University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Michał Dębski

Strong Chromatic Index of Graphs

PhD dissertation

Supervisor

Prof. Dr hab. Jarosław Grytczuk

Faculty of Mathematics and Computer Science
Jagiellonian University

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aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contens of the dissertation have been obtainet by legal means.

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Michał Dębski

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the dissertation is ready to be reviewed

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....
prof. dr hab. Jarosław Grytczuk

Abstract

A **strong edge coloring** of a graph G is a coloring of edges of G such that each color class is an induced matching in G, and the **strong chromatic index** of G, denoted $s'(G)$, is the minimum number of colors in a strong edge coloring of G. We also consider the fractional and topological variant of strong chromatic index, denoted $s'_f(G)$ and $s'_t(G)$ respectively.

Our dream goal is to give a sharp upper bound on the strong chromatic index of a graph G with the given maximum degree Δ. A simple, greedy argument shows that $s'(G) \leq 2\Delta^2$, and the best known bound is $1.93\Delta^2$ (Bruhn and Joos, 2015+). This result is still far from $\frac{5}{4}\Delta^2$, conjectured by Erdős and Nešetřil in 1985 (which would be sharp).

For bipartite graphs the conjectured bound is $s'(G) \leq \Delta^2$ (Faudree, Gyárfás, Schelp and Tuza, 1989) and the best known is $s'(G) \leq 1.93\Delta^2$ (that is, there is no improvement over the mentioned result of Bruhn and Joos); it follows that $s'_t(G) \leq 1.93\Delta^2$. For fractional strong chromatic index, a better bound $1.5\Delta^2$ can be obtained from earlier results.

Our main contribution is "breaking the $1.5\Delta^2$ boundary" – we show that for a bipartite graph G of maximum degree Δ we have $s'_f(G) \leq 1.476\Delta^2$. Moreover, we significantly improve the bound on the topological variant: for a bipartite graph G of maximum degree Δ we show $s'_t(G) \leq 1.703\Delta^2$. We also show that if G is a graph such that every edge of G is in at most $\frac{\Delta^2}{f}$ 4-cycles, then $s'(G) \leq K\frac{\Delta^2}{\ln f}$ for some absolute constant K, and give a bound $s'(G) \leq 4\Delta - 3$ in case when G is chordless.

Keywords: strong chromatic index, induced matching, graph coloring, fractional chromatic number.

AMS Classification: 05C15, 05C35, 05C70, 05C72.

Streszczenie

Silnym kolorowaniem krawędzi grafu G nazywamy kolorowanie krawędzi G, w którym krawędzie w każdym z kolorów tworzą indukowane skojarzenie w G, a **silny indeks chromatyczny** grafu G, oznaczany $s'(G)$, to minimalna możliwa liczba kolorów w silnym kolorowaniu krawędzi G. Rozważamy również ułamkowy i topologiczny odpowiednik silnego indeksu chromatycznego, oznaczany odpowiednio $s'_f(G)$ i $s'_t(G)$.

Zasadniczym, wciąż nieosiągniętym celem, do którego dążymy, jest wyznaczenie dokładnego ograniczenia górnego na silny indeks chromatyczny grafu o zadanym maksymalnym stopniu Δ. Prosty argument, oparty na kolorowaniu zachłannym, pokazuje, że $s'(G) \leq 2\Delta^2$, natomiast najlepsze znane ograniczenie to $1.93\Delta^2$ (Bruhn i Joos, 2015+). Wynik ten jest odległy od ograniczenia $\frac{5}{4}\Delta^2$, które wydaje się prawidłową odpowiedzią (jest to treść hipotezy Erdősa i Nešetřila z roku 1985).

Dla grafów dwudzielnych może być prawdziwe ograniczenie $s'(G) \leq \Delta^2$ (hipoteza Faudree, Gyárfása, Schelpa i Tuzy z roku 1989), jednak najlepszym znanym jest $s'(G) \leq 1.93\Delta^2$ (a więc nie jest znany żaden wynik mocniejszy od twierdzenia Bruhna i Joosa); wynika stąd, że $s'_t(G) \leq 1.93\Delta^2$. Dla ułamkowego silnego indeksu chromatycznego mocniejsze ograniczenie $1.5\Delta^2$ może zostać wywiedzione z wcześniejszych wyników.

Naszym głównym wynikiem jest „przełamanie bariery $1.5\Delta^2$" – pokazujemy, że dla grafu dwudzielnego o maksymalnym stopniu Δ zachodzi $1.476\Delta^2$. Ponadto, istotnie poprawiamy ograniczenie w wariancie topologicznym: dla grafu dwudzielnego G o maksymalnym stopniu Δ dowodzimy, że $s'_t(G) \leq 1.703\Delta^2$. Pokazujemy również, że jeżeli G jest grafem takim, że każda krawędź G zawiera się w co najwyżej $\frac{\Delta^2}{f}$ cyklach o długości 4, wówczas zachodzi $s'(G) \leq K\frac{\Delta^2}{\ln f}$ dla pewnej uniwersalnej stałej K. Dodatkowo, pokazujemy ograniczenie $s'(G) \leq 4\Delta - 3$ w przypadku, gdy G jest grafem bezcięciwowym.

Słowa kluczowe: silny indeks chromatyczny, indukowane skojarzenie, kolorowanie grafów, ułamkowa liczba chromatyczna.

Klasyfikacja AMS: 05C15, 05C35, 05C70, 05C72.

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1 Preliminaries

Basic notions

A **graph** is a pair (V, E), where V is a nonempty finite set and E is a subset of the set of all 2-element subsets of V; we say that elements of V are **vertices** and elements of E – **edges** of the graph. For a graph G, by $E(G)$ (resp. $V(G)$) we denote the set of vertices (resp. edges) of G, and $e(G)$ (resp. $v(G)$) denotes the size of $E(G)$ (resp. $v(G)$).

We use the shorter notation uv to denote the edge $e = \{u, v\}$ of a graph G; we say that vertices u and v are **incident** to an edge uv. If a graph G has an edge uv, we say that u and v are **adjacent** in G and that u is a **neighbor** of v in G (and vice versa).

The number of edges incident to a vertex v in a graph G is called the **degree** of v in G and denoted $\deg_G(v)$, which is shortened to $\deg(v)$ when the considered graph is clear from the context. The **maximum degree** of a graph G, denoted $\Delta(G)$, is the maximum of $\deg(v)$ over all vertices v of G. The **average degree** of G if the average of $\deg(v)$ over all vertices v of G.

The set of neighbors of a vertex v in a graph G is denoted $N_G(v)$, and by $N_G[v]$ we denote the set $N_G(v) \cup \{v\}$. For a set $S \subseteq V(G)$ we define $N_G(S)$ to be the set of vertices of G that are adjecent to at least one vertex from S and not contained in S, and set $N_G[S] = N_G(S) \cup S$. Those notations are shortened to $N(v)$, $N[v]$, $N(S)$ and $N[S]$ when the graph G is clear from the context.

We say that edges uv and wx of a graph G are **joined** in G if $uv \neq wx$ and G contains at least one of the edges uw, ux, vw, vx (in particular, if two edges of G intersect, they are joined in G). This notion is nonstandard.

A graph H is a **subgraph** of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; H is a **spanning** subgraph of G if $V(H) = V(G)$. A subgraph H of a graph G is **induced** if for every two vertices $u, v \in V(H)$ we have $uv \in E(H)$ iff $uv \in E(G)$. For a set $S \subseteq V(G)$ the **subgraph of** G **induced by** S, denoted $G[S]$, is the induced subgraph of G with vertex set S. For a set $F \subseteq E(G)$, the subgraph of G induced by F, denoted $G[F]$, is the graph $(\bigcup F, F)$. For $S \subseteq V(G)$ (resp. $S \subseteq E(G)$) by $G - S$ we denote the grap $G[V(G) \setminus S]$

(resp. $G[E(G) \setminus S]$). By $G - H$ (where H is a graph) we denote the graph $G - E(H)$.

Graphs G and H are **isomorphic** if there is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$; such function f is called **isomorphism**. A **copy of H in G** is a subgraph of G isomorphic to H. A graph G is said to be **H-free** if there is no copy of H in G.

Graph classes and invariants

A **clique** or **complete** graph is a graph G such that for every two vertices $u, v \in V(G)$ we have $uv \in E(G)$. A complete graph on n vertices is denoted K_n. The **clique number** of a graph G, denoted $\omega(G)$, is the number of vertices in the largest complete subgraph of G.

For $n \geq 3$ an **n-cycle**, denoted C_n is a graph isomorphic to the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{i(i+1) : 1 \leq i < n\} \cup \{1n\}$. An n-path, denoted P_n, is a graph isomorphic to the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{i(i+1) : 1 \leq i < n\}$. A graph that is n-cycle (resp. n-path) for some n is called a **cycle** (resp. **path**).

A graph G is **bipartite** if $V(G)$ can be paritioned into two sets X, Y such that every edge of G has one vertex from X and one from Y; X and Y are called the **partition classes** of G. A **complete bipartite graph**, denoted $K_{m,n}$, is a bipartite graph with partition classes X and Y and edge set $\{xy : x \in X, y \in Y\}$.

A graph G is **k-degenerate** if every subgraph of G has a vertex of degree at most k, and the **degeneracy** of a graph is the minimum k such that the graph is k-degenerate. The **k-core** of the graph is the maximum subgraph that has every vertex of degree greater than k. A graph G is **k-regular** if every vertex of G has degree exactly k.

The **line graph** of a graph G, denoted $L(G)$, is a graph L such that $V(L) = E(G)$ and for every two $e, f \in E(G)$ we have $ef \in E(L)$ if and only if e and f intersect. The **square** of a graph G, denoted G^2, is the graph S such that $V(S) = V(G)$ and for every two vertices $u, v \in V(G)$ we have $uv \in E(S)$ iff $uv \in E(G)$ or there is a vertex $z \in V(G)$ such that $uz, vz \in E(G)$. In particular, $L(G)^2$ is that graph with vertex set $E(G)$ such that $ef \in E(L(G)^2)$ iff e and f are joined in G.

A **matching** is a graph M such that no two edges of M intersect; we shall identify a matching with its set of edges. A **perfect matching** in a graph G is a spanning subgraph of G that is a matching with every vertex of degree 1. An **induced matching** in G is a matching that is an induced subgraph of G.

An **independent set** in a graph G is a subset $I \subseteq V(G)$ such that for every two vertices $x, y \in I$, xy is not an edge of G. The size of the largest independent set in G is denoted $\alpha(G)$. Note that an induced matching in G is an independent set in $L(G)^2$.

A **vertex-coloring** (resp. **edge-coloring**) of a graph G is a partition of $V(G)$ (resp. $E(G)$); the sets in the parition are called **color classes** and the number of color classes is shortened to **number of colors**. A k**-vertex-coloring** (resp. k-edge-coloring) is a vertex coloring (resp. edge-coloring) with k colors. The **coloring** and k**-coloring** of a graph is, respectively, vertex-coloring and k-vertex-coloring.

A vertex coloring (resp. edge coloring) of a graph G is **proper** if each color class is an intependent set in G (resp. $L(G)$). The **chromatic number** of G, denoted $\chi(G)$ (resp. **chromatic index** of G, denoted $\chi'(G)$), is the minimum number of colors in a proper vertex coloring (resp. proper edge coloring) of G.

A **fractional** k**-coloring** of a graph G (where k is a real number) is a weighting $w : \mathcal{I}_G \to [0, 1]$ (where $\mathcal{I}_G$ is the set of independent sets in G) such that for every vertex $v \in V(G)$ the sum of weights of sets from $\mathcal{I}_G$ that contain v is equal to 1 and the sum of all weights is at most k (note that a proper k-coloring of G can be seen as a fractional k-coloring of G). The **fractional chromatic number** of G, denoted $\chi_f(G)$, is the minimum k such that there exists a fractional k-coloring of G.

Cauchy-Schwarz inequality

The Cauchy-Schwarz inequality states that for real numbers $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ we have

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \leq \left(\sum_{i=1}^{n} x_i^2\right) \left(\sum_{i=1}^{n} y_i^2\right).$$

We will repetitively use the following consequence of this inequality. Let $x_1, x_2, \ldots, x_n$

be nonnegative real numbers such that $\sum_{i=1}^{n} x_i = S$. We have

$$\sum_{i=1}^{n} x_i^2 \geq \frac{S^2}{n}.$$

To prove the implication, it suffices to set $y_i = 1$ for all i and replace $\sum_{i=1}^{n} x_i y_i$ with S.

2 Introduction

2.1 Two definitions of strong chromatic index

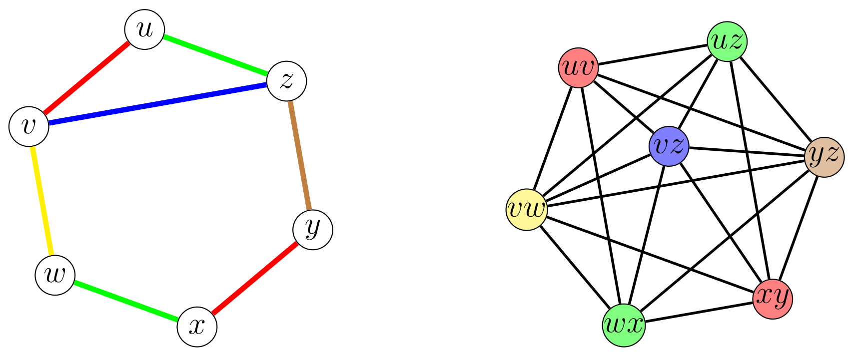
It is worth to note that in a proper edge-coloring of a graph every color class is required to be a matching. Throughout this dissertation we will be considering colorings obeying a stronger condition that the matchings must be induced – recall that a matching M in a graph G is induced if any two edges of M are not joined (by any edge of G).

Definition 2.1.1. *Let G be a graph.*

*A **strong edge-coloring** of G is an edge-coloring of G such that every color class is an induced matching in G.*

*The **strong chromatic index** of G, denoted $s'(G)$, is the minimum possible number of colors in a strong edge-coloring of G.*

As an example, consider the coloring depicted in Figure 1a (edge uv has the same color as xy, uz has the same color as wx and the remaining colors are distinct). It is indeed strong, because matchings $\{uv, xy\}$ and $\{uz, wx\}$ are induced. Moreover, there is no strong edge-coloring that would use 4 colors, because no two edges from $\{uv, uz, vw, vz, yz\}$ can belong to the same induced matching. Therefore, the strong chromatic index of the graph is 5.



(a) Strong edge-coloring of a graph. (b) Square of the line graph.

Figure 1: Two definitions of strong edge-coloring.

We can also think of strong edge-coloring as a proper vertex-coloring of a certain graph. If uv is an edge of G, then its color in a strong edge-coloring of G must be different than the color of any edge that in joined to uv. Conversely, if every two edges that are joined have different colors, then the edge-coloring is strong. Since our notion of joined edges is just an adjacency in the square of the line graph, the following definition is equivalent to Definition 2.1.1.

Definition 2.1.2. *Let G be a graph.*

*A **strong edge-coloring** of G is a coloring of $L(G)^2$.*

*The **strong chromatic index** of G, denoted $s'(G)$, is the chromatic number of $L\left(G\right)^2$.*

Consider our previous example; the square of the line graph is depicted in Figure 1b. The strong chromatic index of the graph on the left is 5 because the chromatic number of the square of its line graph is 5 (note that $\{uv, uz, vw, vz, yz\}$ form a clique in the graph on the right).

The second definition, although indirect, will turn out to be very handy for cosiderations regarding related graph parameters. In particular, the fractional strong chromatic index can be thought of as the fractional chromatic number of the square of the line graph, and it is defined as follows.

Definition 2.1.3. *Let G be a graph.*

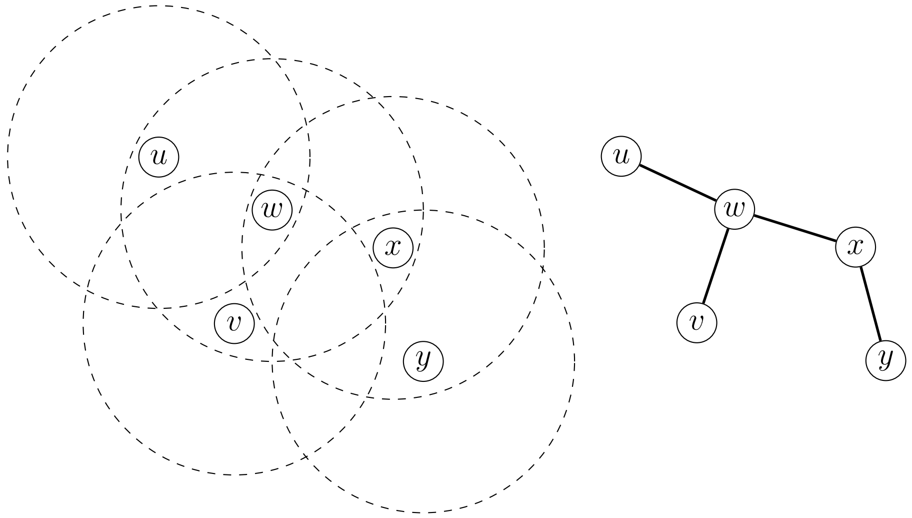
*A **fractional strong edge-coloring** of G is a fractional coloring of $L(G)^2$.*

*The **fractional strong chromatic index** of G, denoted $s'_f(G)$, is the fractional chromatic number of $L\left(G\right)^2$.*

2.2 Motivation

Consider a wireless network, where each node is a transceiver – that is, it can both transmit and receive messages – and that communication possibility is symmetric (if a node x is able to receive transmission from y, then y is also able to receive transmission from x). It naturally defines a graph (with an edge joining every two nodes that can communicate directly); see Figure 2 for an example.

