# On quantum curves, Airy structures and supersymmetry 

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## Author's declaration:

I hereby declare that this dissertation is my own work.

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## Supervisor's declaration:

The dissertation is ready to be reviewed.


#### Abstract

In this thesis we develop theories of quantum curves, topological recursion and quantum Airy structures. These notions and various partition functions that they compute are intricately related with each other. They enable computation of various interesting invariants in mathematics and physics. Two main aspects of this thesis are the following: mathematically rigorous treatment of the quantum curves and generalisation of these notions to the supersymmetric case.

We start our considerations by studying the relationship between Virasoro singular vectors and quantum curves. This association, recently introduced in the context of random matrices and related to the topological recursion, was subsequently examined by me and my collaborators from the point of view of Conformal Field Theory (CFT). A mathematical language of CFT is the theory of Vertex Operator Algebras (VOAs). In this thesis we conduct a mathematically rigorous construction of the quantum curves originating from singular vectors using VOAs. We define wave functions using operators intertwining between three modules over VOA. Action of the Virasoro algebra on the wave functions is defined using the action of this algebra on one of the modules. This action is used to derive the representation of the Virasoro algebra on the space of generalised wave functions, which is a one parameter deformation of the representation derived using CFT methods. Using our definitions we prove Schrödinger equations coming from singular vectors.

Supersymmetric extensions of CFT can be divided into Neveu-Schwarz sector and Ramond sector, equipped with corresponding super extensions of the Virasoro algebra and admitting singular vectors. Having discussed the bosonic (non-supersymmetric) relation between quantum curves and singular vectors, in this thesis we present a supersymmetric extension of the correspondence between singular vectors and quantum curves. This includes cases of the Neveu-Schwarz and Ramond algebras. Here we present CFT approach.

Motivated by the study of quantum curves, their relation to the topological recursion, as well as the supersymmetric extension, we devote the last part of the thesis to supersymmetric generalisation of quantum Airy structures, which provide a reformulation and extension (at least in the nonsupersymmetric case) of the topological recursion. In this formalism one considers a family of quadratic Hamiltonians forming closed algebra under the Poisson bracket. After quantising them the differential operators are obtained, giving rise to the differential equations annihilating the partition function. These operators form a Lie algebra under the usual commutator. We extend the definition of the quantum Airy structures by including Grassman odd variables, defining super quantum Airy structures (SQASs). We prove the theorem about existence and uniqueness of the free energies, being solutions to the equations coming from SQASs. To this aim we use a super analog of the Poincaré lemma. We also present recursion relations on free energies as well as the constrains on the tensors defining the SQASs coming from the Lie superalgebra constraints. QAS can be also examined as symplectic representations of the underlying Lie algebra with some additional structure. We develop theory originating in the decomposition of those representations into weight spaces. We study examples.


Keywords: quantum curves, topological recursion, quantum Airy structures, random matrices, singular vectors, Virasoro algebra, Vertex Operator Algebras, supersymmetry.

AMS Subject Classification: 81R10, 81T40, 05A15, 17B69, 15B52, 17B68, 81Q60, 17B10, 81T30.

## Streszczenie

W pracy tej rozwijamy teorie krzywych kwantowych, topologicznej rekurencji i kwantowych struktur Airy. Pozwalają one na obliczenia różnych niezmienników w matematyce oraz fizyce, zakodowanych w funkcjach partycji. Pojęcia te oraz wspomniane funkcje partycji są pomiędzy sobą misternie powiązane. Dwa główne aspekty tej pracy są następujące: pierwszym jest matematycznie ścisłe wyprowadzenie wyników dotyczących krzywych kwantowych, drugim natomiast supersymmetryczne uogólnienie wspomnianych pojęć.

Nasze rozważania rozpoczynamy od związku pomiędzy wektorami osobliwymi algebry Virasoro oraz krzywymi kwantowymi. Związek ten, wprowadzony pierwotnie w kontekście macierzy losowych i związany z topologiczną rekurencją, był następnie przedstawiony przeze mnie oraz moich współpracowników z punktu widzenia konforemnej teorii pola (CFT). Matematycznym językiem CFT jest teoria algebr operatorów wierzchołkowych (VOAs). W pracy tej przeprowadzamy matematycznie ścisłą konstrukcję krzywych kwantowych pochodzących z wektorów osobliwych używając VOAs. Definiujemy funkcje falowe używając operatorów przeplatających pomiędzy trzema modułami nad VOAs. Działanie algebry Virasoro na funkcji falowej jest zdefiniowane za pomocą działania tej algebry na wektorach jednego z modułów. Jest ono następnie wykorzystane do wyprowadzenia reprezentacji algebry Virasoro na przestrzeni uogólnionych funkcji falowych. Reprezentacja ta stanowi jednoparametrową deformację reprezentacji otrzymanej przy wykorzystaniu metod CFT. Posługując się tymi definicjami dowodzimy pochodzącego z wektorów osobliwych równania Schrödingera.

Supersymetryczne rozszerzenia CFT można podzielić na sektor Neveu-Schwarza oraz sektor Ramonda, wyposażone w odpowiednie super rozszerzenia algebry Virasoro oraz posiadajace własne wektory osobliwe. W pracy tej prezentujemy supersymetryczną wersję odpowiedniości pomiędzy wektorami osobliwymi oraz krzywymi kwantowymi. Dokonujemy tego zarówno w sektorze NeveuSchwarza, jak i w sektorze Ramonda. Ograniczamy się tutaj do podejścia w języku CFT.

Zmotywowani badaniami krzywych kwantowych, ich związku z topologiczną rekurencją, a także rozszerzeniem supersymmetrycznym, ostatnią część tej pracy poświęcamy supersymetrycznemu uogólnieniu kwantowych struktur Airy, które stanowią przeformułowanie oraz uogólnienie (przynajmniej w przypadku nie supersymetrycznym) topologicznej rekurencji. W formalizmie kwantowych struktur Airy rozważana jest rodzina kwadratowych Hamiltonianów stanowiąca algebrę ze względu na nawias Poissona. Po skwantowaniu otrzymujemy operatory różniczkowe oraz zadane przez nie równania różniczkowe anihilujące funkcję partycji. Operatory te tworzą algebrę Liego ze względu na standardowy komutator. Rozszerzamy definicję kwantowych struktur Airy wprowadzając nieparzyste zmienne Grassmana, co prowadzi nas do definicji super kwantowych struktur Airy (SQASs). Dowodzimy twierdznie o istnieniu i jednoznaczności wolnych enegrii, które są rozwiązaniem równań zadanych przez SQASs. W tym celu wykorzystujemy super analog lematu Poincaré. Prezentujemy także równania rekurencyjne na swobodne energie oraz więzy na tensory definiujęce SQASs pochodzące od warunków wynikająch z aksjomatów superalgebr Liego. Kwantowe struktury Airy mogą być także badane jako symplektyczne reprezentacje algebry Liego z pewną dodatkową strukturą. Poprzez rozkład tych reprezentacji na przestrzenie wag rozwijamy teorię tych struktur. Rozważania ilustrujemy odpowiednimi przykładami.

Słowa kluczowe: krzywe kwantowe, topologiczna rekurencja, kwantowe struktury Airy, macierze losowe, wektory osobliwe, algebra Virasoro, algebry operatorów wierzchołkowych, supersymetria.

Klasyfikacja tematyczna AMS: 81R10, 81T40, 05A15, 17B69, 15B52, 17B68, 81Q60, 17B10, 81T30.

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## Chapter 1

## Introduction

This PhD thesis is devoted to the study of various concepts originating in string theory. String theory is a candidate for a theory unifying all known forces in modern physics: electromagnetic interactions, weak and strong interactions and gravitational interactions. However string theory became also a source of inspiration and conjectures in mathematics. Quantities appearing in physics or mathematics can be encoded in the generating series, such as partition functions. Various techniques have been developed in order to compute coefficients of those series. In this thesis we study interrelated theories of quantum curves, topological recursion and quantum Airy structures and various partition functions that they compute.

Topological recursion [25 26] is a procedure which associates recursively to a given spectral curve (a complex curve with a pair of meromorphic functions) a family of invariants. Those invariants are meromorphic symmetric differential forms $\omega_{g, n}$ on the base curve, indexed by two natural numbers $g$ and $n$. They are defined recursively by the formula 2.20. Topological recursion was applied in a variety of problems. Originally it first appeared in the study of the matrix models. Subsequently it turned out that it computes various invariants in string theory and enumerative geometry, such as Gromov-Witten invariants, Hurwitz numbers or volumes of the moduli spaces of curves. It is also conjectured that it can be used to compute knot invariants. Such a broad scope of applications of topological recursion is very surprising and motivates a vivid research in this area.

Quantum Airy structures (QASs) [23], [43] provide a reformulation and generalisation of the topological recursion. In this formalism one considers a family of quadratic Hamiltonians forming a closed algebra under the Poisson bracket. After quantising them differential operators are obtained, giving rise to the differential equations annihilating the partition function. These operators form a Lie algebra under the usual commutator. Particular choice of the QASs, based on the Virasoro algebra, has been shown to compute the invariants encoded in the differential forms $\omega_{g, n}$ coming from the topological recursion.

Quantum curves are objects which arise as a quantisation of complex curves. This can be understood as a passage from the ring of complex functions on the curve to its noncommutative version depending on an additional parameter $\hbar$. If the curve is a zero of a certain polynomial $P(x, y)$ then a quantum curve is a differential operator $\widehat{P}\left(x, \hbar \frac{\partial}{\partial x}\right)$. The wave function $\widehat{\psi}(x)$, which is a solution
to the Schödinger equation $\widehat{P} \widehat{\psi}(x)=0$, encodes the quantities which we are looking for. Among applications of the quantum curves let us mention Hitchin systems 20, 21, Gromov-Witten invariants (of $\mathbb{P}^{1}$ in $[22]$ ) and knot theory $\sqrt[30]{ }$. There exists a conjecture which relates topological recursion with the quantum curves [30]. This relation is given by a formula using which one can compute solution of the quantum curve equations using solution of the topological recursion. It was proven under certain conditions in [11. In the section 2.5.1 we discuss this conjecture in more detail.

Quantum curves can be also associated with the singular vectors in the Verma modules over the Virasoro algebra. Singular vectors appear in the degenerate Verma modules and are a source of the BPZ equations [5], which are differential equations on the correlation functions. In this thesis we analyze the relation between singular vectors and quantum curves form a mathematical point of view. Moreover, we present a generalisation of this relation to the supersymmetric case.

## QFT, CFT and string theory

Contemporary accepted physical models of the reality, such as standard model, are examples of quantum field theories (QFTs). Those theories are composed of quantum fields, which are responsible for the creation or annihilation of the particles. They tell us how to compute correlation functions, which integrated over the spacetime allow for computation of the transition amplitudes, measured in the accelerators. An important ingredients of such theories are symmetries. In classical mechanics Noether theorem relates to a symmetry a quantity, which is conserved during the evolution of the system. Quantum analog of Noether theorem are Ward identities, which are differential equations satisfied by correlation functions.

Conformal field theory (CFT) is a particular kind of QFT, whose symmetry group is composed of the conformal transformation of the spacetime. Exceptionally rich structure in CFT appears when the dimension of the underlying spacetime equals 2 . In this case the spacetime is a holomorphic curve and all holomorphic maps are conformal. It follows that the Witt algebra, the algebra of local conformal transformations, is infinite dimensional. In quantum setting we are interested in projective representations, which correspond to the representations of the central extension of the Witt algebra, called Virasoro algebra. Richness of 2-dimensional conformal symmetry, through Ward identities, gives rise to strong constraints on the correlation functions. Major applications of CFT are string theory and phase transitions in statistical mechanics. String theory replaces particles by tiny vibrating 1-dimensional strings, which as they move through the spacetime trace 2-dimensional surfaces. Transition amplitudes in string theory can be expressed via correlation functions on this surface. Moreover physics of the strings described by the Polyakov action is invariant under conformal transformations of this surface. This is how CFT arises in string theory. In order to obtain perturbative approximation of the correlation functions in QFT one considers sums of terms indexed by Feynman diagrams. In string theory such sum is replaced by an integral over the moduli space of curves $\overline{\mathcal{M}}_{g, n}$, where $g$ stands for the genus of the curve and $n$ for the number of punctures on the curve.

Correlation functions in CFT are related with the random matrices, i.e. sets of matrices with a
probability distribution. Such matrices are used as toy models in physics: they can be thought of as QFT based on spacetime of dimension 0 . They arise also in different dualities. One can also mention the relation of the matrix integrals with the intersection numbers on the moduli spaces of curves $\overline{\mathcal{M}}_{g, n}$ [42].

## Mathematics of CFT

Although experimentally successful, QFTs lacks solid mathematical foundations. On the other hand string theory cannot be currently checked in accelerators, but CFT can be introduced in a mathematically promising way. One of the notions used in this aim are vertex operator algebras (VOAs). Heuristically they can be described as algebras for which multiplication depends on a (formal) parameter $z$. Therefore product of any two vectors gives a formal power series with coefficients belonging to the algebra. Alternatively we can think of this product as a map which maps any vector $v$ to an endomorphism of the algebra defined by multiplying by $v$ (depending on a formal/complex parameter $z$ ): $Y(v, z) w=v \cdot w$. Those objects were first introduced by Borcherds in his proof of the Monstrous Moonshine [8]. Physically speaking the word vertex comes from the vertices in the Feynman diagrams, representing points, where localised interaction between particles takes place. In string theory such vertex is replaced by a punctured sphere. If we consider three punctures $0, z$ and $\infty$ the vertex operator $Y(v, z)$ will tell us how the state incoming at 0 is evolving into an outgoing state at $\infty$, depending on the intermediate state $v$ at $z$. Those states belong to vector spaces associated with each puncture, which are modules over VOA. When those modules are different from the vacuum module, we call the vertex operator also an intertwining operator.

Interactions between other QFTs and mathematics were also fruitful, stimulating research in such areas as knot invariants or the theory of invariants of manifolds: Casson invariant, Donaldson invariant, Floer homology. Let us cite Yi-Zhi Huang [34] from 2016:
"Quantum field theory has become an active research area in mathematics in the last forty years. Among all the quantum field theories, topological quantum field theory is the most successful in mathematics mainly because the state space of a topological quantum field theory is typically finite dimensional. Compared with topological quantum field theory, nontopological quantum field theories and the deep mathematical conjectures derived from these theories are much more difficult and are still quite distant from a complete mathematical understanding.

One of the most famous but also one of the most difficult problems on nontopological quantum field theories is the existence of four-dimensional quantum Yang-Mills theory and the mass gap problem. On the other hand, two-dimensional conformal field theory as the best understood nontopological quantum field theory has in fact been greatly developed and has also directly provided ideas and methods for the successful solutions of mathematical conjectures and problems. The study of two-dimensional conformal field theory will certainly also shed light on the other more difficult nontopological quantum field theories such as the four-dimensional Yang-Mills theory.

In a program of constructing and studying two-dimensional conformal field theories using the
representation theory of vertex operator algebras, the mathematical foundation of two-dimensional conformal field theory has been essentially established."

However, as the author of the above excerpt states later in his note, the construction of 2d CFT satisfying the axioms of Kontsevich, Segal, Moore and Seiberg is an open problem. The genus zero and genus one correlation functions can be obtained from VOAs, however "the construction of the higher-genus correlation functions is the main unsolved problem. If the axioms for conformal field theories are assumed, one can see that if a conformal field theory is constructed, then higher-genus correlation functions can be expanded as series obtained using intertwining operators. Since we have not constructed conformal field theories satisfying all the axioms, even though we can still write down these series using intertwining operators, we cannot use the axioms for conformal field theories to derive the convergence of these series."

## Quantum Curves

Quantum curves can be associated with the singular vectors in the Verma modules over the Virasoro algebra [44]. This relation originally developed from the point of view of matrix models, was later studied from the point of view of CFT. In this thesis we are going to perform a mathematically rigorous construction of the quantum curves originating from such vectors. To this aim we use mathematical language of CFT, namely VOAs. This approach can be motivated also by the following fact: quantum curves are related with the BPZ equations and in fact reduce to them for a special choices of the wave functions. On the other hand BPZ equations were mathematically tracked with a use of VOAs and intertwining operators (see 32 ).

## Matrix models

In [44] a relation between quantum curves and CFT was found. This relation allowed for an extension of the original concept of the quantum curves beyond the operators of the level 2 . This correspondence was introduced in the context of $\beta$-deformed random matrix integrals. Let us describe it here briefly. Let $V(\Delta)$ be the Virasoro Verma module with central charge $c$ and conformal dimension $\Delta$. A singular vector $v_{s} \in V(\Delta)$ is an element linearly independent from the highest weight vector $|\Delta\rangle \in V$, which satisfies equations $L_{n} v_{s}=0$ for $n>0$. Every such vector can be expressed using the Virasoro creation operators $L_{-n}, n>0$ acting on the vector $|\Delta\rangle$ :

$$
v_{s}=A\left(\left\{L_{-n}\right\}_{n>0}\right)|\Delta\rangle,
$$

where $A$ is a polynomial depending on the Virasoro creation operators and constants $c$ and $\Delta$. There exists representation of the Virasoro creation operators in the space of differential operators in the variables $x$ and $\left\{t_{i}\right\}_{i \geq 1}$ which takes the following form

$$
\begin{equation*}
\widehat{L}_{-1}^{M M}=\partial_{x}, \quad \widehat{L}_{-n}^{M M}=\frac{1}{\hbar^{2}(n-2)!}\left(\frac{1}{4} \partial_{x}^{n-2}\left(W^{\prime}(x)\right)^{2}+\frac{Q \hbar}{2} \partial_{x}^{n} W(x)+\partial_{x}^{n-2} \widehat{f}_{f}(x)\right) \quad \text { for } n \geq 2 \tag{1.1}
\end{equation*}
$$

where $\widehat{f}_{t}(x)=\hbar^{2} \sum_{n=0}^{\infty} x^{n} \sum_{m=n+2}^{\infty} m t_{m} \partial_{t_{m-n-2}}$ and $W(x)=\sum_{i=1}^{\infty} t_{i} x^{i}, \hbar$ being a formal parameter. Quantum curves are obtained, where the operator $A$ is expressed in this representation, giving a
differential operator $\widehat{A}$. This operator acts on the wave function

$$
\widehat{\psi}_{\alpha}(x)=\int \Delta\left(z_{1}, \ldots, z_{N}\right)^{2 \beta} \prod_{i=1}^{N}\left(z_{i}-x\right)^{-\frac{2 \alpha \sqrt{\beta}}{\hbar}} e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{i=1}^{N} W\left(z_{i}\right)} \mathrm{d} z_{1} \ldots \mathrm{~d} z_{N}
$$

where $\Delta\left(z_{1}, \ldots, z_{N}\right)=\prod_{i<j}\left(z_{i}-z_{j}\right)$ is the Vandermonde determinant. The above expression is a generalisation of the expectation value of the characteristic polynomial in the random matrix model. It is also called $\alpha / \beta$-deformed matrix integral. Singular vectors in the Virasoro Verma module exists only for a specific value of the parameter $\alpha$ :

$$
\alpha_{r, s}=\frac{r-1}{2 \sqrt{\beta}}-\frac{s-1}{2} \sqrt{\beta},
$$

where $r$ and $s$ are positive integers. For those values of the parameter and for $\Delta=\alpha(Q-\alpha)$, where $Q=\frac{1}{\sqrt{\beta}}-\sqrt{\beta}$ quantum curves annihilate the wave function, giving rise to the Schrödinger equations

$$
\widehat{A}\left(\Delta\left(\alpha_{r, s}\right)\right) \widehat{\psi}_{\alpha_{r, s}}(x)=0
$$

Those equations were checked in [44] up to the order 5 of the level of the singular vectors. The representation (1.1) was also derived there.

## CFT

The correspondence between quantum curves and singular vectors correspondence was revisited from a more formal point of view in [14]. The whole construction was performed using the vertex operators from CFT and their correlation functions. Vertex operators used here are given by the formulas

$$
\begin{equation*}
I(|\alpha\rangle, x)=u^{\alpha} \exp \left(2 \alpha \sum_{j=1}^{\infty} \frac{x^{j}}{j} a_{-j}\right) \exp \left(-2 \alpha \sum_{j=1}^{\infty} \frac{x^{-j}}{j} a_{j}\right) x^{2 \alpha a_{0}}, \tag{1.2}
\end{equation*}
$$

where $a_{j}$ are generators of the Heisenberg algebra. It has been shown in [14] that the wave function introduced in the previous section can be expressed using the product of these vertex operators:

$$
\begin{equation*}
\widehat{\psi}_{\alpha}(x)=\int\langle V| I(|\alpha\rangle, x) \prod_{i=1}^{N} I\left(|-\sqrt{\beta}\rangle, z_{i}\right)|0\rangle \mathrm{d} z_{1} \ldots \mathrm{~d} z_{N}, \tag{1.3}
\end{equation*}
$$

where $\langle V|$ is an appropriately chosen closing state. The action of the Virasoro algebra is defined in the following way

$$
L_{-n} \cdot \widehat{\psi}_{\alpha}(x)=\int\langle V| L_{-n} \cdot I(|\alpha\rangle, x) \prod_{i=1}^{N} I\left(|-\sqrt{\beta}\rangle, z_{i}\right)|0\rangle \mathrm{d} z_{1} \ldots \mathrm{~d} z_{N}
$$

where

$$
L_{-n} \cdot I(|\alpha\rangle, x)=\frac{1}{2 \pi \sqrt{-1}} \int_{C(x)} \frac{1}{(y-x)^{n-1}} T(y) I(|\alpha\rangle, x) \mathrm{d} y
$$

integration is performed over a small circle $C(x)$ around the point $x \in \mathbb{C}$ and $T(y)=\sum_{n \in \mathbb{Z}} y^{n} L_{-n-2}$ is the stress-energy tensor. After tedious calculations this definition gives rise to the representation (1.1). Important ingredient of the construction was a map $S_{\alpha, Q}: V(\Delta) \rightarrow \mathcal{F}(\alpha)$ from the Verma module to the Fock module. This map has the property that it maps singular vectors 0: $S_{\alpha, Q}\left(v_{s}\right)=$ 0 . A special attention should be paid here to the passage from the interpretation of $x$ as a formal parameter to the interpretation as a point on the complex plane.

## VOAs

One of the tasks, which the author of this thesis tried to accomplish, was to present the relation between the Virasoro singular vectors and the quantum curves in a mathematically rigorous way. As we explained in the previous section, a good candidate for a mathematical formulation of CFT are VOAs. Henceforth we are going to use them to reach our aim. Roughly the idea of the application of the VOAs to quantum curves is to replace "vertex operators" appearing in the definition of the wave functions in [14] with the intertwining operators. In fact, particular example of the intertwining operator evaluated at the highest weight vector gives us the vertex operator (1.2) 32. Therefore our notation is consistent. Moreover, we can generalise wave functions by the formula:

$$
\widehat{\psi}_{\alpha}(v, x) \simeq \int\langle V| I(v, x) \prod_{i=1}^{N} I\left(|-\sqrt{\beta}\rangle, z_{i}\right)|0\rangle \mathrm{d} z_{1} \ldots \mathrm{~d} z_{N}
$$

where $I(\cdot, z): M_{1} \rightarrow \operatorname{Hom}\left(M_{2}, M_{3}\right) \otimes \mathbb{C}\{z\}$ are intertwining operators between Fock modules ( $M_{i}=\mathcal{F}(\eta)$ for various values of $\eta$ ) and $v \in \mathcal{F}(\alpha)$ is any vector. An issue necessary for a precise mathematical definition is the specification of an appropriate integration cycle. Such a cycle was constructed in [54]. This result was more briefly presented in [38], which is our direct reference. The cycle $\Gamma$ we use is included in the definition of the screening operator, roughly given by:

$$
\Sigma_{N, \beta, b} \simeq \int_{\Gamma} \prod_{i=1}^{N} I\left(|-\sqrt{\beta}\rangle, z_{i}\right) \mathrm{d} z_{1} \ldots \mathrm{~d} z_{N} .
$$

The precise definition of the generalised wave function is given by the following formula:

$$
\begin{equation*}
\widehat{\psi}_{\alpha}(v, x)=\Phi_{\alpha-N \sqrt{\beta}}^{\hbar} \Sigma_{N, \beta, b} I(v, x)|0\rangle, \tag{1.4}
\end{equation*}
$$

where $\Phi_{\alpha-N \sqrt{\beta}}^{\hbar}$ is appropriate homomorphism playing the rôle of the closing state $\langle V|, v \in \mathcal{F}(\alpha)$ is any vector and $\alpha=Q+N \sqrt{\beta}+b \frac{1}{\sqrt{\beta}}$. Let $\mathcal{W}_{\alpha}$ be the space of the generalised wave functions. Extended domain of the generalised wave functions allows a different definition of the action of the Virasoro algebra. Namely we set

$$
\begin{equation*}
\widehat{L}_{n} \cdot \widehat{\psi}_{\alpha}(x)(v)=\widehat{\psi}_{\alpha}(x)\left(L_{n} v\right) . \tag{1.5}
\end{equation*}
$$

Let $S_{\alpha, Q}: V(\Delta) \rightarrow \mathcal{F}(\alpha)$ be the map from the CFT approach, whose kernel consists of the singular vectors. From our construction it is clear that the generalised wave function evaluated at $S_{\alpha, Q}\left(v_{s}\right)=$ 0 gives $\widehat{\psi}_{\alpha}(x)\left(S_{\alpha, Q}\left(v_{s}\right)\right)=0$. The only difficulty lies in the derivation of the representation. This representation is obtained with the help of two lemmas, where the following representation of the Virasoro algebra is derived $\prod^{1}$

$$
\begin{align*}
I\left(L_{-1} v, z\right) & =\partial_{x} I(v, z) \\
I\left(L_{-n} v, x\right) & =\frac{1}{(n-2)!}: \partial_{x}^{n-2} T(x) I(v, x): \tag{1.6}
\end{align*}
$$

[^0]One can use then the results (1.6) applied to the definitions (1.4) and (1.5) to obtain Schrödinger equations (Theorem 4.2.2). As an outcome we obtain a more general representation (Theorem 4.2.1 that the one presented in 14 . If we denote by $\widehat{L}_{-n}^{C F T}$ the representation obtained using CFT methods (equal to the representation $\widehat{L}_{-n}^{M M}$ from the matrix models 1.1) than our representation takes the following form:

$$
\widehat{L}_{-n}=\widehat{L}_{-n}^{C F T}+\gamma \sum_{m=n-2}^{\infty}\binom{m}{n-2}(m+2) t_{m+2} x^{m-n+2}
$$

where $\gamma=\hbar(\alpha-N \sqrt{\beta})$ is a deformation parameter. As a corollary we obtain a combinatorial identity (Proposition 4.3.1):

$$
\sum_{n=b}^{k-2-a}\binom{k-n-2}{a}\binom{n}{b}(2 n-k+2)=(b-a)\binom{k}{a+b+2} .
$$

for $a, b, k>0$ positive integers and $k \geq a+b+2$. Moreover, approach using VOAs has several other advantages, which we would like to list here:

- We define not only action of the Virasoro algebra, but also a compatible action of the Heisenberg algebra.
- This construction is mathematically clearer and does not need additional physical arguments and assumptions.
- We show in a clear way that the the building blocks are the correct form of the representation of the Virasoro creation operators acting not only on the wave function but also on its descendants. This means that we can apply those building blocks several times one after another. Hence we can apply them to any polynomial expression in the Virasoro creation operators.
- The computation of the building blocks, assuming (1.6) is simpler.

The author believes that the approach using VOAs can be further applied to other algebras with singular vectors, for example $\mathcal{W}$-algebras.

Let us discuss shortly the relation with the BPZ equations. In those equations certain differential operators (coming from the singular vectors) annihilate the correlation of the form

$$
\left\langle w_{1}\right| I(|\alpha\rangle, x) \prod_{i=1}^{N} I\left(|-\sqrt{\beta}\rangle, z_{i}\right)\left|w_{2}\right\rangle,
$$

which is very similar to the definition of the wave function $\widehat{\psi}_{\alpha}(|\alpha\rangle, x)$. However the representation one obtains in this case is different from the representation stated in the Theorem 4.2.1.

## Super Quantum Curves

Supersymmetry is a concept in theoretical physics, which anticipates a duality between elementary particles. In order to be consistent, string theory puts strong constraints on the number of the
dimensions of the spacetime. In the purely bosonic case this number is 26 . Supersymmetry allows reductions of this dimension to 10 . One of the major applications of string theory to mathematics is mirror symmetry, which comes from including supersymmetry. This symmetry postulates duality between varieties, which are 3 complex-dimensional Calabi-Yau manifolds complementary to the macroscopic 4 dimensions of the spacetime. Mathematical description of the supersymmetry includes coordinates, which are anti-commuting. They are called fermionic coordinates, whereas the commuting ones are called bosonic. In the spirit of the non-commutative geometry this means that one is considering ring with anticommutation property, but the underlying space does not necessarily make sense. The real spacetime comes from the bosonic part of the coordinate ring.

Supersymmetric extension of the relation between quantum curves and singular vectors was performed in [50] and [14]. There are various possibilities for such an extension, depending on the super algebra we are considering. In particular there are two $\mathcal{N}=1$ super extensions of the Virasoro algebra: Neveu-Schwarz algebra and Ramond algebra. In the paper [50] the first case was considered from the point of view of the super eigenvalue integrals. In [14] Ramond algebra was introduced into those consideration with the help of CFT formalism.

The author tried to extend the VOAs approach to the quantum curves in the supersymmetric setting, as in the bosonic case. However including fermions turned out to be more difficult task. This difficulty can be exemplified with a no-go theorem (Proposition 5.3.7) which excludes possibility of a certain straightforward approach. Nevertheless we will present in this thesis results from 14 concerning super quantum curves.

## Super quantum Airy structures

There are various attempts at defining supersymmetric version of the topological recursion. As an example one can mention works in the context of super eigenvalue models: [12, [48]. Here we follow a different approach: since QASs are a reformulation of the topological recursion, we generalise them to super quantum Airy structures (SQAS). As was already mentioned QASs consists of the differential operators. They take the following form:

$$
\begin{equation*}
\widehat{L}_{i}=\hbar \partial_{x_{i}}-\frac{1}{2} \sum_{a, b \in I} A_{a, b}^{i} x_{a} x_{b}-\hbar \sum_{a, b \in I} B_{a, b}^{i} x_{a} \partial_{x_{b}}-\frac{1}{2} \hbar^{2} \sum_{a, b \in I} C_{a, b}^{i} \partial_{x_{a}} \partial_{x_{b}}-\hbar D^{i} \tag{1.7}
\end{equation*}
$$

where $i \in I$. They give rise to the differential equations annihilating the partition function $\widehat{L}_{i} e^{F}=0$, where:

$$
F=\sum_{g \geq 0, n \geq 1} \frac{\hbar^{g-1}}{n!} \sum_{i_{1}, \ldots, i_{n} \in I} F_{g, n}\left(i_{1}, \ldots, i_{n}\right) x_{i_{1}} \cdots x_{i_{n}}
$$

The operators $\widehat{L}_{i}$ form a Lie algebra $\mathfrak{g}$ under the usual commutator:

$$
\begin{equation*}
\left[\widehat{L}_{i}, \widehat{L}_{j}\right]=\hbar \sum_{k} f_{i, j}^{k} \widehat{L}_{k} \tag{1.8}
\end{equation*}
$$

for some constants $f_{i j}^{k} \in \mathbb{C}$. Partition function encodes invariants, which one can compute using the topological recursion.

The idea of SQASs is the following: the definitions (1.7) and (1.8) stay the same, however a different meaning is given to the symbols. In (1.7) we allow $x_{a}$ to be Grassman odd anticommuting variables, and in (1.8) we replace the usual commutator with a super commutator. The algebra $\mathfrak{g}$ becomes a super Lie algebra. The partition function annihilated by those operators is required to be of even grading. Existence and uniqueness of the partition function also holds for the SQASs (Theorem 6.1.1). Its proof uses a supersymmetric version of the Poincaré lemma for the 1-forms (Lemma 6.1.1). One advantage, which differentiate SQASs from the even quantum Airy structures is the possibility of introducing supplementary fermionic variable without corresponding operator [10]. This extension allows construction of various examples, for example those based on Neveu-Schwarz algebra. In the section 6.1.4 we present the relations on the tensors $A, B, C$ and $D$ coming from the condition (1.8) and in the section 6.1.3 recursion equations on the coefficients $F_{g, n}$ coming from the condition $\widehat{L_{i} e^{F}}=0$ 10.

QASs can be examined from the point of view of the representation theory. As it has been shown in 23] a classical limit of a QAS corresponds to a symplectic representations $\rho: \mathfrak{g} \rightarrow \operatorname{End}(W)$ of the underlying Lie algebra with a Lagrangian embedding $j: \mathfrak{g} \rightarrow W$ satisfying certain condition. We develop theory originating in the decomposition of $W$ into weight spaces. Together with the assumption that $\mathfrak{g}$ is semi-simple this allows us to prove several lemmas (Lemmas 6.2.4, 6.2.5, 6.2.6, 6.2.3). We use this technique to study classical Airy structures based on the Lie algebra $\mathfrak{s l}(2) \oplus \mathfrak{s l}(2)$. We construct two such structures. Because they are finite-dimensional, they both can be quantised. In [23] an initial step towards classification of the QASs based on simple Lie algebras was done, and many cases of algebras were excluded. The author's intuition was that theory of QASs based on semi-simple Lie algebras is richer than theory based just on simple Lie algebras. In particular, as shown in the example of $\mathfrak{s l}(2) \oplus \mathfrak{s l}(2)$, there are more examples than just direct sums of the representations of the simple Lie algebras. Heuristically the reason is that semi-simple algebras are "more abelian", meaning that the constraints (1.8) are easier to satisfy (for an abelian algebra one can always construct QAS). Study of the QASs based on $\mathfrak{s l}(2) \oplus \mathfrak{s l}(2)$ is also an initial step towards study of the SQASs based on the algebra $\mathfrak{o s p}(3 \mid 2)$. Richness of the QASs based on semi-simple Lie algebras was recently confirmed in [31].

Finally we also study SQASs based on the $\mathfrak{o s p}(1 \mid 2)$ algebra. We decompose irreducible representation of such algebra into weight spaces. By dimensional considerations we show that they cannot support SQAS without additional fermionic degree of freedom. With this addition such SQAS can be constructed, as has been shown in [10].

There are several possibilities, where SAQSs could be applied, besides attempts to define super topological recursion. One idea is the study of the Gromov-Witten invariants of the manifold $X$ with the non vanishing odd cohomologies. Presence of such cohomologies requires fermionic variables to encode GW invariants into generating series. Constraints on such series were derived in [46] in the case where $X$ is a curve. Indeed, we obtain differential operators spanning super Lie algebra. However those operators do not constitute a SQAS, because the number of variables is higher than the number of operators. In particular they do not fix the partition function uniquely.

## Organisation of the thesis

This thesis presents results form the papers [14, [15] and [10], as well as the author's results unpublished before. Moreover it presents theories and results known previously, which serves as an introduction to the reader new into this field of research. This thesis is organised as follows:

- In the chapter 2, which serves as a motivation to what follows, we present known results about random matrix models, topological recursion and quantum curves.
- In the chapter 3 we discuss certain class of Lie algebras, with focus on the Heisenberg and Virasoro algebra. We also introduce Vertex Operator Algebras. Those results are also standard knowledge. Lemma (3.2.3 was proved by the author.
- In the chapter 4 we define wave functions using intertwining operators and screening charges and prove the main result about the construction of quantum curves, which includes deformed representation of the Virasoro algebra. We also obtain certain combinatorial identity. These are author's results and are going to be published separately 49].
- In the chapter 5 we introduce the concept of supersymmetry and extend the topics of the chapter 3 by including fermionic degrees of freedom. This material is also standard. Moreover, we present the results of [14. The new contribution of the author is Theorem 5.3.7 and Section 5.2.2.
- In the chapter 6 we discuss super quantum Airy structures. This chapter is based on [10], but also presents some other unpublished results of the author (presented in the Sections 6.2.2 and 6.2.3. with the exception of the case of $\mathfrak{o s p}(1 \mid 2)$, already discussed in [10]).


## Chapter 2

## Random matrices

In this chapter we give a short overview of the theory of random matrices (also called theory of matrix models). Random matrices serves as an introduction or illustration of the content of this thesis. More precisely our attention will focus on formal random matrix integrals.

In the section 2.1 we give a simple example, thought as motivation for what follows. It is based on Catalan numbers.

In the section 2.2 we give definitions of the perturbative expansions of random matrix integral, as well as of its $\beta$ and $\alpha / \beta$ deformations. We discuss loop equations, which are Ward identities for matrix integrals, and at the same time Virasoro constraints. We also define spectral curves.

In the section 2.3 we study matrix integrals from the point of view of the enumerative geometry. This includes expansion of the random matrix partition function in terms of the graphs and application of those functions in the computation of the intersection numbers on the moduli spaces of curves.

In the section 2.4 we define topological recursion for any spectral curve and show how it can be used to compute correlation functions of random matrices.

In the section 2.5 we briefly discuss quantisations of the spectral curves, which are quantum curves. We also present a conjecture relating quantum curves with topological recursion.

### 2.1 Simple example

Let us start with a simple example. Throughout this work the notion of formal power series reappears. These series provide us a method to collect in one object families of invariants. They often appear as asymptotic expansions of some functions. As an easy example we can consider Laplace transform of a real random variable $X$ :

$$
\varphi_{X}(t)=\mathbb{E} e^{t X}
$$

Under certain assumptions on $X$, asymptotic expansion of this function at 0 encodes in a formal power series all momenta of our variable:

$$
\varphi_{X}(t) \simeq \sum_{t=0}^{\infty} \frac{t^{k}}{k!} \mathbb{E} X^{k} .
$$

Let us specialise to the case of $X$ distributed according to the measure defined using the equation $\mathrm{d} \rho(x)=\frac{2}{\pi} \sqrt{1-x^{2}} \mathbf{1}_{|x| \leq 1} \mathrm{~d} \lambda(x)$, where $\mathrm{d} \lambda(x)$ is Lebesgue measure. This is the limit of the distributions of eigenvalues of Hermitian random matrices of size $n \times n$. Then it is known that its moments are given by the Catalan numbers:

$$
\begin{equation*}
\int x^{2 m} \mathrm{~d} \rho(x)=2^{-2 m} C_{m} \tag{2.1}
\end{equation*}
$$

where

$$
\begin{equation*}
C_{m}=\frac{1}{m+1}\binom{2 m}{m} \tag{2.2}
\end{equation*}
$$

These numbers can be characterised in various combinatorial ways. Let us write down their generating function in a form:

$$
\begin{equation*}
z(x)=\sum_{m=0}^{\infty} C_{m} x^{-2 m-1} \tag{2.3}
\end{equation*}
$$

It can be shown that this function satisfies algebraic equation 21]:

$$
\begin{equation*}
x=z(x)+\frac{1}{z(x)} . \tag{2.4}
\end{equation*}
$$

Therefore if we write down the asymptotic expansion of the inverse function to $x(z): z(x)=$ $\frac{1}{2}\left(x-\sqrt{x^{2}-4}\right)$ we would recover Catalan numbers. What can be seen in this example is the interplay of the theory of random matrices (2.1), combinatorics (2.2), analysis (2.3) and algebraic geometry (2.4). Geometric notions contain information about combinatorial (or enumerative) invariants. Later on we will encounter other examples of such interplay.

### 2.2 Definitions and basic results

Let us introduce basic definitions and results concerning matrix integrals.

### 2.2.1 Hermitian matrix model

In this section we consider the following matrix integral

$$
\int_{E} e^{\operatorname{Tr} V(X)-\operatorname{Tr}\left(X^{2}\right)} \mathrm{d} X,
$$

where $V(x)=\sum_{m=1}^{\infty} t_{m} x^{m}$ is called potential, $t_{i}$ 's are called times, $E \subset \operatorname{Mat}(N)$ is a subset of the space of complex $N \times N$ matrices and $\hbar \in \mathbb{C}$. We assume that this subset is measurable with respect
to the Lebesgue measure $\mathrm{d} X$ on $\operatorname{Mat}(N)$. Convergence of this integral depends on the values of $t_{i} \in \mathbb{C}$. However one can treat $t_{i}$ 's as formal variables and the integral as a formal power series an element of the ring $R=\mathbb{C}\left[\left[t_{1}, t_{2}, \ldots\right]\right]$. In the physics nomenclature expansions in power series with respect to the small parameter of deformation (for example $\frac{1}{N}$ for large matrices, where $N$ is the size of the matrix) is called perturbative approach. We adopt this name here. We will call such an integral partition function.

Definition 2.2.1 $A$ formal random matrix partition function $Z \in R$ for an ensemble $E \subset \operatorname{Mat}(N)$ is defined as:

$$
\begin{align*}
Z & =\sum_{n=0}^{\infty} \sum_{k_{1}, \ldots, k_{n} \geq 1} F\left(k_{1}, \ldots, k_{n}\right) t_{1}^{k_{1}} \ldots t_{n}^{k_{n}}, \\
F\left(k_{1}, \ldots, k_{n}\right) & =\frac{1}{k_{1}!\cdots k_{n}!} \int_{E} \operatorname{Tr}\left(X^{k_{1}}\right) \cdots \operatorname{Tr}\left(X^{k_{n}}\right) e^{-\operatorname{Tr}\left(X^{2}\right)} \mathrm{d} X . \tag{2.5}
\end{align*}
$$

Note that integral appearing in the equation (2.5) is always convergent, as any polynomial growth of the product of traces is suppressed by the Gaussian decay of the term $e^{-\operatorname{Tr}\left(X^{2}\right)}$.

For any measurable function $f: E \rightarrow \mathbb{C}$, assuming convergence, we introduce its expectation value $\mathbb{E} f(X)=\langle f(X)\rangle:$

$$
\begin{aligned}
\langle f(X)\rangle & =\int_{E} f(X) e^{\operatorname{Tr} V(X)-\operatorname{Tr}\left(X^{2}\right)} \mathrm{d} X \\
& =\sum_{n=0}^{\infty} \sum_{k_{1}, \ldots, k_{n} \geq 1} \frac{t_{1}^{k_{1}} \ldots t_{n}^{k_{n}}}{k_{1}!\cdots k_{n}!} \int_{E} f(X) \operatorname{Tr}\left(X^{k_{1}}\right) \cdots \operatorname{Tr}\left(X^{k_{n}}\right) e^{-\operatorname{Tr}\left(X^{2}\right)} \mathrm{d} X
\end{aligned}
$$

Solving a matrix model means computing all of these correlators $\left\langle\operatorname{Tr}\left(X^{k_{1}}\right) \cdots \operatorname{Tr}\left(X^{k_{n}}\right\rangle\right.$. Note that these quantities are encoded in $Z$, and can be obtained by taking appropriate derivatives of $Z$ with respect to the times.

### 2.2.2 $\beta$-generalisation

On the space $\operatorname{Mat}(N)$ we have an action $\rho$ of the unitary group $\mathrm{U}(N)$, defined as $U \cdot X=U X U^{-1}$ for $U \in \mathrm{U}(N)$. Assume that the ensemble $E$ is invariant under this action. Moreover let us assume that all of the elements $X \in E$ are diagonalisable. Then, because the integrand in 2.5 is invariant under the action $\rho$, one can integrate along the $\mathrm{U}(\mathrm{N})$-orbits. This means that we can map the matrix $X$ to the set of its eigenvalues $\Lambda=\left(\lambda_{1}, \ldots, \lambda_{N}\right) \in \mathbb{R}^{N}$. They are real, since $X$ is hermitian. The orbital (or angular) part contribute through the Jacobian, which in this case takes form of the Vandermonde determinant $\Delta(\Lambda)=\prod_{i \neq j}\left(\lambda_{i}-\lambda_{j}\right)$. Let $C=E / \mathrm{U}(\mathrm{N})$. We obtain:

$$
\begin{equation*}
F\left(k_{1}, \ldots, k_{n}\right)=\frac{1}{k_{1}!\cdots k_{n}!} \int_{C} \prod_{i=1}^{n}\left(\lambda_{1}^{k_{i}}+\cdots+\lambda_{N}^{k_{i}}\right) \Delta(\Lambda) e^{-\lambda_{1}^{2}-\cdots-\lambda_{N}^{2}} \mathrm{~d} \lambda_{1} \cdots \mathrm{~d} \lambda_{N} \tag{2.6}
\end{equation*}
$$

This is the starting point for the $\beta$-deformation.

Definition 2.2.2 Let $\beta \in \mathbb{C}$. $\beta$-deformed matrix model partition function $Z_{\beta} \in R$ is defined as

$$
\begin{align*}
Z_{\beta} & =\sum_{n=0}^{\infty} \sum_{k_{1}, \ldots, k_{n} \geq 1} F^{\beta}\left(k_{1}, \ldots, k_{n}\right) t_{1}^{k_{1}} \cdots t_{n}^{k_{n}} \\
F^{\beta}\left(k_{1}, \ldots, k_{n}\right) & =\frac{1}{k_{1}!\cdots k_{n}!} \int_{\Gamma} \prod_{i=1}^{n}\left(\lambda_{1}^{k_{i}}+\cdots+\lambda_{N}^{k_{i}}\right) \Delta(\Lambda)^{\beta} e^{-\sqrt{\beta}\left(\lambda_{1}^{2}-\cdots-\lambda_{N}^{2}\right)} \mathrm{d} \lambda_{1} \cdots \mathrm{~d} \lambda_{N}, \tag{2.7}
\end{align*}
$$

where $\Gamma$ is an appropriately chosen integration cycle (section 2.2.3).

Let us note that for specific values of $\beta$ this is the integral over other sets of random matrices: for $\beta=\frac{1}{2}$ we are integrating over real symmetric matrices and for $\beta=2$ over quaternionic Hermitian matrices. This kind of $\beta$-generalisation is relevant from the point of view of the conformal field theory. It gives rise to systems with more generic central charge than the standard matrix model. Another generalisation was introduced in [44] under the name of $\alpha / \beta$ deformed matrix model.

