University of Warsaw

Faculty of Mathematics, Informatics and Mechanics

Michał Łasica

Parabolic equations with very singular nonlinear diffusion

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Supervisor:
Prof. Piotr Bogusław Mucha
Institute of Applied Mathematics and Mechanics
University of Warsaw

Auxiliary supervisor:
Dr. José Salvador Moll Cebolla
Department of Mathematical Analysis
University of Valencia

Author's declaration:

Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

April 3, 2019 ....
Michał Łasica

Supervisors' declaration:

The dissertation is ready to be reviewed.

April 3, 2019 ....
Prof. Piotr Bogusław Mucha

April 3, 2019 ....
Dr. José Salvador Moll Cebolla

Abstract

This dissertation is a collection of several results in mathematical analysis of solutions to quasilinear parabolic partial differential equations with *very singular diffusion*. By this, we mean that the diffusivity is of order $|\nabla u|^{-1}$, at least near regions where $\nabla u = 0$ (here u is the unknown in the equation). A model example is the total variation gradient flow.

In Chapter 1, we introduce the reader to parabolic equations with very singular diffusion. We present typical features of their solutions on the example of scalar, 1-dimensional total variation flow. Then, we state new results whose demonstrations are contained in the following chapters.

In Chapter 2, we consider the orthotropic total variation flow on a rectangle and prove that the class of functions piecewise constant on grid rectangles (PCR) is preserved by the flow. Consequently, the flow in this case is determined by a finite algorithm. Using this knowledge and density of PCR functions, we show that the flow preserves continuity. This is not necessarily the case if the domain is not convex.

Next, we investigate a model very singular equation with diffusivity equal to $1+\frac{\alpha}{2|u_x|}$, $\alpha > 0$ (in one spatial dimension). We show that the domain can be decomposed into evolving intervals where the solution is constant (facets) and the remaining region, where the solution satisfies the heat equation. We establish some continuity properties of facets.

In Chapter 4, we consider the vector-valued total variation flow on an interval. Given the solution $\boldsymbol{u}$ with initial datum $\boldsymbol{u}_0$ of bounded variation, we show that $|\boldsymbol{u}_x| \leq |\boldsymbol{u}_{0,x}|$ in the sense of measures. This estimate provides a generalization of several results known in the scalar-valued case.

In the last chapter, we establish local well-posedness for *1-harmonic flow*, i. e. the gradient flow of total variation energy of maps into a complete Riemannian manifold, in the class of Lipschitz maps on a convex domain. We assume either that the target manifold is a closed submanifold in $\mathbb{R}^N$, or that it has non-positive sectional curvature. We single out some conditions for global existence of the flow. We show analogous results in the case where the domain is a compact, orientable Riemannian manifold. Finally, we solve the homotopy problem for 1-harmonic maps under some assumptions.

Key words and phrases: very singular parabolic equations, total variation flow, facets, regularity, 1-harmonic map flow, existence, uniqueness, tetris

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Streszczenie

Niniejsza rozprawa stanowi zbiór kilku wyników dotyczących analizy rozwiązań quasiliniowych parabolicznych równań różniczkowych cząstkowych z *bardzo singularną dyfuzją*. To określenie oznacza, że współczynnik dyfuzji jest rzędu $|\nabla u|^{-1}$, przynajmniej w okolicy regionów, gdzie $\nabla u = 0$ (tu u oznacza niewiadomą w równaniu). Typowym przykładem jest potok gradientowy całkowitego wahania.

W Rozdziale 1 zapoznajemy czytelnika z równaniami parabolicznymi z bardzo singularną dyfuzją. Omawiamy typowe własności ich rozwiązań na przykładzie skalarnego, 1-wymiarowego potoku całkowitego wahania. Następnie przedstawiamy nowe wyniki, których dowody znajdują się w kolejnych rozdziałach.

W Rozdziale 2 rozważamy ortotropowy potok całkowitego wahania na prostokącie i dowodzimy, że klasa funkcji kawałkami stałych na kratowych prostokątach (PCR) jest przez niego zachowywana. W konsekwencji, potok jest w tym przypadku wyznaczony przez skończony algorytm. Przy użyciu tego wyniku oraz gęstości funkcji PCR, pokazujemy, że potok zachowuje ciągłość. Gdy dziedzina nie jest wypukła, nie musi tak być.

Następnie badamy modelowe bardzo singularne równanie ze współczynnikiem dyfuzji równym $1 + \frac{\alpha}{2|u_x|}$, $\alpha > 0$ (w jednym wymiarze przestrzennym). Wykazujemy, że dziedzina może być rozłożona na ewoluujące odcinki, na których rozwiązanie jest stałe (fasety), oraz pozostały obszar, gdzie rozwiązanie spełnia równanie ciepła. Dowodzimy pewnych własności regularnościowych faset.

W Rozdziale 4 rozważamy potok całkowitego wahania o wartościach wektorowych na odcinku. Dla rozwiązania $\boldsymbol{u}$ z warunkiem początkowym $\boldsymbol{u}_0$ o wahaniu skończonym wykazujemy, że $|\boldsymbol{u}_x| \leq |\boldsymbol{u}_{0,x}|$ w sensie miar. To oszacowanie uogólnia kilka wyników znanych w przypadku skalarnym.

W ostatnim rozdziale wykazujemy lokalne dobre postawienie dla *potoku 1-harmonicznego*, tj. potoku całkowitego wahania przekształceń w zupełną rozmaitość riemannowską, w klasie przekształceń lipschitzowskich na wypukłej dziedzinie. Zakładamy, że dana rozmaitość jest domkniętą podrozmaitością w przestrzeni euklidesowej, albo że ma niedodatnią krzywiznę przekrojową. Podajemy pewne warunki na globalne istnienie potoku. Pokazujemy analogiczne wyniki w przypadku, gdy dziedzina jest zwartą, orientowalną rozmaitością riemannowską. Na koniec rozwiązujemy problem homotopii dla przekształceń 1-harmonicznych przy pewnych założeniach.

Kluczowe słowa i frazy: bardzo singularne równania paraboliczne, potok całkowitego wahania, fasety, regularność, potok 1-harmoniczny, istnienie, jednoznaczność, tetris

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Chapter 2 is based on paper [56]. The contents of Chapter 4 and section 1.4 are closely related to [35]. Chapter 3 and section 1.3 are planned to appear as a part of a paper, whose previous version is available on arXiv as [55]. Chapter 5, section 1.5 and the appendix virtually coincide with [36]. I would like to thank all the anonymous reviewers, whose work helped improve those papers and, by consequence, this dissertation. I am grateful to Michał Miśkiewicz for reading the manuscript of [56] and sharing his comments.

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Contents

Chapter 1

Introduction

The notoriety of elliptic and parabolic partial differential equations (PDEs) involving the p-Laplace operator

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u),$$

where $p > 1$, in mathematical research of last two decades, has earned it the title of *the mascot of nonlinear analysis* [27]. Considerably less attention was devoted to the boundary case $p = 1$. This is probably due, at least partly, to how distinct this case is. When leaving $p = 2$ for general $p > 1$, one has to abandon the classical, smooth setting and consider weak solutions belonging to Sobolev spaces (which, typically, eventually turn out to have Hölder continuous gradient). The leap from $p > 1$ to $p = 1$ is just as profound. Now, solutions are found in the space of functions of bounded variation, whose derivatives are Radon measures. Furthermore, for $p = 1$, the map $\xi \to |\xi|^{p-2}\xi$ is discontinuous at $\xi = 0$. For these reasons, the notion of solution has to become even more involved.

Investigation of elliptic and parabolic PDEs involving the 1-laplacian

$$\operatorname{div} \tfrac{\nabla u}{|\nabla u|}$$

is interesting not only for the reason of technical subtlety. It appears (up to sign) as the variation of the functional

$$u \mapsto \int |\nabla u|, \tag{1.1}$$

called *total variation.*[1] This functional appears in mathematical mechanics as a term in energy when modeling phase transitions [76] or Bingham fluids [66]. Recently, total variation and its many versions are even more prominent in mathematical imaging science. Minimization problems for energies involving such a term serve as models for solutions to many tasks of image processing, such as denoising [72] or segmentation [16].

Irrespectively of the origin of functional (1.1) (or any of its versions), a natural way of decreasing its value is to follow its steepest descent flow with respect to some notion of distance. A natural choice is the L^2 distance, which gives rise to the *total variation flow*, formally described by a quasilinear parabolic equation

$$u_t = \operatorname{div} \tfrac{\nabla u}{|\nabla u|}. \tag{1.2}$$

[1] In this sentence, the word *variation* appears twice. The first instance refers to variation in the sense of calculus of variations, which can be understood as Fréchet L^2-gradient (or subgradient). The second one is, at least formally, explained by (1.1). Rigorous definitions appear later on.

Before we disclose the statements of this thesis, we would like to acquaint the reader with our setting by recalling some properties of solutions to equation (1.2) in the case of single spatial dimension x, where it assumes the particularly simple form

$$u_t = (\operatorname{sgn} u_x)_x. \tag{1.3}$$

This can be done using only a relatively simple version of technical definitions that will be introduced later on.

1.1 The total variation flow in one dimension

As we are not interested in boundary effects at this point, let us take the 1-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ as the spatial domain for equation (1.3). We can identify $\mathbb{T}$ with the unit interval $[0,1]$ with periodic boundary. We consider the initial problem for (1.3) in $]0,\infty[\times\mathbb{T}$. If the initial datum u_0 is absolutely continuous (i. e. $u_0 \in W^{1,1}(\mathbb{T})$), a unique strong solution u in the class $W^{1,2}_{loc}([0,\infty[; L^2(\mathbb{T})) \cap L^\infty(0,\infty; W^{1,1}(\mathbb{T}))$ can be shown to exist [13], provided that we interpret $\operatorname{sgn} u_x$ as a selection of the multifunction

$$\operatorname{sgn} \circ u_x = \begin{cases} [-1,1] & \text{if } u_x = 0 \\ \left\{\frac{u_x}{|u_x|}\right\} & \text{otherwise.} \end{cases} \tag{1.4}$$

If $u_0 \in L^2(\mathbb{T})$, a kind of solution to the initial problem can still be constructed via monotone operators theory. However, this solution can only be expected to satisfy $u(t,\cdot) \in BV(\mathbb{T})$ for a. e. $t > 0$ instead of $u(t,\cdot) \in W^{1,1}(\mathbb{T})$. In this setting u_x is not a (measurable) function anymore, so (1.4) does not have clear meaning. Let us introduce the proper concept of solution rigorously.

Definition 1.1. Let $u_0 \in L^2(\mathbb{T})$. We say that $u \in C([0,\infty[, L^2(\mathbb{T}))$ satisfying

$$u \in W^{1,2}_{loc}(]0,\infty[, L^2(\mathbb{T})) \cap L^1_{w,loc}(]0,\infty[, BV(\mathbb{T}))$$

is a (strong) solution to the initial problem for (1.3) in $]0,\infty[\times\mathbb{T}$ with datum u_0 if there exists a function $z \in L^2_{loc}(]0,\infty[, W^{1,2}(\mathbb{T}))$ such that for a. e. $t > 0$

$$|z(t,\cdot)| \le 1 \text{ in } \mathbb{T}, \quad z(t,\cdot)\cdot u_x(t,\cdot) = |u_x(t,\cdot)| \text{ as measures on } \mathbb{T}, \tag{1.5}$$

$$u_t(t,\cdot) = z_x(t,\cdot) \text{ in } \mathbb{T}, \tag{1.6}$$

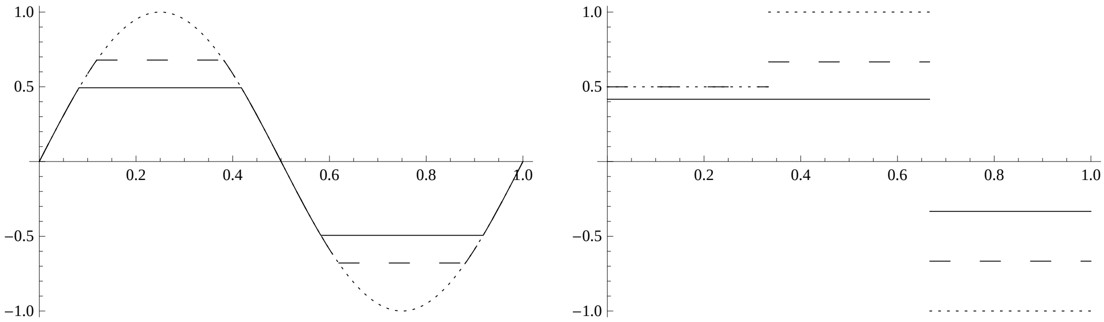
$$u(0,\cdot) = u_0. \tag{1.7}$$

The two conditions (1.5) assume now the role of (1.4). They do indeed make sense for a. e. $t > 0$, as $W^{1,2}(\mathbb{T}) \subset C^{\frac{1}{2}}(\mathbb{T})$ by Morrey's inequality.

Proposition 1.1. *Given any $u_0 \in L^2(\mathbb{T})$, there exists a unique strong solution to the initial problem for* (1.3) *in* $]0,\infty[\times\mathbb{T}$.

This is can be deduced from variational semigroup theory just as Theorem 2.2.

Let us now consider two particular examples of evolution.



(a) Example 1. (b) Example 2.

Figure 1.1: Two cases of evolution described by Examples 1 and 2. Dotted lines represent u_0, dashed lines: $u(1/18, \cdot)$, solid lines: $u(1/9, \cdot)$.

Example 1.1. Let u_0 be a continuous function that changes monotonicity exactly once. This means that there are points $a^0, b^0, a^1, b^1 \in \mathbb{T}$ such that $a^0 < b^0 \le a^1 < b^1 \le a^0$ with respect to the cyclic order on $\mathbb{T}$ and

- u_0 is non-decreasing in $[a^0, a^1]$ and non-increasing in $[a^1, a^0]$,
- u_0 is constant in $[b^0, a^1]$ and $[b^1, a^0]$, and these are maximal (possibly degenerate) intervals with this property.

Points a^0, b^0, a^1, b^1 are uniquely defined. Let us introduce the notation

$$F_0^+ = [b^0, a^1], \quad F_0^- = [b^1, a^0], \quad u_0^\pm = u(F_0^\pm)$$

for the (possibly improper) extrema of u_0. Consider a pair of initial value problems for functions $u^\pm \colon [0, \infty[\to \mathbb{R}$:

$$u_t^+ = -\frac{2}{|F^+|}, \quad \text{where } F^+ = F^+(u^+(t)) = u_0^{-1}\left([u^+(t), +\infty[\right) \quad \text{in }]0, +\infty[, \tag{1.8}$$

$$u^+(0) = u_0^+ \tag{1.9}$$

and

$$u_t^- = \frac{2}{|F^-|}, \quad \text{where } F^- = F^-(u^-(t)) = u_0^{-1}\left(]-\infty, u^-(t)]\right) \quad \text{in }]0, +\infty[, \tag{1.10}$$

$$u^-(0) = u_0^- \tag{1.11}$$

These problems are uniquely soluble in $C([0, +\infty[) \cap C^1(]0, +\infty[)$. The solution u^+ is decreasing with speed not lower than 2, while u^- is increasing with speed not lower than 2. Hence, there is exactly one $t_* > 0$ such that $u^+(t_*) = u^-(t_*)$. For $t \in [0, t_*[$, let us denote

$$I^+(t) = \{x \in \mathbb{T} \colon F^-(t) < x < F^+(t)\}, \quad I^-(t) = \{x \in \mathbb{T} \colon F^+(t) < x < F^-(t)\}$$

and define $u(t, \cdot)$ by

$$u(t,x) = \begin{cases} u^\pm(t) & \text{if } x \in F^\pm(t) \\ u_0(x) & \text{if } x \in I^\pm(t). \end{cases} \tag{1.12}$$

For $t \geq t_*$, we define $u(t,\cdot)$ by

$$u(t,\cdot) \equiv u^+(t_*) \equiv u^-(t_*). \tag{1.13}$$

With such definition,

$$u \in C([0,\infty[, BV(\mathbb{T})) \cap C([0,\infty[\times\mathbb{T}) = C([0,\infty[, W^{1,1}(\mathbb{T}))$$

and

$$\int_{\mathbb{T}} |u_x(t,\cdot)| = 2(u^+(t) - u^-(t))$$

for $t \geq 0$. Next, for $t \in]0, t_*[$ we define $z(t,\cdot)$ as the piecewise affine, continuous function such that

$$z(t,\cdot)\Big|_{I^\pm(t)} \equiv \pm 1, \qquad z(t,\cdot)\Big|_{F^\pm(t)} \text{ is affine.} \tag{1.14}$$

For $t \geq t_*$, we put

$$z(t,\cdot) \equiv 0. \tag{1.15}$$

By (1.14, 1.15), $z(t,\cdot)$ is a selection of $\operatorname{sgn} u_x(t,\cdot)$ for $t \geq 0$, so it satisfies (1.5). Furthermore, by (1.14, 1.12, 1.8, 1.10),

$$z_x(t,\cdot) = -\frac{2}{|F^+(t)|}\mathbf{1}_{F^+(t)} + \frac{2}{|F^-(t)|}\mathbf{1}_{F^-(t)} = u_t(t,\cdot)$$

for $t \in]0, t_*[$, and by (1.15, 1.13)

$$z_x(t,\cdot) \equiv 0 \equiv u_t(t,\cdot)$$

for $t > t_*$. Thus, we have checked that u is a weak solution to the initial problem for (1.6) with datum u_0.

Example 1.2. For $n \geq 2$, let $a^0, a^1, \ldots, a^{n-1} \in \mathbb{T}$ satisfy $a^0 < a^1 < \ldots < a^{n-1} < a^0$. For $k \in \mathbb{Z}_n$, denote $F^k = [a^k, a^{k+1}]$. The family $\{F^k \colon k \in \mathbb{Z}_n\}$ forms a decomposition of $\mathbb{T}$ into non-degenerate closed intervals with pairwise disjoint interiors, ordered so that $F^0 \leq F^1 \leq \ldots \leq F^{n-1} \leq F^0$. Take $u_0 \in BV(\mathbb{T})$ such that

$$u_0 = \sum_{k=0}^{n-1} u_0^k\, \mathbf{1}_{F^k}$$

with $u_0^k \neq u_0^{k+1}$ for $k \in \mathbb{Z}_n$. For $t \geq 0$, define $z(t,\cdot)$ as the continuous function on $\mathbb{T}$ such that for $k \in \mathbb{Z}_n$

$$z(t, a^k) = \operatorname{sgn}(u_0^k - u_0^{k-1}), \qquad z(t,\cdot)\,|_{F^k} \text{ is affine.} \tag{1.16}$$

It follows that, for $t \geq 0$,

$$z_x(t,\cdot) = \sum_{k=0}^{n-1} \frac{\operatorname{sgn}(u_0^{k+1} - u_0^k) - \operatorname{sgn}(u_0^k - u_0^{k-1})}{|F^k|}\, \mathbf{1}_{F^k}. \tag{1.17}$$

Now let us define, for $t \geq 0$,

$$u(t,\cdot) = \sum_{k=0}^{n-1} u^k(t)\mathbf{1}_{F^k}, \qquad u^k(t) = u_0^k + \frac{\operatorname{sgn}(u_0^{k+1} - u_0^k) - \operatorname{sgn}(u_0^k - u_0^{k-1})}{|F^k|}\, t. \tag{1.18}$$

By (1.17, 1.18), clearly (1.6) holds in $]0,+\infty[\times\mathbb{T}$. By definition of z, we have $|z| \leq 1$ in $]0,+\infty[\times\mathbb{T}$. However, the second condition in (1.5) takes form

$$\sum_{k=0}^{n-1} \operatorname{sgn}(u_0^{k+1} - u_0^k)\,(u^{k+1} - u^k)\delta_{a^{k+1}} = \sum_{k=0}^{n-1} |u^{k+1} - u^k|\delta_{a^{k+1}}$$

which holds only as long as

$$\operatorname{sgn}(u^{k+1}(t) - u^k(t)) = \operatorname{sgn}(u_0^{k+1} - u_0^k),$$

i. e. for $t \in [0, t_1[$, where t_1 is the first time instance $t > 0$ such that $u^k(t) = u^{k+1}(t)$ for a $k \in \mathbb{Z}_n$. Thus, u is a solution to the initial problem for (1.6) in $]0, t_1[\times\mathbb{T}$ with datum u_0. At time t_1, either $u(t_1\cdot)$ is constant and evolution stops, or a new decomposition of $\mathbb{T}$ corresponding to $u(t_1, \cdot)$ can be introduced, and the solution can be continued to $[t_1, t_2[$, $t_2 > t_1$, by the same scheme. After at most $n - 1$ such steps, the evolution reaches steady state (a constant function).

Based on these examples, let us discuss typical features of solutions to (1.3), which are also present in higher dimensions.

Facets. In both Examples, in the graphs of solutions $u(t, \cdot)$, $t > 0$ appear flat parts, *facets*. These facets can propagate from the initial datum u_0, as in Example 2, or arise out of local extrema of u_0 in the course of evolution, as in Example 1. In any case, locally (until the moment any pair of them merges) they are well-defined functions of time.

Nonlocality. In the faceted regions (where $u_x(t, \cdot) = 0$), the evolution is nonlocal, i. e. the value of $u_t(t, x)$ for x in such a region depends on the behavior of $u(t, \cdot)$ beyond a small interval centered at x. Indeed, $u_t(t, x)$ depends on the length of the whole facet (1.8, 1.10, 1.18).

Limited regularity. As seen in Example 1, the highest level of regularity on Sobolev/Hölder scale that is propagated by the evolution is $W^{1,\infty}/C^{0,1}$. Indeed, even if (non-constant) u_0 belongs to $C^\infty(\mathbb{T})$, there necessarily exist time instances $t > 0$ such that $u_x(t, \cdot)$ has a jump at $\partial F^\pm(t)$.

(Almost) no regularization. The piecewise constant datum from Example 2 stays piecewise constant throughout the evolution. For $t \in]0, t_1[$ it is not constant. Hence, there is no instantaneous regularization beyond $BV(\mathbb{T})$.

We are now ready to introduce the new results contained in this thesis.

1.2 The orthotropic total variation flow in the plane

As we have seen in Example 2, the class of piecewise constant functions is preserved by the total variation flow in 1D. This has been noticed already in [52]. In fact, more is true: if

$$u_0 = \sum_{k=0}^{n-1} u_0^k\,\mathbf{1}_{F^k}$$

with decomposition $\{F^k\colon k \in \mathbb{Z}_n\}$ of $\mathbb{T}$ as in Example 2, then — whether or not the condition $u_0^k \neq u_0^{k+1}$ for $k \in \mathbb{Z}_n$ holds — we have

$$u(t, \cdot) = \sum_{k=0}^{n-1} u^k(t)\,\mathbf{1}_{F^k}$$

for all $t > 0$, where u^k are continuous, piecewise linear functions that can be found explicitly using the algorithm from Example 2. Qualitatively speaking, facets can *merge*, but they cannot *bend* or *break*. In other words, if $u_{0,x} = 0$ in an open interval U, then $u_x(t, \cdot) = 0$ in U for all $t > 0$.

Already in two dimensions, the situation is more complicated and strongly depends on which generalization of (1.3) is chosen. There are at least two natural candidates. Besides the *isotropic* total variation flow (1.2) (with $|\cdot|$ being the Euclidean norm), which arises as the steepest descent flow of (1.1), one can also consider the *orthotropic* total variation flow given by

$$u_t = (\operatorname{sgn} u_{x_1})_{x_1} + (\operatorname{sgn} u_{x_2})_{x_2}, \tag{1.19}$$

which corresponds to the orthotropic total variation functional

$$\mathrm{TV}_1(u) = \int |u_{x_1}| + |u_{x_2}|. \tag{1.20}$$

Qualitative properties of solutions were investigated first, and most thoroughly, in the isotropic case [9, 10, 3]. In order to avoid contribution of geometry of the domain, which is already present even for the flat two-dimensional torus $\mathbb{T}^2$, the authors of these works considered (1.2) in the plane $\mathbb{R}^2$. In [9], the authors investigate necessary and sufficient conditions for a bounded set of finite perimeter C such that

$$\text{if } u_0 = \mathbf{1}_C, \text{ then } u(t, \cdot) = \widehat{u}(t)\mathbf{1}_C. \tag{1.21}$$

For connected C, they prove that (1.21) holds iff C is convex, ∂C is of class $C^{1,1}$ and

$$\operatorname*{ess\,sup}_{x \in \partial C} \kappa_{\partial C} \leq \frac{P(C)}{\mathcal{L}^2(C)}. \tag{1.22}$$

In (1.22), $\kappa_{\partial C}$ is the curvature of ∂C. The quantity

$$\frac{P(C)}{\mathcal{L}^2(C)} \tag{1.23}$$

is known as the *Cheeger quotient* of C. For any bounded C of finite perimeter, if (1.21) holds, then

$$\frac{P(C)}{\mathcal{L}^2(C)} \leq \frac{P(D)}{\mathcal{L}^2(D)} \tag{1.24}$$

for all $D \subset C$ and

$$\widehat{u}(t) = \left(1 - t\frac{P(C)}{\mathcal{L}^2(C)}\right)_+. \tag{1.25}$$

In [10], a class of solutions to (1.2) of form

$$u(t, \cdot) = \sum_{k=0}^{n-1} u^i(t)\mathbf{1}_{C^i} \tag{1.26}$$

is constructed. The convex sets C^i need to satisfy either $C^i \subset C^j$, $C^i \subset C^j$ or $C^i \cap C^j = \emptyset$ for $i, j = 0, \ldots, n-1$ together with certain bounds on curvature and relative position (essentially, their boundaries cannot be to close to each other). In [3], the evolution of the characteristic function of any bounded convex set E in the plane is described. In particular, if (1.22) is not satisfied, then there is a proper subset C of E satisfying

$$u(t, x) = c(t)$$

for $x \in C$, $t > 0$. It can be shown [50] that the maximal such C satisfies (1.24) for all $D \subset E$ and $c(t) = \left(1 - t\frac{P(C)}{\mathcal{L}^2(C)}\right)_+$. In $E \backslash C$, u decreases faster — *bending* occurs. The result of [3] was generalized in [19] to gradient flow of anisotropic total variation TV_φ, $\mathrm{TV}_\varphi(u) = \int |\nabla u|_\varphi$, where $|\cdot|_\varphi$ is any norm on $\mathbb{R}^2$. In particular, they characterize *convex* sets C such that (1.21) holds. If $u_0 = \mathbf{1}_C$ for a convex C, *breaking* never occurs, i. e. the jump set of $u(t, \cdot)$ is contained in the jump set of u_0. In the isotropic case, this is a special case of a more general fact: breaking does not occur for any initial datum $u_0 \in BV(\mathbb{R}^2)$ [20].

This is no longer the case for (1.19), as evidenced by [65, Example 4]. An offending datum can be constructed as follows. Let $A = [0, 1]^2$ and let $B_a = [0, a] \times [-1, 0]$ for $a \in]0, 1]$. Let $u_0^a = \mathbf{1}_{A \cup B_a}$. Then, whenever $a \in]\frac{1}{2}, 1]$, the solution u^a to (1.19) in $\mathbb{R}^2$ is given by

$$u^a(t, \cdot) = (1 - 4t)_+ \mathbf{1}_A + \left(1 - \frac{2}{a}t\right)_+ \mathbf{1}_{B_a}$$

for $t > 0$. In particular, for $t \in]0, a/2[$, $u^a(t, \cdot)$ has a jump across $[0, a] \times \{0\}$, even though this segment does not belong to the jump set of u_0^a. On the other hand, if $a \in]0, 1/2]$, we have

$$u^a(t, \cdot) = \left(1 - \frac{6}{1 + a}t\right)_+ \mathbf{1}_{A \cup B_a}.$$

Thus, there are non-convex C for which (1.21) holds.

Note, that whether $a > \frac{1}{2}$ or not, $u^a(t, \cdot)$ stays piecewise constant for $t > 0$. Here we generalize this observation. Instead of $\mathbb{R}^2$, we choose as our spatial domain a rectangle $\Omega = [0, a] \times [0, b]$ with Neumann boundary conditions. This is the setting that naturally appears in imaging, it is also consistent with the Cartesian symmetry of (1.19). Now, let $a^0, a^1, \ldots, a^m, b^0, b^1, \ldots, b^n \in \mathbb{R}$ with $0 = a^0 < a^1 < \ldots < a^m = a$, $0 = b^0 < b^1 < \ldots < b^n = b$ and denote

$$F^{ij} = [a^i, a^{i+1}] \times [b^j, b^{j+1}]$$

for $i = 0, \ldots, m-1$ and $j = 0, \ldots, n-1$. Given $u_0^{ij} \in \mathbb{R}$, $i = 0, \ldots, m-1$, $j = 0, \ldots n-1$ let

$$u_0 = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} u_0^{ij} \mathbf{1}_{F^{ij}}, \tag{1.27}$$

and denote by u the solution to (1.19) in $]0, \infty[\times \Omega$ with Neumann boundary conditions and initial datum u_0. We will address the questions of rigorous definition of solution and its existence later on in the Preliminaries.

In Chapter 2, we prove the following

Theorem 1.1. *There exist continuous functions* $u^{ij} \colon [0, \infty[\to \mathbb{R}$ *such that for* $t \geq 0$

$$u(t, \cdot) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} u^{ij}(t) \mathbf{1}_{F^{ij}}.$$

The functions u^{ij} *are finitely piecewise affine, i. e. there exist time instances* $0 = t_0 < t_1 < \ldots < t_k$ *such that* u^{ij} *is affine in* $[t_l, t_{l+1}]$ *for* $l = 0, \ldots, k-1$ *and constant in* $[t_k, \infty[$, $i = 0, \ldots, m-1$, $j = 0, \ldots, n-1$.

The proof of Theorem 1.1 is based on analysis of a more involved, anisotropic version of Cheeger quotient (1.23). In section 2.2, we prove that the quotient is minimized by a rectilinear polygon. An important point in the proof is that, due to the structure of the Cheeger quotient, we are able to construct approximate minimizers that belong to a finite class of rectilinear polygons determined by u_0. As the set of all rectilinear polygons with bounded Cheeger quotient is not closed, this finiteness is essential. In section 2.3 we then use this result to prove Theorem 1.1 by constructing level sets of $u(t, \cdot)$ for $t > 0$. In section 2.4 we then transfer Theorem 1.1 to the case $\Omega = \mathbb{R}^2$.

As $(\operatorname{sgn} u_{x_1})_{x_1} + (\operatorname{sgn} u_{x_2})_{x_2}$ is a monotone operator, for any datum $u_0 \in L^2(\Omega)$ and a sequence $u_{0,n} \in L^2(\Omega)$, $n = 1, 2, \ldots$ such that $u_{0,n} \to u_0$ in $L^2(\Omega)$, solutions $u_n(t, \cdot)$ with data $u_{0,n}$ converge in $L^2(\Omega)$ to the solution $u(t, \cdot)$ with datum u_0 for $t > 0$. It is easy to check that the set of functions of form (1.27) is dense in $L^2(\Omega)$. In fact, it is even strictly dense in $BV(\Omega)$ (in the sense of seminorm $\int_\Omega |\nabla u|_1$), see [18, Theorem 3.4]). Therefore, we do not only construct explicit solution to the Neumann problem for (1.19) when the initial datum is of form (1.27), but we provide a natural approximation to the solution with any initial datum. This is a considerably stronger result than what could be obtained in the isotropic case. In section 2.5 we use it to prove that continuity is preserved by the orthotropic total variation flow on a rectangle.

Theorem 1.2. *Let* Ω *be a rectangle and let* u *be the solution to 1.19 in* $]0, \infty[\times\Omega$ *with Neumann boundary condition and initial datum* $u_0 \in C(\Omega)$. *Then* $u(t, \cdot) \in C(\Omega)$ *in every* $t > 0$. *In fact, if* $\omega_1, \omega_2 \colon [0, \infty[\to [0, \infty[$ *are continuous functions such that*

$$|u_0(x_1, x_2) - u_0(y_1, y_2)| \le \omega_1(|x_1 - y_1|) + \omega_2(|x_2 - y_2|)$$

for each (x_1, x_2), (y_1, y_2) *in* Ω *then we have*

$$|u(t, (x_1, x_2)) - u(t, (y_1, y_2))| \le \omega_1(|x_1 - y_1|) + \omega_2(|x_2 - y_2|)$$

for each $t > 0$, (x_1, x_2), (y_1, y_2) *in* Ω.

Note that if ω is a concave modulus of continuity for u_0 with respect to norm $|\cdot|_1$, then ω_1, ω_2 defined by $\omega_1 = \omega_2 = \omega$ satisfy the assumptions of Theorem 1.2. On the other hand, given ω_1, ω_2 as in the Theorem, $\omega' = \omega_1 + \omega_2$ is a modulus of continuity for u_0 (as well as u). Theorem 1.2 implies for instance that if L is the Lipschitz constant for u_0 with respect to norm $|\cdot|_1$, then the Lipschitz constant of u with respect to norm $|\cdot|_1$ is not greater than L.

In the final section of Chapter 2, we provide several examples illustrating our results. In particular, we show an example of a smooth initial datum on a non-convex rectilinear polygon such that $u(t, \cdot)$ has a jump discontinuity for small enough $t > 0$.

We note that analogous results can be obtained for minimizers of functional

$$u \mapsto \mathrm{TV}_1(u) + \frac{1}{2\lambda} \int (u - u_0)^2 \tag{1.28}$$

with given $u_0 \in L^2(\Omega)$, $\lambda > 0$. Minimization of (1.28), rather than following the flow given by (1.19), is the more common way of decreasing TV_1 in applications to image processing (see e. g. [29, 22]). The paper [56] contains both the results of Chapter 2 and their analogues for minimizers of (1.28). Here we focus on the mathematical side of our results and refer for the discussion of significance in applications to [56].

1.3 Facets in a uniformly parabolic equation

In Chapter 3, we consider the equation

$$u_t = u_{xx} + \tfrac{\alpha}{2}(\operatorname{sgn} u_x)_x \quad \text{on }]0,\infty[\times\mathbb{T}. \tag{1.29}$$

Here, $\mathbb{T}$ is the standard one-dimensional torus, which we identify with the unit interval $[0,1]$ with periodic boundary conditions. Well-posedness of initial value problem for (1.29) may be obtained by viewing it as a parabolic inclusion

$$u_t(t,\cdot) \in \mathcal{L}u(t,\cdot) \quad \text{for a. e. } t > 0, \tag{1.30}$$

where $\mathcal{L} = -\partial\mathcal{J}$ and $\mathcal{J}$ is a functional defined on $L^2(\mathbb{T})$ by

$$\mathcal{J}(u) = \begin{cases} \frac{1}{2}\int_{\mathbb{T}} u_x^2 + \alpha|u_x| & \text{if } u \in H^1(\mathbb{T}), \\ +\infty & \text{otherwise.} \end{cases} \tag{1.31}$$

Clearly, $D(\mathcal{J}) = H^1(\mathbb{T})$ and $\mathcal{J}$ is equivalent to the standard seminorm on $H^1(\mathbb{T})$. Furthermore, $\mathcal{J}$ is convex and lower semicontinuous.

Equipped with above observations concerning $\mathcal{J}$, we can use monotone operators theory to obtain basic existence and regularity result for the inclusion (1.30) [8, Chapter IV, Theorems 2.1 and 2.2].

Proposition 1.2. *Let $u_0 \in L^2(\mathbb{T})$. The problem* (1.30) *with initial condition u_0 has a unique solution*

$$u \in C([0,\infty[; L^2(\mathbb{T})) \cap L^2_{loc}([0,\infty[; H^1(\mathbb{T}))$$

which satisfies

$$u(t,\cdot) \in D(\mathcal{L}) \text{ for all } t \in]0,\infty[.$$

Moreover, we have

$$\frac{\mathrm{d}^+}{\mathrm{d}t}u = \mathcal{L}^0 u \text{ for all } t \in]0,\infty[.$$

Here, $\frac{\mathrm{d}^+}{\mathrm{d}t}u$ denotes the right-sided time derivative of u and $\mathcal{L}^0$ is the minimal selection of $\mathcal{L}$, i. e. for $u \in D(\mathcal{L})$, $\mathcal{L}^0 u$ is the (uniquely defined) element of $\mathcal{L}u$ of minimal norm in $L^2(\mathbb{T})$.

It turns out, that similarly to the case of the total variation flow, flat facets appear in graphs of $u(t,\cdot)$, $t > 0$. However, now their evolution is non-trivial and there is a sort of competition between the faceting effect of the 1-Laplace operator and smoothing due to presence of the laplacian [67]. In Chapter 3, we argue that these facets are well defined functions of time and satisfy certain continuity properties.

We recall that there is a natural cyclic order on $\mathbb{T}$ which is a ternary relation. We will use the notation $a < b < c$ ($a, b, c \in \mathbb{T}$) to represent it. Just as the usual order on $\mathbb{R}$, it can be used to define intervals, e. g.

$$]a,b[= \{x \in \mathbb{T} \colon a < x < b\}$$

for $a, b \in \mathbb{T}$. The cyclic order naturally extends to subsets of $\mathbb{T}$. For $A, B, C \subset \mathbb{T}$, we write

$$A < B < C \text{ if } a < b < c \text{ for all } a \in A, b \in B, c \in C.$$

It is natural to index ordered families of n points or subsets of $\mathbb{T}$ by elements of the cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ consisting of integers $0, \dots n-1$ with addition modulo n.

Let us denote by $\mathfrak{F}(\mathbb{T})$ the set of (non-empty) closed intervals in $\mathbb{T}$. Supplied with Hausdorff distance, this forms a complete metric space. We also introduce complementary notation $\mathfrak{I}(\mathbb{T})$ for the set of (non-empty) open intervals in $\mathbb{T}$. In section 3.2 we prove

Theorem 1.3. *Let $u_0 \in D(\mathcal{L})$ be non-constant and satisfy $\mathcal{L}^0 u_0 \in L^\infty(\mathbb{T})$. There exists an integer m_0 with $2m_0 \le \alpha^{-2} \|\mathcal{L}^0 u\|^2_{L^2(\mathbb{T})}$ and a sequence of time instances*

$$0 = t_0 < t_1 \le t_2 \le \ldots \le t_{m_0}$$

such that for $m = 0, \dots, m_0 - 1$ and $k \in \mathbb{Z}_{2(m_0-m)}$ there exist

$$I^k_m \colon [t_m, t_{m+1}[\to \mathfrak{I}(\mathbb{T}), \quad F^k_m \colon [t_m, t_{m+1}[\to \mathfrak{F}(\mathbb{T})$$

satisfying for $t \in [t_m, t_{m+1}[$:

- $\left\{ I^k_m(t), F^k_m(t) \colon k \in \mathbb{Z}_{2(m_0-m)} \right\}$ *is a disjoint decomposition of $\mathbb{T}$ and*

  $$I^k_m(t) < F^k_m(t) < I^{k+1}_m$$

  for $k \in \mathbb{Z}_{2(m_0-m)}$;
- *$u_x = 0$ in $F^k_m(t)$ for $k \in \mathbb{Z}_{2(m_0-m)}$, if $t \neq 0$ then $u_x > 0$ in $I^k_m(t)$ for k even and $u_x < 0$ for k odd;*
- $|F^k_m(t)| \ge \alpha^2 \|\mathcal{L}^0 u_0\|^{-2}_{L^2(\mathbb{T})}$*;*
- *there holds*

  $$F^k_m(t) = \limsup_{s \to t^-} F^k_m(s) \text{ (for } t \neq t_m) \quad \text{and} \quad F^k_m(t) = \lim_{s \to t^+} F^k_m(s)$$

  in the sense of Kuratowski convergence (equivalently, F^k_m is left upper semicontinuous and right continuous with respect to Hausdorff metric);

Furthermore, for each $m = 1, \dots, m_0$ there exists $k \in \mathbb{Z}_{2(m_0-m)}$ such that

$$\limsup_{t \to t_m^-} F^k_m(t) \cap \limsup_{t \to t_m^-} F^{k+1}_m(t) \neq \emptyset \tag{1.32}$$

and we have

$$u(t, \cdot) \equiv \fint_{\mathbb{T}} u_0 \quad \text{for } t \ge t_{m_0}, \qquad t_{m_0} \le (\alpha\pi)^{-1} \left\| u_0 - \fint_{\mathbb{T}} u_0 \right\|_{L^2(\mathbb{T})}. \tag{1.33}$$

An important ingredient in the proof of Theorem 1.3 is regularity provided by Lemma 3.2 in section 3.2. It also follows from Lemma 3.2 that $\mathcal{L}^0 u \in L^\infty(\mathbb{T})$ in $]0, \infty[$ for the solution u to (1.30) starting from $u_0 \in L^2(\mathbb{T})$. Therefore, Theorem 1.3 in fact describes the generic behavior of the solution to (1.30) starting with any $u_0 \in L^2(\mathbb{T})$.