A pair of adjecent nodes can communicate using some channel only if neither of them is in range of any other transmission on the same channel (because otherwise transmissions



(a) Positions of transceivers. (b) Corresponding graph.

Figure 2: Model of communication in a wireless network.

interfere). In our example, the pairs uw and vw can not use the same channel, because transmissions from u and v would interfere at node w. Similarly, pairs vw and xy can not use the same channel, because a transmission from w to v would interfere at node x with a transmission from y.

We would like to assign a channel to each pair of adjecent nodes in such a way that all of them can communicate at the same time. Note that it is exactly the problem of finding a strong edge-coloring of the corresponding graph (where channels correspond to colors). It is important to use as few channels as possible – and that number is the strong chromatic index of the wireless network graph.

See [5] for a short study of strong edge-colorings in wireless network communication. There are many similar models that take into account the possible assymetry of connections – see [36] for a summary.

2.3 The main focus: upper bound in maximum degree

The main, extremal, question, underlying all parts of this dissertation, is: how big can strong chromatic index of a graph be? Since this parameter grows with maximum degree of the graph, a more precise formulation would be: given an integer Δ, what is the maximum possible strong chromatic index of a graph with maximum degree at most Δ? We are especially interested in an answer for large (that is: sufficiently large) values of Δ.

It is easy to give an upper bound of (roughly) $2\Delta^2$, that follows from bounding the maximum degree in the square of the line graph.

Proposition 2.3.1. *Let G be a graph of maximum degree at most Δ. We have*

$$s'(G) \leq 2\Delta^2 - 2\Delta + 1.$$

Proof. Apply a greedy coloring procedure: order edges of G arbitrarily and perform $e(G)$ steps, where at i-th step assign to e_i (an i-th edge in the order) a color that is not assigned to any edge joined to e_i.

Note that the procedure will always produce a strong edge-coloring. Since, for all i, there are $2\Delta - 2$ neighbors of vertices from e_i (excluding vertices of e_i), each of them incident with at most Δ edges (which totals to at most $2\Delta^2 - 2\Delta$ edges joined to e_i), we will always find a color for e_i, so the procedure will succeed. □

Proposition 2.3.1 can be complemented with a lower bound of (roughly) $\frac{5}{4}\Delta^2$. It is attained by a certain family of graphs – blowups of C_5.

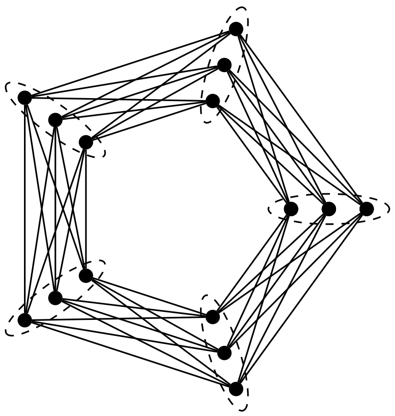
Proposition 2.3.2. *Let Δ be a natural number. There is a graph G such that*

$$s'(G) = \begin{cases} \frac{5}{4}\Delta^2, & \textit{for even } \Delta, \\ \frac{5}{4}\Delta^2 - \frac{2\Delta-1}{4}, & \textit{for odd } \Delta. \end{cases}$$

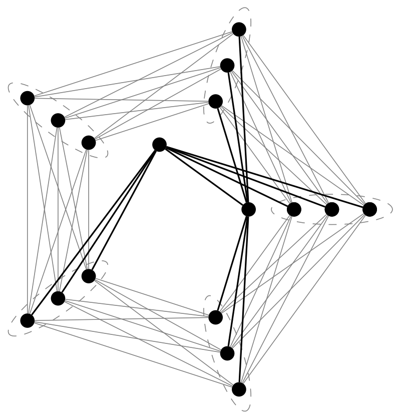
Moreover, $L(G)^2$ is a clique.

Proof. Suppose that Δ is odd. The graph G_Δ is constructed as follows: vertices of G_Δ are (ordered) pairs (i, j), where $i \in \{1, 2, 3, 4, 5\}$ and $j \in \{1, 2, \ldots, \frac{\Delta}{2}\}$. Edges of G_Δ are (unordered) pairs $\{(i, j), (i', j')\}$, where $i' = i \pm 1 \pmod 5$ and $j, j' \in \{1, 2, \ldots, \frac{\Delta}{2}\}$. See Figure 3a for an example.

Note that $L(G_\Delta)^2$ is a clique. Indeed, if we consider any two edges $e = \{(i_e, j_e), (i'_e, j'_e)\}$ and $f = \{(i_f, j_f), (i'_f, j'_f)\}$, without loss of generality we may assume that $i_f = i_e + 1$. Therefore, there in an edge $\{(i_e, j_e), (i_f, j_f)\}$, so e and f are adjacent in $L(G_\Delta)^2$.



(a) $\Delta = 6$ (even case).



(b) $\Delta = 7$ (odd case).

Figure 3: Graphs such that the square of the line graph is a clique.

For an odd Δ we obtain G_Δ from $G_{\Delta-1}$ by adding a double star: we set $V(G_\Delta) = V(G_{\Delta-1}) \cup \{v_{13}, v_{24}\}$ and define $E(G_\Delta)$ to be $E(G_{\Delta-1})$ plus all pairs $\{v_{13}, (1,j)\}$, $\{v_{13}, (3,j)\}$, $\{v_{24}, (2,j)\}$, $\{v_{24}, (4,j)\}$ and the pair $\{v_{13}, v_{24}\}$. See Figure 3b for an example. In this case, $L(G_\Delta)^2$ is also a clique.

Since the number of edges of G_Δ in both cases matches the number contained in the Proposition, the proof is finished. □

2.4 Conjecture of Erdős and Nešetřil

Clearly, at least one of the Propositions 2.3.1 and 2.3.2 is not tight, but improving either of them (even by 1) is a nontrivial task. In 1985, Erdős and Nešetřil conjectured that the upper bound can be strengthened by at least a little bit – that is, there exists an $\epsilon > 0$ such that $s'(G) \leq (2-\epsilon)\Delta^2$ for every graph G of maximum degree Δ.

It took twelve years to give an affirmative answer; it was done by Molloy and Reed in 1997 [32]. The resulting value of ϵ was 0.002, but the authors claim that it can be improved to 0.01 with a little extra effort.

Theorem 2.4.1 (Molloy and Reed, 1997 [32]). *If G is a graph with sufficiently large maximum degree Δ, then*

$$s'(G) \leq 1.998\Delta^2.$$

This result remained the best known for 18 years, until a recent work by Bruhn and Joos [9]. Their ϵ is 0.07, which is a major improvement over 0.002.

Theorem 2.4.2 (Bruhn and Joos, 2015 [9]). *If G is a graph with sufficiently large maximum degree Δ, then*

$$s'(G) \leq 1.93\Delta^2.$$

On the other hand, no improvement over Proposition 2.3.2 (construction that achieves $\frac{5}{4}\Delta^2$) is known; maybe it is not possible. A stronger variant of Erdős and Nešetřil conjecture states that $\frac{5}{4}\Delta^2$ is the correct answer. See [21] for an original appearance of the problem and [4] for a more recent discussion.

Conjecture 2.4.3 (Erdős and Nešetřil, 1985 [21]). *If G is a graph with maximum degree Δ, then*

$$s'(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{for even } \Delta, \\ \frac{5}{4}\Delta^2 - \frac{2\Delta-1}{4}, & \text{for odd } \Delta. \end{cases}$$

The strongest support of this conjecture so far comes from Chung, Gyárfás, Trotter and Tuza [11]. They proved that Proposition 2.3.2 is in some sense the best possible: if $L(G)^2$ is a clique, then the number of edges (and, in consequence, the strong chromatic index) of G is at most $\frac{5}{4}\Delta^2$.

Theorem 2.4.4 (Chung, Gyárfás, Trotter and Tuza, 1990 [11]). *If G is a graph with maximum degree Δ and $L(G)^2$ is a clique, then*

$$e(G) \leq \begin{cases} \frac{5}{4}\Delta^2, & \text{for even } \Delta, \\ \frac{5}{4}\Delta^2 - \frac{2\Delta-1}{4}, & \text{for odd } \Delta. \end{cases}$$

There are a few results concerning graphs of small maximum degree: the Conjecture is trivial for $\Delta \leq 2$, it is proved for $\Delta = 3$ (see [27] and [27]) and there is a bound of 22 colors for $\Delta = 4$ (see [12]; see also [26] for an earlier bound of 23).

2.5 Strong chromatic index of bipartite graphs

The most important class of graphs, for which the (restricted variant of) Conjecture of Erdős and Nešetřil is still unsolved is the class of bipartite graphs. Although those graphs give us a strong structural property to work with, so far we did not manage to use it to find a good upper bound on the strong chromatic index; in fact, Theorem 2.4.2 (a bound of $1.93\Delta^2$) remains the best known even if we restrict our attention to bipartite graphs.

On the other hand, graphs given in Proposition 2.3.2 (that achieve $\frac{5}{4}\Delta^2$) are clearly not bipartite; no examples with strong chromatic index greater than Δ^2 are known, and this bound is attained by $K_{\Delta,\Delta}$. Therefore, maybe the right answer for bipartite graphs is Δ^2 – it was conjectured by Faudree, Gyárfás, Schelp and Tuza in 1989 [22].

Conjecture 2.5.1 (Faudree, Gyárfás, Schelp and Tuza, 1989 [22])**.** *If G is a bipartite graph with maximum degree Δ, then $s'(G) \le \Delta^2$.*

Instead of the maximum degree of a bipartite graph, one may want to consider the degrees in partition classes separately and strengthen Conjecture 2.5.1 by suggesting a bound $\Delta_1\Delta_2$ (where Δ_i is the maximum degree in i-th partition class) instead of Δ^2. This strengthening is due to Brualdi and Quinn [8].

Conjecture 2.5.2 (Brualdi and Quinn, 1993 [8])**.** *If G is a bipartite graph such that the vertices in i-th partition class have degree at most Δ_i (for $i \in \{1,2\}$), then $s'(G) \le \Delta_1\Delta_2$.*

Conjecture 2.5.1 is true for $\Delta = 3$ (see [39]). Conjecture 2.5.2 is true for some special classes of graphs (see [8], [34] and [35]).

2.6 Easier problems

Conjecture 2.5.1 (an upper bound of Δ^2 for bipartite graphs) is supported by much stronger evidence than the Conjecture of Erdős and Nešetřil (Conjecture 2.4.3). In this section we will prove two such results; in fact, both proofs give bounds consistent with Conjecture 2.5.2 (an upper bound of $\Delta_1\Delta_2$ for bipartite graphs) for free – a formulation given here is a bit stronger than in referenced papers.

In an attempt to prove that $\chi\left(L(G)^2\right) \le \Delta^2$, for a bipartite graph G of maximum degree Δ, one may start with an easier task: showing that the maximum clique in $L(G)^2$ has size at most Δ^2. This problem was solved by Faudree, Gyárfás, Schelp and Tuza in 1990 [23].

Theorem 2.6.1 (Faudree, Gyárfás, Schelp and Tuza, 1990 [23]). *If G is a bipartite graph such that the vertices in i-th partition class have degree at most Δ_i (for $i \in \{1, 2\}$), then*

$$\omega(L(G)^2) \leq \Delta_1\Delta_2.$$

Proof. Denote the partition classes of G by V_1 and V_2 (where the maximum degree in V_i is at most Δ_i). Let F be a set of edges that form a clique in $L(G)^2$ and set $H := G[F]$. Define d_i to be the maximum degree in H of a vertex from V_i (for $i = 1, 2$). Without loss of generality we assume that $\Delta_1 d_2 \leq d_1\Delta_2$.

Let v be a vertex from V_1 of degree d_1 in H and denote the neighbors of v in H by $u_1, u_2, \dots u_{d_1}$. We partition F into the set F_{nv} of edges that are incident with a neighbor in G of v and the set F' of the remaining edges from F.

In G there are at most Δ_1 neighbors of v, each incident with at most d_2 edges from F_{nv}, so $|F_{nv}| \leq \Delta_1 d_2 \leq d_1\Delta_2$.

Let e be an edge from F' and w be a vertex from V_1 incident with e. Since G is bipartite, w must be adjecent to every u_i (for $1 \leq d_1$). Therefore, w is adjacent to at most $\Delta_1 - d_1$ vertices from F' (because its degree is at most Δ_1) and there are at most Δ_2 such vertices w (because the degree of u_i is at most Δ_2). If follows that $|F'| \leq (\Delta_1 - d_1)\Delta_2$.

Adding both estimations we get $|F| \leq \Delta_1\Delta_2$, so the proof is finished. □

A coloring witnessing that $\chi(L(G)^2) \leq \Delta^2$ is a partition of edges of G into sets independent in $L(G)^2$ of average size at least $\frac{e(G)}{\Delta^2}$. From this point of view, an easier task would be to show that there exists at least one independent set of this size. This is exactly the result of Faudree, Gyárfás, Schelp and Tuza from 1989 [22].

Theorem 2.6.2 (Faudree, Gyárfás, Schelp and Tuza, 1989 [22]). *If G is a bipartite graph such that the vertices in i-th partition class have degree at most Δ_i (for $i \in \{1, 2\}$), then*

$$\alpha(L(G)^2) \geq \frac{e(G)}{\Delta_1\Delta_2}.$$

Proof. Denote the partition classes of G by V_1 and V_2 (where the maximum degree in V_i is at most Δ_i) and without loss of generality assume that G has no isolated vertices. Let $X = \{x_1, x_2, \dots, x_p\}$ be a minimum subset of V_1 such that $N(X) = V_2$. By the minimality of X, for every x_i there exists some $y_i \in N(v_i)$ such that y_i is not adjacent to any other vertex from X. Since G is bipartite, the set of edges $\{x_1y_1, x_2y_2, \ldots, x_py_p\}$ is an induced matching in G of size p (and, equivalently, an independant set in $L(G)^2$).

Let us count the number of edges of G. Each vertex x_i has at most Δ_1 neighbors, each of them incident with at most Δ_2 edges of G, so we have $e(G) \leq p\Delta_1\Delta_2$. Since $p \leq \alpha\left(L(G)^2\right)$, the proof is complete. □

2.7 Reed's conjecture and strong chromatic index

Reed's conjecture is a fundamental question regarding chromatic number of a graph. If G is a graph of maximum degree Δ, then clearly $\chi(G) \leq \Delta + 1$, but this bound is tight only for complete graphs and odd cycles (and certain disconnected graphs; by Brooks' theorem); and all known examples of graphs with chromatic number close to Δ have cliques of size close to Δ – does it mean a deeper dependency? What is the correct upper bound for chromatic number in terms of both maximum degree and the clique number of a graph? In 1998, Reed posed the following conjecture [37].

Conjecture 2.7.1 (Reed, 1998 [37]). *Let G be a graph with maximum degree Δ and maximum clique of size ω. We have*

$$\chi(G) \leq \left\lceil \frac{\Delta + 1 + \omega}{2} \right\rceil.$$

This conjecture is relevant to our goal; if it was true, it would easily imply a bound on strong chromatic index of bipartite graphs outclassing Theorem 2.4.2 (a bound of $1.93\Delta^2$).

Proposition 2.7.2. *Reed's conjecture would imply that for every bipartite graph G of maximum degree Δ we have*

$$s'(G) \leq 1.5\Delta^2.$$

Proof. Recall that $s'(G) = \chi\left(L(G)^2\right)$. Note that the maximum degree of $L(G)^2$ is at most $2\Delta^2 - 2\Delta$ and, by Theorem 2.6.1, the clique number of $L(G)^2$ is at most Δ^2, so we get the desired implication from Conjecture 2.7.1. □

Reed's conjecture is supported by some partial results (see [37] and [33, Chapter 16]), but the strongest support is that its fractional variant holds, as proved by Molloy and Reed in 2002 [33, Theorem 21.7].

Theorem 2.7.3 (Molloy and Reed, 2002 [33, Theorem 21.7])**.** *Let G be a graph with maximum degree Δ and maximum clique of size ω. We have*

$$s'_f(G) \leq \frac{\Delta + 1 + \omega}{2}.$$

As an immediate consequence of Theorem 2.7.3 we obtain a bound on the fractional strong chromatic index of bipartite graphs.