Definition 2.2.3 For any $\alpha, \beta \in \mathbb{C}$ the wave function $\psi_{\alpha, \beta}(x) \in R(x)$ is defined as

$$
\begin{align*}
\psi_{\alpha, \beta}(x) & =\sum_{n=0}^{\infty} \sum_{k_{1}, \ldots, k_{n} \geq 1} F^{\alpha, \beta}\left(k_{1}, \ldots, k_{n}\right) t_{1}^{k_{1}} \cdots t_{n}^{k_{n}},  \tag{2.8}\\
F^{\alpha, \beta}\left(k_{1}, \ldots, k_{n}\right) & =\frac{1}{k_{1}!\cdots k_{n}!} \int_{\Gamma} \prod_{i=1}^{n}\left(x-\lambda_{i}\right)^{\alpha}\left(\lambda_{1}^{k_{i}}+\cdots+\lambda_{N}^{k_{i}}\right) \Delta(\Lambda)^{\beta} e^{-\sqrt{\beta}\left(\lambda_{1}^{2}-\cdots-\lambda_{N}^{2}\right)} \mathrm{d} \lambda_{1} \cdots \mathrm{~d} \lambda_{N} .
\end{align*}
$$

When $\beta=1$ this expression can be rewritten as expectation value of the $\alpha$-power of the characteristic polynomial: $\psi_{\alpha, 1}(x)=\mathbb{E} \operatorname{det}(x-X)^{\alpha}$. Let us note that the remark concerning the integration cycle $\Gamma$ holds also in this case.

### 2.2.3 Twisted cycles

The functions, which are to be integrated, appearing in the expressions (2.7) and (2.8), are multivalued. Therefore to make the integrals meaningful we need to choose correct integration cycles. They belong to the twisted homology spaces, which we are now going to define. We follow here 1].

Let $\psi$ be a multi-valued function on an open set $U \subset \mathbb{C}^{n}$. We define vector spaces $C_{p}(U, \psi)$ to be freely generated by the expressions

$$
\left.\sigma \otimes \psi\right|_{\sigma}
$$

where $\sigma: \Delta_{p} \rightarrow U$ is a continuous map from a $n$-dimensional simplex, such that on the image $\sigma\left(\Delta_{p}\right)$ one can choose a single branch of $\psi$ and $\left.\psi\right|_{\sigma}$ is a choice of such a branch. We have naturally defined boundary map

$$
\begin{aligned}
\partial_{\psi}^{p}: C_{p}(U, \psi) & \rightarrow C_{p-1}(U, \psi) \\
\partial_{\psi}^{p}\left(\left.\sigma \otimes \psi\right|_{\sigma}\right) & =\left.\partial(\sigma) \otimes \psi\right|_{\partial(\sigma)}
\end{aligned}
$$

which allows us to define twisted homology groups:

$$
H_{p}(U, \psi)=\frac{\operatorname{ker}\left(\partial_{\psi}^{p}\right)}{\operatorname{im}\left(\partial_{\psi}^{p+1}\right)}
$$

Following [1] we can define integration over twisted cycles. If $\left.\sigma \otimes \psi\right|_{\sigma} \in C_{p}(U, \psi)$ for any smooth differential $n$-form $\omega$ we set:

$$
\int_{\left.\sigma \otimes \psi\right|_{\sigma}} \psi \cdot \omega=\int_{\sigma} \psi_{\sigma} \cdot \omega .
$$

Let $\mathcal{H}$ be the space of differentials of the form $F\left(\beta ; z_{1}, \ldots, z_{N}\right) f\left(z_{1}, \ldots, z_{N}\right) \mathrm{d} z_{1} \cdots \mathrm{~d} z_{N}$, where $f \in \mathcal{L}$. The objects we would like to integrate are forms $\omega \in \mathcal{H} \otimes R$, where $R$ is typically a ring of formal power series in infinite number of variables. In such case we extend the integration by defining $\int\left(\sum_{i} \omega_{i} \otimes r_{i}\right)=\sum_{i}\left(\int \omega_{i}\right) \otimes r_{i}$.

### 2.2.4 Loop equations

In analysing (formal) matrix integrals one of the crucial tools are the loop equations. They are differential equations specified by a family of differential operators satisfying commutations relations of the positive part of the Virasoro algebra. For this reason they are called Virasoro constraints.

Proposition 2.2.1 Let $\beta \in \mathbb{N}$. The formal power series $Z_{\beta}\left(t_{1}, t_{2}, \ldots\right)$ satisfies an infinite family of equations $l_{n} Z_{\beta}=0$ for $n \geq-1$, where:

$$
\begin{equation*}
l_{n}=\hbar\left(\sqrt{\beta}-\frac{1}{\sqrt{\beta}}\right)(n+1) \partial_{t_{n}}+\hbar^{2} \sum_{k=0}^{n} \partial_{t_{k}} \partial_{t_{n-k}}+\sum_{k=1}^{\infty} k t_{k} \partial_{t_{n+k}} \tag{2.9}
\end{equation*}
$$

Moreover those differential operators satisfy commutation relations: $\left[l_{n}, l_{m}\right]=(n-m) l_{n+m}$.

Remark. Although $\partial_{t-1}$ seems to appear in the above expression, it is multiplied by 0 . This expression seems to give an example of an Airy structure 6.1.2. However in the last summand $\hbar$ is missing.

Proof. Consider vector fields on $\mathbb{R}^{N}$ defined as $X_{n}\left(z_{1}, \ldots, z_{N}\right)=-\sum_{a=1}^{N} z_{a}^{n+1} \frac{\partial}{\partial z_{a}}$ for $n \geq-1$. Let us denote differential $N$-form $\omega=\Delta\left(z_{1}, \ldots, z_{N}\right)^{\beta} e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^{N} V\left(z_{a}\right)} \mathrm{d} z_{1} \wedge \cdots \wedge \mathrm{~d} z_{N}$. From the Stokes theorem and the fact that the function $\Delta\left(z_{1}, \ldots, z_{N}\right)^{\beta} e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^{N} z_{a}^{2}}$ vanishes exponentially fast when $\left|\left(z_{1}, \ldots, z_{N}\right)\right| \rightarrow \infty$ we deduce that:

$$
\begin{equation*}
\int_{\mathbb{R}^{N}} \mathrm{~d}\left(\iota_{X_{n}} \omega\right)=0 \tag{2.10}
\end{equation*}
$$

where $\iota_{X_{N}} \omega$ is a $N-1$ form defined as $\iota_{X_{N}} \omega\left(v_{2}, \ldots, v_{N}\right)=\omega\left(X, v_{2}, \ldots, v_{N}\right)$. In what follows we
use the notation $\langle f\rangle=\int_{\mathbb{R}^{N}} f \omega$. Equation 2.10 can be expanded into:

$$
\begin{align*}
0 & =\sum_{a=1}^{N}\left\langle(n+1) z_{a}^{n}+2 \beta \sum_{b \neq a} \frac{z_{a}^{n+1}}{z_{b}-z_{a}}-\frac{\sqrt{\beta}}{\hbar} z_{a}^{n+1} \frac{\partial V\left(z_{a}\right)}{\partial z_{a}}\right\rangle \\
& =\left\langle(n+1) \sum_{a=1}^{N} z_{a}^{n}+\beta \sum_{a \neq b} \sum_{k=0}^{n} z_{a}^{k} z_{b}^{n-k}-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^{N} z_{a}^{n+1} \frac{\partial V\left(z_{a}\right)}{\partial z_{a}}\right\rangle \\
& =\left\langle(n+1)(1-\beta) \sum_{a=1}^{N} z_{a}^{n}+\beta \sum_{a, b=1}^{N} \sum_{k=0}^{n} z_{a}^{k} z_{b}^{n-k}-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^{N} \sum_{k=0}^{\infty} k t_{k} z_{a}^{k+n}\right\rangle . \tag{2.11}
\end{align*}
$$

Note that for any function $f$ we have

$$
\left\langle f\left(z_{1}, \ldots, z_{N}\right) \sum_{a=1}^{N} z_{a}^{k}\right\rangle=-\frac{\hbar}{\sqrt{\beta}} \partial_{t_{k}}\left\langle f\left(z_{1}, \ldots, z_{N}\right)\right\rangle .
$$

Therefore the equation (2.11) can be rewritten as a differential equation $l_{n} Z_{\beta}=0$ with $l_{n}$ given by (2.9). The commutation relations $\left[l_{n}, l_{m}\right]=(n-m) l_{n+m}$ can be checked by direct computation.

Remark. Notice that we have an analogous relation for the introduced vector fields:

$$
\left[X_{n}, X_{m}\right]=(n-m) X_{n+m} .
$$

It follows that we have corresponding relation on Lie derivatives: $\left[\mathcal{L}_{X_{n}}, \mathcal{L}_{X_{m}}\right]=(n-m) \mathcal{L}_{X_{n+m}}$, where $\mathcal{L}_{X}=\mathrm{d} \iota_{X}+\iota_{X} \mathrm{~d}$. Notice that $\int_{\mathbb{R}^{N}} \mathrm{~d}\left(\iota_{X_{n}} \omega\right)=\int_{\mathbb{R}^{N}} \mathcal{L}_{X_{n}} \omega$, because $\mathrm{d} \omega=0$. This however does not seem enough to prove the relevant commutation relations.

The identities 2.11) can be also rewritten in a single equation, with a help of a formal variable $x$ :

$$
\left\langle\left(\sum_{a=1}^{N} \frac{1}{\left(x-z_{a}\right)^{2}}-\beta \sum_{1 \leq a<b \leq N} \frac{1}{\left(x-z_{a}\right)\left(x-z_{b}\right)}-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^{N} \frac{V^{\prime}\left(z_{a}\right)}{x-z_{a}}\right\rangle=0 .\right.
$$

This equation is called Ward identity.

### 2.2.5 Spectral curve

In this section let us consider non-formal models, for which $V(x)$ is a polynomial function (that means $t_{i} \in \mathbb{C}$ vanish for sufficiently large $i$ ).

Let $X \in \operatorname{Mat}(N)$ and let $\left\{\lambda_{1}, \ldots, \lambda_{N}\right\}$ be the set of its eigenvalues. We can introduce spectral measure:

$$
L_{X}=\frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_{i}},
$$

where $\delta_{x}$ is the Dirac measure concentrated at $x \in \mathbb{C}$. In other words $L_{X}(A)=\left|\left\{\lambda_{i}: \lambda_{i} \in A\right\}\right|$ for any measurable set $A \subset \mathbb{R}$. If $X$ is a random matrix, $L_{X}$ becomes a random measure. Interesting property of this measure is that (under some assumptions) it becomes deterministic in the limit $N \rightarrow \infty$. A question one may ask is the following: what is this limit? A simple answer can be given for the Hermitian matrix model, that is if in (2.5) we put $V=0$.

Theorem 2.2.1 Define a Borel measure $\rho$ on $\mathbb{R}$ with the following density with respect to the Lebesgue measure $\lambda$ on $\mathbb{R}$ :

$$
\mathrm{d} \rho(x)=\frac{2}{\pi} \sqrt{1-x^{2}} \mathbf{1}_{|x| \leq 1} \mathrm{~d} \lambda(x) .
$$

Let $X$ be a random matrix distributed according to the measure $e^{-\operatorname{Tr}\left(X^{2}\right)} \mathrm{d} X$. Then the measure $L_{X}$ tends weakly to the measure $\rho$ with probability 1. In other words for any continuous, compactly supported function $f: \mathbb{R} \rightarrow \mathbb{R}$ we have

$$
\lim _{N \rightarrow \infty} \int_{\mathbb{R}} f \mathrm{~d} L_{X}=\int_{\mathbb{R}} f(x) \mathrm{d} \rho(x)
$$

which happens for almost all $X \in \operatorname{Mat}(N)$.

We can rewrite density of the measure $\rho$ in the form $\frac{2}{\pi} y(x) \mathbf{1}_{|x| \leq 1}$, where

$$
y(x)^{2}+x^{2}=1
$$

This is an example of a spectral curve. Theorem 2.2.1 justifies this name. We would like to pass to a more general case of random matrices.

Definition 2.2.4 $A$ spectral curve for a matrix model with potential $V(x)$ is given by the equation:

$$
\begin{equation*}
y^{2}-V^{\prime}(x) y+P(x)=0 \tag{2.12}
\end{equation*}
$$

where $x, y \in \mathbb{C}$ and $P(x)$ is a function defined as

$$
\begin{equation*}
P(x)=\lim _{n \rightarrow \infty} \frac{1}{N}\left\langle\sum_{i=1}^{n} \frac{V^{\prime}(x)-V^{\prime}\left(z_{i}\right)}{x-z_{i}}\right\rangle . \tag{2.13}
\end{equation*}
$$

The equation 2.12 can be considered over $\mathbb{R}$ or over $\mathbb{C}$. We will put more attention to the second case, as it will be relevant for the topological recursion. There the importance of the spectral curves will be revealed. We can also rewrite the spectral curve in another variables. New form will be more suitable for us, when we will be considering quantum curves. Let

$$
u(x)=2 y(x)-V^{\prime}(x)
$$

With the help of the new variable equation (2.12) transforms into:

$$
\begin{equation*}
A(x, y)=u(x)^{2}-V^{\prime}(x)^{2}+4 P(x)=0 . \tag{2.14}
\end{equation*}
$$

### 2.3 Enumerative geometry

### 2.3.1 Graph expansion

The correlators introduced in 2.5 can be expressed as a sums over graphs. This gives us a combinatorial interpretation of the matrix integral. Because of the connections to the quantum field theory they are called Feynman graphs. In order to obtain such an expansion one can apply Isserlis theorem (known also as Wick rule) :

Proposition 2.3.1 39 Let $\left(X_{1}, \ldots, X_{2 n}\right)$ be a zero-mean normal random vector in $\mathbb{R}^{2 n}$. Then

$$
\mathbb{E}\left(X_{1} \cdots X_{2 n}\right)=\sum_{P \in \mathcal{P}(2 n)} \prod_{p \in P} \mathbb{E}\left(X_{p_{1}} X_{p_{2}}\right)
$$

where $\mathcal{P}(2 n)$ is the set of pairings of the set $\{1, \ldots, 2 n\}$ and each pairing $p=\left(p_{1}, p_{2}\right)$.

Applying Isserlis theorem we can identify the summation set with a set of suitable graphs.

Definition 2.3.1 $A$ ribbon graph is a finite graph with a cyclic ordering of half-edges adjacent to each vertex.

A natural way of obtaining such ordering is by drawing our graph on an oriented Riemann surface (complex curve). Then half-edges adjacent to any vertex have a cyclic ordering given by going around this vertex clockwise, in such a way that vectors tangent to the two consecutive half-edges and pointing from the vertex form a positively oriented basis of the tangent space at this vertex. Such graphs can be also visualized as fatgraphs. Draw a ribbon graph on a surface and take an $\varepsilon$-neighborhood of it. Its edges are "stripes" glued at vertices:


Figure 2.1: Fatgraph
For any ribbon graph $\Gamma$ let us denote the set of its vertices by $\Gamma_{0}$, the set of its edges by $\Gamma_{1}$ and by $\Gamma_{2}$ the set of its faces (the plaquettes completing the fatgraph to the Riemann surface on which the graph is drawn). From the equation for the Euler characteristic we can deduce genus of the surface:

$$
g=1-\frac{1}{2}\left(\left|\Gamma_{0}\right|-\left|\Gamma_{1}\right|+\left|\Gamma_{2}\right|\right) .
$$

Let us denote by $\mathcal{G}\left(k_{1}, \ldots, k_{n}\right)$ set of ribbon graphs with $n$ vertices of valencies $k_{1}, \ldots, k_{n}$. For each graph $\Gamma \in \mathcal{G}\left(k_{1}, \ldots, k_{n}\right)$ let $\operatorname{Aut}(\Gamma)$ be the set of automorphisms of $\Gamma$ (maps from the set of vertices to itself preserving edge-connectedness).

Proposition 2.3.2 [7] The following expression for the coefficients appearing in 2.5) holds:

$$
\begin{equation*}
F\left(k_{1}, \ldots, k_{n}\right)=\sum_{\Gamma \in \mathcal{G}\left(k_{1}, \ldots, k_{n}\right)} \frac{1}{|\operatorname{Aut}(\Gamma)|} N^{\left|\Gamma_{2}\right|} \tag{2.15}
\end{equation*}
$$

Let us explain how to obtain the above formula. In the expression for $F\left(k_{1}, \ldots, k_{n}\right)$ we can expand each trace as

$$
\operatorname{Tr} X^{k_{a}}=\sum_{\substack{i_{1}^{a}, i_{2}^{a}, \ldots, i_{k a}^{a} \\ \in\{1, \ldots, N\}}} X_{i_{1}^{a} i_{2}^{a}} X_{i_{2}^{a} i_{3}^{a}} \cdots X_{i_{k_{a}}^{a} i_{1}^{a}},
$$

obtaining:

$$
\begin{equation*}
F\left(k_{1}, \ldots, k_{n}\right)=\sum_{A=1, \ldots, n} \sum_{B=1, \ldots, k_{A}} \sum_{i_{B}^{A} \in\{1, \ldots, N\}} \mathbb{E}\left(\prod_{a=1}^{n} \prod_{b=1}^{k_{a}} X_{i_{b}^{a}, i_{b+1}^{a}}\right), \tag{2.16}
\end{equation*}
$$

where we always set $k_{a}+1=1$ for the indices $b$ and $B$. Let us apply Proposition 2.3.1 to the above expression, exchanging the expectation value of the product with a sum over pairings of expectation values of two variables. Those pairings have a nice combinatorial interpretation.


Figure 2.2: Association of pairs of indices $\left(i_{b}^{a}, i_{b+1}^{a}\right)$ to half-edges and pairing.
We associate pair of indices $\left(i_{b}^{a}, i_{b+1}^{a}\right)$ with a half-edge, as in the figure above. We get $n$ vertices with valencies $k_{1}, \ldots, k_{n}$. Each pairing of double indices $\left(\left(i_{b_{1}}^{a_{1}}, i_{b_{1}+1}^{a_{1}}\right),\left(i_{b_{2}}^{a_{2}}, i_{b_{2}+1}^{a_{2}}\right)\right)$ gives us a way to glue corresponding halfedges. Therefore each pairing of all indices gives rise to an element of $\mathcal{G}\left(k_{1}, \ldots, k_{n}\right)$ and each such gluing gives rise to a pairing. Since $X_{i j}$ are independent and
 This means that performing summation over all indices we get contractions: for each loop in the obtained fatgraph we are left with one free index. Therefore the number of the contributions from a given graph is multiplied by $N$ for any plaquette in $\Gamma_{2}$ : it equals to $N^{\left|\Gamma_{2}\right|}$. Factor $\frac{1}{|\operatorname{Aut}(\Gamma)|}$ comes from the symmetry considerations, as explained in [7].

Definition 2.3.2 Connected correlators are defined as

$$
\begin{equation*}
F\left(k_{1}, \ldots, k_{n}\right)^{c}=\sum_{\Gamma \in \mathcal{G}\left(k_{1}, \ldots, k_{n}\right)^{c}} \frac{1}{\operatorname{Aut}(\Gamma) \mid} N^{\left|\Gamma_{2}\right|}, \tag{2.17}
\end{equation*}
$$

where $\mathcal{G}\left(k_{1}, \ldots, k_{n}\right)^{c}$ is the set of connected ribbon graphs with prescribed valencies.

### 2.3.2 Intersection theory

One of the most influential results in the theory of formal random matrices was found by Maxim Kontsevich [42], earlier conjectured by Edward Witten [58]. Identifying two models for 2 dimensional quantum gravity he showed that asymptotic expansion of a suitable matrix integral computes intersection numbers of the moduli spaces of curves. The latter numbers play a crucial role in enumerative geometry. This section is an illustration of the application of the theory of random matrices. Therefore we will give an outline, without introducing formal definitions, which are quite involving. Let $\overline{\mathcal{M}}_{g, n}$ be the compactification of the moduli space of stable Riemann surfaces of genus $g$ and $n$ marked points. This space is an orbifold, whose complex highest dimension of strata is $3 g-3+n$. The intersection numbers (called also Gromov-Witten invariants of a point) are defined by:

$$
\begin{equation*}
\left\langle\tau_{d_{1}} \cdots \tau_{d_{n}}\right\rangle_{g, n}=\int_{\left[\overline{\mathcal{M}}_{g, n}\right]^{v i r}} \prod_{i=1}^{n} \psi_{i}^{d_{i}} \tag{2.18}
\end{equation*}
$$

where $\psi_{i}$ are first Chern classes of bundles $\mathcal{L}_{i}$ defined as follows. Fiber of this bundle at any curve $C$ is the fiber of the cotangent bundle of $C$ at $i^{t h}$ marked point: $\left.\mathcal{L}_{i}\right|_{C}=\left.T^{*} C\right|_{p_{i}}$. The cycle over which the integral is performed $\left[\overline{\mathcal{M}}_{g, n}\right]^{\text {vir }}$ is the virtual fundamental class 17 . In order to ensure matching of the dimensions we need to take $g$ such that $3 g-3+n=\sum_{i=1}^{n} d_{i}$. Those numbers can be collected in a formal power series, called free energy:

$$
Z^{G W}\left(t_{*}\right)=\sum_{\substack{d_{i} \geq 0 \\ \sum d_{i}=d}}\left\langle\tau_{d_{1}} \cdots \tau_{d_{n}}\right\rangle \prod \frac{t_{i}^{k_{i}}}{k_{i}!}
$$

Kontsevich considered following matrix model:

$$
\begin{equation*}
Z^{K M}=\int_{E} \exp \left(\frac{i}{6} \operatorname{Tr} X^{3}-\frac{1}{2} \operatorname{Tr}\left(X^{2} \Lambda\right)\right) c_{\Lambda} \mathrm{d} X, \tag{2.19}
\end{equation*}
$$

where $\mathrm{d} X$ is Lebesgue measure on the set $E$ of Hermitian $N \times N$ matrices, and $c_{\Lambda}$ is a normalisation constant: $c_{\Lambda}^{-1}=\int \exp \left(-\frac{1}{2} \operatorname{Tr}\left(X^{2} \Lambda\right)\right) \mathrm{d} X . \quad Z^{K M}$ depends on a matrix $\Lambda$, and we can restrict to the case when this matrix is diagonal. Using Proposition 2.3 .1 its asymptotic expansion can be expressed as sum over 3 -valent ribbon graphs with labeled set of faces. Let us denote the later set by $\mathcal{G}$. For any $\Gamma \in \mathcal{G}$ let $\Gamma_{0}$ be the set of vertices, $\Gamma_{1}$ set of edges and $\Gamma_{2}$ set of faces. Suppose that $\left|\Gamma_{2}\right|=n$ and genus of $\Gamma$ equals $g$. Labelling means that we have an identification $\Gamma_{2} \simeq\{1, \ldots, n\}$, which also identifies any function $\lambda: \Gamma_{2} \rightarrow \mathbb{C}$ with a vector $\left(\lambda_{1}, \ldots, \lambda_{n}\right) \in \mathbb{C}^{n}$. For any edge $e \in \Gamma_{1}$ denote two faces adjacent to is as $e_{1}$ and $e_{2}$. One can show that asymptotic expansion of 2.19 is given by the formula:

$$
Z^{K M}=\sum_{\Gamma \in \mathcal{G}_{N}} \frac{\left(\sqrt{-\frac{1}{2}}\right)^{\left|\Gamma_{0}\right|}}{|\operatorname{Aut\Gamma }|} \prod_{e \in \Gamma_{1}} \frac{2}{\lambda\left(e_{1}\right)+\lambda\left(e_{2}\right)},
$$

where $\mathcal{G}_{N}$ is the set of equivalence classes of connected nonempty 3 -valent ribbon graphs together with coloring maps $\Gamma_{2} \rightarrow\{1, \ldots, N\}$.

Identification of the two partition functions $Z^{G W}$ and $Z^{K M}$ is possible using main identity proved by Kontsevich:

$$
\sum_{\substack{d_{i} \geq 0 \\ \sum d_{i}=d}}\left\langle\tau_{d_{1}} \cdots \tau_{d_{n}}\right\rangle \prod_{i=1}^{n} \frac{\left(2 d_{i}-1\right)!!}{\lambda_{i}^{2 d_{i}+1}}=\sum_{\Gamma \in \mathcal{G}_{g, n}} \frac{2^{-\left|\Gamma_{0}\right|}}{|\operatorname{Aut} \Gamma|} \prod_{e \in \Gamma_{1}} \frac{2}{\lambda(e)+\lambda(s e)}
$$

and the substitution $t_{i}=-(2 i-1)!!\operatorname{Tr}\left(\Lambda^{-2 i-1}\right)$. The idea of the proof of the above equality is following. Construct a combinatorial model $\mathcal{M}_{g, n}^{c o m b}$ for the space $\mathcal{M}_{g, n}$ as well as for the vector bundles $\mathcal{L}_{i}$ and its Chern classes $\psi_{i}$. This model is found with the help of the Strebel differentials. After performing Laplace transform of the generating function for the intersection numbers of combinatorial classes we obtain expressions arising as asymptotic expansions of Kontsevich matrix integral. This is done by identifying ribbon graphs in the matrix integral with a triangulation of the space $\mathcal{M}_{g, n}^{c o m b}$.

### 2.4 Topological recursion

In order to introduce topological recursion 25,26 let us first give an abstract definition of the spectral curve.

Definition 2.4.1 A spectral curve is an algebraic curve $\Sigma$ equipped with two meromorphic functions $x, y: \Sigma \rightarrow \mathbb{P}^{1}$ such that the zeros of $\mathrm{d} x$ are disjoint from the zeros of $\mathrm{d} y$.

It follows from the definition that for a spectral curve $(\Sigma, x, y)$ the function $(x, y): \Sigma \rightarrow \mathbb{P}^{1} \times \mathbb{P}^{1}$ is an immersion. Suppose that its image is given as a zero set of a polynomial $P(x, y)$. This gives an algebraic characterisation of a spectral curve.

Examples of spectral curves appear in various areas of mathematics and physics. Apart from matrix models there are: A-polynomials in knot theory, spectral curves of Hitchin systems, Seiberg-Witten curves, mirror curves in string theory.

Let $R$ be the ramification divisor of the map $x$ (zeros of $\mathrm{d} x$ ). We will restrict ourselves to the case when all of the zeros are simple (the second derivative of $x$ does not vanish at $R$ ). Then there exists a local coordinate $z_{r}$ on a neighborhood $U_{r}$ of any $r \in R$ such that $x(z)=x_{z_{r}}+\frac{1}{2} z_{r}^{2}$ on $U_{r}$. This enables us to define an involution map $\sigma_{r}: U_{r} \rightarrow U_{r}$ as $\sigma_{r}(z)=-z$, which preserves $x: x\left(\sigma_{r}(z)\right)=x(z)$. Consider the $n^{t h}$ Cartesian product $\Sigma^{n}=\Sigma \times \cdots \times \Sigma$ equipped with the projections into the $i^{\text {th }}$ factors $\pi_{i}: \Sigma^{n} \rightarrow \Sigma$. Denote by $K_{\Sigma}$ the canonical bundle of $\Sigma$ and let $K_{\Sigma}(R)=K_{\Sigma} \otimes \mathcal{O}_{\Sigma}(R)$. Topological recursion says how to any spectral curve associate a family of invariants, called correlators:

$$
\omega_{g, n} \in H^{0}\left(\Sigma^{n}, \pi_{1}^{*} K_{\Sigma}(R) \otimes_{S y m} \cdots \otimes_{S y m} \pi_{n}^{*} K_{\Sigma}(R)\right)
$$

where $g, n \geq 0$ and $2 g+n \geq 2$. More precisely we are only interested in the behavior of $\omega_{g, n}$ on the neighborhoods of the ramification points. We will therefore restrict the definition to the section of the above bundle defined on the set $\left(\bigcup_{r \in R} U_{r}\right)^{n} \subset \Sigma^{n}$. This allows us to trivialise the bundles as well, provided that $U_{r}$ are sufficiently small.

To define the recursion we make a choice of initial conditions, for which we take $\omega_{0,1}=y \mathrm{~d} x \in$ $H^{0}\left(\Sigma, K_{\Sigma}(R)\right)$, and $\omega_{0,2}=B_{\Sigma}$ is the Bergman kernel. It is an element of $H^{0}\left(\Sigma^{2}, \pi_{1}^{*} K_{\Sigma}(R) \otimes_{\text {Sym }}\right.$ $\left.\pi_{2}^{*} K_{\Sigma}(R)\right)$, which satisfies the following conditions:

- it has a pole on the diagonal of the form $\frac{\mathrm{d} z_{1} \mathrm{~d} z_{2}}{\left(z_{1}-z_{2}\right)^{2}}+O(1) \mathrm{d} z_{1} \mathrm{~d} z_{2}$,
- it is symmetric.

This form is not unique, however its arbitrariness does not affect the outcome of the recursion. Define the recursion kernel:

$$
K_{r}\left(z_{1}, z\right)=-\frac{1}{2} \frac{1}{y(z) \mathrm{d} x(z)-y\left(\sigma_{r}(z)\right) \mathrm{d} x\left(\sigma_{r}(z)\right)} \otimes \int_{\sigma_{r}(z)}^{z} \omega_{0,2}\left(z_{1}, \cdot\right)
$$

which is a section of $K_{\Sigma}^{-1} \otimes K_{\Sigma}$ defined over $U_{r} \times \Sigma$. Then the correlators are defined recursively via the formula:

$$
\begin{align*}
\omega_{g, n+1}\left(z_{1}, z_{S}\right) & =\sum_{r \in R} \operatorname{Res}_{z=r} K_{r}\left(z_{1}, z\right)\left(\omega_{g-1, n+2}\left(z, \sigma_{r}(z), z_{S}\right)\right. \\
& \left.+\sum_{\substack{g_{1}+g_{2}=g, I \sqcup J=S}}^{o} \omega_{g_{1}, 1+|I|}\left(z, z_{I}\right) \omega_{g_{2}, 1+|J|}\left(\sigma_{r}(z), z_{J}\right)\right) \tag{2.20}
\end{align*}
$$

equation where the $o$ above the sum means that we are excluding the cases with either $g_{1}=0$, $g_{2}=0, I=\%$ or $J=\%$. This guarantees that the recursion can be computed, since for any $\omega_{g, n+1}$ only correlators $\omega_{h, m}$ with Euler characteristic $\chi=2 h-2+m<2 g-2+n+1$ appear on the right hand side. This also explains the name topological.

Importance of the topological recursion lies in the fact that for specific choices of the spectral curves correlators $\omega_{g, n}$ encode answers to various problems in mathematics and physics. Let us give a few examples:

- Airy curve $\left(\mathbb{P}^{1}, x(z)=\frac{1}{2} z^{2}, y(z)=z\right)$ computes intersection numbers of the moduli space of stable curves with marked points $\overline{\mathcal{M}}_{g, n}$,
- Hurwitz numbers,
- correlators in random matrices (see the next section).

Other curves are conjectured to solve more problems:

- Given a knot $K \subset S^{3}$ its A-polynomial should compute colored Jones polynomials of $K$,
- Computation of the Gromow-Witten invariants of complex varieties $X$ (specifically toric Calabi-Yau manifolds of complex dimension 3), generalising (2.18), which are Gromow-Witten invariants of a point.


### 2.4.1 Topological recursion and random matrices

Let us introduce connected correlators, as functions in variables $x_{i}$ 's:

$$
\begin{equation*}
W_{n}\left(x_{1}, \ldots, x_{n}\right)=\left\langle\prod_{i=1}^{N} \operatorname{Tr}\left(\frac{1}{x_{i}-X}\right)\right\rangle^{c} \tag{2.21}
\end{equation*}
$$

where we are using coefficients (2.17) instead of those appearing in 2.5. To make a connection between random matrices and topological recursion we need to introduce topological expansion of such correlators. Let us recall that in the formula (2.17) we have expressed coefficients of the partition function $F\left(k_{1}, \ldots, k_{n}\right)^{c}$ as functions of the size of the random matrices $N$. This is the starting point for the topological expansion. We define $W_{g, n}$ via the formula:

$$
W_{n}\left(x_{1}, \ldots, x_{n}\right)=\sum_{g=0}^{\infty} N^{2-2 g-n} W_{g, n}\left(x_{1}, \ldots, x_{n}\right) .
$$

Next, let us introduce associated differential forms:

$$
\begin{equation*}
\omega_{g, n}\left(x_{1}, \ldots, x_{n}\right)=\left(W_{g, n}\left(x_{1}, \ldots, x_{n}\right)+\delta_{g=0} \delta_{n=2} \frac{1}{\left(x_{1}-x_{2}\right)^{2}}\right) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{n} \tag{2.22}
\end{equation*}
$$

Theorem 2.4.1 [24] The symmetric differential forms (2.22) satisfy topological recursion with the spectral curve specified by the equation (2.12):

$$
\Sigma=\left\{(x, y) \in \mathbb{C}^{2}: y^{2}-V^{\prime}(x) y+P(x)=0\right\}
$$

and $x, y: \Sigma \rightarrow \mathbb{P}^{1}$ are projections onto the first and the second coordinate respectively.

### 2.5 Quantisation

In this short section let us explain what we mean by quantising algebraic equation like (2.14). Let $A(x, y)$ be an element of $R=\mathbb{C}[x, y]$. In this commutative ring relation $x y=y x$ holds. Quantisation procedure replaces such ring with a family of rings, parametrized by a parameter $\hbar$ (formal or complex), denoted by $R_{\hbar}$. Moreover we would like to recover the initial ring $R$ as a limit of $R_{\hbar}$, when $\hbar$ tends to zero. This limit should be precisely defined, but for our purpose one example is sufficient. More precisely we take Weyl algebra as quantisation of $R$ :

$$
R_{\hbar}=\mathbb{C}\langle x, y\rangle / I,
$$

where $\mathbb{C}\langle x, y\rangle$ is a unital algebra freely generated by elements $x$ and $y$ over $\mathbb{C}$ and $I=\langle x y-y x-\hbar\rangle$ is an ideal generated by a single element. It follows that in $R_{\hbar}$ the relation $[x, y]=\hbar$ holds. One can represent this algebra faithfully as a subalgebra of differential operators on the line acting on smooth functions. Element $x$ becomes multiplication by $x$ operator and $y$ becomes differentiation $\hbar \frac{\mathrm{d}}{\mathrm{d} x}$ operator.

### 2.5.1 Quantum curves and topological recursion

Let $P(x, y)$ be a polynomial function and consider a curve $\Sigma=\left\{(x, y) \in \mathbb{C}^{2}: P(x, y)=0\right\}$. Let $\hat{P}(\hat{x}, \hat{y})$ be a differential operator, called quantum curve, with $\hat{y}=\hbar \frac{\partial}{\partial x}$ and $\hat{x}$ a multiplication by $x$ operator. It arises as a quatisation of the curve $\Sigma$, so that in the semi-classical limit $\hbar \rightarrow 0, \hat{y} \rightarrow y$ it reproduces the polynomial $P$ :

$$
\hat{P}(\hat{x}, \hat{y})=P(\hat{x}, \hat{y})+\sum_{n=1}^{N} \hbar^{n} P_{n}(\hat{x}, \hat{y}) .
$$

In the above equation we assume that each expression $P_{n}(\hat{x}, \hat{y})$, as well as $P(\hat{x}, \hat{y})$, is normally ordered: all operators $\hat{y}$ appear on the left hand side of $\hat{x}$. This ordering allows separation of different terms with respect to the power of the parameter $\hbar$. Moreover, we have assumed that $\hat{P}(\hat{x}, \hat{y})$ is a polynomial expression in the operators $\hat{y}$ and $\hat{x}$. Choice of the ordering is an important problem in the process of quantisation of the curve $\Sigma$.

Consider the equation

$$
\begin{equation*}
\hat{P}(\hat{x}, \hat{y}) \psi(p, \hbar)=0, \tag{2.23}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi(p, \hbar)=\exp \left(\hbar^{-1} S_{0}(p)+S_{1}(p)+\hbar S_{2}(p)+\ldots\right) . \tag{2.24}
\end{equation*}
$$

The equation (2.23), called Schrödinger equation, should be understood as a family of equations for the functions $S_{k}: \Sigma \rightarrow \mathbb{C}$. The conjectural relation between the topological recursion and the quantum curves is the following.

Conjecture. Let $(\Sigma, x, y)$ be a spectral curve given by a polynomial equation $P(x, y)=0$. Let $\omega_{g, n}$ be a family of meromorphic symmetric differentials computed by the topological recursion 2.20 . Define a wave function using the formula 2.24 with the coefficients given by:

$$
S_{k}(p)=\sum_{2 g-1+n=k} \frac{1}{n!} \int^{p} \cdots \int^{p} \omega_{g, n}\left(p_{1}, \ldots, p_{n}\right) .
$$

Then the conjecture states that there exists a quantisation $\hat{P}(\hat{x}, \hat{y})$ of the polynomial $P(x, y)$, which satisfies the Schrödinger equation (2.23).

This conjecture was proved in several cases. For example in [11] it has been shown to be true if the Newton polygon of the polynomial $P(x, y)$ contains no interior points.

## Chapter 3

## Vertex operators and conformal symmetry

In this chapter we introduce mathematical structures related to conformal field theory. It is spit into two parts. The first part (sections from 3.1.1 to 3.1.5 discusses special class of Lie algebras and their representation theory. It is based mainly on the book 38. In the second part (sections from 3.2 .1 to 3.2 .6 we introduce vertex operator algebras. It is based on the books [38], [28] , article [32] (definition of the intertwining operators in the Fock modules: equation 3.24) and computations of the author (Lemmas 3.2 .2 and 3.2 .3 ). A nice introduction to CFT from a physical point of view is 53 .

In the section 3.1.1 we introduce special class of Lie algebras, which we call Lie $\mathbb{Z}$-algebras. Two most important examples are Virasoro algebra and Heisenberg algebra. We study basics of their representation theory, including definitions of the Fock and Verma modules.

In the section 3.1 .2 we introduce bilinear forms (called Shapovalov form) and determinants (including Kac determinant).

In the section 3.1 .3 we introduce special vectors in the Fock/Verma modules, called singular vectors. In the section 3.1.4 we also discuss a particular homomorphism between Fock and Verma module, denoted by $S_{\alpha, Q}$. In this part we follow mainly [38].

In the section 3.1 .5 we discuss duality between Fock modules $\mathcal{F}(\alpha)$ and $\mathcal{F}(Q-\alpha)$, as well as cosingular vectors.

In the section 3.2 .1 we define vertex algebras and normal ordering of fields. We also recall important results such as Goddard's uniqueness theorem and strong reconstruction theorem.

In the section 3.2 .2 we introduce operator product expansion, a notion important in CFT.
In the section 3.2 .3 we define Vertex Operator Algebras (VOAs), which are vertex algebras equipped with a suitable action of the Virasoro algebra.

In the section 3.2 .4 we give two examples of the VOAs.
In the section 3.2.5 we define modules over VOAs and intertwining operators which one can associate to any triple of such modules.

In the section 3.2.6 we discuss intertwining operators between Fock modules, which are most important for us. We prove lemmas related to the action of the Virasoro algebra on the intertwining operators, which we are going to use in the following chapter.

In the section 3.2.7 we discuss some topics not directly related to the main topic of this dissertation, such as Fusion algebra, Verlinde formula and Monstrous Moonshine.

### 3.1 Lie $\mathbb{Z}$-algebras

### 3.1.1 Definitions

Definition 3.1.1 Define a category $\mathcal{C}$ of Lie $\mathbb{Z}$-algebras. Objects of this category are $\mathbb{Z}$-graded Lie algebras $\mathfrak{g}=\oplus_{n \in \mathbb{Z}} \mathfrak{g}^{n}$ such that $\mathfrak{h}=\mathfrak{g}^{0}$ is the Cartan subalgebra and such that $\left[\mathfrak{g}^{m}, \mathfrak{g}^{n}\right] \subset \mathfrak{g}^{m+n}$. Morphisms are homomorphism of Lie algebras, which preserve the grading.

For an element $v \in \mathfrak{g}^{n}$ we will write its grading as $|v|=n$. From the definition it follows that the following subspaces: $\mathfrak{g}^{+}=\bigoplus_{n>0} \mathfrak{g}^{n}, \mathfrak{g}^{\geq}=\bigoplus_{n \geq 0} \mathfrak{g}^{n}, \mathfrak{g}^{-}=\bigoplus_{n<0} \mathfrak{g}^{n}$ and $\mathfrak{g}^{\leq}=\bigoplus_{n \leq 0} \mathfrak{g}^{n}$ are subalgebras of $\mathfrak{g}$. Let us give examples of the Lie $\mathbb{Z}$-algebras, which will be most relevant for us.

Examples. A Virasoro algebra is a graded vector space $\mathcal{V}=\bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_{n} \oplus \mathbb{C} c(c$ is in grading 0$)$ equipped with a Lie bracket:

$$
\begin{equation*}
\left[L_{n}, L_{m}\right]=(n-m) L_{n+m}+\frac{c}{12} m\left(m^{2}-1\right) \delta_{n,-m}, \quad[c, \mathcal{V}]=0 \tag{3.1}
\end{equation*}
$$

It is a unique up to isomorphism central extension of the Witt algebra $\mathcal{W}=\mathbb{C}\left[z, z^{-1}\right] \frac{\partial}{\partial z}$ with the commutator as a Lie bracket. It is algebra of meromorphic vector fields on $\mathbb{C} P^{1}$, with poles only at 0 and $\infty$.

A Heisenberg algebra is a graded vector space $\mathcal{H}=\bigoplus_{n \in \mathbb{Z}} \mathbb{C} a_{n} \oplus \mathbb{C} k(k$ is in grading 0$)$ with a Lie bracket defined as:

$$
\left[a_{n}, a_{m}\right]=\frac{1}{2} n \delta_{n,-m} k, \quad[k, \mathcal{H}]=0
$$

Remark. In the literature one can meet different normalisations of the bracket of the Heisenberg algebra, for example $\left[a_{n}, a_{m}\right]=m \delta_{n,-m} k$. Isomorphisms between such defined algebras can be obtained by rescaling the element $k$. In the chapter 5 we will use different normalisation.

Definition 3.1.2 A module over a Lie $\mathbb{Z}$-algebra $\mathfrak{g}$ is a $\mathbb{Z}$-graded module over the Lie algebra $\mathfrak{g}$, which satisfies $\mathfrak{g}^{n} \cdot M^{m} \subset M^{n+m}$ and is diagonalisable over $\mathfrak{g}^{0}$. Homomorphism between modules is a Lie homomorphism, which preserves the gradings.

Because $\mathfrak{g}^{0} \cdot M^{n} \subset M^{n}$ it follows that the diagonal decomposition is compatible with the $\mathbb{Z}$-grading, in the sense that both can be combined into $\mathbb{Z} \times\left(\mathfrak{g}^{0}\right)^{*}$-grading. Let us introduce a special class of such modules. We denote the universal enveloping algebra of a Lie algebra $\mathfrak{g}$ by $U(\mathfrak{g})$.

Definition 3.1.3 $A$ weight module over $\mathfrak{g}$ with the weight $\lambda \in \mathfrak{h}^{*}$ is a module $M$, such that there exists a vector $v_{\lambda} \in M$ satisfying: $\rho\left(\mathfrak{g}^{+}\right) v_{\lambda}=0, U\left(\mathfrak{g}^{\leq}\right) v_{\lambda}=M$ and $h \cdot v_{\lambda}=\lambda(h) v$ for any $h \in \mathfrak{h}$.

An example of a weight module is the Verma module. Its definition uses a unique one dimensional module $\mathbb{C}_{\lambda}$ over $\mathfrak{g}^{\geq}$defined for any $\lambda \in \mathfrak{h}^{*}$ which satisfies $\mathfrak{g}^{+} \mathbb{C}_{\lambda}=0$ and $h \cdot v=\lambda(h) v$ for $h \in \mathfrak{h}$.

Definition 3.1.4 Let $\mathfrak{p} \subset \mathfrak{g}$ be a Lie sublagebra and let $M$ be a $\mathfrak{p}$-module. For this data we define an induced $\mathfrak{g}$-module:

$$
\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M=U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M
$$

Definition 3.1.5 For any $\lambda \in \mathfrak{h}^{*}$ define Verma module as $M(\lambda)=\operatorname{Ind}_{\mathfrak{g} \geq}^{\mathfrak{g}} \geq \mathbb{C}_{\lambda}$.
Let us denote the weight vector of such Verma module $1 \otimes 1$ as $|\lambda\rangle$. One can describe Verma module for the algebras $\mathcal{V}$ and $\mathcal{H}$ by giving explicitly its basis. The basis can be then written as:

$$
\begin{array}{lr}
L_{-i_{1}} \cdots L_{-i_{k}}|\Delta\rangle & \text { in the case of } \mathcal{V}, \\
a_{-i_{1}} \cdots a_{-i_{k}}|\alpha\rangle & \text { in the case of } \mathcal{H}, \tag{3.3}
\end{array}
$$

where $i_{1} \geq i_{2} \geq \cdots \geq i_{k}>0$ are integers. Those bases are moreover graded-homogeneous, with the above vectors belonging to the subspaces of the grading $i_{1}+\cdots+i_{k}$. Action of an operator ( $a_{n}$ or $L_{n}$ ) on this basis element can be obtained by writing the corresponding operator on the left hand side and commuting it with the operators $L_{i_{q}}$ (or $a_{-i_{q}}$ ) until we are left with a linear combination of the basis elements. In this manner matrix elements of the corresponding operators in this basis can be computed.

The specification of the basis above allows us to compute dimensions of the subspaces of a fixed grading in the Verma modules over Virasoro and Heisenberg algebra. Let us introduce a function $p: \mathbb{Z}_{n>0} \rightarrow \mathbb{Z}_{n>0}$ which computes the number of partitions of a given positive integer into sum of integers:

$$
\begin{equation*}
p(n)=\sum_{k=1}^{n}\left|\left\{\left(i_{1}, \ldots, i_{k}\right) \in \mathbb{Z}_{>0}^{k}: i_{1} \geq i_{2} \geq \cdots \geq i_{k}, i_{1}+\cdots+i_{k}=n\right\}\right| . \tag{3.4}
\end{equation*}
$$

This function can be also described using a generating function:

$$
\sum_{n=0}^{\infty} p(n) x^{n}=\prod_{k=1}^{\infty} \frac{1}{1-x^{k}}
$$

Proposition 3.1.1 The dimensions of the subspaces of a fixed grading of the Verma modules over Virasoro and Heisenberg algebra aee given by $\operatorname{dim} M(\Delta)^{n}=\operatorname{dim} M(\alpha)^{n}=p(n)$.

Verma modules posses the following universal property (Proposition 1.6 [38]):

Proposition 3.1.2 1. For any weight module $M$ with a weight $\lambda \in \mathfrak{h}^{*}$ there exists a surjective homomorphism of modules $M(\lambda) \rightarrow M$ specified by sending the vector $v_{\lambda}$ to a weight vector of the module $M$.
2. There exists a unique $\mathbb{Z} \times \mathfrak{h}^{*}$-graded submodule $J(\lambda)$ such that $L(\lambda)=M(\lambda) / J(\lambda)$ is a simple graded $\mathfrak{g}$-module. It is moreover irreducible lowest weight module with lowest weight $\lambda$.

As we will see Verma module over $\mathcal{H}$ admits also an action of $\mathcal{V}$. In this context we call it Fock module.