1.4 A local estimate for the total variation flow of curves

Let $I =]a, b[$ be an open interval. Given $\boldsymbol{u}_0$ in $BV(I, \mathbb{R}^n)$, let $\boldsymbol{u}(t, \cdot) = S(t)\boldsymbol{u}_0$ for $t \geq 0$, where S is the gradient flow of the convex, lower semicontinuous total variation functional $\mathrm{TV}_I^n \colon L^2(I, \mathbb{R}^n) \to [0, +\infty]$,

$$\mathrm{TV}_I^n(\boldsymbol{u}) = \sup\left\{\int_I \boldsymbol{u} \cdot \boldsymbol{\varphi}_x \colon \boldsymbol{\varphi} \in C_c^1(I, \mathbb{R}^n), |\boldsymbol{\varphi}| \leq 1\right\} = \begin{cases} |\boldsymbol{u}_x|(I) & \text{if } \boldsymbol{u} \in BV(I, \mathbb{R}^n), \\ +\infty & \text{otherwise.} \end{cases}$$

Here and in the following, if $\boldsymbol{v}$ is a vector in a Euclidean space, $|\boldsymbol{v}|$ denotes its Euclidean norm, while if $\boldsymbol{w}$ is a vector measure on I, $|\boldsymbol{w}|$ denotes its variation with respect to the Euclidean norm. As in [6, section 2.2], we deduce that $\boldsymbol{u}$ is the unique element of $C([0, \infty[; L^2(I, \mathbb{R}^n)) \cap L^1_{w,loc}(]0, \infty[; BV(I, \mathbb{R}^n))$ such that there exists

$$\boldsymbol{z} \in L^\infty(]0, \infty[\times I, \mathbb{R}^n) \text{ with } \boldsymbol{z}_x \in L^2(]0, \infty[\times I, \mathbb{R}^n)$$

satisfying

$$\boldsymbol{u}_t = \boldsymbol{z}_x \quad \text{a. e. in }]0, \infty[\times I, \tag{1.34}$$

$$|\boldsymbol{z}| \leq 1 \quad \text{a. e. in }]0, \infty[\times I, \tag{1.35}$$

$$\boldsymbol{z}(t, \cdot) \cdot \boldsymbol{u}_x(t, \cdot) = |\boldsymbol{u}_x(t, \cdot)| \quad \text{for a. e. } t > 0 \text{ in the sense of measures on } I, \tag{1.36}$$

$$\boldsymbol{z}(t, \cdot) = 0 \quad \text{on } \partial I \text{ for a. e. } t > 0, \tag{1.37}$$

$$\boldsymbol{u}(0, \cdot) = \boldsymbol{u}_0. \tag{1.38}$$

Note that the product $\boldsymbol{z}(t, \cdot) \cdot \boldsymbol{u}_x(t, \cdot)$ is indeed well defined for a. e. $t > 0$ as $\boldsymbol{z}(t, \cdot)$ belongs to $C(I, \mathbb{R}^n)$ for a. e. $t > 0$.

It is an essential property of $\boldsymbol{u}$ as a steepest descent curve of TV_I^n, that

$$|\boldsymbol{u}_x(t, \cdot)|(I) \leq |\boldsymbol{u}_{0,x}|(I)$$

for $t > 0$. It turns out that this estimate can be localized in an unusually robust way.

Theorem 1.4. *For a. e. $t > 0$ there holds $|\boldsymbol{u}_x(t, \cdot)| \leq |\boldsymbol{u}_{0,x}|$ in the sense of Borel measures on I, i. e.*

$$|\boldsymbol{u}_x(t, \cdot)|(A) \leq |\boldsymbol{u}_{0,x}|(A) \tag{1.39}$$

for any Borel $A \subset I$.

It is an immediate consequence of Theorem 1.4 that if $\boldsymbol{u}_0$ belongs to any subspace of $BV(I, \mathbb{R}^n)$ defined in terms of a bound on $|\boldsymbol{u}_{0,x}|$, such as $W^{1,p}(I, \mathbb{R}^n)$, $p \in [1, \infty]$ or $SBV(I, \mathbb{R}^n)$, then $\boldsymbol{u}(t, \cdot)$ belongs to the same subspace for a. e. $t > 0$.

Inequality (1.39) is an example of a completely local estimate, in the sense that a local quantity at a time instance $t > 0$ is estimated by precisely the same quantity at a previous time instance. Several results like that are already known for gradient flows functionals similar to TV_I^n. In [13], the authors consider scalar total variation flow on an interval I. By analyzing the evolution of step functions (which is a class preserved by the flow) and using L^q contractivity of the flow, they prove that if u is the solution starting with initial datum $u_0 \in BV(I)$, the size of jumps of $u(t, \cdot)$ is not bigger than the size of corresponding jumps of u_0 and that

$$\operatorname{osc}_J u(t, \cdot) \leq \operatorname{osc}_J u_0, \tag{1.40}$$

where $J \subset I$ is any open interval where u_0 is continuous and $\operatorname{osc}_J v = \sup_J v - \inf_J v$ for $v \in C(J)$. The authors remark that this implies preservation of $W^{1,1}(I)$ and $C^{0,\alpha}(I)$, $\alpha \in]0,1]$ regularity by the flow. Preservation of $W^{1,1}(I)$ regularity has recently been obtained also for gradient flows of more general functionals of linear growth on an interval [68].

On the other hand, in [15] the authors consider the gradient flow of a functional $\boldsymbol{u} \mapsto \int_\Omega |\operatorname{div} \boldsymbol{u}|$, where Ω is a bounded domain in $\mathbb{R}^m$. For a solution $\boldsymbol{u}$ starting with $\boldsymbol{u}_0 \in L^2(\Omega, \mathbb{R}^m)$ such that $\operatorname{div} \boldsymbol{u}_0$ is a Radon measure on Ω, they prove that

$$(\operatorname{div} \boldsymbol{u}(t, \cdot))_\pm \leq (\operatorname{div} \boldsymbol{u}_0)_\pm$$

in the sense of measures for $t > 0$. This coincides with our result in the case $m = n = 1$. Their technique is based on considering the dual problem to the variational semi-discretization of the flow and involves a comparison principle. In the essential lemma, they show certain monotonicity property of level sets of the solution to the dual problem with respect to the parameter of discretization. This does not seem to be readily adaptable to the case where the divergence is a vectorial quantity (which would cover our result for $m = 1$). On the other hand, our technique could be directly transferred to the case of gradient flow of $\int_\Omega |\operatorname{div} \boldsymbol{u}|$. We did not explicitly include this case in order not to obfuscate this Chapter, as it is beside our primary interest.

We note that if $n \geq 2$, the evolution is less obvious than in case $n = 1$. Indeed, as seen in Examples 1.1 and 1.2, in the latter case, it can be made completely explicit for a large class of initial data. This is because if $n = 1$, non-trivial evolution only occurs at local extrema, which is of course no longer true if $n \geq 2$.

At this point, let us mention that our technique allows also to prove analogous version of Theorem 1.4 in the setting where the target of $\boldsymbol{u}$ is a Riemannian submanifold in $\mathbb{R}^n$. In fact, this provides a helpful a priori estimate for the proof of existence of such constrained total variation flow (so-called 1-*harmonic flow*) [37]. Theorem 1.4 can also be generalized to different (e. g. Dirichlet) boundary conditions.

We also mention that in the case of isotropic scalar total variation flow on an m-dimensional domain Ω, a completely local estimate of type (1.39) is available [20] for the jump part of the gradient of the solution u starting with $u_0 \in BV(\Omega) \cap L^\infty(\Omega)$:

$$|\nabla u(t, \cdot)| \llcorner J_u \leq |\nabla u_0| \llcorner J_{u_0}. \tag{1.41}$$

It is also well known that such an estimate does not hold for the absolutely continuous part of $|\nabla u|$. For a counterexample, one can take as u_0 the characteristic function of a *non-calibrable* set, such as a square in the plane [3]. To our knowledge, it is an open question whether an estimate analogous to (1.39) holds for the remaining Cantor part.

On the other hand, as we have already mentioned in section 1.2, for the orthotropic total variation flow, the jump set of solution may expand compared to the initial datum. In fact jumps can appear even if the initial datum has no jumps (Example 2.3).

Finally, we note that a result analogous to Theorem 1.4 can be obtained for minimizers of

$$\boldsymbol{u} \mapsto \mathrm{TV}^n_I(u) + \frac{1}{2\lambda} \int_I (u - u_0)^2. \tag{1.42}$$

This is done in [35]. Theorem 1.4 is then deduced by the authors as a simple consequence.

1.5 Regular 1-harmonic map flow

Let $(\mathcal{N}, g)$ be a complete, connected smooth n-dimensional Riemannian manifold (without boundary). Throughout this section and Chapter 5, without loss of generality [69, 46], we will treat it as an isometrically embedded submanifold in the Euclidean space $\mathbb{R}^N$. Given an open, bounded Lipschitz domain $\Omega \subset \mathbb{R}^m$ we consider the formal steepest descent flow with respect to the L^2 distance of the functional $\mathrm{TV}_\Omega^{\mathcal{N}}$: the total variation functional constrained to functions taking values in $\mathcal{N}$, given for smooth $\boldsymbol{u}$ by

$$\mathrm{TV}_\Omega^{\mathcal{N}}(\boldsymbol{u}) = \int_\Omega |\nabla \boldsymbol{u}|. \tag{1.43}$$

Following the L^2-steepest descent flow is one way of controlled decreasing $\mathrm{TV}_\Omega^{\mathcal{N}}$, which is a problem appearing in image processing. Besides the case $\mathcal{N} \subseteq \mathbb{S}^{N-1}$, which appears in denoising of optical flows [77] or color images [78], other examples of targets appearing in applications include the space of isometries $SO(3) \times \mathbb{R}^3$ [57], the cylinder $\mathbb{R}^2 \times \mathbb{S}^1$ (LCh color space) [82] and the space of positive definite symmetric matrices (diffusion tensors) $Sym_+(3)$ [82]. All of these examples are homogeneous spaces, and therefore have natural invariant metrics. Our main goal in Chapter 5 is to develop a well-posedness theory for the flow in a generality encompassing these cases. As some of these manifolds are non-compact, we refrain from the unnecessary (although convenient) assumption of compactness of $\mathcal{N}$. We underline that in this case well-posedness results are not directly provided by classical semigroup theory, since the spaces of $\mathcal{N}$-valued functions are not even linear.

Given a point $\boldsymbol{p} \in \mathcal{N}$, we denote by

$$\pi_{\boldsymbol{p}} \colon T_{\boldsymbol{p}}\mathbb{R}^N \equiv \mathbb{R}^N \to T_{\boldsymbol{p}}\mathcal{N}$$

the orthogonal projection onto the tangent space of $\mathcal{N}$ at $\boldsymbol{p}$, $T_{\boldsymbol{p}}\mathcal{N}$. Similarly, $\pi_{\boldsymbol{p}}^\perp$ will denote the orthogonal projection of $\mathbb{R}^N$ onto the normal space $T_{\boldsymbol{p}}\mathcal{N}^\perp$. The centered dot will denote the Euclidean scalar product on $\mathbb{R}^m$ or $\mathbb{R}^N$, while k stacked dots will denote the induced scalar product on a Cartesian product of any k-tuple of these spaces. Calculating the first variation of (1.43) at $\boldsymbol{u}$, one obtains that the flow in a time interval $[0, T[$ starting with initial datum $\boldsymbol{u}_0$ is formally given by the system

$$\boldsymbol{u}_t = \pi_{\boldsymbol{u}} \left(\operatorname{div} \tfrac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|} \right) \quad \text{in }]0, T[\times \Omega, \tag{1.44}$$

$$\boldsymbol{\nu}^\Omega \cdot \tfrac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|} = \boldsymbol{0} \quad \text{in }]0, T[\times \partial\Omega, \tag{1.45}$$

$$\boldsymbol{u}(0, \cdot) \equiv \boldsymbol{u}_0. \tag{1.46}$$

The symbol $\boldsymbol{\nu}^\Omega$ denotes the external unit normal of Ω, which is defined $\mathcal{H}^{m-1}$-a. e. on $\partial\Omega$. The meaning of the expression $\frac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|}$ in (1.44, 1.45) deserves a clarification even for smooth $\boldsymbol{u}$: we understand $\frac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|}$ as a multifunction

$$\tfrac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|} \colon (t, \boldsymbol{x}) \mapsto \begin{cases} \frac{\nabla \boldsymbol{u}(t,\boldsymbol{x})}{|\nabla \boldsymbol{u}(t,\boldsymbol{x})|} & \text{if } \nabla \boldsymbol{u}(t, \boldsymbol{x}) \neq \boldsymbol{0} \\ B(0,1) \subset \mathbb{R}^m \times T_{\boldsymbol{u}(t,\boldsymbol{x})}\mathcal{N} & \text{if } \nabla \boldsymbol{u}(t, \boldsymbol{x}) = \boldsymbol{0} \end{cases}$$

and require that (1.44, 1.45) are satisfied for an appropriate selection. This is formalized in the following definition, which is an adapted version of [6, Definition 2.5]. Here and in the following we will use the notation

$$X(U, \mathcal{N}) = \{\boldsymbol{w} \in X(U, \mathbb{R}^N) \colon \boldsymbol{w}(\boldsymbol{y}) \in \mathcal{N} \text{ for a. e. } \boldsymbol{y} \in U\},$$

where U is any domain in $\mathbb{R}^l$ (or a compact l-dimensional Riemannian manifold), $l = 1, 2, \ldots$ and $X(U, \mathbb{R}^N)$ is a subspace of $L^1_{loc}(U, \mathbb{R}^N)$.

Definition 1.2. Let $T \in]0, \infty]$. We say that

$$\boldsymbol{u} \in W^{1,2}_{loc}([0, T[\times \overline{\Omega}, \mathcal{N}) \text{ with } \nabla \boldsymbol{u} \in L^{\infty}_{loc}([0, T[\times \overline{\Omega}, \mathbb{R}^{m \cdot N})$$

is a (regular) solution to (1.44) (in $[0, T[$) if there exists $\boldsymbol{Z} \in L^{\infty}(]0, T[\times \Omega, \mathbb{R}^{m \cdot N})$ with $\operatorname{div} \boldsymbol{Z} \in L^2_{loc}([0, T[\times \overline{\Omega}, \mathbb{R}^N)$ satisfying

$$\boldsymbol{Z} \in \tfrac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|}, \tag{1.47}$$

$$\boldsymbol{u}_t = \pi_{\boldsymbol{u}}(\operatorname{div} \boldsymbol{Z}) \tag{1.48}$$

$\mathcal{L}^{1+m}$ – a. e. in $]0, T[\times \Omega$. We say that a regular solution $\boldsymbol{u}$ to (1.44) satisfies (homogeneous) Neumann boundary condition (1.45) if

$$\boldsymbol{\nu}^{\Omega} \cdot \boldsymbol{Z} = \boldsymbol{0} \tag{1.49}$$

$\mathcal{L}^1 \otimes \mathcal{H}^{m-1}$ – a. e. in $]0, T[\times \partial\Omega$.

Remark. Due to Morrey's embedding theorem, any regular solution to (1.44) has a representative that is locally Hölder continuous on $[0, T[\times \overline{\Omega}$ [47, Theorem 5]. We will identify it with this representative. In particular, the initial condition (1.46) can be understood pointwise. On the other hand, $\boldsymbol{\nu}^{\Omega} \cdot \boldsymbol{Z}$ in (1.49) has to be understood as the normal trace of an L^{∞} vector field with integrable divergence, as defined in [79, 7].

If conditions in Definition 1.2 are satisfied, we will often say that the pair $(\boldsymbol{u}, \boldsymbol{Z})$ is a (regular) solution to (1.44) and/or (1.45). We will often use equivalent (see e. g. the proof of Lemma 5.2) form of (1.48):

$$\boldsymbol{u}_t = \operatorname{div} \boldsymbol{Z} + \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{Z}_i), \tag{1.50}$$

where $\mathcal{A}_{\boldsymbol{p}}$ denotes the second fundamental form of $\mathcal{N}$ at $\boldsymbol{p} \in \mathcal{N}$ and $\boldsymbol{Z} = (\boldsymbol{Z}_1, \ldots, \boldsymbol{Z}_m)$. Here and throughout Chapter 5, we use Einstein's summation convention.

The adjective *regular* in Definition 1.2 is justified by the following considerations. Firstly, $W^{1,\infty}(\Omega)$ is the highest Sobolev regularity that is preserved by the scalar total variation flow [51, 13]. Secondly, such attribute distinguishes the class of solutions in Definition 1.2 from weak (energy) solutions, whose natural spatial regularity is $BV(\Omega)$. However, we note that in the constrained case, even defining a proper concept of solution is non-trivial in the BV setting, the crucial issue being an appropriate identification of the right-hand side of (1.48) or of (1.50). In this regard, the only case considered so far is $\mathcal{N} \subseteq \mathbb{S}^n$, in which (1.50) drastically simplifies due to the isotropy of the sphere:

$$\boldsymbol{u}_t = \operatorname{div} \boldsymbol{Z} + \boldsymbol{u}|\nabla \boldsymbol{u}|.$$

Suitably defined solutions to (1.44, 1.45) have been obtained in [39] when the initial datum is contained in an hyper-octant of $\mathbb{S}^n$ [39]. When $n = 1$, the assumption on u_0 may be relaxed and uniqueness results are available too [38]. A notion of solution extending the one in [38, 39] to $(N-1)$-dimensional manifolds with unique geodesics has been proposed in [25]. Existence of solutions for a discretized Dirichlet problem for (1.44) in the case $\mathcal{N} = \mathbb{S}^n$, $m = 2$ has been obtained in [44]. The validity of Definition 1.2 is supported by the well-posedness results that we obtain. First of all, regular solutions are unique.

Theorem 1.5. *Suppose that $\boldsymbol{u}^1, \boldsymbol{u}^2$ are two regular solutions to (1.44, 1.45) in $[0,T[$, $T \in]0,\infty[$ such that $\boldsymbol{u}^1(0,\cdot) = \boldsymbol{u}^2(0,\cdot) = \boldsymbol{u}_0$. Then $\boldsymbol{u}^1 \equiv \boldsymbol{u}^2$.*

The proof of Theorem 1.5 is different from the proofs of analogous results for p-harmonic flow in [48, 32] in that we do not use strict monotonicity of the p-Laplace operator (since it does not hold for $p = 1$).

Provided that Ω is convex, we are able to construct local-in-time Lipschitz solutions to (1.44, 1.45). We need the assumption of convexity, as we are forced to use global L^p estimates for $\nabla \boldsymbol{u}$. Localization of these estimates is not available due to the strong degeneracy of the 1-Laplace operator $\operatorname{div} \frac{\nabla \boldsymbol{u}}{|\nabla \boldsymbol{u}|}$. In fact, at least in the case of anisotropic total variation flow, there are examples of non-convex Lipschitz Ω, where $W^{1,p}(\Omega)$ regularity classes are not preserved by the flow (Example 2.3). The assumption of convexity is not very restrictive from the point of view of image processing, as typical domains in applications are rectangles (or boxes of different dimensions).

The existential theory depends on the sectional curvature $\mathcal{K}_{\mathcal{N}}$ of $\mathcal{N}$ or, equivalently, on the Riemannian curvature tensor $\mathcal{R}^{\mathcal{N}}$ of $\mathcal{N}$. We denote by $K_{\mathcal{N}}$ the supremum of sectional curvature over $\mathcal{N}$, i. e.

$$K_{\mathcal{N}} = \sup \left\{ \frac{\boldsymbol{v} \cdot \mathcal{R}^{\mathcal{N}}_{\boldsymbol{p}}(\boldsymbol{v}, \boldsymbol{w})\boldsymbol{w}}{|\boldsymbol{v}|^2 |\boldsymbol{w}|^2 - (\boldsymbol{v} \cdot \boldsymbol{w})^2} \,\middle|\, \boldsymbol{p} \in \mathcal{N}, \boldsymbol{v}, \boldsymbol{w} \in T_{\boldsymbol{p}}\mathcal{N} \text{ linearly independent} \right\}. \tag{1.51}$$

Recall that $K_{SO(n)\times\mathbb{R}^n}$ is positive (and finite) and $K_{\mathbb{S}^1\times\mathbb{R}^n}$, $K_{Sym_+(n)}$ are non-positive.

Theorem 1.6. *Suppose that Ω is convex, the embedding of $\mathcal{N}$ in $\mathbb{R}^N$ is closed and $K_{\mathcal{N}} < \infty$. Given $\boldsymbol{u}_0 \in W^{1,\infty}(\Omega, \mathcal{N})$, we denote $T_\dagger = (K_{\mathcal{N}} \|\nabla \boldsymbol{u}_0\|_{L^\infty})^{-1}$ if $K_{\mathcal{N}} > 0$ and $T_\dagger = +\infty$ otherwise. There exists a regular solution $\boldsymbol{u}$ to (1.44, 1.45, 1.46) in $[0, T_\dagger[$ satisfying the energy inequality*

$$\operatorname*{ess\,sup}_{t\in[0,T_\dagger[} \int_\Omega |\nabla \boldsymbol{u}(t,\cdot)| + \int_0^{T_\dagger} \int_\Omega \boldsymbol{u}_t^2 \le \int_\Omega |\nabla \boldsymbol{u}_0|. \tag{1.52}$$

This theorem bears a similarity to [41, Theorem 3.4], where Lipschitz local-in-time solutions to (1.44) are constructed in the case where Ω is a flat torus, i. e. a box with periodic boundary conditions. However, aside from the choice of boundary condition, there are differences between these results — most importantly, in [41], smallness of $\nabla \boldsymbol{u}_0$ in $L^{1+\varepsilon}(\Omega)$ is assumed. This is because in [41], global solutions to p-harmonic flows constructed in [33] for small initial data are used as an approximation. In our case a different approximation scheme is proposed. In fact we cannot use the results in [33] as non-trivial boundary conditions are not handled there.

At least in the case of Dirichlet boundary data, regular solutions to (1.44) can blow up in finite time, as shown by explicit examples in [24, 40]. In our case, we prove that solutions exist globally in time, provided that the range of the initial datum is contained in a small enough ball in $\mathcal{N}$. In fact, in this case they become constant in finite time, similarly as for the scalar total variation flow [42]. Note that in the case of inhomogeneous Dirichlet boundary conditions, the evolution of generic initial data under 1-harmonic flow does not stop in finite time [43], in contrast to what is observed in the scalar total variation flow, at least in 1-dimensional domains [51]. Let us denote by $B_g(\boldsymbol{p}, R)$ the ball centered at $\boldsymbol{p} \in \mathcal{N}$ of radius $R > 0$ with respect to the metric induced by g on $\mathcal{N}$.

Theorem 1.7. *Let $\boldsymbol{p}_0 \in \mathcal{N}$, $\boldsymbol{u}_0 \in W^{1,\infty}(\Omega, \mathcal{N})$ and $\boldsymbol{u}$ be a regular solution to (1.44, 1.45, 1.46) in $[0, T[$. Suppose that $\boldsymbol{u}_0(\Omega) \in \overline{B_g(\boldsymbol{p}_0, R)}$, $R > 0$. There exist*

- *a constant $R_* = R_*(\mathcal{N}, \boldsymbol{p}_0) > 0$ such that if $R < R_*$, then $\boldsymbol{u}(t, \Omega) \in \overline{B_g(\boldsymbol{p}_0, R)}$ for $t \in]0, T[$,*
- *constants $\widetilde{R}_* = \widetilde{R}_*(\mathcal{N}, \boldsymbol{p}_0) \in]0, R_*[$, $C = C(\Omega, \mathcal{N}, \boldsymbol{p}_0) > 0$ and $\boldsymbol{u}_* \in \mathcal{N}$ such that if $R < \min\left(\widetilde{R}_*, \frac{T}{C}\right)$, then $\boldsymbol{u}(t, \cdot) \equiv \boldsymbol{u}_*$ for $t \in]CR, T[$.*

In the particular case $K_{\mathcal{N}} \leq 0$ no blow-up occurs for any Lipschitz datum, and we can obtain a stronger result of global existence. Owing to particularly simple topology of Riemannian manifolds with $K_{\mathcal{N}} \leq 0$, we need not assume the existence of a closed embedding of $\mathcal{N}$ into $\mathbb{R}^N$ in this case.

Theorem 1.8. *Suppose that Ω is convex and $K_{\mathcal{N}} \leq 0$. Let $\boldsymbol{u}_0 \in W^{1,\infty}(\Omega, \mathcal{N})$. There exists a regular solution $\boldsymbol{u}$ to (1.44, 1.45, 1.46) in $[0, \infty[$ satisfying the energy inequality (1.52). There exists $T_* = T_*(\boldsymbol{u}_0) \in [0, \infty[$ and $\boldsymbol{u}_* = \boldsymbol{u}_*(\boldsymbol{u}_0) \in \mathcal{N}$ such that $\boldsymbol{u}(t, \cdot) \equiv \boldsymbol{u}_*$ for $t \geq T_*$. Furthermore,*

$$\operatorname*{ess\,sup}_{t>0} \|\nabla \boldsymbol{u}(t, \cdot)\|_{L^\infty(\Omega)} \leq \|\nabla \boldsymbol{u}_0\|_{L^\infty(\Omega)}.$$

We remark that in the scalar case the preservation of the $W^{1,\infty}$ bound follows from [21, Corollary 5.6]. However, the methods there are not readily adaptable to vectorial problems.

From the point of view of imaging science, the rigorously defined notion of regular solution to (1.44, 1.45, 1.46) provides a theoretical basis for computing a total variation diminishing flow via a finite difference scheme. Our well-posedness results should then be expected to translate to stability results for such a scheme. In these terms, the requirement of Lipschitz regularity of data is not a significant restriction, as it corresponds to the boundedness of difference quotients on the level of discretization. In the case where $\mathcal{N}$ has non-positive sectional curvature (Theorem 1.8), stability propagates indefinitely, even if initial image exhibits prominent contours. However, this is not necessarily the case anymore for general $\mathcal{N}$, as the bound on existence time $T_\dagger$ of the solution constructed in Theorem 1.6 deteriorates with increasing Lipschitz constant of the datum. For this reason, well-posedness for a notion of solution defined on the energy space $BV(\Omega, \mathcal{N})$ would be more desirable. In fact, we plan to use the thery developed here as a basis for treating this problem.

The regular 1-harmonic flow that we consider here is continuous over the spacetime, and hence capable of generating homotopy. For this reason we find it appropriate to discuss in detail the case where the domain is a compact Riemannian manifold $(\mathcal{M}, \gamma)$. In this setting, the total variation functional takes form

$$\mathrm{TV}_{\mathcal{M}}^{\mathcal{N}}(\boldsymbol{u}) = \int_{\mathcal{M}} |\,\mathrm{d}\boldsymbol{u}|_\gamma \,\mathrm{d}\mu_\gamma. \tag{1.53}$$

To explain the notation in (1.53), we introduce local coordinates $\boldsymbol{x} \mapsto (x^1, \ldots, x^m)$ on $\mathcal{M}$ and denote $\gamma(\boldsymbol{v}, \boldsymbol{w}) = \gamma_{ab} v^a w^b$ for any vector fields $\boldsymbol{v}, \boldsymbol{w}$ on $\mathcal{M}$, $\left(\gamma^{ab}\right)_{1\leq a,b\leq m} = (\gamma_{ab})^{-1}_{1\leq a,b\leq m}$. We have $|\,\mathrm{d}\boldsymbol{u}|_\gamma = (\gamma^{ab} u^i_{x^a} u^i_{x^b})^{\frac{1}{2}}$ and $\mathrm{d}\mu_\gamma = |\det(\gamma_{ab})|^{\frac{1}{2}}\,\mathrm{d}\mathcal{L}^m$. In this setting, the system of equations (1.44) representing the flow becomes

$$\boldsymbol{u}_t = \pi_{\boldsymbol{u}}\left(\operatorname{div}_\gamma \tfrac{\mathrm{d}\boldsymbol{u}}{|\,\mathrm{d}\boldsymbol{u}|}\right) \quad \text{in }]0, T[\times\mathcal{M}. \tag{1.54}$$

The expression for div_γ acting on a 1-form $\boldsymbol{\vartheta}$ on $\mathcal{M}$ in coordinates is

$$\mathrm{div}_\gamma \boldsymbol{\vartheta} = |\det(\gamma_{ab})|^{-\frac{1}{2}} \left(|\det(\gamma_{ab})|^{\frac{1}{2}} \gamma^{ab} \vartheta_b \right)_{x^a}.$$

Observe that (1.54) is a formal limit as $p \to 1^+$ of systems

$$\boldsymbol{u}_t = \pi_{\boldsymbol{u}}(\mathrm{div}_\gamma(|\,\mathrm{d}\boldsymbol{u}|^{p-2}\,\mathrm{d}\boldsymbol{u})) \quad \text{in }]0,T[\times\mathcal{M} \tag{1.55}$$

corresponding to p-harmonic map flows between Riemannian manifolds. These were first considered in the case $p = 2$ in connection with the homotopy problem for harmonic maps, i. e. the problem of finding a harmonic map homotopic to a given one. The problem was solved in [28] under the condition that $K_{\mathcal{N}} \leq 0$ by constructing the harmonic map flow. An analogous result was later obtained in [32] for any $p > 1$. We note that there are several non-equivalent notions of p-harmonic maps, among them *weakly p-harmonic maps*, i. e. stationary weak solutions to (1.55).

We introduce the notation

$$\tfrac{\mathrm{d}\boldsymbol{u}}{|\,\mathrm{d}\boldsymbol{u}|_\gamma} \colon (t,\boldsymbol{x}) \mapsto \begin{cases} \frac{\mathrm{d}\boldsymbol{u}(t,\boldsymbol{x})}{|\,\mathrm{d}\boldsymbol{u}(t,\boldsymbol{x})|_\gamma} & \text{if } \mathrm{d}\boldsymbol{u}(t,\boldsymbol{x}) \neq \boldsymbol{0} \\ B_\gamma(0,1) \subset T^*_{\boldsymbol{x}}\mathcal{M} \times T_{\boldsymbol{u}(t,\boldsymbol{x})}\mathcal{N} & \text{if } \mathrm{d}\boldsymbol{u}(t,\boldsymbol{x}) = \boldsymbol{0}. \end{cases}$$

Measurable selections of $\frac{\mathrm{d}\boldsymbol{u}}{|\,\mathrm{d}\boldsymbol{u}|_\gamma}(t,\cdot)$ can be seen as L^∞ sections of the bundle $T^*\mathcal{M} \times \mathbb{R}^N$ over $\mathcal{M}$ for a. e. $t \in]0,T[$, see [70] for reference. As in [70], we let $L^p(T^*\mathcal{M}\times\mathbb{R}^N)$ denote L^p sections of this bundle, $p \in [1,\infty]$. Similarly, we denote by $L^p(]0,T[\times T^*\mathcal{M} \times \mathbb{R}^N)$ the space of L^p sections of the bundle $]0,T[\times T^*\mathcal{M} \times \mathbb{R}^N$ over $]0,T[\times\mathcal{M}$, and by $L^p_{loc}([0,T[\times T^*\mathcal{M} \times \mathbb{R}^N)$ the space of measurable sections of this bundle which are p-integrable locally on $[0,T[\times\mathcal{M}$. We are ready to introduce a concept of solution to (1.54).

Definition 1.3. Let $T \in]0,\infty]$. We say that

$$\boldsymbol{u} \in W^{1,2}_{loc}([0,T[\times\mathcal{M},\mathcal{N}) \text{ with } \mathrm{d}\boldsymbol{u} \in L^\infty_{loc}([0,T[\times T^*\mathcal{M} \times \mathbb{R}^N)$$

is a (regular) solution to (1.54) (in $[0,T[$) if there exists $\boldsymbol{Z} \in L^\infty(]0,T[\times T^*\mathcal{M} \times \mathbb{R}^N)$ with $\mathrm{div}_\gamma \boldsymbol{Z} \in L^2_{loc}([0,T[\times\mathcal{M},\mathbb{R}^N)$ satisfying

$$\boldsymbol{Z} \in \tfrac{\mathrm{d}\boldsymbol{u}}{|\,\mathrm{d}\boldsymbol{u}|_\gamma}, \tag{1.56}$$

$$\boldsymbol{u}_t = \pi_{\boldsymbol{u}}(\mathrm{div}_\gamma \boldsymbol{Z}) \tag{1.57}$$

$\mathcal{L}^{1+m}$ – a. e. in $]0,T[\times\mathcal{M}$.

The strength of our result in this case depends on the sign of the Ricci curvature $\mathcal{R}ic^{\mathcal{M}}$ of $\mathcal{M}$. Opposite to the usual convention, we define it as a $(2,0)$ tensor, i. e.

$$(\mathcal{R}ic^{\mathcal{M}})^{ab} = \gamma^{ac}\gamma^{bd}(\mathcal{R}^{\mathcal{M}})^e{}_{ced} \tag{1.58}$$

in coordinates. We denote

$$Ric_{\mathcal{M}} = \min\left\{ \frac{\mathcal{R}ic^{\mathcal{M}}_{\boldsymbol{p}}(\boldsymbol{\vartheta},\boldsymbol{\eta})}{|\boldsymbol{\vartheta}|_\gamma |\boldsymbol{\eta}|_\gamma} \;\middle|\; \boldsymbol{p} \in \mathcal{M}, \boldsymbol{\vartheta},\boldsymbol{\eta} \in T^*_p\mathcal{M} \setminus \{\boldsymbol{0}\} \right\}.$$

Theorem 1.9. *Let $(\mathcal{M}, \gamma)$ be a compact, orientable Riemannian manifold and let $(\mathcal{N}, g)$ be a compact submanifold in the Euclidean space $\mathbb{R}^N$. Given $\boldsymbol{u}_0 \in W^{1,\infty}(\mathcal{M}, \mathcal{N})$, there exists $T \in]0, \infty]$ and a unique regular solution to (1.54, 1.46) in $[0, T[$.*

If $K_{\mathcal{N}} \leq 0$, the solution exists in $[0, \infty[$. If in addition $Ric_{\mathcal{M}} \geq 0$, there exists a sequence $(t_k) \subset]0, \infty[$, $t_k \to \infty$, $\boldsymbol{u}_* \in W^{1,\infty}(\mathcal{M}, \mathcal{N})$ and $\boldsymbol{Z}_* \in L^\infty(T^*\mathcal{M} \times \mathbb{R}^N)$ with $\mathrm{div}_\gamma \boldsymbol{Z}_* \in L^\infty(\mathcal{M}, \mathbb{R}^N)$ such that

$$\pi_{\boldsymbol{u}_*}(\mathrm{div}_\gamma \boldsymbol{Z}_*) = \mathbf{0}, \qquad \boldsymbol{Z}_* \in \tfrac{\mathrm{d}\boldsymbol{u}_*}{|\mathrm{d}\boldsymbol{u}_*|_\gamma}, \tag{1.59}$$

$$\boldsymbol{u}(t_k, \cdot) \to \boldsymbol{u}_* \text{ in } C(\mathcal{M}, \mathcal{N}). \tag{1.60}$$

As $\boldsymbol{u}$ is continuous and the sequence $(\boldsymbol{u}(t_k, \cdot))$ converges to $\boldsymbol{u}_*$ in $C(\mathcal{M}, \mathcal{N})$, $\boldsymbol{u}_*$ and $\boldsymbol{u}_0$ are homotopic. Thus, we have solved the homotopy problem for (weakly) 1-harmonic maps assuming that $\mathcal{M}$ is orientable with $Ric_{\mathcal{M}} \geq 0$ and $K_{\mathcal{N}} \leq 0$.

The plan of Chapter 5 is as follows. Firstly, in section 5.1, we prove Theorem 1.5. In section 5.2, we obtain well-posedness of an approximating system to (1.44, 1.45, 1.46) and we obtain some a priori estimates (independent of the parameter of approximation) for their solutions. This permits us to prove Theorem 1.6, to which section 5.3 is devoted. The asymptotic behavior is treated in the next sections: in section 5.4, we prove Theorem 1.7 while in section 5.5, we treat the case of non-positive curvature; i.e Theorem 1.8. Section 5.6 is devoted to the case where the domain is a compact Riemannian manifold, in which we prove Theorem 1.9.

Chapter 2

The orthotropic total variation flow in the plane

2.1 Notation and preliminaries

2.1.1 Balls

In this Chapter, by $B_\varphi(\boldsymbol{x}, r)$ we denote the ball in $\mathbb{R}^N$ with respect to norm $|\cdot|_\varphi$, centered at $\boldsymbol{x}$, of radius r. For the ball with respect to the Euclidean norm, we write simply $B(\boldsymbol{x}, r)$. Symbols $B_\varphi(r), B(r)$ stand for balls centered at the origin.

2.1.2 Measures. Lebesgue and Bochner spaces

We denote by $\mathcal{L}^m$ and $\mathcal{H}^{m-1}$ the m-dimensional Lebesgue measure and the $(m-1)$-dimensional Hausdorff measure in $\mathbb{R}^m$, respectively. If $A \subset \mathbb{R}^m$ is a set of positive (possibly infinite) $\mathcal{L}^m$ measure, we denote by $L^p(A)$, $1 \leq p \leq \infty$ the Lebesgue space of functions integrable with power p with respect to $\mathcal{L}^m$. On the other hand, if $A \subset \mathbb{R}^m$ has finite $\mathcal{H}^{m-1}$ measure (e. g. A is the boundary of a Lipschitz domain), $L^p(A)$ denotes the Lebesgue space of functions integrable with power p with respect to $\mathcal{H}^{m-1}$. We adopt similar notation for spaces $L^p(A, \mathbb{R}^k)$, $k = 2, 3, \ldots$ Whenever it is clear, we adopt the convention that an equality or inequality between two measurable functions holds in the sense of Lebesgue spaces, i. e. almost everywhere with respect to the corresponding (implicitly specified) measure, unless otherwise stated.

If $T \in]0, \infty]$ and X is a Banach space, we denote by $L^p(]0, T[, X)$ the usual space of Bochner measurable functions $f \colon]0, T[\to X$ s. t. $\int_0^T \|f\|_X^p < \infty$. By $L^p_w(]0, T[, X)$ we denote the analogous space of weakly measurable functions (see [8, Chapter I]).

2.1.3 Functions of bounded variation and sets of finite perimeter

We use standard definitions of the theory of functions of bounded variation, as outlined in [5, 31], with occasional differences in notation. Let Ω be an open set in $\mathbb{R}^m$. Given $u \in L^1_{loc}(\Omega)$, we define the total variation of u by

$$\mathrm{TV}_\Omega(u) = \sup \left\{ \int_\Omega u \operatorname{div} \boldsymbol{\eta} \, \mathrm{d}\mathcal{L}^m \colon \boldsymbol{\eta} \in C^1_c(\Omega, \mathbb{R}^m), \ |\boldsymbol{\eta}| \leq 1 \right\}.$$

If $\mathrm{TV}_\Omega(u) < +\infty$, we write $u \in BV(\Omega)$. In this case the distributional gradient Du of u is a Radon vector measure, and there holds $\mathrm{TV}_\Omega(u) = |Du|(\Omega)$, where $|Du|$ denotes the variation measure of Du. We write $u^{\pm}(x)$ for the lower and upper approximate limits of u at $x \in \Omega$ which exist $\mathcal{H}^{m-1}$-a. e. and J_u for the jump set of u, i. e. the set of points where $u^+ \neq u^-$. Finally, $\frac{Du}{|Du|}$ denotes the Radon-Nikodym derivative of Du with respect to its variation $|Du|$.

