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# The topology of solution spaces of combinatorial problems

*PhD dissertation*

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Author's declaration:

Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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## Abstract

Graph homomorphism is a notion almost as simple, notationally and conceptually, as graph coloring, but one that gives a rich mathematical structure, allowing for new fruitful connections with algebra and even topology. For combinatorialists, they are already a classical topic, and while not studied as thoroughly as graph coloring problems, they constitute a fundamental tool for a deeper understanding thereof. For computer scientists, graph homomorphisms express an important class of constraint satisfaction problems, exemplifying many relevant questions and techniques.

This thesis presents new applications of topology to graph theory, obtained by studying spaces corresponding to graphs and spaces of homomorphisms between two graphs. First, we consider just the basic topology of the latter space, namely its connected components. That is, we study 'recoloring' or 'reconfiguration' paths between graph homomorphisms, defined as sequences in which consecutive homomorphism differ at only one vertex. A natural topological necessary condition is formulated, which allows to characterize the paths exactly, to strengthen a previous algorithmic result on finding paths between graph 3-colorings, and to generalize it to homomorphisms into any graph without cycles of length four.

We then turn to Hedetniemi's conjecture, a deceptively simple question on the chromatic number of products of graphs, which stands open for more than 50 years. More generally, we consider multiplicative graphs, which are the 'prime' elements in the lattice of graphs partially ordered by graph homomorphisms. The connection of these questions with topology is much less conspicuous, but we apply similar techniques as for reconfiguration. A new proof of the multiplicativity is shown for all graphs where it was previously known. Furthermore, all graphs without cycles of length four are similarly shown to be 'prime', greatly extending this family.

Finally, we show that Hedetniemi's conjecture implies an analogous statement in topology. As the new proofs of multiplicativity show, the underlying principle is based on facts in 1-dimensional topology which do not extend to higher dimensions. Therefore, one may hope to find a counterexample to the topological statement, which would thus immediately refute Hedetniemi's conjecture. On the other hand, this means that a proof in topology should be in principle easier, and could be as easy to extend to graphs as for some of the currently known cases. A partial converse is also given, in fact the topological implication of Hedetniemi's conjecture is shown to be equivalent to another, weaker combinatorial statement.

These final results are obtained using a combinatorial construction that is a certain inverse (formally: the right adjoint) to taking the $k$-th power of a graph (or of its adjacency matrix). The construction turns out to have useful topological properties as well: it preserves the topology corresponding to a graph and refines its geometry, allowing to approximate any continuous map by homomorphisms from such refined graphs. This allows to easily prove the above implication of Hedetniemi's conjecture. It also vastly generalizes a previous result saying that the construction in question preserves the chromatic number when applied to complete graphs.

Hopefully, the results will prove to be not only a significant step in a fundamental conjecture, but also a new chapter at the puzzling intersection of combinatorics and algebraic topology. However, the author also hopes this thesis to be an accessible introduction to some of the connections between these two fields. This is certainly made easier by the fact that the methods used rely mostly on elementary definitions from topology, which can be understood without much engrossment, and that concrete, purely combinatorial conclusions follow.



*Keywords:* graph homomorphisms, Hedetniemi's conjecture, topological combinatorics

*ACM Computing Classification:* Mathematics of computing $\rightarrow$
Graph coloring, Algebraic topology, Graph algorithms

**Streszczenie**

Homomorfizm grafów to pojęcie prawie tak proste jak kolorowanie grafów, dające jednak bogatą matematyczną strukturę, która pozwala na nowe, owocne połączenia z algebrą, a nawet topologią. Dla kombinatoryków są już klasycznym zagadnieniem i choć nie tak pieczołowicie badane jak kolorowania, stanowią niemniej fundamentalne narzędzie dla głębszego ich zrozumienia. Dla informatyków zaś homomorfizmy grafów wyrażają ważną klasę problemów spełniania więzów, ilustrując wiele związanych z nimi pytań i technik.

Rozprawa przedstawia nowe zastosowania topologii w teorii grafów, otrzymane przez badanie przestrzeni odpowiadających grafom oraz przestrzeni homomorfizmów między dwoma grafami. Na początek rozważana jest topologia tej drugiej w najprostszym wymiarze, to jest spójne jej składowe. Analizuję więc proces *przekolorowywania* czy *rekonfiguracji*, czyli ścieżki między homomorfizmami, gdzie kolejne homomorfizmy różnią się na jednym tylko wierzchołku. Sformułowanie naturalnego, topologicznego warunku koniecznego pozwala jednoznacznie scharakteryzować te ścieżki, wzmacniając i uogólniając znany wcześniej algorytm dla znajdowania ścieżek między 3-kolorowaniami do homomorfizmów w dowolny graf bez kwadratów (cykli długości cztery).

W dalszej części badam hipotezę Hedetniemiego: pozornie proste pytanie o liczbę chromatyczną produktu grafów, które pozostaje otwarte od ponad 50 lat. Ogólniej, rozważam tzw. grafy multiplikatywne, czyli elementy 'pierwsze' w kracie grafów (częściowo uporządkowanych relacją istnienia homomorfizmu). Związek tych zagadnień z topologią jest mniej łatwy do dostrzeżenia, ale znajduję zastosowanie dla technik podobnych jak dla rekonfiguracji. Pokazuję nowy dowód multiplikatywności dla wszystkich grafów, gdzie była ona dotychczas znana. Ponadto dowodzę, że podobnie wszystkie grafy bez kwadratów są 'pierwsze', daleko poszerzając tę rodzinę.

Na koniec pokazuję, że hipoteza Hedetniemiego implikuje analogiczne stwierdzenie w topologii. Nowe dowody multiplikatywności unaoczniają, że bazują one na faktach o topologii jednowymiarowej, które do wyższych wymiarów się nie uogólniają; można więc mieć nadzieję na znalezienie topologicznych kontrprzykładów, które zaprzeczyłyby tym samym hipotezie Hedetniemiego. Z drugiej strony znaczy to, że dowód topologicznego wariantu zasadniczo powinien być prostszy, a dowód taki mógłby się stosunkowo łatwo uogólniać na przypadek grafów, tak jak ma to miejsce w obecnie znanych przypadkach. Dowodzę także, że ten topologiczny wniosek z hipotezy Hedetniemiego jest równoważny innemu, słabszemu stwierdzeniu w kombinatoryce.

Te ostatnie wyniki otrzymane są dzięki konstrukcji, która stanowi pewną odwrotność (formalnie: prawe sprzężenie) do operacji brania $k$-tej potęgi grafu (tj. jego macierzy sąsiedztwa). Konstrukcja ta okazuje się mieć również przydatne własności topologiczne: zachowuje topologię przestrzeni odpowiadającej grafowi, a przy tym uszczegóławia jego geometrię, pozwalając na przybliżenie każdej funkcji ciągłej homomorfizmami z tak otrzymanych grafów. Powyższa implikacja hipotezy Hedetniemiego jest wtedy prostym wnioskiem, a jednocześnie otrzymuję dalekie uogólnienie twierdzenia, że rzeczona konstrukcja przyłożona do grafów pełnych zachowuje liczbę chromatyczną.

Wyniki rozprawy można widzieć nie tylko jako krok w badaniach nad ważną hipotezą, ale także jako nowy rozdział na zagadkowym przecięciu kombinatoryki i topologii algebraicznej. Niemniej zamiarem autora rozprawy było również przedstawienie przystępnego wprowadzenia do niektórych połączeń między tymi dziedzinami. Ułatwia to fakt, że użyte metody polegają głównie na elementarnych definicjach z topologii, które można zrozumieć bez przesadnego wysiłku, oraz to, że następują po nich konkretne, kombinatoryczne wnioski.

*Tytuł pracy w języku polskim:*

Topologia przestrzeni rozwiązań problemów kombinatorycznych

*Słowa kluczowe:*

homomorfizmy grafów, hipoteza Hedetniemiego, kombinatoryka topologiczna

*To my parents.*

# Contents

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# Chapter I

# Introduction

The thesis is centered around graph homomorphisms, Hedetniemi's conjecture on the chromatic number of graph products (the statement that $\chi(G \times H) = \min(\chi(G), \chi(H))$), and relations with topology. Let us begin by introducing the notions and notations that appear throughout the thesis, together with some context. Definitions that will be reused are marked on the margin.

## 1. Graph colorings and homomorphisms – basic definitions

Graphs are here understood as pairs $(V(G), E(G))$ where $V(G)$ is a finite set of *vertices* and $E(G)$ is a symmetric binary relation on vertices. For a pair of vertices $u, v \in V(G)$ in the relation, we write this simply as $uv \in E(G)$, we say that $uv$ is an edge of the graph $G$ with endpoints $u$ and $v$, that $u$ and $v$ are *adjacent* to each other and *incident* to the edge $e = uv$. A vertex $v$ may be adjacent to itself; the edge $vv \in E(G)$ is then called a *loop*. The set of vertices adjacent to $v$, $N(v)$
called the *neighborhood* of $v$ is denoted $N_G(v)$ or just $N(v)$; it contains $v$ itself iff there is a loop at $v$. We may also use the more general notion of *digraphs* (for *directed graph*), where the relation $E(G)$ is not necessarily symmetric. A *subgraph* (*induced subgraph*) of $G$ is any graph obtained by removing edges and vertices (resp. only vertices, with all incident edges) from $G$.

A *k-coloring* of a graph $G$, for $k \in \mathbb{N}$, is an assignment of colors to vertices, $f : V(G) \to \{1, \dots, k\}$, coloring
such that adjacent vertices get different colors: $f(u) \neq f(v)$ for $uv \in E(G)$. The *chromatic number* $k$-chrom.
$\chi(G)$ of a graph $G$ is the least $k$ such that a $k$-coloring exists ($\chi(G) = \infty$ if $G$ has a loop). A graph $\chi(G)$
is *$k$-colorable* if a $k$-coloring exists and *$k$-chromatic* if $\chi(G)$ is exactly $k$.

Colorings are of course ubiquitous in graph theory and inspired many of its areas. Their consideration led to a better understanding of a myriad of graph classes and aspects such as sparsity or algebraic graph theory. The study of associated algorithmic problems allows for clear demonstrations of relations between computational complexity and the structure of a problem instance. Colorings also have a multitude of more direct applications. Just to mention one example from the author's work with Bonamy, Kowalik, Nederlof, Pilipczuk, and Socała [Bon+17], a bound on the chromatic number of a certain geometrically-defined class of graphs can be used to bound the number of possible connectivity patterns that can arise in planar digraphs, leading to a faster algorithm.

For two graph $G$, $H$, a *graph homomorphism* from $G$ to $H$, denoted $f\colon G \to H$, is a function $G \to H$
$f\colon V(G) \to V(H)$ such that $uv \in E(G)$ implies $f(u)f(v) \in E(H)$, for all vertices $u, v$. The same definition applies when $G$ and $H$ are digraphs. We write $G \to H$ when any such homomorphism exists. See Figure I.1 for an example. Note that subgraphs of $H$ correspond to injective homomorphisms into $H$. Two graphs are isomorphic, denoted $G \simeq H$, if there are homomorphisms $G \simeq H$
$f : G \to H$ and $g : H \to G$ such that $f \circ g$ and $g \circ f$ are equal to the identity.

Homomorphisms can be thought of as more general colorings, where instead of the condition $f(u) \neq f(v)$, we require that the pair $f(u), f(v)$ is in some fixed relation, which is represented as a graph (or digraph) $H$. Indeed, a $k$-coloring is the same as a homomorphism into the clique $K_n$
graph $K_k$ (defined as $V(K_k) = \{1, \dots, k\}$ and $E(K_k) = \{ij \mid i \neq j \in V(K_k)\}$). In other words, $\chi(G) \leq k$ iff $G \to K_k$. In particular, a graph $G$ is *bipartite* iff $G \to K_2$ (we can take this as the bipartite
definition). For example, the *complete bipartite graph* $K_{n,m}$ has vertex set $\{1_\circ, \dots, n_\circ, 1_\bullet, \dots, m_\bullet\}$ $K_{n,m}$
and edges $i_\circ j_\bullet$ for all $i, j$.

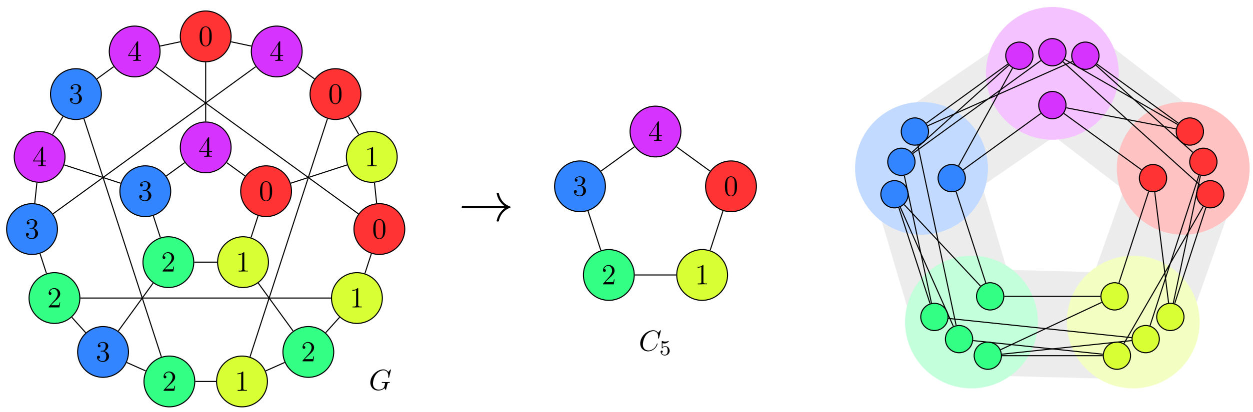



**Figure I.1** A homomorphism from a graph $G$ to the 5-cycle graph $C_5$, depicted as a $C_5$-coloring of $G$ and as an 'embedding' of $G$ in $C_5$. Observe that $C_5 \to G$ also holds.

$H$-col.

Because of this, a homomorphism into a graph $H$ is also called an $H$-coloring, especially when $H$ is fixed in the context. Graph homomorphisms have thus been studied as a generalization of colorings (and many variants thereof), for many of the same reasons. We shall see quite a few applications in combinatorics later, let us mention a few others now. In computer science, the problem of deciding whether a given graph admits a homomorphism into a fixed 'pattern' graph is a basic example of the very general notion of constraint satisfaction problems (CSPs). Graph or digraph homomorphisms correspond to CSPs with just a single type of binary constraint; nevertheless, they can express much of the same. For example, every CSP with a fixed finite family of available constraint types is computationally equivalent to some $H$-coloring problem (for some fixed digraph $H$, decide whether a given digraph is $H$-colorable). See [Bod17; HN08] for details and expositions of the beautiful relations of CSP complexity with universal algebra, presented using graph homomorphisms. In another area, homomorphism problems were the first natural complete problems for the most fundamental classes in algebraic complexity theory [Dur+14; MS16], which allows to define these classes without referring to a computational circuit model. Graph homomorphism are also often used to formalize the notions of 'fitting a pattern' in various settings, eg. in the theory of graph limits [Lov12]. See [HN04] for a monograph on graph homomorphisms or [Neš07] for a survey.

## 2. The structure of graph homomorphisms

A composition of homomorphisms is again a homomorphism, in particular $G \to H$ and $H \to K$ implies $G \to K$. This simple fact entails a rich structure on the notion of graphs and homomorphisms: a poset, a lattice, and a category, which we introduce next. We will not use lattice or category theory in any substantial way, but these structures are home to many constructions and examples, as well as the main context for the questions of this thesis.

### Equivalence and cores

hom. eq.
$G \leftrightarrow H$

Graphs $G$ and $H$ are called *homomorphically equivalent*, denoted $G \leftrightarrow H$, when $G \to H$ and $H \to G$. Two homomorphically equivalent graphs are indistinguishable, as far as we are only concerned with the existence of homomorphisms. We can thus often limit our attention to the smallest graph representing an equivalence class. A graph that is not homomorphically equivalent (core) to any proper subgraph (equivalently: to any graph with fewer vertices or edges) is called a *core graph*. It is not hard to see that every graph $G$ is homomorphically equivalent to a unique (up to isomorphism) core graph, which we call *the core* of $G$. (Note this would not be true in general if we allowed infinite graphs). Furthermore $G$ contains its core as an induced subgraph.

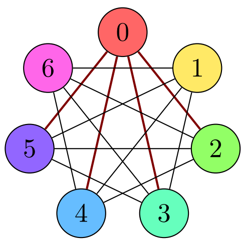
For example, any non-empty graph without edges is (homomorphically) equivalent to $K_1$.

Bipartite graphs are all equivalent to the core graph $K_2$, unless they are edge-less. Similarly, any 3-colorable graph which contains a triangle $K_3$ is equivalent to $K_3$. Finally, any graph trivally admits a homomorphism to the loop graph ($K_1$ with a loop added); hence any graph which contains a loop is equivalent to the loop graph. Examples of core graphs are given by cliques $K_n$ $C_n$
for all $n$ and cycles $C_n$ for odd $n$ (defined as $V(C_n) = \mathbb{Z}_n$ and each $i \in \mathbb{Z}_n$ adjacent to $i \pm 1$).

We remark that the 'equivalence' has to be taken with a grain of salt whenever we talk about algorithms, since testing equivalence or computing the core of a graph is computationally hard (eg. instead of asking whether a graph is 3-colorable, an NP-complete problem, we could equivalently ask whether its disjoint union with $K_3$ is homomorphically equivalent to $K_3$).

### A dense poset

The relation $\to$ defines a preorder on graphs, or a poset (partial order) on equivalence classes of graphs (which can be represented by the unique core graph in the class). It is dense, meaning that for any graphs $G, H$ such that $G < H$ (meaning "$G \to H$ and $H \not\to G$"), there is a graph $K$ such that $G < K < H$ (this holds except for the trivial gaps $K_0 < K_1$ and $K_1 < K_2$).



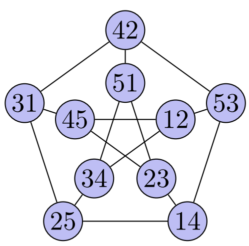

$K_{7/2}$

An example of this which will reappear again and again is the family of *circular cliques*: for $p, q \in \mathbb{N}$ ($p/q > 2$) the graph $K_{p/q}$ has $V(K_{p/q}) = \mathbb{Z}_p$ and $i \in \mathbb{Z}_p$ is adjacent to integers at least $q$ apart: $i+q,\ i+q+1,\ \ldots,\ i+p-q$. They generalize cliques $K_n \simeq K_{n/1}$ as well as odd cycles $C_{2k+1} \simeq K_{(2k+1)/k}$. The basic fact about circular cliques is that $K_{p/q} \to K_{p'/q'}$ if and only if $\frac{p}{q} \le \frac{p'}{q'}$. They thus refine the chain of cliques, corresponding to rational numbers between integers, eg.: $K_{p/q}$

$$K_2 \to \quad \cdots \to C_9 \to C_7 \to C_5 \to C_3 = K_3 \to K_{7/2} \to K_4 \to K_{9/2} \to K_5 \to \cdots$$

It is hence natural to define the *circular chromatic number* $\chi_c(G)$ of a graph $G$: the infimum over $\frac{p}{q}$ such that $G \to K_{p/q}$. Since $\chi(G) = \lceil \chi_c(G) \rceil$, it has been studied as a refinement of the chromatic number; see [Zhu01] for a survey.

Another example in this poset, which will also be useful later, is the existence of incomparable graphs: $G, H$ such that $G \not\to H$ and $H \not\to G$. One way to obtain them is to consider two graph parameters: the chromatic number $\chi$ and the *odd girth*, which is the length of the shortest odd cycle ($\infty$ if there is no odd cycle), which we temporarily denote as $\mathrm{og}(G)$. Since the chromatic number can be defined by homomorphism into cliques, it is easy to see that $G \to H$ implies $\chi(G) \le \chi(H)$. Dually, since the odd girth of $G$ is the least odd integer $k$ such that $C_k \to G$, we have that $G \to H$ implies $\mathrm{og}(G) \ge \mathrm{og}(H)$. Therefore, if $\chi(G) > \chi(H)$ and $\mathrm{og}(G) > \mathrm{og}(H)$, then the graphs $G$ and $H$ are incomparable. Such graphs exist, in fact there are several well known sequences of graphs with strictly increasing $\chi$ and og, giving an infinite set of incomparable graphs (an antichain) in the poset. (This can also be used to show that the poset is dense [HN04, Theorem 3.30]). Perhaps the best known is the randomized construction of Erdős (which in fact yields graphs with arbitrarily high $\chi$ and no short cycles, odd or even).



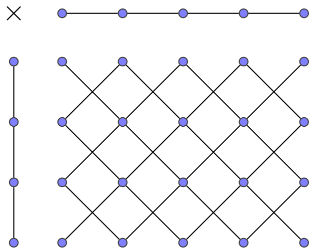

$KG_{5,2}$

A more explicit example is given by *Kneser graphs*: the graph $KG_{n,k}$ has a vertex for every $k$-element subset of $\{1, \ldots, n\}$, with an edge between any two subsets that are disjoint. The figure shows $KG_{5,2}$, also known as the Petersen graph. Showing a lower bound on the odd girth is not trivial already, while the exact lower bound on the chromatic number, $\chi(KG_{n,k}) = n - 2k + 2$, was a conjecture of Kneser, originally phrased without graphs. Suppose we are given objects labeled each by at least $k$ out of $n$ possible labels, and we want to partition the objects so that every two objects in one part share some label. How many parts do we need, in the worst case? The worst case is when we have an object for every $k$-element subset of labels, and the partition then corresponds to a coloring of the Kneser graph. $KG_{n,k}$

The exact bound was ultimately proved by Lovász using topology, as we shall describe later. Incidentally, another classical result in combinatorics, the Erdős–Ko–Rado theorem, describes maximum independent sets (sets of pairwise non-adjacent vertices) in a Kneser graph. Homomorphisms into Kneser graphs also correspond to an independently studied parameter, known as the *fractional chromatic number* $\chi_f(G)$. It can be defined as a fractional variant (the optimum value of a linear programming relaxation) of the standard chromatic number, or as the infimum over $\frac{n}{k}$ such that $G \to KG_{n,k}$.

### The product

The *tensor product* $G \times H$ of graphs $G, H$ is defined as the graph with vertex set $V(G) \times V(H)$ and with $(g, h)$ adjacent to $(g', h')$ whenever $gg' \in E(G)$ and $hh' \in E(H)$. (This corresponds to taking the Kronecker product of adjacency matrices). The figure shows the product of two path graphs (the *path on $n$ vertices* $P_n$ is defined as $C_n$ with one edge removed). $G \times H$ $P_n$



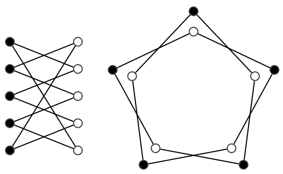
$P_4 \times P_5$

For $v = (g, h) \in G \times H$, the projections $v|_G := g$ and $v|_H := h$ are graph homomorphisms $G \times H \to G$ and $G \times H \to H$, respectively. It is easy to see that $I \to G \times H$ iff $I \to G$ and $I \to H$, for any $I, G, H$. This means the tensor product is the meet (the greatest lower bound) in the poset of (equivalence classes of) graphs. Dually, the disjoint union of graphs, denoted $G \uplus H$, acts as the join (the least upper bound), since $G \uplus H \to I$ iff $G \to I$ and $H \to I$. Together, this is easily checked to be a distributive lattice. $v|_G$ $G \uplus H$

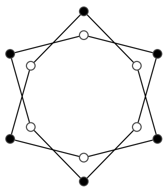
The *category of graphs* has graphs as objects and graph homomorphisms as arrows (morphisms) between them. If we identified all arrows from $G$ to $H$ for each pair $G, H$, disregarding differences between distinct homomorphisms, we would get the above poset (in this context also called the *thin* category), so the category gives a richer structure. The tensor product satisfies the category-theoretic definition of a product (namely, for any graph $I$ with homomorphisms $f : I \to G$ and $f' : I \to H$, there is a unique $f'' : I \to G \times H$ such that $f(\cdot) = f''(\cdot)|_G$ and $f'(\cdot) = f''(\cdot)|_H$). Similarly for the disjoint sum.

The meet in the graph poset is unique up to homomorphic equivalence and the categorical product is unique even up to graph isomorphism; this justifies calling $G \times H$ *the* product, at least in the context of graph homomorphisms, and indeed it is the only graph product we shall consider.

To mention one example, we will often use the tensor product with $K_2$, (also known as the *bipartite double cover* of a graph). Observe that $G \times K_2$ is connected iff $G$ is connected and non-bipartite.



$C_5 \times K_2 \simeq C_{10}$



$C_6 \times K_2 \simeq C_6 \uplus C_6$

### The exponential graph

One way to define a *space of all $K$-colorings* of a graph is the following construction. For two graphs $H, K$, the *exponential graph* $K^H$ has a vertex for each function $V(H) \to V(K)$ (not only homomorphisms), with $f, g : V(H) \to V(K)$ adjacent if they define a $K$-coloring of $H \times K_2$, that is, if $f(u)g(v) \in E(K)$ for all $uv \in E(H)$. The exponential graph was used to study properties of homomorphisms, especially in a category-theoretic setting; several applications are described in [HN04]. $K^H$

Observe that $f : V(H) \to V(K)$ is a homomorphism iff $K^H$ has a loop at the vertex $f$. Any subgraph $G$ of $K^H$ corresponds to a $K$-coloring of $G \times H$, see Figure I.2. In fact, $G \to K^H$ iff $G \times H \to K$, for all graphs $G, H, K$. A graph with this property is unique up to homomorphic

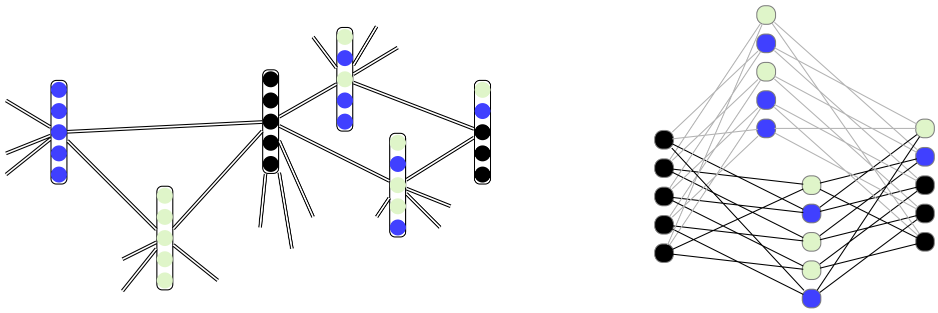


**Figure I.2** Left: a subgraph of the exponential graph $K_3^{C_5}$; each vertex is represented as a column vector with dark red, blue, and light green representing $V(K_3)$. Right: a $K_3$-coloring of $C_4 \times C_5$ corresponding to the $C_4$ subgraph visible in the exponential graph.

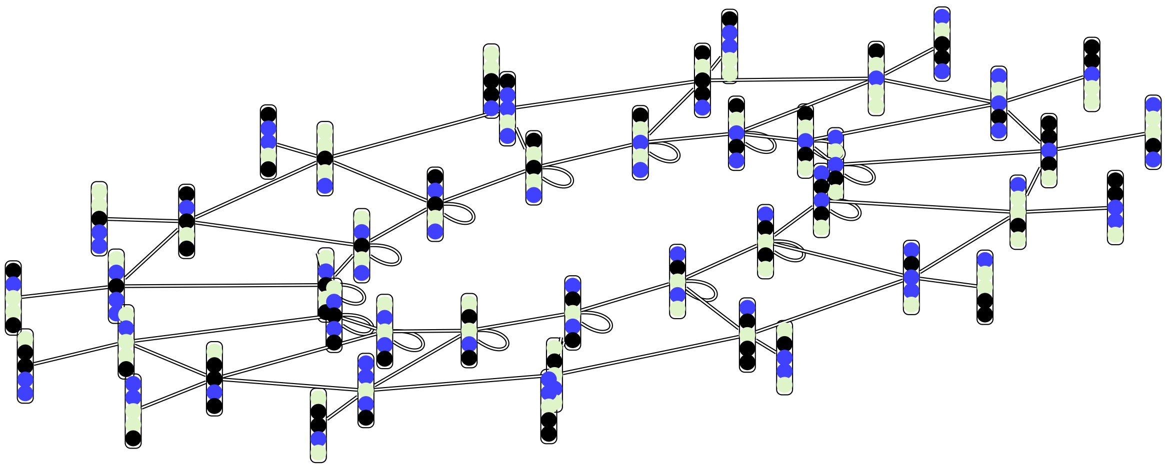


**Figure I.3** One of the three connected components of the exponential graph $K_3^{C_5}$. The 15 looped vertices in the central cycle correspond to proper 3-colorings of $C_5$ that 'wind' once clockwise around $K_3$. Note that consecutive ones only differ at one row.

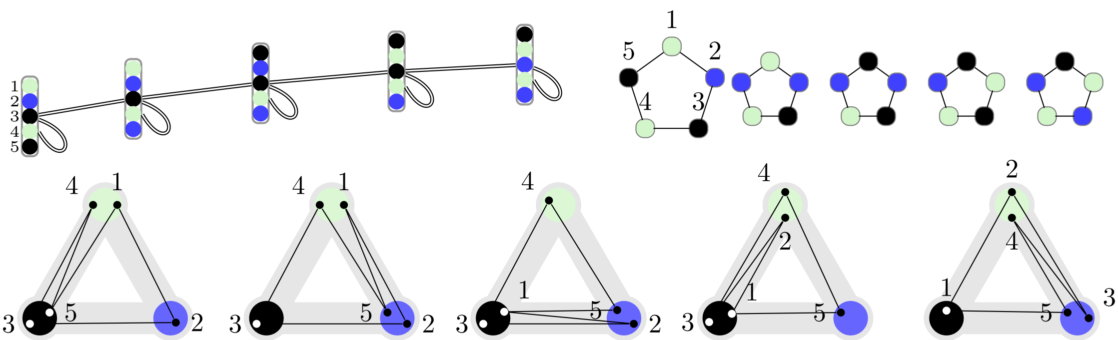


**Figure I.4** A path between homomorphisms $C_5 \to K_3$, as a path on looped vertices in $K_3^{C_5}$, as a recoloring sequence, and as a sequence of 'embeddings' of $C_5$ winding clockwise in $K_3$.

equivalence in the graph lattice; the fact that it always exists makes the lattice a so-called *Heyting algebra*. It also satisfies the category-theoretic definition of an exponential object, which makes it unique up to isomorphism, and gives the category of graphs the structure of a so-called *cartesian closed category*. Many equalities analogous to integers follow, eg. $K^{G \uplus H} \simeq K^G \times K^H$.

Reachability in $K^H$ will turn out to coincide with reachability in another object that represents the space of all $K$-colorings of $H$, this time in the form of a topological space. Moreover, we will later see that paths in $K^H$ correspond to sequences between $K$-colorings of $H$ (or of $H \times K_2$) where subsequent colorings differ at only one vertex, see Figure I.4. Thus already connected components of $K^H$ give us interesting information on which $K$-colorings can be obtained by a simple 'local search' process. The remaining chapters implicitly study the global structure of exponential graphs, in particular we'll see that Hedetniemi's conjecture asks about the $K$-colorability of $K^H$ itself. The main result of Chapter II will be a characterization of reachability in $K^H$ by topological invariants, which will yield an algorithm for finding paths between $K$-colorings. The same understanding will prove useful in later chapters as well, when studying the conjecture.