Theorem 3.1.1 Verma module over $\mathcal{H}$ is irreducible.

Proof. Suppose that $V \subset M(\alpha)$ is a nonzero submodule of the Verma module over $\mathcal{H}$. We need to show that $V=M(\alpha)$. Let $v \in V$ be any nonzero element. It is a linear combination of the basis vectors $v=\sum_{i_{1}, \ldots, i_{k}} c_{i_{1}, \ldots, i_{k}} a_{-i_{1}} \cdots a_{-i_{k}}|\alpha\rangle$ and at least one $c_{j_{1}, \ldots, j_{l}} \neq 0$. Because $V$ is a submodule it follows that $a_{j_{1}} \cdots a_{j_{l}} v \in V$. Notice that $a_{j_{1}} \cdots a_{j_{l}} a_{-i_{1}} \cdots a_{-i_{k}}|\alpha\rangle=0$ if $\left\{j_{1}, \ldots, j_{l}\right\} \neq\left\{i_{1} \ldots, i_{k}\right\}$. We want to show that $a_{j_{1}} \cdots a_{j_{l}} a_{-j_{1}} \cdots a_{-j_{l}}|\alpha\rangle \in \mathbb{C}|\alpha\rangle$ is nonzero. If there are no repetitions among the indices $\left\{j_{1}, \ldots, j_{l}\right\}$, one can easily compute by applying Heisenberg commutation relations and the condition $a_{n}|\alpha\rangle=0$ for $n>0$ that:

$$
\begin{equation*}
a_{j_{1}} \cdots a_{j_{l}} a_{-j_{1}} \cdots a_{-j_{l}}|\alpha\rangle=2^{-l}\left(\prod_{a=1}^{l} j_{a}\right)|\alpha\rangle . \tag{3.5}
\end{equation*}
$$

If there are repetitions commutation relations will generate another terms, which are however always positive. Therefore the coefficient in front of $|\alpha\rangle$ will be no less then $\alpha=2^{-l} \prod_{a=1}^{l} j_{a}$, hence nonzero. It follows that $|\alpha\rangle \in M(\alpha)$, so that $V=M(\alpha)$.

Abusing the notation we denote elements of the space of weights of the Virasoro algebra $\left(\mathcal{V}^{0}\right)^{*} \simeq \mathbb{C}^{2}$ by $(\Delta, c)$, and of the Heisenberg algebra $\left(\mathcal{H}^{0}\right)^{*} \simeq \mathbb{C}^{2}$ by $(\eta, k)$. Hence we denote by $M(\Delta, c)$ the Verma module over $\mathcal{V}$ and by $F(\eta)$ and the Verma module over $\mathcal{H}$ of lowest weight $(\eta, 1)$. The value of $c$ in any representation is called central charge of this representation. We denote corresponding weight states by $|\Delta\rangle$ and $|\eta\rangle$.

One can also define similar representation of the algebra $\mathcal{V}$. Let $c \in \mathbb{C}$ and let $\mathbb{C}_{c}$ be a one dimensional representation of $\mathcal{V}=-1=\bigoplus_{n \geq-1} \mathbb{C} L_{-1} \oplus \mathbb{C} c$ defined as $L_{n} \cdot 1=0$ for $n \geq-1$ and $c \cdot 1=c$. We define new representation as

$$
\begin{equation*}
\mathrm{Vi}_{c}=\operatorname{Ind}_{\mathcal{V} \geq-1}^{\mathcal{V}} \mathbb{C}_{c} \tag{3.6}
\end{equation*}
$$

Let $|0\rangle=1 \otimes 1$. This vector satisfies relations $L_{n}|0\rangle=0$ for $n \geq-1$. This representation will be used in the second section in this chapter to define Virasoro vertex operator algebra.

### 3.1.2 Shapovalov form and determinants

Definition 3.1.6 Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra. An anti-involution map $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$ is a linear map which inverses grading $\sigma\left(\mathfrak{g}^{n}\right) \subset \mathfrak{g}^{-n}, \sigma^{2}=\mathrm{Id}$ and for any $X, Y \in \mathfrak{g}$ we have $\sigma([X, Y])=$ $-[\sigma(X), \sigma(Y)]$.

Since any anti-involution $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$ is a Lie algebra homomorphism, we can extend its action to the enveloping algebra $U(\mathfrak{g})$ by using the equation $\sigma(X Y)=\sigma(Y) \sigma(X)$.

Example 3.1.1 In the case of the algebra $\mathcal{V}$ we can define an anti-involution map by $\sigma_{\mathcal{V}}\left(L_{n}\right)=L_{-n}$ and $\sigma_{\mathcal{V}}(c)=c$. In the case of the algebra $\mathcal{H}$ we have an anti-involution defined by $\sigma_{\mathcal{H}}\left(a_{n}\right)=a_{-n}$ and $\sigma_{\mathcal{H}}(k)=k$. The last example can be generalised to a family of inti-involutions $\sigma_{Q}$ by redefining its action by: $\sigma_{Q}\left(a_{0}\right)=Q-a_{0}$ for any $Q \in \mathbb{C}$. In particular $\sigma_{0}=\sigma_{\mathcal{H}}$.

Definition 3.1.7 Let $M$ be a module over a Lie $\mathbb{Z}$-algebra equipped with an anti-involution map $\sigma: \mathfrak{g} \rightarrow \mathfrak{g}$. A bilinear form $(\cdot, \cdot): M \times M \rightarrow \mathbb{C}$ is called Shapovalov form if for any $X \in \mathfrak{g}$ and $x, y \in M$ we have $(X \cdot x, y)=(x, \sigma(X) \cdot y)$.

Proposition 3.1.3 Given the module $M(\Delta, c)$ and anti-involution $\sigma_{\mathcal{V}}$ there exists a unique Shapovalov form $(\cdot, \cdot)_{\Delta, c}: M(\Delta, c) \times M(\Delta, c) \rightarrow \mathbb{C}$ satisfying $\left(v_{\Delta, c}, v_{\Delta, c}\right)_{\Delta, c}=1$.

There is also unique Shapovalov form on the Verma module over $\mathcal{H}:(\cdot, \cdot)_{\alpha}: \mathcal{F}(\alpha) \times \mathcal{F}(\alpha) \rightarrow$ $\mathbb{C}$ corresponding to the anti-involution $\sigma_{\mathcal{H}}$ and satisfying $\left(v_{\alpha}, v_{\alpha}\right)_{\alpha}=1$. This bilinear form is nondegenerate, i.e. for any nonzero $w \in \mathcal{F}(\alpha)$ there exists $\bar{w} \in \mathcal{F}(\alpha)$ such that $(w, \bar{w})_{\alpha}>0$. Moreover the basis (3.3) is orthogonal.

Proof. To prove the proposition it is enough to determine the value of the Shapovalov form on the basis vectors. In the case of the Virasoro algebra this means we would like to determine the value of the expressions of the form:

$$
\left(L_{-i_{1}} \cdots L_{-i_{k}}|\Delta\rangle, L_{-j_{1}} \cdots L_{-j_{l}}|\Delta\rangle\right)_{\Delta, c}=\left(|\Delta\rangle, L_{i_{k}} \cdots L_{i_{1}} L_{-j_{1}} \cdots L_{-j_{l}}|\Delta\rangle\right)_{\Delta, c}
$$

If $i_{1}+\cdots+i_{k}>j_{1}+\ldots j_{l}$ then the above expression is equal to 0 , since $L_{i_{k}} \cdots L_{i_{1}} L_{-j_{1}} \cdots L_{-j_{l}}|\Delta\rangle=$ 0 . If $i_{1}+\cdots+i_{k}<j_{1}+\ldots j_{l}$ then we arrive at the same conclusion by the symmetry. If $i_{1}+\cdots+i_{k}=j_{1}+\ldots j_{l}$ we can apply Virasoro commutation relations, reducing the expression $L_{i_{k}} \cdots L_{i_{1}} L_{-j_{1}} \cdots L_{-j_{l}}|\Delta\rangle$ to a multiple value of $|\Delta\rangle$. Then the condition $\left(v_{\Delta, c}, v_{\Delta, c}\right)_{\Delta, c}=1$ fixes the Shapovalov form.

The proof of the uniqueness for the Heisenberg algebra is analogous. In order to prove the orthogonality relation notice that if two sets of indices (with repetitions allowed) $\left\{i_{1}, \ldots, i_{k}\right\}$ and $\left\{j_{1}, \ldots, j_{l}\right\}$ are different, when applying the commutation relations while computing the expression

$$
a_{i_{k}} \ldots a_{i_{1}} a_{-j_{1}} \ldots a_{-j_{l}}|\alpha\rangle,
$$

some of the operators among $a_{i_{k}}, \ldots, a_{i_{1}}$ will always hit $|\alpha\rangle$, giving zero. Therefore in such a case

$$
\left(a_{-i_{1} \ldots} \ldots a_{-i_{k}}|\alpha\rangle, a_{-j_{1}} \ldots a_{-j_{l}}|\alpha\rangle\right)_{\alpha}=0 .
$$

Nondegenerancy follows from the fact that for basis vectors we have

$$
\left(a_{-i_{1}} \cdots a_{-i_{k}}|\alpha\rangle, a_{-i_{1}} \cdots a_{-i_{k}}|\alpha\rangle\right)_{\alpha}=\left(|\alpha\rangle, a_{i_{k}} \cdots a_{-i_{1}}|\alpha\rangle, a_{-i_{1}} \cdots a_{-i_{k}}|\alpha\rangle\right)_{\alpha}>2^{-l} \prod_{a=1}^{k} i_{a}>0
$$

where we have used equation (3.5) and following it remark. For a combination of the basis vectors $w=\sum_{i} c_{i} w_{i}$ with $c_{i} \in \mathbb{C}$ we have $(w, \bar{w})_{\alpha}=\sum_{i}\left|c_{i}\right|^{2}\left(w_{i}, w_{i}\right)_{\alpha}>0$, where $\bar{w}=\sum_{i} \bar{c}_{i} w_{i}$ and $\bar{c}_{i}$ is a complex conjugate of $c_{i}$.

As follows from the above proof, the decomposition of the Verma module $M(\Delta, c)=\bigoplus_{n \in \mathbb{Z}} M(\Delta, c)^{n}$ according to the grading is orthogonal with respect to the Shapovalov form. Notice also that this subspaces correspond to the operator's $L_{0}$ eigenspaces $M(\Delta, c)^{n}=\left\{v \in M(\Delta, c): L_{0} \cdot v=(n+\Delta) v\right\}$.

Definition 3.1.8 For any $n \in \mathbb{N}$ let $\operatorname{det}(c, \Delta)_{n}$ be the discriminant of the Shapovalov form restricted to the subspace $M(\Delta, c)^{n} \times M(\Delta, c)^{n}$. It is called Kac determinant.

Theorem 3.1.2 [38] The Kac determinant has the following form

$$
\begin{equation*}
\operatorname{det}(c, \Delta)_{n} \simeq \prod_{\substack{r, s \in \mathbb{Z}>0 \\ r \geq s \\ 1 \leq r s \leq n}} \Phi_{r, s}(c, \Delta)^{p(n-r s)}, \tag{3.7}
\end{equation*}
$$

where

$$
\Phi_{r, s}(c, \Delta)=\left\{\begin{array}{l}
\left(\Delta+\frac{1}{24}\left(r^{2}-1\right)(c-13)+\frac{1}{2}(r s-1)\right) \times \\
\left(\Delta+\frac{1}{24}\left(s^{2}-1\right)(c-13)+\frac{1}{2}(r s-1)\right) \\
+\frac{1}{16}\left(r^{2}-s^{2}\right)^{2} \text { if } r \neq s, \\
\Delta+\frac{1}{24}\left(r^{2}-1\right)(c-13)+\frac{1}{2}\left(r^{2}-1\right) \text { if } r=s .
\end{array}\right.
$$

Idea of the proof is to show that corresponding factors (with appropriate powers) divide the Kac determinant. Then one shows equality of the degrees of the Kac determinant and the right hand side of equation (3.7), as polynomials in $\Delta$. To prove the first step one shows the existence of appropriate singular vectors.

### 3.1.3 Singular vectors

Definition 3.1.9 Let $M$ be a module over a Lie $\mathbb{Z}$-algebra. A vector $v \in M-\{0\}$ is called singular vector if $X \cdot v=0$ for any $X \in \mathfrak{g}^{+}$and the corresponding submodule $U(\mathfrak{g}) v \subset M$ is not equal to $M$ (in other words - proper). It is called null vector if it is orthogonal to any vector with respect to the Shapovalov form.

Existence of a singular vector implies reducibility of the representation. Moreover from the Proposition 3.1.2 it follows that any singular vector in the Verma module $M(\lambda)$ must belong to the submodule $J(\lambda)$.

Any singular vector is automatically a null vector. There exist a null vector in $M(\Delta, c)^{n}$ if and only if $\operatorname{det}(c, \Delta)_{n}=0$. Note that the descendant of any null vector $v$ is again a null vector:

$$
\left(L_{-n} v, w\right)=\left(v, L_{n} w\right)=0,
$$

for any $w \in M$. This is not true in general for the singular vectors.
When talking about singular vectors in the Fock module $\mathcal{F}(\alpha)$ we need to be more careful. Since two algebras are acting on this space we can consider $\mathcal{V}$-singular vectors and $\mathcal{H}$-singular vectors. From the Proposition 3.1 .3 it follows that the second set is empty (Shapovalov form is nondegenerate, hence there are no null vectors and in consequence no singular vectors). Therefore we consider only $\mathcal{V}$-singular vectors and this assumption will be implicit.

Example 3.1.2 Consider the Verma module with $\Delta=0$ (called vacuum module). Then $v_{1}=$ $L_{-1}|0\rangle$ is a singular vector. Clearly $L_{n} v_{1}=0$ for $n \geq 2$. Moreover

$$
L_{1}\left(L_{-1}|0\rangle\right)=L_{-1} L_{1}|0\rangle+\left[L_{1}, L_{-1}\right]|0\rangle=2 L_{0}|0\rangle=0 .
$$

Second example is more interesting. Let $v_{2}=\left(a L_{-2}+L_{-1}^{2}\right)|\Delta\rangle$, where $a \in \mathbb{C}$ is a parameter. Then $L_{n} v_{2}=0$ for $n \geq 3$, whereas

$$
\begin{aligned}
L_{1}\left(v_{2}\right) & =a\left[L_{1}, L_{-2}\right]|\Delta\rangle+a L_{-1}\left[L_{1}, L_{-1}\right]|\Delta\rangle+\left[L_{1}, L_{-1}\right] L_{-1}|\Delta\rangle \\
& =\left(3 a L_{-1}+2 L_{0} L_{-1}+2 L_{-1} L_{0}\right)|\Delta\rangle \\
& =(3 a+2(2 \Delta+1))|\Delta\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
L_{2}\left(v_{2}\right) & =a\left[L_{2}, L_{-2}\right]|\Delta\rangle+L_{-1}\left[L_{2}, L_{-1}\right]|\Delta\rangle+\left[L_{2}, L_{-1}\right] L_{-1}|\Delta\rangle \\
& =\left(4 a L_{0}+a \frac{c}{2}+3 L_{1} L_{-1}\right)|\Delta\rangle \\
& =\left(4 a \Delta+a \frac{c}{2}+6 \Delta\right)|\Delta\rangle .
\end{aligned}
$$

It follows that for $a=-\frac{2}{3}(2 \Delta+1)$ and $c=\frac{18 \Delta}{2 \Delta+1}-8 \Delta$ vector $v_{2}$ is singular.

Theorem 3.1.3 (Corollary 5.2, [38]) Consider a Verma module $M(\Delta(\beta), c(\beta))$ for the Virasoro algebra, where

$$
c(\beta)=1-6 \frac{(\beta-1)^{2}}{\beta}, \quad \Delta(\beta)=\frac{(r \beta-s)^{2}-(\beta-s)^{2}}{4 \beta},
$$

$r, s \in \mathbb{Z}$ and $\beta \in \mathbb{C}^{*}$. Then there exists a unique (up to rescaling) singular vector in $M(\Delta(\beta), c(\beta))$, whose grading is $n=r$.

As a consequence from this theorem we obtain vanishing of the corresponding Kac determinant $\operatorname{det}(c(\beta), \Delta(\beta))_{m}$ in gradings $m \geq n$. This vanishing follows from the existence of the null vectors in those this gradings.

In general any singular (and homogeneous) vector $v_{s} \in M(\Delta, c)^{n}$ can be written in a unique way in the basis of the module:

$$
\begin{equation*}
v_{s}=A_{r, s}(L)|\Delta\rangle=\sum_{\substack{k_{1}>\ldots>k_{m}>0 \\ k_{1}+\cdots+k_{m}=n}} c_{k_{1}, \ldots, k_{m}} L_{-k_{1} \ldots L_{-k_{m}}|\Delta\rangle,} \tag{3.8}
\end{equation*}
$$

where $A_{r, s}(L) \in U\left(\mathcal{V}^{-}\right)$. Computing the exact expressions for the coefficients $c_{k_{1}, \ldots, k_{m}} \in \mathbb{C}$ of this decomposition is in general an open problem. It was solved if $s=1$ (or $r=1$ ), in which case the answer is [6]:

$$
\begin{equation*}
A_{r, 1}=\sum_{p_{1}+p_{2}+\cdots+p_{k}=r} \frac{(r-1)!^{2}}{\prod_{i=1}^{k-1}\left(\sum_{j=1}^{i} p_{j}\right)\left(r-\sum_{j=1}^{i} p_{j}\right)} b^{2(r-k)} L_{-p_{1}} L_{-p_{2}} \ldots L_{-p_{k}} \tag{3.9}
\end{equation*}
$$

Proposition 3.1.4 Let $\mathfrak{g}$ be a Lie $\mathbb{Z}$-algebra and assume that $\phi: M_{1} \rightarrow M_{2}$ is a homomorphism of the weight $\mathfrak{g}$-modules. Suppose that the image of the lowest weight vector $v \in M_{1}$ is nonzero. Then if the weights for $M_{1}$ and $M_{2}$ are equal it is a weight vector. Otherwise it is a singular vector. Moreover nonzero image of a singular vector is again a singular vector or a weight vector.

Proof. Let $v \in M_{1}$ be a weight vector. Observe that $X \cdot \phi(v)=\phi(X \cdot v)=0$ for any $X \in \mathfrak{g}^{+}$. Therefore, if nonzero, $\phi(v)$ is again a singular vectors or a weight vector. Since for any $H \in \mathfrak{h}$ we have $H \cdot \phi(v)=\phi(H \cdot v)=\lambda(H) \phi(v)$, if the weights of the modules $M_{1}$ and $M_{2}$ coincides $\phi(v)$ is a weight vector. Statement of the proposition for the singular vectors follows then from the fact that a singular vector is a weight vector for a module, which it generates.

Proposition 3.1.5 (Proposition 5.2, (38]) Let $\beta \in \mathbb{C}$. Then there exists element $A_{n} \in U(\mathcal{V} \leq)_{-n}$, called Shapovalov element, such that $A_{n}|\Delta(\beta)\rangle$ is a singular vector in the module $M(\Delta(\beta), c(\beta))$.

### 3.1.4 Homomorphism $S_{\alpha, Q}$

The Fock module admits also an action of the Virasoro algebra, which we are now going to describe. Let $Q \in \mathbb{R}$ be a parameter. We also assume in this section that $\alpha \in \mathbb{R}$. Then we can define following representation of $\mathcal{V}$ on $\mathcal{F}(\alpha)$, for which the Virasoro operators take form:

$$
\begin{align*}
L_{0}(Q) & =2 \sum_{n=1}^{\infty} a_{-n} a_{n}+a_{0}\left(a_{0}-Q\right), \\
L_{m}(Q) & =\sum_{n \neq 0, m} a_{m-n} a_{n}+\left(2 a_{0}-(m+1) Q\right) a_{m} \quad \text { for } m \neq 0,  \tag{3.10}\\
c(Q) & =1-6 Q^{2} .
\end{align*}
$$

Although the sums above are infinite, for any given $v \in \mathcal{F}(\alpha)$ only finitely many terms contribute to the value of $L_{m}(Q) v$. Therefore operators (3.10) are well defined elements of $\operatorname{End}(\mathcal{F}(\alpha))$.

Lemma 3.1.1 The action of $\mathcal{V}$ on $\mathcal{F}(\alpha)$ is compatible with the anti-involutions $\sigma_{\mathcal{V}}$ and $\sigma_{Q}$, i.e. $\sigma_{\mathcal{H}}\left(L_{i}(Q)\right)=L_{-i}(Q)$.

Proof. We know that $\sigma_{Q}\left(a_{0}\right)=Q-a_{0}$ and $\sigma_{Q}\left(a_{n}\right)=a_{-n}$ for $n \neq 0$. It follows that $(n \neq 0)$ :

$$
\begin{aligned}
\sigma_{Q}\left(L_{0}(Q)\right) & =\sum_{n=1}^{\infty} a_{-n} a_{n}+\left(Q-a_{0}\right)\left(-a_{0}\right)=L_{0}(Q) \\
\sigma_{Q}\left(L_{m}(Q)\right) & =\sum_{n \neq 0, m} a_{-n} a_{-m+n}+\left(2 Q-2 a_{0}-(m+1) Q\right) a_{-m}=L_{-m}(Q)
\end{aligned}
$$

Let $|\alpha\rangle \in \mathcal{F}(\alpha)$ be the weight state. We can consider a weight module over $\mathcal{V}$ defined using the representation 3.10 by the equation $N(\alpha, Q)=U(\mathcal{V} \leq)|\alpha\rangle \subset \mathcal{F}(\alpha)$. From the universal property (3.1.2) it follows that for $c=1-6 Q^{2}$ and $\Delta=\alpha(\alpha-Q)$ there exists a surjective homomorphism of modules $S_{\alpha, Q}: M(\Delta, c) \rightarrow N(\alpha, Q)$.

Proposition 3.1.6 Let $c=1-6 Q^{2}$ and $\Delta=\alpha(\alpha-Q)$ where $\alpha, Q \in \mathbb{R}$. Then the homomorphism $S_{\alpha, Q}$ enjoys the following properties:

- $S_{\alpha, Q}$ preserves the gradings,
- $S_{\alpha, Q}$ preserves the bilinear forms,
- kernel of $S_{\alpha, Q}$ is the subspace of all null vectors in $M(\Delta, c)$.

Therefore $N(\alpha, Q)=J(c, \Delta)$ is the irreducible weight module from the Proposition 3.1.2.

Proof. The first assertion follows from the definition of the operator $S_{\alpha, Q}$. To simplify notation let $S=S_{\alpha, Q}$. First let us now proof that

$$
\begin{equation*}
(v, w)_{\Delta, c}=(S v, S w)_{\alpha} \tag{3.11}
\end{equation*}
$$

for any $v, w \in M(\Delta, c)$. We will prove by the induction of the (equal) grading of $v$ and $w$. If the gradings are different, both sides vanishes. If they are weight vectors the premise follows from the defining equation $\left(v_{\Delta, c}, v_{\Delta, c}\right)_{\Delta, c}=\left(v_{\alpha}, v_{\alpha}\right)_{\alpha}$. We pass to the induction step. We assume that the statement is true for any $v$ and $w$ for which $|v|=|w| \leq n$. Let $v^{\prime}, w^{\prime} \in M(\Delta)^{n+1}$. Then there exist $i \in \mathbb{Z}_{>0}$ and $v \in M(\Delta)^{n+1-i}$ such that we can write $v^{\prime}=L_{-i} v$. Let From Lemma 3.1.1 and the definition of the form $(\cdot, \cdot)_{\alpha}$ it follows that:

$$
\left(L_{i}(Q) \xi, \eta\right)_{\alpha}=\left(\xi, \sigma_{\mathcal{H}}\left(L_{i}(Q)\right) \eta\right)_{\alpha}=\left(\xi, L_{-i}(Q) \eta\right)_{\alpha}
$$

for any $\xi, \eta \in \mathcal{F}(\alpha)$. Because $L_{i}(Q)\left(S_{\alpha, Q} v\right)=S_{\alpha, Q}\left(L_{i} v\right)$, using the induction hypothesis, we conclude that

$$
\begin{aligned}
\left(v^{\prime}, w^{\prime}\right)_{\Delta, c} & =\left(L_{-i} v, w^{\prime}\right)_{\Delta, c}=\left(v, L_{i} w^{\prime}\right)_{\Delta, c}=\left(S_{\alpha, Q} v, S_{\alpha, Q} L_{i} w^{\prime}\right)_{\alpha} \\
& =\left(S_{\alpha, Q} v, L_{i}(Q) S_{\alpha, Q} w^{\prime}\right)_{\alpha}=\left(L_{-i}(Q) S_{\alpha, Q} v, S_{\alpha, Q} w^{\prime}\right)_{\alpha} \\
& =\left(S_{\alpha, Q} L_{-i} v, S_{\alpha, Q} w^{\prime}\right)_{\alpha}=\left(S_{\alpha, Q} v^{\prime}, S_{\alpha, Q} w^{\prime}\right)_{\alpha} .
\end{aligned}
$$

This gives us the induction step.
To prove the third assertion suppose that $v \in M(\Delta, c)$ is a null vector. Then from the equation (3.11) it follow that $(S v, S w)_{\alpha}=0$ for any $w \in M(\Delta, c)$. Because the coefficients $\alpha$ and $Q$ are real, the matrix of the operator $S$ in the bases (3.3) and (3.2) has real coefficients (it can be calculated using the equations (3.10). Therefore $(S v, S(\bar{v}))_{\alpha}=(S v, \overline{S v})_{\alpha}=0$. By Proposition (3.1.3) this can only happen when $S v=0$, hence $v$ belongs to the kernel of $S$. In the other direction, if $S v=0$ then for any $w \in M(\Delta, c)$ we have $(v, w)_{\Delta, c}=(S v, S w)_{\alpha}=0$, so $v$ is a null vector.

Let us now consider the case of Verma modules containing singular vectors. In such a case the map $S_{\alpha, Q}$ is defined for $c=1-6 Q^{2}, \Delta=\alpha_{r, s}\left(\alpha_{r, s}-Q\right)$ and $\alpha_{r, s}=\frac{r-1}{2} \beta^{-\frac{1}{2}}-\frac{s-1}{2} \beta^{\frac{1}{2}}$, where $r, s \in \mathbb{Z}$. Therefore from Proposition 3.1.6, using the representation of the singular vector (3.8), we obtain directly:

Corollary 3.1.1 In Fock module $\mathcal{F}\left(\alpha_{r, s}\right)$ the relation $A_{r, s}\left(L\left(\alpha_{r, s}\right)\right)\left|\alpha_{r, s}\right\rangle=0$ holds.

### 3.1.5 Duality

Let us notice that for $\alpha^{\prime}=Q-\alpha$ we have $\Delta=\alpha^{\prime}\left(\alpha^{\prime}-Q\right)=\alpha(\alpha-Q)$. This means that the Verma module $M(\Delta, c)$ is the domain of the two homomorphisms: $S_{\alpha, Q}$ and $S_{Q-\alpha, Q}$. In particular from Corollary 3.1.1 we conclude that any singular vector gives rise to two different equations in the two modules: $\mathcal{F}(\alpha)$ and $\mathcal{F}(Q-\alpha)$. Moreover these two modules are dual to each other in the sense of the following definition.

Definition 3.1.10 Let $\mathfrak{g}$ be a Lie $\mathbb{Z}$-algebra and let $M=\bigoplus_{n \in \mathbb{Z}} M^{n}$ be a module over $\mathfrak{g}$ (cf. Definition 3.1.2) for which for any $n \geq 0$ we have $\operatorname{dim}\left(M^{n}\right)<\infty$ and for $n<0$ we have $M^{n}=\{0\}$. We define contragradient dual module as $M^{*}=\bigoplus_{n \geq 0}\left(M^{n}\right)^{*}$, where $\left(M^{n}\right)^{*}$ is a vector space dual to $M^{n}$. The grading is defined as $\left(M^{*}\right)^{n}=\left(M^{n}\right)^{*}$.

Example 3.1.3 Modules $\mathcal{F}(\alpha)$ and $\mathcal{F}(Q-\alpha)$ are contragradient dual to each other.

Notion dual to the singular vector is the cosingular vector.

Definition 3.1.11 Let $M$ be a module over Lie $\mathbb{Z}$-algebra $\mathfrak{g}$ with a weight vector $v_{0}$ (here we does not assume that $v_{0}$ spans $M$ ). We say that $v \in M$ is a cosingular vector if it does not belong to $U(\mathfrak{g}) v_{0}$.

Proposition 3.1.7 There exists a consigular vector in $\mathcal{F}(\alpha)$ if and only if relation

$$
\begin{equation*}
\sum_{k=1}^{n} \sum_{i_{1}+\cdots+i_{k}=n} a_{i_{1}, \ldots, i_{k}} L_{-i_{1}} \cdots L_{-i_{k}}|\alpha\rangle=0 \tag{3.12}
\end{equation*}
$$

holds for some nonzero collection of the coefficients $a_{i_{1}, \ldots, i_{k}}$.

Proof. Let us consider Fock module over $\mathcal{V}$. It is $\mathbb{Z}$-graded $\mathcal{F}(\alpha)=\bigoplus_{n \in \mathbb{Z}} \mathcal{F}(\alpha)^{n}$. Suppose that there exists a cosingular vector $v \in \mathcal{F}(\alpha)$. If we can decompose $v$ into homogeneous parts with respect to this grading at least one homogeneous component need to be consingular as well. Therefore without loss of generality we can assume that $v$ is homogeneous. Let $p(n)$ be the number of the partitions of $n$, as defined in (3.4). Then $\operatorname{dim}\left(\mathcal{F}(\alpha)^{n}\right)=p(n)=\operatorname{dim}\left(U(\mathcal{V})^{n}\right)=p(n)$. Existence of the cosingular vector implies that $\operatorname{dim}\left(U(\mathcal{V})^{n}|\alpha\rangle\right)<\operatorname{dim}\left(U(\mathcal{V})^{n}\right)$. Therefore vectors $L_{-i_{1}} \cdots L_{-i_{k}}|\alpha\rangle$ span a vector space of dimension lower then $p(n)$ and cannot be linearly independent. Hence a linear relation of the form

$$
\sum_{k=1}^{n} \sum_{i_{1}+\cdots+i_{k}=n} a_{i_{1}, \ldots, i_{k}} L_{-i_{1}} \cdots L_{-i_{k}}|\alpha\rangle=0
$$

for some nonzero collection of the coefficients $a_{i_{1}, \ldots, i_{k}}$ must hold.
On the other hand suppose that the relation (3.12) holds. It follows that $\operatorname{dim}\left(U(\mathcal{V})^{n} v_{0}\right)<$ $\operatorname{dim}\left(U(\mathcal{V})^{n}\right)$, so there exists a cosingular vector in grading $n$.

From Corollary 3.1.1 we conclude that there exist cosingular vectors in the modules $\mathcal{F}\left(\alpha_{r, s}\right)$.

### 3.2 Vertex algebras

### 3.2.1 Basic definitions and properties

Let us start with fixing some notation. For any vector space $V$ we denote by $V\left[\left[z, z^{-1}\right]\right]$ the vector space of formal power series with coefficients in $V$, i.e. expressions of the form:

$$
\sum_{i \in \mathbb{Z}} A_{i} z^{i}, \quad A_{i} \in V
$$

whereas by $V((z))$ its subspace consisting of expressions such that there exists $N \in \mathbb{Z}$ for which for all $i<N$ we have $A_{i}=0$. This subspace is an algebra under the usual multiplication:

$$
\left(\sum_{i \in \mathbb{Z}} A_{i} z^{i}\right)\left(\sum_{j \in \mathbb{Z}} B_{j} z^{j}\right)=\sum_{n \in \mathbb{Z}}\left(\sum_{i+j=n} A_{i} B_{j}\right) z^{n},
$$

where because of the vanishing of the coefficients with sufficiently small indices, the sum in the bracket in the right hand side of the above equation is finite, and the whole result belongs to $V((z))$. Similarly we define formal powers series in several variables $V\left[\left[z_{1}, z_{1}^{-1}, \ldots, z_{k}, z_{k}^{-1}\right]\right]$ :

$$
\sum_{i_{1}, \ldots, i_{k} \in \mathbb{Z}} A_{i_{1}, \ldots, i_{k}} z_{1}^{i_{1}} \cdots z_{1}^{i_{k}}, \quad A_{i_{1}, \ldots, i_{k}} \in V
$$

Moreover we use the following convention: $V((z))((w))=(V((z)))((w))$.
An example of a formal power series in two variables is the delta distribution $\delta_{z-w}=\sum_{n \in \mathbb{Z}} z^{-n-1} w^{n}$. We can also define operation

$$
\iota_{z, w}: V\left[\left[(z-w)^{ \pm 1}\right]\right] \rightarrow V\left[\left[z^{ \pm 1}, w^{ \pm 1}\right]\right]
$$

by expanding the rational functions like $\frac{1}{z-w}$ in the domain $|z|>|w|$ :

$$
\frac{1}{z-w}=\sum_{n=0}^{\infty} z^{-n-1} w^{n} .
$$

In what follows we will lso use more general power series:

$$
\mathbb{C}\{z\}=\left\{\sum_{r \in \mathbb{C}} a_{r} z^{r}: \text { only countably many of } a_{r} \in \mathbb{C} \text { are nonzero }\right\} .
$$

Let us define a few operations on this set. For any element $f(z)=\sum_{n \in \mathbb{C}} f_{n} z^{n}$ of $\mathbb{C}\{z\}$ we set $\operatorname{Res}_{z} f(z)=f_{-1}$ and define its derivative: $\partial f(z)=\sum_{n \in \mathbb{C}} n f_{n} z^{n-1}$. Clearly we have a relation $\operatorname{Res}_{z} \partial f(z)=0$. For $f \in V\left[\left[z^{ \pm 1}\right]\right]$ we define its annihilation part: $f(z)_{a}=\sum_{n<0} f_{n} z^{n}$ and the creation part $f(z)_{c}=\sum_{n \geq 0} f_{n} z^{n}$. Those operations commute with taking the derivative: $\partial f(z)_{a}=$ $(\partial f(z))_{a}$ and $\partial f(z)_{c}=(\partial f(z))_{c}$. We also denote the commutator as $[a, b]=a b-b a$.

Suppose that $V=\bigoplus_{n \in \mathbb{Z}}$ is a $\mathbb{Z}$-graded vector space. We say that $\phi \in \operatorname{End} V$ has degree $n$ if $\phi\left(V^{m}\right) \subset V^{n+m}$ for any $m \in \mathbb{Z}$. In such case we write $\operatorname{deg}(\phi)=n$.

Definition 3.2.1 Let $V$ be a vector space. An operator-valued formal power series

$$
A(z) \in \operatorname{End} V\left[\left[z, z^{-1}\right]\right]
$$

is called $a$ field on $V$ if for any $v \in V$ we have $A(z) v \in V((z))$. We denote the space of fields in $V$ by $\mathcal{F}(V)$. Two fields $A(z)$ and $B(w)$ are mutually local if there exists $N \in \mathbb{N}$ such that $(z-w)^{N}[A(z), B(w)]=0$.

Definition 3.2.2 $A$ Vertex algebra consists of a $\mathbb{Z}$-graded vector space $V=\bigoplus_{m \in \mathbb{Z}} V^{m}$, an element $|0\rangle \in V^{0}$ (called vacuum), and a linear map $Y: V \rightarrow \operatorname{End}(V)\left[\left[z, z^{-1}\right]\right]$, such that for any $v \in V$ the expression $Y(v, z)=\sum_{n \in \mathbb{Z}} v_{(n)} z^{-n-1}$ is a field, satisfying the following set of axioms:

- (translation covariance) $[T, Y(v, z)]=\partial Y(v, z)$, where $T(v)=v_{(-2)}|0\rangle$,
- (vacuum) $Y(|0\rangle, z)=I d_{V}, Y(v, z)|0\rangle \in V[[z]]$ and $\left.Y(v, z)|0\rangle\right|_{z=0}=v$,
- (locality) $\forall_{v, w \in V}$ the fields $Y(v, z)$ and $Y\left(w, z^{\prime}\right)$ are mutually local.

Moreover for any $a \in V^{m}$ we have $\operatorname{deg}\left(a_{(n)}\right)=-n+m-1$. In particular $\operatorname{deg}(T)=1$.

Remark 3.2.1 There are different variants of the definition of vertex algebras in the literature. The above version follows [28]. In other places $\mathbb{Z}$-gradation is not required (for example in [38]).

Definition 3.2.3 $A$ homomorphism of vertex algebras $(V,|0\rangle, T, Y) \rightarrow\left(V^{\prime},|0\rangle^{\prime}, T^{\prime}, Y^{\prime}\right)$ is a linear map $\rho: V \rightarrow V^{\prime}$ such that $\rho T=T^{\prime} \rho, \rho(|0\rangle)=|0\rangle^{\prime}$ and

$$
\rho(Y(v, z) w)=Y^{\prime}(\rho(v), z) \rho(w)
$$

for any $v, w \in V$.

Definition 3.2.4 To any vertex algebra $V$ we can associate a group of its automorphisms Aut $(V)$, defined as the group of isomorphism $\phi: V \rightarrow V$.

From the vacuum axiom we know that the map $v \rightarrow Y(v, z)$ is injective. Its image is described by the following theorem.

Theorem 3.2.1 (Goddard's uniqueness theorem) Let $(V,|0\rangle, T, Y)$ be a vertex algebra and let $a \in V$. Assume also that $A(z)$ is a field on $V$ such that:

1. $A(z)|0\rangle \in V[[z]]$ and $\left.A(z)|0\rangle\right|_{z=0}=a$,
2. $\partial_{z} A(z)|0\rangle=T A(z)|0\rangle$,
3. $A(z)$ is mutually local with $Y(b, w)$ for any $b \in V$.

Then $A(z)=Y(a, z)$.

Proof. Consider the following two conditions for an element of $b(z) \in V[[z]]: b(0)=a$ and $\partial_{z} b(z)=$ $T b$. They fix $b(z)$ uniquely, as we can compute its coefficients iteratively (in fact $b(z)=e^{T z} a$ ) and are satisfied by both $A(z)|0\rangle$ and $Y(a, z)|0\rangle$. It follows that $A(z)|0\rangle=Y(a, z)|0\rangle$. Let now $v \in V$ be any element. For large enough $N$ we can write

$$
\begin{aligned}
(z-w)^{N} A(z) Y(v, w)|0\rangle & =(z-w)^{N} Y(v, w) A(z)|0\rangle \\
& =(z-w)^{N} Y(v, w) Y(a, z)|0\rangle \\
& =(z-w)^{N} Y(a, z) Y(v, w)|0\rangle
\end{aligned}
$$

Since $Y(v, w)|0\rangle$ is well defined at $w=0$, evaluating at this point we get $z^{N} A(z) v=z^{N} Y(a, z) v$. Multiplying both sides by $z^{-N}$, and since $v$ was arbitrary, we get the desired result.

Theorem 3.2.2 (Associativity) Let $a, b, c \in V$. The following three elements are expansions of the same element of $V[[z, w]]\left[z^{-1}, w^{-1},(z-w)^{-1}\right]$ :

- $Y(a, z) Y(b, w) c \in V((z))((w))$,
- $Y(b, w) Y(a, z) c \in V((w))((z))$,
- $Y(Y(a, z-w) b, w) c \in V((w))((z-w))$.

To understand this theorem let us see how these elements look in the spaces appearing above:

- in $V((z))((w)): Y(a, z) Y(b, w) c=\sum_{k=k_{0}}^{\infty} \sum_{l=l_{0}\left(k_{0}\right)}^{\infty} a_{(-1-l)} b_{(-k-1)} c z^{l} w^{k}$,
- in $V((w))((z)): Y(b, w) Y(a, z) c=\sum_{l=l_{0}}^{\infty} \sum_{k=k_{0}\left(l_{0}\right)}^{\infty} b_{(-k-1)} a_{(-1-l)} c z^{l} w^{k}$,
- in $V((w))((z-w)): Y(Y(a, z-w) b, w) c=\sum_{l=l_{0}}^{\infty} \sum_{k=k_{0}\left(l_{0}\right)}^{\infty}\left(a_{(-1-l)} b\right)_{(-k-1)} c(z-w)^{l} w^{k}$.

We see that in the first case arbitrarily low powers of $z$ can appear, whereas in the second and in the third cases arbitrarily low powers of $w$.

Definition 3.2.5 We define normal ordered product of two fields $A(z)$ and $B(w)$ by

$$
: A(z) B(w):=A(z)_{c} B(w)+B(w) A(z)_{a}
$$

For more than two fields we set:

$$
\left.: A_{1}\left(z_{1}\right) \cdots A_{n}\left(z_{n}\right):=: A_{1}\left(z_{1}\right)\left(: A_{2}\left(z_{2}\right): \cdots: A_{n-1}\left(z_{n-1}\right) A_{n}\left(z_{n}\right):\right) \cdots:\right): .
$$

More explicit formula is:

$$
: A(z) B(w):=\sum_{n \in \mathbb{Z}}\left(\sum_{m<0} B_{(n)} A_{(m)} z^{-m-1}+\sum_{m \geq 0} A_{(m)} B_{(n)} z^{-m-1}\right) w^{-n-1}
$$

One can check that the operation of the normally ordered product in general is not associative neither commutative. For example for $\phi(x)=\sum_{n \in \mathbb{Z}} a_{n} z^{-n-1}$ we have:

$$
:(: \phi(z) \phi(z):) \phi(z):=: \phi(z)(: \phi(z) \phi(z):):+\frac{\partial^{2}}{\partial z^{2}} \phi(z) .
$$

This gives a counterexample to the Remark 2.2.6 in [28].

Remark 3.2.2 A generatisation of this example of non-associativity will be important afterwards, when we consider action of $\mathcal{V}$ on the space of intertwining operators.

Definition 3.2.6 Let $V$ be a vertex algebra and let $A=\left\{a^{i}(z)\right\}_{i \in I}$ be a family of mutually local fields on $V$. We say that $V$ is strongly generated by the family $A$ if every field of $V$ can be expressed as normally ordered product in the derivatives of the fields $a^{i}(z)$ :

$$
: \partial^{j_{1}-1} a^{i_{1}}(z) \cdots \partial^{j_{n}-1} a^{i_{k}}(z):
$$

Equivalently $V$ is spanned as a vector space by the monomials in $a_{(n)}^{i}(i \in I, n \in \mathbb{Z})$.

Remark. In this notation under the normal ordering sign each derivative acts only on the operator to which it is adjacent.

The following theorem gives sufficient conditions for a family of fields $\left\{a^{i}(z)\right\}_{i \in I}$ to give rise to a vertex algebra.

Theorem 3.2.3 (Strong reconstruction theorem) Given a vector space $V$, a distinguished vector $|0\rangle \in V$, an operator $T \in \operatorname{End} V$, a countable family of mutually local fields $\left\{a^{i}(z)\right\}_{i \in I}$ satisfying:

- $a^{i}(z)|0\rangle=a^{i}+O(z)$ for some vectors $a^{i} \in V$,
- $T|0\rangle=0$ and $\left[T, a^{i}(z)\right]=\partial_{z} a^{i}(z)$,
- $V$ is spanned by the vectors $a_{\left(-j_{1}\right)}^{i_{1}} \ldots a_{\left(-j_{k}\right)}^{i_{k}}|0\rangle, j_{a}>0$,
one can define a unique vertex algebra structure on $V$ such that $Y\left(a^{i}, z\right)=a^{i}(z)$ by setting:

$$
\begin{equation*}
Y\left(a_{\left(-j_{1}\right)}^{i_{1}} \cdots a_{\left(-j_{k}\right)}^{i_{k}}|0\rangle, z\right)=\frac{1}{\left(j_{1}-1\right)!\cdots\left(j_{k}-1\right)!}: \partial^{j_{1}-1} a^{i_{1}}(z) \cdots \partial^{j_{n}-1} a^{i_{k}}(z): . \tag{3.13}
\end{equation*}
$$

Locality of the fields (3.13) follows from the following lemma.

Lemma 3.2.1 (Lemma 2.3.4, [28]) Let $a(z), b(z)$ and $c(z)$ be mutually local fields. Then the fields : $a(z) b(z)$ : and $c(z)$ are mutually local as well.

### 3.2.2 Operator product expansion

One of the important concepts in CFT is the operator product expansion. This notion was introduced in [56] as a tool to deal with ill-defined product of two quantum fields $\phi_{1}\left(z_{1}\right) \phi_{2}\left(z_{2}\right)$. In this approach one considers singularities in $\phi_{1}\left(z_{1}\right) \phi_{2}\left(z_{2}\right)$, which appear when $y$ tends to $x$ :

$$
\phi_{1}\left(z_{1}\right) \phi_{2}\left(z_{2}\right) \sim \sum_{j=0}^{N-1} \frac{C_{12}^{j}\left(z_{2}\right)}{\left(z_{1}-z_{2}\right)^{j+1}}
$$

Operator product expansion equips a vertex algebra with a sort of multiplication. This multiplication can be applied to compute correlators of the theory. This can be done by successively multiplying the fields appearing in the correlators:

$$
\left\langle\phi_{1}\left(z_{1}\right) \phi_{2}\left(z_{2}\right) \ldots \phi_{n}\left(z_{n}\right)\right\rangle=\sum_{j=0}^{N-1} \frac{1}{\left(z_{1}-z_{2}\right)^{j+1}}\left\langle C_{12}^{j}\left(z_{2}\right) \phi_{3}\left(z_{3}\right) \ldots \phi_{n}\left(z_{n}\right)\right\rangle
$$

and henceforth reducing the number of fields appearing therein. The iteration procedure stops since the two-point and the three-point correlators are heavily fixed by the conformal symmetry. Operator product expansion leads also to the bootstrap approach, which we are not going to discuss here.

Proposition 3.2.1 [38] Let $A(z), B(w) \in \mathcal{F}(V)$ be two fields. Then the following statements are equivalent:

- $A(z)$ and $B(w)$ are mutually local,
- There exists fields $C_{j}(w), j=0,1, \ldots N-1$ such that

$$
[A(z), B(w)]=\sum_{j=0}^{N-1} C_{j}(w) \partial_{w}^{(j)} \delta_{z-w},
$$

- $A(z) B(w)$ has an expression

$$
\sum_{j=0}^{N-1} \iota_{z, w} \frac{C_{j}(w)}{(z-w)^{j+1}}+: A(z) B(w):
$$

and $B(w) A(z)$ has an expression:

$$
\sum_{j=0}^{N-1} \iota_{w, z} \frac{C_{j}(w)}{(z-w)^{j+1}}+: A(z) B(w):
$$

- $A(z) B(w)$ converges to the above formula in the domain $|z|>|w|$ and $B(w) A(z)$ does so in the domain $|w|>|z|$.