Given a Lebesgue measurable subset E of $\mathbb{R}^m$, we define its perimeter in Ω by

$$\mathrm{Per}(E, \Omega) = \mathrm{TV}_{\Omega}(\mathbf{1}_E)$$

and write $\mathrm{Per}(E) = \mathrm{Per}(E, \mathbb{R}^m)$.

If E is a set of finite perimeter in Ω, the jump set of $\mathbf{1}_E$ is $\mathcal{H}^{m-1}$-equivalent to the *reduced boundary* $\partial^* E$ defined by the following. We say a point $\boldsymbol{x} \in \Omega$ belongs to $\partial^* E$ if $|D\mathbf{1}_E|(B(\boldsymbol{x}, \varrho)) > 0$ for sufficiently small $\varrho > 0$ and quantity $\frac{D\mathbf{1}_E(B(\boldsymbol{x},\varrho))}{|D\mathbf{1}_E|(B(\boldsymbol{x},\varrho))}$ has a limit that belongs to $\mathbb{S}^{m-1}$ as $\varrho \to 0^+$. If these conditions hold, we denote this limit by $\boldsymbol{\nu}^E(\boldsymbol{x})$. There holds

$$\partial^* E \subset \partial^{\frac{1}{2}} E = \left\{ \boldsymbol{x} \in \Omega \colon \lim_{\varrho \to 0^+} \frac{\mathcal{L}^m(B(\boldsymbol{x}, \varrho) \cap E)}{\mathcal{L}^m(B(\boldsymbol{x}, \varrho))} = \frac{1}{2} \right\},$$

also $\mathcal{H}^{m-1}\left(\partial^{\frac{1}{2}} E \setminus \partial^* E\right) = 0$ and $\mathcal{H}^{m-1}$-almost every point in Ω is either a Lebesgue point for $\mathbf{1}_E$ or belongs to $\partial^* E$.

2.1.4 The anisotropic total variation. The anisotropic perimeter

We recall here the notion of anisotropic total variation introduced in [4]. Given an open set $\Omega \subseteq \mathbb{R}^m$, a norm $|\cdot|_\varphi$ on $\mathbb{R}^m$, and a function $u \in L^1_{loc}(\Omega)$, we define

$$\mathrm{TV}_{\varphi,\Omega}(u) = \sup \left\{ \int_\Omega u \operatorname{div} \boldsymbol{\eta} \, \mathrm{d}\mathcal{L}^m \colon \boldsymbol{\eta} \in C^1_c(\Omega, \mathbb{R}^m), \ |\boldsymbol{\eta}|^*_\varphi \leq 1 \right\}$$

where $|\cdot|^*_\varphi$ denotes the dual norm associated with $|\cdot|_\varphi$. When restricted to $L^2(\Omega)$, this is a proper, convex, lower semicontinuous functional with values in $[0, \infty]$. We have $\mathrm{TV}_{\varphi,\Omega}(u) < +\infty$ iff $u \in BV(\Omega)$, in which case we define a measure $|Du|_\varphi$ by

$$|Du|_\varphi = \left| \frac{Du}{|Du|} \right|_\varphi |Du|.$$

There holds $|Du|_\varphi(\Omega) = \mathrm{TV}_{\varphi,\Omega}(u)$. This is an equivalent seminorm on $BV(\Omega)$.

Given a Lebesgue measurable set E in $\mathbb{R}^m$ we denote $\mathrm{Per}_\varphi(E, \Omega) = \mathrm{TV}_{\varphi,\Omega}(\mathbf{1}_E)$ and $\mathrm{Per}_\varphi(E) = \mathrm{Per}_\varphi(E, \mathbb{R}^m)$. If E has finite perimeter in Ω, then

$$\mathrm{Per}_\varphi(E, \Omega) = \int_{\Omega \cap \partial^* E} |\boldsymbol{\nu}^E|_\varphi \, \mathrm{d}\mathcal{H}^1. \tag{2.1}$$

If ∂E is Lipschitz, we can drop the star in $\partial^* E$, and $\boldsymbol{\nu}^E$ is the pointwise $\mathcal{H}^{m-1}$-a. e. defined outer Euclidean normal to E.

2.1.5 The subdifferential of $\mathrm{TV}_{\varphi,\Omega}$

We consider the space

$$X_\Omega = \left\{ \boldsymbol{\xi} \in L^\infty(\Omega, \mathbb{R}^m) \colon \operatorname{div} \boldsymbol{\xi} \in L^2(\Omega) \right\}. \tag{2.2}$$

In [7, Theorem 1.2], the weak trace on the boundary of a bounded Lipschitz domain Ω of the normal component of $\boldsymbol{\xi} \in X_\Omega$ is defined. Namely, it is proved that the formula

$$\left\langle [\boldsymbol{\xi}, \boldsymbol{\nu}^\Omega], \rho \right\rangle = \int_\Omega \rho \operatorname{div} \boldsymbol{\xi} \, \mathrm{d}\mathcal{L}^m + \int_\Omega \boldsymbol{\xi} \cdot \nabla \rho \, \mathrm{d}\mathcal{L}^m \quad \text{for } \rho \in C^1(\overline{\Omega}) \tag{2.3}$$

defines a linear operator $[\cdot, \boldsymbol{\nu}^\Omega] \colon X_\Omega \to L^\infty(\partial\Omega)$ such that

$$\left\| [\boldsymbol{\xi}, \boldsymbol{\nu}^\Omega] \right\|_{L^\infty(\partial\Omega)} \leq \|\boldsymbol{\xi}\|_{L^\infty(\Omega)} \tag{2.4}$$

for all $\boldsymbol{\xi} \in X_\Omega$ and $[\boldsymbol{\xi}, \boldsymbol{\nu}^\Omega]$ coincides with the pointwise trace of the normal component if $\boldsymbol{\xi}$ is smooth.

In the analysis of differential equations associated with the functional $\mathrm{TV}_{\varphi,\Omega}$, a crucial role is played by the following result characterizing the L^2-subdifferential of $\mathrm{TV}_{\varphi,\Omega}$, whose proof can easily be obtained by adapting that of [6, Proposition 1.10].

Proposition 2.1. *Let Ω be a bounded Lipschitz domain and let $w \in \mathcal{D}(\mathrm{TV}_{\varphi,\Omega}) = BV(\Omega)$. There holds $v \in -\partial \mathrm{TV}_{\varphi,\Omega}(w)$ iff $v \in L^2(\Omega)$ and there exists $\boldsymbol{\xi} \in X_\Omega$ such that $v = \operatorname{div} \boldsymbol{\xi}$ and*

$$|\boldsymbol{\xi}|_\varphi^* \leq 1 \quad \mathcal{L}^m\text{-a. e. in } \Omega, \tag{2.5}$$

$$[\boldsymbol{\xi}, \boldsymbol{\nu}^\Omega] = 0 \quad \mathcal{H}^{m-1}\text{-a. e. on } \partial\Omega. \tag{2.6}$$

$$-\int_\Omega w \operatorname{div} \boldsymbol{\xi} \, \mathrm{d}\mathcal{L}^2 = \int_\Omega |Dw|_\varphi, \tag{2.7}$$

We denote by $X_{\varphi,\Omega}(w)$ the set of $\boldsymbol{\xi} \in X_\Omega$ satisfying (2.5-2.7). Proposition 2.1 holds also with $\Omega = \mathbb{R}^2$. In this case, the Neumann condition (2.6) becomes void.

2.1.6 The orthotropic total variation flow

We are now ready to recall the rigorous definition of *(strong) solution* to the Neumann problem for (1.19), formally given by

$$\begin{aligned} u_t &= (\operatorname{sgn} u_{x_1})_{x_1} + (\operatorname{sgn} u_{x_2})_{x_2} \quad \text{in }]0,\infty[\times \Omega, \\ (\operatorname{sgn} u_{x_1}, \operatorname{sgn} u_{x_2}) \cdot \nu^\Omega &= 0 \quad \text{in }]0,\infty[\times \partial\Omega. \end{aligned} \tag{2.8}$$

which is an adaptation of [65, Definition 4.].

Definition 2.1. A function $u \in C([0,\infty[, L^2(\Omega))$ is called a strong solution to (2.8) if $u_t \in L^2_{loc}(]0,\infty[, L^2(\Omega))$, $u \in L^1_{w,loc}(]0,\infty[, BV(\Omega))$ and there exists $\boldsymbol{z} \in L^\infty(]0,\infty[\times \Omega, \mathbb{R}^2)$ such that for a. e. $t > 0$,

$$u_t = \operatorname{div} \boldsymbol{z} \quad \mathcal{L}^2\text{-a. e. in } \{t\} \times \Omega, \tag{2.9}$$

$$|\boldsymbol{z}|_\infty \leq 1 \quad \mathcal{L}^2\text{-a. e. in } \{t\} \times \Omega, \tag{2.10}$$

$$[\boldsymbol{z}, \boldsymbol{\nu}^\Omega] = 0 \quad \mathcal{H}^1\text{-a. e. on } \{t\} \times \partial\Omega \tag{2.11}$$

and

$$-\int_\Omega u(t,\cdot) \operatorname{div} \boldsymbol{z}(t,\cdot) \, \mathrm{d}\mathcal{L}^2 = \int_\Omega |Du(t,\cdot)|_1. \tag{2.12}$$

By Proposition (2.1) and the theory of semigroups generated by nonlinear maximal monotone operators due to Komura, Brezis and Crandall-Liggett [53, 14, 23], reasoning as in section 2.2 of [6] we obtain the following existence result.

Proposition 2.2. *Let $u_0 \in L^2(\Omega)$. There exists exactly one strong solution u to* (2.8) *such that $u(0,\cdot) = u_0$.*

Here we are concerned with evolution of regular (with respect to the operator $-\partial \mathrm{TV}_{1,\Omega}$) initial data. In such case, semigroup theory yields following result [8, Chapter III].

Proposition 2.3. *Let $u_0 \in \mathcal{D}(\partial \mathrm{TV}_{1,\Omega})$ and let u be the global strong solution to* (2.8) *starting with u_0. Then, every $\boldsymbol{z} \in L^\infty(]0,\infty[\times\Omega, \mathbb{R}^2)$ satisfying (2.9-2.12) has a representative (denoted henceforth $\boldsymbol{z}$) such that*

(1) in every $t \in [0,\infty[$, $\boldsymbol{z}(t,\cdot)$ minimizes

$$\mathcal{F}_\Omega(\boldsymbol{\xi}) = \int_\Omega (\operatorname{div}\boldsymbol{\xi})^2 \, \mathrm{d}\mathcal{L}^2$$

over $X_{\varphi,\Omega}(u(t,\cdot))$ and this condition uniquely defines $\operatorname{div}\boldsymbol{z}(t,\cdot)$*,*

(2) the function

$$[0,\infty[\ni t \mapsto \operatorname{div}\boldsymbol{z}(t,\cdot) \in L^2(\Omega) \quad \textit{is right-continuous,}$$

(3) the function

$$[0,\infty[\ni t \mapsto \|\operatorname{div}\boldsymbol{z}(t,\cdot)\|_{L^2(\Omega)} \quad \textit{is non-increasing,}$$

(4) the function $[0,\infty[\ni t \mapsto u(t,\cdot) \in L^2(\Omega)$ is right-differentiable and

$$\frac{\mathrm{d}}{\mathrm{d}t}^{+} u(t,\cdot) = \operatorname{div}\boldsymbol{z}(t,\cdot) \quad \textit{in every } t \in [0,\infty[.$$

2.1.7 Rectilinear polygons

We denote by $\mathcal{R}$ the set of closed rectangles in the plane whose sides are parallel to the coordinate axes, and by $\mathcal{I}$, the set of all horizontal and vertical closed line segments of finite length in the plane.

We call $F \subset \mathbb{R}^2$ a *rectilinear polygon* if $F = \bigcup \mathcal{R}_F$ with a finite $\mathcal{R}_F \subset \mathcal{R}$. We denote by $\mathcal{F}$ the family of all rectilinear polygons. Similarly, we call $C \subset \mathbb{R}^2$ a *rectilinear curve* if $C = \bigcup \mathcal{I}_C$ with a finite $\mathcal{I}_C \subset \mathcal{I}$. We denote by $\mathcal{C}$ the set of all rectilinear curves.

We call any finite set G of horizontal and vertical lines in the plane a *grid*. If F is a rectilinear polygon, we denote by $G(F)$ the minimal grid such that each side of F is contained in a line belonging to $G(F)$. If C is a rectilinear curve, we denote by $G(C)$ the minimal grid with the property that there exists $\mathcal{I}_C \subset \mathcal{I}$, $C = \bigcup \mathcal{I}_C$ such that all endpoints of intervals in $\mathcal{I}_C$ are vertices of $G(C)$.

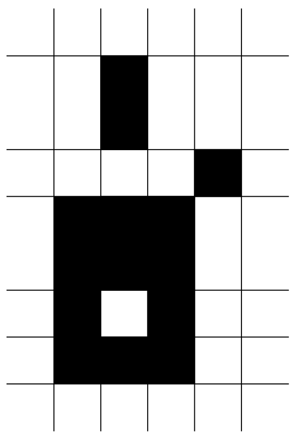


Figure 2.1: An example of a rectilinear polygon F and a grid G. There holds $G = G(F)$ and $F \in \mathcal{F}(G)$.

Given a grid G, we denote

- by $\mathcal{I}(G)$ the set of line segments connecting adjacent vertices of G,
- by $\mathcal{R}(G)$ the set of rectangles whose sides belong to $\mathcal{I}(G)$,
- by $\mathcal{F}(G)$ the set of rectilinear polygons of form $\bigcup \mathcal{R}_F$ with a finite non-empty $\mathcal{R}_F \subset \mathcal{R}(G)$.

Note that all of the above are finite sets.

It is also convenient to introduce the following notions of partitions of rectilinear polygons and signatures for their boundaries. Let Ω be a rectilinear polygon. We say that a finite family $\mathcal{Q}$ of rectilinear polygons with disjoint interiors is a partition of Ω if $\Omega = \bigcup \mathcal{Q}$. If G is a grid, we say that a partition $\mathcal{Q}$ of Ω is subordinate to G if $\mathcal{Q} \subset \mathcal{F}(G)$. Let F be a rectilinear polygon and let G be a grid. We say that $(\partial F^+, \partial F^-) \in \mathcal{C} \times \mathcal{C}$ is a *signature* for ∂F (or for F) if $\partial F^\pm \subset \partial F$ and $\mathcal{H}^1(\partial F^+ \cap \partial F^-) = 0$. We say that a signature $(\partial F^+, \partial F^-)$ for ∂F is subordinate to G if both $\partial F^\pm$ are subordinate to G.

Now, we give a precise definition of the class of functions piecewise constant on rectangles that we will work with. Let Ω be a rectangle and let $w \in L^1(\Omega)$. We write $w \in PCR(\Omega)$ if w has a finite number of level sets of positive $\mathcal{L}^2$ measure, and each one is a rectilinear polygon up to a $\mathcal{L}^2$-null set. We denote the family of level sets of a function $w \in PCR(\Omega)$ by $\mathcal{Q}_w$. $\mathcal{Q}_w$ is a partition of Ω in the sense of the definition in the previous paragraph. Similarly, we will denote by $Q_{(w_1,w_2)}$ the family of level sets of a vectorial function $(w_1, w_2) \in PCR(\Omega)^2$. If $Q \in \mathcal{Q}_w$ and $E \subset Q$ has positive $\mathcal{L}^2$ measure, we denote by $w(E)$ the value taken by w $\mathcal{L}^2$-a. e. on E. Given $w \in PCR(\Omega)$ and $Q \in \mathcal{Q}_w$, we say that $(\partial Q^+, \partial Q^-) \in \mathcal{C} \times \mathcal{C}$ is the *signature induced by w on ∂Q* if

$$\begin{aligned} x \in \partial Q^+ \quad &\text{iff} \quad x \in \partial Q',\ Q' \in \mathcal{Q}_w,\ w(Q') < w(Q), \\ x \in \partial Q^- \quad &\text{iff} \quad x \in \partial Q',\ Q' \in \mathcal{Q}_w,\ w(Q') > w(Q). \end{aligned}$$

Furthermore, we put $G_w = \bigcup_{Q \in \mathcal{Q}_w} G(Q)$, $\mathcal{I}_w = \mathcal{I}(G_w)$, $\mathcal{R}_w = \mathcal{R}(G_w)$, $\mathcal{F}_w = \mathcal{F}(G_w)$. Again, these are all finite sets.

The family $PCR(\Omega)$ is a linear subspace of $BV(\Omega)$. If $w \in PCR(\Omega)$, we have

$$J_w = \bigcup_{Q \in \mathcal{Q}_w} \partial Q^+ = \bigcup_{Q \in \mathcal{Q}_w} \partial Q^-, \qquad w^\pm = w(Q) \quad \mathcal{H}^1\text{-a. e. on } \partial Q^\pm,$$

where $Q \mapsto (\partial Q^+, \partial Q^-)$ is the signature induced by w. Furthermore, we have

$$|Dw|_1 = |Dw| = (w^+ - w^-)\mathcal{H}^1 \llcorner J_w = \sum_{Q \in \mathcal{Q}_w} w(Q)(\mathcal{H}^1 \llcorner \partial Q^+ - \mathcal{H}^1 \llcorner \partial Q^-). \tag{2.13}$$

In particular, if F is a rectilinear polygon, then

$$\mathrm{Per}_1(F, \Omega) = \mathrm{Per}(F, \Omega).$$

We have

Lemma 2.1. *Let Ω be a rectangle, $w \in PCR(\Omega)$ and suppose that $\boldsymbol{\xi} \in X_\Omega$ satisfies*

$$|\boldsymbol{\xi}|_\infty \le 1 \quad \mathcal{L}^2\text{-a. e. in } \Omega, \qquad [\boldsymbol{\xi}, \boldsymbol{\nu}^\Omega] = 0 \quad \mathcal{H}^1\text{-a. e. on } \partial\Omega. \tag{2.14}$$

Then, condition

$$-\int_{\Omega} w \operatorname{div} \boldsymbol{\xi} \, \mathrm{d}\mathcal{L}^2 = \int_{\Omega} |Dw|_1 \tag{2.15}$$

is equivalent to

$$[\boldsymbol{\xi}, \boldsymbol{\nu}^Q] = \mp 1 \quad \mathcal{H}^1\text{-a. e. on } \partial Q^{\pm} \tag{2.16}$$

for each $Q \in \mathcal{Q}_w$, where $\partial Q^{\pm}$ is the signature induced on ∂Q by w.

Furthermore, $X_{1,\Omega}(w)$ is non-empty (equivalently, $w \in D(\partial \mathrm{TV}_{1,\Omega})$).

Proof. First, suppose that $\boldsymbol{\xi} \in X_\Omega$ satisfies $[\boldsymbol{\xi}, \boldsymbol{\nu}^\Omega] = 0$ $\mathcal{H}^1$-a. e. on $\partial\Omega$ and (2.16). Then, using definition of $[\boldsymbol{\xi}, \boldsymbol{\nu}^Q]$ and (2.13), we calculate

$$-\int_{\Omega} w \operatorname{div} \boldsymbol{\xi} \, \mathrm{d}\mathcal{L}^2 = -\sum_{Q \in Q_w} w(Q) \int_{\partial Q} [\boldsymbol{\xi}, \boldsymbol{\nu}^Q] \, \mathrm{d}\mathcal{H}^1 = \int_{J_w} (w^+ - w^-) \, \mathrm{d}\mathcal{H}^1 = \int_{\Omega} |Dw|_1.$$

On the other hand, suppose that $\boldsymbol{\xi} \in X_\Omega$ satisfies (2.14) and (2.15). Then, integrating by parts in each $Q \in \mathcal{Q}_w$ on the l. h. s. of (2.15) and using (2.13) we get

$$\sum_{Q \in \mathcal{Q}_w} w(Q) \int_{\partial Q} [\boldsymbol{\xi}, \boldsymbol{\nu}^Q] \, \mathrm{d}\mathcal{H}^1 = \int_{J_w} (w^+ - w^-) \, \mathrm{d}\mathcal{H}^1.$$

The l. h. s. can be further rewritten as

$$\int_{J_w} \left([\boldsymbol{\xi}, \boldsymbol{\nu}^{Q_{w^+}}] w^+ + [\boldsymbol{\xi}, \boldsymbol{\nu}^{Q_{w^-}}] w^- \right) \, \mathrm{d}\mathcal{H}^1,$$

where $Q_{w^\pm} = w^{-1}(w^\pm)$. Let us briefly verify that

$$[\boldsymbol{\xi}, \boldsymbol{\nu}^{Q_{w^+}}] = -[\boldsymbol{\xi}, \boldsymbol{\nu}^{Q_{w^-}}] \quad \mathcal{H}^1\text{-a. e. on } \partial Q_{w^+} \cap \partial Q_{w^-}.$$

Indeed, for any line segment $I \subset \partial Q_{w^+} \cap \partial Q_{w^-}$ without endpoints and any $\check{\rho} \in C_c^1(I)$, there exists an extension $\rho \in C_c^1(\operatorname{int}(Q_{w^+} \cup Q_{w^-}))$ of $\check{\rho}$. Hence, by (2.3),

$$\begin{aligned}
&\int_I \left([\boldsymbol{\xi}, \boldsymbol{\nu}^{Q_{w^+}}] + [\boldsymbol{\xi}, \boldsymbol{\nu}^{Q_{w^-}}] \right) \check{\rho} \, \mathrm{d}\mathcal{H}^1 \\
&= \int_{Q_{w^+}} \rho \operatorname{div} \boldsymbol{\xi} \, \mathrm{d}\mathcal{L}^2 + \int_{Q_{w^+}} \boldsymbol{\xi} \cdot \nabla\rho \, \mathrm{d}\mathcal{L}^2 + \int_{Q_{w^-}} \rho \operatorname{div} \boldsymbol{\xi} \, \mathrm{d}\mathcal{L}^2 + \int_{Q_{w^-}} \boldsymbol{\xi} \cdot \nabla\rho \, \mathrm{d}\mathcal{L}^2 = \int_{\Omega} \operatorname{div}(\rho \boldsymbol{\xi}) \, \mathrm{d}\mathcal{L}^2 = 0.
\end{aligned}$$

Thus, we have

$$\int_{J_w} [\boldsymbol{\xi}, \boldsymbol{\nu}^{Q_{w^+}}] (w^+ - w^-) \, \mathrm{d}\mathcal{H}^1 = \int_{J_w} (w^+ - w^-) \, \mathrm{d}\mathcal{H}^1.$$

Together with the condition $|\boldsymbol{\xi}|_\infty \leq 1$ and (2.4), this implies (2.16).

One way to construct a field belonging to $X_{1,\Omega}(w)$ is to extend it into each Q from ∂Q, where its normal component is fixed by (2.16) or the Neumann condition, by component-wise linear interpolation. □

It follows from Lemma 2.1, that condition (2.12) in Definition 2.1 can be replaced with

$$[\boldsymbol{z}(t, \cdot), \boldsymbol{\nu}^Q] \Big|_{\partial Q^{\pm}} = \mp 1 \quad \text{for all } Q \in Q_{u(t,\cdot)} \tag{2.17}$$

(where $\partial Q^\pm$ is the signature induced on ∂Q by $u(t, \cdot)$) whenever $u(t, \cdot) \in PCR(\Omega)$.

2.2 Cheeger problems in rectilinear geometry

Let G be a grid. Let F_0 be a rectilinear polygon and let $(\partial F_0^+, \partial F_0^-)$ be a signature for ∂F_0, all subordinate to G.

We introduce a functional $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-}$ with values in $]-\infty,+\infty]$ defined on subsets of F_0 of positive area given by

$$\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-}(E) = \frac{\mathrm{Per}_1(E, \operatorname{int} F_0) + \mathcal{H}^1(\partial^* E \cap \partial F_0^+) - \mathcal{H}^1(\partial^* E \cap \partial F_0^-)}{\mathcal{L}^2(E)},$$

if E has finite perimeter and $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-}(E) = +\infty$ otherwise. Note that for each measurable $E \subset F_0$ of positive area and finite perimeter, we have

$$\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-}(E) = \frac{\mathrm{Per}_1(E) - \mathcal{H}^1(\partial^* E \cap \partial F_0 \setminus \partial F_0^+) - \mathcal{H}^1(\partial^* E \cap \partial F_0^-)\,\mathrm{d}\mathcal{L}^2}{\mathcal{L}^2(E)}.$$

Lemma 2.2. *Let $E \subset F_0$ be a set of finite perimeter with $\mathcal{L}^2(E) > 0$. Then for every $\varepsilon > 0$ there exists a rectilinear polygon $F \in \mathcal{F}(G)$ such that*

$$\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-}(F) < \mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-}(E) + \varepsilon.$$

Proof. Throughout the proof, we write for short $\mathcal{J} = \mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-}$.

Step 1. Smoothing

First, given $\varepsilon > 0$, we obtain a smooth closed set $\widetilde{E} \subset F$ such that $\mathcal{J}(\widetilde{E}) \le \mathcal{J}(E) + \varepsilon$ and $\widetilde{E}$ does not contain any vertices of F_0. For this purpose, we adapt the standard method of smooth approximation of sets of finite perimeter. Namely, we consider superlevels of smooth functions $\psi_\delta * \chi_E$, $\delta > 0$. Here, ψ_δ is a standard smooth approximation of unity. Using Sard's lemma on regular values of smooth functions and the coarea formula for anisotropic total variation [4, Remark 4.4], we obtain, reasoning as in the proof of [45, Theorem 1.24], a number $0 < t < \frac{1}{2}$ and a sequence $\delta_j \to 0^+$ such that

$$\widetilde{E}_j = \{\psi_{\delta_j} * \chi_E \ge t\}$$

is a smooth set for each $j = 1, 2, \ldots$ and

$$\mathcal{L}^2(\widetilde{E}_j \triangle E) \to 0, \qquad \liminf_{j\to\infty} \mathrm{Per}_1(\widetilde{E}_j) = \mathrm{Per}_1(E), \qquad \mathcal{H}^1((\partial^* E) \setminus \widetilde{E}_j) \to 0. \tag{2.18}$$

Here and in the following we denoted by $\triangle$ the symmetric difference. The first two items in (2.18) are covered explicitly in [45]. It remains to justify the last one. Since $\partial^* E \subset \partial^{\frac{1}{2}} E$, for each $\boldsymbol{x} \in \partial^* E$ there is a natural number j_0 such that for every $j > j_0$ there holds $\boldsymbol{x} \in \widetilde{E}_j$. Thus, as $\mathcal{H}^1(\partial^* E) = \mathrm{Per}(E)$ is finite, the assertion follows by continuity of measures.

Perturbing each $\widetilde{E}_j$ a little, we can require that $\partial \widetilde{E}_j$ is transverse to every line in G. Then, $\partial(F_0 \cap \widetilde{E}_j)$ are piecewise smooth curves and it is visible that all items in (2.18) remain true if we substitute $F_0 \cap \widetilde{E}_j$ for $\widetilde{E}_j$. Therefore, for any given $\varepsilon' > 0$ we choose a number j such that

$$\begin{aligned} \mathcal{L}^2(F_0 \cap \widetilde{E}_j) > \mathcal{L}^2(E) - \varepsilon', \qquad \mathrm{Per}_1(F_0 \cap \widetilde{E}_j) < \mathrm{Per}_1(E) + \varepsilon', \\ \mathcal{H}^1(\partial(\widetilde{E}_j \cap F) \cap \partial F_0 \setminus \partial F_0^+) > \mathcal{H}^1(\partial^* E \cap \partial F_0 \setminus \partial F_0^+) - \varepsilon' \\ \text{and} \quad \mathcal{H}^1(\partial(\widetilde{E}_j \cap F) \cap \partial F_0^-) > \mathcal{H}^1(\partial^* E \cap \partial F_0^-) - \varepsilon'. \end{aligned} \tag{2.19}$$

Taking ε' small enough we obtain

$$\mathcal{J}(F_0 \cap \widetilde{E}_j) < \mathcal{J}(E) + \varepsilon. \tag{2.20}$$

Due to transversality, there is at most a finite number of points where the piecewise smooth curve $\partial(F_0 \cap \widetilde{E}_j)$ is not infinitely differentiable. Thus, we can smooth out the set $F_0 \cap \widetilde{E}_j$ in such a way that (2.19), and consequently (2.20), still hold. We denote the resulting set by $\widetilde{E}$. Possibly adjusting $\widetilde{E}$ slighty, we can require that it does not contain any vertices of F_0.

Step 2. Squaring
For each $\boldsymbol{x} \in \partial\widetilde{E}$ there is an open square $U(\boldsymbol{x}) = I(\boldsymbol{x}) \times J(\boldsymbol{x})$ such that $\widetilde{E} \cap U(\boldsymbol{x})$ coincides with the subgraph of a smooth function $g\colon I(\boldsymbol{x}) \to J(\boldsymbol{x})$ or $g\colon J(\boldsymbol{x}) \to I(\boldsymbol{x})$ and that $U(\boldsymbol{x})$ intersects at most one edge of F_0 (contained in the supergraph of g). The family $\{U(\boldsymbol{x})\colon \boldsymbol{x} \in \partial\widetilde{E}\}$ is an open cover of $\partial\widetilde{E}$. We extract a finite cover $\{U_1, \ldots, U_l\}$ out of it. We assume that $\{U_1, \ldots, U_l\}$ is minimal in the sense that none of its proper subsets covers $\partial\widetilde{E}$. Let us take $\widehat{E}_0 = \widetilde{E} \cup \bigcup_{i=1}^{l} W_i$, where $W_i \subset F_0$ is the smallest closed rectangle containing $U_i \cap \widetilde{E}$. The operation of taking a union of $\widetilde{E}$ with W_1 increases volume while not increasing l^1-perimeter. Indeed, denoting $U_1 =]a_1, b_1[\times]a_2, b_2[$ and assuming without loss of generality that $\widetilde{E} \cap U_1$ coincides with the subgraph of a smooth function $g_1\colon]a_1, b_1[\to]a_2, b_2[$, we have

$$\int_{W_1 \cap \partial\widetilde{E}} |\boldsymbol{\nu}^{\widetilde{E}}|_1 \, \mathrm{d}\mathcal{H}^1 = \int_{]a_1, b_1[} 1 + |g_1'| \, \mathrm{d}\mathcal{L}^1 \geq |\sup g_1 - g_1(a_1)| + |b_1 - a_1| + |\sup g_1 - g_1(b_1)|$$

and consequently

$$\mathrm{Per}_1(\widetilde{E}) = \mathcal{H}^1(\partial\widetilde{E} \setminus W_1) + \int_{W_1 \cap \partial\widetilde{E}} |\boldsymbol{\nu}^{\widetilde{E}}|_1 \, \mathrm{d}\mathcal{H}^1 \geq \mathrm{Per}_1(\widetilde{E} \cup W_1).$$

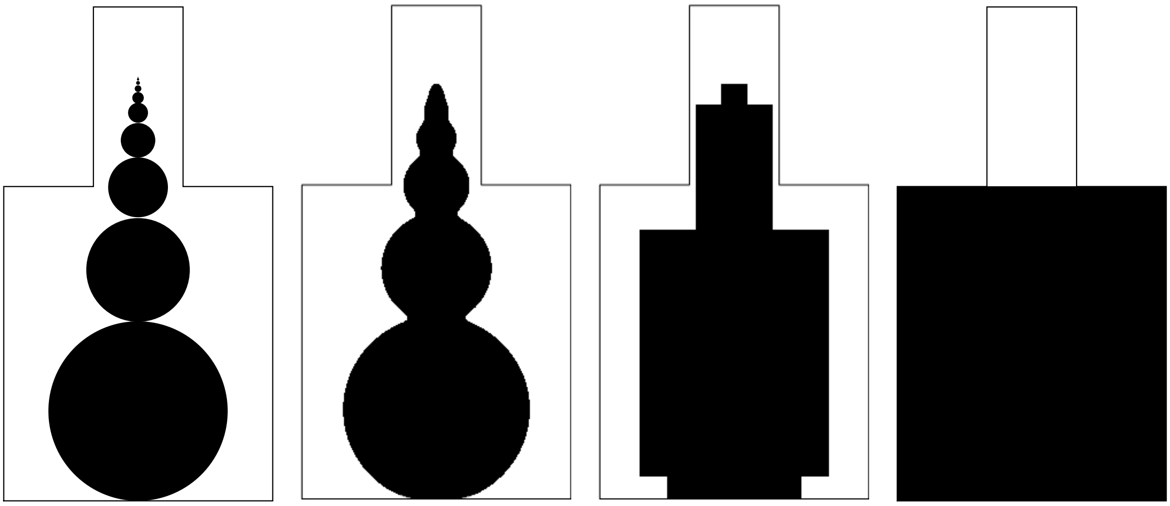
Similarly, we show that taking the union of $\widetilde{E} \cup W_1$ with W_2 does not increase the perimeter, and so on. Furthermore, clearly $\partial\widetilde{E}_0 \cap \partial F_0 \setminus \partial F_0^+ \subset \partial\widehat{E}_0 \cap \partial F_0 \setminus \partial F_0^+$ and $\partial\widetilde{E}_0 \cap \partial F_0^- \subset \partial\widehat{E}_0 \cap \partial F_0^-$. Summing up, we have

$$\begin{aligned} &\mathcal{L}^2(\widehat{E}_0) \geq \mathcal{L}^2(\widetilde{E}), \qquad \mathrm{Per}_1(\widehat{E}_0) \leq \mathrm{Per}_1(\widetilde{E}), \\ &\quad \mathcal{H}^1(\partial\widehat{E}_0 \cap \partial F_0 \setminus \partial F_0^+) \geq \mathcal{H}^1(\partial\widetilde{E} \cap \partial F_0 \setminus \partial F_0^+), \qquad \mathcal{H}^1(\partial\widehat{E}_0 \cap \partial F_0^-) \geq \mathcal{H}^1(\partial\widetilde{E} \cap \partial F_0^-). \end{aligned} \tag{2.21}$$

Thus, $\mathcal{J}(\widehat{E}_0) < \mathcal{J}(E) + \varepsilon$ holds.

Step 3. Aligning
Take any line $L_0 \subset G(\widehat{E}_0)$ that is not contained in G. We assume for clarity that L is horizontal, i. e. $L = \mathbb{R} \times \{y_0\}$, $y_0 \in \mathbb{R}$. We denote $L_0 \cap \partial\widehat{E}_0 = C_0 \times \{y_0\}$ and observe that $C_0 \subset \mathbb{R}$ necessarily contains an interval. Let $L_+ = \mathbb{R} \times \{y_+\}$ and $L_- = \mathbb{R} \times \{y_-\}$ be the lines in $G \cup G(\widehat{E}_0)$ situated above and below L_0 closest to L_0. We have $C_0 \times [y_-, y_+] \subset F_0$. Let us first assume that $\widehat{E}_0 \neq C_0 \times [y_0, y_+]$, $\widehat{E}_0 \neq C_0 \times [y_-, y_0]$. For $y \in [y_-, y_+]$, we define

$$\overline{\mathcal{J}}(y) = \begin{cases} \mathcal{J}(\widehat{E}_0 \triangle (C_0 \times [y_0, y])) & \text{if } y > y_0, \\ \mathcal{J}(\widehat{E}_0 \triangle (C_0 \times [y, y_0])) & \text{otherwise.} \end{cases} \tag{2.22}$$



(a) A BV set E. (b) Smoothed set $\widetilde{E}$. (c) Squared set $\widehat{E}_0$. (d) Aligned set F.

Figure 2.2: The construction in Lemma 2.2 applied to an example set of finite perimeter contained in a rectilinear polygon (in case $f = 0$).

It follows from the choice of L_0 and $L_\pm$, that each one of line segments constituting $\partial C_0 \times [y_-, y_+]$ is contained (up to a finite number of points) in either $\partial F_+ \cup \operatorname{int} F$, ∂F_- or $\partial F \setminus (\partial F^+ \cup \partial F^-)$. Therefore, $\overline{\mathcal{J}}$ is a homographic function and hence monotone on $]y_-, y_+[$.

However, $\overline{\mathcal{J}}$ might be discontinuous at the endpoints of its domain. This is only possible if, as y attains y_+ (or y_-), a pair of edges of $\widehat{E}_0 \triangle (C_0 \times [y_0, y])$ (resp. $\widehat{E}_0 \triangle (C_0 \times [y, y_0])$) vanishes, or an edge touches the boundary of F_0. In either case, there still holds $\lim_{y \to y_\pm} \overline{\mathcal{J}}(y) \geq \overline{\mathcal{J}}(y_\pm)$.

Thus, whether $\overline{\mathcal{J}}$ is continuous or not, either $\overline{\mathcal{J}}(y_+)$ or $\overline{\mathcal{J}}(y_-)$ (or both) is not larger than $\overline{\mathcal{J}}(y_0) = \mathcal{J}(\widehat{E}_0)$. In accordance with that, we denote either $\widehat{E}_1 = \widehat{E}_0 \triangle (C_0 \times [y_0, y_+])$ or $\widehat{E}_1 = \widehat{E}_0 \triangle (C_0 \times [y_-, y_0])$ and perform the same argument with $\widehat{E}_1$ instead of $\widehat{E}_0$.

Now, let us go back to the excluded cases and suppose, without loss of generality, that $\widehat{E}_0 = C_0 \times [y_0, y_+]$. Then, $\overline{\mathcal{J}}$ is still a well-defined homography in $[y_-, y_+[$ and $\lim_{y \to y_+} \overline{\mathcal{J}}(y) = +\infty$. Hence, $\overline{\mathcal{J}}(y_-) \leq \overline{\mathcal{J}}(y_0) = \mathcal{J}(\widehat{E}_0)$ and we put $\widehat{E}_1 = \widehat{E}_0 \cup (C_0 \times [y_-, y_0]) = C_0 \times [y_-, y_+]$ and continue the procedure.

For each i, $G(\widehat{E}_{i+1})$ contains at least one line not contained in G less than $G(\widehat{E}_i)$, so this procedure terminates in a finite number s of steps and we obtain $F = \widehat{E}_s$ whose all edges are contained in G and $\mathcal{J}(F) \leq \mathcal{J}(\widehat{E}_0) < \mathcal{J}(E) + \varepsilon$. $\square$

Theorem 2.1. *The functional $\mathcal{J}_{F_0, \partial F_0^+, \partial F_0^-}$ is bounded from below and is minimized by a rectilinear polygon $F \subset F_0$ such that $F \in \mathcal{F}(G)$.*

Proof. Suppose that $\mathcal{J}_{F_0, \partial F_0^+, \partial F_0^-}(E_n) \to -\infty$. Then, due to Lemma 2.2 there exist rectilinear polygons $F_n \subset F_0$, $n = 1, 2, \ldots$ such that $F_n \in \mathcal{F}(G)$ and $\mathcal{J}_{F_0, \partial F_0^+, \partial F_0^-}(F_n) \to -\infty$, an impossibility.

Now, consider any minimizing sequence (E_n) of $\mathcal{J}_{F_0, \partial F_0^+, \partial F_0^-}$. By means of Lemma 2.2 we find a minimizing sequence of rectilinear polygons $F_n \in F_0$ such that $F \in \mathcal{F}(G)$. As the set of such rectilinear polygons is finite, (F_n) has a constant subsequence $(F_{n_k}) \equiv (F)$. Clearly, F minimizes $\mathcal{J}_{F_0, \partial F_0^+, \partial F_0^-}$. $\square$

Instead of $\mathcal{J}_{F_0,\partial F_0^+,\partial F_0^-}$ we can consider

$$\check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-}(E)\colon \{E \subset F_0 \text{ — measurable s. t. } \mathcal{L}^2(E) > 0\} \to [-\infty, +\infty[$$

defined by

$$\check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-}(E) = \frac{-\mathrm{Per}_1(E, \operatorname{int} F_0) + \mathcal{H}^1(\partial^* E \cap \partial F_0^+) - \mathcal{H}^1(\partial^* E \cap \partial F_0^-)}{\mathcal{L}^2(E)},$$

if E has finite perimeter and $-\infty$ otherwise. Then, noticing that

$$\check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-} = -\mathcal{J}_{F_0,\partial F_0^-,\partial F_0^+}$$

we obtain analogous versions of Lemma 2.2 and Theorem 2.1.