## 3. Hedetniemi's conjecture and multiplicative graphs

In 1966, Stephen T. Hedetniemi [Hed66] made one of the first attempts at a more comprehensive treatment of graph homomorphisms. In his work, he stated the following conjecture:

**3.1 Conjecture.** *$\chi(G \times H) = \min(\chi(G), \chi(H))$, for all graphs $G, H$.*

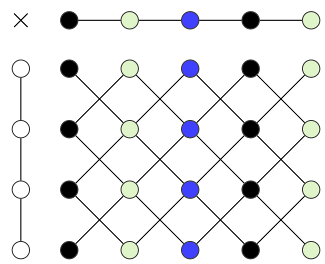
Despite the simplicity of the statement and years of research, only very partial results are known. One of the strongest is a proof for the case when $G \times H$ is 3-colorable, by El-Zahar and Sauer [ES85].[1] We refer the reader to [Zhu98; Sau01; Tar08] for surveys.

The relevance of the conjecture is best seen in the context of graph homomorphisms. Since a $k$-coloring is the same as a homomorphism into the clique $K_k$, the following is an equivalent statement:

$$G \times H \to K_k \text{ if and only if } G \to K_k \text{ or } H \to K_k \quad (\text{for all } G, H, k)$$

The same motivations will apply to the more general question of which graphs satisfy the same property. Namely, we say that a graph $K$ is *multiplicative* if, for all graphs $G, H$: multipl. graph

$$G \times H \to K \text{ if and only if } G \to K \text{ or } H \to K$$



Let us present different aspects of multiplicativity and familiarize ourselves with the concept by showing a few equivalent statements. Since projections give homomorphisms $G \times H \to G$ and $G \times H \to H$, the right side trivially implies the left one; the figure shows how a 3-coloring of a path yields a 3-coloring of a product. In other words, the inequality $\chi(G \times H) \leq \min(\chi(G), \chi(H))$ always holds. Hence multiplicativity is equivalent to the other direction:

$$G \times H \to K \text{ implies } G \to K \text{ or } H \to K$$

There are graphs that are not multiplicative. One construction is the following: take any incomparable graphs $G, H$, that is, $G \not\to H$ and $H \not\to G$ (such as the triangle $K_3$ and the Kneser graph $KG_{8,3}$). Let $K = G \times H$. Then clearly $G \times H \to K$, but say $G \to K$ would imply $G \to G \times H \to H$. In a sense, every example is of this form. Indeed, if $K$ is not multiplicative

[1] Since the direction $\chi(G \times H) \leq \min(\chi(G), \chi(H))$ is trivial, as shown on the figure next, it follows that the conjecture is also true for all 4-colorable graphs $G, H$, as the title of [ES85] indicates. This view may be a bit misleading however, since no insight is gained into the existence of 4-colorings.

for $G, H$, then these are not comparable, since $G \to H$ easily implies $G \to G \times G \to G \times H \to K$. Moreover, $K$ is then homomorphically equivalent to a product of two incomparable graphs, namely $K \leftrightarrow (G \uplus K) \times (H \uplus K)$, as is easy to check. A graph $K$ can be equivalent to the product of two comparable graphs $G \to H$ only in the trivial case when $G$ alone is already equivalent to $K$ (because $G \to G \times G \to G \times H \to G$), hence we may say that a graph $K$ is multiplicative if and only if:

$K$ is not homomorphically equivalent to a non-trivial product.

In other words, multiplicative graphs are those that cannot be factored, the prime elements in the graph lattice (these are called *meet prime* or *meet-irreducible* elements in lattice theory). Note that it is easy to recognize a graph which is (up to isomorphism) a product of two others; the hardness comes from considering graphs up to homomorphic equivalence. In particular, while this formulation is interesting as a motivation, it does not seem useful in *proving* multiplicativity. Dually, we remark that the *join prime* or *join irreducible* elements of the graph lattice are simply the connected graphs, up to homomorphic equivalence (this amounts to the observation that if $K$ is connected then $K \to G \uplus H$ implies $K \to G$ or $K \to H$).

Another way to look at multiplicative graphs is to consider 'obstructions' to colorings. For example, a graph is non-bipartite, $G \not\to K_2$, if and only if it contains an odd cycle (as a homomorphic image or equivalently, in this case, as a subgraph). We can formalize this property by saying that a graph $K$ has *chain duality* (this is not a standard name) if there is a sequence of graphs $O_1 \leftarrow O_2 \leftarrow O_3 \leftarrow \dots$ ('decreasing' in the homomorphism order) such that for all $G$, we have $G \not\to K$ iff $\exists_i\, O_i \to G$. Similar characterizations of homomorphism *to* a graph by obstructions in the form of homomorphisms *from* other graphs have been studied under the name of *dualities* (though usually only for finitely many obstructions); they are extensively discussed in Hell and Nešetřil's monograph [HN04]. It turns out that a graph $K$ is multiplicative if and only if:

$K$ has chain duality.

Indeed, if $K$ has chain duality, then $G \not\to K$ and $G \not\to K$ imply some obstructions $O_i \to G$ and $O_j \to H$; then $O_{\max(i,j)} \to O_i \times O_j \to G \times H$, hence $G \times H \not\to K$. In the other direction, if $K$ is multiplicative, then one can set $O_i$ to be the tensor product of all non-$K$-colorable graphs on at most $i$ vertices; chain duality is then easily checked. Intuitively, Hedetniemi's conjecture is thus the question of whether any two obstructions to $k$-coloring can be generalized by a common obstruction earlier in the homomorphism poset. Unfortunately this view has not been successful, since even the known proofs of multiplicativity of $K_3$ do not display any intuitive obstruction. Moreover, one cannot hope for something as simple as the odd cycles in case of $K_2$-colorings, because this would imply a polynomial-time algorithm for deciding $K$-colorability, whereas the Hell and Nešetřil theorem [HN90] states that this problem is NP-complete for all $K$ except the trivial cases ($K_2, K_1, K_0$, the loop graph, and graphs homomorphically equivalent to them: bipartite graphs and graphs containing loops). However, this does indicate that the most puzzling cases are products of two graphs for which obstructions to $K$-coloring were proved in very different ways. Indeed, various lower bounds on the chromatic number, eg. fractional or topological, are known to extend from two graphs to their product, but mixing different lower bounds, even if they are well understood, seems as hard as the general case of Hedetniemi's conjecture.

Finally, a more helpful restatement comes from the exponential graph. $K$ is multiplicative iff:

$$K^H \to K, \text{ for all } H \text{ such that } H \not\to K.$$

Indeed, the exponential graph has a natural homomorphism $H \times K^H \to K$ (the 'evaluation', $(h, f) \mapsto f(h)$), so multiplicativity of $K$ implies the above statement. In the other direction, if $K$ is not multiplicative, let $G, H$ be a counter-example to its multiplicativity, that is, $G \not\to K$, $H \not\to K$ and $G \times H \to K$. Then the latter means $G \to K^H$, hence $G \not\to K$ implies $K^H \not\to K$.

This proves the other direction and moreover, it shows that if $G \times H$ is a counter-example to the multiplicativity of $K$, then so is $K^H \times H$. In this sense exponential graphs are the hardest cases of Hedetniemi's conjecture!

Recall that $K$-colorings of $H$ correspond to loops in $K^H$, so in particular $H \to K$ iff the graph $K^H$ contains a loop, and is thus homomorphically equivalent to the loop graph. Moreover, there is always a homomorphism $K \to K^H$ (mapping a vertex $v$ of $K$ to the function with constant value $v$). Therefore, $K$ is multiplicative if and only if (letting $C_1$ denote the loop graph):

$$\text{either } K^H \leftrightarrow C_1 \text{ or } K^H \leftrightarrow K, \text{ for all } H.$$

In this sense, Hedetniemi's conjecture is all about the global structure of exponential graphs (sometimes called *coloring graphs* for $K = K_n$). This thesis will present how topological invariants can influence and in many cases characterize connected components of these spaces of homomorphisms, and how this can be used to prove the multiplicativity of different graphs.

Before this thesis, the only non-trivial graphs known to be multiplicative were (up to homomorphic equivalence) $K_3$, as El-Zahar and Sauer [ES85] showed, odd cycles, as generalized by Häggkvist et al. [Häg+88], and circular cliques $K_{p/q}$ with $p/q < 4$, as proved by Tardif [Tar05]. In Chapter III we give a new, uniform proof for all these cases. Furthermore, we show that all square-free graphs (ie. without $C_4$ as a subgraph) are multiplicative. This in particular gives the first multiplicative graphs of chromatic number greater than 4, since graphs of girth at least 5 are square-free, but can have arbitrarily high chromatic number. In Chapter IV we also show as a corollary that 3rd powers of graphs of girth $> 12$ are multiplicative. Nevertheless, the next case of Hedetniemi's conjecture, the multiplicativity of $K_4$, is still frustratingly wide open.

On a side note, Tardif's [Tar05] result can be restated in terms of the circular chromatic number: $\chi_c(G \times H) = \min(\chi_c(G), \chi_c(H))$ whenever $\chi_c(G \times H) < 4$. Zhu [Zhu98] conjectured that this equality, a strengthening of Hedetniemi's conjecture, holds in general. Zhu also later proved the analogous conjecture for the fractional chromatic number $\chi_f$ [Zhu11]. Recently the same has been proven for semi-definite programming relaxations of the chromatic number: the Lovász $\bar{\theta}$ function (also known as the vector chromatic number) [God+16] and Schrijver's $\bar{\theta}'$ function (also known as the strict vector chromatic number) [God+18].

## 4. Topological combinatorics

Many of the proofs in this thesis (all of Chapters II, III) will be formally combinatorial, in the sense that they only refer to finite objects, but the intuitions behind them heavily rely on some basics from algebraic topology, which we introduce here.

By this point the author hopes that an escapade into the field of topology looks like it may be worth the effort, even for the most entrenched combinatorialist. The intersection of the two fields may seem paradoxical at first, especially since what is understood under the name *topological combinatorics* is usually applications of algebraic topology to problems that were at least originally purely combinatorial. We are thus not interested here in other areas of more abstract topology as used in logic, eg., in descriptive set theory. Nor are we interested in planar graphs or other graphs embedded on a surface, as in graph minor theory (though this division is more arbitrary and fuzzy). Still, some more combinatorial problems turn out to benefit from, or perhaps even require, a topological point of view. Books by Matoušek [Mat08] and de Longueville [Lon13] provide an elementary yet extensive introduction to many of them, see also [BMZ17; Loe+17].

The principal example of this, and the one relevant to this thesis, is the area originating from Lovász' surprising proof [Lov78] of Kneser's conjecture. As presented before, Kneser's question was originally about partitioning a family of finite sets, but is conveniently rephrased as follows: can the Kneser graph $KG_{n,k}$ be colored with less than $n - 2k + 2$ colors? In other words, is it true that $KG_{n,k} \to K_{n-2k+1}$? Lovász' negative answer was followed by many alternative proofs, all based on variants of the classical Borsuk-Ulam theorem, the shortest and simplest of which may be by Greene [Gre02]; a combinatorial proof (though more technical and following the same intuitions) was given by Matoušek [Mat04] (see also [Ais+15] for a partial but purely combinatorial proof). The answer spawned a whole area of research on similar topological lower bounds on the chromatic number and other results obtained with related techniques. The modern view on it is usually through equivariant topology and the box complex, which we introduce next.

## Basic definitions in topology

For topological spaces $X, Y$, we call a continuous function $f : X \to Y$ a *map*, for short. Two maps $f, g : X \to Y$ are *homotopic* if they can be continuously transformed into one another; formally: there is a family of maps $\phi_t : X \to Y$ for $t \in [0, 1]$ (called a *homotopy*) such that $\phi_0 = f$, $\phi_1 = g$ and such that the function $(t, x) \mapsto \phi_t(x)$ from $[0, 1] \times X$ to $Y$ is continuous. Two spaces $X, Y$ are *homotopy equivalent* if there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g$ and $g \circ f$ are homotopic to identity maps on $X$ and on $Y$; such spaces are equivalent for all of our purposes.

map

homotop.

homot. eq.

We shall only consider topological spaces described in the following simple combinatorial way. A *(simplicial) complex* $K$ is a family of non-empty finite sets that is downward closed, in the sense that $\emptyset \neq \sigma' \subseteq \sigma \in K$ implies $\sigma' \in K$. The sets in $K$ are called *faces* (or *simplices*) of the complex, while their elements $V(K) := \bigcup_{\sigma \in K} \sigma$ are the *vertices* of the complex. The *geometric realization* $|\sigma|$ of a face $\sigma \in K$ is the subset of $\mathbb{R}^{V(K)}$ defined as the convex hull of $\{e_v \mid v \in \sigma\}$, where $e_v$ is the standard basis vector corresponding to the $v$ coordinate in $\mathbb{R}^{V(K)}$. The geometric realization $|K|$ of $K$ is the topological space obtained as the subspace $\bigcup_{\sigma \in K} |\sigma| \subseteq \mathbb{R}^{V(K)}$. We often refer to $K$ itself as a topological space, meaning $|K|$.

complex

face

vertex

geom. real.

For example, a circle may be represented as the triangle $K = \{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}\}$, meaning that $|K|$, which is the sum of three intervals in $\mathbb{R}^3$, is homotopy equivalent to the unit circle in $\mathbb{R}^2$. Adding the face $\{1, 2, 3\}$ to $K$ would make $|K|$ contractible, that is, homotopy equivalent to the one-point space.

## Equivariant topology – topology with symmetries

It turns out to be easier to work with equivariant topology, that is, considering topological spaces together with their symmetries and symmetry-preserving maps. A $\mathbb{Z}_2$*-space* is a topological space $X$ equipped with a map $\nu : X \to X$, called a $\mathbb{Z}_2$-action on $X$, satisfying $\nu(\nu(x)) = x$ (for all $x \in X$). The main example is the $n$-dimensional *sphere*: the $\mathbb{Z}_2$-space defined as the unit sphere in $\mathbb{R}^{n+1}$ with $\mathbb{Z}_2$-action $x \mapsto -x$ (the antipodal action). We often refer to $X$ itself as a $\mathbb{Z}_2$-space when the $\mathbb{Z}_2$-action is clear from the context.

$\mathbb{Z}_2$-. . .

$\mathcal{S}^n$

A $\mathbb{Z}_2$*-map* from $(X, \nu_X)$ to $(Y, \nu_Y)$ is a map $f : X \to Y$ that preserves the symmetry: $f(\nu_X(x)) = \nu_Y(f(x))$ (this is also called an *equivariant* map). We write $(X, \nu_x) \to_{\mathbb{Z}_2} (Y, \nu_Y)$ if such a map exists. This gives a highly non-trivial relation on spaces (as opposed to the existence of just maps, since mapping everything to one point would always work). In particular, a version of the Borsuk-Ulam theorem says that there is no $\mathbb{Z}_2$-map from a higher-dimensional sphere to a lower-dimensional one: $\mathcal{S}^m \not\to_{\mathbb{Z}_2} \mathcal{S}^n$ for $m > n$. It remains to define how graphs can be turned into $\mathbb{Z}_2$-spaces, so that graph homomorphisms induce $\mathbb{Z}_2$-maps between them: this is the role of the box complex.

$X \to_{\mathbb{Z}_2} Y$

Standard notions extend in a fairly straightforward way to equivariant notions. A $\mathbb{Z}_2$*-complex* is a simplicial complex $K$ together with a function $\nu : V(K) \to V(K)$ such that $\nu(\nu(v)) = v$; the corresponding space $|K|$ is then a $\mathbb{Z}_2$-space with a $\mathbb{Z}_2$-action defined by extending $e_v \mapsto e_{\nu(v)}$ linearly on each face $|\sigma|$ of the geometric realization. The *product* of two $\mathbb{Z}_2$-spaces $X, Y$ is $X \times Y$ with 'simultaneous' $\mathbb{Z}_2$-action $(x, y) \mapsto (\nu_X(x), \nu_Y(y))$. A homotopy $\phi_t$ between $\mathbb{Z}_2$-maps $f, g : X \to Y$ is called a $\mathbb{Z}_2$*-homotopy* if $\phi_t$ is a $\mathbb{Z}_2$-map for all $t \in [0, 1]$; $f, g$ are then $\mathbb{Z}_2$*-homotopic.*

$X \times Y$

We say that two $\mathbb{Z}_2$-spaces $X, Y$ are $\mathbb{Z}_2$-homotopy equivalent, denoted $X \simeq_{\mathbb{Z}_2} Y$, if there are $\mathbb{Z}_2$-maps $f : X \to_{\mathbb{Z}_2} Y$ and $g : Y \to_{\mathbb{Z}_2} X$ such that $f \circ g$ and $g \circ f$ are $\mathbb{Z}_2$-homotopic to the identity on $X$ and on $Y$. Both $f$ and $g$ are then called a $\mathbb{Z}_2$-*homotopy equivalence*. Note this is stronger than just requiring $X \to_{\mathbb{Z}_2} Y$ and $Y \to_{\mathbb{Z}_2} X$; homotopy equivalence is similar to graph isomorphism, not to homomorphic equivalence. $X \simeq_{\mathbb{Z}_2} Y$

## The box complex – the topology of a graph

The box complex is a construction that assigns a topological space $|\mathrm{Box}(G)|$ to a graph $G$. The exact construction will not be important until we get to some more technical proofs, but intuitively, it starts from the edges of $G$ (or rather $G \times K_2$) as a topological space (edges become copies of the unit interval $[0, 1] \subseteq \mathbb{R}$) and then glues faces to each cycle of length 4 and similarly glues higher-dimensional faces to larger complete bipartite subgraphs.

The crucial connection is that a homomorphism $G \to H$ induces a $\mathbb{Z}_2$-map $|\mathrm{Box}(G)| \to_{\mathbb{Z}_2} |\mathrm{Box}(H)|$ (in a straightforward way). This is the modern view of Lovász' proof, which allows to get tight lower bounds on the chromatic number of Kneser graphs. To show that $G$ is not $n$-colorable, suppose to the contrary that $G \to K_n$. Then there is a $\mathbb{Z}_2$-map $|\mathrm{Box}(G)| \to_{\mathbb{Z}_2} |\mathrm{Box}(K_n)|$. One shows that $|\mathrm{Box}(K_n)| \simeq_{\mathbb{Z}_2} \mathcal{S}^{n-2}$ and similarly that $|\mathrm{Box}(G)|$ is or contains a sphere of higher dimension (usually by induction on a parameter defining $G$). Then by the Borsuk-Ulam theorem, one concludes no such map can exist. The same method applies to several families of graphs (eg., Kneser [Lov78], Schrijver [Sch78b], and generalized Mycielski graphs, quadrangulations of the projective plane [You96] and projective spaces [KS15], a certain graph in [GJS04]).

Different constructions can be used in place of the box complex. Lovász [Lov78] originally used just the simplicial complex with $V(G)$ as its vertex set and neighborhoods $N(v)$ for $v \in V(G)$ (and their subsets) as faces, known as the neighborhood complex. He also defined a related, but larger, complex with a $\mathbb{Z}_2$-action. However, Csorba [Cso08] proved that both are equivalent (homotopy and $\mathbb{Z}_2$-homotopy equivalent, respectively) to the box complex, a construction that has the advantages of both: it is small, with only twice as many vertices as $G$, and comes with a $\mathbb{Z}_2$-action.

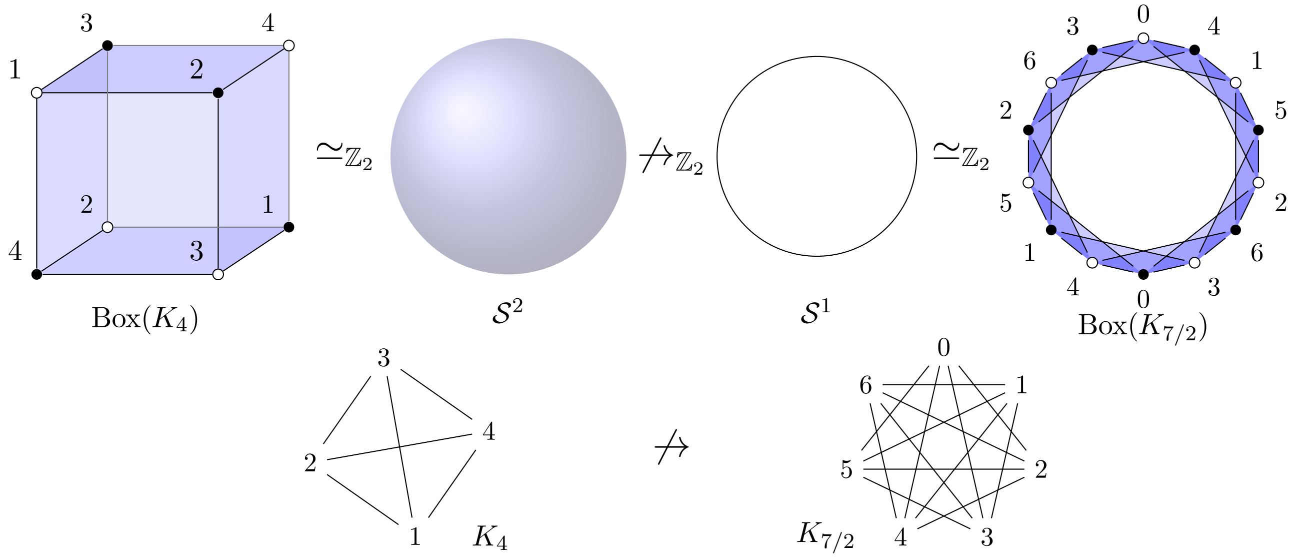



**Figure I.5** The box complex of $K_4$ is the hollow cube (informally speaking; the drawing only shows $K_4 \times K_2$ and the most important faces). It is $\mathbb{Z}_2$-homotopy equivalent to the sphere. The box complex of $K_{7/2}$ is $\mathbb{Z}_2$-homotopy equivalent to the circle. Thus there cannot be a homomorphism from $K_4$ to $K_{7/2}$ (of course in this case it is easier to show this directly).

Formally, the *box complex* $\mathrm{Box}(G)$ of a graph $G$ is a $\mathbb{Z}_2$-complex defined as the family of vertex sets of complete bipartite subgraphs of $G \times K_2$ (with both sides non-empty) and their subsets. In particular it contains all edges of $G \times K_2$ and every $K_{2,2} = C_4$ subgraph. We will write the vertex set of the box complex as $V(G) \times \{\circ, \bullet\}$ (except that isolated vertices, ie. those with no neighbors, are removed, since they do not appear in any face). That is, for every (non-isolated) vertex $v \in V(G)$, the simplicial complex has two vertices, which we denote $v_\circ$ and $v_\bullet$. The $\mathbb{Z}_2$-action $-$ on $\mathrm{Box}(G)$ is defined as $-v_\circ = v_\bullet$ and $-v_\bullet = v_\circ$ for each $v \in V(G)$. See Figure I.5. $\mathrm{Box}(G)$ $v_\circ, v_\bullet$

When drawing examples of box complexes we ignore certain faces (like $K_{1,n}$ subgraphs) and represent 4-vertex faces (which in the geometric realization are formally 3-dimensional tetrahedra) as flat squares for readability. For example in Figure I.5 the face $\{1_\circ, 2_\bullet, 3_\bullet, 4_\bullet\}$ of $\mathrm{Box}(K_4)$ is ignored, while the face $\{1_\circ, 3_\circ\}$ is only implicitly drawn as part of the face $\{1_\circ, 2_\bullet, 3_\circ, 4_\bullet\}$. Formally, the resulting space can be shown to be $\mathbb{Z}_2$-homotopy equivalent by a standard application of discrete Morse theory, which we discuss in Chapter IV.

A graph without loops is *square-free* if it does not have a cycle of length 4 (it excludes $C_4$ as a subgraph, induced or not). In a general graph $G$, we define a *square* to be a quadruple of (possibly equal) vertices $a, b, c, d$ such that $ab, bc, cd, da \in E(G)$. A square is *non-trivial* if $a \neq c$ and $b \neq d$. A graph (with loops allowed) is *square-free* if it has no non-trivial square. Equivalently it has no $C_4$ subgraph, no triangle with a looped vertex, and no edge with both vertices looped. square sq.-free

For square-free graphs $K$, the box complex is essentially just $K \times K_2$ itself. Formally, $\mathrm{Box}(K)$ can be shown to be $\mathbb{Z}_2$-homotopy equivalent to the subcomplex containing only $V(G \times K_2)$ and $E(G \times K_2)$ as faces, giving a 1-dimensional complex. The topology in this case is particularly simple, which is what will allow the result of Chapter II and III to work for square-free graphs $K$ on the right side of homomorphisms $G \to K$ (nevertheless, squares in the left side graph are possible and in fact crucial for the topological arguments). Incidentally, for the same reason topological lower bounds on the chromatic number of square-free graphs are trivial, even though graphs of girth $> 4$ are square-free and may have arbitrarily high chromatic number.

## The Hom complex – the space of homomorphisms

The Hom complex is a construction of a topological space describing all graph homomorphisms between two given graphs. We will not use it in any essential way in proofs, but it may be helpful in understanding the box complex, which it generalizes, as well as the exponential graph, to which it is closely related, justifying calling them *spaces of homomorphisms*. $\mathrm{Hom}(G, H)$ is defined as the simplicial complex with the homomorphisms $G \to H$ as vertices, where a set $\sigma$ of homomorphisms forms a face iff all the ways to mix them still give a homomorphism, that is, $f(u)f'(v) \in E(H)$ for all $uv \in E(G)$ and all $f, f' \in \sigma$.

Observe that a function $f : V(G) \to V(H)$ is a vertex of $\mathrm{Hom}(G, H)$ iff it is a homomorphism iff it is a looped vertex of the exponential graph $H^G$. Moreover, $f, f' : G \to H$ are adjacent in $\mathrm{Hom}(G, H)$ iff they are adjacent in $H^G$. So the reachability relation in $\mathrm{Hom}(G, H)$ is the same as in the subgraph of $H^G$ induced on looped vertices. In fact Kozlov [Koz07] and Dochtermann [Doc09a] showed (among even more general results) that $\mathrm{Hom}(G, H)$ is homotopy equivalent to the looped clique complex of $H^G$ (with looped vertices of $H^G$ as vertices and clique subgraphs as faces).

The complex $\mathrm{Hom}(K_2, G)$ (with a $\mathbb{Z}_2$-action swapping the values of a homomorphism from $K_2$) turns out to be $\mathbb{Z}_2$-homotopy equivalent to the box complex [Cso08], giving yet another description of it, this time with *oriented edges* of $G$ as vertices of the complex.

The Hom complex thus gives an elegant way to connect and generalize these and other notions. For these reasons, in particular in hope of generalizing topological lower bounds on the chromatic number, Hom complexes were extensively studied in topological combinatorics, but also in combinatorial topology, where they give simple description of interesting topological spaces. See eg. Dochtermann and Schultz [Doc09a; DS12], and Kozlov [Koz08a].

One interesting theorem related to the connected components of hom complexes is the following by Brightwell and Winkler [BW04]: if $\mathrm{Hom}(G, H)$ is connected for all $G$ of degree at most $d$, then the chromatic number of $H$ is at least $d/2$ (and is conjectured to be at least $d$).

### Paths between homomorphisms – $\times$-homotopy, recoloring

As mentioned before, different views of a space of homomorphism give the same notion of reachability between graph homomorphisms. For two graphs $G, H$, Dochtermann [Doc09a] defined two homomorphisms $f, g : G \to H$ to be $\times$*-homotopic* if there is a path on looped vertices in $H^G$ between $f$ and $g$. He showed that this is equivalent to being connected by a (topological) path in $\mathrm{Hom}(G, H)$ and more generally: $\times$-homot.

**4.1 Theorem.** ([Doc09a]) *The following are equivalent:*

- $f, g$ *are* $\times$*-homotopic;*
- *For every graph* $T$*, the induced maps* $f_T, g_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ *are homotopic;*
- *The induced maps* $f_G, g_G : \mathrm{Hom}(G, G) \to \mathrm{Hom}(G, H)$ *are homotopic;*
- *For every graph* $T$*, the induced maps* $f^T, g^T : \mathrm{Hom}(H, T) \to \mathrm{Hom}(G, T)$ *are homotopic;*
- *The induced maps* $f^H, g^H : \mathrm{Hom}(H, H) \to \mathrm{Hom}(G, H)$ *are homotopic;*

This reachability relation turns out to be also equivalent to the following simple combinatorial notion, which will bring us back to algorithms and to homomorphisms as constraint satisfaction. We say that $f, g : G \to H$ are $H$*-recolorable* if one can be reached from the other by changing the value at one vertex at a time, that is, if there is a sequence $f = f_1, f_2 \ldots, f_n = g$ such that $f_i : G \to H$ are homomorphisms and for each $i$ there is exactly one $v \in V(G)$ such that $f_i(v) \neq f_{i+1}(v)$ (recall Figure I.4). If $G$ has a loop at $v$ we additionally require that $f_i(v)f_{i+1}(v) \in E(H)$. $H$-recol.

It is easy to see that $f, g$ are $H$-recolorable iff they are $\times$-homotopic. Indeed, a single step between $f_i$ and $f_{i+1}$ in the definition of $H$-recolorability satisfies $f_i(u)f_{i+1}(v) \in E(H)$ for all $uv \in E(G)$, so it corresponds to an edge with looped vertices in $H^G$. Conversely, if $f, g : G \to H$ are looped vertices adjacent in $H^G$, then they are $H$-recolorable by replacing the values of $f$ with values of $g$ one by one, in any order.

We remark that instead of looped vertices in $H^G$, one could consider all of $H^G$. Every oriented edge of $H^G$ corresponds to a $H$-coloring of $G \times K_2$ and reachability between oriented edges in $H^G$ corresponds to $H$-recoloring of $G \times K_2$ (for this to work, an oriented edge $\vec{fg}$ is considered adjacent to $\vec{hg}$, but not to $\vec{gh}$ in general). In practice, this means that any results on the connected components of $\mathrm{Hom}(G, H)$ (equivalently, of the looped subgraph of $H^G$) translate easily to connected components of $H^G$ (or rather $H^G \times K_2$, to account for edge orientations), though we will not use this formally at any point.

### The $H$-Recoloring problem and reconfiguration

$H$-recoloring (in other words, $\times$-homotopy) has been studied by graph theorist and algorithmists as an example of *reconfiguration*: the process of changing a system, usually a solution to some kind of constraint satisfaction problem, by small steps. The motivation here is a better understanding of spaces of solutions, in particular how solutions can be modified by such a 'local search' process to algorithmically find better ones (eg. to gradually satisfy more constraints) or to randomly sample them (eg. to estimate the number of solutions). Similar paths between solutions are studied for SAT or general constraint satisfaction problems and many others, such as Independent Set [HD05; KMM12; BKW14; LM18] or Shortest Path [KMM11; Bon13] reconfiguration; see [Nis17; IS17] for surveys.

Such processes are also studied in statistical physics as modeling particle systems exhibiting interesting phase-transitions (though usually on highly regular graph like grids, but with more quantitative questions). Coloring corresponds to the *zero-temperature anti-ferromagnetic Potts model*, while recoloring corresponds to *Glauber dynamics* in the language of physics, see [BW02]. Physical intuitions in return inspired new approaches in graph theory [BW02] and in algorithmics: see [FV07; Jer98] for surveys on some of them. In particular, spaces of colorings of random graphs (or more generally, spaces of solutions to random constraint satisfaction instances) have been extensively studied, eg. to determine the threshold density for which a random graph becomes $k$-colorable, or for which the space of colorings becomes connected enough to be easily navigated by algorithms using random walks, see eg. [Mol12] for more recent progress. The physics-inspired Survey Propagation and Belief Propagation techniques give some of the most successful heuristics for finding colorings in random instances, see [Bra+06] for an overview.