In a situation like in the proposition above we write

$$
A(z) B(w) \sim \sum_{j=0}^{N-1} \frac{C_{j}(w)}{(z-w)^{j+1}}
$$

and call it operator product expansion, or $O P E$. The way in which right hand side should be expanded is implicit in the order of fields on the left hand side: if $A(z)$ is applied after $B(w)$ we should make the expansion in the domain $|z|>|w|$. The physical interpretation is that $|z|,|w|$ correspond to times of applying our operators, which should be consistent with the order of the application of the operators.

### 3.2.3 Conformal symmetry

In this section we make a link with the first part of the chapter. We introduce Virasoro algebra action on the vertex algebras.

Definition 3.2.7 Vertex operator algebra is a $\mathbb{Z}$-graded vertex algebra with a distinguished conformal vector $\omega \in V$ such that $Y(\omega, z)=\sum_{n \in \mathbb{Z}} L_{n}^{V} z^{-n-2}$ satisfies:

- Operators $L_{n}^{V}$ form a representation of the Virasoro algebra for some value of central charge $c^{V}$,
- $L_{-1}^{V}=T$ and $\left.L_{0}^{V}\right|_{V_{n}}=n \operatorname{Id}_{V_{n}}$ for $n \in \mathbb{Z}$.

Therefore the underlying vector space of the Vertex operator algebra is a module over $\mathcal{V}$. The field $Y(\omega, z)$ is called energy-momentum tensor of the vertex algebra $V$ and usually denoted also by $T(z)$ (which should not be confused with the operator $T$ ). The operator $L_{0}^{V}$ can be interpreted as Hamiltonian and in physical models we demand that the set of its eigenvalues (or spectrum) is real and bounded from below. In Proposition 2.9 this operator was related to the vector field $-\sum_{a} z_{a} \frac{\partial}{\partial z_{a}}$, which generates dilations. If we interpret $\log |z|$ as time, those are time translations, which correspond to the Hamiltonian action.

Definition 3.2.8 Let $\operatorname{Im}(Y) \subset \operatorname{End}(V)[[z]]$ be the image of $Y$. One can define action of $\mathcal{V}$ on this space transporting it from the action on $V$ :

$$
\begin{equation*}
L_{k} \cdot Y(v, z)=Y\left(L_{k}^{V} v, z\right) \tag{3.14}
\end{equation*}
$$

This definition is well defined, since the field $Y(v, z)$, through property $\left.Y(v, z)|0\rangle\right|_{z=0}=v$ from the Definition 3.2.2, indicates the vector $v$ uniquely. Note that this action is not the composition of the field with a linear operator.

### 3.2.4 Examples

## Virasoro vertex algebra

First we equip the representation $\mathrm{Vir}_{c}$ defined in (3.6) with the structure of the vertex operator algebra. We already have defined vacuum $|0\rangle \in \operatorname{Vir}_{c}$ and we set $T=L_{-1}$, so that $T|0\rangle=0$. We would like to use Theorem 3.2 .3 applied to the field $T(z)=\sum_{n \in \mathbb{Z}} z^{-n-2} L_{n}$. Let us check the axioms. Clearly the space $\operatorname{Vir}_{c}$ is spanned by vectors of the form $T_{\left(-j_{1}\right)} \ldots T_{\left(-j_{k}\right)}|0\rangle$, where $T_{(-i)}=L_{-i-1}$. Since we have only one field, the locality condition is void. Moreover we have

$$
[T, T(z)]=\sum_{n \in \mathbb{Z}} z^{-n-2}\left[L_{-1}, L_{n}\right]=\sum_{n \in \mathbb{Z}} z^{-n-2}(-1-n) L_{n-1}=\partial T(z) .
$$

Finally $T(z)|0\rangle=L_{-2}|0\rangle+z L_{-3}|0\rangle+O\left(z^{2}\right)$ is of the desired form. Hence all the assumptions of the Theorem 3.2.3 are satisfied and the equation (3.13) defines a structure of a vertex algebra.

## Free boson algebra

Another example is given by the Fock space representation $\mathcal{F}(0)$. As before vacuum vector $|0\rangle=v_{0}$ is already defined, we set $T=2 \sum_{n=0}^{\infty} a_{-n-1} a_{n}$ and define a field $\phi(z)=\sum_{n \in \mathbb{Z}} a_{n} z^{-n-1}$. Those objects have following properties:

- $T|0\rangle=0$ since $a_{n}|0\rangle=0$ for $n \geq 0$,
- $[T, \phi(z)]=\partial \phi(z)$ follows from the direct computation (by using the relation $[a b, c]=a[b, c]+$ $[a, c] b)$,
- the space $\mathcal{F}(0)$ is spanned by vectors of the form $a_{-j_{1}} \ldots a_{-j_{k}}|0\rangle$, since it is a weight module over $\mathcal{H}$,
- $\phi(z)|0\rangle=a_{-1}|0\rangle+z a_{-2}|0\rangle+\ldots$ is of the desired form.

Therefore from Theorem 3.2 .3 the four $(\mathcal{F}(0), Y, T,|0\rangle)$ with $Y$ given by the equation 3.13 has the structure of the vertex algebra. Moreover define for $Q \in \mathbb{C}$ a representation of $\mathcal{V}$ :

$$
\begin{align*}
L_{0}^{\mathcal{F}(0)} & =2 \sum_{n=1}^{\infty} a_{-n} a_{n}  \tag{3.15}\\
L_{m}^{\mathcal{F}(0)} & =\sum_{n \neq 0, m} a_{m-n} a_{n}-(m+1) Q a_{m} \quad \text { for } m \neq 0 \tag{3.16}
\end{align*}
$$

One can check that those are the modes of the vector $\omega=\frac{1}{2} a_{-1}^{2}+Q a_{-2}$. If we choose this vector as the conformal vector we get the structure of the vertex operator algebra called free boson algebra. In this case the energy-momentum tensor can be also expressed shortly as:

$$
T(z)=Y(\omega, z)=\sum_{n \in \mathbb{Z}} L_{n}^{\mathcal{F}(0)} z^{-n-2}=: \phi(z) \phi(z):+Q \partial \phi(z)
$$

### 3.2.5 Modules and intertwining operators

Definition 3.2.9 A module over a vertex operator algebra $V$ is a vector space $W$ and a linear $\operatorname{map} Y_{W}: V \rightarrow \operatorname{End} W\left[\left[z, z^{-1}\right]\right]$ such that:

- $Y_{W}(|0\rangle, z)=\operatorname{Id}_{W}$,
- modes of $Y_{W}(\omega, z)=\sum_{n \in \mathbb{Z}} L_{n}^{W} z^{-n-2}$ satisfy the commutation relations of the Virasoro algebra with the central charge $c^{V}$,
- for any $v \in V$ we have $Y_{W}(T v, z)=\partial_{z} Y_{W}(v, z)$,
- for any $a, b \in V$ the fields $Y_{W}(a, z)$ and $Y_{W}(b, w)$ are mutually local.

Example 3.2.1 The space $W=\mathcal{F}(\alpha)$ can be equipped with the structure of a module over the free boson vertex algebra $\mathcal{F}(0)$ in the following way. We define

$$
\begin{align*}
Y_{W}(|0\rangle, z) & =\mathrm{Id}_{W}  \tag{3.17}\\
Y_{W}\left(a_{-1}|0\rangle, z\right) & =\sum_{n \in \mathbb{Z}} a_{n} z^{-n-1}  \tag{3.18}\\
Y_{W}\left(a_{-i_{1}} \cdots a_{-i_{n}}|0\rangle, z\right) & =: D^{i_{1}-1} Y_{W}\left(a_{-1}|0\rangle, z\right) \cdots D^{i_{n}-1} Y_{W}\left(a_{-1}|0\rangle, z\right): \tag{3.19}
\end{align*}
$$

The conformal vector $\omega=\frac{1}{2} a_{-1}^{2}+Q a_{-2} \in \mathcal{F}(0)$ is mapped to:

$$
\begin{align*}
Y_{W}(\omega, z) & =\sum_{n \in \mathbb{Z}} L_{n}^{\mathcal{F}(\alpha)} z^{-n-2}  \tag{3.20}\\
L_{0}^{\mathcal{F}(\alpha)} & =2 \sum_{n=1}^{\infty} a_{-n} a_{n}+\alpha(\alpha-Q)  \tag{3.21}\\
L_{m}^{\mathcal{F}(\alpha)} & =\sum_{n \neq 0, m} a_{m-n} a_{n}+(2 \alpha-(m+1) Q) a_{m} \quad \text { for } m \neq 0 \tag{3.22}
\end{align*}
$$

Let us check the axioms of the definition (3.2.9). The first one is obvious. The second one is a result of direct computation. It is sufficient to check the third axiom for $v=a_{-i_{1}} \cdots a_{-i_{n}}|0\rangle$. In such a case we have:

$$
T v=2 \sum_{k=0}^{\infty} a_{-k-1} a_{k} a_{-i_{1}} \cdots a_{-i_{n}}|0\rangle=\sum_{a=1}^{n} i_{a} a_{-i_{1}} \cdots \widehat{a_{-i_{1}}} a_{-1-i_{a}} \cdots a_{-i_{n}}|0\rangle
$$

where $\widehat{a_{-i_{1}}}$ means omission of the corresponding term. Applying the equation (3.19) we get formula $Y_{W}(T v, z)=\partial_{z} Y_{W}(v, z)$ (this is very similar computation as in the Lemma 3.2.2, hence we skip the details). The locality axiom follows from the Dong's lemma (3.2.1).

Definition 3.2.10 $A$ vertex operator algebra $V$ is called rational if every $V$-module is completely reducible.

Definition 3.2.11 $A$ vertex operator algebra $V$ is called $C_{2}$-cofinite if the quotient vector space $V / \operatorname{lin}_{\mathbb{C}}\left\{a_{(-2)} b: a, b \in V\right\}$ is of finite dimension.

Definition 3.2.12 Let $M_{1}, M_{2}$ and $M_{3}$ be modules over a vertex operator algebra $V$. An intertwining operator is a linear map

$$
I(\cdot, z): M_{1} \rightarrow \operatorname{Hom}\left(M_{2}, M_{3}\right) \otimes_{\mathbb{C}} \mathbb{C}\{z\}
$$

satisfying $I\left(L_{-1}^{M_{1}} v, z\right)=\frac{\partial}{\partial z} I(v, z)$ as well as the following intertwining property:

$$
\begin{aligned}
& \operatorname{Res}_{z-w}\left(I\left(Y_{M_{1}}(a, z-w) v, w\right)(z-w)^{m} \iota_{w, z-w}((z-w)+w)^{n}\right) \\
& =\operatorname{Res}_{z}\left(Y_{M_{3}}(a, z) I(v, w) \iota_{z, w}(z-w)^{m} z^{n}\right)-\operatorname{Res}_{z}\left(I(v, w) Y_{M_{2}}(a, z) \iota_{w, z}(z-w)^{m} z^{n}\right)
\end{aligned}
$$

for any $a \in V, v \in M_{1}$ and $m, n \in \mathbb{Z}$. Here Hom denotes the space of linear maps. We will denote the set of such intertwining operators by $\operatorname{Int}\left(M_{1}, M_{2}, M_{3}\right)$.

Definition 3.2.13 We can define an action of $\mathcal{V}$ on the space of intertwining operators in a fashion similar to the one given by the equation (3.14):

$$
\begin{equation*}
\left(L_{k} \cdot I\right)(v, z)=I\left(L_{k}^{M_{1}} v, z\right) \tag{3.23}
\end{equation*}
$$

### 3.2.6 Free boson intertwining operator

We now define intertwining operators which are crucial for defining quantum curves. In this section we use the notation $D^{n}=\frac{1}{n!} \frac{\partial^{n}}{\partial x^{n}}$.

Let us take $M_{1}=\mathcal{F}(\alpha), M_{2}=\mathcal{F}(\beta)$ and $M_{3}=\mathcal{F}(\alpha+\beta)$ and define an element of $\operatorname{Hom}\left(M_{2}, M_{3}\right) \otimes_{\mathbb{C}}$ $\mathbb{C}\{z\}$ :

$$
I(|\alpha\rangle, x)=u^{\alpha} \exp \left(2 \alpha \sum_{j=1}^{\infty} \frac{x^{j}}{j} a_{-j}\right) \exp \left(-2 \alpha \sum_{j=1}^{\infty} \frac{x^{-j}}{j} a_{j}\right) x^{2 \alpha a_{0}}
$$

where $u^{\alpha}: \mathcal{F}(\beta) \rightarrow \mathcal{F}(\alpha+\beta)$ is a mapping such that $u^{\alpha}(|\beta\rangle)=|\alpha+\beta\rangle,\left[u^{\alpha}, a_{n}\right]=0$ for $n \neq 0$ and $\left[a_{0}, u^{\alpha}\right]=\alpha u^{\alpha}$. For descendant states, following [32], we define:

$$
\begin{equation*}
I\left(a_{-j_{1}} \ldots a_{-j_{n}}|\alpha\rangle, x\right)=: D^{j_{1}-1} \phi(x) \cdots D^{j_{n}-1} \phi(x) I(|\alpha\rangle, x):, \tag{3.24}
\end{equation*}
$$

where as before $\phi(z)=\sum_{n \in \mathbb{Z}} a_{n} z^{-n-1}$. Note that although $I(|\alpha\rangle, x)$ can have complex non integer powers of $z$ normal ordering makes sense: $I(|\alpha\rangle, x)$ appears in (3.24) on the right and therefore we are not splitting it into creation and annihilation parts. Let us recall that normal ordering operation is implicitly nested from the right.

Remark 3.2.3 We need to check that the definition (3.24) is well defined. The right hand side of the equation (3.24) is not linearly independent: since the operators $\left\{a_{-i}\right\}_{i>0}$ commute, any permutation of the sequence $\left(a_{-j_{1}}, \ldots, a_{-j_{n}}\right)$ does not change the left hand side of this equation. However the right hand side is also permutation-invariant, since all the operators appearing in the expression : $D^{j_{1}-1} \phi(x) \cdots D^{j_{n}-1} \phi(x) I(|\alpha\rangle, x)$ : are normally ordered (creation operators are put on the left and annihilation operators are put on the right). This ordering does not depend on the order of the sequence $\left(j_{1}, \ldots, j_{n}\right)$.

More explicitly, using $e^{z}=\sum_{n=0}^{\infty} \frac{1}{n!} z^{n}$ and expanding the powers of infinite sums we can write

$$
\begin{align*}
I(|\alpha\rangle, x)= & \sum_{M \in \mathbb{Z}} u^{\alpha} x^{\alpha a_{0}+M} \sum_{k, l=0}^{\infty} \frac{(-1)^{l}}{k!l!}(2 \alpha)^{k+l} \\
& \cdot \sum_{\substack{j_{1}+\cdots+j_{k}=\\
M+m_{1}+\ldots+m_{l} \\
j_{\bullet}, m_{\bullet}>0}} \frac{1}{j_{1} \cdots j_{k} m_{1} \cdots m_{l}} a_{-j_{1} \ldots a_{-j_{k}} a_{m_{1}} \ldots a_{m_{l}} .} . \tag{3.25}
\end{align*}
$$

equation In a shorter way one can also write $I(|\alpha\rangle, x)=: e^{2 \alpha \int \phi(z) \mathrm{d} x}$ :, where the integration constant is understood as " $\frac{1}{2} \log (u)$ ". The following remark will be useful later.

Remark 3.2.4 Notice that we have

$$
I(|\alpha\rangle, x)|0\rangle=\exp \left(2 \alpha \sum_{j=1}^{\infty} \frac{x^{j}}{j} a_{-j}\right)|\alpha\rangle
$$

Therefore it is a power series with strictly positive powers of $x$ and the coefficient at $x=0$ of the vector $I(|\alpha\rangle, x)|0\rangle$ equals to $|\alpha\rangle$. Higher coefficients can be expressed using character polynomials $P_{n}\left(y_{1}, y_{2}, \ldots\right)$ of the irreducible representations (with one row Young diagram of length $n$ ) of the group $\operatorname{GL}(N, \mathbb{C})$, defined using the relation

$$
\exp \left(\sum_{n=1}^{\infty} x^{n} y_{l}\right)=\sum_{l=0}^{\infty} P_{n}\left(y_{1}, y_{2}, \ldots\right) x^{l}
$$

We are not going to check whether $I(v, x)$ defined above satisfies both axioms of the intertwining operators, as this will be not relevant to the construction of the quantum curves. We will check just the first axiom.

Lemma 3.2.2 For any $v \in \mathcal{F}(\alpha)$ we have $I\left(L_{-1}^{\mathcal{F}(\alpha)} v, x\right)=\frac{\partial}{\partial x} I(v, x)$.
Proof. We know that:

$$
L_{-1}^{\mathcal{F}(\alpha)}=2 \sum_{n>0} a_{-1-n} a_{n}+2 a_{-1} a_{0} .
$$

It is enough to check the statement of the lemma for basis vectors $v \in \mathcal{F}(\alpha)$, which can be written in the form $v=\prod_{j=1}^{k} a_{-i_{j}}|\alpha\rangle$. Let us examine how $L_{-1}^{\mathcal{F}(\alpha)}$ acts on such $v$. We have:

$$
\begin{array}{r}
2 \sum_{n>0} a_{-1-n} a_{n} \prod_{j=1}^{k} a_{-i_{j}}|\alpha\rangle=2 \sum_{n>0}\left[a_{-1-n} a_{n}, \prod_{j=1}^{k} a_{-i_{j}}\right]|\alpha\rangle= \\
=\sum_{j=1}^{k} i_{j} a_{-i_{1}} \ldots \widehat{a_{-i_{j}}} a_{-1-i_{j}} \ldots a_{-i_{k}}|\alpha\rangle
\end{array}
$$

where $\widehat{a_{-i_{j}}}$ means the omission of the corresponding term. The above formula follows from the facts that $a_{n}|\alpha\rangle=0$ for $n>0$ that the operators $a_{-1-n} a_{n}$ and $a_{-i_{j}}$ commute among themselves, unless $n-i_{j}=0$, whence $\left[a_{-1-n} a_{n}, a_{-i_{j}}\right]=\frac{1}{2} i_{j} a_{-i_{j}-1}$. Hence each occurrence of $a_{-i_{j}}$ is substituted with $i_{j} a_{-i_{j}-1}$, giving rise to a term in the above sum.

On the other hand we have

$$
\begin{aligned}
\frac{\partial}{\partial x} I\left(\prod_{j=1}^{k} a_{-i_{j}}|\alpha\rangle, x\right) & =\frac{\partial}{\partial x}: D^{i_{1}-1} \phi(x) \cdots D^{i_{k}-1} \phi(x) I(|\alpha\rangle, x):= \\
& =\frac{1}{\left(i_{1}-1\right)!\cdots\left(i_{k}-1\right)!} \sum_{j=1}^{k}: \partial^{i_{1}-1} \phi(x) \cdots \partial^{i_{j}} \phi(z) \cdots \partial^{i_{k}-1} \phi(x) I(|\alpha\rangle, x): \\
& +: D^{i_{1}-1} \phi(x) \cdots D^{i_{k}-1} \phi(x) \frac{\partial}{\partial x} I(|\alpha\rangle, x):
\end{aligned}
$$

We see that the first therm above corresponds exactly to the application of the definition of the operator $I$ to the vector $\sum_{j=1}^{k} i_{j} a_{-i_{1}} \ldots \widehat{a_{-i_{j}}} a_{-1-i_{j}} \ldots a_{-i_{k}}|\alpha\rangle$.
We are left with the other part of the operator $L_{-1}^{\mathcal{F}(\alpha)}$, namely $2 a_{-1} a_{0}$, and with the term involving $\frac{\partial}{\partial z} I(|\alpha\rangle, x)$. This equality corresponds exactly to the statement of the Lemma in the case of $v=|\alpha\rangle$. Since we know that $I(|\alpha\rangle, x)=: e^{2 \alpha \int \phi(x)}$ :, it follows that:

$$
I\left(2 a_{0} a_{-1}|\alpha\rangle, x\right)=2 \alpha: \phi(x) I(|\alpha\rangle, x):=\frac{\partial}{\partial x}: e^{2 \alpha \int \phi(x) \mathrm{d} x}:=\frac{\partial}{\partial x} I(|\alpha\rangle, x)
$$

where the first equation follows from the definition of $I$ and the way in which the operator $a_{0}$ acts.

The following lemma was motivated by the equation (2.2.7) from [53] and expansions of the product of the fields like in Proposition 3.2.1 from 38.

Lemma 3.2.3 For $n \geq 2$ the following formula holds:

$$
\begin{align*}
I\left(L_{-n} v, x\right) & =\frac{1}{(n-2)!}: \partial_{x}^{n-2} T(x) I(v, x): \\
& =\frac{1}{(n-2)!}\left(\partial_{x}^{n-2} T_{3}(x)_{c} I(v, x)+I(v, x) \partial_{x}^{n-2} T_{2}(x)_{a}\right) \tag{3.26}
\end{align*}
$$

Proof. Notice first that it is enough to prove the statement for $n=2$. For higher values of this parameter the conclusion can be than deduced inductively form the commutation relations of the Virasoro algebra. Indeed, for $n>1$ we have

$$
L_{-n-1}=\frac{1}{n-1}\left[L_{-1}, L_{-n}\right]
$$

Therefore, using the fact that $I\left(L_{-1} v, x\right)=\partial_{x} I(v, x)$ we get:

$$
\begin{aligned}
I\left(L_{-n-1} v, x\right) & =\frac{1}{n-1} I\left(L_{-1} L_{-n} v, x\right)-\frac{1}{n-1} I\left(L_{-n} L_{-1} v, x\right) \\
& =\frac{1}{n-1} \partial_{x} \frac{1}{(n-2)!}: \partial_{x}^{n-2} T(x) I(v, x):-\frac{1}{n-1} \frac{1}{(n-2)!}: \partial_{x}^{n-2} T(x) \partial_{x} I(v, x): \\
& =\frac{1}{(n-1)!}: \partial_{x}^{n-1} T(x) I(v, x):
\end{aligned}
$$

Notice that we are using here the following Leibniz rule property of the normal ordering:

$$
\partial_{x}: a(x) b(x):=: \partial_{x} a(x) b(x):+: a(x) \partial_{x} b(x):
$$

which follows from the fact that taking creation (respectively annihilation) part commutes with derivating a field, as was mentioned in the section 3.1.5.

Let us consider now the case of

$$
L_{-2}=2 \sum_{n=1}^{\infty} a_{-n-2} a_{n}+2 a_{-2} a_{0}+a_{-1}^{2}+Q a_{-2}
$$

Therefore from the definition of $I(v, x)$ (equation 3.24$)$ for $v=a_{-i_{1}} \cdots a_{-i_{k}}|\alpha\rangle$ and the commutation relations of the Heisenberg algebra we get:

$$
\begin{align*}
I\left(L_{-2} a_{-i_{1}} \cdots a_{-i_{k}}|\alpha\rangle, x\right) & =\sum_{j=1}^{k} i_{j} I\left(a_{-i_{1}} \cdots a_{-i_{j}-2} \widehat{a_{-i_{j}}} \cdots a_{-i_{k}}|\alpha\rangle, x\right)  \tag{3.27}\\
& +(2 \alpha+Q): \partial \phi(x) I(v, x):+: \phi(x): \phi(x) I(v, x)::
\end{align*}
$$

To prove the lemma we need to show that the above formula equals

$$
: T(x) I(v, x):=:(: \phi(x) \phi(x):+Q \partial \phi(x)) I(v, x):
$$

First let us notice that the $Q$-dependent parts match. Hence we can restrict ourselves to the case $Q=0$. We need to show that the expression

$$
\begin{equation*}
X=:(: \phi(x) \phi(x):) I(v, x):-: \phi(x): \phi(x) I(v, x):: \tag{3.28}
\end{equation*}
$$

exactly equals to the first two terms (for $Q=0$ ) appearing on the right hand side of (3.27). The expression 3.28 can be computed using the Heisenberg algebra commutation relations, although the computation is a little tedious. We have:

$$
\begin{equation*}
:(: \phi(x) \phi(x):) I(v, x):=(: \phi(x) \phi(x):)_{c} I(v, x)+I(v, x)(: \phi(x) \phi(x):)_{a} \tag{3.29}
\end{equation*}
$$

where

$$
\begin{align*}
& (: \phi(x) \phi(x):)_{c}=\sum_{n<0} \sum_{m \leq-n-2} a_{n} a_{m} x^{-n-m-2}+\sum_{n \geq 0} \sum_{m \leq-n-2} a_{m} a_{n} x^{-n-m-2}  \tag{3.30}\\
& (: \phi(x) \phi(x):)_{a}=\sum_{n<0} \sum_{m>-n-2} a_{n} a_{m} x^{-n-m-2}+\sum_{n \geq 0} \sum_{m>-n-2} a_{m} a_{n} x^{-n-m-2} .
\end{align*}
$$

We see that not all the terms appearing in (3.29) are normally ordered, contrary to : $\phi(x)$ : $\phi(x) I(v, x)::$. Moreover, this change of ordering is the only difference between both expressions. The difference can be computed by keeping track of the additional terms, which appears while we would be computing terms appearing in (3.29), while trying to normal order them (moving annihilation operators to the right and creation operators to the left). We only need to compute annihilation operators $a_{n}, n \geq 0$, appearing in $(: \phi(x) \phi(x):)_{c}$ and the creation operators $a_{n}, n<0$, appearing in $(: \phi(x) \phi(x):)_{a}$. Other terms will be skipped by writing $\ldots$. We have:

$$
\begin{aligned}
(: \phi(x) \phi(x):)_{c} & =\sum_{n<0} \sum_{m=0}^{-n-2} a_{-n} a_{m} x^{-n-m-2}+\sum_{m \geq 0} \sum_{n \leq-m-2} a_{n} a_{m} x^{-n-m-2}+\ldots \\
& =\sum_{n=1}^{\infty} \sum_{m=0}^{n-2} a_{-n} a_{m} x^{n-m-2}+\sum_{m \geq 0} \sum_{n \geq m+2} a_{-n} a_{m} x^{n-m-2}+\ldots \\
& =\sum_{n=2}^{\infty} \sum_{m=0}^{n-2} a_{-n} a_{m} x^{n-m-2}+\sum_{n=2}^{\infty} \sum_{m=0}^{n-2} a_{-n} a_{m} x^{n-m-2}+\ldots \\
& =2 \sum_{n=2}^{\infty} \sum_{m=0}^{n-2} a_{-n} a_{m} x^{n-m-2}+\ldots \\
(: \phi(x) \phi(x):)_{a} & =\sum_{n=1}^{\infty} \sum_{m=n-1}^{\infty} a_{-n} a_{m} x^{n-m-2}+\sum_{n=0}^{\infty} \sum_{m=-n-1}^{-1} a_{m} a_{n} x^{-m-n-2}+\ldots \\
& =\sum_{m=1}^{\infty} \sum_{n=m-1}^{\infty} a_{-m} a_{n} x^{m-n-2}+\sum_{n=0}^{\infty} \sum_{m=1}^{n+1} a_{-m} a_{n} x^{m-n-2}+\ldots \\
& =2 \sum_{n=0}^{\infty} \sum_{m=1}^{n+1} a_{-m} a_{n} x^{m-n-2}+\ldots
\end{aligned}
$$

From the two above expressions we get two terms: $X=A+B$, one from moving annihilation operators appearing in $(: \phi(x) \phi(x):)_{c}$ and the second from moving creation operators appearing in $(: \phi(x) \phi(x):)_{a}$ :

$$
A=2 \sum_{n=2}^{\infty} \sum_{m=0}^{n-2} a_{-n}\left[a_{m}, I(v, x)\right] x^{n-m-2}, \quad B=2 \sum_{n=0}^{\infty} \sum_{m=1}^{n+1}\left[I(v, x), a_{-m}\right] a_{n} x^{m-n-2} .
$$

Let us apply the defining equation (3.24) of the operator $I$, form which it follows that:

$$
\begin{aligned}
I(v, x) & =: D^{i_{1}-1} \phi(x): D^{i_{2}-1} \phi(x) \cdots: D^{i_{k}-1} \phi(x) I(|\alpha\rangle, x): \cdots: \\
& =\sum_{n_{1}, \ldots, n_{k} \in \mathbb{Z}}: a_{n_{1}} \cdots a_{n_{k}} I(|\alpha\rangle, x): \prod_{j=1}^{k} x^{-n_{j}-i_{j}} \frac{\left(-n_{j}-1\right)_{\left(i_{j}-1\right)}}{\left(i_{j}-1\right)!},
\end{aligned}
$$

where where $(x)_{(j)}=x(x-1)(x-2) \cdots(x-j+1)$ is the Pochhammer symbol. From the above we infer that while moving operators $a_{n}$ we can get contribution of two types: either it will "hit" another $a_{k}$ or $I(|\alpha\rangle, x)$. According to this we can make corresponding further splittings $A=A_{1}+A_{2}$ and $B=B_{1}+B_{2}$. We get:

$$
\begin{aligned}
& A_{1}=\sum_{j \in\left\{i_{1}, \ldots, i_{k}\right\}} \sum_{n=2}^{\infty} a_{-n} S_{j} x^{n-2-j} \frac{1}{(j-1)!} \sum_{m=1}^{n-2} m(m-1)_{(j-1)} \\
& B_{1}=\sum_{j \in\left\{i_{1}, \ldots, i_{k}\right\}} \sum_{n=0}^{\infty} S_{j} a_{n} x^{-n-2-j} \frac{1}{(j-1)!} \sum_{m=1}^{n+1} m(-m-1)_{(j-1)}
\end{aligned}
$$

where

$$
S_{j}=\sum_{n_{1}, \ldots, \widehat{n_{j}}, \ldots, n_{k} \in \mathbb{Z}}: a_{n_{1}} \cdots \widehat{a_{n_{j}}} \cdots a_{n_{k}} I(|\alpha\rangle, x): \prod_{s \neq j} \frac{\left(-n_{s}-1\right)_{\left(i_{s}\right)}}{i_{s}!} x^{-n_{s}-1}
$$

In computing the combinatorial sums we will use the following identities [19]:

$$
\sum_{m=0}^{n}(m)_{(j)}=\frac{1}{j+1}(n+1)_{(j+1)}, \quad \sum_{m=0}^{n}(-m)_{(j)}=\frac{(-1)^{j}}{j+1}(n+j)_{(j+1)}
$$

Since $m(m-1)_{(j-1)}=(m)_{(j)}$ and $m(-m-1)_{(j-1)}=-(-m)_{(j)}$ we obtain:

$$
\begin{aligned}
A_{1} & =\sum_{j \in\left\{i_{1}, \ldots, i_{k}\right\}} \sum_{n=2}^{\infty} a_{-n} S_{j} x^{n-2-j} \frac{j}{(j+1)!}(n-1)_{(j+1)} \\
& =\sum_{j \in\left\{i_{1}, \ldots, i_{k}\right\}} \sum_{n \leq-2} a_{n} S_{j} x^{-n-2-j} \frac{j}{(j+1)!}(-n-1)_{(j+1)} \\
B_{1} & =\sum_{j \in\left\{i_{1}, \ldots, i_{k}\right\}} \sum_{n=0}^{\infty} S_{j} a_{n} x^{-n-2-j} \frac{(-1)^{j+1} j}{(j+1)!}(n+j+1)_{(j+1)} \\
& =\sum_{j \in\left\{i_{1}, \ldots, i_{k}\right\}} \sum_{n=0}^{\infty} S_{j} a_{n} x^{-n-2-j} \frac{j}{(j+1)!}(-n-1)_{(j+1)}
\end{aligned}
$$

Summing the above two terms we get the right hand side of the equation:

$$
\begin{align*}
& i_{j} I\left(a_{-i_{1}} \cdots a_{-i_{j}-2} \widehat{a_{-i}} \cdots a_{-i_{k}}|\alpha\rangle, x\right)=  \tag{3.31}\\
= & i_{j}: D^{i_{1}-1} \phi(x): D^{i_{2}-1} \phi(x) \cdots: D^{i_{j}+1} \phi(x) \cdots: D^{i_{k}-1} \phi(x) I(|\alpha\rangle, x): \cdots: \\
= & \sum_{n_{1}, \ldots, n_{k} \in \mathbb{Z}}: a_{n_{1}} \cdots a_{n_{k}} I(|\alpha\rangle, x): \frac{\left(-n_{j}-1\right)_{\left(i_{j}+1\right)} i_{j}}{\left(i_{j}+1\right)!} x^{-n_{j}-i_{j}-2} \prod_{s \neq j} \frac{\left(-n_{s}-1\right)_{\left(i_{s}-1\right)}}{\left(i_{s}-1\right)!} x^{-n_{s}-i_{s}} .
\end{align*}
$$

multiplied and summed over $j \in\{1, \ldots, k\}$, which should be compared with (3.27), as desired.
Let us move to $A_{2}$ and $B_{2}$. First we compute commutators for $m>0$ :

$$
\left[a_{m}, I(|\alpha\rangle, x)\right]=\left[a_{m}, 2 \alpha \int \phi(x) \mathrm{d} x\right] I(|\alpha\rangle, x)=\frac{m}{2}\left(2 \alpha \frac{x^{m}}{m}\right) I(|\alpha\rangle, x)=\alpha x^{m} I(|\alpha\rangle, x)
$$

and

$$
\left[I(|\alpha\rangle, x), a_{-m}\right]=I(|\alpha\rangle, x)\left[2 \alpha \int \phi(x) \mathrm{d} x, a_{-m}\right]=-\frac{m}{2}\left(2 \alpha \frac{x^{-m}}{m}\right) I(|\alpha\rangle, x)=-\alpha x^{-m} I(|\alpha\rangle, x)
$$

whereas since $\left[a_{0}, u^{\alpha}\right]=\alpha u^{\alpha}$ we get $\left[a_{0}, I(|\alpha\rangle, x)\right]=\alpha I(|\alpha\rangle, x)$. Therefore we get (for readability we introduce weighted summation operation $\left.\operatorname{sign} \sum_{N}=\sum_{n_{1}, \ldots, n_{k} \in \mathbb{Z}} \prod_{j=1}^{k} x^{-n_{j}-i_{j}}\left(-n_{j}-1\right)_{\left(i_{j}-1\right)}\right)$ :

$$
\begin{aligned}
A_{2} & =2 \sum_{N} \sum_{n=2}^{\infty} \sum_{m=0}^{n-2} a_{-n} x^{n-m-2} x^{m}: a_{n_{1}} \cdots a_{n_{k}}\left[a_{m}, I(|\alpha\rangle, x)\right]: \\
& =2 \alpha \sum_{N} \sum_{n=2}^{\infty}(n-1) a_{-n} x^{n-2}: a_{n_{1}} \cdots a_{n_{k}} I(|\alpha\rangle, x): \\
& =2 \alpha \sum_{n \leq-1}(-n-1) x^{-n-2} a_{n} I(v, x)=2 \alpha \sum_{n \leq-1}^{\infty}\left(\frac{\partial}{\partial x} x^{-n-1}\right) a_{n} I(v, x) \\
B_{2} & =2 \sum_{N} \sum_{n=0}^{\infty} \sum_{m=1}^{n+1}: a_{n_{1}} \cdots a_{n_{k}}\left[I(|\alpha\rangle, x), a_{-m}\right]: a_{n} x^{m-n-2} x^{-m} \\
& =-2 \alpha \sum_{N} \sum_{n=0}^{\infty}(n+1): a_{n_{1}} \cdots a_{n_{k}} I(|\alpha\rangle, x): a_{n} x^{-n-2} \\
& =2 \alpha \sum_{n=0}^{\infty}(-n-1) x^{-n-2} I(v, x) a_{n}=2 \alpha \sum_{n=0}^{\infty}\left(\frac{\partial}{\partial x} x^{-n-1}\right) I(v, x) a_{n}
\end{aligned}
$$

Summing both terms above we get $2 \alpha: \partial \phi(x) I(v, x):$, obtaining the remaining term of the right hand side of the equation 3.27 .

### 3.2.7 Additional topics

A goal of this section is to present interesting and nontrivial notions and results related to the theory of vertex algebras. They also provide a link of the topics discussed here with the author's Master Thesis 13. This section, being a digression form the main topic of the dissertation, is sketchy.

## Fusion algebra

Let $V$ be a vertex operator algebra and let $\operatorname{Rep}(V)$ be the category of its modules. On this category, under some conditions, there exists a (categorical) tensor product $\otimes \operatorname{such}$ that $(V, \otimes)$ becomes braided monoidal tensor category. This product differs from the usual tensor product of representations, and is called fusion product. One difference is that in standard tensor product of representation of $\mathcal{V}$ the central charges would add. Here the central charge of a product equals to (the same) central charge of both modules. Let $W_{i}$ and $W_{j}$ be two modules. One can write the decomposition:

$$
W_{i} \otimes W_{j}=\bigoplus_{k} N_{i j}^{k} W_{k}
$$

where $W_{k} \in \operatorname{Rep}(V)$. The coefficients $N_{i j}^{k}$ (the structure constants of the Grothendieck ring of this category) are called fusion rules. Those coefficients are equal to the dimension of the space of the corresponding intertwining operators. In determining fusion rules singular vectors play an important rôle.

## Verlinde formula

For any module $W$ over $V$ one can define its character $\chi_{W}(\tau)=\operatorname{Tr}_{W}\left(q(\tau)^{L_{0}-c / 24}\right)$, where $q(\tau)=$ $e^{2 \pi \sqrt{-1} \tau}$. Under certain conditions on $V$ (rationality and $C_{2}$-cofinitnes) this character turns out to be a holomorphic function on the upper half plane $\mathbb{H}=\{\tau \in \mathbb{C}: \operatorname{Im}(\tau)>0\}$ [59. For example for the Verma module $V_{\Delta}$ over the Virasoro VOA one has:

$$
\chi_{V_{\Delta}}(\tau)=q(\tau)^{\Delta-c / 24} \prod_{n=1}^{\infty}\left(\frac{1}{1-q(\tau)^{n}}\right)
$$

The vector space spanned by characters of the given $V$, under certain assumptions, is invariant under the natural action of the group $\operatorname{SL}(2, \mathbb{Z})$ and for rational VOA also of finite dimension 59. This group is generated by two elements $T$ and $S$ and the action is given by $T(\tau)=\tau+1$ and $S(\tau)=-\frac{1}{\tau}$. We clearly have $S^{2}=\mathrm{id}$. Consider the matrix of the map $S$ defined by the relation:

$$
\chi_{W_{i}}\left(-\frac{1}{\tau}\right)=\sum_{j} S_{i j} \chi_{W_{j}}(\tau) .
$$

Form the physical perspective $Z(\tau)=\sum_{i} \chi_{W_{i}}(\tau)$ is the partition function of CFT on a torus with the spectrum $\left\{W_{i}\right\}_{i \in I}$ and $\tau$ has an interpretation as the modular parameter.

The famous Verlinde formula relates the $S$-matrix with the fusion rules [55]:

$$
N_{i j}^{k}=\sum_{l} \frac{S_{i l} S_{j l}\left(S^{-1}\right)_{k l}}{S_{0 l}},
$$

where 0 corresponds to the vacuum module $W_{0}$ ( $V$ being a module over itself). For example putting in this formula $i=0$ one can see that $W_{0}$ is the unit of the fusion product: $N_{0 j}^{k}=\delta_{j, k}$. Verlinde formula has many applications. For example Edward Witten has used it to propose a formula for the volume of the moduli space of flat connections on a Riemann surface [57], see also [13].

## Monstrous moonshine

We want to mention another application of the theory presented in this chapter. Recall that forto any vertex algebra we have associated a group of its automorphism. One can construct a vertex algebra $V^{\natural}$, for which $\operatorname{Aut}\left(V^{\natural}\right)$ is the Monster group $\mathbb{M}$ (the largest sporadic finite simple group) 29. Monstrous moonshine asserts that its character is related to the modular $j$-function:

$$
\chi_{V^{\text {घ }}}(\tau)=j(\tau)-744=1728 \frac{g_{2}(\tau)^{3}}{g_{2}(\tau)^{3}-27 g_{3}(\tau)^{2}}-744,
$$

where

$$
\begin{aligned}
& g_{2}(\tau)=60 \sum_{(m, n) \in \mathbb{Z}^{2}-(0,0)} \frac{1}{(m+n \tau)^{4}} \\
& g_{3}(\tau)=140 \sum_{(m, n) \in \mathbb{Z}^{2}-(0,0)} \frac{1}{(m+n \tau)^{6}} .
\end{aligned}
$$

Monstrous moonshine was conjectured in 16 and proved by Borcherds in 8 .

## $\mathcal{W}$-algebras

Let us finish this chapter with a definition of algebras, which extend the conformal symmetry. Instead of only one field, the stress-energy tensor $T(z)$, which strongly generate the vertex algebra, they can possess finite family of such fields.

Definition 3.2.14 $A$ vertex algebra $V$ is called $\mathcal{W}$-algebra if it possesses a minimal finite set of fields $\left\{w_{i}(z)\right\}_{i \in I}$, which strongly generate $V$. If the fields $w_{i}(z)$ have conformal weights $\Delta_{i}$, where $V$ is called $\mathcal{W}$-algebra of type $W\left(\Delta_{1}, \ldots, \Delta_{n}\right)$.

## Chapter 4

## Quantum curves

In this chapter we present the construction of the quantum curves from the singular vectors. The main ingredient is the derivation of the appropriate representation of the Virasoro algebra. This representation is a generalisation of the representation derived in [14. Quantum curves act on the wave function, which are build from the intertwining operators. Algebraic manipulation shows that the wave function corresponds to the $\alpha / \beta$-deformed matrix model (Definition 2.2.3). Therefore wave functions join topics from the Chapter 2 and the Chapter 3.

In the section 4.1 we define the wave function. Our approach is based on the one presented in the paper [14], but uses mathematical notions. We also relate the wave functions with the matrix integral ( $\alpha / \beta$-deformed model). We choose an integration contour, whose existence is showed in [54] (see also our direct source: [38]).

In the section 4.2 we derive the representation necessary for the construction of the quantum curves. The derivation relies on the Lemmas 3.2 .2 and 3.2 .3 from the previous Chapter. As an outcome we obtain a one parameter deformation of the representation from [14]. We also give examples of the quantum curves for the lowest degree singular vectors.

In the section 4.3 we investigate a direct proof of the fact that the operators derived in the Theorem 4.2 .1 are a representation of the Virasoro algebra. As an outcome we obtain also a combinatorial identity (Proposition 4.3.1) as well as a procedure of obtaining new representations of the Virasoro algebra (Proposition 4.3.2).

### 4.1 Wave function

First we will introduce some vector spaces, to which the wave functions would belong and on which quantum curves would act. Since we work with objects defined perturbatively, those are spaces of
formal power series. Let

$$
\begin{aligned}
R & =\mathbb{C}\left[\left[t_{0}, t_{1}, \ldots\right]\right], \\
R_{\hbar} & =\mathbb{C}\left[\left[t_{0}, t_{1}, \ldots\right]\right] \otimes \mathbb{C}\left[\hbar^{ \pm 1}\right], \\
R_{\hbar}(x) & =\mathbb{C}\left[\left[t_{0}, t_{1}, \ldots\right]\right] \otimes \mathbb{C}\left[\hbar^{ \pm 1}\right] \otimes \mathbb{C}\{x\},
\end{aligned}
$$

where as before $\mathbb{C}\{z\}=\left\{\sum_{r \in \mathbb{C}} a_{r} z^{r}\right.$ : only countably many of $a_{r} \in \mathbb{C}$ are nonzero $\}$. Spaces $R$ and $R_{\hbar}$ are algebras, whereas $R_{\hbar}(x)$ is a module over both of them. It is not an algebra, since we cannot multiply elements of $\mathbb{C}\{x\}$. On $R$ we have a family of representations $\rho_{\alpha}$ of $\mathcal{H}$ parametrised by $\alpha \in \mathbb{C}$ and defined by

$$
a_{-i} \cdot f=\frac{i}{2 \hbar} t_{i} f, \quad a_{i} \cdot f=\hbar \frac{\partial}{\partial t_{i}} f \quad \text { for } i>0, \quad a_{0} \cdot f=\alpha f,
$$

for $f \in \mathbb{C}\left[\left[t_{0}, t_{1}, \ldots\right]\right]$. If $V$ is any other vector space over $\mathbb{C}$ then we can extend this representation to $R \otimes V$ in a standard way: $a_{i}(f \otimes v)=\left(a_{i} f\right) \otimes v$. We also introduce a homomorphism:

$$
\Phi_{\alpha}^{\hbar}: \mathcal{F}(\alpha) \rightarrow R_{\hbar}
$$

defined on the basis via

$$
\Phi_{\alpha}^{\hbar}\left(a_{-i_{1}} \cdots a_{-i_{k}}|\alpha\rangle\right)=(2 \hbar)^{-k} i_{1} t_{i_{1}} \cdots i_{k} t_{i_{k}} .
$$

Similarly we extend it to the tensor product with an arbitrary $\mathbb{C}$-vector space $V$ :

$$
\Phi_{\alpha, V}^{\hbar}: \mathcal{F}(\alpha) \otimes_{\mathbb{C}} V \rightarrow R_{\hbar} \otimes_{\mathbb{C}} V
$$

by $\Phi_{\alpha, V}^{\hbar}(a \otimes b)=\Phi_{\alpha}^{\hbar}(a) \otimes b$. By abuse of the notation we will use the same symbol for all $V$ : $\Phi_{\alpha, V}^{\hbar}=\Phi_{\alpha}^{\hbar}$. Those homomorphisms are equivariant with respect to the representations $\rho_{\alpha}$.