Lemma 2.2'. *Let $E \subset F_0$ be a set of finite perimeter with $\mathcal{L}^2(E) > 0$. Then for every $\varepsilon > 0$ there exists a rectilinear polygon $F \in \mathcal{F}(G)$ such that*

$$\check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-}(F) > \check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-}(E) - \varepsilon.$$

Theorem 2.1'. *The functional $\check{\mathcal{J}}_{F_0,\partial F_0^+,\partial F_0^-}$ is bounded from above and is maximized by a rectilinear polygon $F \subset F_0$ such that $F \in \mathcal{F}(G)$.*

2.3 The $\mathrm{TV}_{1,\Omega}$ flow with PCR initial datum

In what now follows, we are concerned with the identification of the evolution of initial datum $w \in PCR(\Omega)$ under (2.8). The result below determines the initial evolution, prescribing possible breaking of initial facets.

Lemma 2.3. *Let $w \in PCR(\Omega)$. There exists a vector field $\boldsymbol{\eta} \in X_{1,\Omega}(w)$ such that $\operatorname{div} \boldsymbol{\eta} \in PCR(\Omega)$ and $G_{\operatorname{div} \boldsymbol{\eta}} \subset G_w$. Furthermore, if $F_0, F_1 \in \mathcal{Q}_{(w,\operatorname{div} \boldsymbol{\eta})}$, then*

$$\begin{aligned}
[\boldsymbol{\eta}, \boldsymbol{\nu}^{F_0}]\Big|_{\partial F_0 \cap \partial F_1} &= -1 \quad \textit{if} \quad w(F_1) < w(F_0) \quad \textit{or} \quad w(F_1) = w(F_0) \textit{ and } \operatorname{div} \boldsymbol{\eta}(F_1) < \operatorname{div} \boldsymbol{\eta}(F_0),\\
[\boldsymbol{\eta}, \boldsymbol{\nu}^{F_0}]\Big|_{\partial F_0 \cap \partial F_1} &= 1 \quad \textit{if} \quad w(F_1) > w(F_0) \quad \textit{or} \quad w(F_1) = w(F_0) \textit{ and } \operatorname{div} \boldsymbol{\eta}(F_1) > \operatorname{div} \boldsymbol{\eta}(F_0).
\end{aligned} \tag{2.23}$$

Proof. We fix $Q \in \mathcal{Q}_w$ and produce a partition $\mathcal{T}_Q$ of Q (that will correspond to level sets of $\operatorname{div} \boldsymbol{\eta}|_Q$) by means of an inductive procedure. Let $(\partial Q^-, \partial Q^+)$ be the signature for ∂Q given by w. By virtue of Theorem 2.1, the functional $\mathcal{J}_{Q,\partial Q^+,\partial Q^-}$ attains its minimum value on a rectilinear polygon $F_1 \in \mathcal{F}(Q)$. We define

$$\partial F_1^- = \partial Q^- \cap \partial F_1 \quad \text{and} \quad \partial F_1^+ = (\partial F_1 \cap \partial Q^+) \cup (\partial F_1 \setminus \partial Q).$$

Next, in k-th step, we put $\check{F}_k = \bigcup_{j=1}^{k-1} F_j$. If $\check{F}_k = Q$ we stop and put $\mathcal{T}_Q = \{F_1, \dots, F_{k-1}\}$. Otherwise we define F_k as any minimizer of

$$\mathcal{J}_{Q\setminus \check{F}_k,\partial Q^+,\partial Q^- \cup \partial \check{F}_k},$$

and

$$\partial F_k^- = \partial F_k \cap \left(\partial Q^- \cup \partial \check{F}_k\right),$$

$$\partial F_k^+ = (\partial F_k \cap \partial Q^+) \cup \left(\partial F_k \setminus \left(\partial Q \cup \partial \check{F}_k\right)\right).$$

$\partial F_k^\pm$ are defined in such a way that each $\mathcal{J}_{F_k,\partial F_k^+,\partial F_k^-}$ is the restriction of $\mathcal{J}_{Q\setminus\check{F}_k,\partial Q^+,\partial Q^-\cup\partial\check{F}_k}$ to subsets of F_k. In particular, F_k is a minimizer of $\mathcal{J}_{F_k,\partial F_k^+,\partial F_k^-}$.

Now, for each $F_k \in \mathcal{T}_Q$, we consider the functional $\mathcal{F}_k$ defined on the set of vector fields $\boldsymbol{\xi} \in L^\infty(F_k, \mathbb{R}^2)$ satisfying

$$\operatorname{div}\boldsymbol{\xi} \in L^2(F_k), \qquad |\boldsymbol{\xi}|_\infty \le 1 \quad \mathcal{L}^2\text{-a. e. on } F_k, \qquad [\boldsymbol{\xi}, \boldsymbol{\nu}^{F_k}]\Big|_{\partial F_k^\pm} = \mp 1, \qquad [\boldsymbol{\xi}, \boldsymbol{\nu}^{F_k}]\Big|_{\partial F_k \cap \partial\Omega} = 0$$

by

$$\mathcal{F}_k(\boldsymbol{\xi}) = \int_{F_k} (\operatorname{div}\boldsymbol{\xi})^2 \,\mathrm{d}\mathcal{L}^2.$$

Proceeding as in [11, Proposition 6.1], we see that $\mathcal{F}_k$ attains a minimum and for any two minimizers $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2$ we have $\operatorname{div}\boldsymbol{\eta}_1 = \operatorname{div}\boldsymbol{\eta}_2$ in F_k. Let us take any minimizer and denote it $\boldsymbol{\eta}_{F_k}$. We now adapt the reasoning in the proof of Theorem 5 in [9] in order to prove that $\operatorname{div}\boldsymbol{\eta}_{F_k}$ is constant. Arguing as in [11, Theorem 6.7] and [12, Theorem 5.3], $\operatorname{div}\boldsymbol{\eta}_{F_k} \in L^\infty(F_k) \cap BV(F_k)$. Let $\nu = \mathcal{J}_{F_k,\partial F_k^+,\partial F_k^-}(F_k)$. By definition of $\partial F_k^\pm$, we have

$$\frac{1}{\mathcal{L}^2(F_k)} \int_{F_k} \operatorname{div}\boldsymbol{\xi}_{F_k} \,\mathrm{d}\mathcal{L}^2 = -\frac{1}{\mathcal{L}^2(F_k)} \left(\mathcal{H}^1(\partial F_k^+) - \mathcal{H}^1(\partial F_k^-)\right) = -\nu.$$

Were $\operatorname{div}\boldsymbol{\eta}_{F_k}$ not constant in F_k, there would exist $\mu < \nu$ such that

$$A_\mu = \left\{\boldsymbol{x} \in F_k \colon\ -\operatorname{div}\boldsymbol{\eta}_{F_k}(\boldsymbol{x}) < \mu\right\}$$

has positive measure and finite perimeter. By virtue of [12, Proposition 3.5],

$$\begin{aligned} &-\nu < -\mu < \frac{1}{\mathcal{L}^2(A_\mu)} \int_{A_\mu} \operatorname{div}\boldsymbol{\eta}_{F_k} \,\mathrm{d}\mathcal{L}^2 \\ &= -\frac{1}{\mathcal{L}^2(A_\mu)} \left(\mathrm{Per}_1(A_\mu, F_k) + \mathcal{H}^1(\partial F_k^+ \cap \partial^* A_\mu) - \mathcal{H}^1(\partial F_k^- \cap \partial^* A_\mu)\right) = -\mathcal{J}_{F_k,\partial F_k^+,\partial F_k^-}(A_\mu), \end{aligned}$$

which would contradict that F_k minimizes $\mathcal{J}_{F_k,\partial F_k^+,\partial F_k^-}$, hence $\operatorname{div}\boldsymbol{\eta}_{F_k}$ is constant in F_k and therefore equal to its mean value:

$$\operatorname{div}\boldsymbol{\eta}_{F_k} = -\frac{1}{\mathcal{L}^2(F_k)} (\mathcal{H}^1(\partial F_k^+) - \mathcal{H}^1(\partial F_k^-)).$$

Next, we repeat the procedure for the rest of $Q \in \mathcal{Q}_w$ and define $\boldsymbol{\eta}$ by $\boldsymbol{\eta}|_{F_k} = \boldsymbol{\eta}_{F_k}$ for every $F_k \in \mathcal{T}_Q$, $Q \in \mathcal{Q}_w$. It is easy to check that $\boldsymbol{\eta}$ satisfies our hypotheses. □

Remark. Instead of considering the minimization problem for $\mathcal{J}$, one can consider at each step the maximization problem for $\check{\mathcal{J}}$ (see Theorem 2.1').

We are now in position to prove Theorem 1.1. We assume that u_0 is not constant. By Lemma 2.3, there exists a vector field $\boldsymbol{z}_0 \in X_{1,\Omega}(u_0)$ such that $\operatorname{div} \boldsymbol{z}_0 \in PCR(\Omega)$, $G_{\operatorname{div} \boldsymbol{z}_0} \subset G_{u_0}$ and

$$\begin{aligned}
[\boldsymbol{z}_0, \boldsymbol{\nu}^F]\Big|_{\partial F \cap \partial F'} &= -1 \quad \text{if} \quad u_0(F') < u_0(F) \quad \text{or} \quad u_0(F') = u_0(F) \text{ and } \operatorname{div} \boldsymbol{z}_0(F') < \operatorname{div} \boldsymbol{z}_0(F), \\
[\boldsymbol{z}_0, \boldsymbol{\nu}^F]\Big|_{\partial F \cap \partial F'} &= 1 \quad \text{if} \quad u_0(F') > u_0(F) \quad \text{or} \quad u_0(F') = u_0(F) \text{ and } \operatorname{div} \boldsymbol{z}_0(F') > \operatorname{div} \boldsymbol{z}_0(F).
\end{aligned} \tag{2.24}$$

for $F, F' \in \mathcal{Q}_{(u_0, \operatorname{div} \boldsymbol{z}_0)}$. We define $\widetilde{\boldsymbol{z}} \colon [0,\infty[\times \Omega \to \mathbb{R}^2$ by $\widetilde{\boldsymbol{z}}(t,\cdot) = \boldsymbol{z}_0$ for $t \geq 0$ and $\widetilde{u} \colon [0,\infty[\times\Omega \to \mathcal{R}$ by

$$\widetilde{u}(0,\cdot) = u_0, \quad \widetilde{u}_t(t,\cdot) = \operatorname{div} \boldsymbol{z}_0 \text{ for } t > 0.$$

The function

$$[0,\infty[\ni t \mapsto \widetilde{u}(t,\cdot) \in BV(\Omega)$$

is affine, in particular it is (strongly) continuous. Clearly, $\widetilde{u}(t,\cdot) \in PCR(\Omega)$ and $G(\widetilde{u}(t,\cdot)) \subset G(u_0, \operatorname{div} \boldsymbol{z}_0)$ for all $t > 0$. Furthermore, given $F, F' \in \mathcal{Q}_{(u_0, \operatorname{div} \boldsymbol{z}_0)}$,

$$\begin{aligned}
&\text{if} \quad u_0(F') < u_0(F) \quad \text{or} \quad u_0(F') = u_0(F) \text{ and } \operatorname{div} \boldsymbol{z}_0(F') < \operatorname{div} \boldsymbol{z}_0(F) \quad \text{then } \widetilde{u}(t,F') \leq \widetilde{u}(t,F), \\
&\text{if} \quad u_0(F') > u_0(F) \quad \text{or} \quad u_0(F') = u_0(F) \text{ and } \operatorname{div} \boldsymbol{z}_0(F') > \operatorname{div} \boldsymbol{z}_0(F) \quad \text{then } \widetilde{u}(t,F') \geq \widetilde{u}(t,F)
\end{aligned} \tag{2.25}$$

for small enough $t > 0$. Let t_1 denote the latest time instance such that (2.25) holds for all $t \in [0, t_1]$ and all $F, F' \in \mathcal{Q}_{(u_0, \operatorname{div} \boldsymbol{z}_0)}$ satisfying $\mathcal{H}^1(\partial F \cap \partial F') > 0$. By virtue of (2.24), Lemma 2.1 and uniqueness of strong solution to (2.8) with initial datum u_0, we obtain $u(t,\cdot) = \widetilde{u}(t,\cdot)$ for $t \in [0, t_1]$. If $u(t_1,\cdot)$ is constant, we have $u(t,\cdot) = u(t_1,\cdot)$ for all $t > t_1$ and the proof is finished. Otherwise, we repeat this reasoning with $u(t_1,\cdot)$ in place of u_0. By iterating this procedure k times, we obtain a sequence of time instances $0 = t_0 < t_1 < \ldots < t_k$ and vector fields $\boldsymbol{z}_0 \in X_{1,\Omega}(u(t_0,\cdot)), \ldots, \boldsymbol{z}_{k-1} \in X_{1,\Omega}(u(t_{k-1},\cdot))$ such that

$$u_t(t,\cdot) = \operatorname{div} \boldsymbol{z}_j(t,\cdot) \in PCR(\Omega) \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in]t_j, t_{j+1}[.$$

Now, let us prove that this procedure terminates after a finite number of steps. For this purpose, we rely on Theorem 2.3. In fact, we prove that there exists a constant $\gamma = \gamma(G) > 0$ such that at each t_j, $j = 1, \ldots, k$ the non-increasing function $t \mapsto \|\operatorname{div} \boldsymbol{z}(t,\cdot)\|_{L^2(\Omega)}$ has a jump of size at least γ. To avoid confusion, we recall that the function $\operatorname{div} \boldsymbol{z} \in L^2(]0,\infty[\times\Omega)$ is uniquely defined for a given strong solution u to (2.8). This however is not necessarily the case for $\boldsymbol{z}$ itself. Although this non-uniqueness does not matter, for mental comfort we can adjust any given $\boldsymbol{z}$ so that

$$\boldsymbol{z}(t,\cdot) = \boldsymbol{z}_j \quad \text{for } \mathcal{L}^1\text{-a.e. } t \in]t_j, t_{j+1}[,$$

$j = 0, \ldots, k-1$.

First, we argue that $\|\operatorname{div} \boldsymbol{z}_j\|_{L^2(\Omega)} < \|\operatorname{div} \boldsymbol{z}_{j-1}\|$ for $j = 0, \ldots, k-1$. We will reason by contradiction. If $\|\operatorname{div} \boldsymbol{z}_j\|_{L^2(\Omega)} = \|\operatorname{div} \boldsymbol{z}_{j-1}\|_{L^2(\Omega)}$, then $\boldsymbol{z}_{j-1}$ is a minimizer of $\mathcal{F}_\Omega$ in $X_{1,\Omega}(u(t_j,\cdot))$ and, by virtue of Theorem 2.3,

$$\operatorname{div} \boldsymbol{z}_{j-1} = \operatorname{div} \boldsymbol{z}_j = \operatorname{div} \boldsymbol{z}(t,\cdot) \quad \text{for } t \in [t_{j-1}, t_{j+1}[. \tag{2.26}$$

According to Lemma 2.1, the minimization problem for $\mathcal{F}_\Omega$ in $X_{1,\Omega}(u(t_j,\cdot))$ is equivalent to minimization of functionals $\mathcal{F}_{Q,\partial Q^-,\partial Q^+}$ defined by

$$\mathcal{F}_{Q,\partial Q^-,\partial Q^+}(\boldsymbol{\eta}) = \int_Q (\operatorname{div} \boldsymbol{\eta})^2 \, \mathrm{d}\mathcal{L}^2$$

on the set of vector fields

$$\boldsymbol{\eta} \in X_Q = \{\boldsymbol{\eta} \in L^\infty(Q, \mathbb{R}^2) \colon \operatorname{div} \boldsymbol{\eta} \in L^2(Q)\}$$

satisfying

$$|\boldsymbol{\eta}|_\infty \leq 1 \ \mathcal{L}^2\text{-a. e. in } Q, \quad [\boldsymbol{\eta}, \boldsymbol{\nu}^Q] = \mp 1 \ \mathcal{H}^1\text{-a. e. on } \partial Q^\pm, \quad [\boldsymbol{\eta}, \boldsymbol{\nu}^Q] = 0 \ \mathcal{H}^1\text{-a. e. on } \partial Q \cap \partial\Omega$$

separately for each $Q \in \mathcal{Q}_{u(t_j,\cdot)}$, where $(\partial Q^+, \partial Q^+)$ is the signature induced on ∂Q by $u(t_j, \cdot)$. Let us take $Q \in \mathcal{Q}_{u(t_j,\cdot)}$ such that there exist F_1, F_2 in $\mathcal{Q}_{(u(t_{j-1},\cdot), \operatorname{div} \boldsymbol{z}_{j-1})}$, $F_1, F_2 \subset Q$, $F_1 \neq F_2$ with $\mathcal{H}^1(\partial F_1 \cap \partial F_2) > 0$. Denote by $\mathcal{Q}_{\mathrm{merge}}$ the maximal subset of $\mathcal{Q}_{(u(t_{j-1},\cdot), \operatorname{div} \boldsymbol{z}_{j-1})}$ with the properties

- F_1, F_2 belong to $\mathcal{Q}_{\mathrm{merge}}$,
- if F belongs to $\mathcal{Q}_{\mathrm{merge}}$ then $F \subset Q$,
- if F belongs to $\mathcal{Q}_{\mathrm{merge}}$ then there exists $F' \in \mathcal{Q}_{\mathrm{merge}}$, $F' \neq F$ with $\mathcal{H}^1(\partial F \cap \partial F') \neq 0$.

In other words, $\mathcal{Q}_{\mathrm{merge}}$ consists of those $F \in \mathcal{Q}_{(u(t_{j-1},\cdot), \operatorname{div} \boldsymbol{z}_{j-1})}$, whose interiors are contained in the same connected component of $\operatorname{int} Q$ as F_1 and F_2. Let now F_0 be a minimizer of $F \mapsto u_t(]t_{j-1}, t_{j+1}[, F) = \operatorname{div} \boldsymbol{z}_{j-1}(F) = \operatorname{div} \boldsymbol{z}_j(F)$ among $F \in \mathcal{Q}_{\mathrm{merge}}$. For all $\sigma \in]t_{j-1}, t_j[$, $\tau \in]t_j, t_{j+1}[$ we have

$$u(\sigma, F_0) > u(\sigma, F), \quad u(\tau, F_0) < u(\tau, F)$$

if $F \in \mathcal{Q}_{\mathrm{merge}}$ and

$$\operatorname{sgn}\,(u(\sigma, F_0) - u(\sigma, E)) = \operatorname{sgn}\,(u(\tau, F_0) - u(\tau, E))$$

if $E \in \mathcal{Q}_{(u(t_{j-1},\cdot), \operatorname{div} \boldsymbol{z}_{j-1})} \setminus \mathcal{Q}_{\mathrm{merge}}$ and $\mathcal{H}^1(E \cap F_0) > 0$. Hence, by (2.17),

$$[\boldsymbol{z}(\sigma, \cdot), \boldsymbol{\nu}^{F_0}] = -1, \quad [\boldsymbol{z}(\tau, \cdot), \boldsymbol{\nu}^{F_0}] = +1 \quad \mathcal{H}^1\text{-a. e. on } \partial F_0 \cap F$$

if $F \in \mathcal{Q}_{\mathrm{merge}}$ and

$$[\boldsymbol{z}(\sigma, \cdot), \boldsymbol{\nu}^{F_0}] = [\boldsymbol{z}(\tau, \cdot), \boldsymbol{\nu}^{F_0}] \quad \mathcal{H}^1\text{-a. e. on } \partial F_0 \cap E$$

if $E \in \mathcal{Q}_{(u(t_{j-1},\cdot), \operatorname{div} \boldsymbol{z}_{j-1})} \setminus \mathcal{Q}_{\mathrm{merge}}$. Consequently,

$$\int_{F_0} \operatorname{div} \boldsymbol{z}(\sigma, \cdot) \, \mathrm{d}\mathcal{L}^2 < \int_{F_0} \operatorname{div} \boldsymbol{z}(\tau, \cdot) \, \mathrm{d}\mathcal{L}^2,$$

which contradicts (2.26).

Next, we observe that there is only a finite set of values, depending only on G, that $\|\operatorname{div} \boldsymbol{z}(t, \cdot)\|_{L^2(\Omega)}$ can achieve. Indeed, for all $t \geq 0$, $\operatorname{div} \boldsymbol{z}(t, \cdot)$ is the unique result of minimization problems for functionals $\mathcal{F}_{Q_i, \partial Q_i^-, \partial Q_i^+}$, $i = 1, \ldots, \#u(t, \Omega)$, where Q_i belong to $\mathcal{Q}_{u(t,\cdot)}$, a rectilinear partition of Ω subordinate to G, and for each i, $(\partial Q_i^-, \partial Q_i^+)$ is a signature induced on ∂Q_i by a function $u(t, \cdot) \in PCR(\Omega)$ subordinate to G. There is only a finite number of these.

It remains to prove finiteness of extinction time of u. Let us first note, that by (2.9, 2.11), we have $\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega u \, \mathrm{d}\mathcal{L}^2 = 0$ $\mathcal{L}^1$-a. e. on $]0, \infty[$. Thus, denoting $m = \frac{1}{\mathcal{L}^2(\Omega)} \int_\Omega u_0 \, \mathrm{d}\mathcal{L}^2$, there holds

$\frac{1}{\mathcal{L}^2(\Omega)} \int_\Omega u(t,\cdot)\,\mathrm{d}\mathcal{L}^2 = m$ for all $t > 0$. Then, using (2.9), (2.12) and Sobolev inequality, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega (u-m)^2\,\mathrm{d}\mathcal{L}^2 = \int_\Omega (u-m)\mathrm{div}\,\boldsymbol{z}\,\mathrm{d}\mathcal{L}^2 = -\int_\Omega |\nabla u|_1\,\mathrm{d}\mathcal{L}^2 \le -C_\Omega \left(\int_\Omega (u-m)^2\,\mathrm{d}\mathcal{L}^2\right)^{\frac{1}{2}}$$

$\mathcal{L}^1$-a. e. on $]0,\infty[$ with a constant $C_\Omega > 0$, and consequently

$$\|u(t,\cdot) - m\|_{L^2(\Omega)} \le \Big(\|u_0 - m\|_{L^2(\Omega)} - C_\Omega t\Big)_+$$

for $t > 0$, whence an estimate on extinction time follows.

2.4 The case $\Omega = \mathbb{R}^2$

In this section we transfer previous results to the case $\Omega = \mathbb{R}^2$. First, we note that all the definitions and theorems in subsection 2.1.6 carry over without change (the Neumann boundary condition becomes void) to this case (see [65]). As for the definitions in subsection 2.1.7, it turns out that the statements of our results transfer nicely to the case of the whole plane if we allow for certain *unbounded rectilinear polygons*. Accordingly, in this section a subset $F \subset \mathbb{R}^2$ will be called a rectilinear polygon if either

- $F = \bigcup \mathcal{R}_F$ with a finite $\mathcal{R}_F \subset \mathcal{R}$ (in which case we say that F is a bounded rectilinear polygon)
- or $F = \overline{\mathbb{R}^2 \setminus \bigcup \mathcal{R}_F}$ with a finite $\mathcal{R}_F \subset \mathcal{R}$ (in which case we say that F is an unbounded rectilinear polygon).

Next, we restrict ourselves to non-negative compactly supported initial data. We say that a non-negative, finitely valued, compactly supported function $w \in BV(\mathbb{R}^2)$ belongs to $PCR_+(\mathbb{R}^2)$ if each of its level sets coincides (up to a $\mathcal{L}^2$-null set) with a rectilinear polygon. Note that in this case $\mathcal{Q}_w$ contains exactly one unbounded set Q_0 and $w|_{Q_0} = 0$.

The essential difficulty in transferring Theorem 1.1 to our current setting lies in constructing $\boldsymbol{z}$ on such unbounded level sets of $u(t,\cdot)$, $t \ge 0$. For this purpose, we need the following

Lemma 2.4. *Let $f \in PCR_+(\mathbb{R}^2)$ and let F be an unbounded rectilinear polygon. Then, there exists a vector field $\boldsymbol{\xi}_F \in X_F$ such that*

$$|\boldsymbol{\xi}_F|_\infty \le 1 \ \mathcal{L}^2\text{-a. e. on } F, \quad [\boldsymbol{\xi}_F, \boldsymbol{\nu}^F] = 1 \ \mathcal{H}^1\text{-a. e. on } \partial F, \quad \mathrm{div}\,\boldsymbol{\xi}_F = 0 \ \mathcal{L}^2\text{-a. e. on } F \tag{2.27}$$

if and only if

$$\mathcal{H}^1(\partial^* E \cap \partial F) \le \mathrm{Per}_1(E, \mathrm{int}\, F) \tag{2.28}$$

for all bounded $E \subset F$ of finite perimeter.

This is a version of [9, Theorem 5 and Lemma 6] where analogous statement is proved for isotropic perimeter. The idea of the proof is to consider auxiliary problem in a large enough ball. The proof of Lemma 2.4 follows along similar lines, however we decided to put it here, also because it seems that there is a small gap in the proof of [9, Theorem 5] that we patch. Namely the first inequality in line 12, page 511 of [9] (corresponding to (2.33) here) does not seem to be satisfied in general.

Proof. If a vector field $\boldsymbol{\xi}_F \in X_F$ satisfies (2.27), then we have for any bounded set $E \subset F$ of finite perimeter

$$0 = \int_E \operatorname{div} \boldsymbol{\xi}_F \, \mathrm{d}\mathcal{L}^2 \geq \mathcal{H}^1(\partial^* E \cap \partial F) - \mathrm{Per}_1(E, \operatorname{int} F).$$

Now assume that (2.28) holds. Let us take $R > 0$ large enough that

$$2 \operatorname{dist}(\partial B_\infty(R), \partial F) \geq \mathcal{H}^1(\partial F). \tag{2.29}$$

Put $c(R) = -\frac{\mathcal{H}^1(\partial F)}{\mathcal{H}^1(\partial B_\infty(R))}$. Denote by $\boldsymbol{\xi}_R$ the minimizer of functional $\mathcal{F}$ defined by $\mathcal{F}(\boldsymbol{\eta}) = \int_{F \cap B_\infty(R)} (\operatorname{div} \boldsymbol{\eta})^2 \, \mathrm{d}\mathcal{L}^2$ on the set of vector fields $\boldsymbol{\eta} \in X_{F \cap B_\infty(R)}$ satisfying

$$|\boldsymbol{\eta}|_\infty \leq 1 \quad \mathcal{L}^2\text{-a.e. on } F \cap B_\infty(R), \qquad [\boldsymbol{\eta}, \boldsymbol{\nu}^F] = 1, \qquad [\boldsymbol{\eta}, \boldsymbol{\nu}^{B_\infty(R)}] = c(R).$$

If $\operatorname{div} \boldsymbol{\xi}_R$ is constant in $F \cap B_\infty(R)$ then, due to choice of $c(R)$, $\operatorname{div} \boldsymbol{\xi}_R \equiv 0$ in $F \cap B_\infty(R)$. Supposing that the opposite is true, we obtain, as in the proof of Lemma 2.3, that there exists $\lambda > 0$ such that

$$Q_\lambda = \{\boldsymbol{x} \in F \cap B_\infty(R) \colon \operatorname{div} \boldsymbol{\xi}_R > \lambda\}$$

is a set of positive measure and finite perimeter, and we have

$$\begin{aligned} -\mathrm{Per}_1(Q_\lambda, \operatorname{int} F \cap B_\infty(R)) + \mathcal{H}^1(\partial^* Q_\lambda \cap \partial F) + c(R)\mathcal{H}^1(\partial^* Q_\lambda \cap \partial B_\infty(R)) \\ \geq \lambda \mathcal{L}^2(Q_\lambda) > 0 \end{aligned} \tag{2.30}$$

which can be rewritten as

$$-\mathrm{Per}_1(Q_\lambda) + 2\mathcal{H}^1(\partial^* Q_\lambda \cap \partial F) + (1 + c(R))\mathcal{H}^1(\partial^* Q_\lambda \cap \partial B_\infty(R)) > 0. \tag{2.31}$$

Assumption (2.29) implies that $c(R) > -1$, so we approximate Q_λ with a closed smooth set as in the proof of Lemma 2.2 in such a way that (2.31) still holds. Due to additivity of left hand side of (2.31), there is a connected component $\widetilde{Q}_\lambda$ of this smooth set that also satisfies (2.31), or equivalently

$$-\mathrm{Per}_1(\widetilde{Q}_\lambda, \operatorname{int} F \cap B_\infty(R)) + \mathcal{H}^1(\partial \widetilde{Q}_\lambda \cap \partial F) + c(R)\mathcal{H}^1(\partial \widetilde{Q}_\lambda \cap \partial B_\infty(R)) > 0. \tag{2.32}$$

If $\partial \widetilde{Q}_\lambda \cap \partial B_\infty(R) = \emptyset$, (2.32) contradicts (2.28). On the other hand, if $\partial \widetilde{Q}_\lambda \cap \partial F = \emptyset$, (2.32) itself is a contradiction (recall that $c(R) \leq 0$). Taking these observations into account, there necessarily holds

$$\mathrm{Per}_1(\widetilde{Q}_\lambda, \operatorname{int} F \cap B_\infty(R)) \geq 2 \operatorname{dist}(\partial B_\infty(R), \partial F), \tag{2.33}$$

whence (2.29) yields a contradiction, *unless* $\widetilde{Q}_\lambda$ is not simply connected in such a way that there is a connected component Γ of $\partial \widetilde{Q}_\lambda$ with following properties:

- $\widetilde{Q}_\lambda$ is inside of Γ (Γ is the exterior boundary of $\widetilde{Q}_\lambda$),
- Γ does not intersect $\partial F \cup \operatorname{supp} f$,
- and Γ intersects all four sides of $B_\infty(R)$.

In this case, let us denote by $\widehat{Q}_\lambda$ the union of $\widetilde{Q}_\lambda$ and the region between Γ and $\partial B_\infty(R)$. We have $\int_{\Gamma\setminus\partial B_\infty(R)} |\boldsymbol{\nu}^{\widetilde{Q}_\lambda}|_1 \,\mathrm{d}\mathcal{H}^1 \geq \mathcal{H}^1(\partial B_\infty(R)\setminus\Gamma)$ and consequently (as $-c(R)<1$)

$$\begin{aligned}\mathrm{Per}_1(\widetilde{Q}_\lambda, \operatorname{int} F\cap B_\infty(R)) - c(R)\mathcal{H}^1(\partial\widetilde{Q}_\lambda\cap\partial B_\infty(R)) \geq \mathrm{Per}_1(\widehat{Q}_\lambda, \operatorname{int} F\cap B_\infty(R))\\ - c(R)\mathcal{H}^1(\partial\widehat{Q}_\lambda\cap\partial B_\infty(R)) = \mathrm{Per}_1(\widehat{Q}_\lambda, \operatorname{int} F\cap B_\infty(R)) + \mathcal{H}^1(\partial F),\end{aligned}$$

a contradiction with (2.32) which implies that $\operatorname{div}\boldsymbol{\xi}_R \equiv 0$.

We define $\boldsymbol{\xi}_F\in X_F$ by

$$\boldsymbol{\xi}_F(x_1,x_2) = \begin{cases} \boldsymbol{\xi}_R(x_1,x_2) & \text{in } F\cap B_\infty(R),\\ (c(R)\operatorname{sgn} x_1, 0) & \text{in } \{|x_1|>R, |x_2|<R\}\\ (0, c(R)\operatorname{sgn} x_2) & \text{in } \{|x_1|<R, |x_2|>R\}\\ (0,0) & \text{in } \{|x_1|>R, |x_2|>R\}.\end{cases} \tag{2.34}$$

□

Next, we argue that Lemma 2.3 holds in the case $\Omega=\mathbb{R}^2$, provided that $w\in PCR_+(\mathbb{R}^2)$. For each bounded $Q\in\mathcal{Q}_w$, we construct the vector field $\boldsymbol{\eta}$ on Q as before. Now, let F be the unbounded rectilinear polygon in $\mathcal{Q}_w$ and let us denote by R_0 the smallest rectangle containing ∂F. Clearly, R_0 is subordinate to G_w. Arguing by approximation of E with smooth sets as in the proof of Lemma 2.1, we observe that

$$\mathrm{Per}_1(E\cap R_0, \operatorname{int} F) - \mathcal{H}^1(\partial^*(E\cap R_0)\cap\partial F) \leq \mathrm{Per}_1(E, \operatorname{int} F) - \mathcal{H}^1(\partial^* E\cap\partial F). \tag{2.35}$$

If there exists a bounded set of finite perimeter $E\subset F$ such that $\mathcal{J}_{F,\emptyset,\partial F}(E)<0$, by (2.35) we have $\mathcal{J}_{F,\emptyset,\partial F}(R_0\cap E)\leq\mathcal{J}_{F,\emptyset,\partial F}(E)$. Thus, we obtain the following alternative:

- either $\mathcal{J}_{F,\emptyset,\partial F}$ is minimized by a bounded rectilinear polygon F_1 subordinate to G,
- or $\mathrm{Per}_1(E, \operatorname{int} F) - \mathcal{H}^1(\partial^* E\cap\partial F)\geq 0$ for each bounded $E\subset F$ of finite perimeter.

In the first case, we produce $\boldsymbol{\eta}|_{F_1}$ as in the proof of Lemma 2.3 and repeat the reasoning above with $F\setminus F_1$ in place of F. In the second case, by virtue of Lemma 2.4 there exists a vector field $\boldsymbol{\xi}_F\in X_F$ such that (2.27) is satisfied. We put $\boldsymbol{\eta}|_F=\boldsymbol{\xi}_F$, and at this point we have $\boldsymbol{\eta}$ defined $\mathcal{L}^2$-a. e. on $\mathbb{R}^2$.

With an equivalent of Lemma 2.3 at hand, the rest of the proof of Theorem 1.1 follows as in the previous section, except a slight difference in the estimate on extinction time t_n. In the case of compactly supported functions on $\mathbb{R}^2$, the BV seminorm controls full L^2 norm, hence for $t>0$ we get

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^2)} \leq \Big(\|u_0\|_{L^2(\mathbb{R}^2)} - Ct\Big)_+,$$

where C is the constant in the Sobolev inequality on $\mathbb{R}^2$.

2.5 Preservation of continuity

We start with a lemma concerning PCR functions on a rectangle, which says, roughly speaking, that the maximal oscillation on horizontal (or vertical) lines, on any given length scale, is not increased by the solution to (2.8) with respect to initial datum $u_0\in PCR(\Omega)$. To make a precise statement, we fix a rectangle Ω and let G be any grid such that Ω is subordinate to G. Further, let m_1 (m_2) be the number of horizontal (vertical) lines of G. For any given integer $0 \le m \le m_1 - 3$ ($m_2 - 3$) we denote by $\mathcal{R}^2_{1,m}(G)$ ($\mathcal{R}^2_{2,m}(G)$) the set of pairs of rectangles that lay in the strip of Ω between any two successive horizontal (vertical) lines of G and are separated by at most m rectangles in $\mathcal{R}(G)$.

Lemma 2.5. *Let Ω be a rectangle, let u be the solution to* (2.8) *with $u_0 \in PCR(\Omega)$ and let G be a grid such that $\mathcal{Q}_{u_0}$ is subordinate to G. Then for $i = 1, 2$ there holds*

$$\max_{(R_1,R_2)\in\mathcal{R}^2_{i,m}(G)} |u(t,R_1) - u(t,R_2)| \le \max_{(R_1,R_2)\in\mathcal{R}^2_{i,m}} |u_0(R_1) - u_0(R_2)| \tag{2.36}$$

in any time instance $t > 0$.

Remark. Taking $m = 0$ in Lemma 2.5 we obtain, for $t > 0$,

$$\mathcal{H}^1\text{-ess}\max_{J_{u(t,\cdot)}}(u_+(t,\cdot) - u_-(t,\cdot)) \le \mathcal{H}^1\text{-ess}\max_{J_{u_0}}(u_{0,+} - u_{0,-}).$$

Proof. The form of solution obtained in Theorem 1.1 implies that the function

$$t \mapsto \max_{(R_1,R_2)\in\mathcal{R}^2_{i,m}(G)} |u(t,R_1) - u(t,R_2)| \tag{2.37}$$

is piecewise linear and continuous, in particular it does not have jumps. Having this observation in mind, let us consider time instance $\tau > 0$ that does not belong to the set of merging times $\{t_1, \dots, t_n\}$. We can assume that $t_l < \tau < t_{l+1}$ ($l \in \{0, \dots, n-1\}$).

For a given $0 \le k \le m_i$ assume we have already proved that the slope of (2.37) is non-positive in $t = \tau$ for each $0 \le m < k$. Take any pair of rectangles $(R_+, R_-) \in \mathcal{R}^2_{i,k}(G)$ that realizes the maximum in (2.37). Let us take rectilinear polygons F_+, F_- in $\mathcal{Q}_{(u(\tau,\cdot),\operatorname{div} \boldsymbol{z}(\tau,\cdot))} = \mathcal{Q}_{(u(t_l,\cdot),\operatorname{div} \boldsymbol{z}(t_l,\cdot))}$ such that $R_\pm \subset F_\pm$.

Now we assume, without loss of generality, that $i = 1$ (i. e. rectangles $R_\pm$ are in the same row of $\mathcal{R}(G)$), $u(\tau, R_+) > u(\tau, R_-)$ and R_- is to the left of R_+. Let us denote by x_- the maximal value of x coordinate of points in R_- and by x_+ the minimal value of x coordinate of points in R_+. Further, let us denote

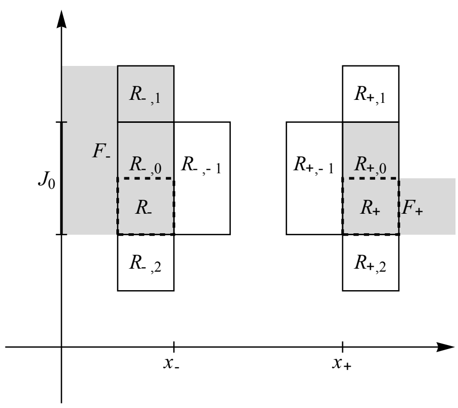


Figure 2.3: Map key for the notation in the proof of Lemma 2.5.

- by J_0 the maximal interval such that $\{x_\pm\} \times J_0 \cap \partial R_\pm \neq \emptyset$ and $\{x_\pm\} \times J_0 \subset \partial F_\pm$,
- by $R_{\pm,0}$ minimal *rectangles* in $\mathcal{F}(G)$ that have $\{x_\pm\} \times J_0$ as one of their sides and contain $R_\pm$,
- by $R_{+,-1}$ (resp. $R_{-,-1}$) the minimal rectangle in $\mathcal{F}(G)$ that has $\{x_+\} \times J_0$ (resp. $\{x_-\} \times J_0$) as one of its sides and does not contain R_+ (resp. R_-),
- by K the number of endpoints of J_0 that do not intersect $\partial\Omega$ ($K \in \{0, 1, 2\}$),
- by $R_{\pm,j}$, $j \in \mathbb{N}$, $j \leq K$ the K pairs of rectangles in $\mathcal{R}(G)$ such that
  - all of $R_{+,j}$, have a common side with $R_{+,0}$ and belong to the same column in $\mathcal{R}(G)$ as R_+,
  - all of $R_{-,j}$, have a common side with $R_{-,0}$ and belong to the same column in $\mathcal{R}(G)$ as R_-,
  - for a fixed j, both $R_{\pm,j}$ belong to the same row in $\mathcal{R}(G)$.