Algorithmically, for a fixed graph $H$, $H$-Recoloring is the problem where given a graph $G$ and two homomorphisms $G \to H$, one asks whether they are $H$-recolorable. For standard $k$-colorings, that is, $H = K_k$, the problem was shown to be computationally hard (NP-hard, in fact PSPACE-complete) for $k \geq 4$ by Bonsma and Cereceda [BC09], but solvable in polynomial time for $k \leq 3$ by Cereceda et al. [CHJ11]. The latter, positive result is quite surprising, since deciding the existence of a 3-coloring is a basic example of an NP-hard problem. Explaining this through a more general result was one of the motivation for the author's work on $H$-recoloring. $H$-Recol.

Previously, the author considered in his master's thesis [Wro14] the generalization of the $K_k$-Recoloring problem to homomorphisms, that is, to $H$-colorings for arbitrary graphs (or digraphs) $H$. This allows to formalize the question of how the complexity of reconfiguration problems depends on the constraint types (the fixed graph $H$) used to define the solution space and on the structure of how constraints are arranged (the input graph $G$). One could also consider more general constraint satisfaction problems in place of graph homomorphisms; we comment on existing results for the boolean case in Chapter II.

One of the results in [Wro14] was that there is a fixed graph $H$ such that the $H$-Recoloring problem is hard even when $G$ is just a cycle. That is, given $n$ and two homomorphisms $C_n \to H$, deciding whether they are $\times$-homotopic is PSPACE-complete (the same would be true for two paths with fixed endpoints in $H$, instead of two cycles). In particular, this implies that for this particular $H$, no algorithmic characterization of paths in $\mathrm{Hom}(C_n, H)$ can exist, and that for some pairs of homomorphisms $C_n \to H$, shortest paths between them are of length super-polynomial in $n$ (assuming NP $\neq$ PSPACE). On the other hand, some initial, purely combinatorial observations were also given for the case where $H$ is a square-free graph (and $G$ is arbitrary), yielding a polynomial-time algorithm in special cases, but these will not be relevant to this thesis.

## 5. Results

With the context and definitions set up, let us conclude this introduction with a more detailed overview of the results in this thesis.

Chapter II studies paths between homomorphisms $G \to H$ for square-free graphs $H$, in particular the $H$-Recoloring problem. The starting point is essentially the observation that if $f, g : G \to H$ are $H$-recolorable ($\times$-homotopic), then the corresponding continuous maps $|\mathrm{Box}(G)| \to_{\mathbb{Z}_2} |\mathrm{Box}(H)|$ are homotopic. For square-free $H$, we show that the converse is almost true, by giving an exact characterization of $H$-recolorability which includes topological homotopy as the main condition. In other words, if instead of the discrete and rigid graph homomorphisms we consider only the corresponding continuous and stretchable maps, the main obstructions to $H$-recolorability become visible as topological invariants.

In the case of square-free $H$ these can also be described using elementary algebraic topology, which allows use to give an algorithm for $H$-Recoloring, generalizing the polynomial-time algorithm for $K_3$-Recoloring and giving a topological interpretation to the invariants originally used. The concrete algebraic and algorithmic description is also a good introduction to the tools we use in the next chapter.

Chapter III gives a new proof of the multiplicativity for all graphs previously known to be multiplicative, and follows with a proof that all square-free graphs are multiplicative. A certain parity invariant used in El-Zahar and Sauer's original proof [ES85] of the main known case of Hedetniemi's conjecture (the multiplicativity of $K_3$) is interpreted as a topological invariant, which allows to directly extend the same proof to circular cliques $K_{p/q}$ with $\frac{p}{q} < 4$. This gives a substantially different proof from Tardif's [Tar05]. We then use these tools to show the multiplicativity of all square-free graphs. This substantially increases the family of known multiplicative graphs, for example giving the first multiplicative graphs of chromatic number greater than 4 (though no direct progress on Hedetniemi's conjecture, the case of cliques, is made).

Chapter IV first takes a detour and considers an operation on graphs $\Omega_k$ which can be defined as a certain inverse (formally, the right adjoint in the graph category) to taking the $k$-th power of a graph (or of its adjacency matrix). This construction has been previously used in Tardif's [Tar05] proof of the multiplicativity of circular cliques, as well as in a few other combinatorial theorems. We show that the operation has remarkable topological properties: it preserves the topology of the box complex (its $\mathbb{Z}_2$-homotopy type) and moreover refines its geometry, allowing to approximate any continuous map from a box complex of $G$ to a box complex $H$ by a graph homomorphism from a refinement $\Omega_k(G)$ (with $k$ sufficiently large) to $H$. This gives a combinatorial characterization of any 'topological' property of graphs, such as topological lower bounds on the chromatic number.

We use this to show that Hedetniemi's conjecture implies an analogous conjecture in equivariant topology. This was recently independently proved by Matushita [Mat17a]. We argue that this is a substantial step in understanding the conjecture and multiplicativity in general: any counter-example to the topological statement would immediately refute Hedetniemi's conjecture, while any proof should be in principle easier and may be a first step to an extension to the combinatorial case, as in Chapter III. While further progress may require more advanced use of algebraic topology, we also discuss some other avenues. In particular, we use $\Omega_k$ in a combinatorial proof showing that powers of graphs of high girth are multiplicative.

The results of Chapter II were presented at STACS 2015 [Wro15], Chapter III was published in J. Comb. Theory B [Wro17b], while Chapter IV is available as a preprint [Wro17a], except for Theorem 1.8, which is joint work with Claude Tardif and will appear in a future publication.

# Chapter II

# Recoloring, or paths between homomorphisms

## 1. Introduction

This chapter considers paths between graph colorings and homomorphisms. For a fixed graph $H$, we consider the algorithmic problem of $H$-RECOLORING: given a graph $G$ and two $H$-colorings of $G$ (two homomorphisms from $G$ to $H$), can one be transformed into the other by changing the color of one vertex at a time? Of course we require maintaining a valid $H$-coloring throughout, see Figure II.1. As introduced in Chapter I, this is the same as finding a path in the $\mathrm{Hom}(G, H)$ complex, a looped path in the exponential graph $H^G$, or deciding whether two given homomorphisms are $\times$-homotopic. In the SHORTEST $H$-RECOLORING problem one is additionally given an integer $\ell$ and the question is whether the transformation can be done in at most $\ell$ steps (single vertex color changes).

For classical colorings, that is, $H = K_k$, $H$-RECOLORING was shown to be NP-hard (in fact PSPACE-complete) when $k \geq 4$ by Bonsma and Cereceda [BC09] and in P (solvable in polynomial time) when $k \leq 3$ by Cereceda et al. [CHJ11]. The latter result was improved by Jonhson et al. [Joh+16] to show that SHORTEST $K_3$-RECOLORING is also in P. Note that this holds despite the fact that deciding the *existence* of a 3-coloring is an emblematic NP-complete problem (and it is NP-complete even on graphs as simple as 4-regular planar graphs [Dai80]). We generalize this positive result by providing an algorithm that solves SHORTEST $H$-RECOLORING in polynomial time for any square-free $H$, even if $H$ is given on input. The algorithm uses a characterization of possible paths between homomorphisms $G \to H$ (Theorem 6.1), whose main part is a purely topological condition, which can be interpreted as requiring corresponding continuous maps to be homotopic.

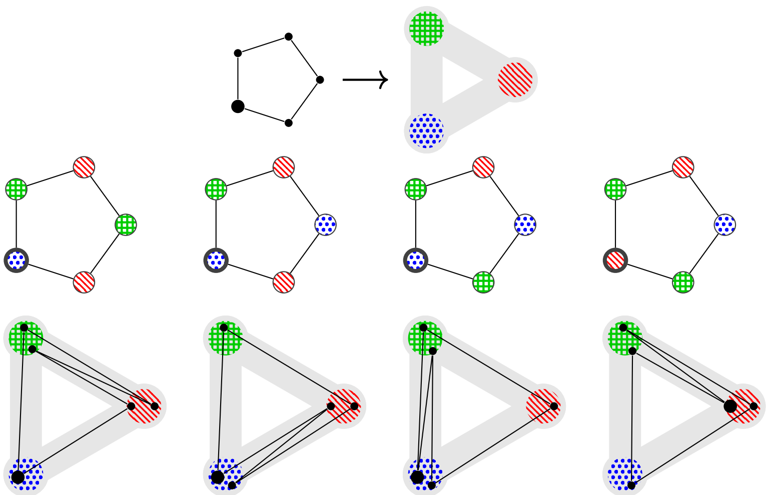


**Figure II.1** A sequence of 3-colorings of $C_5$ and the same sequence seen as a $K_3$-recoloring sequence of homomorphisms from $C_5$ to $K_3$ (a graph with three vertices: striped red, checkered green, dotted blue). One vertex of $C_5$ is thickened for clarity.

Motivations for studying spaces of homomorphisms, from topological combinatorics, graph theory and algorithmics, to statistical physics, have been introduced in Chapter I. While algorithmically finding paths between solutions is, for these purposes, mostly a toy problem, it arises more directly in some settings. Indeed, the Nondeterministic Constraint Logic construction of Hearn and Demaine [HD05], which gives a simple PSPACE-complete reconfiguration problem, allowed to show that many popular puzzles (eg. with sliding blocks) are PSPACE-complete [HD09; Meh14]. More interestingly, Heijltjes and Houston used the construction to prove that deciding the equivalence of proofs in a certain proof system is PSPACE-complete [HH14], answering a question about normal forms of proofs that arose in this context.

Another aim of this study is to give more general statements about reconfiguration, seeing graph homomorphisms as a tool to explore how different constraints influence the complexity of reconfiguration. This approach previously allowed the author to argue in [Wro18] that the only notion of sparseness that can be applied algorithmically to (unparameterized) reconfiguration problems in general is tree-depth, and that many such problems are PSPACE-complete even in graphs of bounded bandwidth (thus pathwidth, treewidth, etc.). The reduction presented there explains why reconfiguration variants of easy combinatorial problems can be hard. Dually, this chapter grew out of an attempt to find reasons for which reconfiguration variants of hard problems can be, quite surprisingly, easy, generalizing the case of 3-colorings.

### Related work

Brewster et al. [Bre+16] recently generalized the dichotomy for classical colorings to circular colorings. Namely, for $H$ being a circular clique $K_{p/q}$ with $p/q < 4$, $H$-Recoloring is in P, while for $p/q \geq 4$ it is PSPACE-complete. Later, Brewster et al. [Bre+17] showed that $H$-Recoloring is PSPACE-complete for odd wheel graphs $H$ (odd-length cycle graphs with an additional vertex adjacent to all others). They also characterized graphs $H$ for which the problem of deciding whether a given graph admits a frozen $H$-coloring is NP-complete. (An $H$-coloring of $G$ is *frozen* if it is isolated in the space of all $H$-colorings, that is, no single vertex can change its color while maintaining a proper $H$-coloring.)

For more general constraint satisfaction problems, mostly the Boolean domain has been considered, that is, the reconfiguration of SAT problems. Here solutions are satisfying assignments of a given formula, and a reconfiguration step flips one variable of the assignment. A dichotomy was shown by Gopalan et al. [Gop+09]: for a fixed set of Boolean constraints $\Gamma$ (that is, Boolean relations, or clause types), the problem of finding paths between solutions of a SAT($\Gamma$) instance (a formula formed from constraints in $\Gamma$) is either in P or PSPACE-complete. In particular it is always in P when the corresponding satisfiability problem is in P (eg. 2-SAT or Horn-SAT), but it is also in P for some $\Gamma$ for which satisfiability is NP-complete. This was slightly corrected (with a further correction in 2015) and extended to several similar problems by Schwerdtfeger [Sch14; Sch16], while a trichotomy was shown for the problem of finding *shortest* paths by Mouawad et al. [Mou+15]. Both [Gop+09] and [Sch14] asked whether their results could be extended to larger domains. This chapter can be seen as a step in this direction, but limited to only one symmetric relation of arity 2 (the adjacency relation of the target graph $H$).

The corresponding dichotomy for satisfiability, that is, for deciding the existence of a solution, was proved by Schaefer [Sch78a]. Generalizing it to CSPs with arbitrary finite domains is a long-standing open problem stated by Feder and Vardi [FV98] (at least two proofs are currently claimed and await peer-review [Zhu17; Bul17]). They showed that the conjecture is unchanged when limited to one relation of arity 2 (digraph homomorphisms). Hell and Nešetřil proved the dichotomy in the case the relation is additionally assumed to be symmetic (graph homomorphism) [HN90]: the problem of deciding the existence of an $H$-coloring of a given graph is in P for $H$ bipartite or containing a loop, and NP-complete otherwise.

### Square-free graphs

In this chapter, $G$ and $H$ are always connected undirected graphs with at least one edge and no loops, and $H$ **is always assumed to be square-free.** Since we do not consider loops, this is equivalent to requiring that $H$ has no $C_4$ subgraph. All results of this chapter extend in a straightforward manner to square-free graphs $H$ with loops allowed; a star graph $K_{1,k}$ with loops added to leaves gives an interesting example. (Recall that in general a graph is *square-free* if does not contain edges $ab, bc, cd, da \in H$ such that $a \neq c$ and $b \neq d$.) However, we choose to omit the few additional group-theoretic details that would be needed in proofs.

Another way to phrase the square-free condition is to require that for every two colors $a, b \in V(H)$, the set of common neighbors $N_H(a) \cap N_H(b)$ contains at most one color. That is, when a vertex of $G$ changes its color from some $a$ to some $b \in V(H)$, all its neighbors are forced to have one common color. Formally, an *$H$-recoloring sequence* is a sequence of $H$-colorings of $G$ in which consecutive colorings differ at one vertex. Consider a step of an $H$-recoloring sequence: a vertex $v \in V(G)$ changes color from $a \in V(H)$ to $b \in V(H)$. Since $G$ is connected, loopless and has an edge, $v$ has a neighbor $w \neq v$. As only $v$ changes its color in the step, $w$ has the same color before and after the step, say $h \in V(H)$. The $H$-coloring before the step implies that $ha \in E(H)$, while the one after the step implies that $hb \in E(H)$. Thus $h \in N_H(a) \cap N_H(b)$. From the assumption that $H$ is square-free we infer that $N_H(a) \cap N_H(b) = \{h\}$. We will often call $h$ '*the color that all neighbors of $v$ have during the step*' (that is, in the $H$-colorings just before and after the step), without arguing its existence and uniqueness anymore.

## 2. The fundamental groupoid

Our main tool will be the fundamental groupoid, a basic notion of algebraic topology, which despite its somewhat intimidating name is a simple group-like structure, describing all paths in a graph or space. Discrete variants like the one defined here were considered in various contexts, the first chapter of [KN07] provides an in-depth reference (including details on graphs with loops). Instead of continuous curves we consider just discrete 'walks'; instead of homotopy we only need to consider 'reducing' pairs of backtracking edges.

### Walks

An *oriented edge* of a graph $H$ is an oriented pair, denoted $uv \in H$, such that $\{u, v\}$ is an edge of $H$; we denote its initial vertex $u$ as $\iota(e)$ and its target vertex $v$ as $\tau(e)$. We write $e^{-1}$ for $\tau(e)\iota(e) \in H$. A *walk* from $u$ to $v$ in a graph $H$ is a sequence of oriented edges $e_1 e_2 \ldots e_n$ of $H$ such that endpoints match: $\iota(e_1) = u$, $\tau(e_n) = v$ and $\tau(e_i) = \iota(e_{i+1})$ for $i = 0, \ldots, n-1$. The edges are not necessarily distinct and a walk may self-intersect. We write $\varepsilon$ for an empty walk (formally there is a different empty walk from $v$ to $v$ for every vertex $v$, but the endpoints of $\varepsilon$ will be clear from the context). The length of a walk $W$, denoted $|W|$, is the number of edges in it. A walk $W_1$ from $u$ to $v$ can be concatenated to a walk $W_2$ from $v$ to $w$ to form a walk $W_1\_W_2$ from $u$ to $w$. We identify an edge with a walk of length 1, by abuse of notation.

walk

$\varepsilon$

$|W|$

$W_1\_W_2$

Note that if $\alpha : G \to H$ is a graph homomorphism and $e = uv$ is an oriented edge in $G$, then by definition $\alpha(e) := \alpha(u)\alpha(v)$ is an oriented edge in $H$. Similarly if $W = e_1 \ldots e_n$ is a walk in $G$, then $\alpha(W) := \alpha(e_1) \ldots \alpha(e_n)$ is a walk in $H$ (in contrast, the image of a path is not necessarily a path).

$\alpha(W)$

### Reducing

We call a walk *reduced* if it contains no two consecutive edges $e_i e_{i+1}$ such that $e_{i+1} = e_i^{-1}$ (in other words, it never backtracks). One can *reduce* a walk by removing any such two consecutive edges from the sequence. It can easily be seen that by iteratively reducing a walk $W$ in any order, one always gets the same reduced walk, which we denote as $\overline{W}$, see Figure II.2. For any two reduced

reduce

$\overline{W}$

$W_1 \cdot W_2$

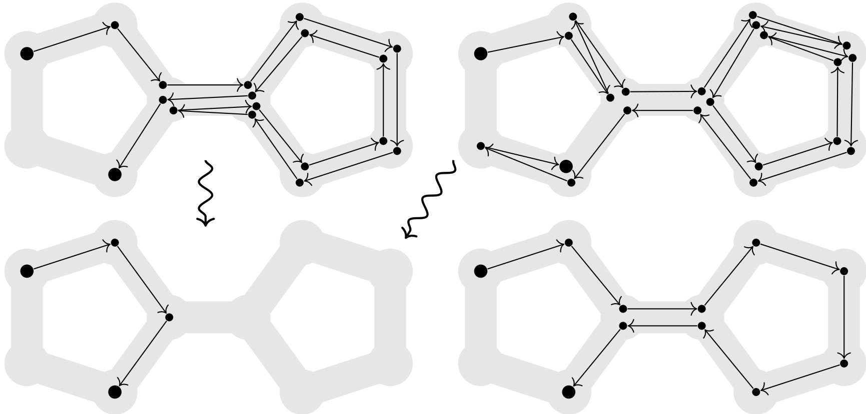


**Figure II.2** Examples of two walks (in a 'dumbbell' graph $H$ on 10 vertices) which reduce to the same, bottom left one. The bottom right one is a different reduced walk; when its endpoints are fixed, it cannot be distorted as a curve to give any of the others.

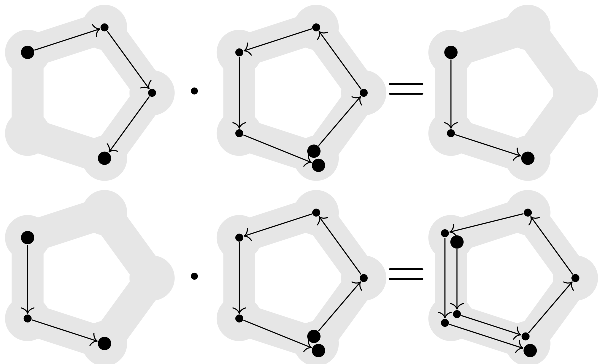


**Figure II.3** Examples of $\cdot$ multiplication in the fundamental groupoid of $H = C_5$.

walks $W_1, W_2$ such that $W_1$ ends where $W_2$ starts, we write $W_1 \cdot W_2$ for $\overline{W_1 \_ W_2}$ and similarly one can observe that $\cdot$ is associative. For any walk $W = e_1 e_2 \dots e_l$ we write $W^{-1}$ for the reversed walk $e_l^{-1} \dots e_2^{-1} e_1^{-1}$. Clearly $\overline{W} \cdot \overline{W}^{-1} = \varepsilon = \overline{W}^{-1} \cdot \overline{W}$ and $\varepsilon \cdot \overline{W} = \overline{W} \cdot \varepsilon = \overline{W}$. $W_1^{-1}$

The (usually infinite) set of reduced walks of a graph forms together with the operations $\cdot$ and $()^{-1}$ a *groupoid*; that is, it satifies all axioms of a group, except that the group operation $\cdot$ is a partial function, defined only when the 'head' of one element matches the 'tail' of the other. (A groupoid can also be defined as a category in which every morphism is invertible.) This particular groupoid is called the *fundamental groupoid* $\pi(H)$ of $H$. See Figure II.3 for an example. groupoid $\pi(H)$

Groupoids behave similarly to groups (much more so than semigroups, for example) and identities such as $(e \cdot f)^{-1} = f^{-1} \cdot e^{-1}$ are easily reproved in groupoids. We could define a group by considering only closed walks starting and ending in a fixed vertex $v$, but this would make formulas less uniform, eg. requiring some tedious additional steps when changing the base point $v$.

### Topological interpretation

Let us comment on how this algebraic structure captures the topology of curves in the graph. We recall some basic facts and definitions, which we do not require formally, but which are helpful, if not crucial, in understanding the results.

A graph $H$ can be naturally associated with a topological space, constructed as copies of the unit interval $[0, 1]$ in $\mathbb{R}$ for each edge, with endpoints merged into vertices accordingly. (In later chapters we will consider the box complex of $H$ instead, to account for squares). A *curve* in this space is a continuous map $f : [0, 1] \to H$, not necessarily injective (self-intersections are allowed).

Two curves $f_0, f_1$ are *homotopic rel endpoints* if one can be continuously transformed into the other with endpoints fixed; that is, there is a family of curves $(\phi_t)_{t\in[0,1]}$ such that $\phi_0 = f_0, \phi_1 = f_1$, endpoints $\phi_t(0)$ and $\phi_t(1)$ are constant, and the mapping $(t, x) \mapsto \phi_t(x)$ is continuous as a function from $[0, 1] \times [0, 1]$ to $H$.

The fundamental groupoid fully describes curves up to homotopy. Two walks between vertices $u, v$ in $H$ give the same walk in $\pi(H)$ after reduction if and only if the corresponding curves in $H$ are homotopic rel endpoints. See again Figure II.2. Instead of fixing endpoints we can also consider closed curves. A closed walk $C_1$ starting and ending at $u$ is *conjugate* to a closed walk $C_2$ starting and ending at $v$ (possibly $u \neq v$), meaning $C_2 = P^{-1} \cdot C_1 \cdot P$ for some $P \in \pi(H)$, if and only if the corresponding closed curves are homotopic (via a homotopy $\phi_t$ such that $\phi_t(1)$ is equal to $\phi_t(0)$ for all $t$, though not necessarily constant anymore).

conjugate

## 3. Vertex walks and realizability

When a vertex $v \in V(G)$ changes colors from $a$ to $b$ in a step of a $H$-recoloring sequence, consider the color $h$ that all neighbors of $v$ have during the change. Then $ah\_hb$ is a walk in $H$ (of length 2). Looking this way at all the sequence of color changes of one vertex gives a walk in $H$ which traces the colors that $v$ had. This walk (for one arbitrarily chosen vertex $v$), even after reducing, will be shown to almost completely describe the $H$-recoloring sequence.

$S(v)$

Formally, consider an $H$-recoloring sequence $S = \sigma_0, \dots, \sigma_l$ of $G$ (so $\sigma_i$ are homomorphisms $G \to H$) and any vertex $v \in V(G)$. We define the *vertex walk* $S(v)$ of $v$ as the following walk in $H$. If $l = 0$, let $S(v) = \varepsilon$. If $l = 1$ ($S$ contains only one reconfiguration step) then $S(v) = \varepsilon$ when $\sigma_0(v) = \sigma_1(v)$ and $S(v) = \sigma_0(v)h\_h\sigma_1(v)$ otherwise, $h$ being the color that all neighbors of $v$ have in $\sigma_0$ and $\sigma_1$. Finally if $l > 1$, then $S(v) = S_0(v)\_S_1(v)\_\dots\_S_{l-1}(v)$, where $S_i$ is the subsequence $\sigma_i, \sigma_{i+1}$ of $S$.

Instead of asking just whether any $H$-recoloring sequence exists, we focus on the following question: which walks in $H$ can be realized, up to reductions, as vertex walks of an arbitrarily fixed vertex $q \in V(G)$ in some solution sequence. For two $H$-coloring $\alpha, \beta$ of $G$ and a chosen $q \in V(G)$ we say a reduced walk $Q \in \pi(H)$ from $\alpha(q)$ to $\beta(q)$ is a *realizable* walk if there is an $H$-recoloring sequence $S = \sigma_0, \dots, \sigma_l$ from $\sigma_0 = \alpha$ to $\sigma_l = \beta$ such that $\overline{S(q)} = Q$. Which elements of $\pi(H)$ are realizable? It is immediate from the definition that $Q$ must have even length (notice that the parity of the length of walks is preserved by reducing, since we only remove pairs of edges $ee^{-1}$). See Figure II.4.

realizable

Parity is one of three conditions that characterize realizable walks. Sections 4, 5 describe the second (topological) and third necessary conditions, respectively. In Section 6 we prove they are sufficient (the characterization), Section 7 describes algorithmically the topological condition, and finally Section 8 uses these to give the main algorithm.

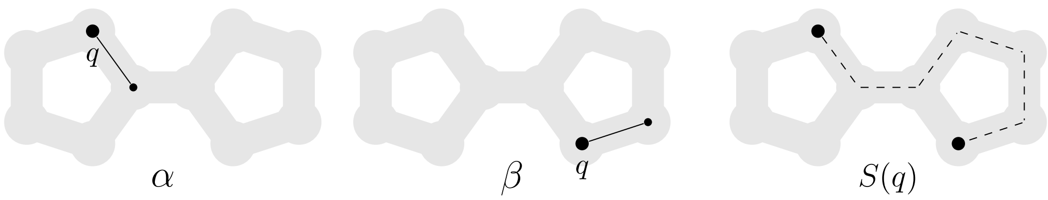



**Figure II.4** A realizable walk for $\alpha, \beta : K_2 \to H$ and $q$. Note that the shortest walk from $\alpha(q)$ to $\beta(q)$ (of length 3) is not realizable because of parity.

## 4. Topological validity

Intuitively, for an $H$-coloring of $G$ we look at the corresponding continuous map from $G$ to $H$ (as on all the figures). A recoloring sequence between two $H$-colorings then corresponds to a homotopy (a continuous transformation) between the maps. While much less constrained, since edges may now be arbitrarily stretched, homotopies still must preserve certain invariants.

We can describe essentially all of these invariants by considering the $H$-recoloring of just walks in $G$, which corresponds to continuously transforming an individual curve within $H$. The following lemma states in one algebraic equation the key implication of this continuity, see Figure II.5.

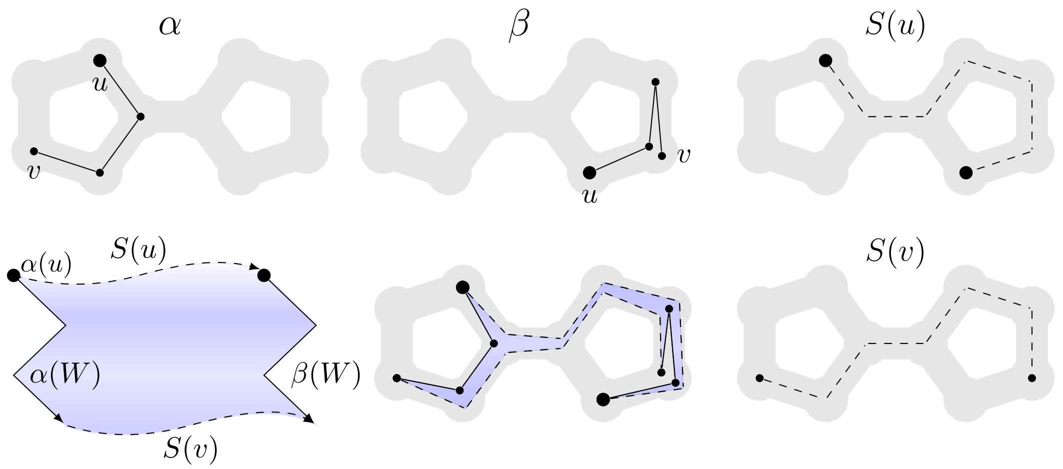



**Figure II.5** When continuously transforming a curve, from $\alpha(W)$ to $\beta(W)$, there is an exact relation between the curves traced by the endpoints $u, v$ of $W$, namely $S(u), S(v)$. One endpoint $v$ traces the same (up to homotopy rel endpoints) as the following: first going to the other endpoint $u$ (along the initial curve's position $\alpha(W)$ in $H$), then tracing $u$, and then going back to $v$ (along the curve's final position $\beta(W)$ in $H$).

**4.1 Lemma.** *Let $S = \sigma_0, \ldots, \sigma_l$ be an $H$-recoloring sequence of $G$ from $\alpha = \sigma_0$ to $\beta = \sigma_l$. Consider any walk $W$ from vertex $u$ to $v$ in $G$. Then $\overline{S(v)} = \overline{\alpha(W)}^{-1} \cdot \overline{S(u)} \cdot \overline{\beta(W)}$.*

Intuitively, if $\alpha$ can be transformed to $\beta$ by $H$-recoloring, then there is a corresponding homotopy $\phi : [0,1] \times [0, |W|] \to H$ from the initial position $\phi(0, \cdot) = \alpha(W)$ to the final position $\phi(1, \cdot) = \beta(W)$ of $W$ in $H$. Let $S(u) = \phi(\cdot, 0)$ and $S(v) = \phi(\cdot, |W|)$ be the paths (vertex walks) traced by the endpoints $u$ and $v$, respectively. Since $\phi$ is a continuous mapping of a rectangle to $H$ and since the boundary of the rectangle can be contracted to a point, the image of this boundary can also be contracted, giving the equality: $\overline{\alpha(W)}^{-1} \cdot \overline{S(u)} \cdot \overline{\beta(W)} \cdot \overline{S(v)}^{-1} = \varepsilon$.

*Proof.* Formally, the proof uses induction and the square-free property of $H$ for the base case. Assume first that $l = 1$, so $S = \sigma_0, \sigma_1$, where $\sigma_1$ is obtained from $\sigma_0$ by recoloring one vertex $w \in V(G)$ from $\sigma_0(w) = a$ to $\sigma_1(w) = b$. Let $h$ be the color that all neighbors of $w$ have in $\sigma_0$ and $\sigma_1$. By definition of vertex walks, $S(w) = ah\_hb$ and all other vertex walks are empty.

If $W = \varepsilon$, then $u = v$ and the claim follows trivially, since $\alpha(W) = \varepsilon$ and $\beta(W) = \varepsilon$.

If $W$ has length one, that is $W$ is a single edge $uv$, then one of the following holds, depending on where the recolored vertex $w$ is:

- $u \neq w$ and $v \neq w$, implying
  $S(u) = S(v) = \varepsilon$ and $\sigma_0(W) = \sigma_l(W)$.
- $u \neq w$ and $v = w$, implying
  $S(u) = \varepsilon$, $S(v) = ah\_hb$ and $\sigma_0(W) = ha$, $\sigma_l(W) = hb$.

- $u = w$ and $v \neq w$, implying
  $S(u) = ah\_hb$, $S(v) = \varepsilon$ and $\sigma_0(W) = ah$, $\sigma_l(W) = bh$.

In each case $\overline{\sigma_l(W)} = \overline{S(u)}^{-1} \cdot \overline{\sigma_0(W)} \cdot \overline{S(v)}$, which is equivalent to the claim.