Secondly, definition of the wave function requires integration of the multi-valued functions such as

$$
F\left(\beta ; z_{1}, \ldots, z_{N}\right)=\Delta\left(z_{1}, \cdots, z_{N}\right)^{2 \beta} \prod_{j=1}^{N} z_{i}^{-\beta(N-1)}=\prod_{i \neq j}\left(1-\frac{z_{i}}{z_{j}}\right)^{\beta} .
$$

In order to perform such integration we need to consider twisted cycles, defined in the Section 2.2.3.
Define a manifold

$$
M_{N}=\left\{\left(z_{1}, \ldots, z_{N}\right) \in\left(\mathbb{C}^{*}\right)^{N}: z_{i} \neq z_{j} \quad \text { for } \quad i \neq j\right\} .
$$

In order to simplify the expressions we will use the notation $\Psi_{\beta}=F\left(\beta ; z_{1}, \ldots, z_{N}\right)$. Let us also define a set

$$
\Omega_{N}=\left\{x \in \mathbb{C}: \forall_{d=1, \ldots, N-1} \quad d(d+1) x \notin \mathbb{Z}, \quad d(N-d) x \notin \mathbb{Z}\right\} .
$$

Theorem 4.1.1 (Theorem 8.4, [38]) Let $\mathcal{L}$ be the algebra of Laurent polynomials $f\left(z_{1}, \ldots, z_{N}\right)$, which are invariant under permutation of $z_{i}$ 's. Then there exists a cycle $\Gamma \in H_{N}\left(M_{N}, \Psi_{\beta}\right)$ such that for any $f \in \mathcal{L}, m \in \mathbb{Z}_{\geq 0}$ and $\beta \in \Omega_{N}$ the integral

$$
\int_{\Gamma} F\left(\beta ; z_{1}, \ldots, z_{N}\right) f\left(z_{1}, \ldots, z_{N}\right) \mathrm{d} z_{1} \cdots \mathrm{~d} z_{N}
$$

is well defined and defines a nontrivial linear mapping $\phi_{\Gamma}(m, \beta): \mathcal{L} \rightarrow \mathbb{C}$. Moreover for any $b_{1}, \ldots, b_{N} \in \mathbb{Z}$

$$
\phi_{\Gamma}(m, \beta)\left(z_{1}^{b_{1}} \cdots z_{N}^{b_{N}}\right)=0
$$

unless $b_{1}+\cdots+b_{N}=-N$.

The cycle from the above theorem can be used to construct a homomorphism of the Fock modules.

Proposition 4.1.1 (Lemma 8.10, [38) Assume tha ${ }^{\text {T }}$

$$
\alpha=Q+N \sqrt{\beta}+b \frac{1}{\sqrt{\beta}},
$$

where $N \in \mathbb{Z}_{>0}, \beta \in \mathbb{C}$ and $b \in \mathbb{Z}$. Then there exists a map $\Sigma_{N, \beta, b}: \mathcal{F}(\alpha) \rightarrow \mathcal{F}(\alpha-N \sqrt{\beta})$, which is a Virasoro homomorphism:

$$
\left[L_{m}, \Sigma_{N, \beta, b}\right]=0 \quad \text { for } m<0
$$

Moreover this map is given by the formula

$$
\Sigma_{N, \beta, b}=\int_{\Gamma} K^{\beta}\left(z_{1}, \ldots, z_{N}\right) \prod_{i=1}^{N} z_{i}^{-b-1} \mathrm{~d} z_{1} \ldots \mathrm{~d} z_{N}
$$

where

$$
K^{\beta}\left(z_{1}, \ldots, z_{N}\right)=E^{\beta}\left(z_{1}\right) \ldots E^{\beta}\left(z_{N}\right) \prod_{i=1}^{N} z_{i}^{-2 \alpha \sqrt{\beta}-(N-1) \beta}
$$

$E^{\beta}\left(z_{i}\right)=I\left(|-\sqrt{\beta}\rangle, z_{i}\right)$ and $\Gamma \in H_{N}\left(M_{N}, \Psi_{\beta}\right)$ is a twisted cycle.

Remark 4.1.1 Assume that $\alpha \neq 0$. Then using the Proposition 4.1.1 we infer that $\Sigma_{N, \beta, b}|0\rangle$, if nontrivial, is a singular vector. In fact $\Sigma_{N, \beta, b}|0\rangle$ is more than just a singular vector: it is annihilated also by the vectors $L_{0}$ and $L_{-1}$, just as the vector $|0\rangle$.

Definition 4.1.1 The wave function is an element of $R_{\hbar}(x)$ defined as

$$
\widehat{\psi}_{\alpha}(x)=\Phi_{\alpha-N \sqrt{\beta}}^{\hbar} \Sigma_{N, \beta, b} I(|\alpha\rangle, x)|0\rangle
$$

where $b=(\alpha-Q) \sqrt{\beta}-N \beta$. Here and in what follows we skip the parameters $N \in \mathbb{Z}_{>0}, \beta \in \mathbb{C}$ and $b \in \mathbb{Z}$ from the notation of the wave function. More generally we can define a wave function for any $v \in \mathcal{F}(\alpha)$ :

$$
\widehat{\psi}_{\alpha}(v, x)=\Phi_{\alpha-N \sqrt{\beta}}^{\hbar} \Sigma_{N, \beta, b} I(v, x)|0\rangle .
$$

We will denote the subspace of $R_{\hbar}(x)$ spanned by wave functions $\widehat{\psi}_{\alpha}(v, x)$ for $v \in \mathcal{F}(\alpha)$ by $\mathcal{W}_{\alpha}$.

[^1]Let us not check under which notation we know that $\widehat{\psi}_{\alpha}(x)$ is nontrivial.

Remark 4.1.2 If $\beta \in \Omega_{N}$, then the homomorphism $\Sigma_{N, \beta, b}$ is nontrivial (section 8.4.3, [38]). Let us look ae the coefficient of $x=0$ of the formal power series $I(|\alpha\rangle, x)$. From Remark 3.2.4 we know that coefficient of $x=0$ of $I(|\alpha\rangle, x)|0\rangle$ equals $|\alpha\rangle$. Henceforth the coefficient of $x=0$ of $\widehat{\psi}_{\alpha}(x)$ equals $\Phi_{\alpha-N \sqrt{\beta}}^{\hbar} \Sigma_{N, \beta, b}|\alpha\rangle$. Assume additionally that $b>0$. Using Corollary 8.1 from 38 . we conclude that $\Sigma_{N, \beta, b}|\alpha\rangle$ is a nonzero (singular) vector. Moreover, from Remark 3.2.4 we see that $\widehat{\psi}_{\alpha}(x)$ is a power series with strictly positive powers of $x$.

The case $b<0$ is more difficult. In order to use Corollary 8.1 from [38] we need to see that one of the coefficients of the series $I(|\alpha\rangle, x)|0\rangle$ is a cosingular vector. To this aim one can check the construction of this vector in [54].

One can also consider interesting case $N=0$, when $\widehat{\psi}_{\alpha}(x)=\exp \left(\frac{\alpha}{\hbar} \sum_{n=1}^{\infty} t_{n} x^{n}\right)$. For such series quantum curves give relations on the character polynomials of the representations of $\mathrm{GL}(N, \mathbb{C})$ (see Remark 3.2.4.

## Integral representation

In this section we will present a representation of the wave function, making a connection with the topics discussed in the Chapter 2. This section is not fully rigorous. Recall that the wave function was defined as

$$
\widehat{\psi}_{\alpha}(x)=\Phi_{\alpha-N \sqrt{\beta}}^{\hbar} \Sigma_{N, \beta, b} I(|\alpha\rangle, x)|0\rangle
$$

In order to make a connection of the above expression with the matrix integral recall that:

$$
\Sigma_{N, \beta, b}=\int_{\Gamma} K^{\beta}\left(z_{1}, \ldots, z_{N}\right) \prod_{i=1}^{N} z_{i}^{-b-1} \mathrm{~d} z_{1} \ldots \mathrm{~d} z_{N}
$$

The parameter $b$ depends on the parameter $\alpha$ in the following form: $b=(\alpha-Q) \sqrt{\beta}-N \beta$. We can put the field $I(|\alpha\rangle, x)$ inside the integral sign, obtaining:

$$
\widehat{\psi}_{\alpha}(x)=\Phi_{\alpha-N \sqrt{\beta}}^{\hbar} \int_{\Gamma} K^{\beta}\left(z_{1}, \ldots, z_{N}\right) I(|\alpha\rangle, x)|0\rangle \prod_{i=1}^{N} z_{i}^{-b-1} \mathrm{~d} z_{1} \ldots \mathrm{~d} z_{N}
$$

We can normally order those terms getting:

$$
K^{\beta}\left(z_{1}, \ldots, z_{N}\right) I(|\alpha\rangle, x)=\Delta\left(z_{1}, \ldots, z_{N}\right)^{2 \beta} \prod_{i=1}^{N}\left(z_{i}-x\right)^{-2 \alpha \sqrt{\beta}}: K^{\beta}\left(z_{1}, \ldots, z_{N}\right) I(|\alpha\rangle, x): .
$$

Note that:

$$
\begin{aligned}
& : K^{\beta}\left(z_{1}, \ldots, z_{N}\right) I(|\alpha\rangle, x):|0\rangle \\
& =\exp \left(2 \alpha \sum_{n=1}^{\infty} \frac{a_{-n}}{n} x^{n}\right) \prod_{i=1}^{N} \exp \left(-2 \sqrt{\beta} \sum_{n=1}^{\infty} \frac{a_{-n}}{n} z_{i}^{n}\right)|\alpha-N \sqrt{\beta}\rangle
\end{aligned}
$$

Applying the homomorphism $\Phi_{\alpha-N \sqrt{\beta}}^{\hbar}$ we get:

$$
\Phi_{\alpha-N \sqrt{\beta}}^{\hbar}\left(: K^{\beta}\left(z_{1}, \ldots, z_{N}\right) I(|\alpha\rangle, x):|0\rangle\right) \exp \left(\frac{\alpha}{\hbar} \sum_{n=1}^{\infty} t_{n} x^{n}\right) \prod_{i=1}^{N} \exp \left(-\frac{\sqrt{\beta}}{\hbar} \sum_{n=1}^{\infty} t_{n} z_{i}^{n}\right) .
$$

Combining the above expressions, and using the notation $W(z)=\sum_{n=1}^{\infty} t_{n} z^{n}$, we get integral representation of the wave function:

$$
\begin{aligned}
\widehat{\psi}_{\alpha}(x)=\exp \left(-\frac{\alpha}{\hbar} \sum_{n=1}^{\infty} t_{n} x^{n}\right) \int_{\Gamma} & \Delta\left(z_{1}, \ldots, z_{N}\right)^{2 \beta} \prod_{i=1}^{N}\left(z_{i}-x\right)^{-2 \alpha \sqrt{\beta}} \prod_{i=1}^{N} z_{i}^{-b-1} \\
& \cdot \prod_{i=1}^{N} \exp \left(-\frac{\sqrt{\beta}}{\hbar} \sum_{n=1}^{\infty} t_{n} z_{i}^{n}\right) \mathrm{d} z_{1} \ldots \mathrm{~d} z_{N}
\end{aligned}
$$

which can be compared with (2.8).

### 4.2 Main construction

We are now ready to pass to the main construction of the quantum curves. Recall that $\mathcal{V}_{\leq}$is the subalgebra of $\mathcal{V}$ generated by $L_{0}, L_{-1}, L_{-2}, \ldots$. The first theorem gives formulas for those generators in a suitable representation, which are constituting building blocks for the quantum curves. What is important, is that the form of the operators $\widehat{L}_{-n}$ below does not depend on the vector $v$.

Theorem 4.2.1 Assume that $N \in \mathbb{Z}_{>0}, \beta \in \mathbb{C}$ and $b \in \mathbb{Z}$. There exists a representation $\rho$ of $\mathcal{V}_{<}$ on the space $\mathcal{W}_{\alpha}$ such that

$$
\begin{equation*}
\widehat{L}_{k} \widehat{\psi}_{\alpha}(v, x)=\widehat{\psi}_{\alpha}\left(L_{-k} v, x\right), \tag{4.1}
\end{equation*}
$$

where $\widehat{L}_{k}=\rho\left(L_{k}\right)$ and $k<0$. This representation can be expressed using the following formulae:

$$
\begin{align*}
& \widehat{L}_{-1}=\partial_{x}, \quad \text { and for } n \geq 2:  \tag{4.2}\\
& \widehat{L}_{-n}=\frac{1}{\hbar^{2}(n-2)!} \partial_{x}^{n-2}\left(\frac{1}{4}\left(W^{\prime}(x)\right)^{2}+\frac{Q \hbar}{2} W^{\prime \prime}(x)+\widehat{f}_{t}(x)+(\alpha-N \sqrt{\beta}) \widehat{d}(x)\right)
\end{align*}
$$

where $\widehat{f}_{t}(x)=\hbar^{2} \sum_{n=0}^{\infty} x^{n} \sum_{m=n+2}^{\infty} m t_{m} \partial_{t_{m-n-2}}$ and $\widehat{d}(x)=\hbar \sum_{m=0}^{\infty}(m+2) t_{m+2} x^{m}$.

Remark 4.2.1 The above representation is a one-parameter deformation of the representation $\widehat{L}_{-n}^{C F T}$ presented in 14 . This deformation comes from the additional term $\widehat{d}(x)$ :

$$
\widehat{L}_{-n}=\widehat{L}_{-n}^{C F T}+\gamma \sum_{m=n-2}^{\infty}\binom{m}{n-2}(m+2) t_{m+2} x^{m-n+2},
$$

where $\gamma$ is an arbitrary deformation parameter.

Proof of the theorem. The case of the operator $\widehat{L}_{-1}$ follows from the Lemma 3.2.2, whereas the case of the operator $\widehat{L}_{0}$ from the fact that vectors of homogeneous grading are eigenvectors of $L_{0}$ and the linearity of $I$. Let us examine the cases when $n \geq 2$. We have:

$$
\begin{equation*}
I\left(L_{-n} v, x\right)=\frac{1}{(n-2)!}: \partial^{n-2} T(x) I(v, x):=D^{n-2} T(x)_{c} I(v, x)+I(v, x) D^{n-2} T(x)_{a} \tag{4.3}
\end{equation*}
$$

Therefore

$$
\widehat{\psi}_{\alpha}\left(L_{-n} v, x\right)=\frac{1}{(n-2)!} \Phi_{\alpha-N \sqrt{\beta}}^{\hbar} \Sigma_{N, \beta, b}\left(\partial^{n-2} T(x)_{c} I(v, x)+I(v, x) \partial^{n-2} T(x)_{a}\right)|0\rangle
$$

Using the formulas $T(x)_{a}|0\rangle=0$ and $\left[\Sigma_{N, \beta, b}, \partial^{n-2} T(x)_{c}\right]=0$ we infer that

$$
\begin{equation*}
\widehat{\psi}_{\alpha}\left(L_{-n} v, x\right)=\frac{1}{(n-2)!} \Phi_{\alpha-N \sqrt{\beta}}^{\hbar} \partial^{n-2} T(x)_{c} \Sigma_{N, \beta, b} I(v, x)|0\rangle \tag{4.4}
\end{equation*}
$$

We concentrate on the case $n=2$, the more general case $n>2$ follows by taking commutators with $\widehat{L}_{-1}$. Let us now look more closely on the field $T(x)_{c}=\sum_{n=0}^{\infty} x^{n} L_{-n-2}$. Since from the equation (3.10)

$$
L_{-m}=2 \sum_{k=1}^{\infty} a_{-k-m} a_{k}+\sum_{l=1}^{m-1} a_{-m+l} a_{-l}+\left(2 a_{0}+(m-1) Q\right) a_{-m},
$$

we have $\Phi_{\alpha-N \sqrt{\beta}}^{\hbar} L_{-m}=\tilde{L}_{-m} \Phi_{\alpha-N \sqrt{\beta}}^{\hbar}$, where

$$
\tilde{L}_{-m}=\sum_{k=1}^{\infty}(k+m) t_{k+m} \frac{\partial}{\partial t_{k}}+\frac{1}{4 \hbar^{2}} \sum_{l=1}^{m-1}(m-l) l t_{m-l} t_{l}+\frac{1}{2 \hbar}(2 \alpha-2 N \sqrt{\beta}+(m-1) Q) m t_{m},
$$

so taking the sum we get:

$$
\begin{align*}
\widehat{L}_{-2}=\Phi_{\alpha}^{\hbar}\left(T(x)_{c}\right) & =\sum_{m=0}^{\infty} x^{m}\left(\sum_{k=1}^{\infty}(k+m+2) t_{k+m+2} \frac{\partial}{\partial t_{k}}+\frac{1}{2 \hbar}(2 \alpha-2 N \sqrt{\beta})(m+2) t_{m+2}\right. \\
& \left.+\frac{1}{4 \hbar^{2}} \sum_{l=1}^{m+1}(m+2-l) l t_{m+2-l} t_{l}+\frac{1}{2 \hbar}(m+1) Q(m+2) t_{m+2}\right) \tag{4.5}
\end{align*}
$$

Notice that for $W(x)=\sum_{i=1}^{\infty} t_{i} x^{i}$ we have

$$
\begin{aligned}
\left(W^{\prime}(x)\right)^{2} & =\sum_{m=0}^{\infty} \sum_{l=1}^{m+1}(m+2-l) l t_{m+2-l} t_{l} x^{m} \\
W^{\prime \prime}(x) & =\sum_{m=0}^{\infty}(m+1)(m+2) t_{m+2} x^{m}
\end{aligned}
$$

which are (up to rescaling) correspondingly the second and the third ingredients appearing in the sum 4.5). The first line of 4.5) corresponds to

$$
\widehat{f}_{t}(x)+(\alpha-N \sqrt{\beta}) \widehat{d}(x)=\hbar^{2} \sum_{n=0}^{\infty} x^{n} \sum_{m=n+2}^{\infty} m t_{m} \partial_{t_{m-n-2}}+\hbar(\alpha-N \sqrt{\beta}) \sum_{m=0}^{\infty}(m+2) t_{m+2} x^{m}
$$

Therefore it follows that

$$
\widehat{L}_{-2}=\frac{1}{4 \hbar^{2}}\left(W^{\prime}(x)\right)^{2}+\frac{Q}{2 \hbar} W^{\prime \prime}(x)+\frac{1}{\hbar^{2}} \widehat{f}_{t}(x)+\frac{(\alpha-N \sqrt{\beta})}{\hbar^{2}} \widehat{d}(x)
$$

To obtain formulas for $\widehat{L}_{-n}$ with $n>2$ notice that it is sufficient to commute the operator $\widehat{L}_{-2}$ sufficiently many times with the operator $\widehat{L}_{-1}$. The fact that formulas 4.2 form a representation of the Virasoro algebra follows straightforwardly from their definition. We need to check that

$$
\left[\widehat{L}_{i}, \widehat{L}_{j}\right] \widehat{\psi}_{\alpha}(v, x)=(i-j) \widehat{L}_{i+j} \widehat{\psi}_{\alpha}(v, x)
$$

for any $v \in \mathcal{F}(\alpha)$. To this aim we use the relation

$$
\left[L_{i}, L_{j}\right]=L_{i} L_{j}-L_{j} L_{i}=(i-j) L_{i+j}
$$

from which it follows that

$$
\begin{equation*}
\left[\widehat{L}_{i}, \widehat{L}_{j}\right] \widehat{\psi}_{\alpha}(v, x)=\widehat{\psi}_{\alpha}\left(\left[L_{i}, L_{j}\right] v, x\right)=\widehat{\psi}_{\alpha}\left((i-j) L_{i+j} v, x\right)=(i-j) \widehat{L}_{i+j} \widehat{\psi}_{\alpha}(v, x) \tag{4.6}
\end{equation*}
$$

Now we proceed to the theorem about quantum curves.

Theorem 4.2.2 Assume that $v_{s}=A_{r, s}|\Delta\rangle$ is a singular vectors, where $\Delta=\alpha_{r, s}\left(Q-\alpha_{r, s}\right)$, $Q=\frac{1}{\sqrt{\beta}}-\sqrt{\beta}, \beta \in \mathbb{R}_{>0}$ and

$$
\alpha_{r, s}=\frac{r-1}{2} \frac{1}{\sqrt{\beta}}-\frac{s-1}{2} \sqrt{\beta} .
$$

Let us take $\alpha=Q-\alpha_{r, s}$. If we assume that $2 N=s-1$ and $2 b=1-r$ ( $N$ and $b$ are implicit parameters of the wave function), then using the representation (4.2) we arrive at the following equation, called quantum curve:

$$
\widehat{A}_{r, s} \widehat{\psi}_{\alpha}(|\alpha\rangle, x)=0 .
$$

Proof. Let $v_{s}=A_{r, s}|\Delta\rangle$ be a singular vector. Then from Proposition 3.1.6 we know that $S_{Q-\alpha_{r, s}, Q}\left(v_{s}\right)=0$ in $\mathcal{F}\left(Q-\alpha_{r, s}\right)=\mathcal{F}(\alpha)$. Recall that $\Delta(\alpha)=\Delta(Q-\alpha)$ (see section 3.1.5). It follows that

$$
\widehat{\psi}_{\alpha}\left(S_{Q-\alpha_{r, s}, Q}\left(v_{s}\right), x\right)=0
$$

On the other hand, since $S_{Q-\alpha_{r, s}, Q}\left(v_{s}\right)=S_{Q-\alpha_{r, s}, Q}\left(A_{r, s}|\Delta\rangle\right)=A_{r, s}|\alpha\rangle$. In order to use Theorem 4.2.1 we need to check that $\alpha=\frac{N-1}{2} \sqrt{\beta}+\frac{b+1}{2} \frac{1}{\sqrt{\beta}}$, where $N \in \mathbb{Z}_{>0}$ and $b \in \mathbb{Z}$. From the condition $\alpha=Q-\alpha_{r, s}$ we obtain the following relations on the parameters:

$$
2 N=s-1, \quad 2 b=1-r .
$$

Therefore we can iteratively move the Virasoro generators appearing in $A_{r, s}$ outside the wave function, obtaining the expression

$$
\widehat{A}_{r, s} \widehat{\psi}_{\alpha}(|\alpha\rangle, x)=0
$$

Here $\widehat{A}_{r, s}$ is a differential equation obtained from the universal formula for the Virasoro singular vectors in the representation 4.2 .

Example 4.2.1 Using the known expressions for the singular vectors one can produce exact formulae for quantum curves in low levels. The first nontrivial examples are

$$
\begin{aligned}
& \left.\widehat{A}_{2}=\partial_{x}^{2}-\frac{\alpha}{\hbar^{4}}\left(\left(W^{\prime}(x)\right)^{2}+2 Q \hbar W^{\prime \prime}(x)+4 \widehat{f}_{t}(x)\right)\right) \\
& \widehat{A}_{3}=\partial_{x}^{3}-4 \frac{\alpha^{2}}{\hbar^{2}} \partial_{x} \widehat{L}_{-2}+2 \hbar^{-4} \alpha^{2}\left(2 \alpha(2 \alpha+Q \hbar)-\hbar^{2}\right) \widehat{L}_{-3}
\end{aligned}
$$

where one needs to substitute using any of the values $\alpha=\alpha_{r, s}$, where $r+s=2$ for $\widehat{A}_{2}$ and $r+s=3$ for $\widehat{A}_{3}$.

### 4.3 Combinatorial identity

One could ask if it is possible to prove that (4.2) form a representation of the Virasoro algebra directly. In [44] it has been shown that

$$
\begin{equation*}
\widehat{L}_{-n}^{C F T}=\frac{1}{\hbar^{2}(n-2)!} \partial_{x}^{n-2}\left(\frac{1}{4}\left(W^{\prime}(x)\right)^{2}+\frac{Q \hbar}{2} W^{\prime \prime}(x)+\widehat{f}_{t}(x)\right) \quad \widehat{L}_{-1}^{C F T}=\partial_{x} \tag{4.7}
\end{equation*}
$$

is a representation of the Virasoro algebra. Therefore we would get a positive answer to our question if the following identity would hold

$$
\left[\widehat{L}_{-n}^{C F T}+\widehat{d}_{m}(x), \widehat{L}_{-n}^{C F T}+\widehat{d}_{n}(x)\right]=(m-n) \widehat{L}_{-n-m}^{C F T}+(m-n) \widehat{d}_{n+m}(x),
$$

where $\widehat{d}_{n}(x)=\frac{1}{\hbar^{2}(n-2)!} \partial_{x}^{n-2} \widehat{d}(x)$. From the above equation, using the fact that $\widehat{L}_{-n}^{C F T}$ is a representation of the Virasoro algebra, we get a condition for a deformation of an algebra:

$$
\left[\widehat{L}_{-n}^{C F T}, \widehat{d}_{m}(x)\right]-\left[\widehat{L}_{-m}^{C F T}, \widehat{d}_{n}(x)\right]=(m-n) \widehat{d}_{n+m}(x)
$$

This condition can be further simplified. Only the operators $\widehat{f}_{t}(x)$ from the terms in $\widehat{L}_{-n}^{C F T}$ contribute to the commutators above. Hence we have

$$
\left[\frac{1}{\hbar^{2}(n-2)!} \partial_{x}^{n-2} \widehat{f}_{t}(x), \widehat{d}_{m}(x)\right]-\left[\frac{1}{\hbar^{2}(m-2)!} \partial_{x}^{m-2} \widehat{f}_{t}(x), \widehat{d}_{n}(x)\right]=(m-n) \widehat{d}_{n+m}(x)
$$

Plugging the definitions of the operators $\widehat{f}_{t}(x)$ and $\widehat{d}_{n}(x)$ we arrive at the following condition:

$$
\begin{align*}
& \sum_{n=a}^{\infty} \sum_{m=b}^{\infty} x^{n+m-a-b} t_{n+m+4}(n+m+4)\binom{n}{a}\binom{m}{b}(m-n) \\
& =\frac{b-a}{(a+b+2)!} \sum_{k=a+b+2}^{\infty}(k+2) t_{k+2} \frac{k!}{(k-a-b-2)!} x^{k-a-b-2} . \tag{4.8}
\end{align*}
$$

Comparing the coefficients of the powers of $x$, the above identity is equivalent to the collection of identities:

$$
\sum_{n=b}^{k-2-a}\binom{k-n-2}{a}\binom{n}{b}(2 n-k+2)=(b-a)\binom{k}{a+b+2} .
$$

Let us argue now why (4.8) is true. We can repeat the reasoning leading to it with a single change: all the operators will be acting on $\widehat{\psi}_{\alpha}(x)$. Since from the equation 4.6 we know that:

$$
\left[\widehat{L}_{-n}, \widehat{L}_{-m}\right] \widehat{\psi}_{\alpha}(x)=(m-n) \widehat{L}_{-n-m} \widehat{\psi}_{\alpha}(x)
$$

we infer that the following identity is true:

$$
\begin{align*}
& \sum_{n=a}^{\infty} \sum_{m=b}^{\infty} x^{n+m-a-b} t_{n+m+4}(n+m+4)\binom{n}{a}\binom{m}{b}(m-n) \widehat{\psi}_{\alpha}(x) \\
& =\frac{b-a}{(a+b+2)!} \sum_{k=a+b+2}^{\infty}(k+2) t_{k+2} \frac{k!}{(k-a-b-2)!} x^{k-a-b-2} \widehat{\psi}_{\alpha}(x) . \tag{4.9}
\end{align*}
$$

From the Remark 4.1.2, taking $N=0, \widehat{\psi}_{\alpha}(x)$ is a nonzero formal power series with only positive powers of $x$. In general the algebra of formal power series is not an integral domain. However, in this case all the series have only positive powers, hence we can divide by $\widehat{\psi}_{\alpha}(x)$. Therefore we have proved the following proposition.

Proposition 4.3.1 Assume that $a, b, k>0$ are positive integers and $k \geq a+b+2$. Then the following identity holds:

$$
\sum_{n=b}^{k-2-a}\binom{k-n-2}{a}\binom{n}{b}(2 n-k+2)=(b-a)\binom{k}{a+b+2} .
$$

As a corollary we get:

Proposition 4.3.2 Assume that a collection of operators $L_{-n}$ for $n>0$ forms a representation of the negative part of the Virasoro algebra: $\left[L_{-n}, L_{-m}\right]=(m-n) L_{-n-m}$. Then a set of the operator series $\widehat{L}_{-n}=\frac{1}{(n-2)!} \partial_{x}^{n-2} T(x)_{c}$ for $n>0$, where $T(x)_{c}=\sum_{m=0}^{\infty} x^{m} L_{-m-2}$ is the creation part of the stress-energy tensor, and $\widehat{L}_{-1}=\partial_{x}$, also forms a representation of the negative part of the Virasoro algebra.

Proof. We have

$$
\widehat{L}_{-a-2}=\sum_{n=a}^{\infty}\binom{n}{a} x^{n-a} L_{-n-2}
$$

from which it follows that

$$
\begin{aligned}
{\left[\widehat{L}_{-a-2}, \widehat{L}_{-b-2}\right] } & =\sum_{n=a}^{\infty} \sum_{m=b}^{\infty}\binom{n}{a}\binom{m}{b} x^{m+n-b-a}\left[L_{-n-a}, L_{-m-2}\right] \\
& =\sum_{n=a}^{\infty} \sum_{m=b}^{\infty}\binom{n}{a}\binom{m}{b} x^{m+n-b-a}(m-n) L_{-n-m-4} .
\end{aligned}
$$

We would like this expression to be equal to

$$
(b-a) \widehat{L}_{-a-b-4}=(b-a) \sum_{k=a+b+2}^{\infty}\binom{k}{a+b+2} x^{k-a-b-2} L_{-k-2} .
$$

Comparing the terms corresponding to the power $x^{k-a-b-2}$ we get an identity:

$$
\sum_{n=b}^{k-2-a}\binom{k-n-2}{a}\binom{n}{b}(2 n-k+2)=(b-a)\binom{k}{a+b+2}
$$

which is true by Proposition 4.3.1.

## Chapter 5

## Supersymmetry

The aim of this chapter is the presentation of the supersymmetric analog of the quantum curves discussed in the previous chapter. Such an extension was presented in the paper from the point of view of the matrix models [50] and in [14] from the point of view of CFT. The initial idea of the author was to present a derivation of the super quantum curves using vertex operator super algebras (VOSAs). This task was unfortunately unaccomplished, and the outcome was a no-go theorem 5.3.7. Therefore, after introducing supersymmetry and presenting super extension of the concepts from the chapter 3 , we briefly explain the construction of the quantum curves from [14]. However, we introduce the language of the twisted modules in this context, using which the structure of the super quantum curves is clearer.

In the section 5.1 basic notions are discussed: super vector spaces, super algebras and integration.
In the section 5.2 we define super eigenvalue integrals. The content of the section 5.2 .2 is due to the author.

In the section 5.3 we introduce super extensions of the Virasoro and the Heisenberg algebras and we study their representations. We also define VOSAs, their modules and the intertwining operators. We also define twisted modules and twisted intertwining operators. Twisting is related with an automorphism of the VOSA. Ramond sector arises as module twisted by the parity automorphism. Finally we introduce intertwining operators with the Grasmann odd variable and prove Theorem 5.3.7. responsible for the difficulties of the derivation of the super quantum curves using VOSAs.

In the section 5.4 we discuss more physical approach to the super quantum curves. Three cases are considered, which correspond to various choices of the twisting of the modules between which intertwining operators act. This section is an excerpt from [14].

### 5.1 Super spaces

In the first section of this chapter we explain what supersymmetry is and how it can be mathematically axiomatised. A good reference for this subject is [45] and [27].

### 5.1.1 Super vector spaces

Definition 5.1.1 $A$ super vector space is a $\mathbb{Z}_{2}$-graded vector space $V=V_{0} \oplus V_{1}$. Elements of the subspaces $V_{0}$ and $V_{1}$ are called homogeneous and for such elements we define their degree: $|v|=i$ if $v \in V_{i}, i \in\{0,1\}$. Elements of degree 0 will be called even and of those of degree 1 will be called odd.

If $\operatorname{dim}(V)<\infty$ we say that a super vector space $V=V_{0} \oplus V_{1}$ is of dimension $\operatorname{dim}\left(V_{0}\right) \mid \operatorname{dim}\left(V_{1}\right)$. An example of a super vector space of dimension $n \mid m$ is $\mathbb{C}^{n \mid m}=\mathbb{C}^{n} \oplus \mathbb{C}^{m}$, that is $\left(\mathbb{C}^{n \mid m}\right)_{0}=\mathbb{C}^{n}$ and $\left(\mathbb{C}^{n \mid m}\right)_{1}=\mathbb{C}^{m}$. For $\tau \in \mathbb{Z}_{2}$ we will use shorter notation $\mathbb{C}^{\tau}=\mathbb{C}^{\tau+1 \mid \tau}$.

If $V$ and $W$ are super vector spaces, a natural structure of a superspace can be defined on $\operatorname{Hom}(V, W)$. We set $\operatorname{Hom}(V, W)=\operatorname{Hom}(V, W)_{0} \oplus \operatorname{Hom}(V, W)_{1}$, where $\operatorname{Hom}(V, W)_{0}$ are those maps, which preserves parity (maps even elements to even elements and odd elements to odd elements) and $\operatorname{Hom}(V, W)_{1}$ those which exchange parity. We will also call the first one even maps and the second odd maps.

The tensor product of two super vector spaces $V=V_{0} \oplus V_{1}$ and $W=W_{0} \oplus W_{1}$ has a structure of a super vector space defined via

$$
\begin{aligned}
& (V \otimes W)_{0}=\left(V_{0} \otimes W_{0}\right) \oplus\left(V_{1} \otimes W_{1}\right) \\
& (V \otimes W)_{1}=\left(V_{0} \otimes W_{1}\right) \oplus\left(V_{1} \otimes W_{0}\right) .
\end{aligned}
$$

### 5.1.2 Superalgebras

Definition 5.1.2 Let $A$ be an algebra, which is also a super vector space. We say that $A$ is supercommutative if for any homogeneous $a, b \in A$ we have

$$
a b=(-1)^{|a||b|} b a .
$$

We say that it is anti-supercommutative if for such $a, b \in A$

$$
a b=-(-1)^{|a||b|} b a .
$$

Example 5.1.1 Let $V$ be a vector space and let $A^{k}(V)=\Lambda^{k} V$. Then $A(V)=\bigoplus_{k=0}^{\infty} A^{k}(V)$ is the exterior algebra. It is a superspace with $A(V)_{0}=\bigoplus_{k=0}^{\infty} \Lambda^{2 k} V$ and $A(V)_{1}=\bigoplus_{k=0}^{\infty} \Lambda^{2 k+1} V$. Because for $\omega \in A^{k}(V)$ and $\eta \in A^{l}(V)$ we have $\omega \wedge \eta=(-1)^{k l} \eta \wedge \omega$, it is a supercommutative algebra. In what follows we will subtract the symbol $\wedge$ from the notation of the multiplication in this algebra. Therefore, if $\theta_{1}, \ldots, \theta_{n}$ is a basis of $V$, elements of the algebra $A$ will be denoted as

$$
\sum_{k=0}^{n} \sum_{i_{1}, \ldots, i_{k}=1}^{n} a_{i_{1}, \ldots, i_{k}} \theta_{i_{1}} \cdots \theta_{i_{k}}
$$

where $a_{i_{1}, \ldots, i_{k}} \in \mathbb{C}$. Elements of $V$, as generators of $A(V)$, will be called fermionic variables. Algebra of $n$ fermionic variables will be denoted by $A\left(\mathbb{C}^{n}\right)$.

Definition 5.1.3 $A$ super Lie algebra $\mathfrak{g}$ is an anti-supercommutative algebra, for which graded version of the Jacobi rule holds:

$$
[a,[b, c]]+(-1)^{|a|(|b|+|c|)}[b,[c, a]]+(-1)^{|c|(|a|+|b|)}[c,[a, b]]=0
$$

for $a, b, c \in \mathfrak{g}$.

As an example we can consider super Lie algebra $\mathfrak{o s p}(1 \mid 2)=\operatorname{span}_{\mathbb{C}}\left\{e, f, h, b_{+}, b_{-}\right\}$, where $e, f, h$ are even elements and $b_{+}, b_{-}$are odd elements. Lie super bracket is then defined as:

$$
\begin{array}{rlrl}
{[e, f]=h,} & {[h, e]} & =2 e, & {[h, f]=-2 f,} \\
{\left[b_{-}, b_{+}\right]} & =h, & {\left[b_{+}, b_{+}\right]=4 e} & = \pm b_{ \pm}, \\
& {\left[b_{-}, b_{-}\right]=-4 f .}
\end{array}
$$

As we can notice even part of this super Lie algebra $\mathfrak{o s p}(1 \mid 2)_{0}$ is isomorphic to $\mathfrak{s l}(2)$.
For any super algebra we can introduce a super Lie bracket by the formula:

$$
\begin{equation*}
[a, b]=a b-(-1)^{|a||b|} b a . \tag{5.1}
\end{equation*}
$$

This definition satisfies super Jacobi rule. In the physical literature commutator of odd elements $a, b$ is called anti-commutator and denoted by $\{a, b\}$. For the super Lie bracket (5.1) we have $\{a, b\}=a b+b a$.

We will say more about super Lie algebras in the Chapter 6, where discussion about their representations is placed.

Definition 5.1.4 Let $A$ be a super algebra. A linear homogeneous map $\phi \in \operatorname{Hom}(A, A)$ is a super derivation if it satisfies the following condition:

$$
\phi(a b)=\phi(a) b+(-1)^{|a||\phi|} a \phi(b),
$$

where $|\phi| \in \mathbb{Z}_{2}$ is the parity of the derivation.

Let $A$ be a supercommutative algebra. Then any odd element $a \in A$ satisfies $a^{2}=0$. We can evaluate any analytic function $f(z)=\sum_{n=0}^{\infty} f_{n} z^{2}$ (or even formal power series) on $a$ : since the series will stop after two steps, we do not need to care for convergence:

$$
\begin{equation*}
f(a)=f(0)+f^{\prime}(0) a . \tag{5.2}
\end{equation*}
$$

### 5.1.3 Integration

In order to define matrix models involving fermionic variables we need to define what does integration with respect to such variables means.

Definition 5.1.5 Let $V$ be a vector space of dimension $n$ and let $A(V)$ be its exterior algebra. Choose a basis $\theta_{1}, \ldots, \theta_{n}$ of $V$. We define integration functional $\int \mathrm{d} \theta: A \rightarrow \mathbb{C}$ as a map

$$
\int\left(\sum_{k=0}^{n} \sum_{i_{1}, \ldots, i_{k}=1}^{n} a_{i_{1}, \ldots, i_{k}} \theta_{i_{1}} \cdots \theta_{i_{k}}\right) \mathrm{d} \theta=a_{1,2, \ldots, n}
$$

where $a_{i_{1}, \ldots, i_{k}} \in \mathbb{C}$ and $\sum_{k=0}^{n} \sum_{i_{1}, \ldots, i_{k}=1}^{n} a_{i_{1}, \ldots, i_{k}} \theta_{i_{1}} \cdots \theta_{i_{k}}$ is any element of $A$.

Remark 5.1.1 If we choose different basis the integration functional will differ by a multiplicative constant, as follows from the properties of the wedge product.

We can extend this definition in a natural way to any algebra of the form $A \otimes R$.

### 5.2 Super matrix models

### 5.2.1 Definitions

The definition of the random matrices can be extended to include fermionic variables. This can be done in the eigenvalue picture. There are several models one can consider, which are related to various super extensions of the Virasoro algebra. Therefore one can consider Neveu-Schwarz super eigenvalue model or Ramond super eigenvalue model. Difference between those models is most clearly visible in the different forms of the following extensions of the Vandermonde determinant. More precisely, they take following forms:

$$
\begin{align*}
\Delta_{N S} & =\prod_{i \neq j}\left(z_{i}-z_{j}-\theta_{i} \theta_{j}\right)  \tag{5.3}\\
\Delta_{R} & =\prod_{i \neq j}\left(z_{i}-z_{j}-\theta_{i} \theta_{j} \sqrt{\frac{z_{j}}{z_{i}}}\right) . \tag{5.4}
\end{align*}
$$

Those expressions are elements of specific algebras: $\Delta_{N S} \in k\left[z_{1}, \ldots, z_{N}\right] \otimes A\left(\mathbb{C}^{N}\right)$ and $\Delta_{R} \in$ $k\left[z_{1}^{ \pm \frac{1}{2}}, \ldots, z_{N}^{ \pm \frac{1}{2}}\right] \otimes A\left(\mathbb{C}^{N}\right)$.

Definition 5.2.1 $\beta$-deformed matrix model partition function in the Neveu-Schwarz sector $Z_{\beta, N S}$ for $\beta \in \mathbb{C}$ is defined as

$$
\begin{align*}
Z_{\beta, N S}= & \sum_{n=0}^{\infty} \sum_{\substack{k_{1}, \ldots, k_{n} \geq 1 \\
l_{1}, \ldots, l_{m} \geq 1}} \frac{t_{1}^{k_{1}} \cdots t_{n}^{k_{n}}}{k_{1}!\cdots k_{n}!} \frac{\xi_{1+\frac{1}{2}}^{l_{1}} \cdots \xi_{m+\frac{1}{2}}^{l_{m}}}{l_{1}!\cdots l_{m}!} F_{N S}^{\beta}\left(k_{1}, \ldots, k_{n}, l_{1}, \ldots, l_{m}\right), \\
F_{N S}^{\beta}\left(k_{1}, \ldots, k_{n}, l_{1}, \ldots, l_{m}\right)= & \int_{C} \prod_{i=1}^{n}\left(z_{1}^{k_{i}}+\cdots+z_{N}^{k_{i}}\right) \prod_{j=1}^{m}\left(z_{1}^{l_{j}} \theta_{1}+\cdots+z_{N}^{l_{j}} \theta_{N}\right) \Delta_{N S}(z, \theta)^{\beta}  \tag{5.5}\\
& \times e^{-\sqrt{\beta}\left(z_{1}^{2}+\cdots+z_{N}^{2}\right)} \mathrm{d} z_{1} \cdots \mathrm{~d} z_{N} \mathrm{~d} \theta_{1} \cdots \mathrm{~d} \theta_{N} . \tag{5.6}
\end{align*}
$$

In a completely parallel way we define Ramond model partition function.

Definition 5.2.2 $\beta$-deformed matrix model partition function in the Ramond sector $Z_{\beta, R}$ for $\beta \in \mathbb{C}$ is defined as

$$
\begin{align*}
Z_{\beta, R}= & \sum_{n=0}^{\infty} \sum_{\substack{k_{1}, \ldots, k_{n} \geq 1 \\
l_{1}, \ldots, l_{m} \geq 1}} \frac{t_{1}^{k_{1}} \ldots t_{n}^{k_{n}}}{k_{1}!\cdots k_{n}!} \frac{\xi_{1+\frac{1}{2}}^{l_{1}} \cdots \xi_{m+\frac{1}{2}}^{l_{m}}}{l_{1}!\cdots l_{m}!} F_{R}^{\beta}\left(k_{1}, \ldots, k_{n}, l_{1}, \ldots, l_{m}\right), \\
F_{R}^{\beta}\left(k_{1}, \ldots, k_{n}, l_{1}, \ldots, l_{n}\right)= & \int_{C} \prod_{i=1}^{n}\left(z_{1}^{k_{i}}+\cdots+z_{N}^{k_{i}}\right) \prod_{j=1}^{m}\left(z_{1}^{l_{j}} \theta_{1}+\cdots+z_{N}^{l_{j}} \theta_{N}\right) \Delta_{R}(z, \theta)^{\beta}  \tag{5.7}\\
& \times e^{-\sqrt{\beta}\left(z_{1}^{2}+\cdots+z_{N}^{2}\right)} \mathrm{d} z_{1} \cdots \mathrm{~d} z_{N} \mathrm{~d} \theta_{1} \cdots \mathrm{~d} \theta_{N} . \tag{5.8}
\end{align*}
$$

Remark 5.2.1 Unlike in the purely bosonic case those expressions for $\beta=2$ cannot be given representation using the integration over random super matrices 51].

### 5.2.2 Integrating out fermionic variables

One can ask following interesting question: what would happen if we would integrate out the fermionic variables, specifying to the case of trivial fermionic potential $\xi_{i}=0$ for any $i \in \mathbb{Z}+\frac{1}{2}$. This section is a digression, as we will not use the results below in the following part of this thesis.

Let $\Lambda=\left\{\lambda_{i j}\right\}_{i, j=1, \ldots, n}$ be an antisymmetric matrix. Notice the following identity:

$$
\prod_{i \neq j}\left(1+\lambda_{i j} \theta_{i} \theta_{j}\right)=\left(\sum_{P \in \mathcal{P}(2 n)} \prod_{p \in P} \lambda_{p_{i} p_{j}}\right) \theta_{1} \cdots \theta_{2 n}+\ldots,
$$

where $\mathcal{P}(2 n)$ is the set of pairings of the set $\{1, \ldots, 2 n\}$ and each pairing $p=\left(p_{1}, p_{2}\right)$, while dots corresponds to the term vanishing under integral. Therefore we have:

$$
\int \prod_{i \neq j}\left(1+\lambda_{i j} \theta_{i} \theta_{j}\right) \mathrm{d} \theta_{1} \cdots \mathrm{~d} \theta_{2 n}=\sum_{P \in \mathcal{P}(2 n)} \prod_{p \in P} \lambda_{p_{i} p_{j}}
$$

This reminds us of the Isserlis Theorem (2.3.1), using which we conclude that:

$$
\int \exp \left(\sum_{i \neq j} \lambda_{i j} \theta_{i} \theta_{j}\right) \mathrm{d} \theta_{1} \cdots \mathrm{~d} \theta_{2 n}=\int \prod_{i \neq j}\left(1+\lambda_{i j} \theta_{i} \theta_{j}\right) \mathrm{d} \theta_{1} \cdots \mathrm{~d} \theta_{2 n}=\mathbb{E}\left(X_{1} \cdots X_{2 n}\right)
$$

where $X_{i}$ are zero-mean Gaussian variables with the correlation matrix given by:

$$
\mathbb{E}\left(X_{i} X_{j}\right)= \begin{cases}\lambda_{i j} & i<j \\ -\lambda_{i j} & i>j, \\ f_{i} & i=j\end{cases}
$$

where $f_{i}$ are any sufficiently large functions of $\left(z_{1}, \ldots, z_{n}\right)$. This largeness condition is necessary to make the matrix $\Lambda$ positively defined, which guarantees the existence of the Gaussian variables with prescribed covariance matrix as above.