Due to the way these are defined, fixing $j \leq K$, at least one of the two rectangles $R_{\pm,j}$ is not contained in $F_+ \cup F_-$, and

$$\text{at least one of inequalities } u(\tau, R_{+,j}) < u(\tau, R_{+,0}),\ u(\tau, R_{-,j}) > u(\tau, R_{-,0}) \text{ hold.} \tag{2.38}$$

If there is a pair of rectangles $R'_\pm$ in $\mathcal{R}^2_{1,m}(G)$, $m < k$ such that $R'_\pm \subset F_\pm$ then we have already proved that

$$|u(\tau, R_+) - u(\tau, R_-)| = |u(\tau, R'_+) - u(\tau, R'_-)| \leq \max_{(R_1,R_2)\in\mathcal{R}^2_{i,k}} |u_0(R_1) - u_0(R_2)|.$$

Therefore, we can assume that

$$u(\tau, R_{+,-1}) < u(\tau, R_{+,0}) \text{ and } u(\tau, R_{-,-1}) > u(\tau, R_{-,0}) \text{ hold.} \tag{2.39}$$

$F_\pm$ is not necessarily a level set of $u(\tau, \cdot)$. However, as τ is not a merging time, $\partial F_\pm$ is contained in the boundary of one. Hence, $\partial F_\pm$ inherits the signature $(\partial F^-_\pm, \partial F^+_\pm)$ induced on it by $u(\tau, \cdot)$. Furthermore, we have

$$F_- \in \arg\min \mathcal{J}_{F_-,\partial F^+_-,\partial F^-_-} \quad \text{and} \quad F_+ \in \arg\max \check{\mathcal{J}}_{F_+,\partial F^+_+,\partial F^-_+}.$$

Therefore, taking into account (2.38, 2.39),

$$\begin{aligned} u_t(\tau, \cdot)|_{F_+} - u_t(\tau, \cdot)|_{F_-} &= -\check{\mathcal{J}}_{F_+,\partial F^+_+,\partial F^-_+}(F_+) + \mathcal{J}_{F_-,\partial F^+_-,\partial F^-_-}(F_-) \\ &\leq -\check{\mathcal{J}}_{F_+,\partial F^+_+,\partial F^-_+}(R_+) + \mathcal{J}_{F_-,\partial F^+_-,\partial F^-_-}(R_-) \leq 0 \end{aligned} \tag{2.40}$$

which concludes the proof. □

Proof. Recall that $\Omega = [0, a] \times [0, b]$. For $k = 1, 2, \ldots$ let

$$G_k = \left(\left\{\frac{ja}{k}, j = 0, 1, \ldots, k\right\} \times \mathbb{R}\right) \cup \left(\mathbb{R} \times \left\{\frac{jb}{k}, j = 0, 1, \ldots, k\right\}\right)$$

and let $u_{0,k} \in PCR(\Omega)$ be defined by

$$u_{0,k}(\boldsymbol{x}) = u_0(\boldsymbol{x}_R) \quad \text{for } \boldsymbol{x} \in R \in \mathcal{R}(G_k),$$

where $\boldsymbol{x}_R$ is the center of R.

For any $k = 1, 2, \ldots$, $i = 1, 2$, $m = 0, \ldots, k-1$, let $(R, R') \in \mathcal{R}^2_{i,m}(G_k)$ with $(x_1, x_2) \in R, (y_1, y_2) \in R'$ we have

$$|u_{0,k}(R) - u_{0,k}(R')| \leq \omega_i(|x_i - y_i| + \tfrac{1}{k}). \tag{2.41}$$

Let us denote by u_k the solution to (2.8) with initial datum $u_{0,k}$. Due to inequality (2.41) and Lemma 2.5 we have for any $(x_1, x_2), (y_1, y_2) \in \Omega$,

$$\begin{aligned} |u_k(t, x_1, x_2) - u_k(t, y_1, y_2)| &\leq |u_k(t, x_1, x_2) - u_k(t, y_1, x_2)| + |u_k(t, y_1, x_2) - u_k(t, y_1, y_2)| \\ &\leq \omega_1(|x_1 - y_1| + \tfrac{1}{k}) + \omega_2(|x_2 - y_2| + \tfrac{1}{k}). \end{aligned}$$

Now, due to monotonicity of $-\partial \mathrm{TV}_{1,\Omega}$ (see [65], Theorem 11.), we have for $t > 0$ $\|u_k(t, \cdot) - u(t, \cdot)\|_{L^2(\Omega)} \leq \|u_{0,k} - u_0\|_{L^2(\Omega)}$. Therefore, there exists a set $N \subset \Omega$ of zero $\mathcal{L}^2$ measure and a subsequence (k_j) such that $u_{k_j}(t, \boldsymbol{x}) \to u(t, \boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega \setminus N$. Now, for each pair $(x_1, x_2), (y_1, y_2) \in \Omega$ take any pair of sequences $((x_{1,n}, x_{2,n})), ((y_{1,n}, y_{2,n})) \subset \Omega \setminus N$ such that $x_{i,n} \to x_i$ and $y_{i,n} \to y_i$. Passing to the limit $j \to \infty$ and then $n \to \infty$ in

$$|u_{k_j}(t, x_{1,n}, x_{2,n}) - u_{k_j}(t, y_{1,n}, y_{2,n})| \leq \omega_1(|x_{1,n} - y_{1,n}| + \tfrac{1}{k_j}) + \omega_2(|x_{2,n} - y_{2,n}| + \tfrac{1}{k_j})$$

we conclude the proof. □

Finally, we note that all the results in this section carry over in a straightforward way to the case $\Omega = \mathbb{R}^2$, provided that in the statement of Lemma 2.5, $PCR(\Omega)$ is replaced with $PCR_+(\mathbb{R}^2)$ and in the statement of Theorem 1.2, $C(\Omega)$ is replaced with $C_{c,+}(\mathbb{R}^2)$ (meaning non-negative, compactly supported continuous functions on $\mathbb{R}^2$). On the other hand, if Ω is a rectilinear polygon different from a rectangle, the continuity is not necessarily preserved as Example 2.3 shows.

2.6 Examples

In this section, we provide several examples illustrating the strength of our results. Theorem 1.1 predicts that the jump set of a function piecewise constant on rectangles may expand under the $\mathrm{TV}_{1,\Omega}$ flow, i. e. facet breaking may occur. Many explicit examples of this kind can be constructed. Here we present a simple one, for which the procedure described in the proof of Theorem 2.3 is concise enough to be presented in detail.

Example 2.1. Let

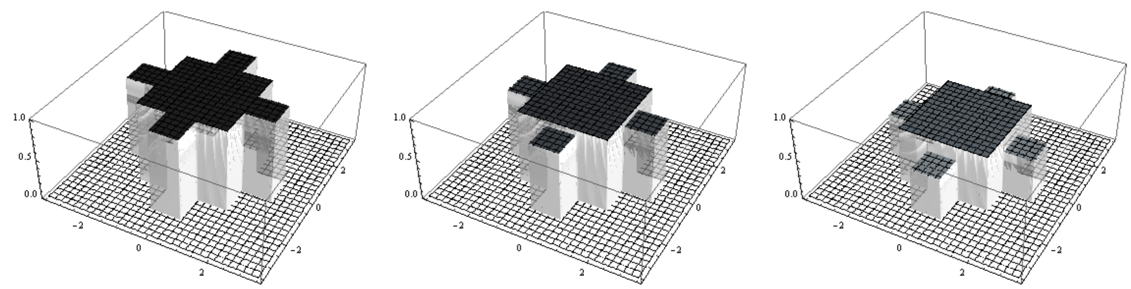
$$u(t, \cdot) = \left(1 - \tfrac{4}{3}t\right)_+ \chi_B + (1 - 2t)_+ \chi_C,$$

where we denoted

$$B = B_\infty\Big((0,0), \tfrac{3}{2}\Big), \quad C = B_\infty\Big((2,0), \tfrac{1}{2}\Big) \cup B_\infty\Big((-2,0), \tfrac{1}{2}\Big) \cup B_\infty\Big((0,2), \tfrac{1}{2}\Big) \cup B_\infty\Big((0,-2), \tfrac{1}{2}\Big).$$

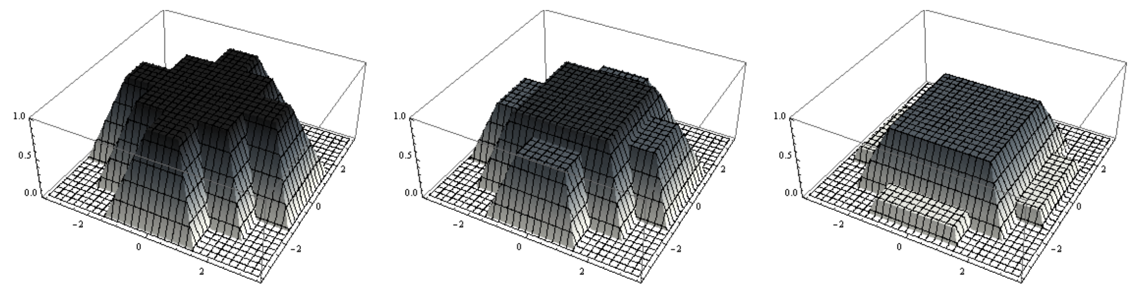
For each $t \geq 0$, $u(t, \cdot) \in PCR_+(\mathbb{R}^2)$ and u solves (1.19) in $]0, \infty[\times \mathbb{R}^2$ with initial datum $u_0 = \chi_{B \cup C}$. To see this, we execute the algorithm described in the proof of Lemma 2.3. Let

$Q_1 = u_0^{-1}(1) = B \cup C$. Due to symmetry, the only plausible largest minimizers of $\mathcal{J}_{Q_1,\partial Q_1,\emptyset}$ are B, C and $B \cup C$ (we only need to consider elements of $\mathcal{F}_{u_0}$ and no subset of square B can produce lower value of the functional than B). We check that the values of $\mathcal{J}_{Q_1,\partial Q_1,\emptyset}$ on these sets are, respectively, $\frac{4}{3}$, 4, and $\frac{20}{13}$, hence B is the minimizer and the initial velocity on B is $-\frac{4}{3}$. Next, we have to find the largest minimizer of $\mathcal{J}_{C,\partial Q_1,\partial B}$. There is only one competitor, C. To find initial velocity on C, we calculate $-\mathcal{J}_{C,\partial Q_1,\partial B}(C) = -2$. Finally, as explained in section 2.4, we need to find the largest minimizer of $\mathcal{J}_{Q_0,\partial R_0,\partial Q_1}$, where we denoted R_0 to be the smallest rectangle (square) containing the support of u_0 and $Q_0 = R_0 \cap u_0^{-1}(0)$. We check that the minimizer is Q_0 itself, with $\mathcal{J}_{Q_0,\partial R_0,\partial Q_1}(Q_0) = 0$.



(a) $t = 0$. (b) $t = 0.08$. (c) $t = 0.24$.

Figure 2.4: Plots of $u(t, \cdot)$ from Example 2.1 in certain time instances t.

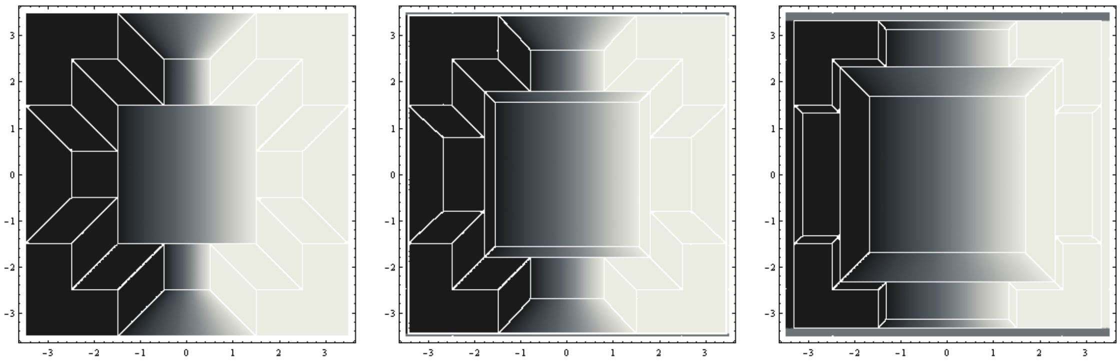


(a) $t = 0$. (b) $t = 0.08$. (c) $t = 0.24$.

Figure 2.5: Plots of $u(t, \cdot)$ from Example 2.2 in certain time instances t.

On the other hand, Theorem 1.2 asserts that if u_0 is (Lipschitz) continuous, the solution u starting with u_0 is (Lipschitz) continuous in every time instance $t > 0$. For instance, if one extends the characteristic function form Example 2.1 continuously outside its support, no jumps will appear in the evolution — another manifestation of nonlocality of the equation.

Example 2.2. Here we present Figure 2.5, depicting evolution u of piecewise linear continuous function u_0 obtained by extending the initial datum from Example 2.1 outside its support up to 0 in such a way that $\nabla u_0 \in \{(0,0), (1,0), (0,1)\}$. The evolution is obtained by explicit identification of corresponding field $\boldsymbol{z} = (z_1, z_2)$ under an ansatz that in each of a finite number of evolving regions either $z_i = \pm 1$ or a z_i is a linear interpolation of boundary values, $i = 1, 2$ (see Figure 2.6). This reduces the problem to a decoupled infinite system of ODEs. The evolution obtained this way is the strong solution starting with u_0 as it satisfies all the requirements in Definition 2.1. Figures 2.5 and 2.6 are obtained by solving numerically the system of ODEs using *Mathematica*'s `NDSolve` function. We omit the quite lengthy details.



(a) $t = 0$. (b) $t = 0.08$. (c) $t = 0.24$.

Figure 2.6: Density plots of $z_1(t, \cdot)$ corresponding to $u(t, \cdot)$ from Example 2.2 in certain time instances t. Black corresponds to value 1, ivory to -1; note the value 0 outside the minimal strip containing the support of $u(t, \cdot)$.

Next we provide an example showing that in non-convex rectilinear polygons (i. e. other than a rectangle) evolution starting with continuous initial datum may develop discontinuities.

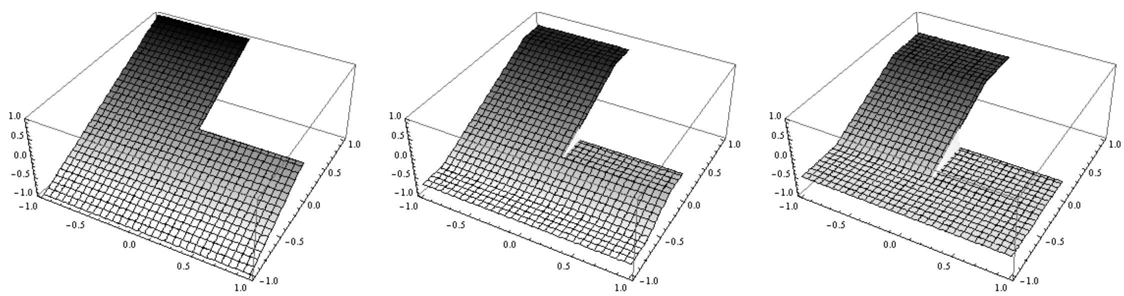
Example 2.3. Let

$$\Omega = \{(x_1, x_2) \colon |(x_1, x_2)|_\infty \leq 1, x_1 \leq 0, x_2 \leq 0\}, \quad u_0(x_1, x_2) = x_2$$

and so $\nabla u(0, \cdot) \equiv (0, 1)$, $\boldsymbol{z}(0, \cdot) \equiv (0, 1)$. The solution can be written explicitly, for $t \leq \frac{1}{8}$ we have

$$u(t, x_1, x_2) = \begin{cases} -1 + \sqrt{2t} & \text{if } x_2 \leq -1 + \sqrt{2t}, \\ -\sqrt{2t} & \text{if } x_1 \geq 0 \text{ and } x_2 \geq -\sqrt{2t}, \\ 1 - \sqrt{2t} & \text{if } x_1 < 0 \text{ and } x_2 \geq 1 - \sqrt{2t}, \\ x_2 & \text{otherwise.} \end{cases}$$

We see that regions where $\nabla u = 0$ appear near the boundary and expand with speed $\frac{1}{\sqrt{2t}}$. In these regions, z_2 is linear interpolation between 0 and 1. Also a jump in the x_2 direction appears near $\boldsymbol{x} = 0$ and grows with the same speed.



(a) $t = 0$. (b) $t = 0.04$. (c) $t = 0.12$.

Figure 2.7: Plots of $u(t, \cdot)$ from Example 2.3 at certain time instances t.

Chapter 3

Facets in solutions to a uniformly parabolic very singular equation

3.1 Characterization of $D(\mathcal{L})$

For $p \in \mathbb{R}$, let us denote

$$J(p) = \tfrac{1}{2}(p^2 + \alpha|p|), \quad L(p) = \partial J(p).$$

It is easy to see that $L(p) = p + \frac{\alpha}{2}\operatorname{sgn} p$, where sgn is understood as a multifunction, i. e.

$$L(p) = \begin{cases} \{p - \frac{\alpha}{2}\} & \text{if } p < 0, \\ [-\frac{\alpha}{2}, \frac{\alpha}{2}] & \text{if } p = 0, \\ \{p + \frac{\alpha}{2}\} & \text{if } p > 0. \end{cases}$$

The operator $\mathcal{L} \equiv -\partial \mathcal{J}$ is given by

Proposition 3.1. *We have*

$$D(\mathcal{L}) \equiv D(\partial \mathcal{J}) = \left\{ \begin{matrix} u \in H^2(\mathbb{T}) \textit{ such that there exists } z \in H^1(\mathbb{T}) \\ \textit{satisfying } z(x) \in L(u_x(x)) \textit{ for } x \textit{ in } \mathbb{T} \end{matrix} \right\}$$

and

$$\mathcal{L}u = \{z_x \colon z \in H^1(\mathbb{T}),\ z(x) \in L(u_x(x)) \textit{ for } x \textit{ in } \mathbb{T}\}.$$

Proposition 3.1 motivates the notation

$$X_{\mathbb{T}}(u) = \{z \in H^1(\mathbb{T}) \colon z(x) \in L(u_x(x)) \text{ for } x \in \mathbb{T}\}$$

for $u \in D(\mathcal{L})$.

Proposition 3.1 can be obtained by adapting the arguments in [14, Examples 2 and 3] (see also [34, Lemma 2.2]). However, in our particularly simple case, a shorter argument is sufficient — we present it here.

Proof. Let $u \in D(\mathcal{J}) = H^1(\mathbb{T})$. Whenever $w \in \partial \mathcal{J}(u)$, we have

$$\mathcal{J}(u + \varphi) \geq \mathcal{J}(u) + \int_{\mathbb{T}} w\varphi \tag{3.1}$$

for any $\varphi \in L^2(\mathbb{T})$. Clearly, it is sufficient to consider φ of form $\varphi = \lambda\psi$ with $\psi \in H^1(\mathbb{T})$, $\lambda > 0$. Then (3.1) becomes

$$\tfrac{1}{2}\int_{\mathbb{T}} |u_x + \lambda\psi_x|^2 + \alpha|u_x + \lambda\psi_x| - \tfrac{1}{2}\int_{\mathbb{T}} |u_x|^2 + \alpha|u_x| \geq \lambda \int_{\mathbb{T}} w\psi \tag{3.2}$$

which we transform and divide by λ to obtain

$$\tfrac{1}{2}\int_{\mathbb{T}} \lambda\psi_x^2 + 2u_x\psi_x + \tfrac{\alpha}{2}\int_{\{u_x=0\}} |\psi_x| + \tfrac{\alpha}{2}\int_{\{u_x\neq 0\}} \tfrac{1}{\lambda}(|u_x + \lambda\psi_x| - |u_x|) \geq \int_{\mathbb{T}} w\psi. \tag{3.3}$$

Next, we pass to the limit $\lambda \to 0^+$. In the limit, the first term of the l. h. s. vanishes. To treat the last one, we notice

$$\begin{aligned}\int_{\{u_x\neq 0\}} &\tfrac{1}{\lambda}(|u_x + \lambda\psi_x| - |u_x|) \\ &= \int_{\{0<|u_x|\leq\lambda|\psi_x|\}} \tfrac{1}{\lambda}(|u_x + \lambda\psi_x| - |u_x|) + \int_{\{\lambda|\psi_x|<|u_x|\}} (\operatorname{sgn} u_x)\psi_x\end{aligned} \tag{3.4}$$

and

$$\int_{\{\lambda|\psi_x|<|u_x|\}} (\operatorname{sgn} u_x)\psi_x \to \int_{\{u_x\neq 0\}} (\operatorname{sgn} u_x)\psi_x, \tag{3.5}$$

$$\left|\int_{0<|u_x|\leq\lambda|\psi_x|} \tfrac{1}{\lambda}(|u_x + \lambda\psi_x| - |u_x|)\right| \leq \int_{0<|u_x|\leq\lambda|\psi_x|} |\psi_x| \to 0 \tag{3.6}$$

as $\lambda \to 0^+$ by virtue of dominated convergence. Therefore, we obtain that if w belongs to $\partial\mathcal{J}(u)$, the inequality

$$\int_{\mathbb{T}} u_x\psi_x + \tfrac{\alpha}{2}\int_{\{u_x=0\}} |\psi_x| + \tfrac{\alpha}{2}\int_{\{u_x\neq 0\}} (\operatorname{sgn} u_x)\psi_x \geq \int_{\mathbb{T}} w\psi \tag{3.7}$$

is satisfied for each $\psi \in H^1(\mathbb{T})$. The converse is also true, as (3.7) implies (3.3). Thus, if $w = -z_x$, where $z \in H^1$ is a selection of the multifunction $L(u_x)$, then $w \in \partial\mathcal{J}(u)$.

On the other hand, take any w that satisfies (3.7). Taking $\psi \equiv 1$ we see that $\int_{\mathbb{T}} w = 0$, and thus w admits a primitive (defined up to a constant, which we will choose in a moment), i. e. $w = -z_x$ for a function $z \in H^1(\mathbb{T})$. Now, noting that

$$\{\psi_x \colon \psi \in H^1(\mathbb{T})\} = \left\{\varphi \in L^2(\mathbb{T}) \colon \int_{\mathbb{T}} \varphi = 0\right\},$$

we take any $\psi \in H^1(\mathbb{T})$ such that $\psi_x = 0$ on $\{u_x = 0\}$. Considering both ψ and $-\psi$ in (3.7) yields

$$\int_{\{u_x\neq 0\}} (u_x + \tfrac{\alpha}{2}\operatorname{sgn} u_x)\psi_x = \int_{\{u_x\neq 0\}} z\psi_x \tag{3.8}$$

and thus, $z = u_x + \frac{\alpha}{2}\operatorname{sgn} u_x$ a. e. in $\{u_x \neq 0\}$ up to a constant (which we now choose to be 0).

Finally, take any $\psi \in H^1(\mathbb{T})$ such that $\psi_x = 0$ on $\{u_x \neq 0\}$. Considering ψ and $-\psi$ in (3.7) yields

$$\left|\int_{\{u_x=0\}} z\psi_x\right| \leq \int_{\{u_x=0\}} |\psi_x| \tag{3.9}$$

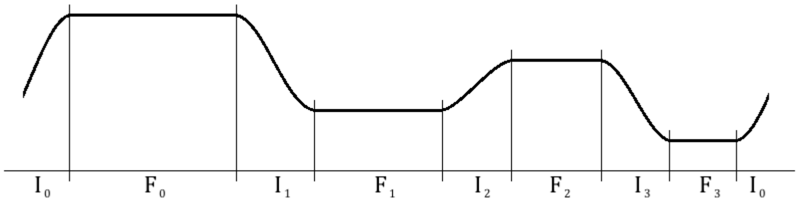


Figure 3.1: Graph of a typical $u \in D(\mathcal{L})$.

which implies that $z = c + z^*$ a. e. in $\{u_x = 0\}$ with $\|z^*\|_{L^\infty(\{u_x=0\})} \le \frac{\alpha}{2}$ and $c \in \mathbb{R}$. Unless u is constant in $\mathbb{T}$, our previous choice of z together with its regularity imply that $c = 0$. If u is constant, we choose $c = 0$.

Now, note that for any $u \in D(\mathcal{L})$, u_x is representable as the composition of the piecewise linear continuous function $p \mapsto \operatorname{sgn} p(|p| - \frac{\alpha}{2})_+$ and any $z \in X_\mathbb{T}(u)$. In particular, the function $u_{xx} = z_x \mathbf{1}_{\{u_x \neq 0\}}$ (defined independently of $z \in X_\mathbb{T}(u)$) is the distributional second derivative of u and belongs to $L^2(\mathbb{T})$. $\square$

Proposition 3.1 states that the elements of $D(\mathcal{L})$ are at least as regular as elements of $D(\Delta)$. However, as the dissipation of $\mathcal{L}$ is essentially stronger than that of the laplacian, they are in fact more regular. This additional regularity is expressed in Proposition 3.1 in a rather convoluted way, as the existence of a regular selection of $L(u_x)$. The following Lemma makes it more explicit. Roughly, it states that a non-constant $u \in H^2(\mathbb{T})$ belongs to $D(\mathcal{L})$ if and only if $\mathbb{T}$ can be divided into a finite number of (non-degenerate) intervals where u is constant and intervals where u is monotone. Furthermore, it identifies $\mathcal{L}^0 u$ as $(L^\circ u_x)_x$, where L° is a certain nonlocal operator introduced in [67]. We recall its definition (in generality sufficient for our purposes) in the statement of the Lemma.

Lemma 3.1. *Let $u \in D(\mathcal{L})$ be non-constant. There exists an even number n and a disjoint decomposition $\{I^k, F^k \colon k \in \mathbb{Z}_n\}$ of $\mathbb{T}$ such that for $k \in \mathbb{Z}_n$:*

(i) F^k *is a non-degenerate closed interval,* I^k *is an open interval and* $I^k < F^k < I^{k+1}$*,*

(ii) $u_x = 0$ *in* F^k*,* F^k *is a maximal interval with this property, and u attains an improper local maximum (resp. minimum) in* F^k *if k is even (resp. odd),*

(iii) *u is non-decreasing (resp. non-increasing) in* I^k *if k is even (resp. odd),*

(iv) $|F^k| \ge \alpha^2 \|\mathcal{L}^0 u\|^{-2}_{L^2(\mathbb{T})}$.

On the other hand, if $u \in H^2(\mathbb{T})$ and a finite disjoint decomposition $\{F_k, I_k \colon k \in \mathbb{Z}_n\}$ of $\mathbb{T}$ satisfies conditions (i, ii, iii) for $k \in \mathbb{Z}_n$, then $u \in D(\mathcal{L})$ and (iv) holds.

Furthermore, there exists a unique element $L^\circ u_x$ of $X_\mathbb{T}(u)$ satisfying

(a) $L^\circ u_x = u_x + \frac{\alpha}{2}$ *in* I^k *for k even,* $L^\circ u_x = u_x - \frac{\alpha}{2}$ *in* I^k *for k odd,*

(b) $(L^\circ u_x)_x = -\frac{\alpha}{|F^k|}$ *in* F^k *for k even,* $(L^\circ u_x)_x = \frac{\alpha}{|F^k|}$ *in* F^k *for k odd.*

There holds $\mathcal{L}^0 u = (L^\circ u_x)_x$.

Proof. Let $z \in X_{\mathbb{T}}(u)$. Due to embedding $H^1(\mathbb{T}) \subset C^{\frac{1}{2}}(\mathbb{T})$ and Proposition 3.1, both u_x and z are continuous. Suppose that u attains a local maximum on a (possibly degenerate) interval $F = [a, b] \subset \mathbb{T}$, and that F is a maximal interval with respect to this property. There are points x, y arbitrarily close to F such that $x < a \leq b < y$ and $u_x(x) > 0$, $u_x(y) < 0$. Hence, $z(a) = \frac{\alpha}{2}$, $z(b) = -\frac{\alpha}{2}$ (in particular $a \neq b$). As the affine function minimizes the functional $z \mapsto \int_F z_x^2$ on $H^1(F)$ with prescribed boundary values, we have

$$\|\mathcal{L}^0 u\|^2_{L^2(\mathbb{T})} \geq \int_F z_x^2 \geq \int_F \left(\frac{z(b) - z(a)}{|F|}\right)^2 = \frac{\alpha^2}{|F|}. \tag{3.10}$$

Analogous argument shows (3.10) if u attains a local minimum on F. Hence, u has only a finite number n of local extrema, in fact $n \leq \alpha^{-2}\|\mathcal{L}^0 u\|^2_{L^2(\mathbb{T})}$ (recall that $|\mathbb{T}| = 1$). We enumerate them $F^0 = [a^0, b^0], \ldots, F^{n-1} = [a^{n-1}, b^{n-1}]$ in a manner consistent with the cyclic order on $\mathbb{T}$ and so that F^0 corresponds to a local maximum of u. Clearly, u is monotone in each of complementary intervals $I^k =]b^{k-1}, a^k[$, $k \in \mathbb{Z}_n$.

Now, assume that $u \in H^2(\mathbb{T})$ and a finite decomposition $\{F^k, I^k\}$ of $\mathbb{T}$ satisfying conditions *(i, ii, iii)* exists. Conditions *(a)* and $L\bar{\circ}u_x \in H^1(\mathbb{T})$ define $L\bar{\circ}u_x$ uniquely in the closure of each interval I^k, with

$$L\bar{\circ}u_x = (-1)^k \tfrac{\alpha}{2} \tag{3.11}$$

at the endpoints. Then, we extend $L\bar{\circ}u_x$ to intervals F^k in the unique way so that condition *(b)* is satisfied and the result is continuous on $\mathbb{T}$. As a continuous, piecewise H^1 function, $L\bar{\circ}u_x \in H^1(\mathbb{T})$ and clearly belongs to $X_{\mathbb{T}}(u)$.

Finally, note that whenever $u_x = 0$ in an open interval $I \subset I^k$, there holds $(L\bar{\circ}u_x)_x = 0$ in I. Taking into account this and (3.10) we see that $(L\bar{\circ}u_x)_x$ minimizes the $L^2(\mathbb{T})$ norm among elements of $D(\mathcal{L})$. Thus, $\mathcal{L}^0 u = (L\bar{\circ}u_x)_x$. □

Remark. It is now an easy observation that $D(\mathcal{L})$ is dense in $L^2(\mathbb{T})$.

3.2 Higher regularity and facets

Formally, we may write

$$z_t = L(u_x)_t = L'(u_x)u_{xt} = L'(u_x)z_{xx}. \tag{3.12}$$

As $L' = 1 + \alpha\delta_0 > 0$ in $\mathcal{D}'(\mathbb{R})$, we could expect (3.12) to yield additional regularity of solutions to (1.30), but due to lack of proper definition of the term $L'(u_x)$ in (3.12) we need to proceed by approximation. Hence, let us denote by J_ε smoothened versions of J given by

$$J_\varepsilon(p) = \tfrac{1}{2}p^2 + \tfrac{\alpha}{2}(\varepsilon^2 + p^2)^{\frac{1}{2}}$$

and by L_ε its derivative

$$L_\varepsilon(p) = J'_\varepsilon(p) = p + \tfrac{\alpha}{2}\tfrac{p}{(\varepsilon^2+p^2)^{\frac{1}{2}}}.$$

In particular we have

$$1 \leq L'_\varepsilon(p) = 1 + \tfrac{\alpha}{2}\tfrac{\varepsilon^2}{(\varepsilon^2+p^2)^{\frac{3}{2}}} \leq 1 + \tfrac{\alpha}{2\varepsilon}. \tag{3.13}$$

Analyzing the approximate problem

$$u^\varepsilon_t = L_\varepsilon(u^\varepsilon_x)_x \quad \text{in } \mathbb{T} \tag{3.14}$$

we obtain the following result.

Lemma 3.2. *Let u be the solution to* (1.30) *with $u_0 \in D(\mathcal{L})$ and let $T > 0$. There holds*

$$\mathcal{L}^0 u \in L^2(0,T;H^1(\mathbb{T})) \cap L^\infty(0,T;L^2(\mathbb{T})).$$

If moreover $\mathcal{L}^0 u_0 \in L^\infty(\mathbb{T})$, then

$$\mathcal{L}^0 u \in L^\infty(]0,T[\times\mathbb{T}) \text{ and } u_x \in C^{\frac{1}{2}}([0,T]\times\mathbb{T}).$$

Proof. Using either monotone operators theory [8] or fixed point methods [54] we obtain the existence of a unique strong solution u^ε to (3.14) with initial datum u_0 in $H^1(0,T;L^2(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T}))$. Testing the problem with u^ε_{xx} and recalling (3.13), we obtain the following estimate independent of ε:

$$\frac{1}{2}\operatorname*{ess\,sup}_{t\in]0,T]} \|u^\varepsilon_x(t,\cdot)\|^2_{L^2(\mathbb{T})} + \|u^\varepsilon_{xx}\|^2_{L^2(0,T;L^2(\mathbb{T}))} \le \|u_{0,x}\|^2_{L^2(\mathbb{T})}. \tag{3.15}$$

We denote $z^\varepsilon = L_\varepsilon(u^\varepsilon_x)$. There holds

$$z^\varepsilon_t = L'_\varepsilon(u^\varepsilon_x) z^\varepsilon_{xx} \quad \text{in } \mathcal{D}'(\mathbb{T}). \tag{3.16}$$

The function $L'_\varepsilon(u^\varepsilon_x)$ is uniformly positive and bounded in $]0,T[\times\mathbb{T}$ for any given $\varepsilon > 0$. Freezing u^ε we may see (3.16) as a linear uniformly parabolic equation for z^ε. Supplying it with initial datum $L^\circ u_{0,x} \in H^1(\mathbb{T})$, we obtain a strong solution in the class $H^1(0,T;L^2(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T}))$, which clearly coincides with z^ε. Testing the problem with z^ε_{xx} we obtain (see [30, 5.9., Theorem 3]) the following estimate independent of ε

$$\frac{1}{2}\operatorname*{ess\,sup}_{t\in]0,T]} \|z^\varepsilon_x(t,\cdot)\|^2_{L^2(\mathbb{T})} + \|z^\varepsilon_{xx}\|^2_{L^2(0,T;L^2(\mathbb{T}))} \le \|\mathcal{L}^0 u_0\|^2_{L^2(\mathbb{T})}. \tag{3.17}$$

Now, we justify the limit passage as $\varepsilon \to 0^+$ in (3.14). We note that $\|u^\varepsilon_{xx}(t,\cdot)\|^2_{L^2(\mathbb{T})} \le \|z^\varepsilon_x(t,\cdot)\|^2_{L^2(\mathbb{T})}$ and $u^\varepsilon_t = z^\varepsilon_x$. It is also easy to check that u^ε and u^ε_x satisfy weak maximum and minimum principles. Summarizing, the following quantities are bounded independently of ε:

$$\|u^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{T})}, \quad \|u^\varepsilon_x\|_{L^\infty(]0,T[\times\mathbb{T})}, \quad \|z^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{T})},$$

$$\|u^\varepsilon_t\|_{L^2(]0,T[\times\mathbb{T})}, \quad \|u^\varepsilon_{xx}\|_{L^2(]0,T[\times\mathbb{T})}, \quad \|u^\varepsilon_{tx}\|_{L^2(]0,T[\times\mathbb{T})}.$$

Due to anisotropic Morrey embedding [47, Theorem 5], we also have a uniform bound on u^ε in $C^{\frac{1}{2}}(]0,T[\times\mathbb{T})$. Hence we can extract from the family $(u^\varepsilon, z^\varepsilon)$ a sequence $(u^{\varepsilon_n}, z^{\varepsilon_n})$ such that there exists a pair $(\overline{u}, \overline{z})$ with $\overline{u} \in H^1(0,T;H^1(\mathbb{T})) \cap L^2(0,T;H^2(\mathbb{T}))$, $\overline{z} \in L^\infty(]0,T[\times\mathbb{T}) \cap L^2(0,T;H^2(\mathbb{T}))$ satisfying

$$u^{\varepsilon_n} \to \overline{u} \text{ in } C([0,T]\times\mathbb{T}), \quad u^{\varepsilon_n}_x \to \overline{u}_x \text{ a.e. in }]0,T[\times\mathbb{T} \quad z^{\varepsilon_n} \overset{*}{\rightharpoonup} \overline{z} \text{ in } L^\infty(]0,T[\times\mathbb{T}),$$

$$\overline{u}_t = \overline{z}_x \text{ a.e. in }]0,T[\times\mathbb{T}.$$

It remains to check that $\overline{z}$ is a selection of $L(\overline{u}_x)$. Indeed, this follows from almost-everywhere convergence of $u^{\varepsilon_n}_x$ and uniform convergence of graphs of L^ε to the graph of multifunction L. Hence, due to uniqueness of solutions to (1.30), we see that $\overline{u} = u$. As a consequence, we get also $\overline{z} = L^\circ u_x$, i. e. $\overline{z}_x = \mathcal{L}^0 u$.

Next, suppose that $\mathcal{L}^0 u_0 \in L^\infty(\mathbb{T})$. We verify that for any $\varepsilon > 0$, z_x^ε satisfies weak maximum and minimum principles: for any $k > 0$ there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}}(z_x^\varepsilon \mp k)_\pm^2 = \langle (L'_\varepsilon(u_x^\varepsilon) z_{xx}^\varepsilon)_x, (z_x^\varepsilon \mp k)_\pm \rangle_{(H^{-1}(\mathbb{T}), H^1(\mathbb{T}))} = -\int_{\{\pm z^\varepsilon > \pm k\}} L'_\varepsilon(u_x^\varepsilon)(z_{xx}^\varepsilon)^2 \le 0$$

a. e. in $]0,T[$. Hence, by a standard argument, $\|z_x^\varepsilon\|_{L^\infty(]0,T[\times\mathbb{T})} \le \|\mathcal{L}^0 u_0\|_{L^\infty(\mathbb{T})}$ and consequently $\|\mathcal{L}^0 u\|_{L^\infty(]0,T[\times\mathbb{T})} \le \|\mathcal{L}^0 u_0\|_{L^\infty(\mathbb{T})}$.

Finally, we recall that $\|u_{xx}\|_{L^\infty(]0,T[\times\mathbb{T})} \le \|\mathcal{L}^0 u\|_{L^\infty(]0,T[\times\mathbb{T})}$. Therefore, again by [47, Theorem 5], $u_x \in C^{\frac{1}{2}}([0,T]\times\mathbb{T})$. □

Remark. If (3.12) was a regular parabolic equation, one would be able to obtain $\mathcal{L}^0 u$ at least in $C([0,T]; L^2(\mathbb{T}))$. The reasoning in the proof of Proposition 3.2 does not lead to such regularity, as the usual uniform estimate on $L^2(0,T;H^{-1}(\mathbb{T}))$ norm of z_x^ε does not hold.

Lemma 3.3. *Let $T^* = (\pi\alpha)^{-1}\|u_0 - \fint_{\mathbb{T}} u_0\|_{L^2(\mathbb{T})}$. The solution u is constant and equal to $\fint_{\mathbb{T}} u_0$ in $[T^*, \infty[\times\mathbb{T}$.*

Proof. Assume first that $\fint_{\mathbb{T}} u_0$ and consequently $\fint_{\mathbb{T}} u = 0$ in a. e. time instance. Testing the problem (1.30) with u we obtain

$$\tfrac{1}{2}\left(\int_{\mathbb{T}} u^2\right)_t = -\int_{\mathbb{T}} u_x^2 + \tfrac{\alpha}{2}|u_x| \tag{3.18}$$

in a. e. instance of time. As

$$2\pi\|u\|_{L^2(\mathbb{T})} \le \|u_x\|_{L^2(\mathbb{T})} \le \int_{\mathbb{T}} |u_x|, \tag{3.19}$$

this yields

$$\left(\|u\|_{L^2(\mathbb{T})}^2\right)_t \le -2\pi\alpha\|u\|_{L^2(\mathbb{T})}. \tag{3.20}$$

Solving this ODE yields

$$\|u(t,\cdot)\|_{L^2(\mathbb{T})} \le \|u_0\|_{L^2(\mathbb{T})} - \pi\alpha t \tag{3.21}$$

as long as $u(t,\cdot) \not\equiv 0$. Hence, $u \equiv 0$ for $t \ge (\pi\alpha)^{-1}\|u_0\|_{L^2(\mathbb{T})}$.