If $W$ is longer, the claim follows inductively by splitting it into shorter walks: $W = W_1\_W_2$ for $W_1$ from $u$ to some vertex $w$ and $W_2$ from $w$ to $v$. It then follows that $\overline{\sigma_l(W)} =$

$$= \overline{\sigma_l(W_1)} \cdot \overline{\sigma_l(W_2)} =$$

$$= \overline{S(u)}^{-1} \cdot \overline{\sigma_0(W_1)} \cdot \overline{S(w)} \cdot \overline{S(w)}^{-1} \cdot \overline{\sigma_0(W_2)} \cdot \overline{S(v)} =$$

$$= \overline{S(u)}^{-1} \cdot \overline{\sigma_0(W_1)} \cdot \overline{\sigma_0(W_2)} \cdot \overline{S(v)} =$$

$$= \overline{S(u)}^{-1} \cdot \overline{\sigma_0(W)} \cdot \overline{S(v)}.$$

It remains to consider the case where $S$ has more than one step. Then the claim follows inductively by writing $S$ as the concatenation of shorter sequences: $S_1$ which ends in $\sigma_i$ and $S_2$ which starts in $\sigma_i$. It then follows that $S(v) = S_1(v)\_S_2(v)$ and $\overline{S(v)} =$

$$= \overline{S_1(v)} \cdot \overline{S_2(v)} =$$

$$= \overline{\sigma_0(W)}^{-1} \cdot \overline{S_1(u)} \cdot \overline{\sigma_i(W)} \cdot \overline{\sigma_i(W)}^{-1} \cdot \overline{S_2(u)} \cdot \overline{\sigma_l(W)} =$$

$$= \overline{\sigma_0(W)}^{-1} \cdot \overline{S_1(u)} \cdot \overline{S_2(u)} \cdot \overline{\sigma_l(W)} =$$

$$= \overline{\sigma_0(W)}^{-1} \cdot \overline{S(u)} \cdot \overline{\sigma_l(W)}.$$

□

The lemma has two important corollaries.

First, in a given instance of $H$-Recoloring, the reduced vertex walk $\overline{S(q)}$ of one vertex $q$ in a $H$-recoloring sequence $S$ determines up to reductions (in other words, up to homotopy) all other vertex walks (since $\alpha$ and $\beta$ are given). Later we will see that in shortest solution sequences all vertex walks are already reduced, so $\overline{S(q)}$ actually determines the sequence exactly, up to reordering color changes of different vertices. This means shortest recoloring sequences can be concisely represented by one realizable element $\overline{S(q)} \in \pi(H)$. (The possible reorderings will be revealed in the proof of the characterization theorem). This is also the reason for which we can focus on one walk and its realizability, instead of trying to describe an entire recoloring sequence.

Second, observe that the equality in the lemma holds for all walks in $G$, even though different walks between the same endpoints could *a priori* give different values. For every closed walk $C$ from $v$ to $v$ in $G$, we infer some equation on $\overline{S(v)}$, namely $\overline{S(v)} = \overline{\alpha(C)}^{-1} \cdot \overline{S(v)} \cdot \overline{\beta(C)}$, which expresses a certain topological condition on how solution sequences look like. We can rearrange this condition as

$$\overline{S(v)}^{-1} \cdot \overline{\alpha(C)} \cdot \overline{S(v)} = \overline{\beta(C)}$$

In group theory we say that $\overline{\alpha(C)}$ and $\overline{\beta(C)}$ are *conjugate* and that $\overline{S(v)}$ is a witness of that. We say a walk is *topologically valid* if it satisfies the above equation for all $C$:

**4.2 Definition.** *Let $\alpha, \beta$ be two $H$-colorings of $G$ and let $q$ be a vertex of $G$. A walk $Q \in \pi(H)$ is* topologically valid *for $\alpha, \beta, q$ if for every closed walk $C$ from $q$ to $q$ we have $\overline{\beta(C)} = Q^{-1} \cdot \overline{\alpha(C)} \cdot Q$.*

**4.3 Corollary.** *If $Q \in \pi(H)$ is realizable for $\alpha, \beta, q$ then $Q$ is topologically valid for $\alpha, \beta, q$.*

We analyze such conjugacy equations in more detail in Section 7, for now let us give their intuitive meaning. The condition that $\overline{\alpha(C)}$ and $\overline{\beta(C)}$ are conjugate means that $C$, during any reconfiguration, always maps around the same cycle (or closed walk) in $H$, up to reductions and rotations (ie. the corresponding closed curves can be continuously transformed into one another, they are homotopic). The number of times the image of $C$ winds around this cycle in $H$ must also remain unchanged (in the special case $H = K_3$ this gives exactly one of the conditions for 3-recoloring given by [CHJ11, Theorem 7 (C2)]). Finally, the condition that the realized walk $\overline{S(v)}$ must be a witness will imply that two realizable walks (two solutions to the recoloring problem) can differ only in the number of times they wind around this cycle, essentially.

In the next, final lemma of this section, we show that equations for closed walks already imply all other equations that would follow from Lemma 4.1. Moreover, one could show that $Q$ is topologically valid for $\alpha, \beta, q$ if and only if there is a homotopy continuously transforming $\alpha$ to $\beta$ (simultaneously on all of $G$, not on a single walk) such that $q$ traces the curve $Q$ throughout this transformation (that is, $\phi_0 = \alpha, \phi_1 = \beta$ and the image of $t \mapsto \phi_t(q)$ is $Q$). This means that Corollary 4.3 is the strongest we can achieve using only this topological setting.

**4.4 Lemma.** *If a walk* $Q \in \pi(H)$ *is* topologically valid *for* $\alpha, \beta, q$, *then for any vertex* $v$ *and any two walks* $W_1, W_2$ *from* $q$ *to* $v$ *in* $G$ *we have* $\overline{\alpha(W_1)}^{-1} \cdot Q \cdot \overline{\beta(W_1)} = \overline{\alpha(W_2)}^{-1} \cdot Q \cdot \overline{\beta(W_2)}$.

*Proof.* $W_1\_W_2^{-1}$ is a closed walk starting and ending in $q$, so $\overline{\beta(W_1\_W_2^{-1})} = Q^{-1} \cdot \overline{\alpha(W_1\_W_2^{-1})} \cdot Q$. Therefore:

$$\begin{aligned} \overline{\alpha(W_2)} \cdot \overline{\alpha(W_1)}^{-1} \cdot Q \cdot \overline{\beta(W_1)} \cdot \overline{\beta(W_2)}^{-1} &= \\ = \overline{\alpha(W_1\_W_2^{-1})}^{-1} \cdot Q \cdot \overline{\beta(W_1\_W_2^{-1})} &= \\ = \overline{\alpha(W_1\_W_2^{-1})}^{-1} \cdot Q \cdot Q^{-1} \cdot \overline{\alpha(W_1\_W_2^{-1})} \cdot Q &= \\ &= Q. \end{aligned}$$

Left-multiplying the equation by $\overline{\alpha(W_2)}^{-1}$ and right-multiplying by $\overline{\beta(W_2)}$ gives the claim. □

## 5. Tight closed walks and frozen vertices

There is one more necessary condition for a walk to be realizable, beside even length and topological validity. Intuitively, closed walks that map to walks tightly stretched around $H$ cannot be recolored in any way.

Formally, in an $H$-coloring $\alpha$ of $G$, a vertex $v$ of $G$ is called *frozen* if for every $H$-recoloring sequence from $\alpha$ the resulting $H$-coloring $\beta$ has $\beta(v) = \alpha(v)$. A closed walk $C = e_1 e_2 \dots e_l$ is *cyclically reduced* if it is reduced ($e_i \neq e_{i+1}^{-1}$) and additionally $e_l \neq e_1^{-1}$. In other words, repeating $C$ gives an infinite reduced walk. A closed walk $C$ is *$\alpha$-tight* if $\alpha(C)$ is cyclically reduced.

frozen

cycl. red.

$\alpha$-tight

**5.1 Lemma.** *Let* $\alpha$ *be an* $H$*-coloring of* $G$ *and let* $C$ *be an* $\alpha$*-tight walk in* $G$*. Then all vertices of* $C$ *are frozen in* $\alpha$*.*

*Proof.* Suppose to the contrary that there is an $H$-recoloring sequence $\sigma_0, \dots, \sigma_l$ from $\alpha$, such that $\sigma_l(C) \neq \sigma_0(C)$. Let $i$ be the least such that $\sigma_i(C) \neq \sigma_0(C)$. Then in $\sigma_{i-1}$ all vertices of $C$ have the same color as in $\alpha = \sigma_0$, so $\sigma_{i-1}(C)$ is cyclically reduced, while $\sigma_i$ is obtained from $\sigma_{i-1}$ by changing the color of some vertex $v \in C$ from $a$ to $b$. Let $h$ be the color that all neighbors of $v$ have in $\sigma_{i-1}$ and $\sigma_i$. Let $u, w$ be the vertices of $C$ just before and just after $v$ on $C$. Since they are neighbors of $v$, they must both have the color $h$ in $\sigma_{i-1}$. But then $\sigma_{i-1}$ maps the subsequent edges $uv\_vw$ of $C$ to $ha\_ah$, contradicting that $\sigma_{i-1}(C)$ is cyclically reduced. □

This generalizes the characterization of frozen vertices in the case of $H = K_3$ from [CHJ11]. In general, frozen vertices can also arise in other situations, see Figure II.6 for an example, but these will not be relevant to the characterization theorem.

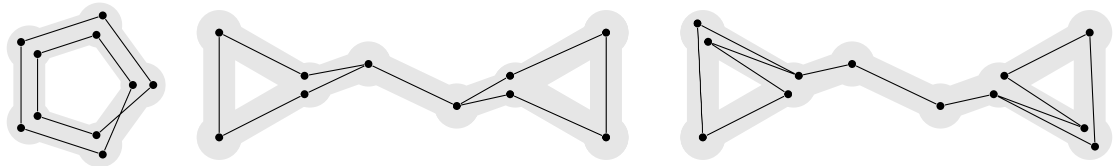


**Figure II.6** Left: walking along the 10 edges of the thin black graph gives a tight walk containing all vertices, so no reconfiguration step is possible. Middle: walking along one cycle, the bridge, the second cycle, and then back along the bridge, gives a tight closed walk containing all vertices, so no reconfiguration step is possible. Right: no closed walk is tight, but the 4 middle vertices are frozen.

Finding any frozen vertex $v$ means $S(v) = \varepsilon$ for any solution sequence $S$. This allows us to limit potentially realizable walks $Q$ to a single one, since even if our arbitrarily chosen vertex $q$ is not frozen, Lemma 4.1 gives $Q = \overline{S(q)} = \overline{\alpha(W)}^{-1} \cdot \overline{S(v)} \cdot \overline{\beta(W)} = \overline{\alpha(W)}^{-1} \cdot \overline{\beta(W)}$, for any walk $W$ from $v$ to $q$. Thus, we have the following necessary condition for a walk to be realizable:

**5.2 Corollary.** *Let $\alpha, \beta$ be two $H$-colorings of $G$ and let $q$ be a vertex of $G$. If $Q \in \pi(H)$ is realizable for $\alpha, \beta, q$, then for any $\alpha(C)$-tight closed walk in $H$, any vertex $v$ on $C$ and any walk $W$ from $v$ to $q$, we have $Q = \overline{\alpha(W)}^{-1} \cdot \overline{\beta(W)}$.*

Finally, we show how to find $\alpha$-tight closed walks by exploring walks $W$ such that $\alpha(W)$ is reduced.

**5.3 Lemma.** *There is an algorithm that given $G, H, \alpha$, finds an $\alpha$-tight walk or concludes there is none, in time $\mathcal{O}(|V(G)| \cdot |E(G)|)$.*

*Proof.* Consider the following directed graph $D$: its vertices are oriented edges of $G$ and there is an arc from $e$ to $e'$ when endpoints match ($\tau(e) = \iota(e')$) and $\alpha(e) \neq \alpha(e')^{-1}$. Then directed cycles in $D$ are $\alpha$-tight closed walks in $G$, and conversely, any $\alpha$-tight closed walk in $G$ gives a directed cycle in $D$ (if some oriented edge of the closed walk is repeated, use only the fragment between the closest two repetitions).

$D$ has $2|E(G)|$ vertices and $\sum_{v \in V(G)} 2\binom{\deg(v)}{2} \leq |V(G)| \cdot \sum_{v \in V(G)} \deg(v) = \mathcal{O}(|V(G)| \cdot |E(G)|)$ arcs, so a directed cycle in $D$ can be found by depth-first search in time $\mathcal{O}(|V(G)| \cdot |E(G)|)$. $\square$

We note that the prefix tree of walks $W$ such that $\alpha(W)$ is reduced (starting from some fixed point $v$), gives a generalization of the layer construction of [CHJ11]. Infinite paths in it are walks that must contain some oriented edge twice, so the fragment between repetitions is an $\alpha$-tight closed walk; conversely, repeating any $\alpha$-tight closed walk gives an infinite walk $W$ such that $\alpha(W)$ is reduced, so an infinite path in the tree. In topology, this tree is known as the universal cover of $H$.

## 6. Characterization of realizable walks

In this section we prove the characterization theorem: the three necessary conditions described in the previous sections are enough to characterize all possible solutions to an $H$-Recoloring instance. This is very unexpected, as it shows we can view the graphs as purely topological structures and the only remaining conditions to remember are a simple parity condition and the condition that tight closed walks are frozen—the fact that edges are actually discrete and cannot be stretched arbitrarily turns out to imply no further obstructions to recoloring (it does however restrict the possible $H$-colorings, which are given on input). The algorithm in Section 8 will use the theorem to find a concise description of the set of all realizable walks, in particular to find one such walk. See Figure II.7 for an example.

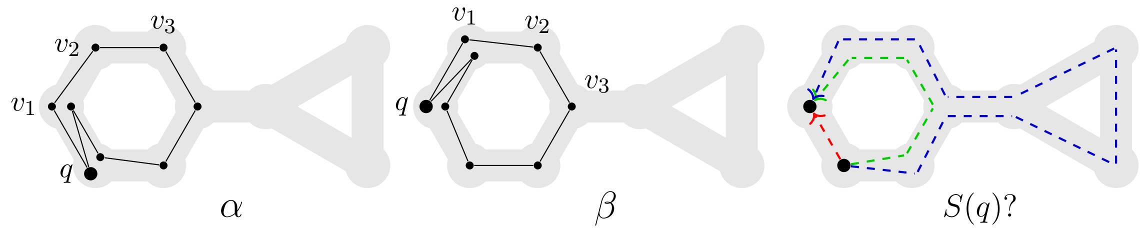



**Figure II.7** Two $H$-colorings $\alpha$, $\beta$ of an 8-cycle, where $H$ is the gray graph on 9 vertices. Even though no vertex is frozen, $\alpha$ cannot be reconfigured to $\beta$. The short (red and green) walks are not realizable for $\alpha, \beta, q$ because of parity. The long (blue) walk has good parity, but is not topologically valid (imagine continuously deforming the 8-cycle by pulling $q$ along this walk—the cycle would necessarily end up stretched around the triangle).

**6.1 Theorem.** *Let $\alpha, \beta$ be two $H$-colorings of $G$. Consider any vertex $q$ of $G$ and let $Q \in \pi(H)$ be a reduced walk in $H$ from $\alpha(q)$ to $\beta(q)$. Then $Q$ is realizable for $\alpha, \beta, q$ if and only if*

- *$Q$ has even length,*
- *$Q$ is topologically valid for $\alpha, \beta, q$, and*
- *for every $\alpha$-tight walk $C$, and any walk $W$ from any vertex on $C$ to $q$, $Q = \overline{\alpha(W)}^{-1} \cdot \overline{\beta(W)}$.*

*Furthermore, there is an $\mathcal{O}(|V(G)|^2 + |V(G)| \cdot |Q|)$-time algorithm that given $G, H, \alpha, \beta$ and given a walk $Q$ satisfying these conditions, outputs a recoloring sequence (as a sequence of color changes) such that $S(q) = Q$, $S(v)$ is reduced and $|S(v)| \leq 2|V(G)| + |Q|$ for all $v \in V(G)$.*

*Proof.* If an $H$-recoloring sequence is given, then the conditions are satisfied by Corollary 4.3 and Corollary 5.2, which proves the 'only if' half.

Consider now a reduced walk $Q \in \pi(H)$ that satisfies the above conditions. For every vertex $v \in V(G)$, let $S_v = \overline{\alpha(W)}^{-1} \cdot Q \cdot \overline{\beta(W)}$ for some walk $W$ from $q$ to $v$; by Lemma 4.4, this does not depend on how $W$ is chosen. In particular $S_q = Q$. We will show an $H$-recoloring sequence $S$ from $\alpha$ to $\beta$ such that $S(v) = S_v$ for all $v \in V(G)$. The idea is that the walks $S_v$ define a correct $H$-recoloring sequence for each edge, which is consistent thanks to topological validity, but it remains to order changes of different vertices into one reconfiguration sequence. Each edge gives a condition on which endpoint should recolor first and it turns out to be enough to respect these conditions. This is impossible if and only if there is a directed cycle of conditions, which turns out to be exactly an $\alpha$-tight cycle.

Formally, observe first that since $|Q|$ is even, $|\alpha(W)| = |W| = |\beta(W)|$, and since reducing preserves parity, we have that each $S_v$ has even length.

Consider two adjacent vertices $u, v \in V(G)$. Let $W$ be any walk from $q$ to $u$. Then $W$ followed by the oriented edge $uv$ is a walk from $q$ to $v$, hence by definition $S_v =$

$$= \overline{\alpha(W\_uv)}^{-1} \cdot Q \cdot \overline{\beta(W\_uv)} =$$

$$= \alpha(v)\alpha(u) \cdot \overline{\alpha(W)}^{-1} \cdot Q \cdot \overline{\beta(W)} \cdot \beta(u)\beta(v) =$$
$$= \alpha(v)\alpha(u) \cdot S_u \cdot \beta(u)\beta(v).$$

Let use write the exact sequence of edges in $S_u$ and $S_v$ (note the use of _ instead of $\cdot$):

$$S_u = a_0a_1 \_ a_1a_2 \_ a_2a_3 \_ a_{n-1}a_n,$$

$$S_v = b_0b_1 \_ b_1b_2 \_ b_2b_3 \_ \dots \_ b_{m-1}b_m.$$

Since $S_u$ is by definition reduced, we have $a_i \neq a_{i+2}$, similarly for $S_v$. Now $\alpha(u) = a_0, \alpha(v) = b_0$ and $\beta(u) = a_n, \beta(v) = b_m$, hence

$$S_v = b_0 a_0 \cdot S_u \cdot a_n b_m.$$

Suppose $S_u$ is non-empty ($n \geq 1$). Then by the parity condition we have $n \geq 2$ and

$$S_v = \overline{b_0 a_0 \_ a_0 a_1} \_ a_1 a_2 \_ \dots \_ a_{n-2} a_{n-1} \_ \overline{a_{n-1} a_n \_ a_n b_m}.$$

There are two cases, depending on whether $\overline{b_0 a_0 \_ a_0 a_1}$ cancels out to $\varepsilon$. Either it does, that is $b_0 a_0 = a_0 a_1^{-1}$, in which case

$$S_v = a_1 a_2 \_ a_2 a_3 \_ \dots \_ a_{n-2} a_{n-1} \_ \overline{a_{n-1} a_n \_ a_n b_m}$$

which means $b_0 = a_1$, $b_1 = a_2$, $\dots$, $b_{n-2} = a_{n-1}$ **(case $\boldsymbol{u \to v}$)**
or it does not, in which case

$$S_v = b_0 a_0 \_ a_0 a_1 \_ a_1 a_2 \_ \dots \_ a_{n-2} a_{n-1} \_ \overline{a_{n-1} a_n \_ a_n b_m}$$

which means $b_1 = a_0$, $b_2 = a_1$, $\dots$, $b_n = a_{n-1}$ **(case $\boldsymbol{u \leftarrow v}$)**.

For two adjacent vertices $u, v \in V(G)$ such that $S_u$ and $S_v$ are non-empty, let us write $u \to v$ in the first case and $u \leftarrow v$ in the other, as defined above. We have $u \to v$ iff $v \leftarrow u$, otherwise $b_0 = a_1 = b_2$ (if $u \to v$ and $v \to u$) or $a_0 = b_1 = a_2$ (if $u \leftarrow v$ and $v \leftarrow u$), contradicting that $S_u, S_v$ are reduced walks.

Furthermore, the $\to$ relation has no cycles. Suppose to the contrary that there exist $v_0$, $v_1$, $\dots$, $v_{l-1}, v_0$ ($l \geq 3$) such that $v_i \to v_{i+1}$ for $i \in \mathbb{Z}_l$. We will write $S_v^j$ for the $j$-th vertex of $S_v$. Then this is an $\alpha$-tight walk: indeed, arrows imply adjacency in $G$, and $v_i \to v_{i+1} \to v_{i+2}$ implies that

$$\alpha(v_{i+2}) = S^0_{v_{i+2}} = S^1_{v_{i+1}} = S^2_{v_i} \neq S^0_{v_i} = \alpha(v_i).$$

Therefore by the last condition we have $Q = \overline{\alpha(W)}^{-1} \cdot \overline{\beta(W)}$ and for any walk $W$ from $v_i$ to $q$,

$$S_{v_i} = \overline{\alpha(W)} \cdot Q \cdot \overline{\beta(W)}^{-1} = \overline{\alpha(W)} \cdot \overline{\alpha(W)}^{-1} \cdot \overline{\beta(W)} \cdot \overline{\beta(W)}^{-1} = \varepsilon.$$

But we did not assign arrows between vertices whose sequences are empty, a contradiction.

Therefore, by the lack of cycles, there is an ordering $v_1, v_2, \dots, v_{|V(G)|}$ of $V(G)$ such that if $v_i \to v_j$ then $i < j$. We claim the following is a valid $H$-recoloring sequence from $\alpha$ to $\beta$. Recolor:
$v_1$ from $S^0_{v_1}$ to $S^2_{v_1}$ , $v_2$ from $S^0_{v_2}$ to $S^2_{v_2}$ , $\dots$ , $v_n$ from $S^0_{v_n}$ to $S^2_{v_n}$,
$v_1$ from $S^2_{v_1}$ to $S^4_{v_1}$ , $v_2$ from $S^2_{v_2}$ to $S^4_{v_2}$ , $\dots$ , $v_n$ from $S^2_{v_n}$ to $S^4_{v_n}$,
$v_1$ from $S^4_{v_1}$ to $S^6_{v_1}$ , $\dots$ $\dots$
We continue in this order (disregarding any undefined recolorings to $S^j_{v_i}$ for $j > |S_{v_i}|$). Because of the parity condition, every vertex $v_i$ eventually gets recolored to the last color in $S_{v_i}$, which is $\beta(v_i)$; that is, the coloring we reach is indeed $\beta$.

To check that it is a valid $H$-recoloring sequence, consider any edge $uv$ of $G$ and define $a_i, b_i$ as above. If both $S_u$ and $S_v$ are empty, then $uv$ gets constantly mapped to the same edge $\alpha(u)\alpha(v)$ of $H$. If exactly one of $S_u, S_v$ is empty, say $S_u$, then $S_v = b_0 a_0 \cdot S_u \cdot a_0 b_m = b_0 a_0 \_ a_0 b_m$ where $b_0 \neq b_m$ (and $m = 2$). Thus $b_1 = a_0$, so $uv$ gets mapped initially to $\alpha(u)\alpha(v) = a_0 b_0$ and then to

$a_0b_2 = b_1b_2$, which is an edge of $H$. If both $S_u$ and $S_v$ are non-empty, then assume without loss of generality $u \to v$ (otherwise swap $u$ and $v$). We have

$$\begin{aligned} S_u &= a_0a_1 \,\text{-}\, a_1a_2 \,\text{-}\, a_2a_3 \,\text{-}\, \dots \,\text{-}\, a_{n-2}a_{n-1} \,\text{-}\, a_{n-1}a_n, \\ S_v &= \phantom{a_0a_1 \,\text{-}\,} a_1a_2 \,\text{-}\, a_2a_3 \,\text{-}\, \dots \,\text{-}\, a_{n-2}a_{n-1} \,\text{-}\, \overline{a_{n-1}a_n \,\text{-}\, a_nb_m} \end{aligned}$$

Thus $uv$ gets mapped initially to $\alpha(u)\alpha(v) = a_0b_0 = a_0a_1$ and then to $a_1a_2$, $a_3a_2$, $a_3a_4$, $\dots$, ending in either $a_na_{n-1}$ or $a_na_{n+1}$ (depending on whether $\overline{a_{n-1}a_n \,\text{-}\, a_nb_m} = \varepsilon$). This is again always an edge of $H$ (because $S_u, S_v$ are walks in $H$). Thus the $H$-coloring condition is never violated on any edge and the sequence is a valid $H$-recoloring sequence.

The algorithm first checks that $|Q|$ is even. Then, it has to choose some arbitrary walks to define $S_v$ for $v \in V(G)$; choosing shortest paths from $q$ (in time $\mathcal{O}(|E(G)|)$) guarantees $|S_v| \leq 2|V(G)| + |Q|$. Then, for each edge $uv$ of $G$, it checks whether $u \to v$ or $v \to u$ holds, by inspecting the first edges of $S_v$ and $S_u$ in constant time, $\mathcal{O}(|E(G)|)$ in total. The ordering $v_1, \dots, v_{|V(G)|}$ (a topological ordering of the arrow graph) is constructed in $\mathcal{O}(|E(G)|)$ time; if none is found, we can output a tight closed walk, in fact a tight cycle. Finally it outputs the sequence of color changes given by $S_v$ in the above order, in time linear in the total number of color changes, which is $\mathcal{O}(|V(G)| \cdot (|V(G)| + |Q|))$. Note that the algorithm can check whether the conditions on $Q$ were really satisfied by checking the consecutive colors on each edge as in the previous paragraph, in total time $\mathcal{O}(|E(G)| \cdot (|V(G)| + |Q|))$; if at some point the check fails, this is a contradiction, which means that $Q$ could not have been topologically valid. If we wanted to output the entire $H$-coloring at each step, this makes the output $|V(G)|$ times larger, requiring $\mathcal{O}(|V(G)|^2 \cdot (|V(G)| + |Q|))$ total time. □

The running time does not depend on $H$ at all, because we only inspect images of edges in $G$; $H$ could indirectly cause $Q$ to be long, but we will construct realizable walks $Q$ of polynomial length.

As the proof of the characterization theorem produces a solution sequence where all vertex walks are reduced, any sequence where this is not true can be shortened.

**6.2 Corollary.** *Let $\alpha, \beta$ be two $H$-colorings of $G$. Let $S = \sigma_0, \dots, \sigma_l$ be an $H$-recoloring sequence between $\sigma_0 = \alpha$ and $\sigma_l = \beta$ such that $l$ is minimized. Then for each vertex $v$ of $G$, $S(v)$ is reduced.*

*Proof.* Suppose $S(q)$ is not reduced for some $q$. Let $Q = \overline{S(q)}$. By the above theorem we know from one side that $Q$ is realizable. From the other side we obtain a solution sequence $S'$ such that $S'(v) = \overline{S'(v)}$ for all $v$ and $\overline{S'(q)} = Q = \overline{S(q)}$. By Lemma 4.1, this implies $S'(v) = \overline{S(v)}$, for all $v$. But $\overline{S(v)}$ is always no longer than $S(v)$, and $\overline{S(q)}$ is strictly shorter than $S(q)$. Since the number of recoloring steps is equal to half the sum of lengths of all $S(v)$, $S$ was not shortest. □

## 7. Calculations in the fundamental groupoid

The goal of this section is to prove Lemma 7.5, which describes algorithmically the set of topologically valid walks. This follows from well-known calculations in the fundamental groupoid of graphs, which we recall here.

Any algorithm will need to limit the number of closed walks considered. The standard way to do that is as follows: fix a vertex $q \in V(G)$ and an arbitrary spanning tree $T$ of $G$ (a minimal connected subgraph that includes all vertices). For each $e \in E(G) \setminus E(T)$ and an arbitrarily fixed orientation $\iota(e)\tau(e)$ of $e$, define the *fundamental cycle* $C_e$ as the closed walk that goes from $q$ to $\iota(e)$ along the unique path that connects them in $T$, then through $e$ to $\tau(e)$, then back to $q$ along the unique path in $T$. There are $|E(G)| - |E(T)| = |E(G)| - |V(G)| + 1$ fundamental cycles and together they generate all other cycles (see for example Lemma 1.2. in [KN07]):

**7.1 Fact.** *Let $C$ be any closed walk from $q$ to $q$ in $G$. Then $\overline{C} = \overline{C_{e_1}}^{s_1} \cdot \ \ldots \ \cdot \overline{C_{e_n}}^{s_n}$ where $e_1, \ldots, e_n \in E(G) \setminus E(T)$ are the consecutive non-tree edges of $C$ and $s_i \in \{-1, +1\}$ are chosen to match their orientation.*

This allows to limit the number of conjugacy equations defining topological validity to polynomially many ($|E(G) \setminus E(T)|$, to be exact). It is also folklore that conjugacy equations can be solved in polynomial time:

**7.2 Fact.** *Given two $H$-colorings $\alpha, \beta$ of a graph $G$ and a vertex $q$, one can find in time $\mathcal{O}(|E(G)| \cdot |V(G)| + |E(H)|)$ a walk $Q \in \pi(H)$ that is topologically valid for $\alpha, \beta, q$, or conclude there is none.*

*Proof.* By definition, $Q$ is topologically valid if and only if for every closed walk $C$ from $q$ to $q$ we have $\overline{\beta(C)} = Q^{-1} \cdot \overline{\alpha(C)} \cdot Q$. By Fact 7.1, this is equivalent to satisfying the equation for each fundamental cycle $C_e$. Let $W$ be any walk from $\beta(q)$ to $\alpha(q)$ in $H$. Then $Q$ satisfies the equations if and only if $Q \cdot W$ satisfies $W \cdot \overline{\beta(C_e)} \cdot W^{-1} = (Q \cdot W)^{-1} \cdot \overline{\alpha(C_e)} \cdot (Q \cdot W)$ for each fundamental cycle $C_e$. In this form, we have polynomially many equations where each of the walks $Q \cdot W$, $\overline{\alpha(C_e)}$ and $W \cdot \overline{\beta(C_e)} \cdot W^{-1}$ is a closed walk from $\alpha(q)$ to $\alpha(q)$ in $H$.

Denote by $\pi(H, \alpha(q))$ the subset of $\pi(H)$ given by closed walks from $\alpha(q)$ to $\alpha(q)$. It is a group (under $\cdot$); moreover, it is the free group generated by the fundamental cycles of $H$ as described in Fact 7.1 (see for example Lemma 1.1., 1.2. in [KN07]). Finding an element $Q \cdot W$ satisfying the above equations in $\pi(H, \alpha(q))$ is therefore the Simultaneous Conjugacy Search Problem in a free group, for which a linear time algorithm is described in Theorem 6.5. of [MU08]. The size of the input to this algorithm can be bounded by the number of equations $|E(G) \setminus E(T)|$ times the length of $\overline{\alpha(C_e)}$ and $W \cdot \overline{\beta(C_e)} \cdot W^{-1}$ in each equation, which is $\mathcal{O}(|V(G)|)$. Additionally, we need to compute a spanning tree of $H$ and give the edges outside of it, in $\mathcal{O}(|E(H)|)$ time, to present $\pi(H, \alpha(q))$ as a free group. $\square$

To describe all valid walks we will need the following. For a non-empty closed walk $C \in \pi(H)$ we define the *primitive root* of $C$ as the unique $R \in \pi(H)$ such that $C = R^n$ for some $n \in \mathbb{N}$ such that $n$ is maximized. Note that if $R$ is a primitive root, then the primitive root of $W \cdot R^n \cdot W^{-1}$ is $W \cdot R \cdot W^{-1}$ (for $n \geq 1$ and $W \in \pi(H)$ such that $W \cdot R$ is defined), for example. It is a routine exercise to check that the primitive root is well defined, can be computed in linear time, and that the following holds (see eg. Lemma 2.1. of [MA80]):

**7.3 Fact.** *Let $C_1, C_2 \in \pi(H)$. Then $C_1$ and $C_2$ commute, ie. $C_1 \cdot C_2 = C_2 \cdot C_1$, if and only if $C_1 = \varepsilon$ or $C_2 = \varepsilon$ or both have the same primitive root or one root is the inverse of the other.*

We now show that whenever a cycle $C$ maps to a non-trivial cycle in $H$, possible solution sequences can only differ in the number of times they wind around this cycle; see Figure II.8.