This result can be extended to the case of a general fermionic potential $W(z)=\sum_{n=0}^{\infty} \xi_{n} z^{n}$. Notice that

$$
\exp \left(\sum_{k} \theta_{k} W\left(z_{k}\right)\right)=\prod_{k}\left(1+\theta_{k} W\left(z_{k}\right)\right)
$$

Therefore the partition function takes the form

$$
Z(\Lambda)=\int \prod_{i \neq j}\left(1+\lambda_{i j} \theta_{i} \theta_{j}\right) \prod_{k}\left(1+\theta_{k} W\left(z_{k}\right)\right) \mathrm{d} \theta_{1} \ldots \mathrm{~d} \theta_{n}
$$

One can expand the second product into the following sum:

$$
\prod_{k}\left(1+\theta_{k} W\left(z_{k}\right)\right)=\sum_{m \in \mathcal{M}(n)} m\left(\theta_{1}, \ldots, \theta_{n}\right) m\left(W\left(z_{1}\right), \ldots, W\left(z_{n}\right)\right)
$$

where $\mathcal{M}(n)$ is the set of all the monomials in $n$ variables of degree less or equal to $n$. For each term in the above sum one needs to find all terms in the expansion of $\prod_{i \neq j}\left(1+\lambda_{i j} \theta_{i} \theta_{j}\right)$ composed of fermionic variables complementary to those appearing in the corresponding monomial. Therefore one obtains:

$$
Z(\Lambda)=\sum_{I \subset\{1, \ldots, n\}} \sum_{P \in \mathcal{P}(\{1, \ldots, n\}-I)} \prod_{p \in P} \lambda_{p_{1} p_{2}} \prod_{i \in I} W\left(z_{i}\right)
$$

The form of the matrix $\Lambda$, which is a function of the variables $\left(z_{1}, \ldots, z_{n}\right)$, depends on the specific model. This can be rewritten further in the form:

$$
\begin{aligned}
Z(\Lambda) & =\sum_{I \subset\{1, \ldots, n\}} \mathbb{E}\left(\prod_{j \in\{1, \ldots, n\}-I} X_{j} \prod_{i \in I} W\left(z_{i}\right)\right) \\
& =\mathbb{E}\left(\sum_{I \subset\{1, \ldots, n\}} \prod_{j \in\{1, \ldots, n\}-I} X_{j} \prod_{i \in I} W\left(z_{i}\right)\right) \\
& =\mathbb{E}\left(\prod_{i \in\{1, \ldots, n\}}\left(X_{i}+W\left(z_{i}\right)\right)\right)=\mathbb{E} \prod_{i \in\{1, \ldots, n\}} \tilde{X}_{i}\left(z_{i}\right)
\end{aligned}
$$

Now $\tilde{X}_{i}$ are Gaussian variables such that $\mathbb{E}\left(\tilde{X}_{i}\right)=W\left(z_{i}\right)$ and

$$
\operatorname{Cov}\left(X_{i}, X_{j}\right)= \begin{cases}\lambda_{i j} & i<j \\ -\lambda_{i j} & i>j \\ f_{i} & i=j\end{cases}
$$

As before, this matrix is positively defined, hence such random variables exist.

### 5.3 Super CFT

This section is parallel to the sections 3.1 and 3.2 . Many results are rewritten without much change, hence we skip proofs.

### 5.3.1 Super Lie algebras

Many notions and statements presented in the Chapter 3 can be extended to the supersymmetric realm. In this section we follow [36, 37], where modules over supersymmetric extensions of the Virasoro are introduced and studied. Intertwining operators are also defined there as maps between those modules (but not the intertwining fields, which are $z$-dependent).

Let us start with the Lie algebras. We will consider super Lie algebras with a $\frac{1}{2} \mathbb{Z}$-gradation: $\mathfrak{g}=\bigoplus_{k \in \frac{1}{2} \mathbb{Z}} \mathfrak{g}^{k}$, where as before $\left[\mathfrak{g}^{n}, \mathfrak{g}^{m}\right] \subset \mathfrak{g}^{n+m}$ and $\mathfrak{g}^{0}$ is the Cartan subalgebra. Moreover we set $\mathfrak{g}^{+}=\bigoplus_{n>0} \mathfrak{g}^{n}, \mathfrak{g}^{\geq}=\bigoplus_{n \geq 0} \mathfrak{g}^{n}, \mathfrak{g}^{-}=\bigoplus_{n<0} \mathfrak{g}^{n}$ and $\mathfrak{g}^{\leq}=\bigoplus_{n \leq 0} \mathfrak{g}^{n}$.

Definition 5.3.1 $A$ lowest module over $\mathfrak{g}$ with the lowest weight $\lambda \in \mathfrak{h}^{*}$ is a representation $\rho$ : $\mathfrak{g} \rightarrow \operatorname{End}(M)$, such that there exists a vector $v_{\lambda} \in M$ satisfying: $\rho\left(\mathfrak{g}^{+}\right) v_{\lambda}=0, U\left(\mathfrak{g}^{\leq}\right) v_{\lambda}=M$ and $h \cdot v_{\lambda}=\lambda(h) v$ for $h \in \mathfrak{h}$.

As before we will also use notation $|\lambda\rangle$ for the weight vector $v_{\lambda}$.
There are two versions of the super Heisenberg algebra: in the Neveu-Schwarz sector we get the algebra $\mathcal{H}_{\frac{1}{2}}$ and in the Ramond sector we get $\mathcal{H}_{0}$. In what follows we will often use common notation for both sectors, distinguished by value of $\varepsilon \in\{0,1\}$. Super Heisenberg algebras are defined as a super vector spaces:

$$
\mathcal{H}_{\varepsilon}=\bigoplus_{k \in \mathbb{Z}+\varepsilon} \mathbb{C} \psi_{k} \oplus \mathbb{C} k \oplus \bigoplus_{n \in \mathbb{Z}} \mathbb{C} a_{n}
$$

where $\psi_{k}$ are odd generators and $a_{n}$ and $K$ are even generators. The super Lie bracket is defined as:

$$
\left[\psi_{k}, \psi_{l}\right]=\delta_{k,-l}, \quad\left[\psi_{k}, a_{n}\right]=0, \quad\left[a_{n}, a_{m}\right]=n \delta_{n+m}, \quad\left[k, \psi_{k}\right]=\left[k, a_{n}\right]=0
$$

Let us introduce analogs of the Fock and Verma modules for those algebras. There is one difference for both sectors, caused by the presence of operator $\phi_{0}$ in the Ramond sector. Its presence makes the vacuum degenerate, i.e. there are two zero energy vectors: $|0\rangle$ and $\phi_{0}|0\rangle$. This is why we need to define fermionic modules in both sectors separately.

We start with the Neveu-Schwarz case. Define a one dimensional module $\mathbb{C}_{\frac{1}{2}}$ over $\left(\mathcal{H}_{\frac{1}{2}}\right)_{\geq 0}$ on which generators act as follows:

$$
\psi_{k}|\alpha\rangle=0, \quad a_{n}|\alpha\rangle=\delta_{n, 0} \alpha|\alpha\rangle, \quad k \cdot|\alpha\rangle=|\alpha\rangle .
$$

Fock module is defined as

$$
\mathcal{F}_{N S}(\alpha)=\operatorname{Ind}_{\mathcal{H}_{\frac{1}{2}}^{2}}^{\left(\mathcal{H}_{\frac{1}{2}}\right)^{2}} \mathbb{C}_{\frac{1}{2}} .
$$

Its basis consists of the vectors of the form:

$$
a_{-j_{1}} \cdots a_{-j_{l}} \psi_{-i_{1}} \cdots \psi_{-i_{k}}|\alpha\rangle
$$

where $i_{\bullet}, j_{\bullet}>0$.

In the Ramond case we define first two dimensional module $\mathbb{C}_{0}=\mathbb{C} \oplus \mathbb{C}$ with the action

$$
\psi_{k} \cdot v=0 \quad \text { for } k>0, \quad a_{n} v=\delta_{n, 0} \alpha v \quad \psi_{0} \cdot(a, b)=\left(\frac{1}{\sqrt{2}} b, a\right), \quad k \cdot v=v
$$

We will use also another basis of $\mathbb{C}_{0}:|0,+\rangle=\frac{1}{2}\left(|0\rangle+\psi_{0}|0\rangle\right)$ and $|0,-\rangle=\frac{1}{2}\left(|0\rangle-\psi_{0}|0\rangle\right)$. The Ramond Fock module is then defined as

$$
\mathcal{F}_{R}(\alpha)=\operatorname{Ind}_{\left(\mathcal{H}_{0}\right) \geq 0}^{\mathcal{H}_{0}} \mathbb{C}_{0} .
$$

Its basis consists of the vectors of the form:

$$
a_{-j_{1}} \cdots a_{-j_{l}} \psi_{-i_{1}} \cdots \psi_{-i_{k}} \psi_{-i_{0}}|\alpha\rangle
$$

where $i_{\bullet}, j_{\bullet}>0$ and $i_{0} \geq 0$.
There are two supersymmetric extensions of the Virasoro algebra: Neveu-Schwarz algebra $\mathcal{V}_{\frac{1}{2}}$ and Ramond algebra $\mathcal{V}_{0}$. There are equipped with additional set of generators $G_{k}$, where $k \in \mathbb{Z}$ in the Ramond case and $r \in \frac{1}{2}+\mathbb{Z}$ in the Neveu-Schwarz case:

$$
\mathcal{V}_{\varepsilon}=\bigoplus_{n \in \mathbb{Z}} \mathbb{C} L_{n} \oplus \mathbb{C} c \oplus \bigoplus_{k \in \mathbb{Z}+\varepsilon} \mathbb{C} G_{k}
$$

where the first two summands give the even part $\left(\mathcal{V}_{\varepsilon}\right)_{0}$ and the last summands give the odd part $\left(\mathcal{V}_{\varepsilon}\right)_{1}$. Moreover, gradation is specified by: $L_{n} \in\left(\mathcal{V}_{\varepsilon}\right)^{n}, G_{k} \in\left(\mathcal{V}_{\varepsilon}\right)^{k}$ and $c \in\left(\mathcal{V}_{\varepsilon}\right)^{0}$. The super Lie bracket is defined by:

$$
\begin{align*}
{\left[L_{n}, G_{k}\right] } & =\left(\frac{n}{2}-k\right) G_{n+k}, \quad\left[G_{k}, G_{r}\right]=2 L_{k+r}+\frac{c}{3}\left(k^{2}-\frac{1}{4}\right) \delta_{k=-r},  \tag{5.9}\\
{\left[L_{n}, L_{m}\right] } & =(n-m) L_{n+m}+\frac{c}{12} m\left(m^{2}-1\right) \delta_{n,-m} .
\end{align*}
$$

In both cases the Cartan subalgebra is $\mathfrak{h}=\mathbb{C} c \oplus \mathbb{C} L_{0}$.
Let us introduce Verma module for those algebras. To this aim, as before, we introduce one dimensional representations of $\left(\mathcal{V}_{\varepsilon}\right)_{\geq}$. Fix $\lambda \in \mathfrak{h}^{*}$ and $\tau \in \mathbb{Z}_{2}$ and define $\mathbb{C}_{\frac{1}{2}, V}(\lambda, \tau)=\mathbb{C}^{\tau}$, with the action

$$
G_{k} \cdot v=L_{k} \cdot v=0 \quad \text { for } k>0, \quad L_{0} \cdot v=\lambda\left(L_{0}\right) v, \quad c \cdot v=\lambda(c) v .
$$

For the Ramond sector we set

$$
\mathbb{C}_{0, V}(\lambda, \tau)=\left\{\begin{array}{lr}
\mathbb{C}^{\tau} & \text { if } \lambda(c)=24 \lambda\left(L_{0}\right) \\
\mathbb{C}^{\tau} \oplus \mathbb{C}^{\tau+1} & \text { in the other case }
\end{array}\right.
$$

with the action

$$
G_{k} \cdot v=L_{k} \cdot v=0 \quad \text { for } k>0, \quad L_{0} \cdot v=\lambda\left(L_{0}\right) v, \quad c \cdot v=\lambda(c) v
$$

and $G_{0} v=0$ if $\lambda(c)=24 \lambda\left(L_{0}\right)$ and $G_{0}(a, b)=\left(\left(\lambda\left(L_{0}\right)-\frac{\lambda(c)}{24}\right) b, a\right)$ otherwise. (This formula follows from the commutation relations, which imply that $G_{0}^{2}=L_{0}-\frac{1}{24} c$.) The first will be called special Ramond case, while the second generic Ramond case.

The Verma module is defined as

$$
M_{\varepsilon}(\lambda, \tau)=\operatorname{Ind}_{\left(\mathcal{V}_{\varepsilon}\right) \geq 0}^{\mathcal{V}_{\varepsilon}} \mathbb{C}_{\varepsilon, V}(\lambda, \tau)
$$

Proposition 5.3.1 [36] 1. For any weight module $M$ with the weight $\lambda \in \mathfrak{h}^{*}$ there exists a surjective homomorphism of modules $M(\lambda) \rightarrow M$
2. There exists a unique $\mathbb{Z} \times \mathfrak{h}^{*}$-graded submodule $J(\lambda)$ such that $L(\lambda)=M(\lambda) / J(\lambda)$ is a simple graded $\mathfrak{g}$-module. It is moreover irreducible lowest weight module with lowest weight $\lambda$.

## Shapovalov form and determinants

Definition 5.3.2 Define an anti-involution map $\sigma: \mathcal{V}_{\varepsilon} \rightarrow \mathcal{V}_{\varepsilon}$ via the following properties: $\sigma\left(L_{n}\right)=$ $L_{-n}, \sigma\left(G_{k}\right)=G_{-k}, \sigma(c)=c$ and $\sigma(X Y)=\sigma(Y) \sigma(X)$. Such a map is unique.

Proposition 5.3.2 There exists a unique bilinear form $(\cdot, \cdot): M(\Delta, c) \times M(\Delta, c) \rightarrow \mathbb{C}$ satisfying $\left(v_{c, h}, v_{c, h}\right)=1$ and for any $X \in U(\mathcal{V})$ and $x, y \in M(\Delta, c)$ we have $(X \cdot x, y)=(x, \sigma(X) \cdot y)$.

We have following decomposition of the Verma module $M(\Delta, c)=\oplus_{n \in \mathbb{Z}} M(\Delta, c)_{n+\Delta}$, where $M(\Delta, c)_{n+\Delta}=\left\{v \in M(\Delta, c): L_{0} \cdot v=(n+\Delta) v\right\}$ are the eigenspaces of the operator $L_{0}$. This decomposition is orthogonal with respect to the Shapovalov form.

Definition 5.3.3 For any $n \in \mathbb{N}$ let $\operatorname{det}(c, \Delta)_{n}$ be the discriminant of the Shapovalov form restricted to the subspace $M(\Delta, c)_{n+\Delta} \times M(\Delta, c)_{n+\Delta}$. It is also called Kac determinant.

Theorem 5.3.1 [36] Let $n \in(1-\varepsilon) \mathbb{Z}_{>0}$. The supersymmetric Fock space determinant has the following form

1. In the special Ramond case we have

$$
\operatorname{det}_{0}(c, \Delta)_{n} \propto \prod_{\substack{r, s \in \mathbb{Z}_{\geq 0} \\ 1 \leq r s \leq 2 n \\ r-s \in 1-2 \varepsilon+2 \mathbb{Z}_{\geq 0}}} \Psi_{r, s}(c, \Delta)^{p_{0}\left(n-\frac{1}{2} r s\right)}
$$

where

$$
\Phi_{r, s}(c, \Delta)=r s c+\frac{3}{2}(2 r-s)(r-2 s)
$$

2. In the Neveu-Schwarz case $\left(\varepsilon=\frac{1}{2}\right)$, as well as in the generic Ramond case $(\varepsilon=0)$ we have

$$
\operatorname{det}_{\varepsilon}(c, \Delta)_{n} \propto\left(\Delta-\frac{1}{24} c\right)^{\delta_{\varepsilon} p_{0}(n)} \prod_{\substack{r, s \in \mathbb{Z} \geq 0 \\ 1 \leq r \leq 2 n \\ r-s \in 1-2 \varepsilon+2 \mathbb{Z}_{\geq 0}}} \Phi_{r, s, \varepsilon}(c, \Delta)^{p_{\varepsilon}\left(n-\frac{1}{2} r s\right)},
$$

where

$$
\Phi_{r, s, \varepsilon}(c, \Delta)=\left\{\begin{array}{lr}
\left(\Delta+\frac{1}{24}\left(r^{2}-1\right)\left(c-\frac{15}{2}\right)+\frac{1}{4}(r s-1)-\frac{1}{8}\left(\frac{1}{2}-\varepsilon\right)\right) \times \\
\left(\Delta+\frac{1}{24}\left(s^{2}-1\right)\left(c-\frac{15}{2}\right)+\frac{1}{4}(r s-1)-\frac{1}{8}\left(\frac{1}{2}-\varepsilon\right)\right)+ \\
\frac{1}{64}\left(r^{2}-s^{2}\right)^{2} & \text { if } r \neq s \\
\Delta+\frac{1}{24}\left(r^{2}-1\right)\left(c-\frac{3}{2}\right) & \text { if } r=s
\end{array}\right.
$$

Here $p_{\varepsilon}=\left(\frac{1}{2}+\varepsilon\right) \operatorname{dim} U\left(\left(\mathcal{V}_{\varepsilon}\right)_{-}\right)_{n}$.

There is a corresponding bilinear form on the Fock module $(\cdot, \cdot): \mathcal{F}(\alpha) \times \mathcal{F}(\alpha) \rightarrow \mathbb{C}$ satisfying $\left(v_{\alpha}, v_{\alpha}\right)=1$ and for any $X \in U(\mathcal{H})$ and $x, y \in \mathcal{F}(\alpha)$ we have $(X \cdot x, y)=(x, \sigma(X) \cdot y)$. This bilinear form is nondegenerate.

## Singular vectors

Definition 5.3.4 Let $M$ be a module over the algebra $\mathcal{V}_{\varepsilon}$. A vector $\xi \in M-\{0\}$ is called singular vector if $\left(\mathcal{V}_{\varepsilon}\right)_{>0} \xi=0$. It is called null vector if it is orthogonal to any nonzero vector with respect to the Shapovalov form.

Any singular vector is also a null vector. There exists a null vector in $M(\Delta, c)_{n+\Delta}$ if and only if $\operatorname{det}(c, \Delta)_{n}=0$. From the Theorem 5.3.1 we infer existence of null vectors in Verma modules.

Note that the descendant of any null vector $v_{n}$ is again a null vector:

$$
\left(L_{-n} v_{n}, w\right)=\left(v_{n}, L_{n} w\right)=0
$$

for any $w \in M$. This is however not true for singular vectors.

Proposition 5.3.3 (38) There exists $A_{n} \in U\left(\mathfrak{g}^{\leq}\right)_{-n}$, called Shapovalov element, such that $A_{n}|0\rangle$ is a singular vector.

Proposition 5.3.4 Assume that $\phi: M_{1} \rightarrow M_{2}$ is a homomorphism of $\mathcal{V}$-modules. Then the image of the weight vector is either 0 , a weight vector or a singular vector.

## Homomorphism $S_{\alpha, Q}$

As in the bosonic case fermionic Fock modules admit action of the corresponding extensions of the Virasoro algebra. For the algebra $\mathcal{V}_{\varepsilon}$ this action is given by the following definitions:

$$
\begin{align*}
L_{0} & =\sum_{m=1}^{\infty} a_{-m} a_{m}+\sum_{k \in \mathbb{Z}_{>0}-\varepsilon} k \psi_{-k} \psi_{k}+\frac{1}{2} \alpha(\alpha-Q)+\delta_{\epsilon, 0} \frac{1}{16}, \\
L_{n} & =\frac{1}{2} \sum_{m \neq 0, n} a_{n-m} a_{m}+\frac{1}{2} \sum_{k \in \mathbb{Z}+\varepsilon} k \psi_{n-k} \psi_{k}+\frac{1}{2}(2 \alpha-(n+1) Q) a_{n} \quad \text { for } n \neq 0,  \tag{5.10}\\
G_{k} & =\sum_{m \neq 0} a_{m} \psi_{k-m}+\left(\alpha-\left(k+\frac{1}{2}\right) Q\right) \psi_{k} \quad \text { for } k \in \mathbb{Z}+\varepsilon .
\end{align*}
$$

Let $|\alpha\rangle \in \mathcal{F}(\alpha)$ be the weight state. We can consider weight module over $\mathcal{V}$ defined using the above representation by $N(\alpha, Q)=U(\mathcal{V} \leq)|\alpha\rangle \subset \mathcal{F}(\alpha)$. From the universal property 5.3.1 it follows that for $c=1-6 Q^{2}$ and $\Delta=\alpha(\alpha-Q)$ there exists a surjective homomorphism of modules $S_{\alpha, Q}: M(\Delta, c) \rightarrow N(\alpha, Q)$.

Proposition 5.3.5 The homomorphism $S_{\alpha, Q}$ enjoys following properties:

- $S_{\alpha, Q}$ preserves the bilinear forms,
- kernel of $S_{\alpha, Q}$ is the subspace of all null vectors in $M(\Delta, c)$,

Therefore $N(\alpha, Q)=L(c, h)$ is the irreducible lowest weight module from Proposition 5.3.1.

Note however that it need not be surjective, since $\mathcal{F}(\alpha)$ is not necessarily a weight module over $\mathcal{V}$. Indeed, for specific values of $\alpha$ this is the case. This follows from the fact that the map $S_{\alpha, Q}$ can have nontrivial kernel and its restriction $S_{\alpha, Q}^{n}: M\left(1-6 Q^{2}, \alpha(Q-\alpha)\right)_{n} \rightarrow \mathcal{F}(\alpha)_{n}$ is a linear mapping between spaces of the same dimension. Therefore it cannot be surjective and hence $\mathcal{F}(\alpha)$ cannot be a weight module over $\mathcal{V}$.

## Examples of singular vectors

Using this representation quantum curves can be obtained from the following operators producing singular vectors. Below are examples for the Ramond sector. First examples of those operators in the grading -1 are:

$$
\begin{align*}
\widehat{A}_{(\cdot \mid 1)}^{1} & =4 \alpha(2 \alpha-Q) G_{-1}-8 L_{-1} G_{0} \\
\widehat{A}_{(1 \mid 0)}^{1} & =4(2 \alpha+Q) L_{-1} G_{0}-(2 \alpha-Q) G_{-1} \\
\widehat{A}_{(\cdot \mid 1,0)}^{1} & =8 \alpha G_{-1} G_{0}-2(2 \alpha-Q) L_{-1}  \tag{5.11}\\
\widehat{A}_{(1 \mid \cdot)}^{1} & =2(2 \alpha-Q)(2 \alpha+Q) L_{-1}-4 G_{-1} G_{0}
\end{align*}
$$

where we need to specialise $\alpha \in\left\{\frac{1}{2} Q, \frac{1}{2} \sqrt{\beta},-\frac{1}{2 \sqrt{\beta}}\right\}$. In the grading -2 , using above operators, we can construct examples:

$$
\begin{aligned}
\widehat{A}_{(1,1 \mid \cdot)}^{2} & =L_{-1}\left(\left(\alpha+\frac{3}{2} Q\right) \widehat{A}_{(1 \mid \cdot)}^{1}+\frac{3}{2} \widehat{A}_{(\cdot \mid 1,0)}^{1}\right)-G_{-1}\left(\frac{9}{2} \widehat{A}_{(1 \mid 0)}^{1}+\left(\alpha+\frac{3}{2} Q\right) \widehat{A}_{(\cdot \mid 1)}^{1}\right), \\
\widehat{A}_{(2 \mid \cdot)}^{2} & =G_{-1}\left(\frac{3}{2} \alpha \widehat{A}_{(1 \mid 0)}^{1}+\left(\alpha^{2}+\frac{3}{2} Q \alpha-\frac{3}{4}\right) \widehat{A}_{(\cdot \mid 1)}^{1}\right)-L_{-1}\left(\frac{3}{2} \widehat{A}_{(1 \mid \cdot)}^{1}+2 \alpha \widehat{A}_{(\cdot \mid 1,0)}^{1}\right), \\
\widehat{A}_{(1 \mid 1,0)}^{2} & =G_{-1}\left(3 \alpha \widehat{A}_{(1 \mid 0)}^{1}+\frac{3}{2} \widehat{A}_{(\cdot \mid 1)}^{1}\right)-L_{-1}\left(\frac{3}{2} \widehat{A}_{(1 \mid \cdot)}^{1}+2 \alpha \widehat{A}_{(\cdot \mid 1,0)}^{1}\right), \\
\widehat{A}_{(\cdot \mid 2,0)}^{2} & =L_{-1}\left(\alpha \widehat{A}_{(1 \mid \cdot)}^{1}+\left(2 \alpha^{2}+Q \alpha-\frac{3}{2}\right) \widehat{A}_{(\cdot \mid 1,0)}^{1}\right)-G_{-1}\left(2 \alpha^{2} \widehat{A}_{(1 \mid 0)}^{1}+\alpha \widehat{A}_{(\cdot \mid 1))}^{1}\right),
\end{aligned}
$$

we need to specialise $\alpha \in\left\{\frac{1}{2} Q, \frac{1}{2} \sqrt{\beta},-\frac{1}{2 \sqrt{\beta}}, \frac{3}{2} \sqrt{\beta},-\frac{3}{2 \sqrt{\beta}}\right\}$.

### 5.3.2 Super vertex algebras

The super vertex algebras can be defined in a similar way as their bosonic counterparts, where more attention has to be put to signs [28,38, 41.

Recall that for a super vector space $V$ there exists a structure of a super vector space of $\operatorname{End}(V)$. Considering the ring $\mathbb{C}\left[\left[z^{ \pm 1}\right]\right]$ as being purely even, we get a super vector space $\operatorname{End}(V)\left[\left[z^{ \pm 1}\right]\right]=$ $\operatorname{End}(V) \otimes \mathbb{C}\left[\left[z^{ \pm 1}\right]\right]$. In other words even operators are those which preserve the grading and odd
operators are those, which reverse the grading. A field $\phi(x)=\sum_{n \in \mathbb{Z}} \phi_{n} z^{n}$ is even (odd) is all of its coefficients $\phi_{n}$ are even (odd).

By [,] we denote the super commutator (5.1) of the operators. Normal ordering should also be considered with grading. For graded-homogeneous fields $\phi_{1}(x)$ and $\phi_{2}(x)$ we have:

$$
: \phi_{1}(x) \phi_{2}(x):=\phi_{2}(x)_{c} \phi_{1}(x)+(-1)^{\left|\phi_{1}\right|\left|\phi_{2}\right|} \phi_{1}(x) \phi_{2}(x)_{a} .
$$

Definition 5.3.5 Let $V$ be a vector super space. Two super fields $A(z)$ and $B\left(z^{\prime}\right)$ on $V$ are called mutually local if $\exists_{N>0}$ such that $\left(z-z^{\prime}\right)^{N}\left[A(z), B\left(z^{\prime}\right)\right]=0$.

Definition 5.3.6 $A$ vertex superalgebra is a super vector space $V=V_{0} \oplus V_{1}$, with a $\frac{1}{2} \mathbb{Z}$-gradation $V=\bigoplus_{k \in \frac{1}{2} \mathbb{Z}} V^{k}$ for which $\bigoplus_{k \in \mathbb{Z}+\frac{1}{2}} V^{k} \subset V_{1}$, with a distinguished vector $|0\rangle \in V_{0}$, and two linear even maps: $T: V \rightarrow V$ and $Y: V \rightarrow \operatorname{End}(V)\left[\left[z^{ \pm 1}\right]\right]$ satisfying the following set of axioms:

- (translation covariance) $[T, Y(v, z)]=\partial Y(v, z)$, where $T(v)=v_{(-2)}|0\rangle$,
- (vacuum) $Y(|0\rangle, z)=I d_{V}, Y(v, z)|0\rangle \in V[[z]]$ and $\left.Y(v, z)|0\rangle\right|_{z=0}=v$,
- (locality) $\forall_{v, w \in V}$ the fields $Y(v, z)$ and $Y\left(w, z^{\prime}\right)$ are mutually local.

Moreover for any $a \in V^{m}$ we have $\operatorname{deg}\left(a_{(n)}\right)=-n+m-1$. In particular $\operatorname{deg}(T)=1$.

As in the bosonic case the operator product expansion holds also for vertex super algebras.

Proposition 5.3.6 [38] Let $A(z)$ and $B(w)$ be two fields. Then the following statements are equivalent:

1. $A(z)$ and $B(w)$ are mutually local,
2. There exist fields $C_{j}(w), j=0,1, \ldots N_{1}$ such that

$$
[A(z), B(w)]=\sum_{j=0}^{N-1} C_{j}(w) \partial_{w}^{(j)} \delta_{z-w}
$$

where $\delta_{z-w}=\sum_{n \in \mathbb{Z}} z^{-n-1} w^{n}$,
3. $A(z) B(w)$ has an expression

$$
\sum_{j=0}^{N-1} \iota_{z, w} \frac{C_{j}(w)}{(z-w)^{j+1}}+: A(z) B(w):
$$

and $(-1)^{|B||A|} B(w) A(z)$ has an expression:

$$
\sum_{j=0}^{N-1} \iota_{w, z} \frac{C_{j}(w)}{(z-w)^{j+1}}+: A(z) B(w):
$$

where $\iota_{z, w}$ means that we expand rational functions like $\frac{1}{z-w}$ in the domain $|z|>|w|$, and likewise for $\iota_{w, z}$,
4. $A(z) B(w)$ converges to the above formula in the domain $|z|>|w|$ and $(-1)^{|B||A|} B(w) A(z)$ does so in the domain $|w|>|z|$.

## Super conformal structure

Super conformal structure of a vertex super algebra is given by two vectors $\omega, \tau \in V$, for which the modes of the corresponding fields satisfy commutation relations of the appropriate super extension of the Virasoro algebra.

Definition 5.3.7 Vertex operator super algebra (VOSA) is a $\frac{1}{2} \mathbb{Z}$-graded vertex super algebra with distinguished conformal vectors $\omega, \tau \in V$ such that $Y(\omega, z)=\sum_{n \in \mathbb{Z}} L_{n}^{V} z^{-n-2}$ and $Y(\tau, z)=$ $\sum_{k \in \mathbb{Z}+\varepsilon} G_{k}^{V} z^{-k-\frac{3}{2}}$ satisfy:

- Operators $L_{n}^{V}, G_{k}^{V}$ form a representation of the $\mathcal{V}_{\varepsilon}$ algebra for some value of central charge $c^{V}$,
- $L_{-1}^{V}=T$ and $\left.L_{0}^{V}\right|_{V_{n}}=n \operatorname{Id}_{V_{n}}$ for $n \in \mathbb{Z}$.

Definition 5.3.8 $A$ module over vertex operator super algebra $V$ is a super vector space $W$ and an even linear map $Y_{W}: V \rightarrow \operatorname{End} W\left[\left[z, z^{-1}\right]\right]$ such that:

- $Y_{W}(|0\rangle, z)=\operatorname{Id}_{W}$,
- modes of $Y_{W}(\omega, z)=\sum_{n \in \mathbb{Z}} L_{n}^{W} z^{-n-2}$ and $Y_{W}(\tau, z)=\sum_{k \in \mathbb{Z}+\frac{1}{2}} G_{k}^{W} z^{-n-\frac{3}{2}}$ satisfy the commutation relations of the algebra $\mathcal{V}_{\varepsilon}$ with the central charge $c^{V}$,
- for any $v \in V$ we have $Y_{W}(T v, z)=\partial_{z} Y(v, z)$,
- for any $a, b \in V$ the fields $Y_{W}(a, z)$ and $Y_{W}(b, w)$ are mutually local.

Examples. As in the bosonic case we can define the structure of a VOSA on the fermionic Fock module $\mathcal{F}_{N S}(0)$. We set $|0\rangle=v_{0}, T=\sum_{n=0}^{\infty} a_{-n-1} a_{n}+\sum_{k=\frac{1}{2}}^{\infty} k \psi_{-1-k} \psi_{k}$ and let $\phi(z)=$ $\sum_{n \in \mathbb{Z}} a_{n} z^{-n-1} \in \operatorname{End} \mathcal{F}_{N S}(0)\left[\left[z, z^{-1}\right]\right]$ and $\psi(z)=\sum_{k \in \mathbb{Z}+\frac{1}{2}} \psi_{k} z^{-k-\frac{1}{2}}$ be fields. The map $Y$ is defined via:

$$
Y\left(a_{-j_{1}} \cdots a_{-j_{l}} \psi_{-i_{1}} \cdots \psi_{-i_{k}}|0\rangle\right)=: D^{j_{1}-1} \phi(z) \cdots D^{j_{l}-1} \phi(z) D^{i_{1}-\frac{1}{2}} \psi(z) \cdots D^{i_{l}-\frac{1}{2}} \psi(z):
$$

Moreover using equations 5.10 , for any $Q \in \mathbb{C}$ we can define structure of a VOSA on the fermionic NS Fock module. In this case the super tensors can be also expressed as:

$$
\begin{align*}
T(x) & =\frac{1}{2}: \phi(x) \phi(x):+\frac{1}{2}: \partial \psi(x) \psi(x):+\frac{1}{2} Q \partial \phi(x)  \tag{5.12}\\
S(x) & =\psi(x) \phi(x)+Q \partial \psi(x) \tag{5.13}
\end{align*}
$$

Notice that in the above definition we do not need to normal order terms appearing in the field $S(x)$, since operators $a_{n}$ and $\psi_{k}$ commute.

The Ramond sector does not arise as a separate VOSA, but rather as a twisted module over the fermionic NS VOSA.

Definition 5.3.9 Let $V$ be a $V O A$ and $\sigma \in \operatorname{End}(V)$ be an automorphism of $V$ of order $N$ (so that $\sigma^{N}=\operatorname{Id}_{V}$ ). We define $\sigma$-twisted module over $V$ as a vector space $M^{\sigma}$ with an operation

$$
\begin{aligned}
Y_{M^{\sigma}}: V & \rightarrow \operatorname{End}\left(M^{\sigma}\right)\left[\left[z^{ \pm \frac{1}{N}}\right]\right] \\
Y_{M^{\sigma}}\left(v, z^{\frac{1}{N}}\right) & =\sum_{n \in \frac{1}{N} \mathbb{Z}} v_{(n)}^{M^{\sigma}} z^{-n-1},
\end{aligned}
$$

which satisfies following conditions:

- $Y_{M^{\sigma}}\left(|0\rangle, z^{\frac{1}{N}}\right)=\operatorname{Id}_{M^{\sigma}}$,
- for any $v \in V$ if $\sigma(v)=e^{\frac{2 \pi i m}{N}} v$ then $v_{(n)}^{M^{\sigma}}=0$ unless $n \in \frac{m}{N}+\mathbb{Z}$,
- for $v \in V$ and $w \in M$ we have $v_{(n)}^{M^{\sigma}} w=0$ for $n$ sufficiently large,
- Jacobi identity [3].

For any VOSA we have a parity automorphism: $p(v)=(-1)^{|v|} v$.

Example 5.3.1 The Ramond Fock module $\mathcal{F}_{R}(\alpha)$ carries the structure of a p-twisted module over $\mathcal{F}_{N S}(0)$ VOSA with the corresponding operation defined as (here $Y_{R}=Y_{\mathcal{F}_{R}(\alpha)^{p}}$ ):

$$
Y_{R}\left(a_{-j_{1}} \cdots a_{-j_{l}} \psi_{-i_{1}} \cdots \psi_{-i_{k}}|0\rangle\right)=: D^{j_{1}-1} \phi(z) \cdots D^{j_{l}-1} \phi(z) D^{i_{1}-\frac{1}{2}} \psi_{R}(z) \cdots D^{i_{l}-\frac{1}{2}} \psi_{R}(z):
$$

where $\psi_{R}(z)=\sum_{n \in \mathbb{Z}} \psi_{n} z^{-n-\frac{1}{2}}$.

### 5.3.3 Twisted intertwining operators

Here we consider intertwining operators for the fermionic Fock modules. There are two ways one can approach this problem. First option is to extend the operator $I(v, x)$ from the bosonic sector to the fermionic one, that is to allow $v \in \mathcal{F}_{N S}(\alpha)$ or $v \in \mathcal{F}_{R}(\alpha)$ :

$$
I(\cdot, z): M_{1} \rightarrow \operatorname{Hom}\left(M_{2}, M_{3}\right)\left[\left[z, z^{-1}\right]\right] \otimes_{\mathbb{C}} \operatorname{lin}_{\mathbb{C}}\left\{z^{\alpha_{1}}, \ldots, z^{\alpha_{n}}\right\},
$$

where $\alpha_{i} \in \mathbb{C}$ and $M_{i} \in\left\{\mathcal{F}_{N S}(\alpha), \mathcal{F}_{R}(\alpha)\right\}$. Its action on the weight vector is

$$
I(|\alpha\rangle, x, \theta)=u^{\alpha} \exp \left(\alpha \sum_{j=1}^{\infty} \frac{x^{j}}{j} a_{-j}\right) \exp \left(-\alpha \sum_{j=1}^{\infty} \frac{x^{-j}}{j} a_{j}\right) z^{\alpha a_{0}},
$$

where $u^{\alpha}: \mathcal{F}_{\bullet}(\beta) \rightarrow \mathcal{F}_{\bullet}(\alpha+\beta)$ is a mapping such that $u^{\alpha}(|\beta\rangle)=|\alpha+\beta\rangle,\left[u^{\alpha}, a_{n}\right]=0$ for $n \neq 0$ and $\left[a_{0}, u^{\alpha}\right]=\alpha u^{\alpha}$. For descendant states we define

$$
\begin{align*}
& I\left(a_{-j_{1}} \cdots a_{-j_{l}} \psi_{-i_{1}} \cdots \psi_{-i_{k}}|0\rangle, x, \theta\right)= \\
& : D^{j_{1}-1} \phi(x) \cdots D^{j_{l}-1} \phi(x) D^{i_{1}-\frac{1}{2}} \psi(x) \cdots D^{i_{l}-\frac{1}{2}} \psi(x) I(|\alpha\rangle, x, \theta): \tag{5.14}
\end{align*}
$$

where as before $\phi(z)=\sum_{n \in \mathbb{Z}} a_{n} z^{-n-1}$. More explicitly we can write

$$
\begin{equation*}
I(|\alpha\rangle, x)=\sum_{M \in \mathbb{Z}} u^{\alpha} x^{\alpha a_{0}+M} \sum_{\substack{k, l=0}}^{\infty} \sum_{\substack{j_{1}+\cdots+j_{k}=\\ M+m_{1}+\cdots+m_{l} \\ j_{\bullet}, m_{\bullet}>0}} \frac{1}{j_{1} \cdots j_{k} m_{1} \cdots m_{l}} a_{-j_{1}} \ldots a_{-j_{k}} a_{m_{1}} \ldots a_{m_{l}} . \tag{5.15}
\end{equation*}
$$

In a shorter way one can also write $I(|\alpha\rangle, x)=: e^{\alpha \int \phi(x) \mathrm{d} x}$ :, where the integration constant is understood as " $\alpha \log (u)$ ".

We can define action of $\mathcal{V}_{\mathcal{\varepsilon}}$ on the space of intertwining operators in a fashion similar to the previous one:

$$
\left(L_{n} \cdot I\right)(v, z)=I\left(L_{n}^{M_{1}} v, z\right) \quad\left(G_{k} \cdot I\right)(v, z)=I\left(G_{k}^{M_{1}} v, z\right)
$$

Important is the following restriction on the possible combination of the modules $M_{i}$ :

Theorem 5.3.2 ( [33], Theorem 4.7) Let I be an intertwining operator between twisted modules $W_{1}, W_{2}$ and $W_{3}$ over VOA $V$ of twistings $\sigma_{1}, \sigma_{2}$ and $\sigma_{3}$ respectively. Then if $V$ is simple and $I \neq 0$ we must have $\sigma_{1}=\sigma_{3} \sigma_{2}$.

The above theorem gives us restrictions on the possible intertwining operators between Ramond and Neveu-Schwarz sectors. More precisely, since $p^{2}=\mathrm{Id}$, we can have following cases:

- untwisted Neveu-Schwarz case: $\sigma_{1}=\sigma_{2}=\sigma_{3}=I d$, i.e. all three modules lie in the Neveu-Shwarz sector,
- twisted Neveu-Schwarz case: $\sigma_{1}=\mathrm{Id}, \sigma_{2}=\sigma_{3}=p$. In this case we again have NeveuSchwarz quantum curves,
- Ramond case: $\sigma_{1}=p, \sigma_{2}=p$ and $\sigma_{3}=\operatorname{Id}$ or $\sigma_{2}=\operatorname{Id}$ and $\sigma_{3}=p$ : in this case we get Ramond quantum curves.

Although both the second and the third cases correspond to the twisted modules over the NeveuSchwarz VOSA, the special rôle of the module $M_{1}$ in the definition of the intertwining operators, which it plays in the construction of the quantum curves, justifies our notation. In 14 those cases are called respectively: Neveu-Schwarz case, Ramond-NS case and Ramond-R case.

Note that the structure of the quantum curves will come from the structure of the singular vectors in the module $M_{1}$. Therefore only in the Ramond case, when $M_{1}$ is twisted, we do get Ramond quantum curves.

### 5.3.4 Intertwining operators with Grassman odd variable

Second extension of the definition of the intertwining operators to the fermionic realm uses additional Grassman odd variable [2, 35]. Those are objects of the type:

$$
\Phi(\cdot, x, \theta): M_{1} \rightarrow \operatorname{Hom}\left(M_{2}, M_{3}\right)\left[\left[z^{ \pm 1}\right]\right] \otimes A(\mathbb{C}) \otimes \operatorname{lin}_{\mathbb{C}}\left\{z^{\alpha_{1}}, \ldots, z^{\alpha_{n}}\right\}
$$

where $\theta$ is the Grassman odd generator of the super algebra $A(\mathbb{C})$. Let us note that following this approach one can also define super vertex algebras with Grassman odd variables 4].

Definition 5.3.10 [35] Given intertwining operator $I(v, x)$ between fermionic Fock modules as in the section 5.3 .3 one can define intertwining operator with Grassman odd variable:

$$
\Phi(v, x, \theta)=I(v, x)+\theta I\left(G_{-\frac{1}{2}}, x\right) .
$$

This definition satisfies following properties:

- $\Phi\left(G_{-\frac{1}{2}} v, x, \theta\right)=\left(\frac{\partial}{\partial \theta}+\theta \frac{\partial}{\partial x}\right) \Phi(v, x, \theta)$,
- $\Phi\left(L_{-1} v, x, \theta\right)=\frac{\partial}{\partial x} \Phi(v, x, \theta)$,
- skew symmetry and Jacobi identity [35].

Using the definition 5.3.10 and taking $I(v, x)$ as in the equation (5.14) we get

$$
\Phi(|\alpha\rangle, x, \theta)=(1+\theta \alpha \psi(x)) I(|\alpha\rangle, x)
$$

Moreover we assume that the relation

$$
\begin{equation*}
\psi \theta=-\theta \psi \tag{5.16}
\end{equation*}
$$

holds.
Unfortunately the following no-go theorem holds: there do not exist intertwining operators satisfying certain conditions. On the other hand those are conditions, which are required for the construction of the super quantum curves. Therefore the straightforward generalization of the method used in the purely bosonic case is not possible after incorporating fermions. The corresponding definitions should be given with a more subtle approach. For example we can restrict the domain of the intertwining operator $\Phi$.

Proposition 5.3.7 The following set of those conditions:

$$
\begin{align*}
\Phi(|\alpha\rangle, x, \theta) & =(1+\theta \alpha \psi(x)) I(|\alpha\rangle, x) \\
\Phi\left(L_{-1} v, x, \theta\right) & =\partial_{x} \Phi(v, x, \theta) \\
\Phi\left(G_{-\frac{1}{2}}|\alpha\rangle, x, \theta\right) & =\left(\partial_{\theta}-\theta \partial_{x}\right) \Phi(|\alpha\rangle, x, \theta)  \tag{5.17}\\
\Phi\left(G_{-\frac{3}{2}}|\alpha\rangle, x, \theta\right) & =: S(x) \Phi(|\alpha\rangle, x, \theta):
\end{align*}
$$

is contradictory.

Proof. Let us recall that

$$
S(x)=\psi(x) \phi(x)+Q \partial_{x} \psi(x)
$$

On the other hand:

$$
G_{-\frac{3}{2}}|\alpha\rangle=a_{-1} \psi_{-\frac{1}{2}}|\alpha\rangle+(\alpha+Q) \psi_{-\frac{3}{2}}|\alpha\rangle .
$$

Hence the fourth relation in (5.17), being valid for any $Q \in \mathbb{C}$, gives rise to the two equations:

$$
\begin{align*}
\Phi\left(a_{-1} \psi_{-\frac{1}{2}}|\alpha\rangle, x, \theta\right) & =:\left(\psi(x) \phi(x)-\alpha \partial_{x} \psi(x)\right) \Phi(|\alpha\rangle, x, \theta): \\
\Phi\left(\psi_{-\frac{3}{2}}|\alpha\rangle, x, \theta\right) & =: \partial_{x} \psi(x) \Phi(|\alpha\rangle, x, \theta): \tag{5.18}
\end{align*}
$$

Now let us notice the following identity, coming from the relations (5.10):

$$
\left[L_{-1}, \psi_{-\frac{1}{2}}\right]=\psi_{-\frac{3}{2}},
$$

so that

$$
L_{-1} \psi_{-\frac{1}{2}}|\alpha\rangle=\psi_{-\frac{1}{2}} L_{-1}|\alpha\rangle+\psi_{-\frac{3}{2}}|\alpha\rangle .
$$

Moreover applying the definition (5.10) we deduce that

$$
L_{-1}|\alpha\rangle=\psi_{-\frac{3}{2}} \psi_{\frac{1}{2}}|\alpha\rangle+\alpha a_{-1}|\alpha\rangle=\alpha a_{-1}|\alpha\rangle
$$

Hence combining the above two results we get

$$
L_{-1} \psi_{-\frac{1}{2}}|\alpha\rangle=\alpha a_{-1} \psi_{-\frac{1}{2}}|\alpha\rangle+\psi_{-\frac{3}{2}}|\alpha\rangle
$$

We can now apply the relations 5.17, 5.18 and $G_{-\frac{1}{2}}|\alpha\rangle=\alpha \psi_{-\frac{1}{2}}|\alpha\rangle$, obtaining:

$$
\begin{aligned}
\frac{1}{\alpha} \partial_{x}\left(\partial_{\theta}-\theta \partial_{x}\right) \Phi(|\alpha\rangle, x, \theta) & =\frac{1}{\alpha} \Phi\left(L_{-1} G_{-\frac{1}{2}}|\alpha\rangle, x, \theta\right)=\Phi\left(\alpha a_{-1} \psi_{-\frac{1}{2}}|\alpha\rangle\right)+\Phi\left(\psi_{-\frac{3}{2}},|\alpha\rangle, x, \theta\right) \\
& =\alpha:\left(\psi(x) \phi(x)-\alpha \partial_{x} \psi(x)\right) \Phi(|\alpha\rangle, x, \theta):+: \partial_{x} \psi(x) \Phi(|\alpha\rangle, x, \theta):
\end{aligned}
$$

We will show that this equation cannot be true. Let $X_{L}$ denote left hand side of the above equation and $X_{R}$ the right hand side. Recall that

$$
\Phi(|\alpha\rangle, x, \theta)=(1+\theta \alpha \psi(x)) I(|\alpha\rangle, x)
$$

Therefore

$$
\frac{1}{\alpha}\left(\partial_{\theta}-\theta \partial_{x}\right) \Phi(|\alpha\rangle, x, \theta)=\psi(x) I(|\alpha\rangle, x)-\theta: \phi(x) I(|\alpha\rangle, x):,
$$

where we have used the fact that $\partial_{x} I(|\alpha\rangle, x)=\alpha: \phi(x) I(|\alpha\rangle, x):$. It follows that

$$
X_{L}=\partial_{x} \psi(x) I(|\alpha\rangle, x)+\alpha \psi(x): \phi(x) I(|\alpha\rangle, x):-\theta: \partial_{x} \phi(x) I(|\alpha\rangle, x):-\alpha \theta: \phi(x)^{2} I(|\alpha\rangle, x): .
$$

We need to compare it with

$$
\begin{aligned}
X_{R} & =\alpha:\left(\psi(x) \phi(x)-\alpha \partial_{x} \psi(x)\right) \Phi(|\alpha\rangle, x, \theta):+: \partial_{x} \psi(x) \Phi(|\alpha\rangle, x, \theta): \\
& =\alpha: \psi(x) \phi(x) I(|\alpha\rangle, x):+\left(1-\alpha^{2}\right): \partial_{x} \psi(x) I(|\alpha\rangle, x): \\
& +\left(1-\alpha^{3}\right): \partial_{x} \psi(x) \psi(x) I(|\alpha\rangle, x): \theta .
\end{aligned}
$$

Let us restrict to the case $\alpha=0$. Note that $I(|0\rangle, x)=1$. Then we have:

$$
X_{L}-X_{R}=: \partial \psi \psi: \theta+\partial \phi \theta,
$$

which is different from zero. Therefore we have arrived at contradiction.
Question: Do there exist values of $\alpha$ for which the equations (5.17) are not contradictory? Are those values related to the existence of the singular vectors?