Finally, we relax the assumption of vanishing mean of u. It suffices to notice that $u - \fint_{\mathbb{T}} u_0$ is the solution to (1.30) with initial datum $u_0 - \fint_{\mathbb{T}} u_0$. □

Armed with this battery of lemmata, we are ready for the

Proof of Theorem 1.3. To prove the existence of functions F_m^k, we use a technique derived from [61]. Let $2m_0$ be the number of local extrema of u_0 (finite due to Lemma 3.1). Let t_{m_0} be the first time instance t such that $u(t,\cdot)$ is constant (well defined due to Lemma 3.3 and continuity of u).

Take s, t with $0 \le s < t < t_{m_0}$. Let $\{I_t^k, F_t^k \colon k \in \mathbb{Z}_{n_t}\}$ be the decomposition of $\mathbb{T}$ produced by Lemma 3.1 given $u(t,\cdot)$. For $k \in \mathbb{Z}_{n_t}$, take $y_k \in I_t^k$ such that $u_x(t,y_k) \neq 0$. Let $U_{s,t} = \{(t,x) \in]s,t[\times\mathbb{T}\colon u_x \neq 0\}$ and let A_k denote the connected component of $U_{s,t}$ such that $(t,y_k) \in \overline{A}_k$. Due to weak maximum principle, there exists $x_k \in \mathbb{T}$ such that $(s,x_k) \in \overline{A}_k$ and $u_x(s,x_k) \neq 0$. As A_k is an open, connected set, both A_k and $\overline{A}_k$ are path-connected. Let $\Gamma_k \colon [0,1] \to \overline{A}_k$ be any continuous path joining x_k and y_k such that $\Gamma_k(]0,1[) \subset A_k$, Clearly, u_x has constant sign along Γ_k; due to the choice of y_k, the sign is positive for k odd and negative for k even. Hence, $u(s,\cdot)$ has a local extremum in $]x_k, x_{k+1}[$ for $k \in \mathbb{Z}_{n_t}$. This proves that the number of local extrema of $u(t,\cdot)$ is not bigger than that of $u(s,\cdot)$.

For $m = 1,\dots,m_0-1$, we denote by t_m the greatest lower bound of the set of time instances t such that $u(t,\cdot)$ has not more than $2(m_0-m)$ local extrema. Due to continuity of u_x, $u(t,\cdot)$ has exactly $2(m_0-m)$ local extrema for $t \in [t_m, t_{m+1}[$, $m = 0,\dots,m_0-1$. For any $m = 0,\dots,m_0-1$ such that $t_m \neq t_{m+1}$, let us now take $s = t_m$, $t \in]t_m, t_{m+1}[$ and construct curves Γ_k, $k \in \mathbb{Z}_{2(m_0-m)}$ as before. As the sign of u_x on Γ_k is alternating with changing k, these curves cannot intersect. Hence, they preserve the cyclic order, i. e.

$$\Gamma_k \cap \{\tau\} \times \mathbb{T} < \Gamma_{k+1} \cap \{\tau\} \times \mathbb{T} < \Gamma_{k+2} \cap \{\tau\} \times \mathbb{T}$$

for $\tau \in [s,t]$. By a counting argument, this implies that $u(\tau,\cdot)$ attains exactly one local extremum between Γ_k and Γ_{k+1}. For $\tau \in [s,t]$, we define $F^k_m(\tau)$ as the set where this extremum is attained. Then, we take $s' = t$ and $t' \in]t, t_{m+1}[$. As before, we construct continuous paths Γ'_k joining points (s', x'_k) and (t', y'_k). We choose the numbering of these points so that $x'_k \in I^k_t$. As before, we use these paths to define $F^k_m(\tau)$ for $\tau \in]s', t']$. We also join Γ'_k with Γ_k by the line segment $\operatorname{conv}\{(t, y_k), (t, x'_k)\}$. Iterating this procedure, we obtain functions $F^k_m\colon [t_m, t_{m+1}[\to \mathfrak{F}(\mathbb{T})$ and continuous paths $\Gamma^k_m\colon [0,1] \to [t_m, t_{m+1}[\times \mathbb{T}$ such that the sets $F^k_{[t_m,t_{m+1}[} = \bigcup_{t\in[t_m,t_{m+1}[} \{t\} \times F^k_m(t)$ are separated from each other by curves Γ^k_m. We also define

$$I^k_m(t) = \{x \in \mathbb{T}\colon F^{k-1}_m(t) < x < F^k_m(t)\}$$

for $t \in [t_m, t_{m+1}[$. This way, we obtain an order-consistent choice of disjoint decompositions $\{I^k, F^k\} = \{I^k_m(t), F^k_m(t)\}$ of $\mathbb{T}$ satisfying conditions (i-iii) from Lemma 3.1 (with $u = u(t,\cdot)$). Hence, $|F^k_m(t)| \geq \alpha^2 \|\mathcal{L}^0 u_0\|^{-2}_{L^2(\mathbb{T})}$.

Let us denote $I^k_{]t_m,t_{m+1}[} = \bigcup_{t\in]t_m,t_{m+1}[} \{t\} \times I^k_m(t)$. Note that for any $t \in]t_m, t_{m+1}[$, we have $u_x \neq 0$ in some neighborhoods in $I^k_m(t)$ of its endpoints. Hence, $I^k_{]t_m,t_{m+1}[}$ coincides with the set of convex combinations of points $\boldsymbol{x} = (t,x)$, $\boldsymbol{x}' = (t,x')$ with $\boldsymbol{x}, \boldsymbol{x}' \in I^k_{]t_m,t_{m+1}[} \cap U_{t_m,t_{m+1}}$. As this latter set is open, it is easy to see that also $I^k_{]t_m,t_{m+1}[}$ is open. By Lemma 3.1, $\mathcal{L}^0 u = u_{xx}$ in $I^k_{]t_m,t_{m+1}[}$. Thus, u is a solution to the heat equation in $I^k_{]t_m,t_{m+1}[}$, in particular $u \in C^\infty(I^k_{]t_m,t_{m+1}[})$. By the strong maximum principle [58, Theorem 2.7], $u_x \neq 0$ in $I^k_{]t_m,t_{m+1}[}$.

Take $t \in]t_m, t_{m+1}[$. Due to continuity of u_x, we have

$$\limsup_{s\to t^-} F^k_m(s) \subset F^k_m(t). \tag{3.22}$$

The difference $F^k_m(t) \setminus \limsup_{s\to t^-} F^k_m(s)$ is contained in the closure of $I^k_{]t_m,t_{m+1}[} \cup I^{k+1}_{]t_m,t_{m+1}[}$ minus the parabolic boundary of this set. Again by the strong maximum principle [58, Theorem 2.7], $F^k_m(t) \setminus \limsup_{s\to t^-} F^k_m(s) = \emptyset$. The same reasoning proves that

$$\limsup_{s\to t^-} \{x \in \mathbb{T}\colon u_x(s,\cdot) = 0\} = \{x \in \mathbb{T}\colon u_x(t,\cdot) = 0\}$$

for any $t \in]0,T]$. For $t = t_m$, this implies (1.32).

Next, we take $t \in [t_m, t_{m+1}[$. Again,

$$\limsup_{s\to t^+} F^k_m(s) \subset F^k_m(t). \tag{3.23}$$

follows from continuity of u_x. For $s \in [t_m, t_{m+1}[$ denote $F_m^k(s) = [b(s), c(s)]$. Take $r > 0$ small enough so that $F_m^{k-1}(t) < b(t) - 2r < F_m^k(t) < c(t) + 2r < F_m^{k+1}(t)$. By virtue of (3.23), there exists $\delta = \delta(r) > 0$ such that

$$F_m^{k-1}(s) < b(t) - 2r < b(t) - r < F_m^k(s) < c(t) + r < c(t) + 2r < F_m^{k-1}(s)$$

if $t \le s \le t+\delta$. Now, consider the solution u^ε to the approximating problem (3.14) in $]t, T[\times \mathbb{T}$ with initial condition $u^\varepsilon(t, \cdot) = u(t, \cdot)$. Let us take a piecewise linear function $\eta \in C(\mathbb{T}, [0, 1])$ such that

$$\begin{aligned} \eta = 1 \text{ in } [b(t) - r, c(t) + r], &\quad \eta = 0 \text{ in } [c(t) + 2r, b(t) - 2r], \\ \eta' = \tfrac{1}{r} \text{ in }]b(t) - 2r, b(t) - r[, &\quad \eta' = -\tfrac{1}{r} \text{ in }]c(t) + r, c(t) + 2r[. \end{aligned}$$

Differentiating (3.16) with respect to x and testing the resulting equation with $z_x^\varepsilon \eta^2$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_{\mathbb{T}} |z_x^\varepsilon|^2\eta^2 = -\int_{\mathbb{T}} L_\varepsilon'(u_x^\varepsilon)|z_{xx}^\varepsilon|^2\eta^2 + 2\int_{\mathbb{T}} L_\varepsilon'(u_x^\varepsilon) z_{xx}^\varepsilon z_x^\varepsilon \eta\eta' \le \int_{\mathbb{T}} L_\varepsilon'(u_x^\varepsilon)|z_x^\varepsilon|^2|\eta'|^2.$$

Integrating this inequality over $[t, s]$ yields

$$\int_{b(t)-r}^{c(t)+r} |z_x^\varepsilon(s, \cdot)|^2 \le \int_{b(t)-2r}^{c(t)+2r} |z_x^\varepsilon(t, \cdot)|^2 + \frac{2}{r^2}\int_t^s \int_{A(t,r)} L_\varepsilon'(u_x^\varepsilon)|z_x^\varepsilon|^2, \tag{3.24}$$

where we have denoted $A(t, r) = [b(t) - 2r, b(t) - r] \cup [c(t) + r, c(t) + 2r]$. We recall that by Lemma 3.2,

$$\|z_x^\varepsilon\|_{L^\infty(]t,t+\delta[\times\mathbb{T})} \le \|\mathcal{L}^0 u(t, \cdot)\|_{L^\infty(\mathbb{T})}.$$

Hence, we can rewrite (3.24) as

$$\int_{b(t)-r}^{c(t)+r} |z_x^\varepsilon(s, \cdot)|^2 \le \int_{b(t)}^{c(t)} |\mathcal{L}^0 u(t, \cdot)|^2 + \left(4r + \frac{4}{r}(s-t)\max_{[t,s]\times A(t,r)} L_\varepsilon'(u_x^\varepsilon)\right)\|\mathcal{L}^0 u(t, \cdot)\|_{L^\infty(\mathbb{T})}. \tag{3.25}$$

Using observations from the proof of Lemma 3.2, we can pass to the limit with $(u^\varepsilon, z^\varepsilon)$ along a suitable sequence ε_n. By virtue of uniqueness of u, the limit coincides with $(u, L\bar{\circ}u)|_{[t,T]\times\mathbb{T}}$. Taking into account that $u_x^{\varepsilon_n}$ converges uniformly to u_x, which is bounded away from 0 on the closed set $[t, t+\delta] \times A(t, r)$, we have uniform convergence $L_{\varepsilon_n}(u_x^{\varepsilon_n}) \to 1$ on this set. Passing to the limit $\varepsilon_n \to 0^+$ in 3.25 yields

$$\operatorname{ess}\sup_{s\in[t,t+\delta]} \int_{b(t)-r}^{c(t)+r} |\mathcal{L}^0 u(s, \cdot)|^2 \le \int_{b(t)}^{c(t)} |\mathcal{L}^0 u(t, \cdot)|^2 + \left(4r + \frac{4\delta}{r}\right)\|\mathcal{L}^0 u(t, \cdot)\|_{L^\infty(\mathbb{T})}.$$

We can assume that $\delta \le r^2$. Recalling that $|\mathcal{L}^0 u(s, \cdot)| = \alpha|F_m^k(s)|^{-1}$ on $F_m^k(s)$,

$$\operatorname{ess}\sup_{s\in[t,t+\delta]} \alpha^2|F_m^k(s)|^{-1} \le \alpha^2|F_m^k(t)|^{-1} + 8r\|\mathcal{L}^0 u(t, \cdot)\|_{L^\infty(\mathbb{T})},$$

which we may further rewrite as

$$\operatorname{ess}\inf_{s\in[t,t+\delta]} |F_m^k(s)| \ge \frac{|F_m^k(t)|}{1 + 8r\|\mathcal{L}^0 u(t, \cdot)\|_{L^\infty(\mathbb{T})}\alpha^{-2}|F_m^k(t)|}. \tag{3.26}$$

As $r > 0$ can be chosen arbitrarily small, this implies right lower semicontinuity of $|F_m^k|$:

$$\liminf_{s\to t^+} |F_m^k(s)| \ge |F_m^k(t)| \tag{3.27}$$

for $t \in [t_m, t_{m+1}[$. Together with (3.23), this implies the desired right continuity of F_m^k.

□

Chapter 4

A local estimate for the total variation flow of curves

The proof of Theorem 1.4 is based on an estimate for a smooth approximation of total variation flow. Suppose for now that $\boldsymbol{u}_0 \in C^\infty(\overline{I})$ and all compatibility conditions $D_x^{2k}\boldsymbol{z}_0 = 0$ on ∂I, $k = 0, 1, \ldots$ hold with $\boldsymbol{z}_0 = \frac{\boldsymbol{u}^\varepsilon_{0,x}}{\sqrt{\varepsilon^2 + |\boldsymbol{u}^\varepsilon_{0,x}|^2}}$. From now on, we stop specifying the codomain in the notation for function spaces. Given $\varepsilon > 0$, let $\boldsymbol{u}^\varepsilon$ in $C^\infty([0,\infty[\times \overline{I})$ be the solution to the system

$$\boldsymbol{u}^\varepsilon_t = \boldsymbol{z}^\varepsilon_x \quad \text{in }]0,\infty[\times I, \tag{4.1}$$

$$\boldsymbol{z}^\varepsilon = \frac{\boldsymbol{u}^\varepsilon_x}{\sqrt{\varepsilon^2 + |\boldsymbol{u}^\varepsilon_x|^2}} \quad \text{in }]0,\infty[\times I \tag{4.2}$$

$$\boldsymbol{z}^\varepsilon(t,\cdot) = 0 \quad \text{on } \partial I. \tag{4.3}$$

$$\boldsymbol{u}^\varepsilon(0,\cdot) = \boldsymbol{u}_0, \tag{4.4}$$

Existence and regularity of the solution to the quasilinear system (4.1-4.4) is known (see e. g. Proposition 5.1).

For $x_0 \in \mathbb{R}$ and $r > 0$, let us denote by $B(x_0, r)$ the **closed** interval $[x_0 - r, x_0 + r]$.

Lemma 4.1. *Let $p \in]1, 3]$ and $t > 0$. Let $x_0 \in I$ and $0 < r < R$ be such that $B(x_0, R) \subset I$. There holds*

$$\int_{B(x_0,r)} \left(|\boldsymbol{u}^\varepsilon_x(t,\cdot)|^2 + \varepsilon^2\right)^{\frac{p}{2}} \le \int_{B(x_0,R)} \left(|\boldsymbol{u}_{0,x}|^2 + \varepsilon^2\right)^{\frac{p}{2}} + \frac{p}{p-1}\frac{2\varepsilon^{p-1}t}{R-r}. \tag{4.5}$$

Proof. In the following calculations we will omit the index ε. First of all, we note that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\boldsymbol{u}_x|^2 = \boldsymbol{u}_x \cdot \boldsymbol{z}_{xx}. \tag{4.6}$$

Now, let φ be the Lipschitz cutoff function such that $\operatorname{supp}\varphi = B(x_0, R)$, $\varphi = 1$ in $B(x_0, r)$ and $|\varphi_x| = \frac{1}{R-r}$ in $B(x_0, R) \setminus B(x_0, r)$. Given $p \ge 1$, we calculate

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \varphi^2 \left(|\boldsymbol{u}_x|^2 + \varepsilon^2\right)^{\frac{p}{2}} &= \int_I \varphi^2 \left(|\boldsymbol{u}_x|^2 + \varepsilon^2\right)^{\frac{p}{2}-1} \boldsymbol{u}_x \cdot \boldsymbol{z}_{xx} = \int_I \varphi^2 \left(|\boldsymbol{u}_x|^2 + \varepsilon^2\right)^{\frac{p}{2}-\frac{1}{2}} \boldsymbol{z} \cdot \boldsymbol{z}_{xx} \\ &= -\int_I \varphi^2 \left(|\boldsymbol{u}_x|^2 + \varepsilon^2\right)^{\frac{p}{2}-\frac{1}{2}} |\boldsymbol{z}_x|^2 - (p-1)\int_I \varphi^2 \left(|\boldsymbol{u}_x|^2 + \varepsilon^2\right)^{\frac{p}{2}-\frac{3}{2}} \boldsymbol{u}_x \cdot \boldsymbol{u}_{xx}\, \boldsymbol{z} \cdot \boldsymbol{z}_x \\ &\quad - 2\int_I \varphi\,\varphi_x \left(|\boldsymbol{u}_x|^2 + \varepsilon^2\right)^{\frac{p}{2}-\frac{1}{2}} \boldsymbol{z} \cdot \boldsymbol{z}_x.\end{aligned} \tag{4.7}$$

We have

$$\boldsymbol{z}\cdot\boldsymbol{z}_x = \frac{1}{2}\left(|\boldsymbol{z}|^2\right)_x = \frac{1}{2}\left(\frac{|\boldsymbol{u}_x|^2}{|\boldsymbol{u}_x|^2+\varepsilon^2}\right)_x = \frac{1}{2}\left(1-\frac{\varepsilon^2}{|\boldsymbol{u}_x|^2+\varepsilon^2}\right)_x = \frac{\varepsilon^2\boldsymbol{u}_x\cdot\boldsymbol{u}_{xx}}{(|\boldsymbol{u}_x|^2+\varepsilon^2)^2}. \tag{4.8}$$

Substituting (4.8) into (4.7) yields

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_I \varphi^2\left(|\boldsymbol{u}_x|^2+\varepsilon^2\right)^{\frac{p}{2}} \leq -(p-1)\varepsilon^2\int_I \varphi^2\left(|\boldsymbol{u}_x|^2+\varepsilon^2\right)^{\frac{p}{2}-\frac{7}{2}}|\boldsymbol{u}_x\cdot\boldsymbol{u}_{xx}|^2 \\ -2\varepsilon^2\int_I \varphi\,\varphi_x\left(|\boldsymbol{u}_x|^2+\varepsilon^2\right)^{\frac{p}{2}-\frac{5}{2}}\boldsymbol{u}_x\cdot\boldsymbol{u}_{xx}\end{aligned} \tag{4.9}$$

We treat the last term in (4.9) with Cauchy's inequality, obtaining

$$\begin{aligned}2\int_I \varphi\,\varphi_x\left(|\boldsymbol{u}_x|^2+\varepsilon^2\right)^{\frac{p}{2}-\frac{5}{2}}\boldsymbol{u}_x\cdot\boldsymbol{u}_{xx} \leq (p-1)\int_I \varphi^2\left(|\boldsymbol{u}_x|^2+\varepsilon^2\right)^{\frac{p}{2}-\frac{7}{2}}|\boldsymbol{u}_x\cdot\boldsymbol{u}_{xx}|^2 \\ +\frac{1}{p-1}\int_I\left(|\boldsymbol{u}_x|^2+\varepsilon^2\right)^{\frac{p}{2}-\frac{3}{2}}\varphi_x^2.\end{aligned}$$

Assuming that $1 < p \leq 3$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_I \varphi^2\left(|\boldsymbol{u}_x|^2+\varepsilon^2\right)^{\frac{p}{2}} \leq \frac{p}{p-1}\frac{2\varepsilon^{p-1}}{|R-r|} \tag{4.10}$$

which, integrated over time, yields (4.5). □

Let us remark that expanding the first term on the right hand side of (4.7) shows it to be of order ε^4, hence it is not useful for obtaining a completely local estimate. For this reason, approximation step with $p > 1$ cannot be omitted in presented reasoning.

With Lemma 4.1 at hand, it is easy to conclude.

Proof of Theorem 1.4. With our assumptions, the families $(\boldsymbol{u}_x^\varepsilon)$ and $\boldsymbol{z}^\varepsilon$ are uniformly bounded in $L^\infty(]0,\infty[\times I)$ and the family $(\boldsymbol{u}_t^\varepsilon)$ is uniformly bounded in $L^2(]0,\infty[\times I)$, see Lemmata 5.1 and 5.3. Hence, there is a sequence (ε_k) and a pair $(\boldsymbol{u},\boldsymbol{z})$ satisfying (1.34-1.38), such that

$$\boldsymbol{u}^{\varepsilon_k}\to\boldsymbol{u} \quad \text{in } C([0,\infty[\times\overline{I}) \quad \text{and} \quad \boldsymbol{z}^{\varepsilon_k}\overset{*}{\rightharpoonup}\boldsymbol{z} \quad \text{in } L^\infty(]0,\infty[\times I).$$

Due to lower semicontinuity of $\int_{B(r,x_0)}|\boldsymbol{u}_x|^p$ with respect to uniform convergence of $\boldsymbol{u}$, passing to the limit $\varepsilon\to 0^+$ with (4.5) yields

$$\int_{B(x_0,r)}|\boldsymbol{u}_x(t,\cdot)|^p \leq \int_{B(x_0,R)}|\boldsymbol{u}_{0,x}|^p \tag{4.11}$$

and, after limit passages $p\to 1^+$, $R\to r^+$,

$$\int_{B(x_0,r)}|\boldsymbol{u}_x(t,\cdot)| \leq \int_{B(x_0,r)}|\boldsymbol{u}_{0,x}|.$$

Next, we remove the smoothness assumption on $\boldsymbol{u}_0$. We take a sequence $(\boldsymbol{u}_0^k)\subset C^\infty(\overline{I})$ converging to $\boldsymbol{u}_0$ strictly in $BV(I)$ as $k\to\infty$. To guarantee that $\boldsymbol{u}_0^k$ satisfy compatibility conditions for (4.1-4.4), we construct them by extending $\boldsymbol{u}_0$ beyond I by even reflection with respect to endpoints and convolving the result with standard even mollifying kernel. The sequence $(\boldsymbol{u}_0^k)$ gives rise to a sequence of solutions $(\boldsymbol{u}^k, \boldsymbol{z}^k)$ to (1.34-1.37) such that $(\boldsymbol{u}^k)$ is uniformly bounded in $L^\infty(0,\infty; BV(I))$, $(\boldsymbol{u}_t^k)$ is uniformly bounded in $L^2(]0,\infty[\times I)$, $(\boldsymbol{z}^k)$ is uniformly bounded in $L^\infty(]0,\infty[\times I)$ and for all $x_0 \in I$, $r > 0$ with $B(x_0, r) \subset I$,

$$\int_{B(x_0,r)} |\boldsymbol{u}_x^k(t,\cdot)| \le \int_{B(x_0,r)} |\boldsymbol{u}_{0,x}^k|. \tag{4.12}$$

We extract a subsequence (renamed immediately to $(\boldsymbol{u}^k, \boldsymbol{z}^k)$) such that

$$\boldsymbol{u}^k \to \boldsymbol{u} \quad \text{in } L^2(]0,\infty[\times I) \quad \text{and} \quad \boldsymbol{z}^k \overset{*}{\rightharpoonup} \boldsymbol{z} \quad \text{in } L^\infty(]0,\infty[\times I),$$

where $(\boldsymbol{u}, \boldsymbol{z})$ is the solution to (1.34-1.38). Fix $t > 0$. Due to lower semicontinuity of total variation, passing to the limit $k \to \infty$ with (4.12) we get

$$|\boldsymbol{u}_x(t,\cdot)|(B(x_0,r)) \le |\boldsymbol{u}_{0,x}|(B(x_0,r)) \tag{4.13}$$

for any ball $B(x_0, r) \subset I$ such that (cf. [5, Proposition 3.7])

$$|\boldsymbol{u}_{0,x}|(\partial B(x_0,r)) = 0 \tag{4.14}$$

Property (4.14) is satisfied for every $x_0 \in I$ and almost every $r > 0$ such that $B(x_0, r) \subset I$.

Hence, by [31, 1.5.2., Corollary 1], up to a set of zero $|\boldsymbol{u}_x(t,\cdot)|$ measure we can fill any open set $U \subset I$ with a countable family of disjoint closed balls $B(x_k, r_k)$ contained in U and satisfying (4.14): hence

$$|\boldsymbol{u}_x(t,\cdot)|(U) = \sum_{k=1}^\infty |\boldsymbol{u}_x(t,\cdot)|\,(B(x_k,r_k)) \le \sum_{k=1}^\infty |\boldsymbol{u}_{0,x}|\,(B(x_k,r_k)) \le |\boldsymbol{u}_{0,x}|(U).$$

Finally, by virtue of [31, 1.1.1., Lemma 1], given a Borel $A \subset I$ and $\varepsilon > 0$ we can find an open U with $A \subset U$ and $|\boldsymbol{u}_{0,x}|(U \setminus A) \le \varepsilon$. Therefore,

$$|\boldsymbol{u}_x(t,\cdot)|(A) \le |\boldsymbol{u}_x(t,\cdot)|(U) \le |\boldsymbol{u}_{0,x}|(U) \le |\boldsymbol{u}_{0,x}|(A) + \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, we are done. □

Chapter 5

Regular 1-harmonic map flow

5.1 Uniqueness

In this section, we give the proof of Theorem 1.5.

Let $(\boldsymbol{u}^1, \boldsymbol{Z}^1)$, $(\boldsymbol{u}^2, \boldsymbol{Z}^2)$ be two regular solutions to (1.44, 1.45). For $i = 1, 2$ there holds

$$\boldsymbol{u}^i_t = \operatorname{div} \boldsymbol{Z}^i + \mathcal{A}_{\boldsymbol{u}^i}(\boldsymbol{u}^i_{x^j}, \boldsymbol{Z}^i_j).$$

Here and in the rest of this section, $\boldsymbol{u}^i_{x^j}$ and $\boldsymbol{Z}^i_j$ denote, respectively, the derivative of $\boldsymbol{u}^i$ in direction of x^j and the j-th component of $\boldsymbol{Z}^i$, $i = 1, 2$, $j = 1, \ldots, m$. We calculate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{u}^1 - \boldsymbol{u}^2|^2 = \int_\Omega (\boldsymbol{u}^1 - \boldsymbol{u}^2) \cdot (\operatorname{div} \boldsymbol{Z}^1 - \operatorname{div} \boldsymbol{Z}^2) \\ + \int_\Omega (\boldsymbol{u}^1 - \boldsymbol{u}^2) \cdot (\mathcal{A}_{\boldsymbol{u}^1}(\boldsymbol{u}^1_{x^j}, \boldsymbol{Z}^1_j) - \mathcal{A}_{\boldsymbol{u}^2}(\boldsymbol{u}^2_{x^j}, \boldsymbol{Z}^2_j)). \quad (5.1)\end{aligned}$$

In the first term on the r. h. s. of (5.1) we integrate by parts, yielding

$$\int_\Omega (\boldsymbol{u}^1 - \boldsymbol{u}^2) \cdot (\operatorname{div} \boldsymbol{Z}^1 - \operatorname{div} \boldsymbol{Z}^2) = -\int_\Omega \left(|\nabla \boldsymbol{u}^1| - \nabla \boldsymbol{u}^1 : \boldsymbol{Z}^2 + |\nabla \boldsymbol{u}^2| - \nabla \boldsymbol{u}^2 : \boldsymbol{Z}^1\right)$$

which is non-positive as $|\boldsymbol{Z}^i| \le 1$, $i = 1, 2$. Next, we note that for any $\boldsymbol{p}^1, \boldsymbol{p}^2 \in \mathcal{N}$ contained in a fixed compact subset K of $\mathcal{N}$ we have

$$|\pi^\perp_{\boldsymbol{p}^i}(\boldsymbol{p}^1 - \boldsymbol{p}^2)| \le C_1(K)|\boldsymbol{p}^1 - \boldsymbol{p}^2|^2$$

for $i = 1, 2$. The exponent two on the right-hand side follows from the second-order Taylor expansion of $\pi^\perp_{\boldsymbol{p}^i}(\boldsymbol{p} - \boldsymbol{p}^i)$ around $\boldsymbol{p}^i$: indeed

$$\boldsymbol{p} - \boldsymbol{p}^i = \exp^{-1}_{\boldsymbol{p}^i} \boldsymbol{p} + O(|\boldsymbol{p} - \boldsymbol{p}^i|^2)$$

in a neighborhood $\mathcal{U} \subset \mathcal{N}$ of $\boldsymbol{p}^i$, where $\exp^{-1}_{\boldsymbol{p}^i} \colon \mathcal{U} \to T_{\boldsymbol{p}^i}\mathcal{N}$ is the logarithmic map of $\mathcal{N}$ at $\boldsymbol{p}^i$ (see e. g. [62, Lemma A.1]). Such exponent is crucial for the following Gronwall-type argument.

As $\boldsymbol{u}^1$, $\boldsymbol{u}^2$ are continuous on $[0, T] \times \overline{\Omega}$ (we assume without loss of generality that T is finite), there is indeed a compact set $K = K(\boldsymbol{u}^1, \boldsymbol{u}^2)$ in $\mathcal{N}$ with $\boldsymbol{u}^i([0, T] \times \overline{\Omega}) \subset K$, $i = 1, 2$.

Therefore, as $\mathcal{A}_{\boldsymbol{u}^i}$ is valued in $T_{\boldsymbol{u}^i}\mathcal{N}^\perp$ $(i = 1, 2)$ there is a constant C_2 depending on K and the norms of $\nabla \boldsymbol{u}^1$, $\nabla \boldsymbol{u}^2$ in $L^\infty(]0, T[\times\Omega, \mathbb{R}^N)$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\boldsymbol{u}^1 - \boldsymbol{u}^2|^2 \le C_2 \int_\Omega |\boldsymbol{u}^1 - \boldsymbol{u}^2|^2$$

for a. e. $t \in]0, T[$. Thus, if $\boldsymbol{u}^1(0, \cdot) = \boldsymbol{u}^2(0, \cdot)$, we have $\boldsymbol{u}^1 \equiv \boldsymbol{u}^2$ due to Gronwall's lemma.

5.2 The approximate system

In this section, $\Omega \subset \mathbb{R}^m$ is assumed to be an open, bounded, smooth, convex domain and $0 < \alpha < 1$. Given $\varepsilon > 0$, $T \in]0, \infty]$ we consider the approximating system for $\boldsymbol{u}^\varepsilon \colon [0, T[\times\Omega \to \mathcal{N}$:

$$\boldsymbol{u}^\varepsilon_t = \pi_{\boldsymbol{u}^\varepsilon}\left(\operatorname{div} \frac{\nabla \boldsymbol{u}^\varepsilon}{\sqrt{\varepsilon^2 + |\nabla \boldsymbol{u}^\varepsilon|^2}}\right) \quad \text{in }]0, T[\times\Omega, \tag{5.2}$$

$$\boldsymbol{\nu}^\Omega \cdot \nabla \boldsymbol{u}^\varepsilon = \mathbf{0} \quad \text{in }]0, T[\times\partial\Omega, \tag{5.3}$$

$$\boldsymbol{u}^\varepsilon(0, \cdot) = \boldsymbol{u}_0. \tag{5.4}$$

Further in this section, we will drop the index ε and denote $\boldsymbol{Z} = \frac{\nabla \boldsymbol{u}}{\sqrt{\varepsilon^2 + |\nabla \boldsymbol{u}|^2}}$.

We will obtain solutions to (5.2, 5.3, 5.4) in parabolic Hölder spaces as defined in [54, Chapter I]. Let us introduce some necessary notation. Given numbers $k = 0, 1, \ldots$, $0 < \alpha < 1$ and an interval I, we write $C^{\frac{k+\alpha}{2}, k+\alpha}(\Omega_I, \mathbb{R}^N)$ for the parabolic Hölder space on $\Omega_I = I \times \Omega$ of order $k + \alpha$. Similarly, we write $\boldsymbol{u} \in C_{loc}^{\frac{k+\alpha}{2}, k+\alpha}(\overline{\Omega}_I, \mathbb{R}^N)$ if $\boldsymbol{u} \in C^{\frac{k+\alpha}{2}, k+\alpha}(\Omega_K, \mathbb{R}^N)$ for every interval K compactly included in I.

5.2.1 Uniform bounds

In this subsection, we prove essential a priori estimates for $\boldsymbol{u} \in C_{loc}^{\frac{3+\alpha}{2}, 3+\alpha}(\overline{\Omega}_{[0,T[}, \mathcal{N})$ solving (5.2, 5.3) with a given $\varepsilon, T > 0$. For brevity, we denote

$$v = \left(|\nabla \boldsymbol{u}|^2 + \varepsilon^2\right)^{\frac{1}{2}}, \qquad v_0 = \left(|\nabla \boldsymbol{u}_0|^2 + \varepsilon^2\right)^{\frac{1}{2}}.$$

The basic energy estimate reflects the gradient flow structure behind (5.2, 5.3).

Lemma 5.1. *Let $\boldsymbol{u} \in C_{loc}^{\frac{3+\alpha}{2}, 3+\alpha}(\overline{\Omega}_{[0,T[}, \mathcal{N})$ satisfy (5.2, 5.3). Then*

$$\sup_{t \in [0,T[} \int_\Omega v(t, \cdot) + \int_0^T \int_\Omega \boldsymbol{u}_t^2 \le \int_\Omega v_0. \tag{5.5}$$

Proof. The estimate follows from the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v = \int_\Omega \boldsymbol{Z} : \nabla \boldsymbol{u}_t = -\int_\Omega \boldsymbol{u}_t^2$$

which holds as $\boldsymbol{u}_t(t, \boldsymbol{x}) \in T_{\boldsymbol{u}(t,\boldsymbol{x})}\mathcal{N}$ for $(t, \boldsymbol{x}) \in]0, T[\times\Omega$. □

In order to derive further uniform bounds, our main tool is the following version of Bochner's identity (see [59, Chapter 1.] for the case of harmonic maps).

Lemma 5.2. *Let* $\boldsymbol{u} \in C_{loc}^{\frac{3+\alpha}{2},3+\alpha}(\overline{\Omega}_{[0,T[},\mathcal{N})$ *satisfy* (5.2)*. Then, on* $]0,T[\times\Omega$*,*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla \boldsymbol{u}|^2 = (\nabla \boldsymbol{u} : \nabla \boldsymbol{Z}_i)_{x^i} - (\pi_{\boldsymbol{u}} \nabla^2 \boldsymbol{u}) \,\vdots\, \nabla \boldsymbol{Z} + \boldsymbol{Z}_i \cdot \mathcal{R}^{\mathcal{N}}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{u}_{x^j})\boldsymbol{u}_{x^j}. \tag{5.6}$$

Proof. Given $t \in]0,T[$, $\boldsymbol{x} \in \Omega$, we choose a local orthonormal frame $(\boldsymbol{N}^k)_{k=1,\ldots,N-n}$ on $\mathcal{N}$ around $\boldsymbol{u}(t,\boldsymbol{x})$. For any $\boldsymbol{p} \in \mathcal{N}$ close enough to $\boldsymbol{u}(t,\boldsymbol{x})$, we express using this frame

$$\pi^{\perp}_{\boldsymbol{p}} = \boldsymbol{N}^k_{\boldsymbol{p}} \otimes \boldsymbol{N}^k_{\boldsymbol{p}}, \quad \mathcal{A}_{\boldsymbol{p}}(\boldsymbol{X},\boldsymbol{Y}) = (\boldsymbol{X} \cdot D_{\boldsymbol{p}} \boldsymbol{N}^k \, \boldsymbol{Y})\boldsymbol{N}^k_{\boldsymbol{p}}, \tag{5.7}$$

where $\boldsymbol{X},\boldsymbol{Y} \in T_{\boldsymbol{p}}\mathcal{N}$, $\boldsymbol{N}^k_{\boldsymbol{p}}$ denotes the value of $\boldsymbol{N}^k$ at $\boldsymbol{p}$ and $D_{\boldsymbol{p}}\boldsymbol{N}^k \colon T_{\boldsymbol{p}}\mathcal{N} \to \mathbb{R}^N$ is the tangent map to $\boldsymbol{N}^k$ at $\boldsymbol{p}$, that is $D_{\boldsymbol{w}}\boldsymbol{N}^k \, \boldsymbol{w}_s = (\boldsymbol{N}^k_{\boldsymbol{w}})_s$ for any C^1 curve $s \mapsto \boldsymbol{w}(s) \in \mathcal{N}$. We recall that $\mathcal{A}_{\boldsymbol{p}}$ is symmetric and does not depend on the choice of $(\boldsymbol{N}^k)$ [74, Chapter 7], and by convention

$$\boldsymbol{X} \otimes \boldsymbol{Y}\, \boldsymbol{u} = (\boldsymbol{Y} \cdot \boldsymbol{u})\boldsymbol{X}, \quad (\boldsymbol{X} \otimes \boldsymbol{Y}\, \boldsymbol{u}) \cdot \boldsymbol{v} = (\boldsymbol{Y} \cdot \boldsymbol{u})(\boldsymbol{X} \cdot \boldsymbol{v}) \tag{5.8}$$

for any $\boldsymbol{X},\boldsymbol{Y},\boldsymbol{u},\boldsymbol{v} \in \mathbb{R}^N$.

First, we calculate

$$-\boldsymbol{N}^k_{\boldsymbol{u}} \otimes \boldsymbol{N}^k_{\boldsymbol{u}} \operatorname{div} \boldsymbol{Z} = -\boldsymbol{N}^k_{\boldsymbol{u}} \underbrace{(\boldsymbol{N}^k_{\boldsymbol{u}} \cdot \boldsymbol{Z}_j)}_{=0}{}_{x^j} + \boldsymbol{N}^k_{\boldsymbol{u}}((\boldsymbol{N}^k_{\boldsymbol{u}})_{x^j} \cdot \boldsymbol{Z}_j) = \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^j}, \boldsymbol{Z}_j)$$

which allows us to rewrite (5.2) as

$$\boldsymbol{u}_t = \operatorname{div} \boldsymbol{Z} + \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^j}, \boldsymbol{Z}_j). \tag{5.9}$$

Using (5.9), we obtain

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla \boldsymbol{u}|^2 &= \nabla \boldsymbol{u} : \nabla \operatorname{div} \boldsymbol{Z} + \nabla \boldsymbol{u} : \nabla \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{Z}_i) \\ &= (\nabla \boldsymbol{u} : \nabla \boldsymbol{Z}_i)_{x^i} - \nabla^2 \boldsymbol{u} \,\vdots\, \nabla \boldsymbol{Z} + \big(\underbrace{\boldsymbol{u}_{x^j} \cdot \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{Z}_i)}_{=0}\big)_{x^j} - \Delta \boldsymbol{u} \cdot \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{Z}_i) \end{aligned} \tag{5.10}$$

where in the last line we used that $\mathcal{A}_{\boldsymbol{u}}$ is orthogonal to $\boldsymbol{u}_{x^j} \in T_{\boldsymbol{u}}\mathcal{N}$.