Proposition 2.7.4. *For every bipartite graph G of maximum degree Δ we have $s'_f(G) \leq 1.5\Delta^2$.*

Proof. The maximum degree of $L(G)^2$ is at most $2\Delta^2 - 2\Delta$ and, by Theorem 2.6.1, the clique number of $L(G)^2$ is at most Δ^2. By Theorem 2.7.3, the chromatic number of $L(G)^2$ is at most $1.5\Delta^2$, which completes the proof. □

Proposition 2.7.4 is proved by using Theorems 2.7.3 and 2.6.1 as "black boxes" (note that both of them are sharp); is it possible to improve it by delving deeper into the structure of the problem? We answer this question in Chapter 5.

In this section we focused on bipartite graphs, but a similar discussion can be repeated in general case if instead of Theorem 2.6.1 we use a recent result by Śleszyńska-Nowak that for every graph G of maximum degree Δ, the clique number of $L(G)^2$ is at most $1.5\Delta^2$ [40] (note that Theorem 2.4.4, a bound of $\frac{5}{4}\Delta^2$ on the number of edges of a graph G such that $L(G)^2$ is a clique, is too weak for this purpose).

2.8 Graphs with strong chromatic index much lower than Δ^2

As we have seen in section 2.3, the upper bound on the strong chromatic index of a graph with maximum degree Δ, where Δ tends to infinity, must be of order Δ^2. However, some classes of graphs admit much smaller values of this parameter; a trivial example is a star – we have $s'(K_{1,n}) = \Delta(K_{1,n}) = n$.

In 2000, Mahdian [30] identified a nice structural property that makes the strong chromatic index much smaller than Δ^2. If a graph has no cycles of length 4 and large maximum degree Δ, then its strong chromatic index is of order at most $\frac{\Delta^2}{\ln \Delta}$.

Theorem 2.8.1 (Mahdian, 2000 [30]). *For any $\epsilon > 0$ there is Δ_0 such that for every C_4-free graph G we have*

$$s'(G) \leq (2+\epsilon)\,\frac{\Delta^2}{\ln \Delta}.$$

We will discuss the arising questions – why does lack of C_4's help to find a strong edge-coloring with few colors and what happens when when a graph contains a small number of C_4's – in Chapter 3. We will see that sparsity of the graph is the "real" reason.

Surprisingly, strong chromatic index can be much smaller than Δ^2 even in very dense graphs. In 2012, Alon, Moitra and Sudakov showed that there are almost complete graphs on n vertices of strong chromatic index as small as $n^{1+\epsilon}$ (note that the strong chromatic index of a complete graph is of order n^2).

Theorem 2.8.2 (Alon, Moitra and Sudakov, 2012 [2]). *For every $\epsilon > 0$ there exists $\delta > 0$ such that for every sufficiently large n there exists a graph G on n vertices that satisfies*

$$s'(G) \leq n^{1+\epsilon} \quad \textit{and} \quad e(G) \geq \binom{n}{2} - n^{2-\delta}.$$

The next step is to investigate graphs with strong chromatic index that is linear in maximum degree (note that is some sense it is the final step: we always have $s'(G) \geq \Delta(G)$). We discuss this case in Chapter 4.

3 Graphs with small number of C_4's

3.1 Number of C_4's and strong chromatic index

At first glance, Mahdian's Theorem (Theorem 2.8.1: a bound of $(2+\epsilon)\frac{\Delta^2}{\ln \Delta}$ for C_4-free graphs) may seem surprising. Intuitively, C_4's should help us in finding a strong edge-coloring – in the proof of Proposition 2.3.1 (a bound of $2\Delta^2$ for all graphs, that follow from greedy coloring) we neglect the fact that some of the neighboring edges may be counted twice, which would allow us to improve the bound; if every edge of a graph is in at least $\frac{3}{4}\Delta^2$ 4-cycles, then its chromatic index is at most $\frac{5}{4}\Delta^2$, so the graph satisfies the conjecture of Erdős and Nešetřil (Conjecture 2.4.3). Does Mahdian's Theorem suggest that 4-cycles make a strong edge-coloring with not many colors harder, instead of easier, to find?

We would like to find out what happens if we relax the assumption – what is the maximum possible chromatic index of a graph, of maximum degree Δ, in which every edge is in a small number of 4-cycles? The "small number" may range from a constant, through $O(\Delta)$, up to $\epsilon\Delta^2$ for some small constant ϵ.

Besides our general considerations we will investigate a specific class of graphs, unit distance graphs, that have a nice geometric representation and rich, but manageable structure. Informally, vertices of a unit distance graph are points in $\mathbb{R}^d$ and edges join all pairs of points at distance 1.

Definition 3.1.1. *A graph G is a* ***unit distance graph in*** $\mathbb{R}^d$ *iff there is an injective function $f : V(G) \to \mathbb{R}^d$ such that uv is an edge of G if and only if the Euclidean distance between $f(u)$ and $f(v)$ is* 1.

What is the maximum possible strong chromatic index of a unit distance graph in $\mathbb{R}^2$ of maximum degree Δ? Every edge of such a graph is in at most Δ 4-cycles, so this question is a special case of the above problem. How abot unit distance graphs in higher dimensions?

3.2 Strengthening Mahdian's Theorem

A (potential) surprise, discussed in the start of Section 3.1, becomes less surprising after we understand the "real" reason that makes Mahdian's Theorem true. It is related to the local sparsity of the square of the line graph; Alon, Krivelevich and Sudakov proved in 1999 that a locally sparse graphs have chromatic number much smaller then their maximum degree (see [1]).

Theorem 3.2.1 (Alon, Krivelevich and Sudakov, 1999 [1])**.** *There exists a constant c such that the following is true. Let G be a graph with maximum degree Δ such that for every vertex $v \in V(G)$ the subgraph of G induced by $N(v)$ has at most $\frac{\Delta^2}{f}$ edges, where $1 < f \leq \Delta$. Then the chromatic number of G is at most $c\frac{\Delta}{\ln f}$.*

As an easy corollary of Theorem 3.2.1 we can obtain a strengthening of Theorem 2.8.1 (but with a worse constant); note that a graph is C_4-free if every two vertices have at most one common neighbor, so the following Corollary is a strengthening indeed.

Corollary 3.2.2 (Vu, 2002 [41])**.** *There exists a constant K such that the following holds. Let G be a graph of maximum degree Δ such that every two vertices of G have at most $\frac{\Delta}{g}$ common neighbors, where $1 < g \leq \Delta$. Then, we have*

$$s'(G) \leq K\frac{\Delta^2}{\ln g}.$$

Proof. Let uv be an edge of G. By the assumption on G, every vertex of G, except u and v, have at most $\frac{2\Delta}{g}$ neigbours in $N(uv)$. It follows that every edge incident to a vertex from $N(uv)$ can be joined to at most $\frac{4\Delta^2}{g} + 2\Delta$ other such edges. Therefore, every neighborhood in $L(G)^2$ spans at most $\frac{4\Delta^4}{g} + 2\Delta^3$ edges, so the result follows from Theorem 3.2.1. □

This corollary immediately gives us an upper bound on the strong chromatic index of unit distance graphs in the plane.

Corollary 3.2.3. *There is a constant K such that if G is a unit distance graph in $\mathbb{R}^2$ of maximum degree Δ, then*

$$s'(G) \leq K\frac{\Delta^2}{\ln \Delta}.$$

Proof. Note that every two vertices of G have at most 2 common neighbors, because neighbors of any vertex of G lie on a circle and any two circles intersect in at most 2 points. Therefore, if we set $g = \frac{\Delta}{2}$, the result follows by Corollary 3.2.2. $\square$

We would like to show a similar bound for higher dimensions. However, corollary 3.2.2 is too weak for this purpose – the assumption may not be satisfied even in $\mathbb{R}^3$, and the counterexample is $K_{2,\Delta}$.

Proposition 3.2.4. *For every Δ, $K_{2,\Delta}$ is a unit distance graph in $\mathbb{R}^3$.*

Proof. Pick any two points x, y at distance less than 2. Note that there are infinitely many points at distance 1 from both x and y (and they lie on the circle that is the intersection of the two spheres of radius 1); pick Δ such points $v_1, v_2, \dots, v_\Delta$. The resulting unit distance graph is a bipartite graph with partition classes $\{x, y\}$ and $\{v_1, v_2, \dots, v_\Delta\}$. $\square$

Our main contribution in this chapter is a theorem that significantly improves Corollary 3.2.2 and implies a bound on the strong chromatic index of unit distance graphs in $\mathbb{R}^3$. The Theorem is proved in section 3.3.

Theorem 3.2.5 (MD, 2015+)**.** *There exists a constant K such that the following holds. Let G be a graph of maximum degree Δ such that every edge of G is in at most $\frac{\Delta^2}{g}$ cycles of length 4, where $1 < g \leq \Delta^2$. We have*

$$s'(G) \leq K \frac{\Delta^2}{\ln g}.$$

In order to apply our result, we need an estimation on the number of edges of a unit distance graph in $\mathbb{R}^3$. We use a Theorem of Erdős from 1960 [20]; this result has been improved over last 55 years (see [6, Chapter 5.2] for the summary), but the improvements are not relevant for our purpose.

Theorem 3.2.6 (Erdős, 1960 [20])**.** *There is a constant K' such that if G is a unit dstance graph in $\mathbb{R}^3$ with n vertices, then*

$$e(G) \leq K' n^{\frac{5}{3}}.$$

Now, we are ready to bound the strong chromatic index of unit distance graphs in $\mathbb{R}^3$.

Corollary 3.2.7. *There is a constant K such that if G is a unit distance graph in $\mathbb{R}^3$ of maximum degree Δ, then*

$$s'(G) \le K\frac{\Delta^2}{\ln \Delta}.$$

Proof. Let $e = uv$ be an edge of G. Note that every 4-cycle in G that contains e corresponds to an edge between a vertex from $N(u) \setminus \{v\}$ and a vertex from $N(v) \setminus \{u\}$ (and vice versa). By Theorem 3.2.6, the subgraph of G induced by $N(uv)$ has at most $K'\Delta^{\frac{5}{3}}$ edges, so there are at most $K'\Delta^{\frac{5}{3}}$ 4-vertex cycles containing e. Therefore, the result follows from Theorem 3.2.5 by setting $g = \frac{\Delta^{\frac{2}{5}}}{K'}$. □

3.3 Proof of Theorem 3.2.5

We start with the technical lemma that will be used to bound the number of edges of certain bipartite graphs.

Lemma 3.3.1. *Let G be a bipartite graph such that each partition class of G has at most n vertices. Then there is an edge $e \in E(G)$ that is contained in at least*

$$\frac{e(G)^3}{n^4} + \frac{3n^3}{e(G)}$$

cycles of length 4.

Proof. Let A and B be partition classes of G. Let n_{P_3} be the number of 3-vertex paths in G with middle vertex in A. Since each vertex v from A is in $\frac{1}{2}(\deg(v) - 1)\deg(v)$ such paths, we have

$$n_{P_3} = \frac{1}{2}\sum_{v\in A}\deg(v)^2 - \frac{1}{2}\sum_{v\in A}\deg(v).$$

Note that the second sum is equal to the number of edges of G. By applying the Cauchy-Schwarz inequality to the first sum and using the equality $\sum_{v\in A}\deg(v) = 2e(G)$ we obtain

$$n_{P_3} \ge \frac{e(G)^2}{2n} - \frac{e(G)}{2} \ge \frac{e(G)^2}{2n} - \frac{n^2}{2}.$$

Now, let n_{C_4} be the number of 4-cycles in G. Let $d(u, v)$ denote the number of common neighbors of vertices u and v. For every two vertices $\{u, v\} \subseteq B$, the number of 4-vertex cycles in G containing both u and v is $\frac{1}{2}d(u,v)\,(d(u,v)-1)$, so we have

$$n_{C_4} = \frac{1}{2}\sum_{\{u,v\}\subseteq B} d(u,v)^2 - \frac{1}{2}\sum_{\{u,v\}\subseteq B} d(u,v).$$

The sum $\sum_{\{u,v\}\subseteq B} d(u,v)$ is equal to n_{P_3} (and we have $n_{P_3} \leq n^3$), so by applying the Cauchy-Schwarz inequality we get

$$n_{C_4} \geq \frac{1}{2\binom{n}{2}}\left(\frac{e(G)^2}{2n} - \frac{n^2}{2}\right)^2 - \frac{n^3}{2} = \frac{e(G)^4 - n^3 e(G)^2 + n^6 - 2n^6(n-1)}{4n^3(n-1)}.$$

Since $\frac{1}{n-1} > \frac{1}{n}$, $e(G) \leq n^2$ and $1 \leq n$, we have

$$n_{C_4} \geq \frac{e(G)^4}{4n^4} - \frac{3}{4}n^3.$$

Every 4-vertex cycle has 4 edges, so by the pigeonhole principle we get the desired result. □

Proof of Theorem 3.2.5. We start with two rather technical Claims that will allow us to use Theorem 3.2.1. Claim 1 will be used in the proof of more important Claim 2.

Claim 1. *In G, for every $e \in E(G)$ there are at most*

$$\sqrt[3]{4}\frac{\Delta^2}{\sqrt[3]{g}} + 4\Delta^{\frac{7}{4}}$$

edges with both endpoints in $N_G(e)$.

Proof of Claim 1. We use an auxiliary bipartite graph H with partition classes $N_G(e)$ and $\overline{N_G(e)}$ (two copies of $N_G(e)$), such that for every edge $xy \in E(G)$, where $x, y \in N_G(e)$, we have $x\bar{y}, \bar{x}y \in E(H)$.

Let e_N be the number of edges of G with both endpoints in $N_G(e)$. Note that H has $2e_N$ edges and at most 2Δ vertices in each partition class. By Lemma 3.3.1 we get that some edge $x\bar{y}$ in H is in at least

$$\frac{(2e_N)^3}{(2\Delta)^4} - \frac{3(2\Delta)^3}{2e_N}$$

4-vertex cycles in H. Note that the edge xy is in the same number of 4-vertex cycles in G, so by our assumption we get

$$\frac{e_N^3}{2\Delta^4} - \frac{12\Delta^3}{e_N} \leq \frac{\Delta^2}{g}.$$

Note that if $e_N \geq 4\Delta^{\frac{7}{4}}$, then we have $\frac{12\Delta^3}{e_N} \leq \frac{e_N^3}{4\Delta^4}$, so we can rewrite the inequality as

$$\frac{e_N^3}{2\Delta^4} - \frac{e_N^3}{4\Delta^4} \leq \frac{\Delta^2}{g},$$

$$e_N^3 \leq 4\frac{\Delta^6}{g},$$

$$e_N \leq \sqrt[3]{4}\frac{\Delta^2}{\sqrt[3]{g}}.$$

In the remaining case by definition we have $e_N \leq 4\Delta^{\frac{7}{4}}$ and the Claim follows by summing both upper bounds on e_N. □

Claim 2. *For every $e \in E(G)$ there are at most*

$$7\sqrt[3]{4}\frac{\Delta^4}{\sqrt[3]{g}} + 44\Delta^{\frac{15}{4}}$$

edges in $L(G)^2$ with both endpoints in $N_{L(G)^2}(e)$.

Proof of Claim 2. Let G' be a subgraph of G induced by neighbors of e in $L(G)^2$. The edges in $L(G)^2$ with both endpoints in $N_{L(G)^2}(e)$ correspond to certain pairs of edges of G' (pairs $\{p, q\}$, $p, q \in E(G')$, such that p and q have at least one endpoint in $N_G(e)$ and pq is an edge of $L(G)^2$); there are at most $n_1 + n_2 + n_3 + n_4 + n_5$ of them, where we define $n_1, \ldots, n_5$ such that there are:

- n_1 pairs of intersecting edges of G',
- n_2 pairs of edges of G' such that at least one of them intersects e,
- n_3 pairs of edges of G' joined by an edge of G with both endpoints in $N_G(e)$,
- n_4 pairs of edges of G' such that at least one of them has both endpoints in $N_G(e)$,
- n_5 pairs of edges of G' such that both of them have at least one endpoint outside $N_G(e)$ and they are joined by an edge with at least one endpoint outside $N_G(e)$.

There are at most $2\Delta^2$ edges of G', each intersecting at most 2Δ other edges, so we get $n_1 \leq 4\Delta^3$. Similarly, there are at most 2Δ edges intersecting e and each of them is joined to at most $2\Delta^2$ other edges of G', so we have $n_2 \leq 4\Delta^3$.

By Claim 1, there are at most

$$\sqrt[3]{4}\frac{\Delta^2}{\sqrt[3]{g}} + 4\Delta^{\frac{7}{4}}$$

edges of G with both endpoints in $N_G(e)$, and each of them joins at most Δ^2 edges of G' and is joined to at most $2\Delta^2$ edges of G', so we have

$$n_3 + n_4 \leq 3\sqrt[3]{4}\frac{\Delta^4}{\sqrt[3]{g}} + 12\Delta^{\frac{15}{4}}.$$

Now, for an edge $p \in E(G')$ let $X_p = N_G(p) \setminus N_G(e)$. Let $H_p \subseteq G'$ be an auxiliary bipartite graph such that partition classes of H_p are X_p and $N_G(e)$, and edges of H_p are all edges of G' with one endpoint in each partition class. Let e_p be the number of edges of H_p.