We should address here one more question: does one need to consider intertwining operators with Grassman odd variables? Cannot we construct wave function and prove the Schrödinger equations using only operators $I(|\alpha\rangle, x)$ ? Indeed one can define the wave function in the following way:

$$
\widehat{\psi}_{\alpha}(x)=\int \psi\left(z_{1}\right) I\left(|\gamma\rangle, z_{1}\right) \ldots \psi\left(z_{N}\right) I\left(|\gamma\rangle, z_{N}\right)|0\rangle \mathrm{d} z_{1} \ldots \mathrm{~d} z_{N}
$$

where $\gamma=-\sqrt{\beta}$. Here we have integrated out fermionic variables from the standard definition. However, in this case it is not clear how to proceed with the proof of the Schrödinger equations.

### 5.4 Super quantum curves

In this section we are going to present some of the results from [14. We are not planning to be complete, but rather to give an outline of the construction of the super quantum curves from the point of view of CFT. Although it is probably possible, we do not present an analog of the construction using Vertex Operator Superalgebras.

We split our discussion into three parts, corresponding to the untwisted Neveu-Schwarz case, twisted Neveu-Schwarz case and the Ramond case.

This part is written with less mathematical rigour.

### 5.4.1 Untwisted Neveu-Schwarz

Recall the definition of the super intertwining operator with the Grassman odd variable:

$$
\Phi^{u}(|\alpha\rangle, x, \theta)=\left(1+\theta \alpha \psi^{u}(x)\right) I(|\alpha\rangle, x) .
$$

Normally ordering intertwining fields we get:

$$
\begin{equation*}
\Phi^{u}(|\alpha\rangle, x, \theta) \Phi^{u}\left(\left|\alpha^{\prime}\right\rangle, x^{\prime}, \theta^{\prime}\right)=\left(x-x^{\prime}-\theta \theta^{\prime}\right)^{\alpha \alpha^{\prime}}: \Phi(|\alpha\rangle, x, \theta) \Phi\left(\left|\alpha^{\prime}\right\rangle, x^{\prime}, \theta^{\prime}\right): \tag{5.19}
\end{equation*}
$$

Those super intertwining operators satisfy following commutation relations:

$$
\begin{aligned}
& {\left[L_{m}, \Phi^{u}(|\alpha\rangle, x, \theta)\right]=x^{m}\left(x \frac{\partial}{\partial x}+(m+1)\left(\Delta_{\alpha}+\frac{1}{2} \theta \frac{\partial}{\partial \theta}\right) \Phi^{u}(|\alpha\rangle, x, \theta)\right.} \\
& {\left[G_{k}, \Phi^{u}(|\alpha\rangle, x, \theta)\right]=\alpha x^{k-\frac{1}{2}}\left(\theta\left(x \frac{\partial}{\partial x}+2 \Delta_{\alpha}\left(k+\frac{1}{2}\right)\right)-x \frac{\partial}{\partial \theta}\right) \Phi^{u}(|\alpha\rangle, x, \theta) .}
\end{aligned}
$$

where $k \in \mathbb{Z}+\frac{1}{2}$ in the untwisted case and $k \in \mathbb{Z}$ in the twisted case. Let us look on the creation and annihilation parts of the tensors (here instead of subscripts $a$ and $c$ we use + and - respectively):

$$
\begin{array}{ll}
T_{+}^{u}(x)=\sum_{m=-1}^{\infty} x^{-m-2} L_{m}, & T_{-}^{u}(x)=\sum_{m=2}^{\infty} x^{m-2} L_{-m}, \\
S_{+}^{u}(x)=\sum_{k=-\frac{1}{2}}^{\infty} x^{-k-\frac{3}{2}} G_{k}, & S_{-}^{u}(x)=\sum_{k=\frac{3}{2}}^{\infty} x^{k-\frac{3}{2}} G_{-k} .
\end{array}
$$

Those fields satisfy:

$$
T_{+}^{u}(x)|0\rangle=0, \quad S_{+}^{u}(x)|0\rangle=0
$$

Moreover it follows that the following commutation relations hold:

$$
\begin{aligned}
& {\left[T_{+}(y), \Phi^{u}(|\alpha\rangle, x, \theta)\right]=\frac{\Delta_{\alpha}+\frac{1}{2} \theta \frac{\partial}{\partial \theta}}{(y-x)^{2}} \Phi(|\alpha\rangle, x, \theta)+\frac{1}{y-x} \frac{\partial}{\partial x} \Phi^{u}(|\alpha\rangle, x, \theta)} \\
& {\left[S_{+}(y), \Phi^{u}(|\alpha\rangle, x, \theta)\right]=\frac{2 \Delta_{\alpha} \theta}{(y-x)^{2}}+\frac{1}{y-x}\left(\theta \frac{\partial}{\partial x}-\frac{\partial}{\partial \theta}\right) \Phi^{u}(|\alpha\rangle, x, \theta)}
\end{aligned}
$$

Let us pass now to the definition of the super wave function. First we introduce notation

$$
K^{\beta, u}(\vec{z}, \vec{\theta})=\Phi^{u}\left(|\beta\rangle, z_{1}, \theta_{1}\right) \cdots \Phi^{u}\left(|\beta\rangle, z_{N}, \theta_{N}\right)
$$

where to shorten the formulas we used the notation $\vec{z}=\left(z_{1}, \ldots, z_{N}\right)$ and $\vec{\theta}=\left(\theta_{1}, \ldots, \theta_{N}\right)$. From the relation (5.19) it follows that:

$$
K^{\beta, u}(\vec{z}, \vec{\theta})=\Delta_{N S}(\vec{z}, \vec{\theta})^{\beta}: \Phi^{u}\left(|\beta\rangle, z_{1}, \theta_{1}\right) \cdots \Phi^{u}\left(|\beta\rangle, z_{N}, \theta_{N}\right):
$$

where we introduced supersymmetric versions of the Vandermond determinants respectively in the untwisted and twisted sectors:

$$
\Delta_{N S}^{u}(\vec{z}, \vec{\theta})=\prod_{a<b}\left(z_{a}-z_{b}-\theta_{a} \theta_{b}\right)
$$

Those commutation relations allow us to introduce the integrands of the wave functions as well:

$$
\begin{aligned}
\Phi^{u}(|\alpha\rangle, x, \theta) K^{\beta, u}(\vec{z}, \vec{\theta})= & \Delta_{N S}(\vec{z}, \vec{\theta})^{\beta} \prod_{i=1}^{N}\left(x-z_{i}-\theta \theta_{i}\right)^{-\frac{\alpha \sqrt{\beta}}{\hbar}} \\
& : \Phi^{u}(|\alpha\rangle, x, \theta) \Phi^{u}\left(|\beta\rangle, z_{1}, \theta_{1}\right) \cdots \Phi^{u}\left(|\beta\rangle, z_{N}, \theta_{N}\right):
\end{aligned}
$$

We introduce super wave function in the untwisted Neveu-Schwarz sector using the formula

$$
\begin{equation*}
\widehat{\chi}(x, \theta)=\int \Phi^{u}(|\alpha\rangle, x, \theta) K^{\beta, u}(\vec{z}, \vec{\theta})|0\rangle \mathrm{d} \vec{z} \mathrm{~d} \vec{\theta} . \tag{5.20}
\end{equation*}
$$

On this function we define action of the Neveu-Schwarz algebra

$$
\begin{align*}
\widehat{L}_{-n} \cdot \widehat{\chi}(x, \theta) & =\int_{C(x)} \frac{\mathrm{d} y}{2 \pi i} \frac{1}{(y-x)^{n-1}} \int T^{u}(y) \Phi^{u}(|\alpha\rangle, x, \theta) K^{\beta, u}(\vec{z}, \vec{\theta})|0\rangle \mathrm{d} \vec{z} \mathrm{~d} \vec{\theta}  \tag{5.21}\\
\widehat{G}_{-k} \cdot \widehat{\chi}(x, \theta) & =\int_{C(x)} \frac{\mathrm{d} y}{2 \pi i} \frac{1}{(y-x)^{k-\frac{1}{2}}} \int S^{u}(x) \Phi^{u}(|\alpha\rangle, x, \theta) K^{\beta, u}(\vec{z}, \vec{\theta})|0\rangle \mathrm{d} \vec{z} \mathrm{~d} \vec{\theta} \tag{5.22}
\end{align*}
$$

Theorem 5.4.1 [14] The representation of the Neveu-Schwarz algebra is given by the following differential operators:

$$
\begin{gathered}
\widehat{L}_{0}=\Delta_{\frac{\alpha}{\hbar}}+\frac{1}{2} \theta \partial_{\theta}, \quad \widehat{L}_{-1}=\partial_{x}, \\
\widehat{L}_{-n}=\frac{1}{2 \hbar^{2}(n-2)!}\left(\partial_{x}^{n-2}\left(W_{B}^{\prime}(x)\right)^{2}+\partial_{x}^{n-2}\left(W_{F}^{\prime}(x) W_{F}(x)\right)+Q \hbar \partial_{x}^{n} W_{B}(x)+2 \partial_{x}^{n-2} \widehat{f}(x)\right),
\end{gathered}
$$

and

$$
\begin{gathered}
\widehat{G}_{\frac{1}{2}}=2 \theta \Delta_{\frac{\alpha}{\hbar}}, \quad \widehat{G}_{-\frac{1}{2}}=\theta \partial_{x}-\partial_{\theta}, \\
\widehat{G}_{-k}=\frac{1}{\hbar^{2}\left(k-\frac{3}{2}\right)!}\left(\partial_{x}^{k-\frac{3}{2}}\left(W_{F}(x) W_{B}^{\prime}(x)\right)+Q \hbar \partial_{x}^{k-\frac{1}{2}} W_{F}(x)+\partial_{x}^{k-\frac{3}{2}} \widehat{h}(x)\right) .
\end{gathered}
$$

Those operators can be used to construct super quantum curves. To this aim one should express the Neveu-Schwarz singular vectors in this representation.

### 5.4.2 Twisted Neveu-Schwarz

Twisted Neveu-Schwarz intertwining operator takes form

$$
\begin{equation*}
\Phi^{t}(|\alpha\rangle, x, \theta)=\left(1+\alpha \psi^{t}(x) \theta\right) I(|\alpha\rangle, x), \tag{5.23}
\end{equation*}
$$

where $\psi^{t}(x)=\sum_{k \in \mathbb{Z}} \psi_{k} x^{-k-\frac{1}{2}}$. Normally ordering intertwining fields we get in the twisted sector:

$$
\Phi^{t}(|\alpha\rangle, x, \theta) \Phi^{t}\left(\left|\alpha^{\prime}\right\rangle, x^{\prime}, \theta^{\prime}\right)=\left(x-x^{\prime}-\theta \theta^{\prime} \sqrt{\frac{x^{\prime}}{x}}\right)^{\alpha \alpha^{\prime}}: \Phi(|\alpha\rangle, x, \theta) \Phi\left(\left|\alpha^{\prime}\right\rangle, x^{\prime}, \theta^{\prime}\right):
$$

Twisted Neveu-Schwarz intertwining operators satisfy commutation relations

$$
\begin{aligned}
& {\left[L_{m}, \Phi^{t}(|\alpha\rangle, x, \theta)\right]=x^{m}\left(x \frac{\partial}{\partial x}+(m+1)\left(\Delta_{\alpha}+\frac{1}{2} \theta \frac{\partial}{\partial \theta}\right) \Phi^{t}(|\alpha\rangle, x, \theta)\right.} \\
& {\left[G_{k}, \Phi^{t}(|\alpha\rangle, x, \theta)\right]=\alpha x^{k-\frac{1}{2}}\left(\theta\left(x \frac{\partial}{\partial x}+2 \Delta_{\alpha}\left(k+\frac{1}{2}\right)\right)-x \frac{\partial}{\partial \theta}\right) \Phi^{t}(|\alpha\rangle, x, \theta)}
\end{aligned}
$$

We define tensors:

$$
\begin{array}{ll}
T_{+}^{t}(x)=\sum_{m=0}^{\infty} x^{-m-2} L_{m}, & T_{-}^{t}(x)=\sum_{m=1}^{\infty} x^{m-2} L_{-m}, \\
S_{+}^{t}(x)=\sum_{k=0}^{\infty} x^{-k-\frac{3}{2}} G_{k}, & S_{-}^{t}(x)=\sum_{k=1}^{\infty} x^{k-\frac{3}{2}} G_{-k} .
\end{array}
$$

Note that here the splitting is different than that into creation and annihilation parts. This is the reason why we use different notation. These fields satisfy:

$$
T_{+}(x)|0, \pm\rangle=\frac{1}{16} x^{-2}|0, \pm\rangle, \quad S_{+}(x)|0, \pm\rangle=-\frac{1}{2 \sqrt{2}} Q|0, \mp\rangle
$$

The following commutation relations hold:

$$
\begin{aligned}
& {\left[T_{+}^{t}(y), \Phi(|\alpha\rangle, x, \theta)\right]=\frac{\Delta_{\alpha}+\frac{1}{2} \theta \frac{\partial}{\partial \theta}}{(y-x)^{2}} \Phi(|\alpha\rangle, x, \theta)+\frac{\partial}{\partial x}\left(\frac{1}{(y-x)}-\frac{1}{y}\right) \Phi(|\alpha\rangle, x, \theta)} \\
& {\left[S_{+}^{t}(y), \Phi(|\alpha\rangle, x, \theta)\right]=\frac{\left(\Delta_{\alpha}-\frac{1}{2}\right)(x+y) \theta}{\sqrt{x y}(y-x)^{2}} \Phi(|\alpha\rangle, x, \theta)+\left(\theta \frac{\partial}{\partial x}-\frac{\partial}{\partial \theta}\right) \frac{\sqrt{x}}{\sqrt{y}(y-x)} \Phi(|\alpha\rangle, x, \theta) .}
\end{aligned}
$$

In order to define wave function in the twisted sector we introduce notation:

$$
K^{\beta, t}(\vec{z}, \vec{\theta})=\Phi^{t}\left(|\beta\rangle, z_{1}, \theta_{1}\right) \cdots \Phi^{t}\left(|\beta\rangle, z_{N}, \theta_{N}\right)
$$

where to shorten the formulas we used the notation $\vec{z}=\left(z_{1}, \ldots, z_{N}\right)$ and $\vec{\theta}=\left(\theta_{1}, \ldots, \theta_{N}\right)$. From the commutation relations it follows that:

$$
K^{\beta, u}(\vec{z}, \vec{\theta})=\Delta_{N S}(\vec{z}, \vec{\theta})^{\beta}: \Phi^{t}\left(|\beta\rangle, z_{1}, \theta_{1}\right) \cdots \Phi^{t}\left(|\beta\rangle, z_{N}, \theta_{N}\right):,
$$

where we introduced supersymmetric versions of the Vandermonde determinant in the twisted sector:

$$
\Delta_{N S}^{t}(\vec{z}, \vec{\theta})=\prod_{a<b}\left(z_{a}-z_{b}-\theta_{a} \theta_{b} \sqrt{\frac{z_{b}}{z_{a}}}\right) .
$$

Those commutation relations allow us to introduce the integrands of the wave functions as well:

$$
\begin{aligned}
I^{t}(|\alpha\rangle, x, \theta) K^{\beta, u}(\vec{z}, \vec{\theta})= & \Delta_{N S}(\vec{z}, \vec{\theta})^{\beta} \prod_{i=1}^{N}\left(x-z_{i}-\theta \theta_{i} \sqrt{\frac{z_{i}}{x}}\right)^{-\frac{\alpha \sqrt{\beta}}{\hbar}} \\
& : I^{t}(|\alpha\rangle, x, \theta) \Phi^{t}\left(|\beta\rangle, z_{1}, \theta_{1}\right) \cdots \Phi^{t}\left(|\beta\rangle, z_{N}, \theta_{N}\right):
\end{aligned}
$$

We introduce super wave function in the untwisted Neveu-Schwarz sector using the formula

$$
\begin{equation*}
\widehat{\chi}(x, \theta)=\int \Phi^{t}(|\alpha\rangle, x, \theta) K^{\beta, u}(\vec{z}, \vec{\theta})|0\rangle \mathrm{d} \vec{z} \mathrm{~d} \vec{\theta} . \tag{5.24}
\end{equation*}
$$

On this function we define action of the Neveu-Schwarz algebra

$$
\begin{align*}
& \widehat{L}_{-n} \cdot \widehat{\chi}(x, \theta)=\int_{C(x)} \frac{\mathrm{d} y}{2 \pi i} \frac{1}{(y-x)^{n-1}} \int T^{t}(y) \Phi^{t}(|\alpha\rangle, x, \theta) K^{\beta, u}(\vec{z}, \vec{\theta})|0\rangle \mathrm{d} \vec{z} \mathrm{~d} \vec{\theta}  \tag{5.25}\\
& \widehat{G}_{-k} \cdot \widehat{\chi}(x, \theta)=\int_{C(x)} \frac{\mathrm{d} y}{2 \pi i} \frac{1}{(y-x)^{k-\frac{1}{2}}} \int S^{t}(x) \Phi^{t}(|\alpha\rangle, x, \theta) K^{\beta, u}(\vec{z}, \vec{\theta})|0\rangle \mathrm{d} \vec{z} \mathrm{~d} \vec{\theta} \tag{5.26}
\end{align*}
$$

Theorem 5.4.2 In the twisted case representation of the Neveu-Schwarz algebra takes form:

$$
\begin{aligned}
\widehat{G}_{\frac{1}{2}} & =2 \Delta_{\frac{\alpha}{\hbar}} \theta, \quad \widehat{G}_{-\frac{1}{2}}=\theta \partial_{x}-\partial_{\theta} \\
\widehat{G}_{-\frac{3}{2}} & =\frac{\Delta_{\frac{\alpha}{\hbar}} \theta}{4 x^{2}}-\frac{1}{2 x}\left(\theta \partial_{x}-\partial_{\theta}\right) \quad+\frac{x^{-1 / 2}}{\hbar^{2}}\left(\left(W_{B}^{\prime}(x)-\frac{Q \hbar}{2 x}\right)\left(W_{F}(x)-\frac{1}{2} \hbar^{2} \partial_{\xi_{0}}\right)+Q \hbar W_{F}^{\prime}(x)+\widehat{h}(x)\right) \\
\widehat{L}_{0} & =\Delta_{\frac{\alpha}{\hbar}}+\frac{1}{2} \theta \partial_{\theta} \quad \widehat{L}_{-1}=\partial_{x}, \\
\widehat{L}_{-2} & =\frac{1}{16 x^{2}}-\frac{1}{x} \partial_{x}+\frac{1}{\hbar^{2}}\left\{\frac{1}{x} \widehat{f}(x)+\frac{1}{2}\left(\left(W_{B}^{\prime}(x)\right)^{2}+Q \hbar W_{B}^{\prime \prime}(x)+\frac{1}{x} W_{F}^{\prime}(x)\left(W_{F}(x)-\frac{\hbar^{2}}{2} \partial_{\xi_{0}}\right)\right)\right\} .
\end{aligned}
$$

### 5.4.3 Ramond

In this case the quantum curves have the structure of the Ramond singular vectors. We restrict our attention to the case, where intertwiners map twisted module to the untwisted module (and not the other way round). Since we have two linearly independent vectors of the lowest conformal dimension in the Ramond Fock module, we can define two intertwining operators:

$$
R_{ \pm}(|\alpha\rangle, x)=I(|\alpha, \pm\rangle, x)=E^{\alpha}(x) \sigma_{ \pm}(x),
$$

where $\sigma_{ \pm}(x)$ is the chiral spin field. Its definition using operator product expansion was recalled in (14). The wave function for the Penner potential is defined as an expression

$$
\begin{equation*}
\chi_{\alpha}^{\mathrm{R}}(x, \xi)=x^{1 / 8}(x-w)^{-\frac{\alpha \gamma}{\hbar^{2}}} e^{-\frac{\gamma \xi \eta}{\hbar^{2}} \frac{\sqrt{x}}{\sqrt{w(x-w)}}} \int \mathrm{d}^{N} z \mathrm{~d}^{N} \theta \Psi(x, \xi, z, \theta), \tag{5.27}
\end{equation*}
$$

where

$$
\begin{equation*}
\Psi(x, \xi, z, \theta)=\Psi_{+}(x, z, \theta)+\frac{\sqrt{2}}{\hbar} e^{\frac{i \pi}{4}} \xi \Psi_{-}(x, z, \theta) \tag{5.28}
\end{equation*}
$$

and

$$
\begin{equation*}
\Psi_{ \pm}(x, x, \theta)=\left\langle\alpha_{0}\right| \mathrm{R}_{ \pm}^{\frac{\alpha}{\hbar}}(x) \Phi^{\frac{\gamma}{\hbar}}(w, \eta) \prod_{a=1}^{N} \Phi^{-\sqrt{\beta}}\left(z_{a}, \theta_{a}\right)\left|\sigma_{+}\right\rangle, \quad \frac{\alpha+\gamma}{\hbar}-N \sqrt{\beta}=Q-\frac{\alpha_{0}}{\hbar} \tag{5.29}
\end{equation*}
$$

The action of the Ramond operators is defined by

$$
\begin{equation*}
\widehat{G}_{m} \widehat{\chi}_{ \pm, \alpha}^{\mathrm{R}}(x)=\int d^{N} z d^{N} \theta\left\langle\alpha_{0}\right| G_{m} \cdot \mathrm{R}_{ \pm}^{\frac{\alpha}{\hbar}}(x) \Phi^{\frac{\gamma}{\hbar}}(w, \eta) \prod_{a=1}^{N} \Phi^{-\sqrt{\beta}}\left(z_{a}, \theta_{a}\right)\left|\sigma_{+}\right\rangle, \tag{5.30}
\end{equation*}
$$

where

$$
\begin{equation*}
S(y) \mathrm{R}_{ \pm}^{\alpha}(x)=\sum_{m \in \mathbb{Z}} \frac{1}{(y-x)^{m+\frac{3}{2}}} G_{m} \cdot \mathrm{R}_{ \pm}^{\alpha}(x) . \tag{5.31}
\end{equation*}
$$

The expression

$$
\begin{equation*}
\frac{\alpha}{\hbar}\left(\frac{2 \alpha}{\hbar}-Q\right) G_{-1}-2 L_{-1} G_{0} \tag{5.32}
\end{equation*}
$$

for $\alpha=\frac{\hbar Q}{2}, \alpha=\frac{\hbar \sqrt{\beta}}{2}$, and $\alpha=-\frac{\hbar}{2 \sqrt{\beta}}$ gives null vectors. Using it we obtain differential equations

$$
\begin{align*}
& \left(\frac{\partial}{\partial x}+\frac{1}{8 x}\right) \hat{\chi}_{+, \alpha}^{\mathrm{R}}(x)=-\mathrm{e}^{\frac{i \pi}{4}} \frac{\alpha}{\hbar} \sqrt{\frac{2 w}{x(x-w)}}\left(\frac{\left.\Delta_{\frac{\gamma}{\hbar} \eta x}^{w(x-w)}+\eta \partial_{w}-\partial_{\eta}\right) \widehat{\chi}_{-, \alpha}^{\mathrm{R}}(x), ~}{w, ~}\right.  \tag{5.33}\\
& \left(\frac{\partial}{\partial x}+\frac{1}{8 x}-\frac{Q \alpha}{\hbar x}\right) \hat{\chi}_{-, \alpha}^{\mathrm{R}}(x)=\mathrm{e}^{-\frac{i \pi}{4}} \frac{\alpha}{\hbar} \sqrt{\frac{2 w}{x(x-w)}}\left(\frac{\left.\Delta_{\frac{\gamma}{\hbar} \eta x}^{w(x-w)}+\eta \partial_{w}-\partial_{\eta}\right) \widehat{\chi}_{+, \alpha}^{\mathrm{R}}(x), ~}{x}\right.
\end{align*}
$$

which are valid for $\alpha=\frac{\hbar}{2} \sqrt{\beta}$ or $-\frac{\hbar}{2 \sqrt{\beta}}$.

## Chapter 6

## Super quantum Airy structures

In this part of our dissertation we return to the concept introduced in the beginning: topological recursion. Recently it has been shown that topological recursion can be reformulated in the language of quadratic differential operators [23,43. Those operators are required to form a closed Lie algebra under the commutator as the Lie bracket. They annihilate a unique partition function. Coefficients of this function can be related with the differential forms $\omega_{g, n}$ computed using the topological recursion. Notion of quantum Airy structures is however more general: there are examples which do not arise from the spectral curves.

There are various attempts to derive supersymmetric version of the topological recursion. In the context of super eigenvalue models such study was done in [12] and [48]. In this chapter we pursue a different approach: we generalise quantum Airy structures (QASs) to super quantum Airy structures (SQASs). Question about the relation of this new concept with the super spectral curves remains open. However, as was show in [30 quantum curves and topological recursion are related with each other. As QASs are a reformulation and generalisation of topological recursion we hope that QSAs (or SQASs) defined in this thesis would allow to express in this language the theory of quantum curves.

Let us also note that in the context of string theory quantisation of quadratic Hamiltonians appeared even earlier in the work of Givental, in the context of topological recursion for example in 18,47 .

In the section 6.1 we define SQAS and its solution (free energy). We also prove existence and uniqueness result, derive relations on the tensors defining SQAS and write down recursion relations on the coefficients of the free energy. We give also two examples of the infinite dimensional SQASs.

In the section 6.2 we discuss classical limit and its connection with the symplectic representations. We show how they can be used as obstructions to the existence of the SQAS. We also sketch a method based on the weight spaces, which can be used in analysing classical Airy structures. We illustrate those considerations in examples.

### 6.1 Definitions and Theorem

### 6.1.1 Definitions

Let $V=V_{0} \oplus V_{1}$ be a super vector space over a field $k$ of characteristic 0 . We also use the notation $\tilde{V}=V \oplus k^{0 \mid 1}$ (here $k^{0 \mid 1}$ is a vector super space over $k$ of dimension $0 \mid 1$ ), $W=V \oplus V^{*}$ and $\tilde{W}=\tilde{V} \oplus \tilde{V}^{*}$.

Definition 6.1.1 Given a vector super space $V$ we define its Weyl algebra as a quotient of the tensor algebra tensored with polynomials in a formal variable $\hbar$ :

$$
\mathcal{W}(V)=\left(T\left(V \oplus V^{*}\right) / I\right) \otimes k[\hbar],
$$

where the ideal $I \subset T\left(V \oplus V^{*}\right)$ is generated by the elements

$$
\begin{array}{r}
y \otimes x-x \otimes y-(-1)^{|x||y|} \hbar y(x) \\
x_{1} \otimes x_{2}-(-1)^{\left|x_{1}\right|\left|x_{2}\right|} x_{2} \otimes x_{1} \\
y_{1} \otimes y_{2}-(-1)^{\left|y_{1}\right|\left|y_{2}\right|} y_{2} \otimes y_{1}
\end{array}
$$

for $x, x_{1}, x_{2} \in V$ and $y, y_{1}, y_{2} \in V^{*}$.

Let us choose a basis $\left\{x_{i}\right\}_{i \in I}$ of $V$ and let $\left\{y_{i}\right\}_{i \in I}$ be a dual set (this means that $\left.y_{j}\left(x_{i}\right)=\delta_{i, j}\right)$. We assume that the basis elements are homogeneous elements with respect to the grading on the vector super space $V$. We denote the images of those elements in the Weyl algebra $\mathcal{W}_{V}$ using symbols $x_{i}$ and $\hbar \frac{\partial}{\partial x_{j}}$ respectively. Hence, in this notation we have a usual commutation relation $\left[\hbar \frac{\partial}{\partial x_{j}}, x_{i}\right]=\hbar \delta_{i, j}$.
Let us denote by $A$ the subalgebra of $\mathcal{W}_{V}$ generated by the set $\left\{x_{i}\right\}_{i \in I}$ and the identity 1 .

Definition 6.1.2 Let $V$ be a vector super space and let $\left\{x_{i}\right\}_{i \in I}$ be its basis. Assume that $V^{*}$ has a super Lie algebra structure and let $\left\{y_{i}\right\}_{i \in I}$ be a dual set. A super quantum Airy structure (SQAS) based on $V$ is an even homomorphism of super Lie algebras

$$
\begin{equation*}
\widehat{L}: V^{*} \rightarrow \mathcal{W}(V) \tag{6.1}
\end{equation*}
$$

such that operators $\widehat{L}_{i}=\widehat{L}\left(y_{i}\right)$ satisfy following conditions:

- $\widehat{L}_{i}$ have the from:

$$
\begin{equation*}
\widehat{L}_{i}=\hbar \partial_{x_{i}}-\frac{1}{2} \sum_{a, b \in I} A_{a, b}^{i} x_{a} x_{b}-\hbar \sum_{a, b \in I} B_{a, b}^{i} x_{a} \partial_{x_{b}}-\frac{1}{2} \hbar^{2} \sum_{a, b \in I} C_{a, b}^{i} \partial_{x_{a}} \partial_{x_{b}}-\hbar D^{i}, \tag{6.2}
\end{equation*}
$$

for some $A_{a, b}^{i}, B_{a, b}^{i}, C_{a, b}^{i}, D^{i} \in k$ (coefficients of the tensors $A, B, C$ and $D$ ).

- Those operators are required to span a vector super space, which equipped with the super commutator is a super Lie algebra. In other words there exist coefficients $f_{i, j}^{k} \in k$ such that $\left[\widehat{L}_{i}, \widehat{L}_{j}\right]=\hbar \sum_{k} f_{i, j}^{k} \widehat{L}_{k}$. for any $i, j \in I$.

This definition can be extended to the one containing also an additional fermionic variable. More precisely the extended SQAS is a map

$$
\widehat{L}: V^{*} \rightarrow \mathcal{W}(\tilde{V})
$$

Notice that this is not the simple change of $V$ for $\tilde{V}$, since the domain of the map $\widehat{L}$ stays the same. This remark is an important observation, as it allows constructions of new examples. In particular we will see that the superalgebra $\mathfrak{o s p}(1 \mid 2)$ cannot be a domain of a SQAS without this additional fermionic variable. One can also make a further extension of the definition, including for example operators of higher degree that 2 . This extension however will not be relevant for us.

Definition 6.1.3 We define free energy as an even element $F \in A[[\hbar]]$. We can expand it in the following way

$$
F=\sum_{g \geq 0, n \geq 1} \frac{\hbar^{g-1}}{n!} \sum_{i_{1}, \ldots, i_{n} \in I} F_{g, n}\left(i_{1}, \ldots, i_{n}\right) x_{i_{1}} \cdots x_{i_{n}},
$$

where $F_{g, n}\left(i_{1}, \ldots, i_{n}\right) \in k$ are the coefficients. Note that the sums

$$
\sum_{i_{1}, \ldots, i_{n} \in I} F_{g, n}\left(i_{1}, \ldots, i_{n}\right) x_{i_{1}} \cdots x_{i_{n}}
$$

are always finite, from the definition of the algebra $A$.

We are interested in the solutions to the equations $\widehat{L}_{i} e^{F}=0$, where $F$ is the free energy. This form of the equation is however not very convenient: due to the presence of $\hbar^{-1}$ the expression $e^{F}$ is ill-defined. One can however proceed in a different way, defining an automorphism of the algebra $\mathcal{W}_{V}$ associated with the free energy $F$.

Definition 6.1.4 For any $F \in A[[\hbar]]$ define an automorphism $T_{F}: \mathcal{W}_{V} \rightarrow \mathcal{W}_{V}$ given on generators by $x_{i} \rightarrow x_{i}$ and $\hbar \frac{\partial}{\partial x_{i}} \rightarrow \hbar \frac{\partial}{\partial x_{i}}+\hbar \frac{\partial}{\partial x_{i}}(F)$.

The map $T: A[[\hbar]] \rightarrow \operatorname{Aut}\left(\mathcal{W}_{V}\right)$ is a homomorphism of groups, where group structure on $A[[\hbar]]$ is given by addition. Its kernel is equal to $k[[\hbar]]$. Every operator $T_{F}$ preserves Lie bracket in $\mathcal{W}_{V}$, hence is maps sub-Lie algebras to sub-Lie algebras.
Now we arrive at the definition of the solution to the equations $\widehat{L}_{i} e^{F}=0$.
Definition 6.1.5 We say that $Z=\exp (F)$ is a solution to the system of equations $\widehat{L}_{i} Z=0$ if $T_{F}\left(\widehat{L}_{i}\right)(1)=0$ for all $i \in I$.

In what follows we also use the following technical assumption. It is always satisfied if the dimension of $V$ is finite, but includes also infinite dimensional cases.

Definition 6.1.6 We say that $S Q A S\left\{\widehat{L}_{i}\right\}_{i \in I}$ is of finite type if for any polynomial $f \in \hbar^{-1} A \otimes \mathbb{C}[\hbar]$ in the variables $x_{a}$ and $\hbar^{ \pm 1}$ the expression $T_{f}\left(\widehat{L}_{i}\right) 1$ is also a polynomial for any $i \in I$ and it is nonzero only for a finite set of indices $i$ (where this finite set of indices depends on $f$ ).

### 6.1.2 Existence and uniqueness theorem

In this section we prove that a solution to SQAS of finite type always exists.

Theorem 6.1.1 Suppose that operators $\widehat{L}_{i}$ form a SQAS of finite type. Then there exists exactly one collection of elements $F_{g, n} \in A$, which satisfy equations $\widehat{L}_{i} \exp \left(\sum_{g \geq 0} \sum_{n \geq 1} \hbar^{g-1} F_{g, n}\right)=0$ in the sense of the Definition 6.1.5 and such that $F_{0,2}=0$.

Here we give a proof in the finite dimensional case. For more general case see [10.
Lemma 6.1.1 Suppose that $H_{1}, \ldots, H_{n} \in A$ satisfy relations $\frac{\partial}{\partial x_{i}} H_{j}=(-1)^{|i||j|} \frac{\partial}{\partial x_{j}} H_{i}$. Then there exists an element $S \in A$ such that for all $i$ we have $\frac{\partial}{\partial x_{i}} S=H_{i}$.

Proof. It is sufficient to give a proof in the case in which all $H_{i}$ are homogeneous polynomials of the same degree, which we denote by $m$. Define $S=\frac{1}{m+1} \sum_{i=1}^{n} x^{i} H_{i}$.

We use the fact that the operator $\sum_{j} x_{j} \frac{\partial}{\partial x_{j}}$ acts on homogeneous polynomials by multiplication by the degree. To see this it is enough to consider its action on the monomials. For any occurrence of $x_{j}$ in a monomial $a x_{j} b$ we have

$$
x_{j} \frac{\partial}{\partial x_{j}}\left(a x_{j} b\right)=x_{j}(-1)^{|j||a|} a b+\cdots=(-1)^{2|j||a|} a x_{j} b+\cdots=a x_{j} b+\ldots
$$

Here the dots represent action of the derivative on other occurrences. It follows that the operator $\sum_{j} x_{j} \frac{\partial}{\partial x_{j}}$ acts by multiplying by the number of any occurrences of any variable, which is the degree. Hence, using the assumption $\frac{\partial}{\partial x_{i}} H_{j}=(-1)^{|i||j|} \frac{\partial}{\partial x_{j}} H_{i}$ we can write

$$
\begin{aligned}
(m+1) \frac{\partial}{\partial x_{i}} S & =H_{i}+\sum_{j} x_{j}(-1)^{|i||j|} \frac{\partial}{\partial x_{i}} H_{j}=H_{i}+\sum_{j} x_{j}(-1)^{2|j||i|} \frac{\partial}{\partial x_{j}} H_{i} \\
& =H_{i}+\sum_{j} x_{j} \frac{\partial}{\partial x_{j}} H_{i}=(m+1) H_{i} .
\end{aligned}
$$

This completes the proof.
Proof of the theorem. Uniqueness follows from the explicit recursion relations shown in the Section 6.1.3. To show the existence we can proceed by induction as in the proof of the Theorem 2.4.2 in [KS].

First let us introduce some notation. We say that a monomial in variables $x_{a}$ of degree $m$ and multiplied by $\hbar^{g}$ is of type $(m, g)$. We can introduce lexicographic order on the set of types: we say that $(m, g)<\left(m^{\prime}, g^{\prime}\right)$ if $g<g^{\prime}$ or $g=g^{\prime}$ and $m<m^{\prime}$. Let us split the free energy with respect to the types: $F=\sum_{g, n} F_{g, n}$. We construct the summands $F_{g, n}$ of type ( $n, g-1$ ) inductively. Recall that the operator $T_{F}$ acts on the generators in the following way: $x_{i} \rightarrow x_{i}$ and $\hbar \frac{\partial}{\partial x_{i}} \rightarrow \hbar \frac{\partial}{\partial x_{i}}+\hbar \frac{\partial}{\partial x_{i}}(F)$.

The proof is inductive. The induction starts with the trivial $F_{0,2}=0$, as assumed in the statement of the theorem. The induction step goes as follows: suppose that we have constructed solution up
to the element of type $(n, g-1)$ :

$$
S_{g, n}=\sum_{g^{\prime}<g, n^{\prime}} F_{g^{\prime}, n^{\prime}}+\sum_{n^{\prime} \leq n} F_{g, n^{\prime}}
$$

In the inductive step we want to construct next term $F_{g, n+1}$. We achieve this by looking on the equations coming from $T_{F}\left(\widehat{L}_{i}\right) 1=0$, which it must satisfy. Let us denote $R_{g, n}=F-S_{g, n}$ the hypothetical rest of the solution (not yet constructed). Using this notation we can write

$$
\begin{align*}
T_{F}\left(\widehat{L}_{i}\right) 1= & \hbar \partial_{i}\left(S_{g, n}+R_{g, n}\right)-\frac{1}{2} A_{a, b}^{i} x^{a} x^{b}-\hbar B_{a, b}^{i} x^{a}\left(\partial_{x^{b}} S_{g, n}+\partial_{x^{b}} R_{g, n}\right) \\
& -\frac{1}{2} \hbar^{2} C_{a, b}^{i}\left(\partial_{x^{a}} S_{g, n}+\partial_{x^{a}} R_{g, n}+\partial_{x^{a}}\right)\left(\partial_{x^{b}} S_{g, n}+\partial_{x^{b}} R_{g, n}\right) 1 \tag{6.3}
\end{align*}
$$

Let us note that the rest $R_{g, n}$ gives contribution to terms appearing on the right hand side of the above equation of which lowest type is $(n, g)$. This lowest type term comes only from $\hbar \partial_{i} F_{g, n+1}$. On the other hand $S_{g, n}$ contributes also to the terms of lower type. However, since we have assumed that it is a partial solution, those terms must vanish. Let us denote the term of type ( $n, g$ ) coming from $S_{g, n}$ by $\hbar^{g} H_{i}(g, n)$. From (6.3) in the type $(n, g)$ we obtain the following equation:

$$
\begin{equation*}
\hbar^{g} \partial_{i} F_{g, n+1}+\hbar^{g} H_{i}(g, n)=0 \tag{6.4}
\end{equation*}
$$

In order to solve it we use Lemma 6.1.1. We need to check that functions $H_{i}(g, n)$ satisfy appropriate condition. Notice that

$$
\begin{equation*}
T_{S_{g, n}}\left(\widehat{L}_{i}\right)=\hbar \frac{\partial}{\partial x_{i}}+\hbar^{g} H_{i}(g, n)-\hbar B_{a, b}^{i} x^{a} \partial_{x^{b}}-\frac{1}{2} \hbar^{2} C_{a, b}^{i} \partial_{x^{a}} \partial_{x^{b}}-\hbar^{2} C_{a, b}^{i} \partial_{x^{a}} S_{g, n} \partial_{x^{b}}+\ldots, \tag{6.5}
\end{equation*}
$$

where dots represent terms of type higher than $(n, g)$. The operation $T_{S_{g, n}}$ preserves commutator of the operators, hence we can write:

$$
\left[T_{S_{g, n}}\left(\widehat{L}_{i}\right), T_{S_{g, n}}\left(\widehat{L}_{j}\right)\right] \cdot 1=\hbar \sum_{k} f_{i, j}^{k} T_{S_{g, n}}\left(\widehat{L}_{k}\right) \cdot 1
$$

Since $S_{g, n}$ is a partial solution on the right hand side of the above formula there do not appear terms of type $(n-1, g+1)$ or lower, but on the left hand side a priori there are such term. Therefore they must vanish and we obtain the following equation:

$$
\frac{\partial}{\partial x_{i}} H_{j}(g, n)-(-1)^{|i||j|} \frac{\partial}{\partial x_{j}} H_{i}(g, n)=0 .
$$

Those are equations necessary in order to use Lemma 6.1.1. Hence we arrive at the conclusion that there exist $F_{g, n+1}$ satisfying equation (6.4). This gives us next term in our inductive step $(g, n) \rightarrow(g, n+1)$. Moreover if $S_{g, n}$ was a polynomial, since our SQAS is of finite type, $F_{g, n+1}$ is again a polynomial (because only finitely many of functions $H_{i}(g, n)$ are nonzero).

To finish the proof we also need to show how to obtain a term with a higher power $g$ of the parameter $\hbar$. Notice that adding a constant to the free energy does not change the equations $T_{F}\left(\widehat{L}_{i}\right) 1=0$, since it does not change the transformation $T_{F}$. Therefore we can always add term of type $(0, g)$ to our solution, rising the power of $\hbar$ appearing in the free energy.

Remark 6.1.1 From the theorem and lemma we also get different version of the recursion for $F_{g, n}: F_{g, n+1}=\frac{1}{n+1} \sum_{i} x_{i} H_{i}(g, n)$, where $S_{g, n}=\sum_{(m, h) \leq(n, g)} F_{m, h}$.

### 6.1.3 Recursion

In this section we are going to rewrite the equations $\widehat{L}_{i} e^{F}=0$ using the coefficients $F_{g, n}$ of the free energy. We will see that those differential equations are equivalent to the recursions relations for those coefficients. Those relations shows that the free energies are unique.

Before proceeding let us change a little bit the perspective and instead of considering the numbers $F_{g, n}\left(i_{1}, \ldots, i_{n}\right)$ consider $F_{n, m}^{g}(I \mid J)$. They appear if we make a segregation of the coefficients with respect to the grading. We assume that indices in the set $I$ correspond to the even variables (which we denote by $x$ 's) and the indices in the set $J$ to the odd variables (which we denote by $\theta$ 's). We also segregate the operators: we assume that $L_{i}$ are even operators and that $G_{j}$ are the odd ones. We also need to split the tensors $A, B, C$ and $D$. New tensors arise as coefficients in the following equations:

$$
G_{i}=\hbar \partial_{\theta_{i}}+\sum_{a \in I, b \in J} P_{a, b}^{i} \theta_{a} x_{b}+\hbar \sum_{a \in I, b \in J} Q_{a, b}^{i} x_{a} \partial_{\theta_{b}}+\hbar \sum_{a \in J, b \in I} R_{a, b}^{i} \theta_{a} \partial_{x_{b}}+\hbar^{2} \sum_{a \in I, b \in J} S_{a, b}^{i} \partial_{x_{a}} \partial_{\theta_{b}}
$$

and

$$
\begin{aligned}
L_{i} & =\hbar \partial_{x_{i}}-\frac{1}{2} \sum_{a, b \in I} A_{a, b}^{i} x_{a} x_{b}-\hbar \sum_{a, b \in I} B_{a, b}^{i} x_{a} \partial_{x_{b}}-\frac{1}{2} \hbar^{2} \sum_{a, b \in I} C_{a, b}^{i} \partial_{x_{a}} \partial_{x_{b}}-\hbar D^{i} \\
& -\frac{1}{2} \sum_{a, b \in J} E_{a, b}^{i} \theta_{a} \theta_{b}-\hbar \sum_{a, b \in J} K_{a, b}^{i} \theta_{a} \partial_{\theta_{b}}-\frac{1}{2} \hbar^{2} \sum_{a, b \in J} H_{a, b}^{i} \partial_{\theta_{a}} \partial_{\theta_{b}} .
\end{aligned}
$$

Remark 6.1.2 Note that the tensors $A, B, C$ and $D$ here have a different meaning comparing to the definition 6.1.2.