**7.4 Lemma.** *Let $\alpha, \beta$ be two $H$-colorings of $G$ and let $q$ be a vertex of $G$. Let $P \in \pi(H)$ be topologically valid for $\alpha, \beta, q$. Then $Q \in \pi(H)$ is topologically valid for $\alpha, \beta, q$ if and only if for every closed walk $C$ in $G$ starting and ending in $q$ such that $\overline{\alpha(C)} \neq \varepsilon$, we have*

$$Q = R^n \cdot P$$

*for some $n \in \mathbb{Z}$, where $R$ is the primitive root of $\overline{\alpha(C)}$.*

*Proof.* Suppose $Q$ is topologically valid and let $C$ be a closed walk starting and ending in $q$. Then by definition we have $\overline{\beta(C)} = P^{-1} \cdot \overline{\alpha(C)} \cdot P$ and $\overline{\beta(C)} = Q^{-1} \cdot \overline{\alpha(C)} \cdot Q$. Therefore

$$P^{-1} \cdot \overline{\alpha(C)} \cdot P = Q^{-1} \cdot \overline{\alpha(C)} \cdot Q$$

$$Q \cdot P^{-1} \cdot \overline{\alpha(C)} = \overline{\alpha(C)} \cdot Q \cdot P^{-1}$$

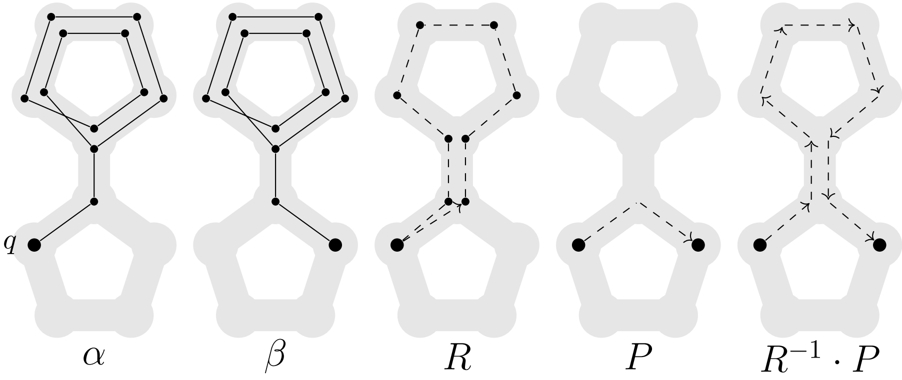



**Figure II.8** In this example, let $C$ be the shortest non-trivial walk from $q$ to $q$ in the thin black graph $G$. Its image $\alpha(C)$ and $\beta(C)$ winds twice around the root $R$ ($\overline{\alpha(C)} = R^2$). The topologically valid paths are exactly $\{R^n \cdot P \mid n \in \mathbb{Z}\}$ (think about deforming $\alpha$ by pulling $q$: one can pull it once or more around the top cycle by rotating all of $\alpha$, but this is impossible for the bottom cycle if we want to end at $\beta$).

So $\overline{\alpha(C)}$ commutes with $Q \cdot P^{-1}$. Therefore, if $\overline{\alpha(C)} \neq \varepsilon$ and $R$ is the primitive root of $\overline{\alpha(C)}$, then $Q \cdot P^{-1} = R^n$ for some $n \in \mathbb{Z}$ by Fact 7.3.

For the other side, suppose that for every closed walk $C$ from $q$ to $q$ with $\overline{\alpha(C)} \neq \varepsilon$ and a primitive root $R$ of $\overline{\alpha(C)}$, there is an $n \in \mathbb{Z}$ such that $Q = R^n \cdot P$. Then for every closed walk $C$ from $q$ to $q$, $\overline{\alpha(C)}$ commutes with $Q \cdot P^{-1}$, because either $\overline{\alpha(C)} = \varepsilon$ or $\overline{\alpha(C)} = R^k$ and $Q \cdot P^{-1} = R^n$ for some $R \in \pi(H)$ and $n, k \in \mathbb{Z}$. Thus $\overline{\beta(C)} =$

$$= P^{-1} \cdot \overline{\alpha(C)} \cdot P =$$

$$= Q^{-1} \cdot Q \cdot P^{-1} \cdot \overline{\alpha(C)} \cdot P =$$

$$= Q^{-1} \cdot \overline{\alpha(C)} \cdot Q \cdot P^{-1} \cdot P =$$

$$= Q^{-1} \cdot \overline{\alpha(C)} \cdot Q,$$

which shows the topological validity of $Q$. □

The above lemma allows us to describe the set of all topologically valid walks:

**7.5 Lemma.** *Let $\alpha, \beta$ be $H$-colorings of $G$ and $q$ a vertex of $G$. Consider the (possibly infinite) set $\Pi \subseteq \pi(H)$ of topologically valid walks for $\alpha, \beta, q$. One of the following holds:*

*0.* $\Pi = \emptyset$.

*1.* $\Pi = \{Q\}$ *for some* $Q \in \pi(H)$.

*2.* $\Pi = \{R^n \cdot P \mid n \in \mathbb{Z}\}$ *for some* $R, P \in \pi(H)$.

*3.* $\Pi$ *contains all reduced walks from* $\alpha(q)$ *to* $\beta(q)$.

*Moreover, there is an algorithm that given $G, H, \alpha, \beta, q$ decides in time $\mathcal{O}(|E(G)| \cdot |V(G)| + |E(H)|)$ which case holds and outputs $Q$ and $R, P$ in cases 1 and 2.*

*Proof.* Use Fact 7.2 to compute a topologically valid walk $P \in \pi(H)$ for $\alpha, \beta, q$ in time $\mathcal{O}(|E(G)| \cdot |V(G)| + |E(H)|)$. If there is none, we immediately answer *case 0*. Fix an arbitrary spanning tree, compute the elements $\overline{\alpha(C_e)}$ for all fundamental cycles of $G$ (in total time $\mathcal{O}(|E(G)| \cdot |V(G)|)$) and if some for $e$ it is non-empty, check if it commutes with all other elements (again in time $\mathcal{O}(|E(G)| \cdot |V(G)|)$, since $|\overline{\alpha(C_e)}| = \mathcal{O}(|V(G)|)$). One of the following holds:

a) For every $C_e$, $\overline{\alpha(C_e)} = \varepsilon$. Then by Fact 7.1, for every closed walk $C$ from $q$ to $q$, $\overline{\alpha(C)} = \varepsilon$. By Lemma 7.4, vacuously, every walk $Q \in \pi(H)$ from $\alpha(q)$ to $\beta(q)$ is topologically valid.

b) There is a $C_e$ such that $\overline{\alpha(C_e)} \neq \varepsilon$ and for every $C_f$, $\overline{\alpha(C_f)}$ commutes with $\overline{\alpha(C_e)}$. Then by Fact 7.1, for every closed walk $C$ starting and ending in $q$, $\overline{\alpha(C)}$ commutes with $\overline{\alpha(C_e)}$. Let $R$ be the primitive root of $\overline{\alpha(C_e)}$. $R$ (or its inverse) is also the primitive root of every non-empty $\overline{\alpha(C)}$, so by Lemma 7.4, $Q$ is topologically valid iff $Q = R^n \cdot P$ for some $n \in \mathbb{Z}$.

c) There are $C_e, C_f$ such that $\overline{\alpha(C_e)}$ and $\overline{\alpha(C_f)}$ do not commute. Then we show $\Pi = \{P\}$. Clearly $\overline{\alpha(C_i)} \neq \varepsilon$, so let $R_i$ be the primitive root of $\overline{\alpha(C_i)}$ for $i \in \{e, f\}$. Suppose $Q \in \pi(H)$ is topologically valid. Then by Lemma 7.4, $Q = R_e^{n_e} \cdot P$ and $Q = R_f^{n_f} \cdot P$ for some $n_e, n_f \in \mathbb{Z}$. Thus $R_e^{n_e} = R_f^{n_f}$. If this element has a primitive root ($n_e, n_f \neq 0$), then it is equal to both $R_e$ and $R_f$, implying that $\overline{\alpha(C_e)}$ and $\overline{\alpha(C_f)}$ have the same primitive root, contradicting Fact 7.3. Therefore $n_e = n_f = 0$, so $Q$ must be equal to $P$.

We hence output, respectively, *case 3.*, *case 2.* with $R, P$, or *case 1.* with $P$. □

## 8. The main algorithm

In this section we give the main algorithm, which returns a description of all solution sequences, in particular telling whether there is one. It follows directly from the algorithm for describing topologically valid walks in Lemma 7.5 by simply checking the two other conditions of Theorem 6.1.

**8.1 Theorem.** *Let $\alpha, \beta$ be $H$-colorings of $G$ and $q$ a vertex of $G$. Consider the set $\Pi' \subseteq \pi(H)$ of realizable walks for $\alpha, \beta, q$. One of the following holds:*

*0.* $\Pi' = \emptyset$.

*1.* $\Pi' = \{Q\}$ *for some* $Q \in \pi(H)$.

*2.* $\Pi' = \{R^n \cdot P \mid n \in \mathbb{Z}\}$ *for some* $R, P \in \pi(H)$.

*3.* $\Pi'$ *contains all reduced walks of even length from* $\alpha(q)$ *to* $\beta(q)$.

*Moreover, there is an algorithm that given $G, H, \alpha, \beta, q$ decides in time $\mathcal{O}(|E(G)| \cdot |V(G)| + |E(H)|)$ which case holds and outputs $Q$ or $R, P$ in cases 1,2.*

*Proof.* First, find any $\alpha$-tight closed walk and if there is one, let $Q$ be the only possibly realizable walk as in the last condition of Theorem 6.1. By running the algorithm from Theorem 6.1 we can check whether it is indeed realizable and return either $\Pi' = \emptyset$ or $\Pi' = \{Q\}$.

Assume now that there is no $\alpha$-tight walk. Run the algorithm of Lemma 7.5 to get a description of topologically valid walks $\Pi$ and consider the following cases:

0. $\Pi = \emptyset$. Then also $\Pi' = \emptyset$ (see Theorem 6.1).
1. $\Pi = \{Q\}$ for some $Q \in \pi(H)$. Return $\Pi' = \{Q\}$ if $Q$ has even length and $\Pi' = \emptyset$ otherwise.
2. $\Pi = \{R^n \cdot P \mid n \in \mathbb{Z}\}$ for some $R, P \in \pi(H)$. The only remaining condition is parity, so one of the following holds:
   - $R$ is even and $P$ is odd: then $\Pi' = \emptyset$,
   - $R$ is even and $P$ is even: then $\Pi' = \Pi$,
   - $R$ is odd and $P$ is even: then $\Pi' = \{R^{2n} \cdot P \mid n \in \mathbb{Z}\}$,
   - $R$ is odd and $P$ is odd: then $\Pi' = \{R^{2n} \cdot (R \cdot P) \mid n \in \mathbb{Z}\}$.
3. $\Pi$ contains all reduced walks from $\alpha(q)$ to $\beta(q)$. Then $\Pi'$ contains all reduced walks of even length from $\alpha(q)$ to $\beta(q)$. □

The set of even walks from $\alpha(q)$ to $\beta(q)$ in $H$ is empty if and only if $H$ is bipartite and $\alpha(q), \beta(q)$ are on different sides of a bipartition. We can construct an even walk or conclude there is none in linear time. Thus in each case we decide whether there is a realizable walk $Q$ and if so, construct one of length bounded by the total running time, $\mathcal{O}(|E(G)| \cdot |V(G)| + |E(H)|)$. From $Q$, the algorithm of Theorem 6.1 can compute an actual recoloring sequence (as a sequence of color changes) in time $\mathcal{O}(|V(G)|^2 + |V(G)| \cdot |Q|) = \mathcal{O}(|E(G)| \cdot |V(G)|^2 + |E(H)| \cdot |V(G)|)$. In particular, whenever some sequence exists, we output a sequence of polynomial length. (Note that for non-square-free $H = K_4$, examples where shortest recoloring sequences have length exponential in $|V(G)|$ are known [BC09]).

**8.2 Corollary.** *For square-free graphs $H$, $H$-*Recoloring *can be decided in time $\mathcal{O}(|E(G)| \cdot |V(G)| + |E(H)|)$.*

Shortest recoloring sequences can also be found in polynomial time with some more care.

**8.3 Theorem.** *For square-free graphs $H$,* Shortest *$H$-*Recoloring *can be solved in time polynomial in the size of $G$ and $H$.*

*Proof.* By Corollary 6.2, it suffices to choose a walk $Q \in \Pi'$ from Theorem 8.1 minimizing

$$\sum_{v \in V(G)} \overline{S(v)} = \sum_{v \in V(G)} |\overline{\alpha(W_v)}^{-1} \cdot Q \cdot \overline{\beta(W_v)}|, \tag{1}$$

where $W_v$ is a walk from $q$ to $v$ (arbitrarily chosen). In cases 0. and 1. this is trivial. In case 2. ($Q = R^n \cdot P$, for any $n \in \mathbb{N}$) it is easy to see that repeating $R$ will eventually lengthen all summands of (1), hence $|n| \leq 2|V(G)| + |P|$ in shortest sequences. It thus suffices to compute (1) for all these possibilities for $n$.

In case 3., consider a realizable walk $Q$, ie., any reduced walk of even length from $\alpha(q)$ to $\beta(q)$. Let $P_1$ be the longest common prefix of $Q$ and $\overline{\alpha(W_v)}$, choosing $v \in V(G)$ to maximize its length. That is, $P_1$ is longest such that all of $P_1$ will reduce with $\overline{\alpha(W_v)}^{-1}$ in some summand of (1). Analogously, let $P_2$ bet the longest common suffix of $Q$ and some $\overline{\beta(W_v)}^{-1}$. Either $P_1$ and $P_2$ overlap, or $Q = P_1 Q' P_2$, for some $Q' \in \pi(H)$. In the latter case, since by definition no element of $Q'$ will be reduced in any summand of (1), it can be written as

$$\sum_{v \in V(G)} |\overline{\alpha(W_v)}^{-1} \cdot Q \cdot \overline{\beta(W_v)}| = \sum_{v \in V(G)} \left( |\overline{\alpha(W_v)}^{-1} \cdot P_1| + |Q'| + |P_2 \cdot \overline{\beta(W_v)}| \right).$$

Thus we can guess $P_1$ by enumerating all prefixes of all $\overline{\alpha(W_v)}$, similarly guess $P_2$ and guess how much they overlap. In case they do not overlap, the sum is minimized by taking $Q'$ to be an arbitrary shortest path of appropriate parity from the tail of $P_1$ to the head of $P_2$ in $H$. Enumerating all possibilities for (the length of) $P_1$, $P_2$ and the overlap can be done in polynomial time, and a shortest path of given parity in $H$ can be found by duplicating every vertex, ie., finding a shortest path in the tensor product $H \times K_2$. □

## 9. Conclusions and future work

### The case $H = K_3$

Our result generalizes the algorithm for $K_3$-Recoloring of [CHJ11] and recovers many of its features in a more general and perhaps more intuitive setting. When limited to $H = K_3$ (a 3-cycle), there is only one possible root $R$ for closed walks in $H$ (and its inverse), so they all commute. This means that either no walk is topologically valid (that is, $\alpha, \beta$ are not homotopic), or all are. (In the proof of Lemma 7.5, case 1. is impossible, while case 2. is the same as case 3.).

This allows to simplify the algorithm for $K_3$-RECOLORING substantially. Given an instance $\alpha, \beta$, if there is any solution sequence, any realizable walk, then we can find it as follows, knowing that all walks are topologically valid. Either there is some frozen vertex, which implies $Q_1 = \varepsilon$ is realizable for this vertex, or no vertex is frozen, which implies that all even walks are realizable, in particular the walk $Q_2$ from $\alpha(q)$ to $\beta(q)$ of length 0 or 2. Thus we do not need to perform any of the calculations in Section 7, it suffices to run the simple algorithms of Lemma 5.3 and Theorem 6.1 (with either $Q_1$ or $Q_2$) and check whether the resulting sequence is a valid $H$-recoloring sequence (if not, the assumption that some realizable walk exists was false).

Similarly, we can easily deduce the following purely graph-theoretic observation:

**9.1 Theorem.** *Let $G$ be a graph with no cycles of length divisible by 3. Then $G$ is 3-colorable.*

*Proof.* The proof is by induction on the number of edges: let $\alpha$ be a 3-coloring of $G - e$, for an arbitrary $e = uv \in E(G)$. If $\alpha(u) \neq \alpha(v)$, then this is a 3-coloring of $G$.

Otherwise, define $\beta(x) = \alpha(x) + 1 \mod 3$ (where the colors, or vertices of $K_3$, are $\{0, 1, 2\}$). This is another 3-coloring of $G - e$, obtained just by rotating $\alpha$, so homotopic to $\beta$. That is, we can choose $q \in V(G)$ arbitrarily and let $Q$ be the walk of length 2 from $\alpha(q)$ to $\beta(q)$: it has even length, it is easily checked to be topologically valid, and by the assumption that $G$ has no cycles of length divisible by 3, there are no $\alpha$-tight cycles. Hence $Q$ is realizable for $\alpha, \beta, q$.

Therefore, there is a $H$-recoloring sequence from $\alpha$ to $\beta$. But then at some point $u$ or $v$ changes its color for the first time, so it becomes different from the color of $v$ or $u$, respectively, giving a 3-coloring of $G$. $\square$

The statement already follows from a stronger theorem of Chen and Saito [CS94], that graphs with no cycles of length divisible by 3 are in fact 2-degenerate (all their subgraphs have a vertex of degree $\leq 2$, see also a strengthening by Gauthier [Gau17]). But at least in principle, this shows we can deduce the existence of homomorphism using recoloring.

If we only exclude cycles of length divisible by 3 as *induced* subgraphs, it is an open problem whether such a graph is 3-colorable, but Bonamy et al. [BCT14] recently showed that the chromatic number is bounded. In [Dvo+16], Bonamy points out that 3-colorability would follow from the same proof as Theorem 9.1 if the following were true:

**9.2 Conjecture.** *Every graph $G$ without induced cycles of length divisible by 3 has an edge $e$ such that $G - e$ still has no induced cycles of length divisible by 3.*

Curiously, the chromatic number of graphs with no induced cycles of length divisible by 3 is related to a very different connection between colorings and topology conjectured by Gil Kalai and Roy Meshulam, see [BCT14]. See also [Bre+16] and [SS17] for further results on coloring graphs with few cycles of prescribed mod $k$ length.

## The box complex

In this chapter we focused on combinatorial and algebraic descriptions, avoiding in particular box or Hom complexes. However, they may offer a nicer and perhaps deeper view into the characterization in Theorem 6.1.

First, we could get rid of the parity condition, that realizable walks must have even length, by using $\pi(H \times K_2)$ instead of $\pi(H)$. Intuitively, this is simply because a vertex of $H \times K_2$ accounts for not only our position in $H$, but also for the parity of the walk we used to get there. Recall that the box complex of $H$ is constructed by taking $H \times K_2$ and gluing faces to squares (and more generally, to complete bipartite subgraphs). So for square-free $H$, the box complex of $H$ is just

$H \times K_2$ (as a simplicial complex, up to $\mathbb{Z}_2$-homotopy equivalence). This way, the parity condition is already accounted for topologically, if we use the box complex.

If we use Hom complexes, we can rephrase all the three conditions of Theorem 6.1 into one. Indeed, recall that the box complex of $G$ is equivalent to $\mathrm{Hom}(K_2, G)$. Furthermore, the condition on tight cycles can also be described more topologically by noticing that a cyclically reduced closed walk of length $n$ in $G$ is exactly an isolated vertex of $\mathrm{Hom}(C_n, G)$. An $H$-coloring $\alpha$ of $G$ induces a continuous map from $\mathrm{Hom}(C_n, G)$ to $\mathrm{Hom}(C_n, H)$, and an $\alpha$-tight cycle is an isolated vertex of $\mathrm{Hom}(C_n, G)$ that is mapped to an isolated vertex of $\mathrm{Hom}(C_n, H)$. Thus with some effort one can rephrase Theorem 6.1 as follows:

**9.3 Theorem.** *Let $\alpha, \beta$ be two $H$-colorings of $G$ (where $H$ is a square-free graph). Then $\alpha, \beta$ are $\times$-homotopic (connected by a path in $\mathrm{Hom}(G, H)$) if and only if*

- *the induced maps $\alpha_T, \beta_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ are homotopic for $T = K_2$ and $T = C_n$, $n \le |V(G)|$.*

Recall that the same holds (almost trivially) for $T = G$ (see Chapter I, Theorem 4.1), but the point is that the topology of paths in $\mathrm{Hom}(G, H)$ is governed by a few much simpler Hom complexes. This is also essentially what allows us to get a polynomial-time algorithm. It would be interesting to see if this can be extended to all of $\mathrm{Hom}(G, H)$, not just paths in it, or to non-square-free graphs $H$. The hardness result for $H = K_4$ by Bonsma [BC09] suggests that useful topological invariants might not exist already in this case. On the other hand, it is also possible that such invariants do exist, but are algorithmically more complex only because they involve higher dimensions or additional discrete conditions.

### Statistical physics

Interesting, partly related properties of homomorphisms to square-free graphs, with motivations in statistical thermodynamics, were found independently by Chandgotia [Cha17]. One of the results there states that if two homomorphisms from the infinite grid graph $\mathbb{Z}^d$ with $d \ge 2$ to a square-free graph $H$ differ at only finitely many vertices, then they can be recolored into one another.

### Generalizations

We note that none of the proofs in this chapter used any structural properties of $H$. If we consider $H$-Recoloring for any graph $H$, but only allow recoloring a vertex if all of its neighbors have one common color, the same results will follow.

An obvious question is how far can our results be extended to more general constraint satisfaction problems: to the asymmetric (directed) case, to multiple constraint types, to hypergraphs (relations of arbitrary arity)? Is there any connection with the tractable cases of generalized SAT reconfiguration problems?

Another question is whether the problems of graph homomorphism reconfiguration exhibit a dichotomy. For which graphs $H$ is $H$-Recoloring in P or PSPACE-complete? For the hard side, it is known that $K_4$-Recoloring is PSPACE-complete even for $G$ bipartite [BC09]. This is equivalent to saying that $H$-Recoloring is PSPACE-complete for $H$ the cube graph $K_4 \times K_2$, which similarly implies that $H$-Recoloring is PSPACE-complete for $H$ the 4-cycle $C_4$ with all loops added, for example. It is not known whether for every loop-free graph $H$ containing $K_4$ the problem is hard. However, an easy reduction (known as *folding*, see [FL12]) allows us to focus on so called *stiff* graphs. These statements are discussed in more detail in [Wro14].

Instead of an algorithmic dichotomy, we may also ask for which $H$ is it true that shortest $H$-recolorings of $G$ have polynomial length in $|V(G)|$? In other words, which $H$ guarantee that the connected components of $\mathrm{Hom}(G, H)$ have polynomially small diameter?

Answering these questions could be a way to get a better understanding of the category of graphs and the topological combinatorics of graph homomorphisms. As we will see in the following chapter, the tools that characterize paths in $\mathrm{Hom}(G, H)$ are also very useful in the study of Hedetniemi's conjecture.

# Chapter III

# Multiplicativity of circular cliques and square-free graphs

## 1. Introduction

Recall that a graph $K$ is *multiplicative* if $G \times H \to K$ implies $G \to K$ or $H \to K$, for all graphs $G, H$. They are the prime elements in the lattice of graphs, and as presented in Chapter I, Hedetniemi's conjecture states that all cliques $K_n$ are multiplicative. However, the only non-trivial graphs known to be multiplicative are $K_3$, odd cycles, and still more generally, circular cliques $K_{p/q}$ with $\frac{p}{q} < 4$.

We make no progress for cliques, but in this chapter we show that all square-free graphs are multiplicative. This answers a question of Tardif [Tar08] and greatly extends the family of known multiplicative graphs. For example, it gives the first multiplicative graphs of chromatic number higher than 4 (since graph of girth at least five have no squares, but have arbitrarily high chromatic number). Generalizing, in terms of the box complex, the topological insight behind existing proofs for odd cycles, we also give a different proof for circular cliques with $\frac{p}{q} < 4$.

**Previous work** That $K_2$ is multiplicative—ie., a product of two graphs is bipartite iff one of the factors is—follows easily from the fact that a graph is bipartite iff it has no odd-length cycle. $K_3$ was proved to be multiplicative by El Zahar and Sauer [ES85]. Their proof was generalized to odd cycles by Häggkvist et al. [Häg+88]. Much later, Tardif [Tar05] used the fact that odd cycles are multiplicative to extend the result to circular cliques $K_{p/q}$, for any integers $p, q$ satisfying $2 \leq \frac{p}{q} < 4$. His method was very different, deducing the multiplicativity of one graph from another one by using general constructions: powers and inverse powers of graphs (which we study in the next chapter). The new proof presented here is based on the original approach from [ES85; Häg+88], but extends it to circular cliques by making the topological intuitions therein more general and explicit. This shows that this approach applies to all graphs currently known to be multiplicative. However, it makes it all the more interesting to ask whether this can be connected with the approach in [Tar05], which appears to be more general.

One relatively recent partial result on multiplicative graphs that is especially relevant here, is a proof by Delhommé and Sauer [DS02] that if $G$ and $H$ are connected graphs each containing a triangle and $K$ is a square-free graph, then $G \times H \to K$ implies $G \to K$ or $H \to K$. Our result is thus lifting the requirement on containing triangles, but beside similarities to the original proof for $K_3$, their approach is combinatorial and substantially different.

**Squares in graph products** Formally, all the proofs in this chapter are again self-contained and combinatorial, not requiring any knowledge of topology and not involving continuous spaces. However, the intuitions behind proofs heavily rely on some basic algebraic topology.

The proofs on multiplicativity in [ES85; Häg+88; DS02] rely on a common parity invariant, which turns out to be essentially just the parity of the winding number of certain cycles. Informally, this was already noted before, but by making the topological interpretation more explicit we are able to extend the proof to circular cliques and to find a cleaner argument for the crucial step where certain parities are shown to be different. For square-free graphs, we make use of a stronger invariant, the homotopy type of these cycles (describing where the cycle is mapped to, in the square-free graph, up to continuous transformations).

We will hence use the fundamental groupoid $\boldsymbol{\pi}(K)$ of a graph $K$, as defined in Chapter II, to describe these invariants. The results from Chapter II are not applicable directly, but we will also use the general intuition that if we can do something via continuous transformations, then we can probably do this via recoloring, at least in square-free graphs.

However, it will be crucial to take squares ($C_4$ subgraphs) into account, when discussing the topology of graphs, especially of tensor product of graphs $G \times H$ (which have plenty of squares). On an intuitive level, we will be looking at the box complex of graphs (instead of at graphs directly as 1-dimensional simplicial complexes) and use the fact that the box complex of $G \times H$ is equivalent to the product of box complexes of $G$ and $H$ (as a $\mathbb{Z}_2$-spaces). Recall that the box complex has a face for every square in the graph (and similarly, higher-dimensional faces for every complete bipartite subgraph). This means if the vertices $v_0, v_1, v_2, v_3$ form a square, then a path going from $v_0$ through $v_1$ to $v_2$ can be continuously transformed to a path going from $v_0$ through $v_3$ to $v_2$. Formally, this will give an equivalence relation $\sim$ between walks, making one side of a square equivalent to the other, for each square in a graph. This results in a coarser groupoid $\boldsymbol{\pi}(G)_{/\sim}$, essentially quotienting $\boldsymbol{\pi}(G)$ by squares. The crucial property is that $\boldsymbol{\pi}(G \times H)_{/\sim}$ is isomorphic to $\boldsymbol{\pi}(G)_{/\sim} \times \boldsymbol{\pi}(H)_{/\sim}$ (up to some parity issues). Formal definitions and proofs are given in Section 3.

This coarser groupoid is essentially the fundamental groupoid of the box complex of $G$, as defined in topology, except that instead of using $G \times K_2$ (as in the definition of the box complex), we will use $G$ and just consider the parity of the length of walks directly. Since we only look at the fundamental groupoid, so walks and cycles, the higher-dimensional faces of the box complex turn out to be irrelevant. We note that combinatorial definitions of so called higher homotopy groups of the box complex (and general Hom complexes) have been given by Dochtermann [Doc09b].

**Proof outline** Consider a homomorphism $\mu : G \times H \to K$. A cycle in $G$ yields a cycle in $G \times H$, which is mapped to a closed walk in $K$ (precise definitions will come in the main text). In Section 3 we deduce two facts from the properties of coarse groupoids of graph products. (1.) Closed walks in $K$ coming from odd (=odd-length) cycles in $G$ and $H$ must all wind around the same cycle in $K$. (2.) Such closed walks can be composed into an image of an odd cycle in $G \times H$. An image of an odd cycle in $G \times H$ must wind an odd number of times around an odd cycle in $K$. Thus any two closed walks from $G$ and from $H$ must have a total winding number that is odd.

Section 4 then considers the case $K$ is a circular clique, first showing that indeed $K$ is topologically a circle (formally, that $\boldsymbol{\pi}(K)_{/\sim}$ is isomorphic to $\mathbb{Z}$). From the above (2.) it will easily follow that all closed walks in $K$ coming from odd cycles in either $G$ or $H$, say $G$, have odd winding numbers. This odd parity then implies that every odd cycle in $G$ has an edge $g_0g_1$ such that the $K$-coloring $\mu$ maps the edge $(g_0, h_0)(g_1, h_1)$ of $G \times H$ close to its antipode $(g_0, h_1)(g_1, h_0)$ (for some arbitrary fixed edge $h_0h_1$ of $H$). Such edges of $G$ can be disregarded, and we get a bipartite subgraph of $G$, which we color with either $\mu(\cdot, h_0)$ or $\mu(\cdot, h_1)$ according to a bipartition. This gives a $K$-coloring of $G$, concluding the proof for circular cliques.

A reader comfortable with topology may for a moment jump to Chapter IV, Lemmas 6.1–6.3 (pages 75–76), where a purely topological analogue is given; as many combinatorial details become unnecessary, the proof there takes little more than a page.

For the case of square-free $K$, in the last theorem of Section 3, we show that the above (1.) implies that one of three possibilities holds: either all closed walks in $K$ coming from cycles in $G$ are topologically trivial (equivalent to $\varepsilon$), or the same holds for $H$ instead, or all cycles in $G \times H$ map to closed walks winding around the same cycle in $K$. This is the starting point for Section 5, where we aim to improve the $K$-coloring of $G \times H$ by recoloring vertices one by one, so that it reflects its topological type more directly. An improved $K$-coloring of $G \times H$ then turns out to be in fact just a $K$-coloring of $G$ composed with a projection, or a $K$-coloring of $H$ composed with a projection, or a homomorphism from $G \times H$ to a cycle in $K$. Multiplicativity of cycles then implies $G \to K$ or $H \to K$, concluding the proof.