Note that for every pair $\{p, q\}$ that contributes to n_5 we have $p \in E(H_q)$ or $q \in E(H_p)$. If follows that n_5 is at most $2\Delta^2$ times the maximum value of e_p over all edges p; now we will bound this maximum.

Since both X_p and $N_G(e)$ have size at most 2Δ, by Lemma 3.3.1 some edge of H_p is in at least

$$\frac{e_p^3}{16\Delta^4} - \frac{24\Delta^3}{e_p}$$

4-vertex cycles. Therefore, by our assumption we get

$$\frac{e_p^3}{16\Delta^4} - \frac{24\Delta^3}{e_p} \leq \frac{\Delta^2}{g}.$$

Now (as in the proof of Claim 1), we either have $e_p \leq 12\Delta^{\frac{7}{4}}$ or $e_p \geq 12\Delta^{\frac{7}{4}}$. In the second case we have $\frac{24\Delta^3}{e_p} \leq \frac{e_p^3}{32\Delta^4}$, so the above inequality gives

$$\frac{e_p^3}{16\Delta^4} - \frac{e_p^3}{32\Delta^4} \leq \frac{\Delta^2}{g},$$

Therefore, after solving for e_p and taking into account the first case, we obtain

$$e_p \leq 2\sqrt[3]{4}\frac{\Delta^2}{\sqrt[3]{g}} + 12\Delta^{\frac{7}{4}},$$

and n_5 is at most $2\Delta^2$ times this upper bound. Note that $\Delta^3 \leq \Delta^{\frac{15}{4}}$, so the claim follows by summing all the upper bounds on $n_1, \ldots, n_5$. $\square$

First, consider the case $g \leq \Delta^{\frac{3}{4}}$ and define $f = \frac{\sqrt[3]{g}}{56}$. Note that $\frac{\Delta^4}{\sqrt[3]{g}} \geq \Delta^{\frac{15}{4}}$, so by Claim 2 we get that a subgraph of $L(G)^2$ induced by $N_{L(G)^2}(e)$ for any $e \in V(L(G)^2)$ has at most $\frac{4\Delta^4}{f}$ edges, so by Theorem 3.2.1 we get

$$\chi\left(L(G)^2\right) \leq K' \frac{2\Delta^2}{\ln f}$$

(where K' is a constant from Theorem 3.2.1). Since $\ln g$ is equal to some constant times $\ln f$, we get that

$$s'(G) \leq K_1 \frac{\Delta^2}{\ln g}.$$

In the remaining case, we take $f = \frac{\sqrt[4]{\Delta}}{56}$ and by the same argument (with an exception that from Claim 2 we deduce the bound $56\Delta^{\frac{15}{4}}$) we get

$$s'(G) \leq K_2 \frac{\Delta^2}{\ln g}.$$

Finally, by taking K as maximum of K_1 and K_2, the Theorem follows.

□

3.4 Further investigations

After we establish Corollary 3.2.7 (a bound of $K\frac{\Delta^2}{\ln \Delta}$ for unit distance graph in $\mathbb{R}^3$), a natural question arises: can we show a similar bound on strong chromatic index of unit distance graphs in higher dimensions? This problem is much easier, and the answer is negative; every complete bipartite graph is a unit distance graph in $\mathbb{R}^4$.

Proposition 3.4.1 (Lenz, see [6, Chapter 5.2]). *For every Δ, $K_{\Delta,\Delta}$ is a unit distance graph in $\mathbb{R}^4$.*

Proof. Consider two orthogonal circles C_1, C_2 of radius $\frac{\sqrt{2}}{2}$, centered at $(0, 0, 0, 0)$ (that is, C_1 satisfies $x_1^2 + x_2^2 = \frac{1}{2}$ and $x_3 = x_4 = 0$; and C_2 satisfies $x_3^2 + x_4^2 = \frac{1}{2}$ and $x_1 = x_2 = 0$). Note that each point of C_1 is at distance 1 from each point of C_2, so by picking Δ points from each circle we obtain $K_{\Delta,\Delta}$. □

Another natural question is the one regarding optimality of the results. Theorem 3.2.5 (a bound of $K\frac{\Delta^2}{\ln g}$ for graphs with every edge in at most $\frac{\Delta^2}{g}$ 4-cycles) is tight up to a factor

$O(\frac{\ln \Delta}{\ln g})$, because there are C_4-free graphs with chromatic index at least $\Theta\left(\frac{\Delta^2}{\ln \Delta}\right)$ (see [30]). Note that this factor is constant when g is at least some power of Δ; we believe that it should be constant also for lower values of g.

On the other hand, mentioned construction of C_4-free graphs with high strong chromatic index is probabilistic and (probably) does not give unit distance graphs in $\mathbb{R}^2$ or $\mathbb{R}^3$; we do not know any nontrivial estimation on tightness of Corollaries 3.2.3 and 3.2.7 (the bounds of $K\frac{\Delta^2}{\ln \Delta}$ for unit distance graph in $\mathbb{R}^2$ and $\mathbb{R}^3$). Can those bounds be lowered to at least $\Delta^{2-\epsilon}$ for some constant $\epsilon > 0$? We suspect that it is the case, but such an improvement would require a totally different approach, not using Theorem of Alon, Krivelevich and Sudakov (Theorem 3.2.1).

Finally, Theorem 3.2.5 has a nice consequence regarding the conjecture of Erdős and Nešetřil (Conjecture 2.4.3): the conjecture is true if we assume that every edge of the graph is in at most $\epsilon\Delta^2$ 4-cycles (for some absolute constant $\epsilon > 0$). As we mentioned in Section 3.1, it is also true if every edge of the graph is in at least $\frac{3}{4}\Delta^2$ 4-cycles. It would be interesting to prove the conjecture in the mixture of those cases – that is, when some edges of the graph are in at most $\epsilon\Delta^2$ 4-cycles, and the remaining ones are in at least $\frac{3}{4}\Delta^2$ 4-cycles.

4 Linear bounds on strong chromatic index

4.1 Short history of a conjecture of Chang and Narayanan

In Section 2.6 we omitted one result that supports Conjecture 2.5.1 (an upper bound of Δ^2 for bipartite graphs). Recall that a graph is bipartite if and only if the length of every cycle is even; Faudree, Gyárfás, Schelp and Tuza proved in 1990 that if we require the length of every cycle to be divisible by 4 instead, then the conjecture is true.

Theorem 4.1.1 (Faudree, Gyárfás, Schelp and Tuza, 1990 [23]). *If G is a graph of maximum degree Δ and the length of every cycle in G is divisible by 4, then $s'(G) \leq \Delta^2$.*

However, the authors suspected that the result is not tight – that strong chromatic index of such graphs should be much smaller, probably even linear in maximum degree – because of a special structure of graphs in question. If all cycles in a graph have length divisible by 4, then it must be very sparse; one may prove that such graphs are 2-degenerate.

The suspicion was confirmed in 2012 by Chang and Narayanan [10]: they showed that the strong chromatic index of every 2-degenerate graph of maximum degree Δ is at most $10\Delta - 10$. The authors conjectured that the result can be generalized to k-degenerate graphs for all k.

Conjecture 4.1.2 (Chang and Narayanan, 2012 [10]). *There exists a constant c such that for any k-degenerate graph G of maximum degree Δ we have $s'(G) \leq ck\Delta$.*

This statement was proved by the author of this dissertation, Grytczuk and Śleszyńska-Nowak (see [17]). However, the conjecture was in fact confirmed in 2006, even before it was stated, by Barrett, Kumar, Marathe, Thite, Istrate and Thulasidasan [5]. Their proof contains a small mistake which infuences the resulting constant (they claimed that the main term is $(4k-3)\Delta$, while it should be $(4k-1)\Delta$); here, we state the corrected version of their theorem and give a valid proof.

Theorem 4.1.3 (Barrett et al., 2006 [5], corrected)**.** *Let G be a k-degenerate graph of maximum degree Δ, where $k \leq \Delta$. We have*

$$s'(G) \leq (4k-1)\Delta - 2k^2 - k + 1.$$

Proof. We start by enumerating vertives of G as $v_1, v_2 \ldots, v_n$ in such a way that v_i have at most k neigbours in $\{v_1, v_2, \ldots, v_{i-1}\}$, for every i (such ordering can be found iteratively: at each step we pick a vertex of degree at most k, add it before the first element of already built list and remove it from the graph).

We will color edges of G greedily, starting with edges incident to v_1, then proceeding with edges incident to v_2, and so on. We need to show that each edge, at the time when it is assigned a color, is joined to at most $(4k-1)\Delta - 2k^2 - k$ other edges; it guarantees that $(4k-1)\Delta - 2k^2 - k + 1$ colors will always suffice to complete the coloring.

Suppose that we are coloring an edge v_iv_j, where $i < j$. Colored edges joined to v_iv_j are either (A) incident to a neigbour of v_i other than v_j or (B) incident to a neighbor of v_j other that v_i.

(A) Note that at most k neigbours of v_i precede it in our ordering, and clearly each of them is incident to at most Δ colored edges. Every other neigbour of v_i is incident to at most k colored edges (because by our ordering a vertex $v_{i'}$, for $i' > i$, have at most k neigbours among $\{v_1, v_2, \ldots, v_i\}$). Since thare are at most $\Delta - 1$ neigbours of v_i that are not v_j, it totals to at most $k\Delta + (\Delta - 1 - k)k$ colored edges.

(B) There are at most $k-1$ neigbours of v_j that precede v_i in the ordering, each incident to at most Δ colored edges. Remeining neigbours of v_j, except v_i, are incident to at most k coloring edges (by the same argument as above). It totals to at most $(k-1)\Delta + (\Delta - 1 - (k-1))\, k$.

Summing the estimations (A) and (B) we get $(4k-1)\Delta - 2k^2 - k$, so the proof is complete. $\square$

4.2 Chordless graphs

We say that a graph G is **chordless** if every cycle in G is induced (that is, there is no edge joining two nonconsecutive vertices on a cycle – such an edge is called a chord).

This notion generalizes both graphs with every cycle length divisible by 4 and minimally 2-connected graphs, and forms a (proper) subclass of 2-degenerate graphs.

Chang and Narayanan [10] proved that $s'(G) \leq 8\Delta - 6$ if G is a chordless graph of maximum degree Δ. Although the result is never stronger than Theorem 4.1.3 (by Theorem 4.1.3, when $k = 2$, we get $7\Delta - 9$), their approach is different: it involves recoloring some previously colored edges with additional colors and relies on a useful structural lemma.

Lemma 4.2.1 (Chang and Narayanan, 2012 [10])**.** *Every chordless graph G contains some vertex v such that at least $\deg(v) - 1$ of its neighbors have degree at most 2.*

Our main contribution in this chapter is an improved bound on strong chromatic index of chordless graphs. The proof uses Lemma 4.2.1 and contains all key ideas used in the (mentioned in the previous section) proof of the conjecture of Chang and Narayanan by MD, Grytczuk and Śleszyńska-Nowak.

Theorem 4.2.2 (MD, Grytczuk, Śleszyńska-Nowak, 2015 [17])**.** *If G is a chordless graph of maximum degree Δ, then*

$$s'(G) \leq 4\Delta - 3.$$

Given a graph G, we say that a vertex $v \in V(G)$ is **nice** (in G) if it has at most one neigbour of degree greater than 2 and at least one neighbor of degree at most 2.

Lemma 4.2.3. *Let G be a chordless graph with at least one edge and take X to be the set of vertices of degree 1 in G. Then either G or $G - X$ contains a nice vertex.*

Proof. Without loss of generality we assume that G has no isolated vertices. If G has no vertices of degree 1, the result follows by Lemma 4.2.1. Otherwise, we have three cases: (a) $G - X$ has no vertices, (b) $G - X$ has a vertex of degree at most 1 and (c) $G - X$ has no vertices of degree 0 or 1. Now, in (a) every vertex of G satisfies the desired property and in (b) we pick a vertex of degree at most one in $G - X$. For (c) note that a nice vertex in $G - X$ is also nice in G, so the result again follows by Lemma 4.2.1. □

Proof of Theorem 4.2.2. We will in fact show that $L(G)^2$ is $(4\Delta - 4)$-degenerate by ordering edges of G in such a way that each edge is joined to at most $4\Delta - 4$ edges that preceed it in the order. We will construct an (ordered) list of egdes of G starting from the end – that is, whenever we add an edge to the list, it is added before the first element of our list – and ensure that each added edge is joined (in G) to at most $4\Delta - 4$ edges that are not on the list yet. We will proceed in steps and define L_i to be the list obtained after i-th step, starting with L_0 being an empty list.

Suppose that L_i, with $i \geq 0$, is defined, and let $I_i \subset E(G)$ be the set of edges that appear in L_i (where I stands for "Inactive edges"). Let $H_i = (V(G), E(G) \setminus I_i)$ (that is, H_i is a subgraph of G induced by active edges), and let X_i be the set of vertices of degree 1 in H_i. Take v_i to be a vertex that is nice in H_i or in $H_i \setminus X_i$; note that, by Lemma 4.2.3, such a vertex must exist (the vincinity of v_i is depicted in Figure 4).

(A) If v_i is nice in H_i, we put $A_i = \emptyset$ (and A_i stands for "edges added to the list in part (A) of the i-the step"). Otherwise, we set A_i to be the set of all edges incident to a vertex of degree 1 in H_i and a neigbour (in $H_i \setminus X_i$) of v_i that has degree at most 2 in $H_i \setminus X_i$. Now, we set L_i' to be the list obtained by adding to L_i all edges from A_i in an arbitrary order. Let $H_i' = H_i \setminus A_i$. Note that v_i is nice in H_i'.

(B) Let B_i be the set of edges of H_i' incident with v_i and a vertex of degree at most 2, that is,

$$B_i = \{v_i y \in E(H_i') : \deg_{H_i'}(y) \leq 2\}$$

(where B_i stands for "edges added to the list in part (B) of the i-the step"). Take L_{i+1} to be the list obtained from L_i' by appending edges from B_i in any order. Clearly, B_i is nonempty, so for some s the list L_s contains all edges of G.

We will show that the following invariant holds: the number of active edges incident to each vertex of an inactive edge is at most 1. More formally, we shall prove the following claim.

Claim 3. *For every $i = 0, 1, \ldots, s$, and every vertex $v \in V(G)$, if v is incident to at least one edge from I_i, then it is incident to at most 1 edge outside I_i.*

Proof. We use induction on i. For $i = 0$ we have $I_i = \emptyset$, so there is nothing to prove.

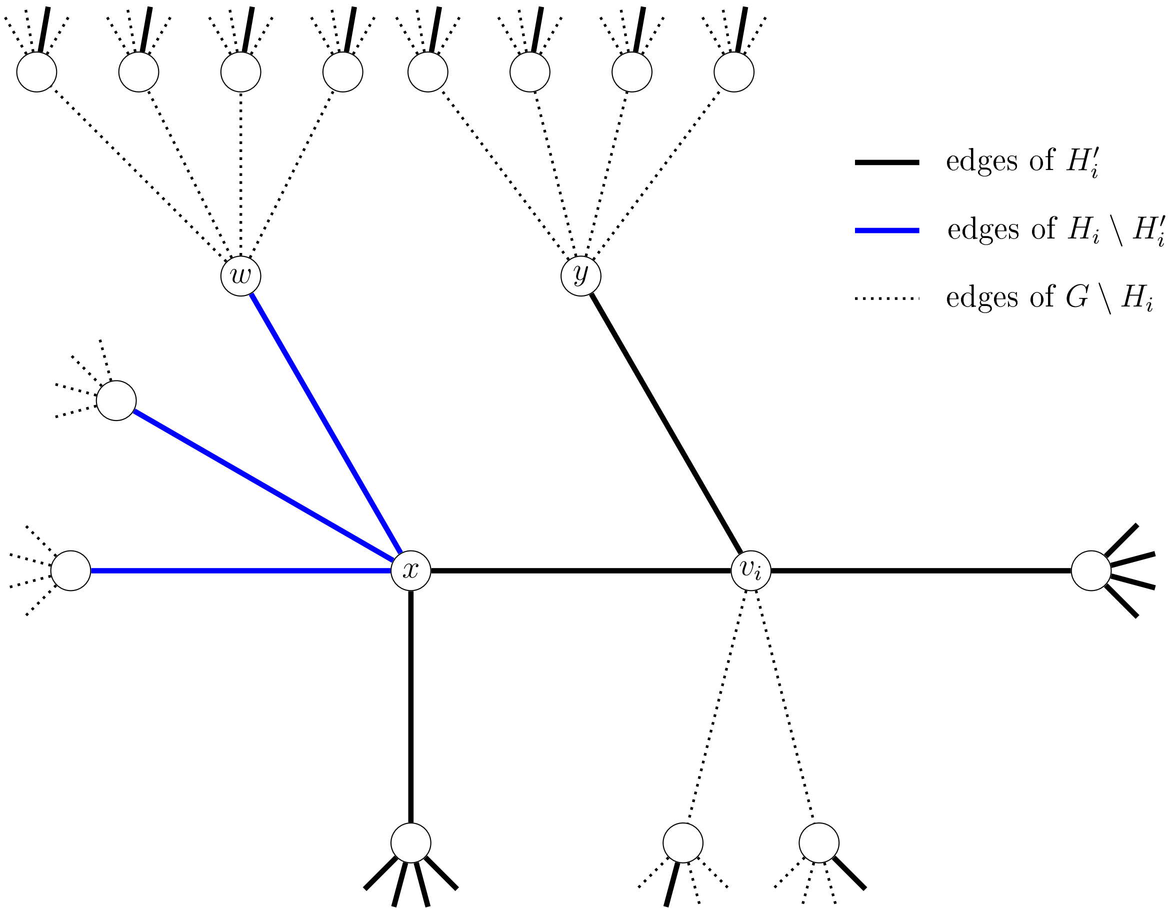


Figure 4: Proof of Theorem 4.2.2, vicinity of the vertex v_i that is nice in H'_i (all edges incident with depicted vertices are shown).