We define free energy as a formal power series in the variables $x_{i}$ and $\theta_{j}$ :

$$
F=\sum_{\substack{g \geqslant 0 \\ m, n \geqslant 0}} \frac{\hbar^{g-1}}{(2 m)!n!} \sum_{\substack{i_{1}, \ldots, i_{n} \in I \\ j_{1}, \ldots, j_{2 m} \in J}} F_{n, 2 m}^{g}\left(i_{1}, \ldots, i_{n} \mid j_{1}, \ldots, j_{2 m}\right) x_{i_{i}} \cdots x_{i_{n}} \theta_{j_{1}} \cdots \theta_{j_{2 m}} .
$$

From the equation $\widehat{G}_{i} Z=0$ we get the following recursion relations on the coefficients of the free energy (for any sets of indices $N \subset I^{n}$ and $M \subset J^{m-1}$ ):

$$
\begin{align*}
F_{n, m}^{g}(N \mid i, M)= & -\sum_{k=1}^{n} \sum_{i_{k} \in N, a \in J} Q_{i_{k}, a}^{i} F_{n-1, m}^{g}\left(N-i_{k} \mid a, M\right) \\
& -\sum_{k=1}^{m-1} \sum_{i_{k} \in M, a \in I} R_{i_{k}, a}^{i} F_{n+1, m-2}^{g}\left(a, N \mid M-i_{k}\right)+\sum_{a \in I, b \in J} S_{a, b}^{i}\left(F_{n+1, m}^{g-1}(a, N \mid b, M)\right. \\
& \left.+\sum_{\substack{g_{1}+g_{2}=g \\
\{a\} \cup N \cup N_{2}=N \\
\{b\} \cup M_{1} \cup M_{2}=M}}^{*} F_{\left|I_{1}\right|+1,\left|J_{1}\right|}^{g_{1}}\left(a, N_{1} \mid M_{1}\right) F_{\left|N_{2}\right|,\left|M_{2}\right|+1}^{g_{2}}\left(N_{2} \mid b, M_{2}\right)\right) \tag{6.6}
\end{align*}
$$

Form the equations $\widehat{L}_{i} e^{F}=0$ we infer that (for any sets of indices $N \subset I^{n-1}$ and $M \subset J^{m}$ ):

$$
\begin{align*}
& F_{n, m}^{g}(i, N \mid M)=\sum_{k=1}^{n-1} \sum_{i_{k} \in N, a \in I} B_{i_{k}, a}^{i} F_{n-1, m}^{g}\left(a, N-i_{k} \mid M\right) \\
& +\sum_{k=1}^{m} \sum_{i_{k} \in M, a \in I} K_{i_{k}, a}^{i} F_{n-1, m}^{g}\left(N \mid a, M-i_{k}\right)+\frac{1}{2} \sum_{a, b \in I} C_{a, b}^{i}\left(F_{n+1, m}^{g-1}(a, b, N \mid M)\right. \\
& \left.+\sum_{\substack{g_{1}+g_{2}=g \\
\{a\} \cup N_{1} \cup N_{2}=N \\
\{b\} \cup M_{1} \cup M_{2}=M}}^{*} F_{1+\left|N_{1}\right|,\left|M_{1}\right|}^{g_{1}}\left(a, N_{1} \mid M_{1}\right) F_{1+\left|N_{2}\right|,\left|M_{2}\right|}^{g_{2}}\left(b, N_{2} \mid M_{2}\right)\right)  \tag{6.7}\\
& +\frac{1}{2} \sum_{a, b \in J} H_{a, b}^{i}\left(F_{n-1, m+2}^{g-1}(N \mid b, a, M)\right. \\
& \left.+\sum_{\substack{g_{1}+g_{2}=g \\
\{a\} \cup N_{1} \cup N_{2}=N \\
\{b\} \cup M_{1} \cup M_{2}=M}}^{*} F_{\left|N_{1}\right|, 1+\left|M_{1}\right|}^{g_{1}}\left(N_{1} \mid a, M_{1}\right) F_{\left|N_{2}\right|, 1+\left|M_{2}\right|}^{g_{2}}\left(N_{2} \mid b, M_{2}\right)\right) .
\end{align*}
$$

Sign $\sum^{*}$ means, that from the summation two cases are excluded: $\left(g_{1}=0, I_{1}=\emptyset, J_{1}=\emptyset\right)$ and $\left(g_{2}=0, I_{2}=\emptyset, J_{2}=\emptyset\right)$. Moreover we get the following relations in low Euler characteristic:

$$
\begin{align*}
F_{3,0}^{0}(i, j, k \mid-) & =A_{j, k}^{i}, \quad F_{1,0}^{1}(i \mid-)=D^{i},  \tag{6.8}\\
F_{1,2}^{0}(i \mid j, k) & =E_{j, k}^{i}=-P_{k, j}^{i} .
\end{align*}
$$

We see that the above equations give recursion with respect to the Euler characteristic $e=2 g-$ $2+n+m$ with the initial conditions given by tensors $A, D$ and $E \sim-P$. However it is not obvious that coefficients satisfying those equation exists and are symmetric/antisymmetric.

### 6.1.4 Tensor relations

The condition of the closing of the super Lie algebra generated by the operators $\widehat{L}_{i}$ 's and $\widehat{G}_{j}$ 's can be also rewritten as specific relations on the tensors defining those operators. Here we suppose that the dimension of $V$ is finite. Closing of the superalgebra requires that: $\left\{\widehat{G}_{i}, \widehat{G}_{j}\right\}=\hbar g_{i j}^{k} \widehat{L}_{k}$. This gives us relations (here and below we use Einstein summation convention, meaning that the sum over repeating indices is implicit):

$$
\begin{aligned}
P_{j, a}^{i}+P_{i, a}^{j} & =0 \\
R_{j, a}^{i}+R_{i, a}^{j} & =g_{i, j}^{a} \\
P_{a b}^{i} Q_{c a}^{j}+P_{a c}^{i} Q_{b a}^{j}-R_{j k}^{i} A_{b c}^{k} & =-(i \leftrightarrow j) \\
Q_{b a}^{i} R_{a c}^{j}+P_{a b}^{i} S_{c a}^{j}-R_{j k}^{i} B_{b c}^{k} & =-(i \leftrightarrow j) \\
R_{a b}^{i} S_{c a}^{j}+R_{a c}^{i} S_{b a}^{j}-R_{j k}^{i} C_{b c}^{k} & =-(i \leftrightarrow j) \\
P_{c a}^{i} R_{b a}^{j}-P_{b a}^{i} R_{c a}^{j}+R_{j k}^{i} E_{b c}^{k} & =-(i \leftrightarrow j) \\
Q_{a c}^{i} R_{b a}^{j}-P_{b a}^{i} S_{a c}^{j}+R_{j k}^{i} K_{b c}^{k} & =-(i \leftrightarrow j) \\
Q_{a c}^{i} S_{a b}^{j}-Q_{a b}^{i} S_{a c}^{j}+R_{j k}^{i} H_{b c}^{k} & =-(i \leftrightarrow j) \\
P_{a b}^{i} S_{b a}^{j}-R_{j k}^{i} D_{k}= & =-(i \leftrightarrow j)
\end{aligned}
$$

Here and below the symbol $(i \leftrightarrow j)$ means the left hand side of the respective equation with a substitution $i$ for $j$ and vice versa. From the condition $\left[\widehat{L}_{i}, \widehat{L}_{j}\right]=\hbar f_{i j}^{k} \widehat{L}_{k}$ we get the relations from the paper 23 for tensors $A, B$ and $C$, as well as additional relations:

$$
\begin{aligned}
-E_{a b}^{i} K_{c a}^{j}-E_{a c}^{i} K_{b a}^{j}+B_{j k}^{i} E_{c b}^{k} & =(i \leftrightarrow j) \\
E_{a b}^{i} H_{c a}^{j}-K_{b a}^{i} K_{a c}^{j}+B_{j k}^{i} K_{b c}^{k} & =(i \leftrightarrow j) \\
K_{a b}^{i} H_{a c}^{j}+K_{a c}^{i} H_{b a}^{j}+B_{j k}^{i} H_{b c}^{k} & =(i \leftrightarrow j)
\end{aligned}
$$

The condition for the $D$ vector gets modified comparing to 23 :

$$
\frac{1}{2} E_{a b}^{i} H_{a b}^{j}+B_{j k}^{i} D^{k}+\frac{1}{2} C_{a b}^{i} A_{a b}^{j}=(i \leftrightarrow j)
$$

Finally the commutation relation $\left[L_{i}, G_{j}\right]=\hbar h_{i j}^{k} G_{k}$ gives us the relations:

$$
\begin{align*}
h_{i j}^{k} & =Q_{i k}^{j}+K_{j k}^{i} \\
0 & =P_{a i}^{j}+E_{j a}^{i} \\
h_{i j}^{k} P_{c b}^{k} & =R_{c a}^{j} A_{a b}^{i}+E_{a c}^{i} Q_{b a}^{j}-K_{c a}^{i} P_{a b}^{j}-B_{b a}^{i} P_{c a}^{j} \\
h_{i j}^{k} S_{b d}^{k} & =B_{a b}^{i} S_{a d}^{j}-C_{b a}^{i} Q_{a d}^{j}+K_{a d}^{i} S_{b a}^{j}+H_{a d}^{i} R_{a b}^{j} \\
H_{i j}^{k} R_{c b}^{k} & =B_{a b}^{i} R_{c a}^{j}-C_{b a}^{i} P_{c a}^{j}+E_{a c}^{i} S_{b a}^{j}-K_{c a}^{i} R_{a b}^{j} \\
h_{i j}^{k} Q_{c b}^{k} & =A_{c a}^{i} S_{a b}^{j}-B_{c a}^{i} Q_{a b}^{j}+K_{a b}^{i} Q_{c a}^{j}+H_{a b}^{i} P_{a c}^{j} \tag{6.9}
\end{align*}
$$

Remark 6.1.3 In [10] it has been shown that these relations can be used in order to prove existence of the free energy solving the SQASs equations. That proof is computational and it shows that recursion relations 6.6, 6.7) define coefficients that are graded-symmetric.

### 6.1.5 Examples

In this section we present two examples of the SQASs based on the Virasoro algebra or its supersymmetric extension. Those are examples of infinite dimension and consists of operators, which do not fit in the definition 6.1.2. They require considering a completion $\widehat{\mathcal{W}}(V)$ of the space $\mathcal{W}(V)$. Details can be found in 10 .

## Virasoro algebra

The first one is an important example of a purely bosonic SQAS, whose free energies count intersection numbers on the moduli space of curves [9]. It is based on a vector space spanned by the
elements $x_{0}, x_{1}, x_{2}, \ldots$ and the corresponding operators have the form:

$$
\begin{align*}
\widehat{L}_{0} & =\hbar \frac{\partial}{\partial x_{0}}-\frac{1}{2} x_{0}^{2}-\sum_{k=0}^{\infty} \hbar x_{k} \frac{\partial}{\partial x_{k+1}} \\
\widehat{L}_{1} & =\hbar \frac{\partial}{\partial x_{1}}-\hbar \sum_{k=0}^{\infty} \frac{2 k+1}{3} x_{k} \frac{\partial}{\partial x_{k}}-\frac{1}{24} \hbar  \tag{6.10}\\
\widehat{L}_{i} & =\hbar \frac{\partial}{\partial x_{i}}-\hbar \sum_{k=0}^{\infty} \frac{(2 i+2 k-1)!!(2 k+1)}{(2 i+1)!!(2 j+1)!!} x_{k} \frac{\partial}{\partial x_{i+k-1}} \\
& -\frac{1}{2} \hbar^{2} \sum_{k=0}^{i-2} \frac{(i-k-2)!!(2 k+1)!!}{(2 k+1)!!} \frac{\partial}{\partial x_{k}} \frac{\partial}{\partial x_{i-2-k}} \quad \text { for } \quad i \geq 2
\end{align*}
$$

Those operators satisfy $\hbar$-deformed Virasoro algebra:

$$
\left[\widehat{L}_{i}, \widehat{L}_{j}\right]=\hbar(i-j) \widehat{L}_{i+j}
$$

The free energies are given by $F_{g, n}\left(d_{1}, \ldots, d_{n}\right)=\hbar^{g-1}\left\langle\tau_{d_{1}} \cdots \tau_{d_{n}}\right\rangle_{g, n}$, where the intersection numbers $\left\langle\tau_{d_{1}} \cdots \tau_{d_{n}}\right\rangle_{g, n}$ were defined in 2.18.

## Neveu-Schwarz algebra

The next example of a SQAS is based on Neveu-Schwarz algebra. It is an example coming from a Vertex Operator Super Algebra [10]. Here we consider one particular case of such SQAS presented in an explicit form. The generators are given by the equations:

$$
\begin{aligned}
\widehat{L}_{i} & =\hbar \frac{\partial}{\partial x_{i}}+\frac{\hbar^{2}}{2} \sum_{k=0}^{i} \frac{\partial}{\partial x_{k}} \frac{\partial}{\partial x_{i-k}}+\hbar \sum_{k=i+1}^{\infty}(k-i) x_{k-i} \frac{\partial}{\partial x_{k}}+\frac{\hbar}{2} \sum_{r=\frac{1}{2}}^{\infty}\left(r+\frac{i}{2}\right) \theta_{r+\frac{1}{2}} \frac{\partial}{\partial \theta_{i+r+\frac{1}{2}}} \\
& +\frac{1}{2} \hbar^{2} \sum_{r=\frac{1}{2}}^{i-\frac{1}{2}} \frac{\partial}{\partial \theta_{r+\frac{1}{2}}} \frac{\partial}{\partial \theta_{i-r+\frac{1}{2}}}+\frac{1}{2} \hbar \sum_{r=i+\frac{1}{2}}^{\infty}\left(-r+\frac{i}{2}\right) \theta_{r-i+\frac{1}{2}} \frac{\partial}{\partial \theta_{r+\frac{1}{2}}}+\hbar D \delta_{i=0}, \\
\widehat{G}_{r} & =\hbar \frac{\partial}{\partial \theta_{r+\frac{1}{2}}}+\frac{1}{2} \hbar^{2} \sum_{m=0}^{r-\frac{1}{2}} \frac{\partial}{\partial \theta_{r+\frac{1}{2}-m}} \frac{\partial}{\partial x_{m}}+\frac{1}{2} \hbar \sum_{m=1}^{\infty} x_{m} \frac{\partial}{\partial \theta_{r+\frac{1}{2}+m}} \\
& +\frac{1}{2} \hbar \sum_{m=r+\frac{1}{2}}^{\infty} \theta_{m-r+\frac{1}{2}} \frac{\partial}{\partial x_{m}},
\end{aligned}
$$

with gradings $\left|\widehat{L}_{i}\right|=0$ and $\left|\widehat{G}_{r}\right|=1,\left|x_{i}\right|=0$ and $\left|\theta_{r}\right|=1$, where the indices $i \in \mathbb{Z}_{\geq 0}, r \in \mathbb{Z}_{\geq 0}+\frac{1}{2}$ and $D \in \mathbb{C}$ is a parameter. Those operators satisfy the following super commutation relations:

$$
\begin{aligned}
& {\left[\widehat{L}_{i}, \widehat{L}_{j}\right]=\hbar(i-j) \widehat{L}_{i+j}} \\
& {\left[\widehat{L}_{i}, \widehat{G}_{r}\right]=\hbar\left(\frac{1}{2} i-r\right) \widehat{G}_{i+r}} \\
& \left\{\widehat{G}_{r}, \widehat{G}_{q}\right\}=2 \hbar \widehat{L}_{r+q}
\end{aligned}
$$

### 6.2 Super Airy structures and representations

In the paper [23] another view on classical Airy structures was presented using the theory of the representations of the Lie algebras, which we are going to present here. Let $V$ be a vector super space, such that $V^{*}=\mathfrak{g}$ has a structure of a super Lie algebra. We also use notation $W=V \oplus V^{*}$.

Definition 6.2.1 Let $\widehat{L}: \mathfrak{g} \rightarrow \mathcal{W}_{V}$ be a SQAS. We define its classical limit $L: \mathfrak{g} \rightarrow k[W]$ as a composition of $\widehat{L}$ with a map $\mathcal{W}_{V} \rightarrow \mathcal{W}_{V} / \hbar \mathcal{W}_{V} \simeq k\left[V \oplus V^{*}\right]$, where we identify variables $y_{i}$ on $V^{*}$ with the images of $\hbar \frac{\partial}{\partial x_{i}}$ and $x_{i}$ 's goes to $x_{i}$ 's.

The classical limit can be expressed in the following form:

$$
\begin{equation*}
L_{i}=y_{i}-\frac{1}{2} \sum_{a, b \in I} A_{a, b}^{i} x_{a} x_{b}-\sum_{a, b \in I} B_{a, b}^{i} x_{a} y_{b}-\frac{1}{2} \sum_{a, b \in I} C_{a, b}^{i} y_{a} y_{b} \tag{6.11}
\end{equation*}
$$

The space $W$ carries a natural symplectic structure defined as:

$$
\begin{align*}
\omega\left(x_{1}, x_{2}\right) & =0 & & \text { for } x_{1}, x_{2} \in V \\
\omega\left(y_{1}, y_{2}\right) & =0 & & \text { for } y_{1}, y_{2} \in V^{*}  \tag{6.12}\\
\omega(x, y) & =y(x) & & \text { for } x \in V, y \in V^{*} .
\end{align*}
$$

Therefore the ring $k[W]$ can be equipped with a Poisson bracket $\{\cdot, \cdot\}$. If $e_{i}$ is a basis of $\mathfrak{g}$ then we obtain a relation

$$
\left\{L\left(e_{i}\right), L\left(e_{j}\right)\right\}=\sum_{k} f_{i, j}^{k} L\left(e_{k}\right),
$$

where $f_{i, j}^{k}$ are the structure constants of the algebra $\mathfrak{g}:\left[e_{i}, e_{j}\right]=\sum_{k} f_{i, j}^{k} e_{k}$. Therefore classical Airy structure is a homomorphism of the Lie algebras, where the structure of a super Lie algebra on $k\left[V \oplus V^{*}\right]$ is given by the super Poisson bracket.
Let us assume that the collection of the operators $\widehat{L}_{i}$ for $i=1, \ldots, n$ forms a SQAS based on a vector space $V$ and such that the structure constants $f_{i j}^{k}$ define a super Lie algebra $\mathfrak{g}$. Let $L_{i} \in k[W]$ be its classical limit. With this limit we can associate a representation of the super Lie algebra $\mathfrak{g}$ on the vector space $W$.

Recall that a SQAS is a map $\widehat{L}: \mathfrak{g} \rightarrow \mathcal{W}_{V}$ 6.1). Using this map we can consider the adjoint representation ad : $\mathfrak{g} \rightarrow \operatorname{End}\left(\mathcal{W}_{V}\right)$ defined as $\operatorname{ad}(v)=\hbar^{-1}[\widehat{L}(v), \cdot]$. Recall that the image of the map $\widehat{L}$ consists only of the operators of degree at most two. Super-commutator of two operators of degree at most two is again of such degree. Therefore the representation ad preserves the subspace $\mathcal{W}_{\bar{V}}^{\leq 2} \subset \mathcal{W}_{V}$. We can make a further splitting:

$$
\mathcal{W}_{\bar{V}}^{\leq 2}=\mathcal{W}_{V}^{0} \oplus \mathcal{W}_{V}^{1} \oplus \mathcal{W}_{V}^{2}
$$

according to the degree of the operator, where $\mathcal{W}_{V}^{0} \simeq \mathbb{C}$ are the constants and $\mathcal{W}_{V}^{1} \simeq \mathfrak{g} \oplus \mathfrak{g}^{*}$. From a similar argumentation as before it follows that ad restricts also to a representation

$$
\phi: \mathfrak{g} \rightarrow \operatorname{End}\left(\mathcal{W}_{V}^{0} \oplus \mathcal{W}_{V}^{1}\right)
$$

Let us compose this map with the projection onto the second summand $\pi: \mathcal{W}_{V}^{0} \oplus \mathcal{W}_{V}^{1} \rightarrow \mathcal{W}_{V}^{1}$ and injection $\iota: \mathcal{W}_{V}^{1} \rightarrow \mathcal{W}_{V}^{0} \oplus \mathcal{W}_{V}^{1}$, obtaining

$$
\varphi(\cdot)=\pi \circ \phi(\cdot) \circ \iota: \mathfrak{g} \rightarrow \operatorname{End}\left(\mathcal{W}_{V}^{1}\right)
$$

We can examine this map in more detail by looking how it acts on the basis of $\operatorname{End}\left(\mathcal{W}_{V}^{1}\right)$. Using the explicit form (6.2) we have:

$$
\begin{aligned}
& {\left[\widehat{L}_{i}, x_{c}\right]=\hbar \delta_{i, c}-\hbar B_{a, c}^{i} x^{a}-\hbar^{2} C_{a, c}^{i} \partial_{x^{a}}} \\
& {\left[\widehat{L}_{i}, \hbar \partial_{x_{c}}\right]=\hbar A_{a, c}^{i} x^{a}+\hbar^{2} B_{c, b}^{i} \partial_{x^{b}}}
\end{aligned}
$$

Using the above equations we conclude that the matrix of the map $\varphi\left(e_{i}\right)$ (where $e_{i}$ is the basis element of $\mathfrak{g}$ such that $\left.\rho\left(e_{i}\right)=\widehat{L}_{i}\right)$ has the following form:

$$
\varphi\left(e_{i}\right)=\left[\begin{array}{cc}
-B^{i} & A^{i}  \tag{6.13}\\
-C^{i} & \left(B^{i}\right)^{t}
\end{array}\right]
$$

Let us notice that all of the operators $\varphi\left(e_{i}\right)$ are infinitesimal symplectomorphisms with respect to the symplectic structure 6.12 , i.e. they satisfy relation

$$
\omega\left(\varphi\left(e_{i}\right) v_{1}, v_{2}\right)+\omega\left(v_{1}, \varphi\left(e_{i}\right) v_{2}\right)=0
$$

for any $v_{1}, v_{2} \in \mathcal{W}_{V}^{1}$.

Proposition 6.2.1 The map $\varphi$ is a representation of the super Lie algebra $\mathfrak{g}$.

Proof. The conclusion of the proposition follows if the following conditions are satisfied:

$$
\left[\varphi\left(e_{i}\right), \varphi\left(e_{j}\right)\right]=\varphi\left(\left[e_{i}, e_{j}\right]\right)=\sum_{k} f_{i, j}^{k} \varphi\left(e_{k}\right)
$$

where the second equation is a property of the Lie algebra $\mathfrak{g}$. Let us notice that

$$
\begin{aligned}
{\left[\varphi\left(e_{i}\right), \varphi\left(e_{j}\right)\right] } & =\varphi\left(e_{i}\right) \varphi\left(e_{j}\right)-\varphi\left(e_{i}\right) \varphi\left(e_{j}\right) \\
& =\pi \phi\left(e_{i}\right) \iota \pi \phi\left(e_{j}\right) \iota-\pi \phi\left(e_{i}\right) \iota \pi \phi\left(e_{j}\right) \iota \\
& =\pi\left(\phi\left(e_{i}\right) \iota \pi \phi\left(e_{j}\right)-\phi\left(e_{i}\right) \iota \pi \phi\left(e_{j}\right)\right) \iota \\
& \left.=\pi\left(\phi\left(e_{i}\right) \phi\left(e_{j}\right)-\phi\left(e_{i}\right) \phi\left(e_{j}\right)\right]\right) \iota \\
& =\pi\left[\phi\left(e_{i}\right), \phi\left(e_{j}\right)\right] \iota=\sum_{k} f_{i, j}^{k} \pi \phi\left(e_{k}\right) \iota=\sum_{k} f_{i, j}^{k} \varphi\left(e_{k}\right)
\end{aligned}
$$

where in the forth equality we have used the fact that $\varphi=\pi \phi$ differs from $\phi$ by elements of $\mathcal{W}_{V}^{0}$, which lie in the kernel of $\phi$ (supercommute with any other element of $\mathcal{W}_{V}$ ). Therefore it is enough to show that $\phi$ is a representation of $\mathfrak{g}$. This is clear since $\phi$ is a subrepresentation of ad.

Note that representation $\varphi$ can be also obtained from the classical limit (Definition 6.2.1). Let $L^{c l} \mathfrak{g} \rightarrow k\left[\mathfrak{g} \oplus \mathfrak{g}^{*}\right]$ be such a classical limit of SQAS $\widehat{L}$. Then we define representation $\theta: \mathfrak{g} \rightarrow$ End $\left(\mathfrak{g} \oplus \mathfrak{g}^{*}\right)$ using the Poisson bracket:

$$
\theta(v) w=\left\{L^{c l}(v), w\right\}
$$

One can check that indeed $\theta=\varphi$.
Another important map arising from a quantum Airy structure is a Lagrangian embedding $j: \mathfrak{g} \rightarrow$ $\mathcal{W}_{V}^{1}$. In fact, as shown in [23, there is a one-to-one correspondence between pairs of symplectic representations $\varphi$ as described above and Lagrangian embeddings which satisfy the following relation:

$$
\begin{equation*}
\varphi\left(v_{1}\right) j\left(v_{2}\right)-\varphi\left(v_{2}\right) j\left(v_{1}\right)=j\left(\left[v_{1}, v_{2}\right]\right) \tag{6.14}
\end{equation*}
$$

for $v_{1}, v_{2} \in \mathfrak{g}$. This embedding is simply given by the map

$$
j\left(e_{i}\right)=\hbar \frac{\partial}{\partial x_{i}} .
$$

Existence of (super) symplectic representation of dimension twice that of the underlying super Lie algebra is therefore a necessary condition for the existence of the SQAS without additional fermionic variable. As we will see this condition gives obstruction for existence of such structures for certain algebras. With additional fermionic variable we can add one more dimension.

### 6.2.1 Relation between super and even Airy structures

Let $\mathfrak{g}=\mathfrak{g}_{0} \oplus \mathfrak{g}_{1}$ be a super Lie algebra. Let us denote $W=\mathfrak{g} \oplus \mathfrak{g}^{*}$, which is a vector super space $W=W_{0} \oplus W_{1}$. In particular $W_{0}=\mathfrak{g}_{0} \oplus \mathfrak{g}_{0}^{*}$.

Proposition 6.2.2 Given any super classical Airy structure $L^{\mathrm{cl}}: \mathfrak{g} \rightarrow k[W]$, one can always produce a bosonic one based on $\mathfrak{g}_{0}$.

Proof Consider a restriction of this map to $L_{0}^{\mathrm{cl}}: \mathfrak{g}_{0} \rightarrow k[W]$. Since $L^{\mathrm{cl}}$ is even, so is its restriction. Therefore image of $L_{0}^{\mathrm{cl}}$ consists of even elements. Those elements are linear combinations of monomials, which can be of two types: either have no fermionic variables or have an even number of them. Observe that both of these subsets are closed under the Poisson bracket. Let $\pi_{0}: k[W] \rightarrow k\left[W_{0}\right]$ be the projection onto the subspace spanned by those elements which have no fermionic variables. We define a bosonic Airy structure as a composition $L^{\mathrm{cl}, b}=\pi_{0} \circ L_{0}^{\mathrm{cl}}: \mathfrak{g}_{0} \rightarrow k\left[W_{0}\right]$. The subspace spanned by those monomials in $k[W]$, which have even and nonzero number of fermionic variables, is a left ideal with respect to the Poisson bracket. Therefore the condition $\left\{L_{i}^{\mathrm{cl}}, L_{j}^{\mathrm{cl}}\right\}=\hbar \sum_{k} f_{i j}^{k} L_{k}^{\mathrm{cl}}$, for some $f_{i j}^{k} \in k[W]$, implies that $\left\{L_{i}^{\mathrm{cl}, b}, L_{j}^{\mathrm{cl}, b}\right\}=\hbar \sum_{k} \pi_{0}\left(f_{i j}^{k}\right) L_{k}^{\mathrm{cl}, b}$. This proves that $L^{\mathrm{cl}, b}$ is a classical Airy structure.

A generalisation of the above proposition is the following lemma.

Lemma 6.2.1 Suppose that $L: \mathfrak{g} \rightarrow \operatorname{End}\left(\mathfrak{g} \oplus \mathfrak{g}^{*}\right)$ is a symplectic representation giving rise to a classical Airy structure. Let $\mathfrak{j} \subset \mathfrak{g}$ be a subalgebra such that $\rho(\mathfrak{j})\left(\mathfrak{j} \oplus \mathfrak{j}^{*}\right) \subset \mathfrak{j} \oplus \mathfrak{j}^{*}$. Then L restricted to $\mathfrak{j}$ gives rise to a classical Airy structure.

Proof. Quadratic Lagrangians (6.11) corresponding to the the vectors in $\mathfrak{j}$ can be decomposed as $L(v)=L^{\mathfrak{j}}(v)+R(v)$, where $L^{\mathfrak{j}}$ depends only on $\mathfrak{j} \oplus \mathfrak{j}^{*}$ and $R_{v}$ only on some coordinates complementary
to the first part. We assume that this splitting of the coordinates is compatible with the symplectic structure. This is possible by the assumption $\rho(\mathfrak{j})\left(\mathfrak{j} \oplus \mathfrak{j}^{*}\right) \subset \mathfrak{j} \oplus \mathfrak{j}^{*}$. Mixed terms in $L(v)$ cannot appear, since such terms after acting on elements of $j \oplus j^{*}$ would give something outside $j \oplus j^{*}$ (recall that this action is given by the Poisson commutation). The two ingredients of the decomposition Poisson-commute, hence it follows that they both give a separate representations of $\mathfrak{j}$.

As a corollary we get:

Proposition 6.2.3 If we have a representation $\mathfrak{g} \rightarrow \operatorname{End}(W)$ coming from classical super Airy structure, then it can be restricted to a representation $\mathfrak{g}_{0} \rightarrow \operatorname{End}\left(W_{0}\right)$ corresponding to an even classical Airy structure.

Proof. We need to check the condition $\rho\left(\mathfrak{g}_{0}\right)\left(\mathfrak{g}_{0} \oplus \mathfrak{g}_{0}^{*}\right) \subset \mathfrak{g}_{0} \oplus \mathfrak{g}_{0}^{*}$. This condition follows from the fact that the action of the subalgebra $\mathfrak{g}_{0}$ preserves the $\mathbb{Z}_{2}$ decomposition of the space $W$.

From this criteria it follows that if we want to construct a classical super Airy structure of $\mathfrak{g}$, its even part should be an even classical Airy ctructure of $\mathfrak{g}_{0}$, and the super symplectic representation of $\mathfrak{g}$ should be an extension of the symplectic representation of $\mathfrak{g}_{0}$. List of possible Airy sStructures on simple Lie algebras is given in Proposition 6.9 in [23].

### 6.2.2 Airy structures and weight space decompositions.

In this section we study what are the consequences of an existence of an Airy structure for a given representation of a semisimple Lie algebra from the point of view of weight space decomposition of the space $W$. We restrict ourselves to the even Airy structures of finite dimension. By $\mathfrak{h}$ we denote the Cartan subalgebra of $\mathfrak{g}$. For any Lie superagebra $\mathfrak{g}$ we denote by $\mathfrak{g}_{0}$ and $\mathfrak{g}_{1}$ even and odd part correspondingly. We also decompose our vector space into even and odd part: $W=W^{0} \oplus W^{1}$.

Definition 6.2.2 An Airy structure is called (semi) simple if the underlying Lie algebra is (semi) simple.

In this note we study finite dimensional (semi)simple (super) Airy structures. Let us recall the Whitehead lemma 1

Lemma 6.2.2 (Whitehead lemma) 40] Assume that $\mathfrak{g}$ is a semi-simple Lie algebra of finite dimension, $\alpha: \mathfrak{g} \rightarrow \operatorname{End}(M)$ a finite-dimensional module and $\beta: \mathfrak{g} \rightarrow M$ a map satisfying the condition

$$
\alpha\left(v_{1}\right) \beta\left(v_{2}\right)-\alpha\left(v_{2}\right) \beta\left(v_{1}\right)=\beta\left(\left[v_{1}, v_{2}\right]\right)
$$

for any $v_{1}, v_{2} \in \mathfrak{g}$. Then there exists an element $m \in M$ such that $\beta(v)=\alpha(v) m$ for any $v \in \mathfrak{g}$.

[^2]Using the above lemma, recalling the that classical Airy structure is a pair consisting of the symplectic representation and a Lagrangian embedding $j$ satisfying the equation (6.14), we conclude that semisimple classical Airy structure is a symplectic representation $\rho: \mathfrak{g} \rightarrow \operatorname{End}(W)$ and a choice of a vector $\Omega \in W$ such that $j(\cdot)=\rho(\cdot) \Omega$ is an embedding onto a Lagrangian subspace. Note that such definition of $j$ always satisfies the condition (6.14).

Let us introduce a few useful notions. We say that a nonzero vector $v \in W$ is a weight vector if it is an eigenvector for all $\rho(h), h \in \mathfrak{h}$. (Note that this notation differs from the notion of the weight vector in the Chapter 3.) For any weight vector $v \in W$ we denote by $\lambda_{v} \in \mathfrak{h}^{*}$ the corresponding weight, i.e. we have $\rho(h) v=\lambda_{v}(h) v$ for any $h \in \mathfrak{h}$. We say that $a, b \in W$ are conjugated if $\omega(a, b) \neq 0$. We say that $\sigma: W \rightarrow W$ is a symplectic involution if for any $v \in W$ vectors $v$ and $\sigma(v)$ are conjugated. A weight diagram is a graph, whose nodes represent weight vectors in $W$ forming a basis and arrows represent the action of some basis of $\mathfrak{g}$. We also denote by $\Lambda \subset \mathfrak{h}^{*}$ the weight lattice of the representation $\rho$ i.e. the set of all weights.

Lemma 6.2.3 Suppose that $a, b \in W$ are weight vectors which are conjugate. Then $\lambda_{a}=-\lambda_{b}$.

Proof. For any $h \in \mathfrak{h}$ we have $\lambda_{a}(h) \omega(a, b)=\omega(\rho(h) a, b)=-\omega(a, \rho(h) b)=-\lambda_{b}(h) \omega(a, b)$.
This lemma gives a strong constraint for the possible symplectic structures. Since $\omega$ is nondegenerate, it follows that $\Lambda$ is symmetric, i.e. $-\Lambda=\Lambda$. Let $W=\bigoplus_{\lambda \in \Lambda} V_{\lambda}$ be the weight space decomposition, and denote by $\omega_{\lambda}$ the restriction of the symplectic form to $V_{\lambda} \oplus V_{-\lambda}$ (or $V_{0}$ for $\lambda=0$ ). It is again a symplectic form and we have $\omega=\sum_{\lambda \in \Lambda_{+}} \omega_{\lambda}+\omega_{0}$, where $\Lambda_{+}$is the set of positive weights.

Using the decomposition of $W$ into weight spaces we can write a unique decomposition of the vector generating Lagrangian subspace $\Omega=\sum_{\lambda \in \Lambda} \Omega_{\lambda}$, where $\Omega_{\lambda} \in V_{\lambda}$.

Lemma 6.2.4 The set $X=\left\{\lambda \in \Lambda: \Omega_{\lambda} \neq 0\right\}$ spans $\mathfrak{h}^{*}$.

Proof. Since the map $j(x)=\rho(x) \Omega$ is an embedding, it must also be an embedding when restricted to $\mathfrak{h}$. Using the weight space decomposition $\Omega=\sum_{\lambda \in \Lambda} \Omega_{\lambda}=\sum_{\lambda \in X} \Omega_{\lambda}$ this embedding can be expressed as

$$
j(h)=\rho(x) \Omega=\sum_{\lambda \in X} \rho(h) \Omega_{\lambda}=\sum_{\lambda \in X} \lambda(h) \Omega_{\lambda} .
$$

If there would be a covector $\mu \in \mathfrak{h}^{*}$ linearly independent from $\lambda \in X$ we could find a nonzero $h \in \mathfrak{h}$ such that $\mu(h) \neq 0$ and $\lambda(h)=0$ for all $\lambda \in X$. In that case $j(h)=0$, which is a contradiction with the fact that $j$ is an embedding.

This lemma means in particular that we must have at least $\operatorname{dim}\left(\mathfrak{h}^{*}\right)=\operatorname{dim}(\mathfrak{h})$ elements in the decomposition of $\Omega$. Therefore $\Omega$ can be a weight vector only if $\operatorname{dim}(\mathfrak{h})=1$.

Lemma 6.2.5 For any $x, y \in \mathfrak{g}$ the vectors $\Omega$ and $\rho(x) \rho(y) \Omega$ cannot be conjugate, as well as $\Omega$ and $\rho(x) \Omega$.

Proof. This follows directly from the definition of the Lagrangian subspace: for any $x, y \in \mathfrak{g}$ we have $0=\omega(\rho(x) \Omega, \rho(y) \Omega)=-\omega(\Omega, \rho(x) \rho(y) \Omega)$. The second statement follows from the semisimplicity: any $x \in \mathfrak{g}$ can be represented as $x=\left[y_{1}, y_{2}\right]$ and we can use the relation $\rho(x)=\rho\left(y_{1}\right) \rho\left(y_{2}\right)-$ $\rho\left(y_{2}\right) \rho\left(y_{1}\right)$.

Graphically this Lemma means that $\Omega$ cannot lie on the "border" of the Lagrangian.

Lemma 6.2.6 Suppose that $a, b \in W$ are conjugate and there exists a path in the weight diagram connecting $a$ and $b$, whose arrows represent pairwise commuting operators. Then the length of this path is odd.

Proof. Let $E_{1}, \ldots, E_{k}$ be the operators as in the statement of the lemma and let $b=E_{1} \cdots E_{k} a$. Then

$$
\begin{aligned}
\omega(a, b) & =\omega\left(a, E_{1} \cdots E_{k} a\right)=(-1)^{k} \omega\left(E_{k} \cdots E_{1} a, a\right)=(-1)^{k} \omega\left(E_{1} \cdots E_{k} a, a\right) \\
& =(-1)^{k} \omega(b, a)=(-1)^{k+1} \omega(a, b) .
\end{aligned}
$$

Since $\omega(a, b) \neq 0$, we arrive at the conclusion.
The above lemma is useful for the description of the Airy structure of a direct sum of two algebras $\mathfrak{g}_{1} \oplus \mathfrak{g}_{2}$. Any representation of such a sum decomposes into direct sum of the tensor products of the corresponding representations of $\mathfrak{g}_{1}$ and $\mathfrak{g}_{2}: V=\bigoplus V_{i} \oplus V_{j}$. If we take any $X \in \mathfrak{g}_{1}$ and $Y \in \mathfrak{g}_{2}$ then any word in $X$ and $Y$ gives us a path in the weight diagram. Since $X$ and $Y$ commute, we can apply the lemma.

### 6.2.3 Examples

Let us pass to the examples illustrating above considerations. We discuss only algebras of finite dimension.

## Case of $\mathfrak{o s p}(1 \mid 2)$

Let us recall that super Lie algebra $\mathfrak{o s p}(1 \mid 2)$ has five dimensional basis $e, f, h, b_{+}, b_{-}$subject to the relations:

$$
\begin{array}{cc}
{[e, f]=h, \quad[h, e]=2 e, \quad[h, f]=-2 f,} & {\left[h, b_{ \pm}\right]= \pm b_{ \pm}} \\
\left\{b_{-}, b_{+}\right\}=h, \quad\left\{b_{+}, b_{+}\right\}=4 e, & \left\{b_{-}, b_{-}\right\}=-4 f .
\end{array}
$$

Following [52] we describe irreducible representations of $\mathfrak{o s p}(1 \mid 2)$. Since $\mathfrak{s l}(2) \subset \mathfrak{o s p}(1 \mid 2)$ is a subalgebra, any representation of $\mathfrak{o s p}(1 \mid 2)$ splits into direct sum of irreducible representations of $\mathfrak{s l}(2)$. If we assume that representation of $\mathfrak{o s p}(1 \mid 2)$ is irreducible, then two representations can appear in the decomposition and the weight diagram have the form:


The two rows correspond to the two irreducible representations of $\mathfrak{s l}(2)$. Each node represents one dimensional subspace, which is an eigenspace of the operator $h$ spanning Cartan subalgebra. Each left arrow represents an action of the element $e$ and should be paired with a right arrow representing $f$, going in opposite direction. Also for any $b_{+}$we have an arrow going in opposite direction representing $b_{-}$. The diagram is commutative up to a scalar multiplication.

Proposition 6.2.4 The algebra $\mathfrak{o s p}(1 \mid 2)$ does not support a $S Q A S$ without an additional fermionic variable.

Proof. We show that there does not exist a classical Airy structure for $\mathfrak{o s p}(1 \mid 2)$ without an additional fermionic variable. In order to construct such a structure we need a 10 dimensional representation. Since the even part $\mathfrak{o s p}(1 \mid 2)_{0} \simeq \mathfrak{s l}(2)$, using the Proposition 6.2 .2 we would get a classical Airy structure of $\mathfrak{s l}(2)$ as one of the components. From [23] we know that such structure must be a 6 -dimensional irreducible representation. In the diagram as above this could be the lower or the upper row. However in both cases the minimal dimension of the representation of $\mathfrak{o s p}(1 \mid 2)$ is 11 . Therefore we can not get a 10 dimensional representation, hence no representation fits into the SQAS formalism.

Remark 6.2.1 SQAS based on $\mathfrak{o s p}(1 \mid 2)$ exists if one adds additional fermionic variable, as has been shown in 10. In this case we have a 11 dimensional representation.

Case of $\mathfrak{s l}(2) \oplus \mathfrak{s l}(2)$

The next natural choice for a SQAS would be the algebra $\mathfrak{o s p}(3 \mid 2)$. Its even part is a direct sum $\mathfrak{s l}(2) \oplus \mathfrak{s l}(2)$. Therefore if we are looking for a SQAS based on $\mathfrak{o s p}(3 \mid 2)$ we can first classify Airy structures for $\mathfrak{s l}(2) \oplus \mathfrak{s l}(2)$. Such a classification is interesting also because it shows that considering semisimple Lie algebras gives us more freedom for the existence of Airy structures then restricting to the simple algebras. This is intuitively clear: semisimple algebras are "more abelian" - "higher percentage" of the commutators vanish - hence there are "less constraints" for the operators defining Airy structures. Another reason for considering semisimple Lie algebras is that it gives nice illustration for application of the method presented in the Section 6.2.2.

Any finite dimensional representation of the algebra $\mathfrak{s l}(2) \oplus \mathfrak{s l}(2)$ decomposes as $W=\oplus_{i, j} V_{i} \otimes V_{j}$, where $V_{i}$ is the $i$ 'th dimensional irreducible representation of $\mathfrak{s l}(2)$.

Remark 6.2.2 From the Lemma 6.2.6 it follows that if $V_{i} \otimes V_{j}$ is a symplectic submodule of $W$, then $i+j$ must be odd (weight diagram is rectangular, from Lemma 2 diagonal opposite nodes are
conjugate (since they correspond to the maximal and minimal weights) and the length of the path connecting them is $i+j-2$ ). From the symplecticity $i \cdot j$ must be even, so we can conclude that $i$ is odd and $j$ is even or the other way round.

Using this statements we can exclude some cases in the decomposition of $W$. In what follows we assume that each submodule $V_{i} \otimes V_{j}$ in the decomposition is symplectic. It follows that $L$ must have nontrivial intersection with each of the spaces in the decomposition. Let us first assume there are two ingredients. The dimensions of the possible decompositions are: $0+12,2+10,4+8$ and $6+6$.

- $\mathbf{0 + 1 2}$. Because of the Remark 6.2 .2 we must have $W \simeq V_{3} \otimes V_{4}$ or $V_{1} \otimes V_{12}$. The second one can be excluded, because the map $j(\cdot)=\varphi(\cdot) \Omega: \mathfrak{h} \rightarrow W$ cannot give an embedding for any $\Omega \in W$ (the action of the first summand of $\mathfrak{s l}(2) \oplus \mathfrak{s l}(2)$ is trivial). The first case is left for further study.
- $\mathbf{2 + 1 0}$. In this case let $v_{1}$ and $v_{2}$ be two basis weight vectors in the space $V_{2} \otimes V_{1}$. We can assume that $\Omega=c_{1} v_{1}+c_{2} v_{2}+\epsilon$, where $c_{1}, c_{2} \in \mathbb{C}$ and $\epsilon \in V_{1} \otimes V_{10}$. Let us notice that the first summand in $\mathfrak{s l}(2) \oplus \mathfrak{s l}(2)$, when acting on $\Omega$ kills the vector $\epsilon$. It follows that the space $\rho(\mathfrak{s l}(2) \oplus 0) \Omega$ is a subspace of $V_{2} \otimes V_{1}$, hence can have at most dimension 2. Therefore the map $j(\cdot)=\rho(\cdot) \Omega$ cannot be in such case an embedding. This case if excluded.
- $4+8$. From the Remark 6.2 .2 in this situation the only possible case is $V_{8} \otimes V_{1}+V_{1} \otimes V_{4}$, which can be excluded because Lemma 1 implies that it would give an Airy structure of dimension 4 for $\mathfrak{s l}(2)$.
- 6+6. Because of the Remark 6.2 .2 possible options are: $V_{1} \otimes V_{6}+V_{6} \otimes V_{1}, V_{1} \otimes V_{6}+V_{2} \otimes V_{3}$, $V_{1} \otimes V_{6}+V_{3} \otimes V_{2}, V_{2} \otimes V_{3}+V_{2} \otimes V_{3}$ or $V_{2} \otimes V_{3}+V_{3} \otimes V_{2}$.
A. Trivial example is $V_{1} \otimes V_{6}+V_{6} \otimes V_{1}$, which is a direct sum of two $\mathfrak{s l}(2)$ Airy structures.
B. More interesting one is $V_{2} \otimes V_{3}+V_{3} \otimes V_{2}$. The weight diagram in this case takes the form:


The vector corresponding to $(i, j)^{t h}$ node will be denoted by $v_{i j}$ for the left part of the diagram and by $w_{i j}$ for the right part. The symplectic form is given by

$$
\begin{aligned}
\omega= & \mathrm{d} v_{11} \wedge \mathrm{~d} v_{31}-\mathrm{d} v_{21} \wedge \mathrm{~d} v_{22}+\mathrm{d} v_{31} \wedge \mathrm{~d} v_{13}+ \\
& \mathrm{d} w_{12} \wedge \mathrm{~d} w_{23}-\mathrm{d} w_{12} \wedge \mathrm{~d} w_{22}+\mathrm{d} w_{13} \wedge \mathrm{~d} w_{21},
\end{aligned}
$$

and $\Omega=v_{12}+w_{21}$. The subspace $\rho(\mathfrak{g}) \Omega=\operatorname{lin}\left(\left\{v_{11}, v_{12}, v_{22}, w_{11}, w_{12}, w_{22}\right\}\right)$ is clearly a Lagrangian with respect to $\omega$. The condition (6.14) is automatically satisfied.

Remark 6.2.3 As has been shown in [43] every finite dimensional classical Airy structure can be quantised. Therefore we have shown existence of two QASs based on the Lie algebra $\mathfrak{s l}(2) \oplus \mathfrak{s l}(2)$.

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[^0]:    ${ }^{1}$ First of these equalities follows from the definition of the intertwining operator, hence first lemma is a check whether chosen $I$ is such an operator. It is not know to the author if the second lemma also follows from general definition of the intertwining operators.

[^1]:    ${ }^{1}$ Notice that comparing to 38 we use a different notation, differences can be summarised as follows: $\mu=-\sqrt{2 \beta}$, $a=N, \lambda=\frac{Q}{\sqrt{2}}, \mathcal{F}^{\eta}=\mathcal{F}\left(\frac{\eta}{\sqrt{2}}\right)$.

[^2]:    ${ }^{1}$ Relevance of the Whitehead lemma in the context of quantum Airy structures was noted by Błażej Ruba.