Next we perform the following calculations:

$$\begin{aligned} (\pi^{\perp}_{\boldsymbol{u}} \nabla^2 \boldsymbol{u}) \,\vdots\, \nabla \boldsymbol{Z} &\overset{(5.7)}{=} (\boldsymbol{N}^k_{\boldsymbol{u}} \otimes \boldsymbol{N}^k_{\boldsymbol{u}} \nabla^2 \boldsymbol{u}) \,\vdots\, \nabla \boldsymbol{Z} = (\boldsymbol{N}^k_{\boldsymbol{u}} \otimes \boldsymbol{N}^k_{\boldsymbol{u}}\, \boldsymbol{u}_{x^i x^j}) \cdot \boldsymbol{Z}_{i,x^j} \overset{(5.8)_2}{=} \boldsymbol{N}^k_{\boldsymbol{u}} \cdot \boldsymbol{u}_{x^i x^j}\, \boldsymbol{N}^k_{\boldsymbol{u}} \cdot \boldsymbol{Z}_{i,x^j} \\ &= \big((\underbrace{\boldsymbol{N}^k_{\boldsymbol{u}} \cdot \boldsymbol{u}_{x^i}}_{=0})_{x^j} - (\boldsymbol{N}^k_{\boldsymbol{u}})_{x^j} \cdot \boldsymbol{u}_{x^i}\big)\big((\underbrace{\boldsymbol{N}^k_{\boldsymbol{u}} \cdot \boldsymbol{Z}_i}_{=0})_{x^j} - (\boldsymbol{N}^k_{\boldsymbol{u}})_{x^j} \cdot \boldsymbol{Z}_i\big) \\ &= \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{u}_{x^j}) \cdot \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^j}, \boldsymbol{Z}_i) \end{aligned}$$

and similarly

$$\pi^{\perp}_{\boldsymbol{u}} \Delta \boldsymbol{u} \overset{(5.8)_1}{=} (\boldsymbol{N}^k_{\boldsymbol{u}} \cdot \boldsymbol{u}_{x^j x^j})\boldsymbol{N}^k_{\boldsymbol{u}} = \big((\underbrace{\boldsymbol{N}^k_{\boldsymbol{u}} \cdot \boldsymbol{u}_{x^j}}_{=0})_{x^j} - (\boldsymbol{N}^k_{\boldsymbol{u}})_{x^j} \cdot \boldsymbol{u}_{x^j}\big)\boldsymbol{N}^k_{\boldsymbol{u}} = -\mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^j}, \boldsymbol{u}_{x^j}),$$

so that

$$\Delta \boldsymbol{u} \cdot \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{Z}_i) = \pi^{\perp}_{\boldsymbol{u}} \Delta \boldsymbol{u} \cdot \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{Z}_i) = -\mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^j}, \boldsymbol{u}_{x^j}) \cdot \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{Z}_i).$$

Hence, (5.10) may be rewritten as

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla \boldsymbol{u}|^2 = (\nabla \boldsymbol{u} : \nabla \boldsymbol{Z}_i)_{x^i} - \left(\pi_{\boldsymbol{u}} \nabla^2 \boldsymbol{u}\right) \vdots \nabla \boldsymbol{Z} \\ - \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{u}_{x^j}) \cdot \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^j}, \boldsymbol{Z}_i) + \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^j}, \boldsymbol{u}_{x^j}) \cdot \mathcal{A}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{Z}_i).$$

Finally, we recall the Gauss-Codazzi equation

$$\boldsymbol{W} \cdot \mathcal{R}^{\mathcal{N}}_{\boldsymbol{p}}(\boldsymbol{X}, \boldsymbol{Y})\boldsymbol{Z} = \mathcal{A}_{\boldsymbol{p}}(\boldsymbol{Y}, \boldsymbol{Z}) \cdot \mathcal{A}_{\boldsymbol{p}}(\boldsymbol{X}, \boldsymbol{W}) - \mathcal{A}_{\boldsymbol{p}}(\boldsymbol{X}, \boldsymbol{Z}) \cdot \mathcal{A}_{\boldsymbol{p}}(\boldsymbol{Y}, \boldsymbol{W})$$

for any quadruple of vectors $\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}, \boldsymbol{W} \in T_{\boldsymbol{p}}\mathcal{N}$, $p \in \mathcal{N}$, which finishes the proof. □

We are now ready to derive uniform Lipschitz bounds.

Lemma 5.3. *Let* $\boldsymbol{u} \in C^{\frac{3+\alpha}{2}, 3+\alpha}_{loc}(\overline{\Omega}_{[0,T[}, \mathcal{N})$ *satisfy (5.2-5.4).*

(i) If $K_{\mathcal{N}} \in]0, \infty[$*, then*

$$\|v(t,\cdot)\|_{L^\infty} \leq \frac{\|v_0\|_{L^\infty}}{1 - t\, K_{\mathcal{N}}\, \|v_0\|_{L^\infty}} \tag{5.11}$$

for $t \in]0, \min(T_\dagger, T)[$*, where* $T_\dagger := (K_{\mathcal{N}}\, \|v_0\|_{L^\infty})^{-1}$*.*

(ii) If $K_{\mathcal{N}} \leq 0$*, then for* $0 < t < T < T_\dagger := +\infty$ *there holds*

$$\|v(t,\cdot)\|_{L^\infty} \leq \|v_0\|_{L^\infty}\,. \tag{5.12}$$

Proof. Given a finite $p \geq 1$, using (5.6) and integrating by parts, we calculate

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega v^p = \int_\Omega v^{p-2} \nabla \boldsymbol{u} : \nabla \boldsymbol{u}_t \\ = -\int_\Omega v^{p-2} (\pi_{\boldsymbol{u}} \nabla^2 \boldsymbol{u}) \vdots \nabla \boldsymbol{Z} - (p-2) \int_\Omega v^{p-4} \nabla \boldsymbol{u} : \nabla^2 \boldsymbol{u} \cdot \nabla \boldsymbol{Z} : \nabla \boldsymbol{u} \\ + \int_{\partial\Omega} v^{p-2} (\nabla \boldsymbol{u} : \nabla \boldsymbol{Z}_i)(\boldsymbol{\nu}^\Omega)^i + \int_\Omega v^{p-3} \boldsymbol{u}_{x^i} \cdot \mathcal{R}^{\mathcal{N}}_{\boldsymbol{u}}(\boldsymbol{u}_{x^i}, \boldsymbol{u}_{x^j}) \boldsymbol{u}_{x^j}. \tag{5.13}$$

We have

$$Z^i_{j,x^k} = v^{-1}\left(u^i_{x^j x^k} - Z^i_j (\nabla \boldsymbol{u}_{x^k} : \boldsymbol{Z})\right)$$

and

$$\nabla \boldsymbol{Z}_j : \nabla \boldsymbol{u} = v^{-1} (\nabla \boldsymbol{u} : \nabla \boldsymbol{u}_{x^j} - Z^i_j\, Z^i_k\, \nabla \boldsymbol{u}_{x^k} : \nabla \boldsymbol{u})$$

for $i = 1, \ldots, N$ and $j, k = 1, \ldots, m$. Thus, we can rewrite

$$\nabla \boldsymbol{u} : \nabla^2 \boldsymbol{u} \cdot \nabla \boldsymbol{Z} : \nabla \boldsymbol{u} = v^{-1} \nabla \boldsymbol{u} : \nabla \boldsymbol{u}_{x^j} (I^m_{jk} - Z^i_j Z^i_k) \nabla \boldsymbol{u}_{x^k} : \nabla \boldsymbol{u} \tag{5.14}$$

(we use the notation $\mathbf{I}^l = (I^l_{jk} : j, k = 1, \ldots, l)$ for the l-dimensional identity matrix). On the other hand,

$$(\pi_{\boldsymbol{u}} \nabla^2 \boldsymbol{u}) \vdots \nabla \boldsymbol{Z} = v^{-1} (\pi_{\boldsymbol{u}} \nabla \boldsymbol{u}_{x^j}) : (\mathbf{I}^m \otimes \mathbf{I}^N - \boldsymbol{Z} \otimes \boldsymbol{Z}) : \nabla \boldsymbol{u}_{x^j}. \tag{5.15}$$

From (5.14), (5.15) and the fact that $|\boldsymbol{Z}| \leq 1$, it is clear that, provided $p \geq 2$, the first two terms on the r. h. s. of (5.13) are non-positive. To treat the remaining boundary term, we extend $\boldsymbol{\nu}^\Omega$ to a normal tubular neighbourhood of $\partial\Omega$ in such a way that it is constant in the fibers, and calculate (at points in $\partial\Omega$)

$$(\nabla \boldsymbol{u} : \nabla \boldsymbol{Z}_i)(\boldsymbol{\nu}^\Omega)^i = \nabla u^j \cdot \nabla (Z_i^j (\boldsymbol{\nu}^\Omega)^i) - \nabla u^j \cdot \nabla (\boldsymbol{\nu}^\Omega)^i \, Z_i^j = -v^{-1} \boldsymbol{\nu}^\Omega \cdot \mathcal{A}^{\partial\Omega}(\nabla u^i, \nabla u^i). \quad (5.16)$$

The term $\nabla u^j \cdot \nabla (Z_i^j (\boldsymbol{\nu}^\Omega)^i)$ vanishes because, due to (5.3), $\nabla u^j \in T_{\boldsymbol{x}}\partial\Omega$ and $\nabla (Z_i^j (\boldsymbol{\nu}^\Omega)^i) \in (T_{\boldsymbol{x}}\partial\Omega)^\perp$ for $j = 1, \dots N$. By $\mathcal{A}^{\partial\Omega}$ we denoted the second fundamental form of hypersurface $\partial\Omega$,

$$\mathcal{A}^{\partial\Omega}_{\boldsymbol{x}}(\boldsymbol{X}, \boldsymbol{Y}) = (\boldsymbol{X} \cdot D_{\boldsymbol{x}} \boldsymbol{\nu}^\Omega \, \boldsymbol{Y}) \boldsymbol{\nu}^\Omega_{\boldsymbol{x}}$$

for $\boldsymbol{x} \in \partial\Omega$, where index $\boldsymbol{x}$ on $\mathcal{A}^{\partial\Omega}$, $\boldsymbol{\nu}^\Omega$ denotes evaluation at $\boldsymbol{x}$ and $D_{\boldsymbol{x}}\boldsymbol{\nu}^\Omega \colon T_{\boldsymbol{x}}\partial\Omega \to \mathbb{R}^m$ is the tangent map of $\boldsymbol{\nu}^\Omega$ at $\boldsymbol{x}$ (see the remark after (5.7)). As Ω is convex, $\boldsymbol{\nu}^\Omega \cdot \mathcal{A}^{\partial\Omega}$ is non-negative. This ends the proof of (5.12) in the case $K_{\mathcal{N}} \leq 0$.

Now, assume that $K_{\mathcal{N}} \in]0, \infty[$. By virtue of the previous calculations and (1.51), we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\int_\Omega v^p \right)^{\frac{1}{p}} \leq \left(\int_\Omega v^p \right)^{\frac{1}{p} - 1} K_{\mathcal{N}} \int_\Omega v^{p+1} \leq K_{\mathcal{N}} \left(\int_\Omega v^p \right)^{\frac{1}{p}} \|v\|_{L^\infty}.$$

Passing to the limit $p \to \infty$ we obtain, at least in a weak sense,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|v\|_{L^\infty} \leq K_{\mathcal{N}} \|v\|^2_{L^\infty}$$

which implies (5.11). □

5.2.2 Existence for the approximate system

In order to prove existence of solutions to the approximate system we proceed similarly as in [48, Section 3.]. The assumption that the embedding of $\mathcal{N}$ into $\mathbb{R}^N$ is closed enables us to construct a metric h on $\mathbb{R}^N$ such that $(\mathcal{N}, g)$ is a totally geodesic Riemannian submanifold of $(\mathbb{R}^N, h)$ (see Lemma A.1 in the appendix), i. e.,

- the restriction of h to $T\mathcal{N}$ coincides with g, that is $h_{\boldsymbol{p}}|_{T_{\boldsymbol{p}}\mathcal{N} \times T_{\boldsymbol{p}}\mathcal{N}} \equiv g_{\boldsymbol{p}}$ for $\boldsymbol{p} \in \mathcal{N}$,
- there is a tubular neighborhood $\mathcal{T}$ of $\mathcal{N}$ in $\mathbb{R}^N$ such that the involution $\tau \colon \mathcal{T} \to \mathcal{T}$ given by multiplication by -1 in the fibers of $\mathcal{T}$ is an isometry.

The gradient flow of the unconstrained functional $\int_\Omega |\nabla \boldsymbol{u}|_h$ defined for any regular enough function $\boldsymbol{u} \colon \Omega \to \mathbb{R}^N$ is expressed by the system

$$u^i_t = \operatorname{div} \frac{\nabla u^i}{\sqrt{\varepsilon^2 + |\nabla \boldsymbol{u}|^2_h}} + \frac{1}{\sqrt{\varepsilon^2 + |\nabla \boldsymbol{u}|^2_h}} \Gamma^i_{jk}(\boldsymbol{u}) u^j_{x^l} u^k_{x^l}, \quad (5.17)$$

$$\boldsymbol{\nu}^\Omega \cdot \nabla u^i = 0, \quad (5.18)$$

where $i = 1, \dots, N$ and Γ^i_{jk} are the Christoffel symbols of $(\mathbb{R}^N, h)$. As h restricted to $T\mathcal{N}$ coincides with g, the system (5.17, 5.18) is identical to (5.2, 5.18) as long as the range of $\boldsymbol{u}$ is contained in $\mathcal{N}$. In order for $C^{\frac{3+\alpha}{2}, 3+\alpha}_{loc}(\Omega_{[0,T[}, \mathcal{N})$ solutions to the system (5.17, 5.18) with initial datum $\boldsymbol{u}_0$ to exist, the following compatibility conditions

$$\boldsymbol{\nu}^\Omega \cdot \nabla u^i_0 = 0 \quad (5.19)$$

$$\boldsymbol{\nu}^\Omega \cdot \nabla \left(\operatorname{div} \frac{\nabla u^i_0}{\sqrt{\varepsilon^2 + |\nabla \boldsymbol{u}_0|^2_h}} + \frac{1}{\sqrt{\varepsilon^2 + |\nabla \boldsymbol{u}_0|^2_h}} \Gamma^i_{jk}(\boldsymbol{u}_0) u^j_{0,x^l} u^k_{0,x^l} \right) = 0 \quad (5.20)$$

on $\partial\Omega$ for $i = 1, \dots, N$ need to be satisfied.

Proposition 5.1. *Suppose that $K_{\mathcal{N}} < \infty$ and $\alpha \in]0,1[$. Let $\boldsymbol{u}_0 \in C^{3+\alpha}(\Omega, \mathcal{N})$ satisfy (5.19, 5.20). Then for any $\varepsilon > 0$ the system (5.2-5.4) has a unique solution*

$$\boldsymbol{u} \in C_{loc}^{\frac{3+\alpha}{2}, 3+\alpha}(\overline{\Omega}_{[0,T_\dagger[}, \mathcal{N})$$

where $T_\dagger = T_\dagger(\|\nabla \boldsymbol{u}_0\|_{L^\infty}, K_{\mathcal{N}}) \in]0, \infty]$ is defined in Lemma 5.3.

Note that $T_\dagger$ in Proposition 5.1 does not depend on ε.

The expressions on the right hand side of (5.17) make sense without assuming a priori that the range of $\boldsymbol{u}$ is contained in $\mathcal{N}$. This fact enables us to obtain a local-in-time solution using known results for parabolic systems. For that purpose, the authors in [48] or in [32] combine a general existence result from [60] with sectoriality estimates from [81]. On the other hand, in [63] the author employs estimates from [73] and [64]. However, both [81] and [73] can only be applied to the system with Dirichlet boundary condition, or to the case with no boundary. As we are dealing with homogeneous Neumann boundary condition, we appeal instead to a result of Acquistapace and Terreni [1, Theorem 1.1.] for quasilinear systems with general boundary conditions.

To justify its applicability to our problem, let us briefly check the assumptions. We can rewrite the divergence part of the right hand side of (5.17) as $\mathbf{A}_{kl}(\nabla \boldsymbol{u})\, \boldsymbol{u}_{x^k x^l}$, where $\mathbf{A}_{kl} \colon \mathbb{R}^N \to \mathbb{R}^N$ is given by

$$\mathbf{A}_{kl}(\mathbf{P}) = \tfrac{1}{\sqrt{\varepsilon^2 + |\mathbf{P}|_h^2}} \left(I_{kl}^m \, \mathbf{I}^N - \tfrac{\boldsymbol{P}_k}{\sqrt{\varepsilon^2 + |\mathbf{P}|_h^2}} \otimes \tfrac{\boldsymbol{P}_l}{\sqrt{\varepsilon^2 + |\mathbf{P}|_h^2}} \right),$$

for $k, l = 1, \ldots, m$ with $\mathbf{P} = (\boldsymbol{P}_1, \ldots, \boldsymbol{P}_m)$. $(\mathbf{A}_{kl})$ defines a locally uniformly strongly elliptic operator (see e. g. [2]) and therefore satisfies assumption (0.2) from [1]. It is easy to check that (5.18) satisfies the complementarity condition (0.3) from [1], and that the system satisfies regularity condition (0.4) from [1].

Thus, as $\boldsymbol{u}_0 \in C^{2+\alpha}(\Omega, \mathcal{N})$ satisfies compatibility condition (5.19), we obtain for any $p > m$ the existence of unique solution to (5.17, 5.3) with initial datum $\boldsymbol{u}_0$ in $C^{1+\frac{\alpha}{2}}([0, T_0[, L^p(\Omega, \mathbb{R}^N)) \cap C^{\frac{\alpha}{2}}([0, T_0[, W^{2,p}(\Omega, \mathbb{R}^N))$ for some $T_0 > 0$. We choose p so that $W^{2,p}(\Omega) \subset C^{1,\alpha}(\Omega)$. Then, we can treat the system (5.17, 5.3) as a linear system with $C^{\frac{\alpha}{2},\alpha}$ coefficients and apply [54, Theorem VII.10.1] to obtain $\boldsymbol{u} \in C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega_{[0,T_0[})$. As long as $\boldsymbol{u}(t, \cdot) \in C^{2+\alpha}(\Omega, \mathbb{R}^N)$, we can extend the solution via Acquistapace-Terreni theorem. Therefore, there exists a maximal time $T_* \leq \infty$ such that

- $\boldsymbol{u}$ exists in $C_{loc}^{1+\frac{\alpha}{2}, 2+\alpha}(\overline{\Omega}_{[0,T_*[}, \mathbb{R}^N)$,
- the norm of $\boldsymbol{u}$ in $C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega_{[0,t[}, \mathbb{R}^N)$ blows up as $t \to T_*^-$ if $T_* < \infty$.

Since $\boldsymbol{u} \in C_{loc}^{1+\frac{\alpha}{2}, 2+\alpha}(\overline{\Omega}_{[0,T_*[}, \mathbb{R}^N)$, the coefficients of (5.17), seen as a linear equation, belong to $C_{loc}^{\frac{1+\alpha}{2}, 1+\alpha}(\overline{\Omega}_{[0,T_*[})$. Therefore, provided $\boldsymbol{u}_0 \in C^{3+\alpha}(\Omega, \mathbb{R}^N)$ and the additional compatibility condition (5.20) is satisfied, we may appeal once more to [54, Theorem VII.10.1] and conclude that $\boldsymbol{u} \in C_{loc}^{\frac{3+\alpha}{2}, 3+\alpha}(\overline{\Omega}_{[0,T_*[}, \mathbb{R}^N)$.

We now argue that $\boldsymbol{u}(t, \Omega) \subset \mathcal{N}$ for all $t \in [0, T_*[$. Suppose, to the contrary, that there is $t \in]0, T_*[$ with $\boldsymbol{u}(t, \Omega) \not\subset \mathcal{N}$. Let $T_{\mathcal{N}}$ be the first time instance such that $\boldsymbol{u}(t, \Omega) \not\subset \mathcal{N}$ for $T_{\mathcal{N}} < t < T_{\mathcal{N}} + \delta$ with some $\delta > 0$. Possibly diminishing δ we can assume that $\boldsymbol{u}(t, \Omega) \subset \mathcal{T}$

for $t \in [0, T_{\mathcal{N}} + \delta[$. Then $\tau \circ \boldsymbol{u}$ is a solution to (5.17) different to $\boldsymbol{u}$ with the same initial and boundary conditions, thus violating uniqueness. Therefore, $\boldsymbol{u}(t, \Omega) \subset \mathcal{N}$ for all $t \in [0, T_*[$.

It remains to show that $T_* \geq T_\dagger$, where $T_\dagger$ is defined in Lemma 5.3. Suppose that $T_* < T_\dagger$. Lemma 5.3 yields

$$\sup_{t \in [0, T_*[} \|\nabla \boldsymbol{u}(t, \cdot)\|_{L^\infty(\Omega)} < \infty. \tag{5.21}$$

Let now $q > \frac{m+2}{1-\alpha}$. According to [54, Theorem VII.10.4 and Lemma II.3.3], there holds $\boldsymbol{u} \in W^{1,q}(]0, T_*[, L^q(\Omega, \mathbb{R}^N)) \cap L^q(]0, T_*[, W^{2,q}(\Omega, \mathbb{R}^N))$ and consequently $\nabla \boldsymbol{u} \in C^{\frac{\alpha}{2}, \alpha}(\Omega_{[0,T_*[}, \mathbb{R}^{m \cdot N})$. Now, [54, Theorem VII.10.1] yields $\boldsymbol{u} \in C^{1+\frac{\alpha}{2}, 2+\alpha}(\Omega_{[0,T_*[}, \mathbb{R}^N)$, a contradiction.

5.3 Local existence

In this section we prove Theorem 1.6.

Step 1. We assume that Ω is smooth and the initial datum $\boldsymbol{u}_0 \in C^{3+\alpha}(\Omega)$ satisfies the compatibility conditions (5.19), (5.20). We want to pass to the limit $\varepsilon \to 0^+$ in (5.2-5.4). Owing to Lemmata 5.1 and 5.3, we have uniform bounds on $\boldsymbol{u}_t^\varepsilon$ in $L^2(]0, T[\times\Omega)$ and on $\nabla \boldsymbol{u}^\varepsilon$ in $L^\infty(]0, T[\times\Omega)$ for any $T < T_\dagger$. Consequently, we also have a uniform bound on $\boldsymbol{u}^\varepsilon$ in $C^{\frac{1}{n+1}}(]0, T[\times\Omega)$ [47]. All these imply that we can extract a sequence $(\boldsymbol{u}_k) = (\boldsymbol{u}^{\varepsilon_k})$ from $(\boldsymbol{u}^\varepsilon)$ such that

$$\boldsymbol{u}_k \to \boldsymbol{u} \text{ in } C([0, T] \times \overline{\Omega}), \qquad \nabla \boldsymbol{u}_k \rightharpoonup \nabla \boldsymbol{u} \text{ in } L^2(]0, T[\times\Omega).$$

Due to definition of $\boldsymbol{Z}^\varepsilon$, we have $\|\boldsymbol{Z}^\varepsilon\|_{L^\infty} \leq 1$, hence

$$\boldsymbol{Z}_k \overset{*}{\rightharpoonup} \boldsymbol{Z} \text{ in } L^\infty(]0, T[\times\Omega) \text{ with } |\boldsymbol{Z}| \leq 1 \text{ a.e. in }]0, T[\times\Omega \tag{5.22}$$

for a sequence $(\boldsymbol{Z}_k) = (\boldsymbol{Z}^{\varepsilon_k})$. Furthermore, by virtue of the strong convergence of $\boldsymbol{u}_k$,

$$0 = \pi_{\boldsymbol{u}_k}^\perp \boldsymbol{Z}_k \overset{*}{\rightharpoonup} \pi_{\boldsymbol{u}}^\perp \boldsymbol{Z} \text{ in } L^\infty(]0, T[\times\Omega). \tag{5.23}$$

Next, note that due to the Hölder bound, the family $\boldsymbol{u}^\varepsilon$ is contained in a compact subset of $\mathcal{N}$. Rewriting (5.2) as

$$\boldsymbol{u}_t^\varepsilon = \operatorname{div} \boldsymbol{Z}^\varepsilon + \mathcal{A}_{\boldsymbol{u}^\varepsilon}(\boldsymbol{u}_{x^i}^\varepsilon, \boldsymbol{Z}_i^\varepsilon), \tag{5.24}$$

we deduce a uniform bound on $\operatorname{div} \boldsymbol{Z}^\varepsilon$ in $L^2(]0, T[\times\Omega)$. By a standard div-curl reasoning,

$$\nabla \boldsymbol{u}_k : \boldsymbol{Z}_k \rightharpoonup \nabla \boldsymbol{u} : \boldsymbol{Z} \text{ in } L^2(]0, T[\times\Omega). \tag{5.25}$$

A simple calculation shows that

$$\nabla \boldsymbol{u}^\varepsilon : \boldsymbol{Z}^\varepsilon = \tfrac{|\nabla \boldsymbol{u}^\varepsilon|^2}{\sqrt{\varepsilon^2 + |\nabla \boldsymbol{u}^\varepsilon|^2}} \geq |\nabla \boldsymbol{u}^\varepsilon| - \varepsilon. \tag{5.26}$$

Hence, by lower semicontinuity of $|\cdot|$ with respect to weak convergence, we get

$$\nabla \boldsymbol{u} : \boldsymbol{Z} \geq |\nabla \boldsymbol{u}|. \tag{5.27}$$

Collecting (5.22, 5.23, 5.25, 5.27) we obtain that $\nabla \boldsymbol{u}$ and $\boldsymbol{Z}$ satisfy (1.47). Boundedness of $\operatorname{div} \boldsymbol{Z}^\varepsilon$ in $L^2(]0, T[\times\Omega)$ together with strong convergence of $\boldsymbol{u}_k$ is enough to pass to the limit in (5.2, 5.3), obtaining that $\nabla \boldsymbol{u}$ and $\boldsymbol{Z}$ satisfy (1.48, 1.49).

Step 2. Now, we relax the regularity assumption on the initial datum to $\boldsymbol{u}_0 \in W^{1,\infty}(\Omega, \mathcal{N})$. Take a sequence $(\boldsymbol{u}_{0,j}) \subset C^\infty(\overline{\Omega}, \mathcal{N})$ such that $\boldsymbol{u}_{0,j}$ converges uniformly to $\boldsymbol{u}_0$, satisfies the compatibility conditions (5.19, 5.20) and

$$\|\nabla \boldsymbol{u}_{0,j}\|_{L^\infty} \to \|\nabla \boldsymbol{u}_0\|_{L^\infty}. \tag{5.28}$$

Such a sequence is produced in Lemma A.2. By the previous step, there exists a regular solution $(\boldsymbol{u}_j, \boldsymbol{Z}_j)$ to (1.44, 1.45) with initial datum $\boldsymbol{u}_{0,j}$. Recall that due to the form of estimates in Lemmata 5.1 and 5.3 the norms of $\boldsymbol{u}_{j,t}$ in $L^2(]0,T[\times\Omega, \mathbb{R}^N)$ and of $\nabla \boldsymbol{u}_j$ in $L^\infty(]0,T[\times\Omega, \mathbb{R}^{m\cdot N})$ are controlled by $\|\nabla \boldsymbol{u}_{0,j}\|_{L^\infty}$. By virtue of (5.28), this control is uniform with respect to j. Hence, we can extract a subsequence converging to a regular solution to (1.44, 1.45, 1.46) following the same argument as in the previous step, with $(\boldsymbol{u}^\varepsilon, \boldsymbol{Z}^\varepsilon)$ replaced by $(\boldsymbol{u}_j, \boldsymbol{Z}_j)$, except that now we have $\nabla \boldsymbol{u}_j : \boldsymbol{Z}_j = |\nabla \boldsymbol{u}_j|$ instead of (5.26).

Step 3. Next, we lift the smoothness assumption on the domain. A convex domain Ω can be approximated with respect to the Hausdorff distance by smooth convex domains $\Omega_k \subset \Omega$, $k = 1, 2, \ldots$. For a proof of this result using the signed distance function of Ω, see Lemma A.3 in the appendix. The reasoning in the previous paragraph yields a sequence of pairs $(\boldsymbol{u}_k, \boldsymbol{Z}_k)$, with k-th one satisfying (1.47, 1.48, 1.49) in $]0,T[\times\Omega_k$ with initial datum $\boldsymbol{u}_0|_{\Omega_k}$. The estimates provided by Lemmata 5.1 and 5.3 are uniform with respect to k. Hence, we can use them as before together with a diagonal argument to extract subsequences of $(\boldsymbol{u}_k)$, $(\boldsymbol{Z}_k)$ that converge on compact subsets of $[0,T[\times\Omega$ to a regular solution $(\boldsymbol{u}, \boldsymbol{Z})$ to (1.44, 1.46) in $]0,T[\times\Omega$.

Finally, we argue that the boundary condition (1.49) is satisfied. Let us fix $\varphi \in C^1(]0,T[\times\overline{\Omega})$. We have

$$\int_0^T \int_{\partial\Omega} \varphi\, \boldsymbol{\nu}^\Omega \cdot \boldsymbol{Z} = \int_0^T \int_\Omega \varphi \operatorname{div} \boldsymbol{Z} + \nabla\varphi \cdot \boldsymbol{Z},$$

$$0 = \int_0^T \int_{\partial\Omega_k} \varphi\, \boldsymbol{\nu}^{\Omega_k} \cdot \boldsymbol{Z}_k = \int_0^T \int_{\Omega_k} \varphi \operatorname{div} \boldsymbol{Z}_k + \nabla\varphi \cdot \boldsymbol{Z}_k.$$

Let us denote $f = \varphi \operatorname{div} \boldsymbol{Z} + \nabla\varphi \cdot \boldsymbol{Z}$, $f_k = \varphi \operatorname{div} \boldsymbol{Z}_k + \nabla\varphi \cdot \boldsymbol{Z}_k$. By virtue of Hausdorff convergence, for a given $\varepsilon > 0$, we are allowed to choose $K \subset \Omega$ and k_0 so that $|]0,T[\times(\Omega \setminus K)| \leq \varepsilon^2$ and $K \subset \Omega_k$ for $k \geq k_0$. Recalling (5.24), we note that $\|f_k\|_{L^2(]0,T[\times\Omega_k)}$ is controlled in terms of norms $\|\boldsymbol{u}_{k,t}\|_{L^2(]0,T[\times\Omega_k)}$ and $\|\nabla \boldsymbol{u}_k\|_{L^\infty(]0,T[\times\Omega_k)}$ and hence is uniformly bounded. We can assume that $(f_k|_K)_{k=k_0}^\infty$ converges weakly to $f|_K$ in $L^2(]0,T[\times K)$. Thus, we can choose $k \geq k_0$ large enough so that $\left|\int_0^T \int_K f - f_k\right| \leq \varepsilon$. We estimate

$$\begin{aligned}\left|\int_0^T \int_{\partial\Omega} \varphi\, \boldsymbol{\nu}^\Omega \cdot \boldsymbol{Z}\right| &\leq \left|\int_0^T \int_K f - f_k\right| + \left|\int_0^T \int_{\Omega\setminus K} f\right| + \left|\int_0^T \int_{\Omega_k\setminus K} f_k\right| \\ &\leq (1 + \|f\|_{L^2(]0,T[\times\Omega)} + \|f_k\|_{L^2(]0,T[\times\Omega_k)})\varepsilon.\end{aligned}$$

As ε and φ are arbitrary, we are done. □

5.4 Finite extinction time

In order to prove Theorem 1.7 we will work directly with regular solutions to (1.44, 1.45, 1.46) in local coordinates $\boldsymbol{p} \mapsto (p^1, \ldots, p^n)$ on $\mathcal{N}$, in which (1.48) is expressed [28] as

$$u_t^i = \operatorname{div} Z^i + \Gamma^i_{jk}(\boldsymbol{u}) u^j_{x^l} Z^k_l, \quad i = 1, \ldots, n, \tag{5.29}$$

where Γ^i_{jk} are the Christoffel symbols of the chosen coordinate system. For $\boldsymbol{p}_0 \in \mathcal{N}$ we denote

$$R_*(\boldsymbol{p}_0) = \min\left\{\sup\left\{R > 0\colon R \le \frac{\pi}{2}\left[K_{B_g(\boldsymbol{p}_0,R)}\right]_+^{-\frac{1}{2}}\right\}, \frac{\ell(\boldsymbol{p}_0)}{4}\right\}, \tag{5.30}$$

where $[K_{B_g(\boldsymbol{p}_0,R)}]_+$ is the supremum of sectional curvature over $B_g(\boldsymbol{p}_0, R)$ (compare with (1.51)) or $+0$ if the supremum is negative, $\ell(\boldsymbol{p}_0)$ is the infimum of lengths of maximal closed geodesics in $\mathcal{N}$ passing through $\boldsymbol{p}_0$, and π is the length of a circle of radius $\frac{1}{2}$. $R_*(\boldsymbol{p}_0)$ is positive and lower than both the convexity radius and half of the injectivity radius $\mathrm{inj}_{\mathcal{N}}(\boldsymbol{p}_0)$ [71, section 6.3.2].

First, we prove

Lemma 5.4. *Let $\boldsymbol{p}_0 \in \mathcal{N}$, $\boldsymbol{u}_0 \in W^{1,\infty}(\Omega)$. If $\boldsymbol{u}_0(\Omega) \subset \overline{B_g(\boldsymbol{p}_0, R)}$ with $R \in]0, R_*(\boldsymbol{p}_0)[$, then $\boldsymbol{u}(t,\Omega) \subset \overline{B_g(\boldsymbol{p}_0, R)}$, $t \in]0,T[$.*

Proof. We proceed by contradiction. Let $T_* = \inf\{t \in [0,T[\colon \boldsymbol{u}(t,\Omega) \not\subset \overline{B_g(\boldsymbol{p}_0, R)}\}$. Due to continuity of $\boldsymbol{u}$, there is a $\delta > 0$ such that $\boldsymbol{u}(t,\Omega) \subset B_g(\boldsymbol{p}_0, R_*(\boldsymbol{p}_0))$ for $t \in [0, T_* + \delta[$. We choose on $B_g(\boldsymbol{p}_0, R_*(\boldsymbol{p}_0))$ a polar coordinate system $\boldsymbol{p} \mapsto (p^r, p^{\vartheta^1}, \ldots, p^{\vartheta^{n-1}})$ centered at $\boldsymbol{p}_0$. Due to the block diagonal form of the metric in these coordinates, (5.29) for the radial coordinate p^r takes the form

$$u^r_t = \operatorname{div} Z^r - \tfrac{1}{2} g_{\vartheta^i\vartheta^j,r}(\boldsymbol{u}) u^{\vartheta^i}_{x^l} Z^{\vartheta^j}_l. \tag{5.31}$$

Here and in the following, $g_{\vartheta^i\vartheta^j,r}$ denotes the derivative of $g_{\vartheta^i\vartheta^j}$ (a function on $\mathcal{N}$) in direction p^r. Equation (5.31) is satisfied a. e. in the open set $\{(t,\boldsymbol{x}) \in]0, T_* + \delta[\times\Omega\colon \boldsymbol{u}(t,\boldsymbol{x}) \ne \boldsymbol{p}_0\}$. Furthermore, there holds (see the proof of Corollary 2.4 in [71, Chapter 6])

$$\left(g_{\vartheta^i\vartheta^j,r}(\boldsymbol{p})\right)_{i,j=1}^{n-1} \ge \tfrac{2}{p^r} \cos\left(\left[K_{B_g(\boldsymbol{p}_0,R)}\right]_+^{\frac{1}{2}} p^r\right) \left(g_{\vartheta^i\vartheta^j}(\boldsymbol{p})\right)_{i,j=1}^{n-1} \quad \text{for } \boldsymbol{p} \in \mathcal{N} \tag{5.32}$$

as quadratic forms. Taking into account (5.31, 5.32, 1.47, 1.49) and recalling that $\boldsymbol{u}_{x^l}$ is a non-negative multiple of $\boldsymbol{Z}_l$ for $l = 1, \ldots, m$ we calculate

$$\begin{aligned}
\tfrac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u^r - R)_+^2 &= \int_\Omega (u^r - R)_+ u^r_t \\
&\le -\int_{\{\boldsymbol{x}\in\Omega\colon u^r(\boldsymbol{x})>R\}} |\nabla u^r| - \int_\Omega \frac{(u^r - R)_+}{u^r}\left(\cos\frac{\pi}{2}\right) g_{\vartheta^i\vartheta^j}(\boldsymbol{u}) u^{\vartheta^i}_{x^l} Z^{\vartheta^j}_l \le 0.
\end{aligned} \tag{5.33}$$

□

Next, we recall the notion of Riemannian center of mass. Let $R < R_*(\boldsymbol{p}_0)$, $\boldsymbol{p}_0 \in \mathcal{N}$. We say that $\boldsymbol{p}_c \in \overline{B_g(\boldsymbol{p}_0, R)}$ is a center of mass of a Radon measure μ on $\overline{B_g(\boldsymbol{p}_0, R)}$ if $\boldsymbol{p}_c$ is a minimizer of the function $f_\mu\colon \overline{B_g(\boldsymbol{p}_0, R)} \to [0,\infty[$ given by

$$f_\mu(\boldsymbol{p}) = \tfrac{1}{2}\int_{\mathcal{N}} \mathrm{dist}_g(\cdot, \boldsymbol{p})^2 \,\mathrm{d}\mu.$$

A unique center of mass exists for any Radon measure on $\overline{B_g(\boldsymbol{p}_0, R)}$ and we have

$$0 = \mathrm{d}f_\mu(\boldsymbol{p}_c) = \int_{\overline{B_g(\boldsymbol{p}_0,R)}} \exp^{-1}_{\boldsymbol{p}_c} \mathrm{d}\mu, \tag{5.34}$$

where $\exp_{\boldsymbol{p}_c}^{-1}\colon B_g(\boldsymbol{p}_c, \mathrm{inj}_{\mathcal{N}}(\boldsymbol{p}_c)) \to T_{\boldsymbol{p}_c}\mathcal{N}$ denotes the logarithmic map at $\boldsymbol{p}_c$. In (5.34), we identified elements of $T^*_{\boldsymbol{p}_c}\mathcal{N}$ and $T_{\boldsymbol{p}_c}\mathcal{N}$ via g [49, Section 1]. For $\boldsymbol{p}_0 \in \mathcal{N}$, we denote

$$\widetilde{R}_*(\boldsymbol{p}_0) = \tfrac{1}{2}\inf\left\{R_*(\boldsymbol{p})\colon \boldsymbol{p} \in B_g\left(\boldsymbol{p}_0, R_*(\boldsymbol{p}_0)\right)\right\}. \tag{5.35}$$

We are ready to state

Lemma 5.5. *Suppose that $\boldsymbol{u}_0 \in W^{1,\infty}(\Omega)$ satisfies $\boldsymbol{u}_0(\Omega) \subset \overline{B_g\left(\boldsymbol{p}_0, R\right)}$, $\boldsymbol{p}_0 \in \mathcal{N}$, $0 < R < \widetilde{R}_*(\boldsymbol{p}_0)$. Let $\boldsymbol{p}_c(t)$ be the center of mass of the pushforward measure $\mu(t) = \boldsymbol{u}(t,\cdot)_{\#}\mathcal{L}^m$ on $\overline{B_g(\boldsymbol{p}_0, R)}$. There exists $C_0 = C_0(\Omega, \mathcal{N}, \boldsymbol{p}_0) > 0$ such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} f_\mu(\boldsymbol{p}_c) \le -C_0 R^{\frac{2}{m}-1} f_\mu(\boldsymbol{p}_c)^{1-\frac{1}{m}} \tag{5.36}$$

for $t > 0$.