Now, suppose that the claim holds for some i (where $0 \leq i < s$), and consider some vertex v incident with at least one edge from I_{i+1}. Note that if v is incident to an edge from I_i, then the statement follows from induction hypothesis, as $I_i \subset I_{i+1}$. In the remaining case, v is either (i) a vertex of degree 1 in H_i, (ii) a neigbour of v_i in H_i of degree at most 2 in H'_i or (iii) v_i. In case (i), all edges incident to v are in I_{i+1}, in case (ii) we have $vv_i \in B_i$, so there remains at most one edge outside I_{i+1} incident to v, and in case (iii), by v being nice in H'_i, at most one edge incident to v in H_i is outside B_i, so the claim follows. □

Now, for any edge $e \in E(G)$, we will count the number of edges that are joined to e and preceed e in L_s; see Figure 4 for an illustration. We either have (i) $e \in A_i$ or (ii) $e \in B_i$, for some i.

In case (i) we need to count the number of edges of H_i joined (in G) to e. Let $e = wx$, where w is a vertex of degree 1 in H_i and w is a vertex of degree at most 2 in H'_i. Note that x have at most $\Delta - 1$ neigbours in G other than w and, by Claim 3, each of them is incident to at most one edge of H_i. By definition of A_i, w have at most 2 neigbours in H_i that are incident to more than one edge of H_i and, by Claim 3, all other neigbours of w in G (other than x) are incident to at most one edge from H_i, which totals to at most $3\Delta - 3$ edges. Therefore, e is joined to at most $4\Delta - 4$ edges of H_i.

In case (ii) we need to count the number of edges of H'_i joined (in G) to e. Let $e = v_i y$, where y have degree at most 2 in H'_i. Either y has degree 2 in H'_i and (by Claim 3) is not incident with any edge from I_i or it has degree 1 in H'_i and (again by Claim 3) every neigbour of y in G, other than v_i, have degree at most 1 in H_i. Therefore, neigbours of y in G, other than v_i, are incident to a total of at most Δ edges of H'_i. By Claim 3 and choice of v_i, there is at most one neigbour of v_i in G incident to more than two edges of H'_i, so neighbors of v_i in G other than y are incident to at most $3\Delta - 4$ edges of H'_i. Therefore, e is joined to at most $4\Delta - 4$ edges of H'_i.

It follows that $4\Delta - 3$ colors will suffice to find a strong edge coloring of G with a greedy coloring in order given by L_s, so the proof is complete. $\square$

4.3 Discussion on linear bounds on strong chromatic index

Sections 4.1 and 4.2 suggest the following question: are k-degenerate graphs (for constant k) the only graphs that have the strong chromatic index at most linear in maximum degree? The answer is negative: there exist graphs with arbitrarily large maximum (and even: average) degree Δ that have strong chromatic index at most 2Δ.

Proposition 4.3.1 (see [17]). *If $n = 2^k$, for an integer $k \geq 1$, there exists a $(\log_2 n)$-regular graph with n vertices such that $s'(G) = 2\log_2 n$.*

Proof. For every $n = 2^k$ we construct a regular graph G_k of degree k and $\chi'_s(G) \leq 2k$. The set of vertices of G_k is the set of all binary sequences of length k, with two vertices being adjacent when their Hamming distance is exactly $k-1$ (that is they agree in exactly one position). Now we will color the edges of G_k using the set of pairs $C = \{(i, j) : i \in$

$\{1, 2, \ldots, k\}, j \in \{0, 1\}\}$ as the set of colors, in the following way: the color of an edge uv is the pair (i, j) if they agree on the ith coordinate whose value is j. Clearly, the set of edges in a fixed color (i, j) forms a strong matching in G_k. □

Brešar and Kraner Šumenjak in [7] conjectured that an upper bound 2Δ is also valid for median graphs (G is a median graph if for every three vertices $u, v, w \in V(G)$ there exists exactly one vertex that lies on a shortest path from u to v, a shortest path from v to w and a shortest path from u to w; such graphs on n vertices can have average degree of order $\log n$, hypercube is an example).

Can we find denser graphs with small strong chromatic index? When asking such a question, it is reasonable to define "small" using the average degree instead of maximum degree of the graph (note that if G is a disjoint union of a dense graph and s star $K_{1,r}$, for large enough r, we have $s'(G) = \Delta(G)$, and such construction is not very insightful).

For a fixed constant $c > 0$, let $\mathcal{F}_c$ denote the family of graphs G satisfying $\chi'_s(G) \leq cd(G)$. Let $f_c(n) = \max\{d(G) : G \in \mathcal{F}_c, |V(G)| \leq n\}$. We shall demonstrate that $c = 2$ is the smallest constant for which the class $\mathcal{F}_c$ is not empty.

Proposition 4.3.2 (see [17]). *For every graph G we have*

$$s'(G) \geq 2d(G) - 1.$$

Proof. Let G be a simple graph on the set of n vertices $V(G) = \{v_1, v_2, \ldots, v_n\}$. For any edge $e = uv$, let $s(e) = d(u) + d(v)$ denote the sum of degrees of its ends u and v. Let $M = \max\{s(e) : e \in E(G)\}$. Since $s'(G) \geq M - 1$, it suffices to show that $M \geq 2d(G)$. First notice that

$$\sum_{e \in E(G)} s(e) = \sum_{v \in V(G)} d(v)^2.$$

Next consider two n-dimensional vectors $x = (d(v_1), d(v_2), \ldots, d(v_n))$ and $y = (1, 1, \ldots, 1)$. Applying the Cauchy-Schwarz inequality to x and y gives

$$\sum_{v \in V(G)} d(v) \leq \sqrt{n} \cdot \sqrt{\sum_{v \in V(G)} d(v)^2}.$$

Hence, we get

$$2\,|E(G)| \leq \sqrt{n} \cdot \sqrt{\sum_{e \in E(G)} s(e)} \leq \sqrt{n} \cdot \sqrt{M\,|E(G)|}.$$

By squaring we get $4\left|E(G)\right|^2 \leq nM\left|E(G)\right|$, which implies that $4\left|E(G)\right|/n \leq M$. This proves the desired relation $2d(G) \leq M$. $\square$

By Proposition 4.3.1, $f_2(n)$ is unbounded, but what is its order of magnitue? How about $f_c(n)$ for larger values of c? Finding the right answer reamins an open problem.

Problem 4.3.3 (see [17]). *What is the order of magnitude of $f_c(n)$?*

5 Fractional strong chromatic index of bipartite graphs

5.1 The main Theorem

Is it possible to strengthen Proposition 2.7.4 (an upper bound $1.5\Delta^2$ on fractional strong chromatic index of bipartite graphs)? Although the bound $1.5\Delta^2$ is far from the one conjectured by Faudree, Gyárfás, Schelp and Tuza (Δ^2; Conjecture 2.5.1), it follows from two theorems that are tight. Is it a premise that the mentioned conjecture is wrong and that 1.5 is the right constant?

A careful examination of the proof of Proposition 2.7.4 reveals that Theorem 2.7.3 (which confirms fractional relaxation of Reed's conjecture) is applied to squares of line graphs of bipartite graphs – although Theorem 2.7.3 is tight, it is not clear whether it is tight for this particular class of graphs. Another weakness of the proof of Proposition 2.7.4 is that Theorem 2.6.1 (which says that maximum clique in a square of the line graph of a bipartite graph G of maximum degree Δ is at most Δ^2), although tight, is not the best possible – if a clique in $L(G)^2$ has maximum possible size Δ^2, then it must consist of vertices of degree $\Delta^2 - 1$ (that is, roughly half of the maximum possible); it prompts that a better tradeoff between clique size and vertex degrees in $L(G)^2$ can be found and used.

We show that the constant 1.5 can be improved to 1.476. In the proof we replace Theorem 2.6.1 with Lemma 5.2.2, which concerns both the cliqes and degrees in $L(G)^2$, and an immediate consequence of Theorem 2.7.3 which is "compatible" with it. The proof is in Sections 5.2 and 5.3.

Theorem 5.1.1 (MD, 2015+ [14]). *Let G be a bipartite graph of maximum degree Δ. We have*

$$s'_f(G) \leq \frac{31}{21}\Delta^2 + \Delta^{1.5}.$$

We consider Theorem 5.1.1 the main result of this dissertation. Note that the constant is improved by only a little (less than 0.24) and the most important message is that it can be improved at all.

It is worth to note that if Reed's conjecture (Conjecture 2.7.1) is true, we will get a strengthening of Theorem 5.1.1, concerning strong chromatic index instead of fractional strong chromatic index, for free.

Remark 5.1.2. *Reed's conjecture would imply that for every bipartite graph G of maximum degree Δ we have*

$$s'(G) \leq \frac{31}{21}\Delta^2 + \Delta^{1.5}.$$

5.2 Outline of the proof

Our general goal is to apply Theorem 2.7.3 (inequality $\chi_f(H) \leq \frac{\Delta(H)+\omega(H)+1}{2}$) to obtain a fractional coloring of the graph $H = L(G)^2$ with strictly less than $\frac{\Delta(H)+\omega(H)+1}{2}$ colors. In order to succeed, we need our graph H to satisfy a certain property, which is an assumption of Lemma 5.2.1 (think of x as not much larger than $\Delta(H)$).

Lemma 5.2.1. *Let H be a graph such that each complete subgraph of H of order r contains a vertex of degree at most $x - r$. Then we have $\chi_f(H) \leq \frac{x+\Delta(H)}{3} + 1$.*

Proof. We start by finding an induced subgraph of H, denoted H', such that $\omega(H') \leq \frac{2x-\Delta(H)+1}{3}$. We define H' to be a graph obtained from H by deleting all vertices of degree less than $\frac{x+\Delta(H)-1}{3}$.

Now, we show that $\omega(H') \leq \frac{2x-\Delta(H)+1}{3}$. Suppose for the contrary that H' contains a complete subgraph S of order greater than $\frac{2x-\Delta(H)+1}{3}$. Since it is also a complete subgraph of H, by our assumption on H we deduce that S contains a vertex of degree (in H) less than $x - \frac{2x-\Delta(H)+1}{3} = \frac{x+\Delta(H)-1}{3}$, which contradicts the choice of H'.

Applying Theorem 2.7.3 to H' we get that

$$\chi_f(H') \leq \frac{\frac{2x-\Delta(H)+1}{3} + \Delta(H) + 1}{2} \leq \frac{x + \Delta(H)}{3} + 1.$$

Any $(\frac{x+\Delta(H)}{3} + 1)$-coloring of H' can be extended to a $(\frac{x+\Delta(H)}{3} + 1)$-coloring of H (by assingning colors to the remaining vertices greedily), so the proof is complete. □

The main difficulty of the proof is showing that $L(G)^2$ satisfies the desired property, with $x \approx \frac{17}{7}\Delta(G)^2$.

Lemma 5.2.2. *Let G be a bipartite graph and H a subgraph of G such that $E(H)$ is a clique in $L(G)^2$. There exists an edge $e \in E(H)$ such that*

$$deg_{L(G)^2}(e) + e(H) \leq \frac{17}{7}\Delta(G)^2 + \sqrt{2}\Delta(G)^{1.5}.$$

Theorem 5.1.1 is an immediate consequence of the two lemmas.

Proof of Theorem 5.1.1. Let G be a bipartite graph. By Lemma 5.2.2, each complete subgraph of $L(G)^2$ of order r contains a vertex of degree at most $x - r$, where $x = \frac{17}{7}\Delta(G)^2 + \sqrt{2}\Delta(G)^{1.5}$. Therefore, the Theorem follows by Lemma 5.2.1 applied for $L(G)^2$. □

5.3 Proof of Lemma 5.2.2

We will need the following lemma, which is a quantitive form of the statement "if there is a large clique in $L(G)^2$, then G contains a large number of 3-vertex paths". The result will be useful for $|A|$ of order $\Delta(G)$ and $e(H)$ of order $\Delta(G)^2$ ($|B|$ can be larger, which makes the proof more complicated than it would be otherwise).

Lemma 5.3.1. *Let G be a bipartite graph with partition classes A, B and H a subgraph of G such that $E(H)$ is a clique in $L(G)^2$. There are at least*

$$\frac{|A|^2 \Delta(G)}{2}\left(1 - \sqrt{1 - \frac{e(H)^2}{|A|^2 \Delta(G)^2}}\right) - \frac{\sqrt{2}}{2}\Delta(G)\,|A|\,\sqrt{|A|}$$

copies of P_3 in G that have two vertices in A.

Proof. We will count the number of unordered pairs $\{e, f\}$ such that e and f are adjacent in $L(G)^2$ (where $e, f \in E(G)$). Let us denote this number by n_p. On one hand, $n_p \geq \frac{e(H)^2 - e(H)}{2}$ (because $E(H)$ is a clique in $L(G)^2$).

Now consider a vertex $u \in B$. The number of pairs $\{e, f\}$, where $u \in e$ and f contains a neighbor (in G) of u, that contribute to n_p is at most $\deg_G(u)^2\Delta(G) - \binom{\deg_G(u)}{2} - \deg_G(u)$ (there are $\deg_G(u)$ choices for e and at most $\deg_G(u)\Delta(G)$ choices for f, so we get $\deg_G(u)^2\Delta(G)$. However, this way we double-count $\binom{\deg_G(u)}{2}$ pairs $\{e, f\}$ where both

e and f are incident to u and $\deg_G(u)$ adjacencies of some edge to itself, so we substract respective numbers). We will say that such pairs are counted by u.

