let us set one more notation, namely the k-dimensional Hausdorff measure will be denoted by $\mathcal{H}^k$. Now we are ready to formulate the classical Federer Theorem [56, Theorem 4.5.11].

Theorem 4.6.35 (Theorem 4.5.11 in [56])**.** *. Let E be a set of finite perimeter in Ω. Then*

$$\mathcal{F}E \cap \Omega \subset E^{1/2} \subset \partial^\star E$$

and

$$\mathcal{H}^{n-1}\big(\Omega \setminus (\mathcal{F}E \cup E^0 \cup E^1)\big) = 0.$$

At the end, we present a partial statement of the Federer Theorem, just dealing with the part of our applications, i.e. we restrict our attention to the topological boundary.

Corollary 4.6.36 (Proposition 3.62 in [9])**.** *. Let E be open and $\mathcal{H}^{n-1}(\partial E) < +\infty$. Then E has a finite perimeter and $|D\chi_E| = f(x)\mathcal{H}^{n-1}_{|\partial E}$, where the latter stays for the measure $\mathcal{H}^{n-1}$ cut to ∂E and $f(x) \leq 1$ almost everywhere in the sense of $\mathcal{H}^{n-1}_{|\partial E}$. Moreover, whenever E has a Lipschitz boundary, we have $f(x) \equiv 1$ a.e. in the sense of $\mathcal{H}^{n-1}_{|\partial E}$.*

Chapter 5

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