Many steps of the proof generalize to other graphs $K$, but in the end we need the topology of its box complex to be essentially 1-dimensional, and even that is not sufficient. The results still strongly rely on analyzing odd cycles, which may exclude interesting generalizations, in particular any applicability to Hedetniemi's conjecture. We comment more on this issue in Chapter IV.

## 2. Definitions

**Graphs** Throughout this chapter we again consider only graphs without loops (undirected, simple), this time because the questions we consider have trivial answers for graphs with loops. Similarly, we assume that all graphs have at least two vertices and every vertex has a neighbor (that is, there are no isolated vertices—otherwise we could handle them trivially).

Recall from Chapter I that for a vertex set $S \subseteq V(G)$, its *neighborhood* $N_G(S)$ is defined as $\{v \mid uv \in E(G), u \in S\}$. We write $N_G(v)$ for $N_G(\{v\})$ and $N_G^2(S)$ for $N_G(N_G(S)) \setminus S$, often skipping the subscript $G$.

$N_G()$
$N_G^2()$

The *tensor product* $G \times H$ is the graph with $V(G \times H) = V(G) \times V(H)$ and $(g,h)(g',h') \in E(G \times H)$ iff $gg' \in E(G)$ and $hh' \in E(H)$. For a vertex $v = (g,h) \in V(G \times H)$, $v|_G = g$ denotes its projection to $G$. For $h_0h_1 \in H$, we write $G \times h_0h_1$ for the subgraph of $G \times H$ induced on $V(G) \times \{h_0, h_1\}$, isomorphic to $G \times K_2$. Note that $C_n \times K_2$ is a cycle for odd $n$. Whenever the connectivity of some graph is needed, we frequently use the fact that $G \times H$ is connected if and only if $G, H$ are connected and at least one of them is not bipartite.

$G \times H$
project.
$v|_G$
$G \times h_0h_1$

**The fundamental groupoid** From Chapter II, recall that we write $uv \in G$ and $vu = uv^{-1} \in G$ for oriented edges. A walk $W$ is a sequence of oriented edges with matching endpoints. Its length $|W|$ is the number of edges in it, $\varepsilon$ is the empty walk, $W\_W'$ denotes concatenation of walks and $W^{-1}$ denotes the reverse walk. Reducing a walk means removing consecutive edges $e_i, e_{i+1}$ such that $e_{i+1} = e_i^{-1}$ and $\overline{W}$ denotes the result of iteratively reducing $W$ as long as possible. We write $W \cdot W'$ for the $\overline{W\_W'}$. The set of all reduced walks in a graph with $\cdot$ and $()^{-1}$ operations is the *fundamental groupoid* of $G$ and we denote it $\boldsymbol{\pi}(G)$. For a vertex $v \in V(G)$, the subset of all reduced walks which are closed, starting and ending at $v$, forms a group, called the *fundamental group* and denoted $\boldsymbol{\pi}_v(G)$. It is easy to check that any walk $W$ from $u$ to $v$ gives an isomorphism between $\boldsymbol{\pi}_u(G)$ and $\boldsymbol{\pi}_v(G)$ by mapping $C$ to $W^{-1} \cdot C \cdot W$.

$|W|, \varepsilon$
$W\_W'$
$\overline{W}$
$W \cdot W'$
$\boldsymbol{\pi}(G)$
$\boldsymbol{\pi}_v(G)$

If $\mu : G \to K$ is a graph homomorphism and $W$ is a walk in $G$, then $\mu(W)$ is a walk in $K$. In particular projections are homomorphisms, $|_G \colon G \times H \to G$, so if $W$ is a walk in $G \times H$, then $W|_G$ is a walk in $G$. Clearly a graph homomorphism $\mu : G \to H$ induces a groupoid homomorphism $W \mapsto \overline{\mu(W)}$ from $\boldsymbol{\pi}(G)$ to $\boldsymbol{\pi}(H)$. Formally, a *groupoid homomorphism* $\phi : \Pi \to \Pi'$ is a function such that $\phi(P^{-1}) = \phi(P)^{-1}$, and if $P \cdot Q$ is defined in $\Pi$, then $\phi(P) \cdot \phi(Q)$ is defined and equal to $\phi(P \cdot Q)$ in $\Pi'$.

$W|_G$

## 3. Topological invariants of cycles

Recall that a *square* in a graph $G$ is a quadruple of vertices $a, b, c, d$ such that $ab, bc, cd, da \in G$. A square is *trivial* if $a = c$ or $b = d$, *non-trivial* otherwise (since we work with graphs without loops only, a square is non-trivial if and only if its vertices are pairwise different). A graph is *square-free* if it has no non-trivial squares.

square

The definition of the fundamental groupoid $\boldsymbol{\pi}(G)$ turns out to be too fine-grained. One intuitive reason is that we would like $\boldsymbol{\pi}(G\times H)$ to have something in common with $\boldsymbol{\pi}(G)\times\boldsymbol{\pi}(H)$: intuitively, the product of two cycles should behave like a torus, see later Figure III.1. Another reason is recoloring: if $a,b,c,d$ is a square, a walk going through $a,b,c$ can be changed to go through $a,d,c$ instead, by changing just one value, so we want to allow such a replacement in $\boldsymbol{\pi}(G)$ too.

For a graph $G$, we define $\sim$ to be the smallest equivalence relation between walks in $G$ which makes a walk $W$ equivalent to its reduction $\overline{W}$, and makes $W\_ab\_bc\_W'$ equivalent to $W\_ad\_dc\_W'$ for all walks $W,W'$ and every non-trivial square $a,b,c,d$ in $G$. In other words, two walks are equivalent under $\sim$ if and only if one can be obtained from the other by a series of *elementary steps*, each step consisting of either deleting or introducing a subwalk of the form $e\_e^{-1}$ (for some edge $e$) or replacing a subwalk $ab\_bc$ with $ad\_dc$ (for some non-trivial square $a,b,c,d$ in $G$). Note that the words 'non-trivial' can be dropped without changing the definition, because if $b=d$, a subwalk $ab\_bc$ is of course already equal to $ad\_dc$, and if $a=c$, then a subwalk $ab\_bc$ can be deleted and a subwalk $ad\_dc$ can be introduced (in other words, they both reduce to $\varepsilon$). $\sim$ elem. step

We denote the equivalence class of $P$ under $\sim$ as $[P]$. Two walks equivalent under $\sim$ must have the same initial and final vertex. Since $P\sim P'$ and $Q\sim Q'$ implies $P\_Q\sim P'\_Q'$ for walks $P,P',Q,Q'$, and since $P\sim\overline{P}$, we have that $P\sim P'$ and $Q\sim Q'$ implies $P\cdot Q\sim P'\cdot Q'$ for reduced walks. Also $P\sim P'$ implies $P^{-1}\sim P'^{-1}$. Therefore the quotient groupoid $\boldsymbol{\pi}(G)_{/\sim}$ (and group $\boldsymbol{\pi}_v(G)_{/\sim}$) can be defined naturally on equivalence classes of walks under $\sim$. We note that similar quotients were considered in [STW17] and [STW16]. $[P]$ $\boldsymbol{\pi}(G)_{/\sim}$

The equivalence of $\boldsymbol{\pi}$ and $\boldsymbol{\pi}_{/\sim}$ for square-free graphs follows from definitions:

**3.1 Lemma.** *Let $K$ be a square-free graph. Then the function from $\boldsymbol{\pi}(K)_{/\sim}$ to $\boldsymbol{\pi}(K)$ mapping $[W]$ to $\overline{W}$ for any walk $W$ in $K$ is well defined and is a groupoid isomorphism.*

Quite naturally, a graph homomorphism implies a groupoid homomorphism for $\boldsymbol{\pi}_{/\sim}$ as well.

**3.2 Lemma.** *Let $\mu : G\to K$ be a graph homomorphism. Then the function from $\boldsymbol{\pi}(G)_{/\sim}$ to $\boldsymbol{\pi}(K)_{/\sim}$ mapping $[W]$ to $[\mu(W)]$ for any walk $W$ in $G$ is well defined and is a groupoid homomorphism.*

*Proof.* We want to show that if $W\sim W'$ for two walks in $G$, then $\mu(W)\sim\mu(W')$. It suffices to check this when $W$ and $W'$ differ by one elementary step. If $W'$ is obtained from $W$ by deleting (or introducing) a subsequence $g_0g_1\_g_1g_0$ for some $g_0g_1\in G$, then $\mu(W')$ is obtained from $\mu(W)$ by deleting (or introducing) the subsequence $\mu(g_0g_1)\_\mu(g_1g_0)$. Hence $\mu(W)\sim\mu(W')$. If $W'$ is obtained from $W$ by replacing a subsequence $g_1g_2\_g_2g_3$ with $g_1g_4\_g_4g_3$ for some square $g_1,g_2,g_3,g_4$ in $G$, then $\mu(W')$ is obtained from $\mu(W)$ by replacing the corresponding images of $\mu$. Since $\mu$ is a graph homomorphism, $\mu(g_1),\mu(g_2),\mu(g_3),\mu(g_4)$ is a square in $K$, and hence $\mu(W)\sim\mu(W')$. Thus $W\sim W'$ implies $\mu(W)\sim\mu(W')$, meaning the function is well defined. It is indeed a groupoid homomorphism, because $\mu(W^{-1})=\mu(W)^{-1}$, and $\mu(W\cdot W')\sim\overline{\mu(W\cdot W')}=\overline{\mu(\overline{W\_W'})}=\overline{\mu(W\_W')}=\overline{\mu(W)\_\mu(W')}=\mu(W)\cdot\mu(W')$. □

A crucial observation is that if $\overline{W}=\overline{W'}$ or more generally $W\sim W'$, then the lengths of $W$ and $W'$ have the same parity (this follows immediately by considering elementary steps). We can hence speak of the parity of an element of $\boldsymbol{\pi}(G)_{/\sim}$. length parity

We would like to think of $\boldsymbol{\pi}(G\times H)_{/\sim}$ as being isomorphic to $\boldsymbol{\pi}(G)_{/\sim}\times\boldsymbol{\pi}(H)_{/\sim}$, but one may see that by projecting a walk in $G\times H$ to $G$ and $H$, we can never get a pair of walks of different parity. Except for this problem (which could be resolved by taking $G\times K_2$ instead of $G$), they are in fact equivalent, and we will only need the following slightly weaker lemma. The intuitive meaning is that when considering the equivalence class (the homotopy type) of a walk in $G\times H$, it suffices to look at the projections to $G$ and $H$ independently. Unfortunately the proof is quite technical. Note the lemma would not be true if we considered $\boldsymbol{\pi}(G)$ instead of $\boldsymbol{\pi}(G)_{/\sim}$.

**3.3 Lemma.** *Let $G, H$ be graphs. The function $\phi : \boldsymbol{\pi}(G \times H)_{/\sim} \to \boldsymbol{\pi}(G)_{/\sim} \times \boldsymbol{\pi}(H)_{/\sim}$ mapping $[W]$ to $([W|_G], [W|_H])$ for any walk $W$ in $G \times H$ is a well defined injective groupoid homomorphism.*

*Proof.* We first show that if $W \sim W'$ for two walks $W, W'$ in $G \times H$, then $W|_G \sim W'|_G$ (and symmetrically $W|_H \sim W|'_H$). It suffices to show this when $W$ and $W'$ differ by an elementary step. If $W'$ is obtained from $W$ by a single reduction deleting (or introducing) a subwalk $(g_1, h_1)(g_2, h_2)\_(g_2, h_2)(g_1, h_1)$, then $W'|_G$ is obtained from $W|_G$ by a single reduction deleting (or introducing) $g_1 g_2\_g_2 g_1$, and hence $W|_G \sim W'|_G$. If $W'$ is obtained from $W$ by a replacing a subwalk $v_1 v_2\_v_2 v_3$ with $v_1 v_4\_v_4 v_3$ for some square $v_1, v_2, v_3, v_4$ in $G \times H$, then $W'|_G$ is obtained from replacing the corresponding subsequences after projection, where $v_1|_G, v_2|_G, v_3|_G, v_4|_G$ is a square in $G$, and hence $W|_G \sim W'|_G$. Thus $W \sim W'$ implies $W|_G \sim W'|_G$ and $W|_H \sim W'|_H$, so $\phi([W]) = ([W|_G], [W|_H])$ unambiguously defines a function from $\boldsymbol{\pi}(G \times H)_{/\sim}$ to $\boldsymbol{\pi}(G)_{/\sim} \times \boldsymbol{\pi}_h(H)_{/\sim}$. Clearly $(W \cdot W')|_G \sim \overline{(W \cdot W')|_G} = \overline{\overline{W\_W'}|_G} = \overline{W\_W'|_G} = \overline{\overline{W}|_G\_\overline{W'}|_G} = \overline{W}|_G \cdot \overline{W'}|_G$ and $W^{-1}|_G = W|_G^{-1}$, so $\phi$ defines a groupoid homomorphism. It remains to show that $\phi$ is injective.

For walks $P$ in $G$ and $Q$ in $H$ such that $|P| = |Q| \mod 2$, define $\texttt{join}(P, Q)$ as the following walk in $G \times H$. If $|P| \geq |Q|$, let $\texttt{join}(P, Q)$ be the walk whose projection to $G$ is $P$ and whose projection to $H$ is $Q\_e^{-1}\_e\_\dots\_e^{-1}\_e$, where $e$ is an arbitrary edge ending in the same vertex as $Q$, repeated $|P| - |Q|$ times here. Otherwise, if $|P| < |Q|$, define $\texttt{join}(P, Q)$ analogously, extending $P$ with an arbitrary edge $e$ so that it's length matches the length of $Q$, that is, $\texttt{join}(P, Q)|_G = P\_e^{-1}\_e\dots\_e^{-1}\_e$ and $\texttt{join}(P, Q)|_H = Q$.

Observe that $\overline{\texttt{join}(P, Q)|_G} = \overline{P}$, $\overline{\texttt{join}(P, Q)|_H} = \overline{Q}$ for every pair $P, Q$ for which it is defined, and $W = \texttt{join}(W|_G, W|_H)$ for every walk $W$ in $G \times H$. We claim that for any walks $P, P', Q, Q'$, if $P \sim P'$ and $Q \sim Q'$ then $\texttt{join}(P, Q) \sim \texttt{join}(P', Q')$ if both joins are defined. It suffices to show this in the case $P$ differs from $P'$ with an elementary step and $Q' = Q$ (we can show the case $Q$ differs from $Q'$ and $P = P'$ in a symmetric way).

If $P'$ is obtained from $P$ by replacing a subwalk $g_1 g_2\_g_2 g_3$ with $g_1 g_4\_g_4 g_2$ for some square $g_1, g_2, g_3, g_4$ in $G$, then $|P| = |P'|$, so $\texttt{join}(P', Q)$ is obtained from $\texttt{join}(P, Q)$ by replacing a subwalk $(g_1, h_1)(g_2, h_2)\_(g_2, h_2)(g_3, h_3)$ with $(g_1, h_1)(g_4, h_2)\_(g_4, h_2)(g_3, h_3)$ for vertices $h_1, h_2, h_3 \in V(H)$ (that is, the projections to $G$ are as described and projections to $H$ are unchanged). Then $(g_1, h_1), (g_2, h_2), (g_3, h_3), (g_4, h_2)$ is a square in $G \times H$, so $\texttt{join}(P, Q) \sim \texttt{join}(P', Q)$.

If $P'$ is obtained from $P$ by introducing a subwalk $g_1 g_2\_g_2 g_1$ for some $g_1, g_2 \in V(G)$, then let $P = P_1\_P_2$ and $P' = P_1\_g_1 g_2\_g_2 g_1\_P_2$ for some walks $P_1, P_2$ in $G$. We prove the claim by induction on the length $P_2$. For the inductive step, let $P_2$ be non-empty, so $P_2 = g_1 g_x\_P_3$ for some edge $g_1 g_x$ and walk $P_3$ of $G$. Define an intermediate walk $P'' = P_1\_g_1 g_x\_g_x g_1\_P_2$. Then $P'$ and $P''$ have the same lengths, so $\texttt{join}(P', Q) = \texttt{join}(P_1\_g_1 g_2\_g_2 g_1\_P_2, Q)$ is obtained from $\texttt{join}(P'', Q)$ by replacing a subwalk $(g_1, h_1)(g_2, h_2)\_(g_2, h_2)(g_1, h_3)$ with a subwalk $(g_1, h_1)(g_x, h_2)\_(g_x, h_2)(g_1, h_3)$ for some $h_1, h_2, h_3 \in V(H)$. Note $(g_1, h_1), (g_2, h_2), (g_1, h_3), (g_x, h_2)$ is a square in $G \times H$, so $\texttt{join}(P', Q) \sim \texttt{join}(P'', Q)$. Since $P'' = P_1\_g_1 g_x\_g_x g_1\_g_1 g_x\_P_3$ can be obtained from $P = P_1\_g_1 g_x\_P_3$ by introducing $g_x g_1\_g_1 g_x$ before $P_3$, whose length is shorter than $P_2$, we know by inductive assumption that $\texttt{join}(P, Q) \sim \texttt{join}(P'', Q)$ and hence $\texttt{join}(P, Q) \sim \texttt{join}(P', Q)$.

For the basis of the induction assume now that $P_2$ is empty. Suppose first that $P = P_1$ is strictly shorter than $Q$. Then by definition of $\texttt{join}$, $\texttt{join}(P, Q) = \texttt{join}(P\_e^{-1}\_e, Q)$ for some edge $e$ with the same endpoint as $P$, that is, $e = (g_x, g_1)$ for some $g_x \in V(G)$. In this case $\texttt{join}(P', Q) = \texttt{join}(P\_g_1 g_2\_g_2 g_1, Q)$ can be obtained from $\texttt{join}(P, Q)$ by replacing the subwalk whose projection to $G$ is $e^{-1}\_e$, namely $(g_1, h_1)(g_x, h_2)\_(g_x, h_2)(g_1, h_3)$, with a walk whose projection to $G$ is the introduced fragment (and the projection to $H$ is unchanged), that is, $(g_1, h_1)(g_2, h_2)\_(g_2, h_2)(g_1, h_3)$. Note $(g_1, h_1), (g_x, h_2), (g_1, h_3), (g_2, h_2)$ is a square in $G \times H$, so $\texttt{join}(P, Q) \sim \texttt{join}(P', Q)$.

Suppose now that $P_2$ is empty and $P = P_1$ is at least as long as $Q$. Then $\mathtt{join}(P', Q) = \mathtt{join}(P\_g_1g_2\_g_2g_1, Q)$ is obtained from $\mathtt{join}(P, Q)$ by appending $(g_1, h_1)(g_2, h_2)\_(g_2, h_2)(g_1, h_1)$ to it, where $e = h_1h_2$ is the edge with which $Q$ would be extended in the definition of $\mathtt{join}$. Thus $\mathtt{join}(P, Q) \sim \mathtt{join}(P', Q)$.

This concludes the proof that for any walks $P, P', Q, Q'$, if $P \sim P'$ and $Q \sim Q'$, then $\mathtt{join}(P, Q) \sim \mathtt{join}(P', Q')$, if both joins are defined. Thus we can unambiguously define the function $\mathtt{join}'([P], [Q]) := [\mathtt{join}(P, Q)]$ for walks $P, Q$ whose lengths have the same parity. Since $\mathtt{join}'(\phi([W])) = [\mathtt{join}(W|_G, W|_H)] = [W]$, the function $\mathtt{join}'$ is the inverse of $\phi$. Thus $\phi$ is an injection. In fact, $\phi$ gives an isomorphism between $\boldsymbol{\pi}_{(g,h)}(G \times H)_{/\sim}$ and the subgroup of $\boldsymbol{\pi}_g(G)_{/\sim} \times \boldsymbol{\pi}_h(H)_{/\sim}$ formed by those pairs $([P], [Q])$ where $|P|$ and $|Q|$ have the same parity. □

The only cases for which we will use the above 'product lemma' are the next two corollaries, focusing on closed walks of $G$ and $H$. For a closed walk $C$ in a graph $G$ and an oriented edge $h_0h_1$ in a graph $H$, we define $C \otimes h_0h_1$ as the closed walk in $G \times H$ whose projection to $G$ is $C\_C$ and $C \otimes h_0h_1$
whose projection to $H$ is $h_0h_1\_h_1h_0$ repeated $|C|$ times (this is just a way to represent a trivial closed walk $\varepsilon$ in $H$). For a closed walk $D$ in $H$ and $g_0g_1 \in G$ we define $g_0g_1 \otimes D$ symmetrically. Elements of the form $[\mu(C \otimes h_0h_1)]$ in $\boldsymbol{\pi}(K)_{/\sim}$ will be our central tool; see Figure III.1. The 'product lemma' then translates to the following corollary.

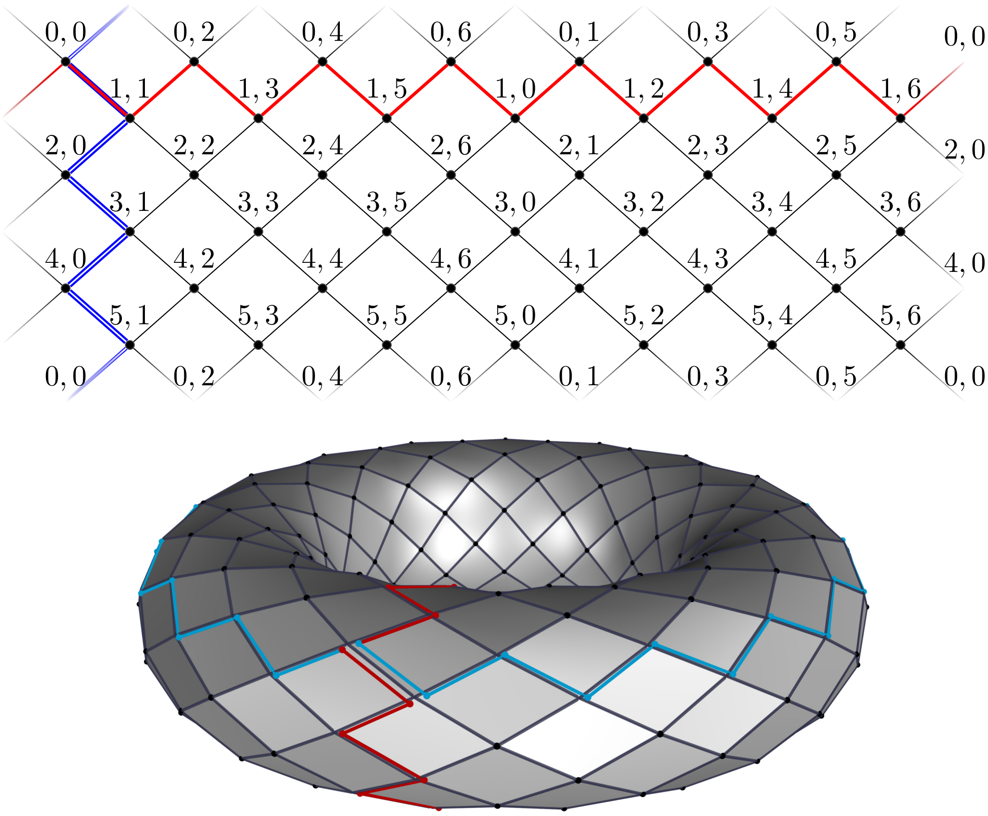



**Figure III.1** *Top:* the graph $C_6 \times C_7$. For $h \in V(H)$, the vertices $(g, h)$ are drawn in two columns depending on the parity of $g + h$ to make the structure more apparent. The red line shows $0\,1 \otimes C_7$. The blue line shows $C_6 \otimes 0\,1$ (visiting each edge twice, since $C_6$ is even).

*Bottom:* a larger example, $C_{16} \times C_{17}$. The red cycle $0\,1 \otimes C_{16}$ and the the blue cycle $C_{17} \otimes 0\,1$ represent different elements (they are not equivalent under $\sim$). However, they commute, ie. $(0\,1 \otimes C_{16})\_(C_{17} \otimes 0\,1) \sim (C_{17} \otimes 0\,1)\_(0\,1 \otimes C_{16})$. Intuitively, this means the first walk, $(0\,1 \otimes C_{16})\_(C_{17} \otimes 0\,1)$, can be continuously transformed along the surface to get the second one (with the starting point fixed).

**3.4 Corollary.** *Let $\mu : G \times H \to K$. Let $g_0g_1 \in G$ and $h_0h_1 \in H$. Let $C \in \boldsymbol{\pi}_{(g_0,h_0)}(G \times H)$. Then $[\mu(C)]^{\cdot 2} = [\mu(C|_G \otimes h_0h_1)] \cdot [\mu(g_0g_1 \otimes C|_H)]$.*

*Proof.* Since $h_0h_1\_h_1h_0 \sim \varepsilon$ in $H$ and $\phi$ is a groupoid homomorphism by Lemma 3.3, we have

$$\begin{aligned} \phi([C^2]) = ([C^2|_G],[C^2|_H]) = ([C^2|_G],[\varepsilon]) \cdot ([\varepsilon],[C^2|_H]) = \\ = ([C|_G{}^2],[h_0h_1\_h_1h_0\_\ldots]) \cdot ([g_0g_1\_g_1g_0\_\ldots],[C|_H{}^2]) = \\ = \phi([C|_G \otimes h_0h_1]) \cdot \phi([g_0g_1 \otimes C|_H]) = \phi([C|_G \otimes h_0h_1] \cdot [g_0g_1 \otimes C|_H]). \end{aligned}$$

Lemma 3.3 also says that $\phi$ is injective, and hence $[C]^{\cdot 2} = [C|_G \otimes h_0h_1] \cdot [g_0g_1 \otimes C|_H]$. By Lemma 3.2, $[\mu(C)]^{\cdot 2} = [\mu(C|_G \otimes h_0h_1)] \cdot [\mu(g_0g_1 \otimes C|_H)]$. □

The crucial property of the tensor product is that if $X$ is a closed walk of the form $C \otimes h_0h_1$ and $Y$ is of the form $g_0g_1 \otimes D$, then $X$ and $Y$ commute in $\boldsymbol{\pi}(G \times H)_{/\sim}$, that is $X \cdot Y = Y \cdot X$. Therefore, by Lemma 3.2, $[\mu(X)]$ commutes with $[\mu(Y)]$, which will give us a lot of information (in the case of square-free $K$).

**3.5 Corollary.** *Let $\mu : G \times H \to K$. Let $g_0g_1 \in G$ and $h_0h_1 \in H$. Let $C \in \boldsymbol{\pi}_{g_0}(G)$ and $D \in \boldsymbol{\pi}_{h_0}(H)$. Then $[\mu(C \otimes h_0h_1)]$ commutes with $[\mu(g_0g_1 \otimes D)]$.*

*Proof.* Since $h_0h_1\_h_1h_0 \sim \varepsilon$ in $H$, $\phi([C \otimes h_0h_1]) = ([C^2|_G],[\varepsilon])$ and similarly $\phi([g_0g_1 \otimes D]) = ([\varepsilon],[D^2|_H])$. As $[\varepsilon]$ commutes with any element of $\boldsymbol{\pi}_{g_0}(G)_{/\sim}$ (and similarly for $H$), $([C^2|_G],[\varepsilon])$ commutes with $([\varepsilon],[D^2|_H])$. By Lemma 3.3, $\phi$ is an injective homomorphism and hence $[C \otimes h_0h_1]$ commutes with $[g_0g_1 \otimes D]$. Since $\mu$ is a graph homomorphism, by Lemma 3.2, $[\mu(C \otimes h_0h_1)]$ commutes with $[\mu(g_0g_1 \otimes D)]$. □

We now consider the information given by $\boldsymbol{\pi}(K)_{/\sim}$ more precisely. For the graphs $K$ we consider, $\boldsymbol{\pi}_v(K)_{/\sim}$ is a free group, for any $v \in V(K)$. For square-free graphs this follows from the fact that $\boldsymbol{\pi}_v(K)$ is always a free group (see eg. [KN07]). For circular cliques this follows from the fact that $\boldsymbol{\pi}_v(K_{p/q})_{/\sim}$ is isomorphic to $\mathbb{Z}$, for $2 < \frac{p}{q} < 4$ (Lemma 4.1). The property of free groups we need (and which can easily be checked directly for $\boldsymbol{\pi}_v(K)$ and $\mathbb{Z}$) is that primitive roots can be unambiguously defined and that primitive roots of commuting elements are equal, up to inversion. For an element $O$ of a free group $\boldsymbol{\pi}$ other than the trivial element $\varepsilon$, its *primitive root* is the unique $R \in \boldsymbol{\pi}$ such that $O = R^n$ for some $n \in \mathbb{N}$ with $n$ maximized (see eg. [MA80] for a linear time algorithm computing $R$). primitive root

**3.6 Fact.** *Let $O_1, O_2$ be elements of a free group. Then $O_1$ and $O_2$ commute, ie. $O_1 \cdot O_2 = O_2 \cdot O_1$, if and only if $O_1 = \varepsilon$ or $O_2 = \varepsilon$ or their primitive roots are equal or the inverse of each other.*

Turning our attention to square-free graphs $K$, from Corollary 3.5 (and Lemma 3.1) we have that $\overline{\mu(C \otimes h_0h_1)}$ commutes with $\overline{\mu(g_0g_1 \otimes D)}$ for any cycles $C, D$ in $G, H$, and therefore they have the same primitive root (up to inversion), if they are both non-$\varepsilon$ elements of $\boldsymbol{\pi}_{\mu(g_0,h_0)}(K)$. Intuitively, this means that if we take any cycle in $G$ and any cycle in $H$, then $\mu$ maps them to closed walks that wind around the same cycles (or the same sequence of cycles) in $K$, though they may wind a different number of times and in opposite directions.

Since this is true for any pair of cycles, this implies that either all cycles in $G$ map to $\varepsilon$, or all cycles in $H$ map to $\varepsilon$, or all cycles in $G$ and $H$ map to closed walks winding around one common cycle of $K$. We make this more formal in the following proof. The theorem captures all we need from this section for the case of square-free $K$. In the first and second case we will later be able to directly obtain a graph homomorphism from $G$ and $H$, respectively, while in the third case, we will reduce our problem by obtaining a homomorphism $G \times H \to C_n$, where $C_n \to K$ corresponds to the common primitive root.

**3.7 Theorem.** *Let $\mu : G \times H \to K$ for a square-free graph $K$. Let $g_0g_1 \in G, h_0h_1 \in H$. Then one of the following holds:*

- $\overline{\mu(C)} = \varepsilon$ *for every closed walk $C$ from $(g_0, h_0)$ in $G \times h_0h_1$,*
- $\overline{\mu(D)} = \varepsilon$ *for every closed walk $D$ from $(g_0, h_0)$ in $g_0g_1 \times H$,*
- *there is an $R \in \boldsymbol{\pi}(K)$ such that for every closed walk $C'$ from $(g_0, h_0)$ in $G \times H$, $\overline{\mu(C')} = R^{\cdot i}$ for some $i \in \mathbb{Z}$.*

*Proof.* Suppose first that $\overline{\mu(C \otimes h_0h_1)} = \varepsilon$ for all $C \in \boldsymbol{\pi}_{g_0}(G)$. Let $C'$ be any closed walk from $(g_0, h_0)$ in $G \times h_0h_1$. Then $C'.C' = C'|_G \otimes h_0h_1$, and hence $\overline{\mu(C')}^{\cdot 2} = \overline{\mu(C'|_G \otimes h_0h_1)} = \varepsilon$, implying $\overline{\mu(C')} = \varepsilon$. So the first case of our claim holds. Symmetrically, if $\overline{\mu(g_0g_1 \otimes D)} = \varepsilon$ for all $D \in \boldsymbol{\pi}_{h_0}(H)$, then the second case of our claim holds.