Proof. We have

$$f_{\mu(t)}(\boldsymbol{p}_c(t)) = \tfrac{1}{2}\int_\Omega \mathrm{dist}_g(\boldsymbol{u}(t,\cdot), \boldsymbol{p}_c(t))^2 = \tfrac{1}{2}\int_\Omega u^r(t,\cdot)^2,$$

where we have chosen polar coordinates centered at $\boldsymbol{p}_c(t)$. Employing (5.34, 5.31, 5.32, 1.47, 1.49) and observing that $\cos\left(\left[K_{B_g(\boldsymbol{p}_c,R)}\right]_+^{\frac{1}{2}} R\right) \ge \cos\left(\left[K_{B_g(\boldsymbol{p}_c,R_*(\boldsymbol{p}_c))}\right]_+^{\frac{1}{2}} \frac{R_*(\boldsymbol{p}_c)}{2}\right) \in \left[\frac{\sqrt{2}}{2}, 1\right]$,

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t} f_\mu(\boldsymbol{p}_c) &= \left\langle \mathrm{d}f_\mu(\boldsymbol{p}_c), \boldsymbol{p}_{c,t}\right\rangle_{T^*_{\boldsymbol{p}_c}\mathcal{N}, T_{\boldsymbol{p}_c}\mathcal{N}} + \int_\Omega u^r u^r_t \\ &\le -\int_\Omega |\nabla u^r| - \cos\left(\left[K_{B_g(\boldsymbol{p}_c,R)}\right]_+^{\frac{1}{2}} R\right)\int_\Omega g_{\vartheta^i\vartheta^j}(\boldsymbol{u}) u^{\vartheta^i}_{x^l} Z^{\vartheta^j}_l \le -\tfrac{\sqrt{2}}{2}\int_\Omega |\nabla \boldsymbol{u}|_g.\end{aligned} \tag{5.37}$$

This equation is rigorously justified by passing to the limit $R \to 0^+$ in the weak formulation of (5.33) using Lebesgue monotone convergence theorem. Now, we choose on $B(\boldsymbol{p}_c, R_*(\boldsymbol{p}_c))$ coordinate system $\boldsymbol{p} \mapsto \exp_{\boldsymbol{p}_c(t)}^{-1}\boldsymbol{p} = (p^1, \dots, p^n)$. From (5.37) we obtain that there exists a constant $C_1 = C_1(\mathcal{N}, \boldsymbol{p}_0) > 0$ such that (recall that $p^r = \sqrt{p^i p^i}$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^i u^i \le -C_1 \int_\Omega \sqrt{u^i_{x^j} u^i_{x^j}}. \tag{5.38}$$

Finally, applying Sobolev-Poincaré inequality (recall (5.34)):

$$\left(\int_\Omega u^i u^i\right)^{1-\frac{1}{m}} \le R^{1-\frac{2}{m}}\left(\int_\Omega \left(\sqrt{u^i u^i}\right)^{\frac{m}{m-1}}\right)^{1-\frac{1}{m}} \le C_2 R^{1-\frac{2}{m}}\int_\Omega \sqrt{u^i_{x^j} u^i_{x^j}} \tag{5.39}$$

with $C_2 = C_2(\Omega) > 0$. Estimates (5.38, 5.39) add up to (5.36). □

Proof of Theorem 1.7. First of all, by Lemma 5.4, we obtain the bound $\boldsymbol{u}(t,\Omega) \subset \overline{B_g(\boldsymbol{p}_0, R)}$ if $\boldsymbol{u}_0(\Omega) \subset \overline{B_g(\boldsymbol{p}_0, R)}$ for $R < \widetilde{R}_*(\boldsymbol{p}_0)$ and any $t \in [0, T[$. Next, we deduce the estimate on extinction time from (5.36) by solving the ordinary differential inequality, which yields

$$f_{\mu(t)}(\boldsymbol{p}_c(t))^{\frac{1}{m}} \le \left(f_{\mu(0)}(\boldsymbol{p}_c(0))^{\frac{1}{m}} - \tfrac{1}{m} C_0 R^{\frac{2}{m}-1} t\right)_+,$$

where

$$f_{\mu(t)}(\boldsymbol{p}_c(t)) = \int_\Omega \mathrm{dist}(\boldsymbol{u}(t,\cdot), \boldsymbol{p}_c)^2.$$

As $f_{\mu(0)}(\boldsymbol{p}_c(0)) \le \frac{1}{2}|\Omega| R^2$, there is $\boldsymbol{u}_* \in \mathcal{N}$ such that $\boldsymbol{u}(t,\cdot) \equiv \boldsymbol{u}_*$ for $t \ge CR$, where $C = m\left(\frac{|\Omega|}{2}\right)^{\frac{1}{m}} C_0^{-1}$. □

5.5 Non-positive sectional curvature of the target

This section is devoted to the proof of Theorem 1.8.

Let $T > 0$ and suppose that Ω is convex and $\mathcal{N}$ is a complete Riemannian manifold with $K_{\mathcal{N}} \leq 0$. In order to prove Theorem 1.8 without the assumption that there is a closed embedding of $\mathcal{N}$ into $\mathbb{R}^N$, we introduce a universal cover $\gamma\colon \widetilde{\mathcal{N}} \to \mathcal{N}$ of $\mathcal{N}$ with a Riemannian manifold $(\widetilde{\mathcal{N}}, \widetilde{g})$. As a simply-connected Riemannian manifold of non-positive curvature, $\widetilde{\mathcal{N}}$ is diffeomorphic to $\mathbb{R}^n$ via the exponential map (this is the content of Cartan-Hadamard theorem [26]). In other words, there is a global coordinate system on $\widetilde{\mathcal{N}}$, $\widetilde{\boldsymbol{p}} \mapsto \exp^{-1}_{\widetilde{\boldsymbol{p}}_0} \widetilde{\boldsymbol{p}} = (\widetilde{p}^1, \dots, \widetilde{p}^n)$. As Ω is topologically trivial, any function $\boldsymbol{u}_0 \in C(\Omega, \mathcal{N})$ can be lifted preserving any Sobolev or Hölder regularity to $\widetilde{\boldsymbol{u}}_0 \in C(\Omega, \widetilde{\mathcal{N}})$ such that $\boldsymbol{u}_0 = \gamma \circ \widetilde{\boldsymbol{u}}_0$. Then, assuming that Ω and $\boldsymbol{u}_0$ are of class $C^{3+\alpha}$ and $\boldsymbol{u}_0$ satisfies the compatibility conditions (5.19,5.20) for $i = 1, \dots, n$, we consider the system

$$\widetilde{u}^{\varepsilon,i}_t = \operatorname{div} \tfrac{\nabla \widetilde{u}^{\varepsilon,i}}{\sqrt{\varepsilon^2 + |\nabla \widetilde{\boldsymbol{u}}^\varepsilon|^2_{\widetilde{g}}}} + \tfrac{1}{\sqrt{\varepsilon^2 + |\nabla \widetilde{\boldsymbol{u}}^\varepsilon|^2_{\widetilde{g}}}} \widetilde{\Gamma}^i_{jk}(\widetilde{\boldsymbol{u}}^\varepsilon) \widetilde{u}^{\varepsilon,j}_{x^l} \widetilde{u}^{\varepsilon,k}_{x^l} \text{ in }]0, T_*[\times \Omega,$$

$$\nabla \widetilde{u}^{\varepsilon,i} \cdot \boldsymbol{\nu}^\Omega = 0 \text{ in }]0, T_*[\times \partial\Omega,$$

$$\widetilde{u}^{\varepsilon,i}(0, \cdot) = \widetilde{u}^i_0,$$

$i = 1, \dots, n$. This system satisfies the assumptions of the Aquistapace-Terreni existence theorem (see subsection 5.2.2), hence a unique solution exists for some $T_* > 0$. Vector lengths $|\widetilde{\boldsymbol{u}}^\varepsilon_t|_{\widetilde{g}}$ and $|\nabla \widetilde{\boldsymbol{u}}^\varepsilon|_{\widetilde{g}}$ are invariant under local isometries of the target manifold, and any Riemannian manifold is locally isometric to a submanifold in a Euclidean space. Therefore, we can repeat the proofs of Lemmata 5.1, 5.2 and 5.3 performing the computations in a neighbourhood of any point, obtaining bounds on $\|\widetilde{\boldsymbol{u}}^\varepsilon_t\|_{L^2(]0,T_*[\times\Omega)}$ and $\|\nabla \widetilde{\boldsymbol{u}}^\varepsilon\|_{L^\infty(]0,T_*[\times\Omega)}$ independent on T_*. Reasoning as in subsection 5.2.2, the solution can be prolonged up to the arbitrary given T. Then, taking $\boldsymbol{u}^\varepsilon = \gamma \circ \widetilde{\boldsymbol{u}}^\varepsilon$, we obtain a solution to (5.2-5.4). Using the uniform bounds, we pass to the limit as in section 5.3 obtaining a regular solution $\boldsymbol{u}$ to (1.44-1.46) with any $\boldsymbol{u}_0 \in W^{1,\infty}(\Omega)$ in any convex Ω.

Finally, we consider any lifting $\widetilde{\boldsymbol{u}}\colon \Omega \to \widetilde{\mathcal{N}}$ of $\boldsymbol{u}$ with $\widetilde{\boldsymbol{u}}_t \in L^2(]0, T[\times\Omega, \mathbb{R}^N)$, $\nabla \widetilde{\boldsymbol{u}} \in L^\infty(]0, T[\times\Omega, \mathbb{R}^N)$. As $R_* = +\infty$ for $\widetilde{N}$, arguments from section 5.4 imply that $\widetilde{\boldsymbol{u}}$ becomes constant in finite time (if we take large enough T), and consequently the same holds for $\boldsymbol{u} = \gamma \circ \widetilde{\boldsymbol{u}}$.

5.6 The case where the domain is a Riemannian manifold

Throughout this section, we assume that $(\mathcal{M}, \gamma)$ is an orientable, compact Riemannian manifold. Our aim is to prove Theorem 1.9.

Similarly as in section 5.2, given $\varepsilon, T > 0$ we first consider the following approximate system for $\boldsymbol{u}^\varepsilon\colon [0, T[\times\mathcal{M} \to \mathcal{N}$:

$$\boldsymbol{u}^\varepsilon_t = \pi_{\boldsymbol{u}^\varepsilon}\left(\operatorname{div}_\gamma \tfrac{\mathrm{d}\boldsymbol{u}^\varepsilon}{\sqrt{\varepsilon^2 + |\,\mathrm{d}\boldsymbol{u}^\varepsilon|^2_\gamma}}\right) \quad \text{in }]0, T[\times\mathcal{M}, \tag{5.40}$$

$$\boldsymbol{u}^\varepsilon(0, \cdot) = \boldsymbol{u}_0. \tag{5.41}$$

Again, in what follows we drop the index ε and denote

$$\boldsymbol{Z} = \tfrac{\mathrm{d}\boldsymbol{u}}{\sqrt{\varepsilon^2 + |\,\mathrm{d}\boldsymbol{u}|^2_\gamma}}, \qquad v = \left(|\,\mathrm{d}\boldsymbol{u}|^2_\gamma + \varepsilon^2\right)^{\frac{1}{2}}, \qquad v_0 = \left(|\,\mathrm{d}\boldsymbol{u}_0|^2_\gamma + \varepsilon^2\right)^{\frac{1}{2}}$$

Lemma 5.6. *We have*

$$\sup_{t\in[0,T[}\int_{\mathcal{M}} v(t,\cdot)+\int_0^T\int_{\mathcal{M}} \boldsymbol{u}_t^2 \le \int_{\mathcal{M}} v_0. \tag{5.42}$$

There exists $T_\dagger = T_\dagger(Ric_{\mathcal{M}}, K_{\mathcal{N}}, \|v_0\|_{L^\infty}) \in]0,\infty]$ and a non-decreasing function

$$M_{Ric_{\mathcal{M}},K_{\mathcal{N}},\|v_0\|_{L^\infty}} :]0,T_\dagger[\to]0,\infty[$$

such that for $t\in]0,\min(T,T_\dagger)[$ there holds

$$\|v(t,\cdot)\|_{L^\infty} \le M_{Ric_{\mathcal{M}},K_{\mathcal{N}},\|v_0\|_{L^\infty}}(t). \tag{5.43}$$

If $K_{\mathcal{N}}\le 0$, $T_\dagger=+\infty$. If moreover $Ric_{\mathcal{M}}\ge 0$, for $t\in]0,T[$ there holds $\|v(t,\cdot)\|_{L^\infty}\le\|v_0\|_{L^\infty}$.

Proof. We start by deriving a version of the Bochner formula (5.6) in our current setting. We calculate:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\,\mathrm{d}\boldsymbol{u}|_\gamma^2 = \gamma^{ab}\boldsymbol{u}_{x^a}\cdot(\pi_{\boldsymbol{u}}\mathrm{div}_\gamma\boldsymbol{Z})_{x^b} = \gamma^{ab}\boldsymbol{u}_{x^a}\cdot(\mathrm{div}_\gamma\boldsymbol{Z})_{x^b}+\gamma^{ab}\boldsymbol{u}_{x^a}\cdot(\pi_{\boldsymbol{u}})_{x^b}\mathrm{div}_\gamma\boldsymbol{Z}. \tag{5.44}$$

Let us recall the expression of div_γ as the trace of covariant derivative [17, Lemma 2.6],

$$\mathrm{div}_\gamma\vartheta = \gamma^{ab}\vartheta_{a;x^b} = (\gamma^{ab}\vartheta_a)_{;x^b} \tag{5.45}$$

(note that covariant derivative of the metric vanishes) and the Ricci identity [75, Chapter 5]

$$\vartheta_{a;x^bx^c}-\vartheta_{a;x^cx^b} = \vartheta_d(\mathcal{R}^{\mathcal{M}})^d{}_{abc} \tag{5.46}$$

for the commutator of covariant derivatives of a 1-form ϑ on $\mathcal{M}$. Using (5.46), we obtain

$$(\gamma^{cd}\boldsymbol{Z}_{c;x^d})_{x^b} = \gamma^{cd}\boldsymbol{Z}_{c;x^dx^b} = \gamma^{cd}\boldsymbol{Z}_{c;x^bx^d}+\gamma^{cd}\boldsymbol{Z}_e(\mathcal{R}^{\mathcal{M}})^e{}_{cdb}. \tag{5.47}$$

By antisymmetry of the Riemannian tensor and (1.58),

$$\gamma^{cd}(\mathcal{R}^{\mathcal{M}})^e{}_{cdb} = -\gamma^{ed}(\mathcal{R}^{\mathcal{M}})^c{}_{dcb} = -(\mathcal{R}ic^{\mathcal{M}})^{ef}\gamma_{fb}. \tag{5.48}$$

An application of (5.45) yields

$$\begin{aligned}\gamma^{ab}\boldsymbol{u}_{x^a}\cdot\gamma^{cd}\boldsymbol{Z}_{c;x^bx^d} &= (\gamma^{ab}\boldsymbol{u}_{x^a}\cdot\gamma^{cd}\boldsymbol{Z}_{c;x^b})_{;x^d}-\gamma^{ab}\boldsymbol{u}_{x^a;x^d}\cdot\gamma^{cd}\boldsymbol{Z}_{c;x^b}\\ &= \mathrm{div}_\gamma(\gamma^{ab}\boldsymbol{u}_{x^a}\cdot\boldsymbol{Z}_{;x^b})-\gamma^{ab}\gamma^{cd}\boldsymbol{u}_{x^d;x^a}\cdot\boldsymbol{Z}_{c;x^b}.\end{aligned} \tag{5.49}$$

Combining (5.45), (5.47), (5.48) and (5.49) we obtain

$$\gamma^{ab}\boldsymbol{u}_{x^a}\cdot(\mathrm{div}_\gamma\boldsymbol{Z})_{x^b} = \mathrm{div}_\gamma(\gamma^{ab}\boldsymbol{u}_{x^a}\cdot\boldsymbol{Z}_{;x^b})-\gamma^{ab}\gamma^{cd}\boldsymbol{u}_{x^d;x^a}\cdot\boldsymbol{Z}_{c;x^b}-\mathcal{R}ic^{\mathcal{M}}(\boldsymbol{Z}^i,\,\mathrm{d}u^i). \tag{5.50}$$

From (5.44) and (5.50) we derive, proceeding as in the proof of Lemma 5.2, a Bochner-type formula involving only coordinate-invariant expressions:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\,\mathrm{d}\boldsymbol{u}|_\gamma^2 &= \mathrm{div}_\gamma(\gamma^{ab}\boldsymbol{u}_{x^a}\cdot\boldsymbol{Z}_{;x^b})-\gamma^{ab}\gamma^{cd}(\pi_{\boldsymbol{u}}\boldsymbol{u}_{x^d;x^a})\cdot\boldsymbol{Z}_{c;x^b}\\ &\quad-\mathcal{R}ic^{\mathcal{M}}(\boldsymbol{Z}^i,\,\mathrm{d}u^i)+\gamma^{ab}\gamma^{cd}\boldsymbol{Z}_a\cdot\mathcal{R}^{\mathcal{N}}_{\boldsymbol{u}}(\boldsymbol{u}_{x^b},\boldsymbol{u}_{x^c})\boldsymbol{u}_{x^d}.\end{aligned} \tag{5.51}$$

We take any $p > 2$ and calculate $\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}|\,\mathrm{d}\boldsymbol{u}|_\gamma^p$. Proceeding as in the proof of Lemma 5.3, appealing to (5.45) and the fact that covariant derivatives of the metric vanish (or just working in normal coordinates) we obtain

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}v^p &\leq \mathrm{div}_\gamma(v^{p-2}\gamma^{ab}\boldsymbol{u}_{x^a}\cdot\boldsymbol{Z}_{;x^b}) - v^{p-3}\mathcal{R}ic^{\mathcal{M}}(\,\mathrm{d}u^i,\,\mathrm{d}u^i) + v^{p-3}\gamma^{ab}\gamma^{cd}\boldsymbol{u}_{x^a}\cdot\mathcal{R}^{\mathcal{N}}_{\boldsymbol{u}}(\boldsymbol{u}_{x^b},\boldsymbol{u}_{x^c})\boldsymbol{u}_{x^d}\\ &\leq \mathrm{div}_\gamma(v^{p-2}\gamma^{ab}\boldsymbol{u}_{x^a}\cdot\boldsymbol{Z}_{;x^b}) - Ric_{\mathcal{M}}v^{p-1} + K_{\mathcal{N}}v^{p+1}. \quad (5.52)\end{aligned}$$

Next, we integrate (5.52) over $\mathcal{M}$. As $\mathcal{M}$ is compact and orientable, the term

$$\int_{\mathcal{M}} \mathrm{div}_\gamma(v^{p-2}\gamma^{ab}\boldsymbol{u}_{x^a}\cdot\boldsymbol{Z}_{;x^b})\,\mathrm{d}\mu_\gamma$$

vanishes due to Stokes theorem. We are led to the following estimate:

$$\begin{aligned}\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{M}} v^p\,\mathrm{d}\mu_\gamma &\leq -Ric_{\mathcal{M}}\int_{\mathcal{M}} v^{p-1}\,\mathrm{d}\mu_\gamma + K_{\mathcal{N}}\int_{\mathcal{M}} v^{p+1}\,\mathrm{d}\mu_\gamma\\ &\leq -Ric^-_{\mathcal{M}}\,\mu_\gamma(\mathcal{M})^{\frac{1}{p}}\left(\int_{\mathcal{M}} v^p\,\mathrm{d}\mu_\gamma\right)^{1-\frac{1}{p}} + K^+_{\mathcal{N}}\|v\|_{L^\infty}\int_{\mathcal{M}} v^p\,\mathrm{d}\mu_\gamma,\end{aligned}$$

where we have used Hölder inequality and denoted $Ric^-_{\mathcal{M}} = \min(Ric_{\mathcal{M}}, 0)$, $K^+_{\mathcal{N}} = \max(K_{\mathcal{N}}, 0)$. Thus,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{M}} v^p\,\mathrm{d}\mu_\gamma\right)^{\frac{1}{p}} \leq -Ric^-_{\mathcal{M}}\mu_\gamma(\mathcal{M})^{\frac{1}{p}} + K^+_{\mathcal{N}}\|v\|_{L^\infty}\left(\int_{\mathcal{M}} v^p\,\mathrm{d}\mu_\gamma\right)^{\frac{1}{p}}.$$

Passing to the limit $p \to \infty$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|v\|_{L^\infty} \leq -Ric^-_{\mathcal{M}} + K^+_{\mathcal{N}}\|v\|^2_{L^\infty}. \quad (5.53)$$

We let $M_{Ric_{\mathcal{M}},K_{\mathcal{N}},\|v_0\|_{L^\infty}}$ be the locally existing solution to

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -Ric^-_{\mathcal{M}} + K^+_{\mathcal{N}}M^2$$

with initial datum $\|v_0\|_{L^\infty}$, and let $T_\dagger$ be the maximal time of existence of $M_{Ric_{\mathcal{M}},K_{\mathcal{N}},\|v_0\|_{L^\infty}}$, completing the proof. □

Proposition 5.2. *Let $\boldsymbol{u}_0 \in C^{3+\alpha}(\mathcal{M},\mathcal{N})$. There exist $T_\dagger = T_\dagger(Ric_{\mathcal{M}}, K_{\mathcal{N}}, \|\nabla\boldsymbol{u}_0\|_{L^\infty}) > 0$ and unique solution $\boldsymbol{u} \in C^{\frac{3+\alpha}{2},3+\alpha}_{loc}([0,T_\dagger[\times\mathcal{M},\mathcal{N})$ to the system (5.40, 5.41).*

Proof. Let $\boldsymbol{u}_0 \in C^{3+\alpha}(\mathcal{M},\mathcal{N})$. As in [48, section 3], we show that there exists $T > 0$ and unique solution $\boldsymbol{u} \in C^1([0,T], C^\alpha(\mathcal{M},\mathcal{N})) \cap C([0,T], C^{2+\alpha}(\mathcal{M},\mathcal{N}))$ to (5.40, 5.41). Using linear theory [54], we rise regularity of the solution to $C^{\frac{3+\alpha}{2},3+\alpha}([0,T[\times\mathcal{M},\mathcal{N})$. Then, using the uniform bound on $\mathrm{d}\boldsymbol{u}$ in L^∞ from Lemma 5.6 we show that the solution can be extended to $[0,T_\dagger[$ as in the proof of Proposition 5.1. □

Proof of Theorem 1.9. The proof of uniqueness follows along the lines of the proof of Theorem 1.5. An important point is that integration by parts is allowed because $\mathcal{M}$ is orientable.

Given any initial datum $\boldsymbol{u}_0 \in W^{1,\infty}(\mathcal{M},\mathcal{N})$, we take an approximating family $(\boldsymbol{u}_0^\varepsilon) \subset C^{3+\alpha}(\mathcal{M},\mathcal{N})$ such that $\boldsymbol{u}_0^\varepsilon \to \boldsymbol{u}_0$ as $\varepsilon \to 0^+$ in $C(\mathcal{M},\mathcal{N})$ and $\|\,\mathrm{d}\boldsymbol{u}_0^\varepsilon\|_{L^\infty} \to \|\,\mathrm{d}\boldsymbol{u}_0\|_{L^\infty}$. Proposition 5.2 generates a family $(\boldsymbol{u}^\varepsilon)$, where $\boldsymbol{u}^\varepsilon$ solves (5.40) with initial datum $\boldsymbol{u}_0^\varepsilon$. This family satisfies uniform bounds on $(\boldsymbol{u}_t^\varepsilon)$ in $L^2(]0,T_\dagger[\times\mathcal{M},\mathbb{R}^N)$ and on $(\,\mathrm{d}\boldsymbol{u}^\varepsilon)$ in $L^\infty_{loc}([0,T_\dagger[\times T^*\mathcal{M}\times\mathbb{R}^N)$. Using these bounds, we pass to the limit as in section 5.3 and obtain the regular solution $(\boldsymbol{u},\boldsymbol{Z})$ to (1.54) in $[0,T_\dagger[$. Recall that if $K_\mathcal{N} \le 0$, $T_\dagger = +\infty$.

Now we assume that $K_\mathcal{N} \le 0$ and $Ric_\mathcal{M} \ge 0$. In this case we have

$$\boldsymbol{u}_t \in L^2(]0,\infty[\times\mathcal{M},\mathbb{R}^N), \qquad \|\,\mathrm{d}\boldsymbol{u}(t,\cdot)\|_{L^\infty} \le \|\,\mathrm{d}\boldsymbol{u}_0\|_{L^\infty} \text{ in a. e. } t>0. \tag{5.54}$$

Therefore, we can choose a sequence of time instances $(t_k) \subset]0,\infty[$, $t_k \to \infty$ such that there exists $\boldsymbol{u}_* \in W^{1,\infty}(\mathcal{M},\mathcal{N})$ with

$$\boldsymbol{u}(t_k,\cdot) \to \boldsymbol{u}_* \text{ in } C(\mathcal{M},\mathcal{N}), \qquad \boldsymbol{u}_t(t_k,\cdot) \rightharpoonup \boldsymbol{0} \text{ in } L^2(\mathcal{M},\mathbb{R}^N), \tag{5.55}$$

and

$$\boldsymbol{u}_t(t_k,\cdot) = \pi_{\boldsymbol{u}(t_k,\cdot)}\left(\mathrm{div}_\gamma \boldsymbol{Z}(t_k,\cdot)\right), \quad \boldsymbol{Z}(t_k,\cdot) \in \tfrac{\mathrm{d}\boldsymbol{u}}{|\,\mathrm{d}\boldsymbol{u}|_\gamma}(t_k,\cdot) \quad \mu_\gamma\text{-a. e. in } \mathcal{M}. \tag{5.56}$$

The first item in (5.56) can be rewritten as

$$\boldsymbol{u}_t(t_k,\cdot) = \mathrm{div}_\gamma \boldsymbol{Z}(t_k,\cdot) + \gamma^{ab}\mathcal{A}_{\boldsymbol{u}(t_k,\cdot)}(\boldsymbol{u}_{x^a}(t_k,\cdot),\boldsymbol{Z}_b(t_k,\cdot)),$$

hence (5.54) implies that the sequence $\mathrm{div}_\gamma\boldsymbol{Z}(t_k,\cdot)$ is uniformly bounded in $L^2(\mathcal{M},\mathbb{R}^N)$. The second item in (5.56) is equivalent to

$$\pi^\perp_{\boldsymbol{u}(t_k,\cdot)}\boldsymbol{Z}(t_k,\cdot) = \boldsymbol{0}, \quad |\boldsymbol{Z}(t_k,\cdot)|_\gamma \le 1,$$
$$\gamma^{ab}\boldsymbol{u}_{x^a}(t_k,\cdot)\cdot\boldsymbol{Z}_b(t_k,\cdot) = |\,\mathrm{d}\boldsymbol{u}(t_k,\cdot)|_\gamma \qquad \mu_\gamma\text{-a. e. in } \mathcal{M}.$$

Hence, there exists $\boldsymbol{Z}_* \in L^\infty(T^*\mathcal{M}\times\mathbb{R}^N)$ satisfying $\mathrm{div}_\gamma\boldsymbol{Z}_* \in L^\infty(\mathcal{M},\mathbb{R}^\mathcal{N})$ and (possibly decimating the sequence (t_k))

$$\boldsymbol{Z}(t_k,\cdot) \overset{*}{\rightharpoonup} \boldsymbol{Z}_* \text{ in } L^\infty(T^*\mathcal{M}\times\mathbb{R}^N), \qquad \mathrm{div}_\gamma\boldsymbol{Z}(t_k,\cdot) \rightharpoonup \mathrm{div}_\gamma\boldsymbol{Z}_* \text{ in } L^2(\mathcal{M},\mathbb{R}^N), \tag{5.57}$$

$$\pi^\perp_{\boldsymbol{u}_*}\boldsymbol{Z}_* = \boldsymbol{0}, \quad |\boldsymbol{Z}_*|_\gamma \le 1 \qquad \mu_\gamma\text{-a. e. in } \mathcal{M}. \tag{5.58}$$

Using a standard div-curl reasoning and weak-star convergence of $\boldsymbol{u}(t_k,\cdot)$ in $W^{1,\infty}(\mathcal{M},\mathcal{N})$ we also obtain

$$|\,\mathrm{d}\boldsymbol{u}_*|_\gamma \le \liminf|\,\mathrm{d}\boldsymbol{u}(t_k,\cdot)|_\gamma = \gamma^{ab}\boldsymbol{u}_{*,x^a}\cdot\boldsymbol{Z}_{*,b} \le |\,\mathrm{d}\boldsymbol{u}_*|_\gamma \qquad \mu_\gamma\text{-a. e. in } \mathcal{M}.$$

This together with (5.58) yields the second item of (1.59). The first item of (1.59) is produced by passing to the limit in the first item of (5.56) using (5.55, 5.57). □

Appendix: Technical lemmata

Lemma A.1. *Let* $(\mathcal{N}, g)$ *be a closed embedded Riemannian submanifold in the Euclidean space* $\mathbb{R}^N$*. There exists a Riemannian metric* h *on* $\mathbb{R}^N$ *such that* $(\mathcal{N}, g)$ *is a totally geodesic Riemannian submanifold of* $(\mathbb{R}^N, h)$*.*

Proof. Let $R > 0$. As $\mathcal{N}$ is a closed submanifold of $\mathbb{R}^N$, $\mathcal{N} \cap \overline{B(0,R)}$ is compact. Hence, there is a non-increasing function $R \mapsto \delta_R \in]0,1[$ such that

$$N_{R,\delta} = \{\boldsymbol{y} + \boldsymbol{n} \colon \boldsymbol{y} \in \mathcal{N} \cap B(0,R), \boldsymbol{n} \in T_{\boldsymbol{y}}\mathcal{N}^{\perp}, |\boldsymbol{n}| < \delta\}$$

is a tubular neighborhood of $\mathcal{N} \cap B(0,R)$ in $\mathbb{R}^N$ that does not intersect $\mathcal{N} \setminus B(0,R)$ for $\delta \in]0, \delta_R[$. Identifying $T_{\boldsymbol{y}+\boldsymbol{n}} N_{R,\delta_R}$ with $T_{\boldsymbol{y}}\mathcal{N} \times \mathbb{R}^{N-n}$, we define a Riemannian metric h^R on N_{R,δ_R} as follows:

$$h^R_{\boldsymbol{p}+\boldsymbol{n}}(\boldsymbol{w}_1 + \boldsymbol{w}_1', \boldsymbol{w}_2 + \boldsymbol{w}_2') = g_{\boldsymbol{p}}(\boldsymbol{w}_1, \boldsymbol{w}_2) + \boldsymbol{w}_1' \cdot \boldsymbol{w}_2'$$

for $\boldsymbol{p} \in \mathcal{N} \cap B(0,R)$, $|\boldsymbol{n}| < \delta_R$, $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_{\boldsymbol{p}}\mathcal{N}$, $\boldsymbol{w}_1', \boldsymbol{w}_2' \in \mathbb{R}^{N-n}$. Next, we define the tubular neighborhood of $\mathcal{N}$

$$\mathcal{T} = \bigcup_{k=1}^{\infty} N_{k, \frac{1}{2}\delta_{k+1}}$$

so that

$$\{\mathbb{R}^N \setminus \overline{\mathcal{T}}, N_{1,\delta_1}, N_{2,\delta_2}, \ldots\}$$

is an open cover of $\mathbb{R}^N$. Indeed, if $\boldsymbol{z} \notin \mathbb{R}^N \setminus \overline{\mathcal{T}}$, i.e. $\boldsymbol{z} \in \overline{\mathcal{T}}$, then letting k_0 be the smallest integer bound of $|\boldsymbol{z}|$, we have

$$\boldsymbol{z} \in \overline{\mathcal{T}} \cap B(0, k_0+1) \subset \overline{\bigcup_{k=1}^{k_0+1} N_{k,\frac{1}{2}\delta_{k+1}}} = \bigcup_{k=1}^{k_0+1} \overline{N_{k,\frac{1}{2}\delta_{k+1}}}.$$

Here, we used the fact that $U \cap \overline{\bigcup_{k=1}^{\infty} A_k} \subset \overline{\bigcup_{k=1}^{\infty} U \cap A_k}$ for any sequence of sets A_k and open set U. Hence, by definition of k_0, $\boldsymbol{z} \in \overline{N_{k_0,\frac{1}{2}\delta_{k_0+1}}}$. Therefore

$$\boldsymbol{z} = \boldsymbol{y} + \boldsymbol{n} \quad \text{with} \quad |\boldsymbol{n}| \leq \frac{1}{2}\delta_{k_0+1} < \delta_{k_0+1} \quad \text{and} \quad \boldsymbol{y} \in \overline{B(0,k_0)} \subset B(0, k_0+1),$$

that is, $\boldsymbol{z} \in N_{k_0+1,\delta_{k_0+1}}$.

We take a smooth partition of unity $\{\varphi_0, \varphi_1, \varphi_2, \ldots\}$ subordinate to this cover (a construction of a partition of unity subordinate to an infinite open cover can be found in [80, Appendix C]) and define

$$h_{\boldsymbol{y}}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \varphi_0(\boldsymbol{y})\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 + \sum_{k=1}^{\infty} \varphi_k(\boldsymbol{y}) h^k_{\boldsymbol{y}}(\boldsymbol{v}_1, \boldsymbol{v}_2)$$

for $\boldsymbol{y} \in \mathbb{R}^N$. It is easy to check that $(\mathcal{N}, g)$ is a totally geodesic submanifold in $(\mathbb{R}^N, h)$. $\square$

Lemma A.2. *Let $\boldsymbol{u}_0 \in W^{1,\infty}(\Omega, \mathcal{N})$. There exists a family $(\boldsymbol{u}_{0,\varepsilon}) \subset C^\infty(\overline{\Omega}, \mathcal{N})$, $\varepsilon \in]0, \varepsilon_0[$, $\varepsilon_0 > 0$ such that*

- $\boldsymbol{u}_{0,\varepsilon} \to \boldsymbol{u}_0$ *in* $C(\overline{\Omega}, \mathbb{R}^N)$ *as* $\varepsilon \to 0^+$,
- $\|\nabla \boldsymbol{u}_{0,\varepsilon}\|_{L^\infty} \to \|\nabla \boldsymbol{u}_0\|_{L^\infty}$ *as* $\varepsilon \to 0^+$,
- $\boldsymbol{u}_{0,\varepsilon}$ *satisfy compatibility conditions (5.19, 5.20) for* $\varepsilon \in]0, \varepsilon_0[$.

Proof. As $\partial\Omega$ is a compact smooth submanifold of $\mathbb{R}^m$, there is $\varepsilon_0' > 0$ and a tubular neighbourhood of $\partial\Omega$

$$T = \{\boldsymbol{y} + r\boldsymbol{\nu}^\Omega, \boldsymbol{y} \in \partial\Omega, r \in]-\varepsilon_0', \varepsilon_0'[\}.$$

We extend $\boldsymbol{u}_0$ to $\boldsymbol{w} \in W^{1,\infty}(\Omega \cup T, \mathcal{N})$ putting

$$\boldsymbol{w}(\boldsymbol{y} + r\boldsymbol{\nu}^\Omega(\boldsymbol{y})) = \boldsymbol{y}$$

for $r \in [0, \varepsilon_0'[$. For any $\varepsilon \in]0, \varepsilon_0'[$ we define

$$\Omega_\varepsilon = \{\boldsymbol{x} \in \Omega \colon \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > \varepsilon\}.$$

Mollifying $\boldsymbol{w}$ as in [49, Theorems 4.4, 4.6] we produce a family of maps $(\boldsymbol{w}_\varepsilon)_{\varepsilon \in]0,\varepsilon_0[}$, $\varepsilon_0 \in]0, \varepsilon_0'[$, $\boldsymbol{w}_\varepsilon \in C^\infty(\overline{\Omega}, \mathcal{N})$ such that $\boldsymbol{w}_\varepsilon \to \boldsymbol{u}_0$ in $C(\overline{\Omega}, \mathcal{N})$ and $\|\nabla \boldsymbol{w}_\varepsilon\|_{L^\infty} \to \|\nabla \boldsymbol{u}_0\|_{L^\infty}$ as $\varepsilon \to 0^+$.

Now, let $\eta_\varepsilon \in C^\infty(]0, \varepsilon[,]0, \varepsilon[)$ satisfy the conditions

- $\eta_\varepsilon(r) = r$ for $r \in [\frac{\varepsilon}{2}, \varepsilon[$,
- $\eta_\varepsilon'(r) = 0$ for $r \in]0, \frac{\varepsilon}{4}]$,
- $0 \leq \eta' \leq 1$.

We define $\Phi_\varepsilon \in C^\infty(\Omega, \Omega)$ by

$$\Phi_\varepsilon(\boldsymbol{x}) = \begin{cases} \boldsymbol{y} - \eta_\varepsilon(r)\boldsymbol{\nu}^\Omega & \text{if } \boldsymbol{x} = \boldsymbol{y} - r\boldsymbol{\nu}^\Omega \in \Omega \setminus \Omega_\varepsilon, \\ \boldsymbol{x} & \text{if } \boldsymbol{x} \in \Omega_\varepsilon. \end{cases}$$

It is easy to see that $\boldsymbol{u}_{0,\varepsilon} = \boldsymbol{w}_\varepsilon \circ \Phi_\varepsilon$ satisfies the desired conditions. □

Lemma A.3. *Let $\Omega \subset \mathbb{R}^m$ be open and convex. There exists a family (Ω_ε) of open, convex sets with smooth boundary such that $\Omega_\varepsilon \subset \Omega$ for $\varepsilon \in]0, \varepsilon_0[$, $\varepsilon_0 > 0$ and the Hausdorff distance of Ω_ε from Ω tends to zero as $\varepsilon \to 0^+$.*

Proof. Let d denote the signed distance function of Ω, i. e.

$$d(\boldsymbol{x}) = \operatorname{dist}(\boldsymbol{x}, \Omega) - \operatorname{dist}(\boldsymbol{x}, \mathbb{R}^m \setminus \Omega) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^m.$$

This function is convex and satisfies

$$|d(\boldsymbol{x}) - d(\boldsymbol{y})| \leq |\boldsymbol{x} - \boldsymbol{y}| \quad \text{for } \boldsymbol{x}, \boldsymbol{y} \text{ in } \mathbb{R}^N. \tag{5.59}$$

Let $(\varphi_\varepsilon)_{\varepsilon > 0}$ be a standard family of mollifying kernels such that

$$\operatorname{supp} \varphi_\varepsilon \subset B(\boldsymbol{0}, \varepsilon) \tag{5.60}$$

and denote $d_\varepsilon = \varphi_\varepsilon * d$. It is easy to check that d_ε is smooth and convex. Let us further denote

$$\Omega_\varepsilon = \{\boldsymbol{x} \in \mathbb{R}^m \colon d_\varepsilon(\boldsymbol{x}) < -\varepsilon\}.$$

As a sublevel set of a convex function, Ω_ε is convex. Now, denote by r_Ω the inradius of Ω, equivalently $r_\Omega = |\min d|$. Take $\varepsilon_0 = \frac{r_\Omega}{3}$ and assume $\varepsilon < \varepsilon_0$. Suppose that $d(\boldsymbol{x}) \geq 0$. Due to (5.60, 5.59), we have

$$d_\varepsilon(\boldsymbol{x}) = \int_{B(\boldsymbol{x},\varepsilon)} \varphi_\varepsilon(\boldsymbol{x} - \boldsymbol{y}) d(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} > -\varepsilon.$$

Hence, $\overline{\Omega}_\varepsilon \subset \Omega$. Similarly, if $d(\boldsymbol{x}) \leq -2\varepsilon$, then $d_\varepsilon(\boldsymbol{x}) < -\varepsilon$. This in turn implies that

$$\operatorname{dist}(\partial\Omega_\varepsilon, \partial\Omega) = \min\{-d(\boldsymbol{x}) \colon d_\varepsilon(\boldsymbol{x}) = -\varepsilon\} < 2\varepsilon. \tag{5.61}$$

Denoting by $\boldsymbol{x}_\Omega$ the center of any circle inscribed in $\overline{\Omega}$,

$$\min d_\varepsilon \leq \int_{B(\boldsymbol{x}_\Omega,\varepsilon)} \varphi_\varepsilon(\boldsymbol{x}_\Omega - \boldsymbol{y}) d(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} < -r_\Omega + \varepsilon < -2\varepsilon. \tag{5.62}$$

Recall that a critical point of a smooth convex function on $\mathbb{R}^m$ is necessarily its global (possibly improper) minimum. Hence, by virtue of (5.61, 5.62), Ω_ε does not contain critical points of d_ε, and so it is a smooth hypersurface. Finally, (5.61) implies the Hausdorff convergence of Ω_ε to Ω as $\varepsilon \to 0^+$.

□

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