Note that every pair $\{e, f\}$ that contributes to n_p, is counted by some vertex $u \in B$ (because G is bipartite). Moreover, if there is a copy of C_4 in G on vertices v_1, u_2, v_3, u_4 (with edges $\{v_1u_2, u_2v_3, v_3u_4, u_4v_1\}$; where $u_2, u_4 \in B$), then pairs $\{v_1u_2, v_3u_4\}$ and $\{u_2v_3, u_4v_1\}$ are counted by both u_2 and u_4. Therefore, if we set n_{C_4} to be the number of copies of C_4 in G, we obtain

$$\frac{e(H)^2 - e(H)}{2} \leq n_p \leq \sum_{u \in B} \left(\deg_G(u)^2 \Delta(G) - \binom{\deg_G(u)}{2} - \deg_G(u) \right) - 2n_{C_4}. \quad (1)$$

Now we will find a lower bound on n_{C_4}. For $v, w \in A$, where $v \neq w$, let $y_{\{v,w\}}$ be a number of common neighbors (in G) of v and w. There are $\binom{y}{2} = \frac{y_2}{2} - \frac{y}{2}$ copies of C_4 in G that contain vertices v and w, so we have

$$n_{C_4} = \frac{1}{2} \sum_{v,w \in A, v \neq w} y^2_{\{v,w\}} - \frac{1}{2} \sum_{v,w \in A, v \neq w} y_{\{v,w\}}. \quad (2)$$

Note that the sum of all $y_{\{v,w\}}$ (over $v, w \in A, v \neq w$) is equal to $\frac{1}{2}\sum_{u \in B} \deg_G(u)(\deg_G(u) - 1)$ (because u is a common neighbor of every two of its neighbors). Therefore, considering $(y_{\{v,w\}})$ as an $\binom{|A|}{2}$-dimensional vector, by the Cauchy-Schwarz inequality we obtain

$$\sum_{v,w \in A} y_{\{v,w\}} \geq \frac{\left(\frac{1}{2}\sum_{u \in B} \deg_G(u)(\deg_G(u) - 1)\right)^2}{\binom{|A|}{2}}.$$

Now, from (2) (and by $\binom{|A|}{2} \leq \frac{|A|^2}{2}$) we obtain

$$n_{C_4} \geq \frac{\left(\sum_{u \in B} \deg_G(u)(\deg_G(u) - 1)\right)^2}{4\,|A|^2} - \frac{\sum_{u \in B} \deg_G(u)(\deg_G(u) - 1)}{4}. \quad (3)$$

For convenience, we set $x = \sum_{u \in B} \deg_G(u)(\deg_G(u) - 1)$. Now, from (1) and (3) we get

$$\frac{e(H)^2 - e(H)}{2} \leq \Delta(G)x + \Delta(G)\sum_{u \in B} \deg_G(u) - \frac{x}{2} - \sum_{u \in B} \deg_G(u) - \frac{x^2}{2\,|A|^2} + \frac{x}{2}. \quad (4)$$

Note that we have $e(H) \leq \sum_{u \in B} \deg_G(u)$ and $\sum_{u \in B} \deg_G(u) \leq |A|\,\Delta(G)$ so we can rewrite (4) as

$$0 \leq -e(H)^2 + 2\Delta(G)x + 2\Delta(G)^2\,|A| - \frac{x^2}{|A|^2}. \quad (5)$$

The right side of (5) is a quadratic function in x that attains its maximum value for $x_{\max} = |A|^2 \Delta(G)$ (and this value is at least 0 by $e(H) \leq |A| \Delta(G)$). Moreover, in point $x_- = |A|^2 \Delta(G)(1 - \sqrt{1 - \frac{e(H)^2}{|A|^2 \Delta(G)^2}}) - \sqrt{2}\Delta(G) |A| \sqrt{|A|}$ this function is negative (and equal to $-2\sqrt{2}\sqrt{1 - \frac{e(H)^2}{|A|^2 \Delta(G)^2}}\Delta(G)^2 |A| \sqrt{|A|}$). Therefore, inequality (5) implies that

$$x \geq |A|^2 \Delta(G) \left(1 - \sqrt{1 - \frac{e(H)^2}{|A|^2 \Delta(G)^2}}\right) - \sqrt{2}\Delta(G) |A| \sqrt{|A|}. \tag{6}$$

Since each vertex $u \in B$ is a central vertex of $\frac{\deg_G(u)(\deg_G(u)-1)}{2}$ copies of P_3 in G that have two vertices in A, by definition of x inequality (6) finishes the proof. □

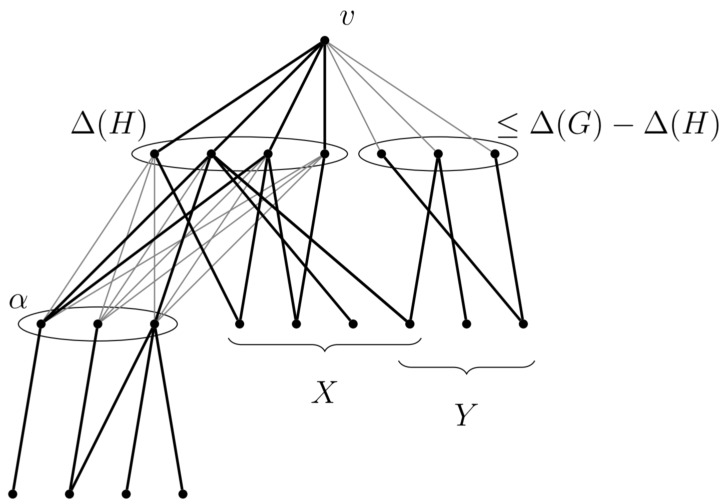


Figure 5: Notation used in the proof of Lemma 5.2.2 (edges outside H are grayed out or not shown)

Proof of Lemma 5.2.2. To shorten the notation, we will write Δ_G instead of $\Delta(G)$ and Δ_H instead of $\Delta(H)$.

Let $v \in V(H)$ be a vertex of degree (in H) equal to Δ_H. We will now introduce a few notions, see Figure 5 for a visualization.

Let V_α be the set of vertices of H that are adjacent (in G) to a neighbor of v and incident to at least one edge of H that is not incident to a neighbor (in G) of v. Define α to be the size of V_α. Moreover, we will denote the set of edges of H incident to a vertex from V_α by E_α.

Let X (resp. Y) denote the set of egdes of H that are incident to a vertex from $N_H(v)$ (resp. $N_G(v) \setminus N_H(v)$) and not incident to any vertex from N_α. Set x (resp. y) to be the size of X (resp. Y).

Since $E(H)$ is a clique in $L(G)^2$, each edge of H is incident to a neighbor of v or a vertex from V_α. Moreover $E_\alpha \leq \alpha\Delta_H$, so we have

$$e(H) \leq \alpha\Delta_H + x + y + \Delta_H. \tag{7}$$

Now we will count the number of copies (in G) of directed P_3 that start in $N_H(v)$ and end in $N_G(v)$ (a directed P_3 is a P_3 with one leaf marked as a startpoint, and the other as an endpoint; each copy of P_3 correponds to two directed P_3's). Let us denote this number by $\#P_3$.

Note that every vertex from V_α is adjacent to every vertex from $N_H(v)$ (by definition of V_α and G being bipartite). Therefore, there are (exactly!) $\Delta_H(\Delta_H - 1)\alpha$ directed P_3's that start and end in $N_H(v)$ and have a middle point in V_α.

By Lemma 5.3.1 (applied for a graph G' that is a subgraph of G induced by $N_H(v) \cup N_H^2(v)$, its subgraph $H' = (V(G'), X)$ and $A = N_H(v)$) we get that there are at least

$$\Delta_H^2\Delta_G\left(1 - \sqrt{1 - \frac{x^2}{\Delta_H^2\Delta_G^2}}\right) - \sqrt{2}\Delta_G\Delta_H\sqrt{\Delta_H}$$

directed P_3's that start and end in $N_H(v)$, and have a middle point outside V_α.

Moreover, for every two edges $e \in X$ and $f \in Y$ there is a directed P_3 that starts in $e \cap N_H(v)$, ends in $f \cap (N_G(v) \setminus N_H(v))$ and starts with e or ends with f (because $E(H)$ is a clique in $L(G)^2$ and G is bipartite). Since each such path corresponds to at most $2\Delta_H$ pairs (e, f), we have at least $\frac{xy}{2\Delta_H}$ directed P_3's that start in $N_H(v)$ and end in $N_G(v) \setminus N_H(v)$.

Now we have an estimation of the number of directed P_3's that start in $N_H(v)$, end in $N_G(v)$ and have a middle vertex other than v. By the above estimations and the pigeonhole principle we deduce that there is a vertex $u \in N_H(v)$ that is the start of at least

$$(\Delta_H - 1)\alpha + \Delta_H\Delta_G\left(1 - \sqrt{1 - \frac{x^2}{\Delta_H^2\Delta_G^2}}\right) - \sqrt{2}\Delta_G\sqrt{\Delta_H} + \frac{xy}{2\Delta_H^2} \tag{8}$$

directed P_3's that start in $N_H(v)$, end in $N_G(v)$ and have a middle vertex other than v. It remains to show that the edge vu satisies the condition ot the Lemma.

Each directed P_3 that starts in u, ends in $N_G(v)$ and have middle vertex other that v corresponds to a copy of C_4 in G that contains the edge uv. Since $\deg_{L(G)^2}(uv)$ is at most $2\Delta_G^2 - 2\Delta_G$ minus the number of copies of C_4 in G that contain uv, by combining (7) and (8) we obtain

$$\deg_{L(G)^2}(uv) + e(H) \le 2\Delta_G^2 - 2\Delta_G - (\Delta_H - 1)\alpha - \Delta(H\Delta_G\left(1 - \sqrt{1 - \frac{x^2}{\Delta_H^2\Delta_G^2}}\right)$$
$$+\sqrt{2}\Delta_G\sqrt{\Delta_H} - \frac{xy}{2\Delta_H^2} + \alpha\Delta_H + x + y + \Delta_H. \quad (9)$$

Observing that $\alpha \le \Delta_G$ and $\Delta_H \le \Delta_G$, we simplify this to

$$\deg_{L(G)^2}(uv) + e(H) \le 2\Delta_G^2 - \Delta_H\Delta_G + \sqrt{\Delta_H^2\Delta_G^2 - x^2} + x - \frac{xy}{2\Delta_H^2} + y + \sqrt{2}\Delta_G^{1.5}. \quad (10)$$

By $y \le \Delta_H(\Delta_G - \Delta_H)$ and $x \le \Delta_H^2$ we obtain

$$\deg_{L(G)^2}(uv) + e(H) \le 2\Delta_G^2 + \sqrt{\Delta_H^2\Delta_G^2 - x^2} + x - \frac{x(\Delta_G - \Delta_H)}{2\Delta_H} - \Delta_H^2 + \sqrt{2}\Delta_G^{1.5}. \quad (11)$$

Now we introduce γ so that $x = \gamma\Delta_G\Delta_H$ (note that $0 \le \gamma \le 1$) and we get

$$\deg_{L(G)^2}(uv) + e(H) \le 2\Delta_G^2 - \frac{\gamma\Delta_G^2}{2} + \Delta_G\Delta_H\left(\sqrt{1-\gamma^2} + \frac{3\gamma}{2}\right) - \Delta_H^2 + \sqrt{2}\Delta_G^{1.5}. \quad (12)$$

The right side of inequality 12 is a quadratic function in Δ_H that attains its maximum in point $\frac{2\sqrt{1-\gamma^2}+3\gamma}{4}\Delta_G$, which implies that

$$\deg_{L(G)^2}(uv) + e(H) \le \Delta_G^2\left(\frac{9}{4} - \frac{1}{2}\gamma + \frac{5}{16}\gamma^2 + \frac{3}{4}\gamma\sqrt{1-\gamma^2}\right) + \sqrt{2}\Delta_G^{1.5}. \quad (13)$$

Note that we have $\sqrt{1-\gamma^2} \le \frac{29}{21} - \frac{20}{21}\gamma$ (graph of the function on the left is a part of a circle, and the graph of the function on the right is a line tangent to the circle at point $\gamma = \frac{20}{29}$). Therefore, (13) implies

$$\deg_{L(G)^2}(uv) + e(H) \le \Delta_G^2\left(\frac{9}{4} + \frac{45}{84}\gamma - \frac{135}{336}\gamma^2\right) + \sqrt{2}\Delta_G^{1.5}. \quad (14)$$

The function in brackets is quadratic in γ and attains its maximum value $\frac{51}{21}$ for $\gamma = \frac{2}{3}$, so the proof is complete. $\square$

5.4 Possible improvements

Consider the relation between Lemma 5.2.2 and Theorem 2.6.1 – the lemma implies that the maximum clique in $L(G)^2$ is not larger than (rougly) $\frac{17}{14}\Delta(G)^2$ (for a bipartite graph G), which is weaker than the (tight) bound $\Delta(G)^2$ given by the theorem, and the theorem does not does not give any information on the degrees of vertices in cliques in $L(G)^2$. Hence, those two results are incomparable (that is, neither of them is an easy cosequence of the other).

We believe that Lemma 5.2.2 can be strengthened so that its consequence would match Theorem 2.6.1. Note such a strengthening, apart from being interesting on its own, would translate to an essentianl improvement of Theorem 5.1.1 – the constant $\frac{31}{21}$ would improve to $\frac{4}{3}$. We conjecture the following.

Conjecture 5.4.1 (MD, 2015 [14]). *Let G be a bipartite graph and H a subgraph of G such that $E(H)$ is a clique in $L(G)^2$. There exists an edge $e \in E(H)$ such that*

$$\deg_{L(G)^2}(e) + e(H) \leq 2\Delta(G)^2.$$

Our work would also benefit from a certain strengthening of Theorem 2.7.3. Note that the quantity x in assumption of Lemma 5.2.1 corresponds to $\omega(G) + \Delta(G) + 1$ (and is smaller than or equal to that number). Although we do not dare to pose it as a conjecture, we would like to see a stronger version of Theorem 2.7.3 with x insted of $\omega(G)+\Delta(G)+1$ (in fact, Lemma 5.2.1 is a small step in this direction). For a discussion of variants of Theorem 2.7.3 see the paper by Edwards and King [19]; in particular, their Conjecture 4 is related to our suggestion and would probably imply a strengthening of Theorem 5.1.1.

Finally, we believe that the same ideas can be used to give an upper bound on fractional strong chromatic index of all (not necessarily bipartite) graphs – note that the only missing part is a non-bipartite analog of Lemma 5.2.2. Recall that if G is a blowup of C_5 (depicted in Figure 3a), then $L(G)^2$ is a clique of order $\frac{5}{4}\Delta(G)^2$; we conjecture that it is the worst possible and that the desired analog of Lemma 5.2.2 should be as follows.

Conjecture 5.4.2. *Let G be a graph and H a subgraph of G such that $E(H)$ is a clique* *in* $L(G)^2$*. There exists an edge* $e \in E(H)$ *such that*

$$\deg_{L(G)^2}(e) + e(H) \leq 2\frac{1}{2}\Delta(G)^2.$$

6 Topological strong chromatic index of bipartite graphs

6.1 Topological analog of chromatic number

In this section we define a topological equivalent of the chromatic number of a graph (which is relevant to the topic of the dissertation – recall that strong chromatic index is the chromatic number of a certain graph, according to Definition 2.1.2). There is a number of parameters that may be considered a topological counterpart of the chromatic number, and among those we focus only on the largest one (being an upper bound for all other). We refer to the paper by Simonyi and Zsbán [38] for the discussion of other similar notions.

A $\mathbb{Z}_2$**-space** is a pair (X, v), where X is a topological space and $v : T \to T$ is a continuous function satisfying $v \circ v = id_X$ (that is, $v(v(x)) = x$ for all x; such v is called a $\mathbb{Z}_2$**-action**). We say that a $\mathbb{Z}_2$-space (X, v) is **free** if $v(x) \neq x$ for all $x \in X$. We apply this notion to topological spaces arising from simplicial complexes. We say that a simplicial complex F equipped with a simplicial map $f : V(F) \to V(F)$ is a **free** $\mathbb{Z}_2$**-complex** if $(||F||, ||f||)$ is a free $\mathbb{Z}_2$-space, where $||F||$ is a geometric realization of F and $||f||$ is a natural extension of f to a continuous function on $||F||$.

A $\mathbb{Z}_2$**-map** between two $\mathbb{Z}_2$-spaces (X, v) and (Y, u) is a continous map $m : (X, v) \to (Y, u)$, such that $m(v(x)) = u(m(x))$ for any $x \in X$. The $\mathbb{Z}_2$**-index** of a free $\mathbb{Z}_2$-space (X, v) is the minimum d such that there exists a $\mathbb{Z}_2$-map $m : (X, v) \to (\mathbb{S}^d, -)$, where $\mathbb{S}^d$ is a d-dimensional sphere and $-$ is a natural antipodal operation. We define the $\mathbb{Z}_2$**-index** of a free $\mathbb{Z}_2$-complex (F, f) to be the $\mathbb{Z}_2$-index of the underlying $\mathbb{Z}_2$-space and denote it by $\operatorname{ind}(F)$.

We define the **box complex** of a graph G (denoted by $B(G)$) to be a free $\mathbb{Z}_2$-complex on two copies of vertices of G, $V(G)$ and $\overline{V(G)}$, where $A \cup \overline{B}$ is a face if and only if either G contains a complete bipartite subgraph with partition classes A and B (for A, B being nonempty) or all vertices in A and B have at least one common neighbor (for A or B being an empty set). A $\mathbb{Z}_2$-action v is defined by $v(x) = \overline{x}$ and $v(\overline{x}) = x$ for $x \in V(G)$.

The $\mathbb{Z}_2$-index of the box complex of G (plus 2) can be thought of as a topological analog of the chromatic number of G. We have $\chi(G) \geq \mathrm{ind}(B(G)) + 2$ and in many cases this lower bound turns out to be sharp. In particular, Kneser graph with parameters n, k satisfy $\mathrm{ind}(B(K_{n,k})) + 2 = n - 2k + 2$ and have chromatic number equal to $n - 2k + 2$, while the fractional chromatic number is $\frac{n}{k}$, which shows that $\mathrm{ind}(B(G))+2$ can be greater than the fractional chromatic number of G. Results concerning this chromatic parameter provide supporting evidences for a number of conjectures on chromatic number of specific graphs [38].

We will use two properties of the $\mathbb{Z}_2$-index. The first one is a topological counterpart of the observation that adding a new vertex of degree d to a graph cannot increase its chromatic number above $d + 1$. It is implicitly proved in the book by Matoušek [31] (see Proposition 5.3.2).

Lemma 6.1.1 (see [31, Proposition 5.3.2])**.** *Let G be a graph and take $G' = G - v$, where v is a vertex of G of degree d. We have*

$$\mathrm{ind}(B(G)) + 2 \leq \max(\mathrm{ind}(B(G')) + 2, d + 1).$$

The second tool is the so-called $K_{l,m}$-theorem of Csorba *et. al* [13] stating that a graph of large $\mathbb{Z}_2$-index must contain large complete bipartite subgraphs (note that it is not the case with chromatic number).