If neither of the above two possibilities holds, then there is a $C_0 \in \boldsymbol{\pi}_{g_0}(G)$ with $\overline{\mu(C_0 \otimes h_0h_1)} \neq \varepsilon$ and a $D_0 \in \boldsymbol{\pi}_{h_0}(H)$ with $\overline{\mu(g_0g_1 \otimes D_0)} \neq \varepsilon$. Let $R$ be the primitive root of $\overline{\mu(C_0 \otimes h_0h_1)}$, $R \neq \varepsilon$.

Let $D$ be any element of $\boldsymbol{\pi}_{h_0}(H)$. By Corollary 3.5 and Lemma 3.1, $\overline{\mu(C_0 \otimes h_0h_1)}$ commutes with $\overline{\mu(g_0g_1 \otimes D)}$. By Fact 3.6, this implies that the primitive root of $\overline{\mu(g_0g_1 \otimes D_0)}$ is $R$, up to inversion. That is, for every $D \in \boldsymbol{\pi}_{h_0}(H)$, $\overline{\mu(g_0g_1 \otimes D)} = R^{\cdot i}$ for some $i \in \mathbb{Z}$. Using $D_0$, we can symmetrically show that for every $C \in \boldsymbol{\pi}_{g_0}(G)$, $\overline{\mu(C \otimes h_0h_1)} = R^{\cdot i}$ for some $i \in \mathbb{Z}$.

For any $C' \in \boldsymbol{\pi}_{(g_0,h_0)}(G \times H)$, by Corollary 3.4 (and Lemma 3.1), $\overline{\mu(C')}^{\cdot 2} = \overline{\mu(C'|_G \otimes h_0h_1)} \cdot \overline{\mu(g_0g_1 \otimes C'|_H)}$ and hence $\overline{\mu(C')}^{\cdot 2} = R^{\cdot i}$ for some $i \in \mathbb{Z}$. So either $\overline{\mu(C')}$ is empty or it has the same primitive root as $\overline{\mu(C')}^{\cdot 2}$, and in both cases $\overline{\mu(C')} = R^{\cdot i}$ for some $i \in \mathbb{Z}$. $\square$

## 4. The case when $K$ is circular

We begin this section by showing that circular cliques with $2 < \frac{p}{q} < 4$ behave like circles, topologically, and so the (coarse) fundamental group just describes an integer: the winding number. (An analogous fact is well known for the box complex: formally, it is homotopy equivalent to a circle and its fundamental group is isomorphic to $\mathbb{Z}$ for circular cliques with $2 < \frac{p}{q} < 4$). For simplicity, we only consider odd $p$; this includes the case of odd cycles in particular (as $C_{2n+1}$ is isomorphic to $K_{2n+1/n}$), and will still allow us to conclude the general case.

**4.1 Lemma.** *Let $p, q$ be integers such that $2 < \frac{p}{q} < 4$ and $p$ is odd. Then $\boldsymbol{\pi}_v(K_{p/q})_{/\sim}$ is a group isomorphic to $\mathbb{Z}$, for any $v \in V(K_{p/q})$.*

*Proof.* We begin by defining a more intuitive view of a circular clique (so that edges will join numbers that are close enough, instead of far enough). We need the following definitions:

- For $i, j \in \mathbb{Z}_{2p}$, define $\vec{d}(i-j)$ to be the integer (in $\mathbb{Z}$) in the set $\{-(p-1), -(p-2), \ldots, p-1, p\}$ which is equivalent to $i - j \mod 2p$ (this depends only on $i - j$, but we think of it as a signed distance between $i$ and $j$ in the circle $\mathbb{Z}_{2p}$).
- Let $K'$ be the graph with $V(K') = \{0, 1, \ldots, 2p-1\}$, $E(K') = \{ij \mid \vec{d}(i-j) \text{ is odd and } |\vec{d}(i-j)| \leq p - 2q\}$ (that is, $K'$ is the Cayley graph of $\mathbb{Z}_{2p}$ with generators $\pm 1, \pm 3, \ldots, \pm p - 2q$).
- Define $\phi : K_{p/q} \times K_2 \to K'$ as $\phi(i, 0) = 2i \mod 2p$ and $\phi(i, 1) = 2i + p \mod 2p$. This is easily seen to be a graph isomorphism. Indeed, vertices whose difference is $(i - j) \in \{q, q+1, \ldots, \lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil, \ldots, p - q\} \mod p$, will map to vertices whose difference is
$$2(i-j) + p \in \{2q + p, 2q + 2 + p, \ldots, 2\lfloor \tfrac{p}{2} \rfloor + p, 2\lceil \tfrac{p}{2} \rceil + p, \ldots, 2(p-q) + p\} =$$
$$= \{-(p-2q), -(p-2q-2), \ldots, -1, 1, \ldots, p-2q\} \mod 2p.$$

- For a walk $W$ in $K_{p/q}$ whose $(i+1)$-th edge is $w_i w_{i+1}$, define $\varphi(W)$ to be the walk in $K'$ of the same length whose $(i+1)$-th edge is $\phi(w_i, i \mod 2)\phi(w_{i+1}, i+1 \mod 2)$. Note that $\varphi$ maps closed walks of odd length beginning and ending in $w_0 \in V(K_{p/q})$ to walks between $\phi(w_0, 0)$ and $\phi(w_0, 1) = \phi(w_0, 0) + p \mod 2p$, which are not closed.
- For a walk $W'$ in $K'$ whose $(i+1)$-th edge is $w_i w_{i+1}$, define $\Delta(W) = \sum_{i=0}^{|W|-1} \vec{d}(w_{i+1} - w_i)$. Intuitively, $\Delta$ measures how far $W$ went winding around $K'$, and we should think of a closed walk $W$ as winding $d$ times if $\Delta(W) = d \cdot 2p$.

We claim that $\Delta$ defines a functor from $\boldsymbol{\pi}(K')_{/\sim}$ to the group $\mathbb{Z}$, that is, $\Delta(W\_W') = \Delta(W) + \Delta(W')$ and $W \sim W'$ implies $\Delta(W) = \Delta(W')$ for any two walks $W, W'$ in $K'$. The first is clear, so consider two walks $W \sim W'$ in $K'$. It suffices to show that $\Delta(W) = \Delta(W')$ when $W$ and $W'$ differ by an elementary step. If $W'$ is obtained from $W$ by introducing/deleting a subsequence $w_1w_2\_w_2w_1$ for some $w_1w_2 \in E(K')$, then $\Delta(W')$ is obtained from $\Delta(W)$ by adding/subtracting $\Delta(w_1w_2\_w_2w_1) = \vec{d}(w_2 - w_1) + \vec{d}(w_1 - w_2) = 0$, so the two values are indeed equal (note that $\vec{d}(w_2 - w_1) = -\vec{d}(w_1 - w_2)$ unless both are equal to $p$, which is impossible when $w_1w_2 \in E(K')$). If $W'$ is obtained from $W$ by replacing a subsequence $ab\_bc$ by $ad\_dc$, for some square $a, b, c, d$ in $K'$, then $\Delta(W')$ differs from $\Delta(W)$ by $(\vec{d}(b-a)+\vec{d}(c-b))-(\vec{d}(d-a)+\vec{d}(c-d)) = \vec{d}(b-a) + \vec{d}(a-d) + \vec{d}(d-c) + \vec{d}(c-b)$. Since $ab$ is an edge of $K'$, we have $|\vec{d}(b-a)| \leq p - 2q$ and similarly $|\vec{d}(a-d)| \leq p - 2q$. Hence $|\vec{d}(b-a)| + |\vec{d}(a-d)| \leq 2p - 4q < p$, so $a, b, d$ are contained in an interval of length less than $p$ in $\mathbb{Z}_{2p}$, implying $\vec{d}(b-a) + \vec{d}(a-d) = \vec{d}(b-d)$ and $|\vec{d}(b-d)| < p$. Similarly $\vec{d}(d-c) + \vec{d}(c-b) = \vec{d}(d-b)$. Therefore the difference between $\Delta(W')$ and $\Delta(W)$ is $\vec{d}(b-d) + \vec{d}(d-b) = 0$, so they are in fact equal.

Furthermore, $\Delta \circ \varphi$ is a functor from $\boldsymbol{\pi}(K_{p/q})_{/\sim}$ to the group $\mathbb{Z}$, that is, $\Delta(\varphi(W\_W')) = \Delta(\varphi(W)) + \Delta(\varphi(W'))$ and $W \sim W'$ implies $\Delta(\varphi(W)) = \Delta(\varphi(W'))$. The first follows from the definitions and the fact that $\vec{d}((p+i) - (p+j)) = \vec{d}(i-j)$ for $i, j \in \mathbb{Z}_{2p}$. The second follows from the fact that an elementary step showing $W \sim W'$ in $K_{p/q}$ corresponds to an elementary step showing $\phi(W) \sim \phi(W')$ in $K'$; in particular, if $a, b, c, d$ is a square in $K_{p/q}$, then $\phi(a, i), \phi(b, 1-i), \phi(c, i), \phi(d, 1-i)$ is a square in $K'$ for $i = 0, 1$.

Let us define a generator for $\boldsymbol{\pi}_0(K_{p/q})_{/\sim}$. Define $O$ as the closed walk of length $p$ in $K_{p/q}$ whose $(i+1)$-th edge is $(i \cdot \lceil \frac{p}{2} \rceil \mod p, (i+1) \cdot \lceil \frac{p}{2} \rceil \mod p)$. Then $\varphi(O)$ is a walk of length $p$ in $K'$ going from $0$ to $p$, whose $(i+1)$-th edge is $(i \mod 2p, i+1 \mod 2p)$; indeed, for even $i$, it is by definition $(2 \cdot i \lceil \frac{p}{2} \rceil \mod 2p, 2 \cdot (i+1) \lceil \frac{p}{2} \rceil + p \mod 2p) = (i \cdot (p+1) \mod 2p, (i+1) \cdot (p+1) + p \mod 2p) = (i \mod 2p, i+1 \mod 2p)$, and similarly for odd $i$. Thus $\Delta(\varphi(O)) = \sum_{i=0}^{|O|} 1 = |O| = p$.

We claim that for every closed walk $W$ in $K_{p/q}$ from $0$ to $0$, there is a $d \in \mathbb{Z}$ such that $W \sim O^d$. First, we use elementary steps to transform $W$ so that $w_{i+1} - w_i \in \{\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil\}$ (as elements in $\mathbb{Z}_p$) for any edge $w_i w_{i+1}$ of $W$. Indeed, if say $w_{i+1} - w_i \in \{q, q+1, \ldots, \lfloor \frac{p}{2} \rfloor - 1\}$, then letting $x = w_i + \lfloor \frac{p}{2} \rfloor$ and $y = w_i - 1$, we see that $w_i - x = \lfloor \frac{p}{2} \rfloor$, $x - y = \lceil \frac{p}{2} \rceil$, and $w_{i+1} - y = w_{i+1} - w_i + 1 \mod p$. In particular, $w_i, x, y, w_{i+1}$ is a square in $K_{p/q}$, so $w_i w_{i+1} \sim w_i w_{i+1}\_w_{i+1} y\_y w_{i+1} \sim w_i x\_xy\_y w_{i+1}$. Hence we can replace the subwalk $w_i w_{i+1}$ in $W$ by $w_i x\_xy\_y w_{i+1}$. Since this introduced two edges with difference between endpoints in $\{\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil\}$ and changed this difference for the third edge to be closer to $\lfloor \frac{p}{2} \rfloor$, we can do such replacements until we get a walk $W' \sim W$ with $w'_{i+1} - w'_i \in \{\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil\}$ for any edge $w'_i w'_{i+1}$ of $W'$. Then, if two consecutive edges have a different difference, say, $w'_{i+1} - w'_i = \lfloor \frac{p}{2} \rfloor$ and $w'_{i+2} - w'_{i+1} = \lceil \frac{p}{2} \rceil$, then in fact $w'_{i+2} = w'_i + \lfloor \frac{p}{2} \rfloor + \lceil \frac{p}{2} \rceil = w'_i$, so they reduce, that is, the subwalk $w'_i w'_{i+1}\_w'_{i+1} w'_{i+2}$ can be deleted in an elementary step. We do this until we get a walk $W'' \sim W$ such that $w''_{i+1} - w''_i = c$ for all edges $w''_{i+1} w''_i$ of $W''$, for some constant $c \in \{\lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil\} = \{\lfloor \frac{p}{2} \rfloor, -\lfloor \frac{p}{2} \rfloor\}$. Then the $i$-th vertex of the walk is $w''_i = i \cdot c$. Since $W''$ is a closed walk, it must be that $|W''| \cdot c = 0 \mod p$. Therefore, $p$ must divide $|W''|$ and $W'' = O^{|W|/p}$ or $W'' = O^{-|W|/p}$.

The above paragraph shows that every element of $\boldsymbol{\pi}_0(K_{p/q})_{/\sim}$ is of the form $[O^d]$ for some $d \in \mathbb{Z}$. Since $\Delta(\varphi(O^d)) = d \cdot p$, these are pairwise different elements, for different $d$. Clearly $[O^d] \cdot [O^{d'}] = [O^{d+d'}]$, hence $\boldsymbol{\pi}_0(K_{p/q})_{/\sim}$ is a group isomorphic to $\mathbb{Z}$. For any $v \in V(K_{p/q})$, let $P$ be any walk from 0 to $v$ in $K_{p/q}$. Then $[O] \mapsto [P] \cdot [O] \cdot [P]^{-1}$ is easily checked to be a group isomorphism between $\boldsymbol{\pi}_v(K_{p/q})_{/\sim}$ and $\boldsymbol{\pi}_0(K_{p/q})_{/\sim}$. □

Next, with give a very short proof of a parity argument used in [ES85; Häg+88; DS02]. For a cycle $C$ in $G$ or $H$, if $[\mu(C \otimes h_0h_1)] = X^{\cdot 2}$ for some $X \in \boldsymbol{\pi}(K)_{/\sim}$, define the *half-parity* of $C$ as the parity of $|X|$. The following lemma shows that the half-parity of each odd-length cycle in $G$ is defined and different from the half-parity of each odd-length cycle in $H$. half--parity

**4.2 Lemma.** *Let $\mu : G \times H \to K$, $g_0g_1 \in G$, $h_0h_1 \in H$, and assume $\boldsymbol{\pi}_{\mu(g_0,h_0)}(K)_{/\sim}$ is a free group. Let $C$ be a closed walk from $g_0$ in $G$ and let $D$ be a closed walk from $h_0$ in $H$, with $|C|$,$|D|$ odd. Then $[\mu(C \otimes h_0h_1)] = R^{\cdot 2i}$ and $[\mu(g_0g_1 \otimes D)] = R^{\cdot 2j}$ for some $R \in \boldsymbol{\pi}_{\mu(g_0,h_0)}(K)_{/\sim}$ of odd length and $i, j \in \mathbb{Z}$ such that $i + j$ is odd.*

*Proof.* Let $J$ be the closed walk from $(g_0, h_0)$ in $G \times H$ whose projection to $G$ is $C^{|D|}$ and whose projection to $H$ is $D^{|C|}$. Then $J$ has length $|C| \cdot |D|$, which is odd, in particular $[\mu(J)] \neq \varepsilon$. Let $R$ be the primitive root of $[\mu(J)]$, that is, $[\mu(J)] = R^{\cdot k}$, for some $k \in \mathbb{Z}$. It follows that $R$ and $k$ are odd. By Corollary 3.4,

$$R^{\cdot 2k} = [\mu(J)]^{\cdot 2} = [\mu((C^{|D|}) \otimes h_0h_1)] \cdot [\mu(g_0g_1 \otimes (D^{|C|}))] = [\mu(C \otimes h_0h_1)]^{\cdot |D|} \cdot [\mu(g_0g_1 \otimes D)]^{\cdot |C|}$$

By Corollary 3.5, $[\mu(C \otimes h_0h_1)]$ and $[\mu(g_0g_1 \otimes D)]$ commute. Hence they both commute with $R^{\cdot 2k}$ and by Fact 3.6, they are equal to $R^{\cdot 2i}$ and $R^{\cdot 2j}$ respectively, for some $i, j \in \mathbb{Z}$ (the exponents must be even because $R$ is odd and $|C \otimes h_0h_1|$ is even). Then $R^{\cdot 2k} = R^{\cdot(2i|D|+2j|C|)}$, so $i \cdot |D| + j \cdot |C| \equiv i + j \mod 2$ must be odd. □

This implies that either the half-parity is odd for all odd-length cycles in $G$ and even for all odd-length cycles in $H$, or vice versa. (As a side note, let us mention this conclusion could be reached more generally, even when $\boldsymbol{\pi}(K)_{/\sim}$ is not free, as long as the edges of $K$ admit an orientation such that no square $a, b, c, d$ of $K$ is oriented $a \to b \to c \to d$ and $a \to d$; if such an orientation exists, the algebraic length mod 4 of a walk turns out to be a suitable invariant.)

The key to the parity approach is however in the next lemma, and in the corollary following it. It will show that if an odd-length cycle in $G$ has odd half-parity, then $\mu(g, h_0)$ is equal or close to $\mu(g, h_1)$ for some vertex $g$ of the cycle. Excluding such a vertex (or edge) from every odd cycle, we will later make a large subgraph of $G$ bipartite.

Intuitively, the lemma reflects the topological fact that in a map from a circle to a circle $\mu : S^1 \to S^1$ winding an odd number of times, there must be a pair of antipodal points that maps to antipodal points, that is, a point $x \in S^1 \subseteq \mathbb{R}^2$ satisfying $\mu(x) = -\mu(-x)$. The idea is then that given a cycle $C$ in $G$, we can view $\mu$ as a map from $C \otimes h_0h_1$ to $K \times K_2$, which can be extended piece-wise linearly to a continuous map from a circle to the topological space corresponding to $K \times K_2$. If $C$ has odd half-parity, then this map will be winding an odd number of times. The above fact then implies that some antipodal points map to antipodes in $K \times K_2$ and hence the to the same point in $K$. If $K$ is an odd cycle and $|C|$ is odd, it can be shown that such antipodal points will occur as vertices of $C \otimes h_0h_1$ (instead of some general position in the continuous extension). This is not true for circular cliques, but we can still show a slight relaxation.

**4.3 Lemma.** *Let $O = k_0k_1 \_ k_1k_2 \_ \ldots \_ k_{2n-1}k_0$ be a closed walk of length $2n$ in $K_{p/q}$, for $n, p$ odd and $2 < \frac{p}{q} < 4$. If $[O] = [R]^{\cdot 2}$ for some walk $R$ of odd length in $K_{p/q}$, then there is an index $i \in \mathbb{Z}_{2n}$ such that $k_ik_{i+n+1}$ and $k_{i+1}k_{i+n}$ are edges of $K_{p/q}$.*

*Proof.* We reuse the definitions of $\vec{d}, K', \Delta, \varphi$ of the proof of Lemma 4.1. Let us first translate the statement in these terms. In particular, $\varphi(O)$ is a closed walk in $K'$ of length $2n$. Since $|R|$ is odd, $\Delta(\varphi(R))$ is odd too (from the definitions, it is a sum of $|R|$ summands, each of which corresponds to an edge of $K'$ and hence is an odd integer). We showed that $O \sim R^2$ implies $\Delta(\varphi(O)) = \Delta(\varphi(R^2)) = 2 \cdot \Delta(\varphi(R))$ and hence $\Delta(\varphi(O)) \equiv 2 \mod 4$, which is all we need to know about $\varphi(O)$.

Let $\varphi(O) = c_0 c_1 \text{\_} c_1 c_2 \text{\_} \dots \text{\_} c_{2n-1} c_0$. We wish to show that for some $i \in \mathbb{Z}_{2n}$, $k_i k_{i+n+1}$ and $k_{i+1} k_{i+n}$ are edges of $K_{p/q}$. This is the same as saying that the difference between endpoints is in $\{q, q+1, \dots, \lfloor \frac{p}{2} \rfloor, \lceil \frac{p}{2} \rceil, \dots, p-q\} \mod p$, which by definition of $\varphi$ is equivalent to saying that the difference between endpoints of $c_i c_{i+n+1}$ and $c_{i+1} c_{i+n}$ is in

$$\{2q, 2q+2, \dots, 2\lfloor \frac{p}{2} \rfloor, 2\lceil \frac{p}{2} \rceil, \dots, 2(p-q)\} = \{2q, 2q+2, \dots, p-1, -(p-1), \dots, -2q\},$$

that is, $|\vec{d}(c_i - c_{i+n+1})| \geq 2q$ and $|\vec{d}(c_{i+1} - c_{i+n})| \geq 2q$. We can forget $K_{p/q}$ and focus on the walk $\varphi(O)$ in $K'$ from now on.

The statement we want to prove is now the following: if $\varphi(O) = c_0 c_1 \text{\_} c_1 c_2 \text{\_} \dots \text{\_} c_{2n-1} c_0$ is a walk in $K'$ of length $2n$ for $n$ odd such that $\Delta(\varphi(O)) \equiv 2 \mod 4$, then there is an $i \in \mathbb{Z}_{2n}$ such that $|\vec{d}(c_i - c_{i+n+1})| \geq 2q$ and $|\vec{d}(c_{i+1} - c_{i+n})| \geq 2q$.

Suppose to the contrary that for all $i \in \mathbb{Z}_{2n}$, $|\vec{d}(c_i - c_{i+n+1})| < 2q$ or $|\vec{d}(c_{i+1} - c_{i+n})| < 2q$. We claim that for all $i \in \mathbb{Z}_{2n}$,

$$\vec{d}(c_{i+1} - c_i) - \vec{d}(c_{i+n+1} - c_{i+n}) = \vec{d}(c_{i+1} - c_{i+n+1}) - \vec{d}(c_i - c_{i+n}) \qquad (*)$$

Fix $i \in \mathbb{Z}_{2n}$ and assume first that $|\vec{d}(c_i - c_{i+n+1})| < 2q$. Then $c_i c_{i+1} \in E(K')$ means $|\vec{d}(c_{i+1} - c_i)| \leq p - 2q$ and hence $c_i, c_{i+1}$ and $c_{i+n+1}$ are contained in an interval of length less than $p$ of $\mathbb{Z}_{2p}$, which implies $\vec{d}(c_{i+1} - c_i) + \vec{d}(c_i - c_{i+n+1}) = \vec{d}(c_{i+1} - c_{i+n+1})$. Similarly $c_{i+n+1} c_{i+n} \in E(K')$ implies that $\vec{d}(c_i - c_{i+n+1}) + \vec{d}(c_{i+n+1} - c_{i+n}) = \vec{d}(c_i - c_{i+n})$. Subtracting the two gives (*). Note also that $|\vec{d}(c_i - c_{i+n})| < 2q + p - 2q = p$. The proof is analogous in the other case, when $|\vec{d}(c_{i+1} - c_{i+n})| < 2q$.

Let us now sum (*) over $i = 0, 1, \dots, n-1$. The left side then amounts to $\sum_{i=0}^{n-1} \vec{d}(c_{i+1} - c_i) - \sum_{i=n}^{2n-1} \vec{d}(c_{i+1} - c_i)$, while the right side telescopes to simply $-\vec{d}(c_0 - c_{0+n}) + \vec{d}(c_{n-1+1} - c_{n-1+n+1}) = -\vec{d}(c_0 - c_n) + \vec{d}(c_n - c_0) = 2\vec{d}(c_n - c_0)$ (the last equality follows from $|\vec{d}(c_0 - c_n)| < p$). Since $n$ is odd, $c_n$ and $c_0$ belong to different sides of the bipartition of $K'$ and hence $\vec{d}(c_n - c_0)$ is odd. Therefore $\sum_{i=0}^{n-1} \vec{d}(c_{i+1} - c_i) - \sum_{i=n}^{2n-1} \vec{d}(c_{i+1} - c_i) = 2\vec{d}(c_n - c_0) \equiv 2 \mod 4$. Since similarly $\vec{d}(c_{i+1} - c_i)$ is odd for all $i$, and $n$ is odd, we have $2 \cdot \sum_{i=n}^{2n-1} \vec{d}(c_{i+1} - c_i) \equiv 2 \mod 4$. Together, this implies $\sum_{i=0}^{2n-1} \vec{d}(c_{i+1} - c_i) \equiv 0 \mod 4$. But this contradicts our assumption that $\Delta(\varphi(O)) = \sum_{i=0}^{2n-1} \vec{d}(c_{i+1} - c_i) \equiv 2 \mod 4$. □

**4.4 Corollary.** *Let $\mu : G \times H \to K_{p/q}$ for $2 < \frac{p}{q} < 4$ and $p$ odd. Let $h_0 h_1 \in H$. Let $C$ be an odd-length closed walk in $G$. If $C$ has odd half-parity, then there is an edge $gg'$ of $C$ such that $\mu(g, h_0)\mu(g', h_0)$ and $\mu(g, h_1)\mu(g', h_1)$ are edges in $K_{p/q}$.*

*Proof.* Recall $C$ having odd half-parity means $[\mu(C \otimes h_0 h_1)] = X^{\cdot 2}$ for some odd $X \in \boldsymbol{\pi}(K_{p/q})_{/\sim}$. The claim follows then from Lemma 4.3 applied to the closed walk $\mu(C \otimes h_0 h_1)$: it has length $2|C|$, where $|C|$ is odd, and vertices indexed with $i$, $i + |C| + 1$, $i + 1$ and $i + |C|$ are $\mu(g, h_j), \mu(g', h_j), \mu(g', h_{1-j})$ and $\mu(g, h_{1-j})$ respectively, for some $gg' \in C$ and $j \in \{0, 1\}$. □

Finally, we use what we obtained to get a graph homomorphism $G \to K$, similarly as in [ES85], except for using the relaxed condition on edges instead of a condition on vertices.

**4.5 Lemma.** *Let $\mu : G \times H \to K$. Let $g_0g_1 \in G$, $h_0h_1 \in H$. If every odd-length closed walk in $G$ has an edge $gg'$ such that $\mu(g,h_0)\mu(g',h_0) \in K$ and $\mu(g,h_1)\mu(g',h_1) \in K$, then $G \to K$.*

*Proof.* Let $G'$ be the subgraph of $G$ obtained by removing those edges $gg' \in G$ which satisfy $\mu(g,h_0)\mu(g',h_0) \in K$ and $\mu(g,h_1)\mu(g',h_1) \in K$. Then the assumption says that $G'$ is bipartite. Fix a bipartition of $G'$ and let $\delta(g) = h_0$ for $g \in V(G)$ on one side of it and $\delta(g) = h_1$ for $g$ on the other side. In other words, $\delta$ is a graph homomorphism from $G'$ to $h_0h_1$, a subgraph of $H$ isomorphic to $K_2$.

Define $\gamma : V(G) \to V(K)$ as $\gamma(g) = \mu(g, \delta(g))$ for $g \in V(G)$. To show that $\gamma$ is a graph homomorphism, consider any edge $gg' \in G$. If $gg' \in G'$, then $\delta(gg') \in H$ (in fact $\delta(gg') = h_0h_1$) and $gg' \in G$, which implies $\gamma(gg') = \mu((g,\delta(g))(g',\delta(g'))) \in K$. If $gg' \notin G'$, then either $\delta(gg') \in H$ and $\gamma(gg') \in E(K)$ follows as before, or $\delta(g) = \delta(g') = h_i$ for some $i \in \{0,1\}$, which implies $\gamma(gg') = \mu(g,h_i)\mu(g',h_i)$, which is an edge of $K$ by construction of $G'$. □

Since the above lemma is the one that gives the final graph homomorphism, we note a potentially interesting generalization which follows straightforwardly from the same proof: let $\mu : G \times H \to K$, let $H'$ be an induced subgraph of $H$, and let $G'$ be the subgraph of $G$ obtained by removing those edges $gg' \in E(G)$ such that $\forall_{h,h' \in V(H')} \mu(g,h)\mu(g',h') \in K$; then $G' \to H'$ implies $G \to K$.

**4.6 Theorem.** *The circular clique $K_{p/q}$ is multiplicative, for $2 \le \frac{p}{q} < 4$.*

*Proof.* We first show the claim for $p$ odd (in particular $2 < \frac{p}{q}$). Let $\mu : G \times H \to K_{p/q}$ for some graphs $G, H$ and let $g_0g_1 \in G$, $h_0h_1 \in H$. We can assume $G$ and $H$ are connected and non-bipartite. By Lemma 4.1, $\boldsymbol{\pi}_{\mu(g_0,h_0)}(K_{p/q})_{/\sim}$ is isomorphic to $\mathbb{Z}$ and hence a free group. By Lemma 4.2, for any odd-length closed walks $C, D$ in $G, H$ from $g_0, h_0$ respectively, the half-parities of $C$ and $D$ are different. Assume without loss of generality that the half-parity is odd for all odd-length closed walks $C$ from $g_0$ in $G$ (otherwise swap $G$ and $H$).

If $C'$ is an odd-length closed walk from $g'$ in $G$, we claim $C'$ has odd half-parity too. Indeed, taking any even-length walk $W$ from $g_0$ to $g'$, $W\_C'\_W^{-1}$ is an odd-length closed walk from $g_0$ in $G$. It hence has odd half-parity, meaning $[\mu((W\_C'\_W^{-1}) \otimes h_0h_1)] = X^{\cdot 2}$ for some odd $X \in \boldsymbol{\pi}(K_{p/q})_{/\sim}$. Thus $[\mu(C' \otimes h_0h_1)] = Y^{\cdot 2}$ for $Y = [\mu(W')]^{-1} \cdot X \cdot [\mu(W')]$, where $W'$ is the walk with $W'|_G = W$ and $W'|_H = (h_0h_1\_h_1h_0)^{|W|/2}$. Hence $C'$ has odd half-parity too.

Therefore by Corollary 4.4 every odd-length closed walk in $G$ has an edge with the property from the claim and hence Lemma 4.5 gives a homomorphism $G \to K_{p/q}$.