Theorem 6.1.2 (Csorba, Lange, Schurr and Wassmer, 2004 [13])**.** *If G is a graph satisfying $\mathrm{ind}(B(G)) + 2 \geq t$, then for every possible $l, m \in \mathbb{N}$ with $l + m = t$, the complete bipartite graph $K_{l,m}$ appears as a subgraph of G.*

6.2 An upper bound on topological strong chromatic index of bipartite graphs

We define a topological analog of strong chromatic index of a graph in a way consistent with Definition 2.1.2.

Definition 6.2.1. *The **topological strong chromatic index** of a graph G, denoted $s'_t(G)$, is $\mathrm{ind}(B(L(G)^2)) + 2$.*

Note that $s'_t(G) \leq s'(G)$ (as mentioned in Section 6.1), so if we prove an upper bound on $s'(G)$, it is also an upper bound on $s'_t(G)$. On the other hand, an upper bound on $s'_t(G)$ would not yield any formal consequences regarding $s'(G)$, but it may indicate that a similar bound on $s'(G)$ exists. In view of the apparent difficulty of Conjectures 2.4.3 and 2.5.1, it is worthwhile to investigate their topological variants.

Conjecture 6.2.2 (Erdős and Nešetřil, topological variant)**.** *For any graph G of maximum degree Δ we have $s'_t(G) \leq \frac{5}{4}\Delta^2$.*

Conjecture 6.2.3 (Faudree, Gyárfás, Schelp and Tuza, topological variant)**.** *For any bipartite graph G of maximum degree Δ we have $s'_t(G) \leq \Delta^2$.*

Conjectures 6.2.2 and 6.2.3 would be sharp, as witnessed by a blowup of C_5 (see Proposition 2.3.2 for definition) and $K_{\Delta,\Delta}$ respectively.

Recall that for a bipartite graph G of maximum degree Δ, the best known bound on $s'(G)$ is $1.93\Delta^2$ (Theorem 2.4.2), which implies that $s'_t(G) \leq 1.93\Delta^2$. We improve the constant to 1.703.

Theorem 6.2.4 (MD, 2015 [15])**.** *Let G be a bipartite graph of maximum degree Δ. We have*

$$s'_t(G) \leq 1.703\Delta^2.$$

6.3 Proof of Theorem 6.2.4

The main idea of our argument goes as follows. First we consider the case when G contains no complete bipartite subgraph with $z\Delta^2$ edges, where z is some constant, to be revealed later. Once we establish Corollary 6.3.3, the rest of the proof goes by a simple greedy coloring argument.

A complete bipartite subgraph of $L(G)^2$ can be viewed as two disjoint sets of edges of G, such that each two edges e and f, where e belongs to the first set and f to the second one, are joined. We refer to those sets as red edges (denoted R) and blue edges (denoted B). We use the term **red degree** (respectively, **blue degree**) of $v \in V(G)$, denoted $d_r(v)$ (resp. $d_b(v)$), which is defined to be the number of red (blue) edges that are incident to

v (in G). Finally, by the **second red degree** (the **second blue degree**) of $v \in V(G)$, denoted $d_r^{(2)}(v)$ ($d_b^{(2)}(v)$) we mean the number of red (blue) edges incident to at least one neighbor of v.

We refer to a pair (R, B) of subsets of $V(L(G)^2)$ as a **selection** (in G), if $|R| = |B|$, $R \cap B = \emptyset$ and each edge in R is within distance 1 from each edge in B. The **order** of a selection (R, B) is defined as $|R| + |B|$.

Lemma 6.3.1. *Let G be a bipartite graph of maximum degree Δ which contains no complete bipartite subgraph with at least $z\Delta^2$ edges, and let (R, B) be a selection in G. If $u, v \in V(G)$ are in different partition classes of G, then*

$$d_r^{(2)}(u) + d_b^{(2)}(v) < (1 + 2z - z^2)\Delta^2.$$

Proof. Note that in a bipartite graph $d_c^{(2)}(w)$ (where c equals r or b) is equal to the sum $\sum_{i=1}^{\Delta} d_c(w_i)$, where $w_1, \ldots, w_\Delta$ are neighbors of w. For $i = 1, 2, \ldots, \Delta$, define β_i to be the red degree of the i-th neighbor of v and let γ_i be the blue degree of the i-th neighbor of u, setting $\beta_i = 0$ ($\gamma_i = 0$) if there are fewer than i neighbors. We denote the neighbors of v by $v_1, v_2, \ldots$ and neighbors of u by $u_1, u_2, \ldots$.

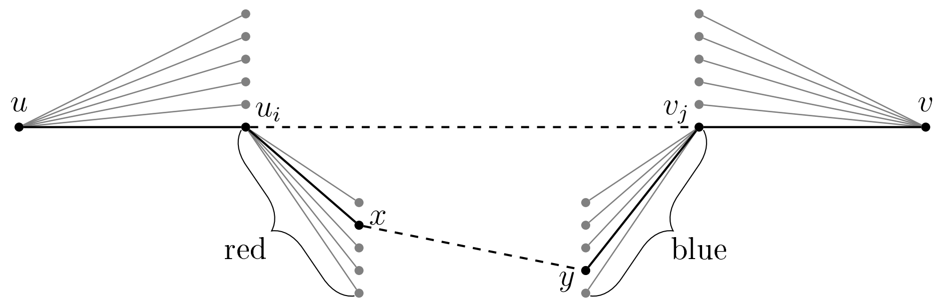


Figure 6: Part of the proof of Lemma 6.3.1: we have $u_iv_j \in E(G)$ or $xy \in E(G)$

Note that if we have $\beta_i\gamma_j \geq z\Delta^2$ for some i and j, then $v_iu_j \in E(G)$ (see Figure 6). Indeed, if $v_iu_j \notin E(G)$, then every neighbor x of v_i connected to it by a red edge must be adjacent to each neighbor y of u_i connected to it by a blue edge (since G is bipartite we have $x \neq y$, so xy remains the only possible link between those two edges), so G contains a complete bipartite subgraph with $\beta_i\gamma_j$ edges, which contradicts our assumption on G.

Clearly, the sum $d_r^{(2)}(u) + d_b^{(2)}(v)$ is equal to $\sum_i \beta_i + \sum_i \gamma_i$. For any $\alpha \in (0, 1]$, let $n_\alpha = |\{i : \beta_i \geq \alpha\Delta\}||\{i : \gamma_i \geq \frac{z}{\alpha}\Delta\}|$ (that is, the number of pairs (i, j) such that $\beta_i\gamma_j \geq z\Delta^2$ and $\beta_i \geq \alpha\Delta$). By the above claim, for each such pair, there is an edge v_iu_j, so G contains a complete bipartite subgraph with at least n_α edges. Therefore, we must have

$$n_\alpha < z\Delta^2. \tag{15}$$

Our aim is to show that the condition (15) implies the upper bound of $(1+2z-z^2)\Delta^2$ on the sum $S = \sum_i \beta_i + \sum_i \gamma_i$ for any real numbers $\beta_1, \ldots, \beta_\Delta, \gamma_1, \ldots, \gamma_\Delta$ that are nonnegative and at most Δ. Note that this claim immediately completes the proof.

Our first step is to prove that for any configuration (i.e. the choice of values β_i and γ_i satisfying (15) for all α) there is a configuration at least as good (with not lower value of S), in which there is no β_i nor γ_i in the interval $[z\Delta, \Delta)$. Indeed, suppose that there is some $\beta_k \in [z\Delta, \Delta)$ and there exists γ_m strictly smaller than Δ. Without loss of generality we may assume that β_k is the lowest among such β_i, γ_m is the highest among such γ_i and $\beta_k \leq \gamma_m$. Note that if we decrease β_k to $\frac{\beta_k\gamma_m}{\Delta}$ and increase γ_m to Δ, S will increase. Since no β_i lies in the interval $[z\Delta, \beta_k)$, the increase of γ_m will not result in increasing any n_α, so the condition (15) would hold. If for every i we have $\gamma_i = \Delta$ then replacing β_k by Δ does not change n_α (for any α). Note that the same argument applies if we exchange β with γ. By repeating this process we obtain the desired configuration.

Hence, we may and shall assume that all β_i's and γ_j's are either equal to Δ or smaller than $z\Delta$. Let c_r be the number of β_i that are equal to Δ ($c_r = |\{i : \beta_i = \Delta\}|$) and let c_b stand for the number of γ_i equal to Δ. Then

$$S \leq \Delta(c_r + c_b) + z\Delta((1 - c_r) + (1 - c_b)) = \Delta(1 - z)(c_r + c_b) + 2z\Delta.$$

Note that $0 < c_r, c_b \leq \Delta$ and, by (15), we have $c_rc_b < z\Delta^2$, so we obtain $S < (1 + 2z - z^2)\Delta^2$, as desired. $\square$

Lemma 6.3.2. *Let G be a graph of maximum degree Δ which contains no complete bipartite subgraph with at least $z\Delta^2$ edges. Then, the order $|R| + |B|$ of each selection (R, B) is at most*

$$\max((2 - z), 2f(1 + 2z - z^2, z), 2f(1 - z/2, z))\Delta^2,$$

where $f(\alpha, z) = \alpha/2 + 2z/\alpha - z$.

Proof. Let w, t be a pair of vertices from different partition classes of G maximizing $d_b^{(2)}(t) + d_r^{(2)}(w)$. Take $S = d_b^{(2)}(t) + d_r^{(2)}(w)$ and without loss of generality suppose that $d_r^{(2)}(w) \geq \frac{S}{2}$. Let v be a neighbor of w of highest red degree. Clearly, $d_r(v) \geq \frac{d_r^{(2)}(w)}{\Delta}$. Consider $d_r(v)$ neighbors of v that are connected to v by a red edge, denoted $u_1, u_2, \ldots, u_{d_r(v)}$. We start by proving the following claim.

Claim 4. *The number of blue edges is at most* $S/2 + 2z\Delta^4/S - z\Delta^2 = f(S/(2\Delta^2), z)$*.*

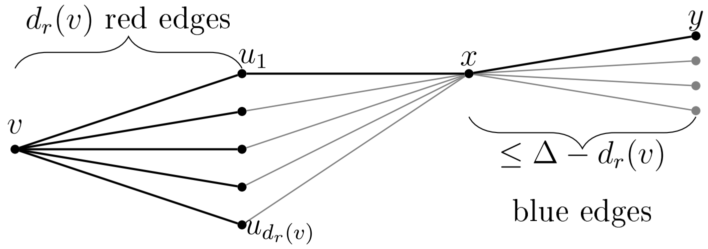


Figure 7: Part of the proof of Lemma 6.3.2: we have at most $\Delta - d_r(v)$ blue edges incident with x and not incident with a neighbor of v

Proof. Let us count the number of blue edges incident to a neighbor of some u_i and not incident with any neighbor of v. Consider a vertex x at distance 2 from v and distance 1 from u_1 (see Figure 7). If there is a blue edge xy, where y is different than all u_i, then either y is at distance 1 from v (so we do not count xy) or all u_i must be adjacent to x (because xu_i is the only possible link between xy and vu_i by G being bipartite). There are clearly at most $\Delta - d_r(v)$ blue edges incident with x and not incident with any neighbor of v. As G has no complete bipartite subgraph on $z\Delta^2$ edges, there are at most $z\Delta^2/d_r(v)$ such vertices x (incident with at least one blue edge not incident to a neighbor of v), so the number of edges in question is at most $(\Delta - d_r(v))z\Delta^2/d_r(v)$.

To bound the number of all blue edges, we need to add the number of blue edges incident with a neighbor of v. Consequently, there are at most

$$d_b^{(2)}(v) + (\Delta - d_r(v))\frac{z\Delta^2}{d_r(v)} \leq d_b^{(2)}(v) + \frac{z\Delta^4}{d_r^{(2)}(w)} - z\Delta^2$$

blue edges. Since v and w belong to different partition classes, $d_r^{(2)}(w) \geq S/2$ and $d_b^{(2)}(v) \leq S/2$ which concludes the proof of the claim. $\square$

Claim 5. *If $S \leq (1 - z/2)\Delta^2$, then the order of selection (R, B) is at most $(2 - z)\Delta^2$.*

Proof. Let $m_{i,c}$, where $i \in \{1, 2\}$ and $c \in \{r, b\}$ denote the maximum value of $d_c^{(2)}(u)$ for a vertex u in i-th partition class of G. Without loss of generality we may assume that $m_{2,b} \leq m_{2,r}$. For any red edge $ab \in R$ (where a is in the first partition class of G) the number of blue edges is at most

$$|B| \leq d_b^{(2)}(a) + d_b^{(2)}(b) \leq m_{1,b} + m_{2,b} \leq m_{1,b} + m_{2,r} \leq S.$$

Consequently, since $|R| = |B|$, the order of selection (R, B) is at most $2S$, which proves Claim 5. $\square$

Now note that as a function of S, when z and Δ are fixed, the function $S/2+2z\Delta^4/S - z\Delta^2$ is unimodal, so if we could bound S both from above and below, it would result in an upper bound on the number of blue edges by Claim 4.

If S is smaller than $(1-z/2)\Delta^2$, then the result follows by Claim 5. Therefore, we may assume that $S/\Delta^2 \geq 1 - z/2$. Moreover, by Lemma 6.3.1 we have $S/\Delta^2 < 1 + 2z - z^2$. Using the observation from the above paragraph and Claim 1 we have that the number of blue edges is at most $\max(f(1 - z/2, z), f(1 + 2z - z^2, z))\Delta^2$, which completes the proof. $\square$

After plugging in the value $z = 0.298$, we immediately get the following Corollary.

Corollary 6.3.3. *Let G be a graph of maximum degree Δ which contains no complete bipartite subgraph with at least $0.298\Delta^2$ edges. Then, the order of each selection (R, B) is less than $1.703\Delta^2$.*

Proof of Theorem 6.2.4. Suppose that G is a minimal counterexample to Theorem 6.2.4. If G has no complete bipartite subgraphs with at least $0.298\Delta^2$ edges, then by Corollary 6.3.3 the maximum order of a selection in G (equal to maximum order of a bipartite subgraph of $L(G)^2$ with partition classes of equal order) is smaller than $1.703\Delta^2$. By Theorem 6.1.2 we get that $s_t'(G) \leq 1.703\Delta^2$, a contradiction.

In the remaining case, let H be a complete bipartite subgraph of G with at least $0.298\Delta^2$ edges and consider the graph $G' = G \setminus V(H)$. Note that, by our choice of G, vertices of $L(G')^2$ can be colored using $1.703\Delta^2$ colors. Thus, to complete the proof it is enough to verify that $L(G)^2$ can be obtained from $L(G')^2$ by adding to it vertices one by one in such a way that each added vertex has degree less than $1.702\Delta^2$ in the existing graph, and use Lemma 6.1.1. We start with vertices of $L(G)^2$ corresponding to edges having exactly one end in $V(H)$, and then proceed with edges of H.

Consider an edge vw of G, where $v \in V(H)$ and $w \notin V(H)$. Clearly, the degree of vw in $L(G)^2 \setminus E(H)$ (and all its subgraphs) is at most $2\delta^2 - |E(H)| \leq 1.702\Delta^2$.

Now let $e = uv \in E(H)$ be an edge of $L(G)^2$. There are at most $2\Delta - 2$ vertices adjacent to u or v (and not equal neither to u nor to v), each incident with at most Δ edges, so the degree of e (in $L(G)^2$) is at most $2\Delta^2 - 2\Delta$ minus the number of edges (other than e) incident to both a neighbor of u and a neighbor of v. The latter number is at least the number of edges of H that are not incident with neither u nor v, so it is strictly greater than $0.298\Delta^2 - 2\Delta$. consequently, the degree of any edge $e = uv \in E(H)$ in $L(G)^2$ is less than $1.702\Delta^2$.

This completes the proof of Theorem 6.2.4. □

6.4 Possible and impossible improvements

It seems that further exploration of our ideas may lead to results stronger than Theorem 6.2.4. The first possibility is to strengthen Lemma 6.3.1, but even if we could replace the constant $(1 + 2z - z^2)$ by 1, it would result in strengthening of Theorem 6.2.4 by only $0.014\Delta^2$. The obstacle to greater improvement is hidden in the proof of Lemma 6.3.2, where we must carefully consider the case when s is small: if the lower bound on s provided by Claim 2 would be a bit weaker (where **a bit** stands for at least $0.061\Delta^2$), then it would lead to worse constants in Corollary 6.3.3.

Other possible way of strengthening the result is to directly bound the number of red and blue edges, like in the proof of Lemma 6.3.1, getting rid of the weaknesses of Lemma 6.3.2 mentioned above. There is also some hope that this approach will let us remove the assumption that G is bipartite.

Note that our proof relies on Theorem 6.1.2. As useful as it is, it is not strong enough to confirm Conjectures 6.2.2 and 6.2.3. Indeed, if we take the graph $K_{\Delta,\Delta}$ and subdivide each edge of some small complete bipartite subgraph, we can produce a graph G' such that $L(G')^2$ contains the bipartite subgraph $K_{l,m}$, for any given l, m such that $l+m = (1+\epsilon)\Delta^2$ (where ϵ is some constant around 0.05). Similar statement holds for the blowup of C_5 (we can achieve $l + m = (\frac{5}{4} + \epsilon)\Delta^2$ for ϵ around 0.02).

7 Side problem: matching subsets

7.1 Mazur's conjecture

In this chapter we discuss a problem that is not related to strong chromatic index of graphs, except a rather loose connection; recall that in a strong edge coloring we require each color to be an induced matching – here we consider perfect matchings in certain graphs (the graphs are not explicitly used, which simplifies the formulation).

Let S be a finite set and let $\mathcal{P}(S)$ be the family of all subsets of S. A red-blue coloring of $\mathcal{P}(S)$ is called **antipodal** if each set is colored differently than its complement. A **perfect matching respecting given coloring of** $\mathcal{P}(S)$ is a partition of $\mathcal{P}(S)$ into a number of pairs, such that for every pair $\{A, B\}$, A and B are inclusion related subsets of different colors.

There are many interesting and notoriously difficult problems involving matching properties in the subset lattice (cf. for instance [18], [28], [29]). The following Conjecture constitutes an extension of the problem posed by Przemysław Mazur [25].

Conjecture 7.1.1. *For every antipodal coloring of $\mathcal{P}(S)$ there is a perfect matching respecting this coloring.*

The original formulation of this problem concerns matching in the set of all possible products of a given set of prime numbers, where the smaller half is red and the rest is blue.

We prove the conjecture in the **monotone** case: one of the color classes is closed on taking subsets (in consequence, the other color class is closed on taking supersets), which solves the original problem. Our result is slightly stronger and concerns partial colorings of $\mathcal{P}(S)$. A rigorous formulation and the proof is found in Section 7.2.

We may generalize Conjecture 7.1.1 to lattices of divisors of an integer. Since notions of antipodal coloring and a perfect matching can be easily extended, the generalization may be formulated in exactly the same words. We show that this (obviously stronger) version of Conjecture 7.1.1 is equivalent to its original form.

Remark 7.1.2. *Conjecture 7.1.1 would imply that for every antipodal coloring of the set of divisors of an integer n there is a perfect matching respecting this coloring*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ be the standard factorization into primes. Note that the set of divisors of the form $p_1^k m$ and $p_1^{\alpha_1 - k} m$ form a sublattice closed under taking complements, that is isomorphic to the set of divisors of $n' = p_1 p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ (when $\alpha_1 \neq 2k$) or $n' = p_2^{\alpha_2} \ldots p_r^{\alpha_r}$ (when $\alpha_1 = 2k$). The result follows by induction. $\square$

Note that the same argument can be used to prove an analog of our Theorem 7.2.1 for these lattices.

The study of matchings that respect certain constraints is also related to the union-closed sets conjecture, posed by P. Frankl in 1979, stating that for any union-closed family $\mathcal{F}$ of sets from $\mathcal{P}(S)$ there is an element $a \in S$ that belongs to at least half of the sets in the family (cf. [24]).

Remark 7.1.3. *Let $\mathcal{F}$ be a family of sets from $\mathcal{P}(S)$. If there exists a perfect matching M between $\mathcal{F}$ and $\overline{\mathcal{F}}$ such that for every $\{A, \overline{B}\} \in M$ we have $\overline{B} \subseteq A$ (where $A, B \in \mathcal{F}$), then every element of S is contained in at least half of the sets in $\mathcal{F}$*

Proof. Consider a directed graph G on a vertex set $\mathcal{F}$, in which AB is an edge whenever $\{A, \overline{B}\}$ is contained in M. It is easy to see that if $a \notin A$, then $a \in B$. As G is a union of disjoint cycles, the result follows. $\square$

Note that this remark, together with Theorem 7.2.1, proves Frankl conjecture in case when $\mathcal{F}$ is closed under taking supersets (however, it this special case the conjecture can be proved by a simple, direct argument).

7.2 Aa affirmative result

Let $c : \mathcal{P}(S) \to \{-1, 0, 1\}$ be a coloring of the subsets of S. We interpret -1 as the red color, 1 as the blue color, and 0 as white. A coloring c is called **antipodal** if $c(A) = -c(\overline{A})$, and **monotone** if $A \subseteq B$ implies $c(A) \leq c(B)$, for all $A, B \subseteq S$. A **matching** is a collection of disjoint pairs $\{A, B\}$ such that either $A \subseteq B$ or $B \subseteq A$. We say that a matching M **respects coloring** c, if $c(A) = -c(B)$ and $c(A) \neq 0$, for every pair $\{A, B\} \in M$. Finally, a matching M is **perfect** (with respect to c) if every subset A with $c(A) \neq 0$ occurs in some pair of M.

Theorem 7.2.1 (MD, 2012 [16]). *For any antipodal, monotone coloring c of $\mathcal{P}(S)$ there is a perfect matching respecting c.*

Before proceeding to the proof we need more notation. Assume that $S = \{1, 2, \ldots, n\}$ and let $S' = S \setminus \{n\}$. In order to apply inductive argument we use two types of reduction of a coloring c from $\mathcal{P}(S)$ to $\mathcal{P}(S')$. Let $c : \mathcal{P}(S) \to \{-1, 0, 1\}$ be any coloring of $\mathcal{P}(S)$. The **weak reduction** of c is a coloring $c_w : \mathcal{P}(S') \to \{-1, 0, 1\}$ defined by

$$c_w(A) = sgn(c(A) + c(A \cup \{n\}))$$

for every $A \subseteq S'$. So, $c_w(A) = 0$ if and only if both sets A and $A \cup \{n\}$ are white, or one of them is red and the other one is blue (in coloring c). In all remaining cases the color of A is inherited from a non-white member of the pair $(A, A \cup \{n\})$. The **strong reduction** of c is the coloring $c_s : \mathcal{P}(S') \to \{-1, 0, 1\}$ given by

$$c_s(A) = c(A) + c(A \cup \{n\}) - c_w(A).$$

For convenience we collect all possibilities in the table below.

$c(A)$	$c(A \cup \{n\})$	$c_w(A)$	$c_s(A)$
0	0	0	0
$+$	$+$	$+$	$+$
$-$	$-$	$-$	$-$
$+$	$-$	0	0
$-$	$+$	0	0
0	$+$	$+$	0
0	$-$	$-$	0
$+$	0	$+$	0
$-$	0	$-$	0

(16)

The following lemmas show that both types of reduction preserve the desired properties.

Lemma 7.2.2. *Let c be a monotone antipodal coloring of $\mathcal{P}(S)$. Then the weak reduction c_w is a monotone antipodal coloring of $\mathcal{P}(S')$.*

Proof. Let A be any subset of S'. Since A is at the same time a subset of S, we have to distinguish between the complements of A in S and in S', which will be denoted simply as $S \setminus A$ and $S' \setminus A$, respectively. By antipodality of c we may write

$$\begin{aligned} c_w(A) &= sgn\,(c(A) + c(A \cup \{n\})) = sgn(-c(S \setminus A) - c(S \setminus (A \cup \{n\}))) \\ &= -sgn(c(S \setminus A) + c(S' \setminus A)) = -sgn(c((S' \setminus A) \cup \{n\}) + c(S' \setminus A)) \\ &= -c_w(S' \setminus A). \end{aligned}$$

This shows that c_w is antipodal. Monotonicity of c_w follows easily from the monotonicity of the function $sgn(x)$. $\square$

Similar lemma holds for the strong reduction.

Lemma 7.2.3. *Let c be a monotone antipodal coloring of $\mathcal{P}(S)$. Then the strong reduction c_s is a monotone antipodal coloring of $\mathcal{P}(S')$.*

Proof. Let A be a subset of S'. Then by antipodality of c and c_w we may write

$$\begin{aligned} c_s(A) &= c(A) + c(A \cup \{n\}) - c_w(A) = -c(S \setminus A) - c(S - (A \cup \{n\})) + c_w(S' \setminus A) \\ &= -c((S' \setminus A) \cup \{n\}) - c(S' \setminus A) + c_w(S' \setminus A) \\ &= -c_s(S' \setminus A). \end{aligned}$$

Let A and B be subsets of S', with $A \subseteq B$. Put $k = c(B) + c(B \cup \{n\}) - c(A) - c(A \cup \{n\})$. By monotonicity of c we have $k \geq 0$. Hence, the inequality $sgn(x) \geq sgn(x+k) - k$ holds for any real number x, and we may write

$$\begin{aligned} c_s(A) &= c(A) + c(A \cup \{n\}) - sgn(c(A) + c(A \cup \{n\})) \\ &\leq c(A) + c(A \cup \{n\}) + k - sgn(c(A) + c(A \cup \{n\}) + k) \\ &= c(B) + c(B \cup \{n\}) - sgn(c(B) + c(B \cup \{n\})) \\ &= c_s(B). \end{aligned}$$

This proves the lemma. $\square$

The following lemma gives a construction of the desired matching from matchings related to reduced colorings.

Lemma 7.2.4. *Let $c : \mathcal{P}(S) \to \{-1, 0, 1\}$ be a monotone antipodal coloring of $\mathcal{P}(S)$, and let c_w and c_s be the weak reduction and the strong reduction of c, respectively. Let M_w and M_s be perfect matchings of $\mathcal{P}(S')$ such that M_w respects c_w and M_s respects c_s. Then, there exists a perfect matching that respects c.*

Proof. Let G denote a graph on the vertex set $\mathcal{P}(S)$, in which AB forms an edge whenever A and B are inclusion related. Let $H = M_w \cup M_s$ denote a subgraph of G consisting of the edges of matchings M_w and M_s. Denote for convenience $A^+ = A \cup \{n\}$, and for a connectivity component C of the graph H, let $C^+ = \{A^+ : A \in C\}$. Clearly, C may be a path or an even cycle. We shall define a new matching M on the red-blue part of $\mathcal{P}(S)$ separately for each set $C \cup C^+$. We distinguish two cases.

Case 1. *C is an even cycle.*

Let $A_1, \ldots, A_{2k}$, $k \geq 2$, be the consecutive vertices of the cycle C, where the edges $A_1A_2, A_3A_4, \ldots, A_{2k-1}A_{2k}$ belong to the matching M_s. This implies that $c_s(A_j) \neq 0$ for every $j = 1, 2, \ldots, 2k$. In consequence, $c(A_j) = c(A_j^+) \neq 0$ for all j (see table 16). This means that the color pattern on the cycle C is the same as on its shifted copy C^+. Thus, we may extend the matching M_s by adding pairs $A_1^+A_2^+, A_3^+A_4^+, \ldots A_{2k-1}^+A_{2k}^+$. Clearly, each new pair respects inclusion.

Case 2. *C is a path with at least one edge.*

Let the vertices of the path C be denoted as $A_1, \ldots, A_m$. First we claim that $c_s(A_1) = 0$. Indeed, if $c_s(A_1)$ is nonzero, then $c_s(A_j) \neq 0$ for all $j = 1, 2, \ldots, m$, as the matching M_s covers all of C. This implies (see the table) that also $c_w(A_j) \neq 0$ for all j. Hence, the matching M_w covers all vertices of C, too. But this is impossible as the end vertices of the path have degree one. For the last vertex A_k we may argue similarly, hence $c_s(A_m) = 0$. In consequence, the first and the last edge of C must belong to the matching M_w, which implies that $m = 2k$. So, the edges $A_1A_2, \ldots, A_{2k-1}A_{2k}$ belong to M_w, while $A_2A_3, \ldots, A_{2k-2}A_{2k-1}$ belong to M_s. Therefore all values $c_s(A_j)$ are nonzero except for

$j = 1$ and $j = 2k$. Analyzing possible sign patterns on the path C (see Table 16) and remembering that coloring c is monotonic, we get the following picture:

$$\begin{array}{|c|ccccccc|}
\hline
 & A_1 & A_2 & A_3 & A_4 & \cdots & A_{2k-1} & A_{2k} \\
\hline
c_w(A_j) & - & + & - & + & \cdots & - & + \\
c_s(A_j) & 0 & + & - & + & \cdots & - & 0 \\
c(A_j) & - & + & - & + & \cdots & - & 0 \\
c(A_j^+) & 0 & + & - & + & \cdots & - & + \\
\hline
\end{array}. \tag{17}$$

(The case $c_w(A_1) = 1$ gives a symmetric table.) Notice that $c(A_1^+) = c(A_{2k}) = 0$, so these vertices will not be covered by our new matching M. Now, by monotonicity of the weak reduction c_w it must be $A_1 \subseteq A_2$. Hence $A_1 \subseteq A_2^+$ and we may include the edge $A_1A_2^+$ to the matching M. Next we take all edges of the matching M_s, that is $A_2A_3, \ldots, A_{2k-2}A_{2k-1}$, and the shifted edges $A_3^+A_4^+, \ldots, A_{2k-1}^+A_{2k}^+$. Clearly, all new edges respect inclusion.

Finally, we have to take care of those vertices of G which are colored red or blue by coloring c, but became white in both reductions. Then we simply match A with $A \cup \{n\}$, which is correct, as in this case the sets have opposite colors (see table 16). This concludes the proof of the lemma. □

Proof of Theorem 7.2.1. Use induction on n and apply Lemma 7.2.4 (that can be aplied due to Lemmas 7.2.2 and 7.2.3) to construct a desired matching. □

7.3 Further generalizations

Is it likely that (an analog of) Conjecture 7.1.1 or Theorem 7.2.1 holds for posets other than the one induced by $\mathcal{P}(S)$? As the antipodality restriction on the coloring is essential, we shall focus on posets in which every element has a natural complement. Apart from lattices of divisors of a natural number, mentioned in Remark 7.1.2, we may consider lattices of subspaces of a finite vector space (where the unique complement of a subspace is the largest orthogonal subspace). We believe that at least the following statement is true.

Conjecture 7.3.1. *For every monotone antipodal* 2*-coloring of a lattice of subspaces of a finite vector space there is a perfect matching respecting this coloring.*

Remark 7.1.3 suggests a broader area of interest. Are there any properties of antipodal (partial) colorings of $\mathcal{P}(S)$ (other than monotonicity) that assure the existance of a red-blue perfect matching? The antipodality alone is not enough (consider a partial coloring where only two sets are colored). On the other hand, the case when coloring is antipodal and all sets of size k and $n-k$ are non-white seems promising.

Conjecture 7.3.2. *For every partial antipodal coloring of $\mathcal{P}(S)$, that assigns color to all sents of order k and $|S|-k$, there is a perfect matching respecting this coloring.*

8 Conclusions

Note that all our results concerning (variants of) the strong chromatic index of graphs share one fundamental trait: they rely on not-too-high local density of the graph – In Theorem 3.2.5 we require that every edge of the considered graph G is in a small number of 4-cycles, Theorem 6.2.4 is proved by showing that $L(G)^2$ do not contain a large complete bipartite subgraph, Theorem 5.1.1 follows from restricting the degree of vertices in large cliques in $L(G)^2$ and Theorem 4.2.2 is proved by coloring edges around nice vertices (vertices that have a low number of edges in their vicinity). Therefore, our techniques can be thought of as a development of earlier ideas: the proofs of Theorem of Bruhn and Joos (Theorem 2.4.2) and Theorem of Molloy and Reed (Theorem 2.4.1) use the fact that neighborhoods in $L(G)^2$ are sparse.

It is not clear how to relate our results on topological and fractional strong chromatic index to the original variant of the problem. We proved $s'_t(G) \leq 1.703\Delta^2$ and $s'_f(G) \leq 1.476\Delta^2$ for a bipartite graph G of maximum degree Δ, and both of theese bounds are far stronger than $1.93\Delta^2$ that follows from Theorem 2.4.2 (recall that $2\Delta^2$ is trivial), but both $s'_t(G)$ and $s'_f(G)$ are *lower* bounds on $s'(G)$. The improvements seem even more exciting if one believes that $s'_t(G)$, $s'_f(G)$ and $s'(G)$ are always close to each other – note that it is the case with known extremal graphs (blowups of C_5 in Proposition 2.3.2 and complete bipartite graphs) and that there are no premises to think otherwise.

The main weakness of our findings is that they are not tight. However, one should not reasonably hope for a tight result on the topological, fractional or original variant of the strong chromatic index, when much easier problems remain unsolved – we would like to highlight two of them. The first is a very relaxed variant of Conjecture of Erdős and Nešetřil (Conjecture 2.4.3): show that the clique number of $L(G)^2$is at most $\frac{5}{4}\Delta^2$ (where G is a graph of maximum degree Δ; see also [40]). The second one emerges from our considerations in Chapter 5: what is the minimum $x = x(\Delta)$ such that for every graph G of maximum degree Δ and every clique in $L(G)^2$ the size of that clique plus the minimum degree of a vertex in that clique is at most x (Conjecture 5.4.2)?

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