Consider now $K_{p/q}$ with $p$ even. Suppose $G \times H \to K_{p/q}$. Then $G \times H \to K_{p'/q'}$ for any $p', q'$ with $\frac{p}{q} < \frac{p'}{q'}$ and thus $G \to K_{p'/q'}$ or $H \to K_{p'/q'}$ for $p'$ odd. Since the set of rationals $2 < \frac{p'}{q'} < 4$ with $p'$ odd is dense in the interval $(2,4)$, and since $\chi_c(G) = \inf\{\frac{p'}{q'} : G \to K_{p'/q'}\}$ is known to be attained [Zhu01], it follows that $G \to K_{p/q}$ or $H \to K_{p/q}$. □

## 5. The case when $K$ is square-free

As sketched in the introduction, the proof will rely on inductively improving a $K$-coloring $\mu$ of $G \times H$ by recoloring. Recall that we say a $K$-coloring $\mu$ of a graph $G$ can be *recolored* to $\mu^*$ if there is a sequence $\mu_0, \dots, \mu_n$ of $K$-colorings of $G$ with $\mu_0 = \mu, \mu_n = \mu^*$, where $\mu_{i+1}$ differs from $\mu_i$ for at most one value $g \in V(G)$. Note that if $\mu^*$ is obtained from $\mu$ by changing colors at some independent set of vertices (a set $S \subseteq V(G)$ such that $S \times S \cap E(G) = \emptyset$), then $\mu^*$ can be obtained by recoloring (considering vertices of $S$ one by one, in any order). Recoloring can be thought as a discrete homotopy, it preserves the topological invariants we defined before; we will need this only in the following case (see the previous chapter for a more constructive statement; note also this works for general $K$ by taking $\boldsymbol{\pi}(K)_{/\sim}$ instead of $\boldsymbol{\pi}(K)$). recolor

**5.1 Lemma.** *Let $\mu, \mu^* : G \to K$ for $K$ square-free. Assume $\mu$ can be recolored to $\mu^*$. Let $C$ be any closed walk in $G$. Then $\overline{\mu(C)}$ and $\overline{\mu^*(C)}$ are conjugate, that is, there is a $Q \in \boldsymbol{\pi}(K)$ such that $\overline{\mu(C)} = Q \cdot \overline{\mu^*(C)} \cdot Q^{-1}$.*

*Proof.* It suffices to prove the lemma in the case $\mu^*$ is obtained in a single step, changing the color of $g \in V(G)$ only. Let $C = c_0c_1\_c_1c_2\_\dots\_c_{n-1}c_0$. For any $i \in \mathbb{Z}_n$ such that $c_i = g$, since $G$ is loop-free, $c_{i-1}$ and $c_{i+1}$ are different from $g$. Thus $\mu(c_{i-1}) = \mu^*(c_{i-1})$ and $\mu(c_{i+1}) = \mu^*(c_{i+1})$, which means $\mu(c_{i-1}), \mu(c_i), \mu(c_{i+1}), \mu^*(c_i)$ is a square in $K$. Since $K$ is square-free, this implies $\mu(c_{i-1}) = \mu(c_{i+1})$ and thus $\overline{\mu(c_{i-1}c_i\_c_ic_{i+1})} = \varepsilon = \overline{\mu^*(c_{i-1}c_i\_c_ic_{i+1})}$. Hence, if $c_0 \neq g$, $\overline{\mu(C)} = \overline{\mu^*(C)}$, while if $c_0 = g$, then

$$\overline{\mu(C)} = \overline{\mu(c_0c_1)} \cdot \overline{\mu(c_1c_2\_\dots\_c_{n-2}c_{n-1})} \cdot \overline{\mu(c_{n-1}c_0)} = \overline{\mu(c_0c_1)} \cdot \overline{\mu^*(c_1c_2\_\dots\_c_{n-2}c_{n-1})} \cdot \overline{\mu(c_{n-1}c_0)} =$$

$$= \overline{\mu(c_0c_1)} \cdot \overline{\mu^*(c_0c_1)}^{-1} \cdot \overline{\mu^*(C)} \cdot \overline{\mu^*(c_{n-1}c_0)}^{-1} \cdot \overline{\mu(c_{n-1}c_0)} = Q \cdot \overline{\mu^*(C)} \cdot Q^{-1}$$

for $Q = ab\_bc$ where $a = \mu(c_0)$, $b = \mu(c_1) = \mu^*(c_1) = \mu(c_{n-1}) = \mu^*(c_{n-1})$ and $c = \mu^*(c_0)$. $\square$

The above lemma, together with the observation that $Q \cdot R^{\cdot i} \cdot Q^{-1} = (Q \cdot R \cdot Q^{-1})^{\cdot i}$, implies that if any case of Theorem 3.7 is true for $\mu$, then it is also true for any $K$-coloring reachable from it by recoloring. We use this to improve a given $K$-coloring without losing the conclusions of Theorem 3.7.

By *$H$-improving* a $K$-coloring $\mu$ of $G \times H$, we mean recoloring $\mu$ to make $\mu(\cdot, h)$ as constant as possible, for every $h \in V(H)$. Formally, $\mu^* : G \times H \to K$ $H$-improves over $\mu$ if the number of triples $g, g' \in V(G), h \in V(H)$ such that $g, g'$ have a common neighbor in $G$ and $\mu^*(g,h) \neq \mu^*(g',h)$ is lower than for $\mu$. We say $\mu$ can be $H$-improved by recoloring if there is a $\mu^*$ to which it can be recolored and which $H$-improves over $\mu$. $H$-impr.

For readers familiar with covering spaces in topology, the intuitions behind 'improving' can be explained in the following terms (which in fact could be made formal using the theory of graph coverings presented in [KN07]). Consider any base vertex $(g_0, h_0)$ of $G \times H$ with any edge $h_0h_1 \in H$. In the first case of Theorem 3.7, when all cycles in $G \times h_0h_1$ map to closed walks in $K$ that are topologically trivial, we can lift the $K$-coloring $\mu$ to a graph homomorphism mapping $G \times h_0h_1$ to the universal cover of $K$, which is a tree (its nodes are the reduced walks based at $\mu(g_0, h_0)$). This graph homomorphism to a tree can then be folded until it becomes a homomorphism to an edge, constant on $V(G) \times \{h_0\}$ and on $V(G) \times \{h_1\}$. We fold it by finding extremal vertices in $G \times h_0h_1$—those which map the furthest from a fixed base vertex in the universal cover—and changing the mapping so that they map closer. For example, if $\mu$ maps a walk starting at the base point to $ab\_bc\_cd\_dc\_cb\_ba$, then we recolor the extremal vertex (colored $d$) so that the walk maps to $ab\_bc\_cb\_bc\_cb\_ba$; then we recolor the new extremal vertices (colored $c$) to reach $ab\_ba\_ab\_ba\_ab\_ba$.

We proceed similarly in the last case of Theorem 3.7, when all cycles of $G \times H$ map to closed walks winding around the same root $R$ of $K$. Instead of the universal cover, we can only lift to a covering space whose fundamental group is (instead of the trivial group) the subgroup of $\boldsymbol{\pi}(K)$ generated by $R$. In other words, we measure for each vertex, using any walk from the base vertex to it, how far this walk (as mapped in $K$) goes outside $R$. Folding vertices extremal in this sense, we eventually reach a $K$-coloring that maps all such walks within $R$, which means there is a graph homomorphism to a cycle which admits a homomorphism to $K$. Using the multiplicativity of cycles concludes the proof.

Formally, for $\mu : G \times H \to K$, an *$H$-extremal set* is a pair $(S, h_0h_1)$ where $h_0h_1 \in H$ and $S$ is a subset of $V(G) \times \{h_1\}$ that is monochromatic, whose neighborhood is monochromatic, and whose second neighborhood is non-empty, with colors different from the color of $S$ (Figure III.2, left). That is, $\mu(S) = \{a\}$, $\mu(N_{G \times h_0h_1}(S)) = \{b\}$, $N^2_{G \times h_0h_1}(S) \neq \emptyset$ and $a \notin \mu(N^2_{G \times h_0h_1}(S))$, for some $a, b \in V(K)$. $H$-extr.

The following technical lemma gives our basic inductive argument. Intuitively, if we find an $H$-extremal set $(S, h_0h_1)$, then we can $H$-improve $\mu$ by recoloring $S$ to match some color in its second neighborhood. If this is not immediately possible, because the colors would conflict with some $\mu(\cdot, h_2)$, then by square-freeness we will find that the conflicting values give a smaller $H$-extremal set.

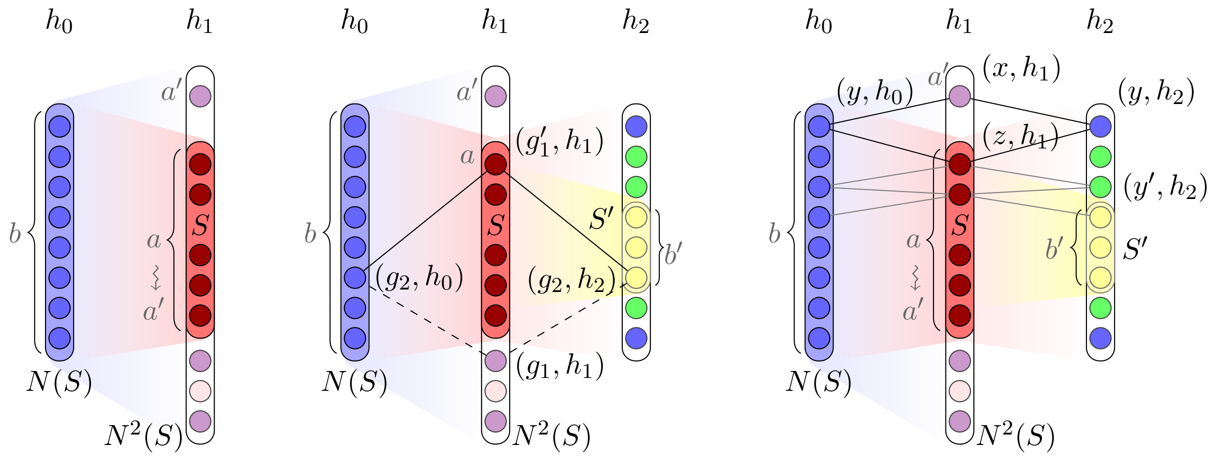



**Figure III.2** Illustration for the proof of Lemma 5.2: $K$-colored vertices of $G \times H$ (arranged in columns according to their $H$ coordinate). Left: an $H$-extremal set (dark red) $S$, its neighborhood (blue) and second neighborhood in $G \times h_0h_1$. Middle: $S'$ cannot have a neighbor outside $S$. Right: $S'$ has a non-empty second neighborhood.

**5.2 Lemma.** *Let $\mu : G \times H \to K$ for $K$ square-free, $G$ connected and non-bipartite. If there is an $H$-extremal set, then $\mu$ can be $H$-improved by recoloring.*

*Proof.* Choose an $H$-extremal set $(S, h_0h_1)$ minimizing $|S| + |N_{G\times h_0h_1}(S)|$. Let $a'$ be any color in $\mu(N^2_{G\times h_0h_1}(S))$, $a' \neq a$. Consider recoloring $S$ from $a$ to $a'$, that is, consider the assignment $\mu^* : V(G \times H) \to V(K)$ obtained from $\mu$ by setting $\mu^*(g, h_1) := a'$ for $(g, h_1) \in S$. It is easy to see that $\mu^*$ $H$-improves over $\mu$ (indeed, the relation $\mu(g, h) \stackrel{?}{=} \mu(g', h)$ could change only for pairs with $(g, h) \in S, (g', h) \notin S$; so $h = h_1$ and $\mu(g, h_1) = a \neq \mu(g', h_1)$, a non-equality, could only change to an equality, namely $\mu^*(g, h_1) = a' = \mu^*(g', h_1)$, which indeed happened for at least one $g'$).

If $\mu^*$ is a $K$-coloring then we are done, so assume otherwise. There must be some $(g, h_2) \in N_{G\times H}(S)$ with a color $b' := \mu^*(g, h_2)$ such that $a'b' \notin K$. Since $H$ is loop-free, $h_2 \neq h_1$, hence $\mu^*(g, h_2) = \mu(g, h_2) = b'$. By definition of $H$-extremal, $N_{G\times h_0h_1}(S)$ is mapped to one color, say $b$. It must be that $a'b \in K$ (as $a'$ appears on $N^2_{G\times h_0h_1}(S)$ and $\mu$ was a $K$-coloring) and thus $b \neq b'$ and in particular $h_2 \neq h_0$.

Let $S' = N_{G\times h_2h_1}(S) \cap \mu^{-1}(\{b'\})$. By the above, $S'$ is non-empty. We want to show $(S', h_1h_2)$ should have been chosen instead of $(S, h_0h_1)$.

We claim that $N_{G\times h_2h_1}(S') \subseteq S$. Suppose to the contrary that $(g_1, h_1)(g_2, h_2) \in G \times H$ for some $(g_2, h_2) \in S'$ and $(g_1, h_1) \notin S$. By definition of $S'$, $(g_2, h_2)$ also has a neighbor $(g_1', h_1) \in S$. Consider now $(g_2, h_0)$—it must be a neighbor of $(g_1, h_1)$ and $(g_1', h_1)$ as well. Hence $(g_1, h_1)$ is in $N^2_{G\times h_0h_1}(S)$, implying $a'' := \mu(g_1, h_1) \neq a$. But then $\mu(g_2, h_0) = b$, $\mu(g_1', h_1) = a$, $\mu(g_2, h_2) = b'$, and $\mu(g_1, h_1) = a''$, which gives a square in $K$ with $b \neq b'$, $a \neq a''$, a contradiction.

Hence $N_{G\times h_2h_1}(S') \subseteq S$. Thus, we have a non-empty set $S' \subseteq G \times \{h_2\}$ such that $\mu(S') = \{b'\}$, $\mu(N_{G\times h_2h_1}(S')) = \{a\}$, and $\mu(N^2_{G\times h_2h_1}(S')) \subseteq \mu(N_{G\times h_2h_1}(S) \setminus S') \not\ni b'$ by choice of $S'$.

To show that $N^2_{G\times h_2h_1}(S') \neq \emptyset$, let $(x, h_1) \in N^2_{G\times h_0h_1}(S)$, let $(y, h_0)$ be its neighbor in $N_{G\times h_0h_1}(S)$, and let $(z, h_1)$ be a neighbor of $(y, h_0)$ in $S$. Then $(y, h_2)$ is also a neighbor of $(x, h_1)$ and $(z, h_1)$. Since $\mu(x, h_1) \neq a = \mu(z, h_1)$, it must be that $\mu(y, h_2) = \mu(y, h_0)$ (by square-freeness of $K$) and hence $\mu(y, h_2) = b \neq b'$. The set $S \cup N_{G\times h_0h_1}(S)$ must be connected in $G \times h_0h_1$, otherwise we could limit $S$ to one of the connected components at the beginning. Thus $S \cup N_{G\times h_2h_1}(S)$ is connected in $G \times h_2h_1$ as well, which means it contains a path from $S'$ to $(y, h_2)$. The first vertex $(y', h_2)$ on this path such that $\mu(y', h_2) \neq b'$ then exists and is in $N^2_{G\times h_2h_1}(S')$, showing its non-emptiness. Hence $(S', h_1h_2)$ is an $H$-extremal set.

It remains to show that $|S'| + |N_{G\times h_2h_1}(S')| < |S| + |N_{G\times h_0h_1}(S)|$. We have already proved $N_{G\times h_2h_1}(S') \subseteq S$, so $|N_{G\times h_2h_1}(S')| \leq |S|$. The inclusion $S' \subseteq N_{G\times h_2h_1}(S)$ is strict because of $(y, h_2)$, hence $|S'| < |N_{G\times h_2h_1}(S)| = |N_{G\times h_0h_1}(S)|$. Adding the inequalities gives the claim, so $(S', h_1h_2)$ indeed should have been chosen in place of $(S, h_0h_1)$ at the beginning. □

A $K$-coloring that cannot be improved further has no $H$-extremal sets, which we use in the following lemma to strengthen the outcomes of Theorem 3.7. For the first and second outcome we will use $H' = h_0h_1$ instead of $H$, for the third outcome we apply this lemma directly.

**5.3 Lemma.** *Let $\mu : G \times H \to K$ for $K$ square-free, $G, H$ connected, and $G$ non-bipartite. Suppose $\mu$ has no $H$-extremal sets and suppose there is an $R \in \boldsymbol{\pi}_{\mu(g_0,h_0)}(K)$ such that for every closed walk $C$ from $(g_0, h_0)$ in $G \times H$, $\overline{\mu(C)} = R^{\cdot i}$ for some $i \in \mathbb{Z}$. Then either:*

- *$\mu$ is constant on $V(G) \times \{h\}$ for some $h \in V(H)$, or*
- *for every walk $W$ in $G \times H$ starting at $(g_0, h_0)$, $\overline{\mu(W)}$ is a prefix of $R^{\cdot i}$ for some $i \in \mathbb{Z}$.*

*Proof.* For a reduced walk $W$ in $G \times H$ starting at $(g_0, h_0)$, define $\texttt{pre}(W)$ as the longest prefix of $\overline{\mu(W)}$ which is also a prefix of $R^{\cdot i}$ for some $i \in \mathbb{Z}$ (note it might be a prefix of $R$ and $R^{-1}$ at the same time). Define $\texttt{ext}(W)$ as the remaining suffix: $\overline{\mu(W)} = \texttt{pre}(W)\_\texttt{ext}(W)$.

We claim for two walks $W, W'$ with same endpoints, $\texttt{ext}(W) = \texttt{ext}(W')$. Indeed, since $W\_W'^{-1}$ is a closed walk starting and ending in $(g_0, h_0)$, we have $R^{\cdot i} = \overline{\overline{\mu(W)} \cdot \overline{\mu(W')}^{-1}} = \overline{\texttt{pre}(W)\_\texttt{ext}(W)\_\texttt{ext}(W')^{-1}\_\texttt{pre}(W')^{-1}}$ for some $i \in \mathbb{Z}$. All edges in $\texttt{ext}(W)$ (and possibly more) must be reduced by $\texttt{ext}(W')^{-1}$ in the above expression, as otherwise $\texttt{pre}(W)\_e$ would be a prefix of the reduced expression, where $e$ is the first edge of $\texttt{ext}(W)$—this is impossible because $\texttt{pre}(W)\_e$ is not a prefix of $R^{\cdot i}$ (by definition of $\texttt{pre}$). Hence $\texttt{ext}(W)$ is a suffix of $\texttt{ext}(W')$. Symmetrically, $\texttt{ext}(W')$ is a suffix of $\texttt{ext}(W)$, so the two are equal.

Therefore, we can unambiguously define $\texttt{ext}(v)$ for $v \in G \times H$ as $\texttt{ext}(W)$ for any walk $W$ from $(g_0, h_0)$ to $v$. If $\texttt{ext}(v) = \varepsilon$ for all $v \in V(G \times H)$, then the second case of the claim holds.

Assume then that $\texttt{ext}(v)$ is not always $\varepsilon$. Choose $(g^*, h^*) \in V(G\times H)$ maximizing $|\texttt{ext}((g^*, h^*))|$. Let $\texttt{ext}((g^*, h^*)) = a_0a_1\_a_1a_2\_\dots\_a_{n-1}a_n$ for $a_i \in V(K)$, where $n \geq 1$ by assumption. Let $S$ be the set of vertices $s$ in $V(G) \times \{h^*\}$ with $\texttt{ext}(s) = \texttt{ext}((g^*, h^*))$. As $\texttt{ext}(s)$ is a walk ending at $\mu(s)$, this implies $\mu(S) = \{a_n\}$ (see Figure III.3).

We claim that $\mu(N_{G\times H}(S)) = \{a_{n-1}\}$. Indeed, let $x \in V(G \times H)$ be a neighbor of some $s \in S$. Let $W$ be a walk from $(g_0, h_0)$ to $s$. Since $\overline{\mu(W)} = \texttt{pre}(W)\_\texttt{ext}(s)$, we have $\overline{\mu(W\_sx)} = \overline{\texttt{pre}(W)\_\texttt{ext}(s)\_\mu(s)\mu(x)}$. Since $W\_sx$ is a walk to $x$ and $|\texttt{ext}(x)| \leq |\texttt{ext}(s)|$, the last edge of $\texttt{ext}(s)$ must reduce with $\mu(s)\mu(x)$. This implies $\mu(x) = a_{n-1}$ as claimed.

Let $h'$ be any neighbor of $h^*$ in $H$. Now either $N^2_{G\times h^*h'}(S)$ is empty or not. In the first case, by connectedness of $G \times h^*h'$ this means that $S$ and $N_{G\times h^*h'}(S)$ cover all of $G \times h^*h'$. Since $S \subseteq V(G) \times \{h^*\}$, $S$ must be equal to the side $V(G) \times \{h^*\}$ of the bipartition of $G \times h^*h'$, and $N_{G\times h^*h'}(S)$ must be equal to the other. As $\mu$ is constant on $S$, the first case of the claim holds.

In the second case, if $N^2_{G\times h^*h'}(S)$ is not empty, we show that $a_n \notin \mu(N^2_{G\times h^*h'}(S))$. Suppose to the contrary $\mu(y) = a_n$ for some $y \in N^2_{G\times h^*h'}(S)$. Let $x \in N_{G\times h^*h'}(S)$ be a neighbor of $y$ and let

$s \in S$ be a neighbor of $x$. As argued before, $\overline{\mu(s) = a_n}$, $\overline{\mu(x) = a_{n-1}}$. Since $\mu(y) = a_n$ too, for any walk $W$ from $(g_0, h_0)$ to $x$ we have $\overline{\mu(W\_xy)} = \overline{\mu(W\_xs)}$ and hence $\texttt{ext}(y) = \texttt{ext}(s)$. As $y$ is on the same side of the bipartition of $G \times h^*h'$ as $s$, this means it must have been in $S$ (by choice of $S$), a contradiction. Thus in fact $a_n \notin \mu(N^2_{G\times h_0h_1}(S))$, so $(S, h_0h_1)$ would be an $H$-extremal set, meaning it is never the case that $N^2_{G\times h_0h_1}(S)$ is not empty. □

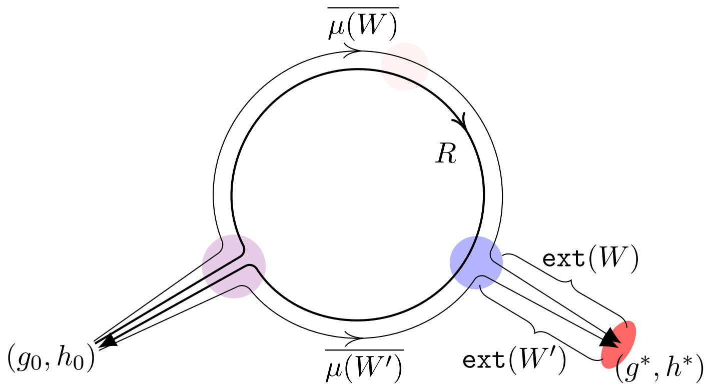



**Figure III.3** The images in $K$ of two walks from $(g_0, h_0)$ to $(g^*, h^*)$. Their final vertex, extending out of $R$, defines an $H$-extremal set: it is mapped to $a_n$ (the red color), it's neighbors are mapped to $a_{n-1}$ (the blue color), while second neighbors are not red. We could hence improve the mapping by moving $(g^*, h^*)$ to the violet color, say.

The first outcome of Lemma 5.3 is easily strengthened, giving a homomorphism $H \to K$:

**5.4 Lemma.** *Let $\mu : G \times H \to K$ for $K$ square-free, $G, H$ connected, and $G$ non-bipartite. If $\mu$ has no $H$-extremal sets and is constant on $V(G) \times \{h\}$ for some $h \in V(H)$, then $\mu = \gamma \circ \delta$, where $\delta : G \times H \to H$ is the projection to $H$ and $\gamma : H \to K$ is a graph homomorphism.*

*Proof.* We first show that $\mu$ is constant on $V(G) \times \{h\}$ for every $h \in V(H)$. Suppose the contrary holds. Then by connectivity of $H$ there is an edge $h_0h_1 \in E(H)$ such that $\mu$ is constant on $V(G) \times \{h_0\}$ and is not constant on $V(G) \times \{h_1\}$. Let $a \in \mu(V(G) \times \{h_1\})$ and let $S = \mu^{-1}(a) \cap (V(G) \times \{h_1\})$. Then by connectivity of $G \times h_0h_1$ it is easy to see that $(S, h_0h_1)$ is an $H$-extremal set, contradicting the assumption on $\mu$.

Thus we can define $\gamma : V(H) \to V(K)$ by letting $\gamma(h)$ be the unique value in $\mu(V(G) \times \{h\})$. Clearly $\mu = \gamma \circ \delta$, where $\delta : G \times H \to H$ is the projection to $H$ and $\gamma : H \to K$. □

For the other outcome of Lemma 5.3, we first need to show that $R$ is not only reduced, but is *cyclically reduced*, meaning $R\_R$ is reduced. This follows easily by temporarily considering a different base point. cyclically reduced

**5.5 Lemma.** *Let $\mu : G \times H \to K$ for $K$ square-free, $G$ and $H$ connected and non-bipartite. Suppose $\mu$ has no $H$-extremal sets. Suppose $R \in \boldsymbol{\pi}_{\mu(g_0,h_0)}(K)$ is such that for every closed walk $C$ from $(g_0, h_0)$ in $G \times H$, $\overline{\mu(C)} = R^{\cdot i}$ for some $i \in \mathbb{Z}$. If $\mu$ is not constant on $V(G) \times \{h\}$ for any $h \in V(H)$, then $R$ is cyclically reduced.*

*Proof.* Suppose to the contrary that $R = e\_R'\_e^{-1}$ for some $e = (k_0, k_1) \in E(K)$ (where $k_0 = \mu(g_0, h_0)$) and $R' \in \boldsymbol{\pi}_{k_1}(K)$. Let $C$ be any closed walk from $(g_0, h_0)$ in $G \times H$ of odd length (it exists by assumptions on $G$ and $H$). Then $\overline{\mu(C)}$ must be odd too, so in particular $\overline{\mu(C)} = R^{\cdot i}$ for some $i \in \mathbb{Z}$ other than 0. Thus the first edge of $\overline{\mu(C)}$ is $e$. Let $C = W_1\_W_2$, where $W_1$ is the longest prefix of $C$ such that $\overline{\mu(W_1)} = e$. Let $(g', h')$ be the last vertex of $W_1$ (and first of $W_2$). For any closed walk $C'$ from $(g', h')$ in $G \times H$, since $W_1\_C'\_W_1^{-1}$ is a closed walk from $(g_0, h_0)$ in $G \times H$, we have $\overline{\mu(C')} = \overline{\mu(W_1)}^{-1} \cdot \overline{\mu(W_1 \_ C' \_ W_1^{-1})} \cdot \overline{\mu(W_1)} = e^{-1} \cdot R^{\cdot j} \cdot e = e^{-1} \cdot (e \cdot R' \cdot e^{-1})^{\cdot j} \cdot e = R'^{\cdot j}$ for some $j \in \mathbb{Z}$. Therefore the premises of Lemma 5.3 are true for $(g', h')$ and $R'$ too (instead of $(g_0, h_0)$ and $R$). The first outcome of the lemma does not hold by assumption, so the second outcome is true, implying in particular that $\overline{\mu(W_1^{-1})}$ is a prefix of $R'^{\cdot k}$ for some $k \in \mathbb{Z}$. However, $\overline{\mu(W_1^{-1})} = e^{-1}$ and this cannot be the first edge of $R'$ nor $R'^{-1}$, because $R = e \_ R' \_ e^{-1}$ is a reduced walk, a contradiction. □

The next lemma (used for $F = G \times H$) gives the final conclusion of the second outcome of Lemma 5.3. The proof describes the homomorphisms and then just checks their validity.

**5.6 Lemma.** *Let $\mu : F \to K$. Suppose there is an $R \in \boldsymbol{\pi}_{\mu(f_0)}(K)$ such that $R$ is cyclically reduced and for every closed walk $C$ from $f_0$ in $F$, $\overline{\mu(C)} = R^{\cdot i}$ for some $i \in \mathbb{Z}$. Suppose for any walk $W$ in $F$ starting at $f_0$, $\overline{\mu(W)}$ is a prefix of $R^{\cdot i}$ for some $i \in \mathbb{Z}$. Then there exist graph homomorphisms $\gamma : F \to C_{|R|}$ and $\delta : C_{|R|} \to K$ such that $\mu = \delta \circ \gamma$.*

*Proof.* For a walk $W$ in $F$ starting at $f_0$, let $i \in \mathbb{Z}$ be such that $\overline{\mu(W)}$ is a prefix of $R^{\cdot i}$ and define $\gamma(W) = \mathrm{sgn}(i) \cdot |\overline{\mu(W)}| \mod |R|$. Note this is unambiguous, as $\overline{\mu(W)}$ either has zero length (so the choice of $i$ is irrelevant), or cannot be both a prefix of $R^{\cdot i}$ for positive and negative $i$, because we assumed $R$ is cyclically reduced (so $\mathrm{sgn}(i)$ does not depend on the choice of $i$).

For any two walks $W, W'$ from $f_0$ to the same endpoint, we want to show that $\gamma(W) = \gamma(W')$. Indeed, $\overline{\mu(W')} = \overline{\mu(W')} \cdot \overline{\mu(W)}^{-1} \cdot \overline{\mu(W)} = \overline{\mu(W' \_ W^{-1})} \cdot \overline{\mu(W)} = R^{i} \cdot \overline{\mu(W)}$ for some $i \in \mathbb{Z}$ (since $W' \_ W^{-1}$ is a closed walk). Then one of the following holds, in each case implying $\gamma(W) = \gamma(W')$:

- $\overline{\mu(W)}$ is empty, and then $\overline{\mu(W')} = R^{\cdot i}$ has length $0 \mod |R|$ too;
- $i = 0$, implying $\overline{\mu(W')} = \overline{\mu(W)}$ and hence $\gamma(W) = \gamma(W')$ trivially;
- $\overline{\mu(W)}$ is a prefix of $R^{\cdot j}$ for $j \in \mathbb{Z} \setminus \{0\}$ with $\mathrm{sgn}(j) = \mathrm{sgn}(i)$, in which case $\overline{\mu(W')} = R^{\cdot i} \cdot \overline{\mu(W)} = R^{\cdot i} \_ \overline{\mu(W)}$, which is a prefix of $R^{\cdot i+j}$ with the same sign and length mod $|R|$;
- $\overline{\mu(W)}$ is a prefix of $R^{\cdot j}$ for $j \in \mathbb{Z} \setminus \{0\}$ with $\mathrm{sgn}(j) = -\mathrm{sgn}(i)$ and $|\overline{\mu(W)}| > |R^{\cdot i}|$, in which case $\overline{\mu(W')} = R^{\cdot i} \cdot \overline{\mu(W)}$ is a prefix of $R^{\cdot j}$ of length $|\overline{\mu(W)}| - |R^{\cdot i}| = |\overline{\mu(W)}| \mod |R|$;
- $\overline{\mu(W)}$ is a prefix of $R^{\cdot j}$ for $j \in \mathbb{Z} \setminus \{0\}$ with $\mathrm{sgn}(j) = -\mathrm{sgn}(i)$ and $|\overline{\mu(W)}| \leq |R^{\cdot i}|$, in which case $\overline{\mu(W')} = R^{\cdot i} \cdot \overline{\mu(W)}$ is a prefix of $R^{\cdot i}$ of length $|R^{\cdot i}| - |\overline{\mu(W)}| = -|\overline{\mu(W)}| \mod |R|$.

Therefore, we can unambiguously define $\gamma : V(F) \to \{0, \ldots, |R|-1\}$ as $\gamma(f) = \gamma(W)$ for any walk $W$ from $f_0$ to $f$. This is a graph homomorphism from $F$ to $C_{|R|}$, because if $\{f, f'\}$ is an edge of $F$, then $\gamma(f') = \gamma(W \_ f f') = \gamma(W) \pm 1$ for any walk $W$ from $f_0$ to $f$. The last equality holds because $|\overline{\mu(W \_ f f')}| = |\overline{\mu(W)}| \pm 1$ and the sign in the definition of $\gamma$ can only change when one of $\overline{\mu(W \_ f f')}, \overline{\mu(W)}$ is empty.

Let $R = r_0 r_1 \_ r_1 r_2 \_ \ldots \_ r_{|R|-1} r_0$ for $r_i \in V(K)$. Define $\delta : \{0, \ldots, |R|-1\} \to V(K)$ as $\delta(i) = r_i$. Since $R$ is a closed walk in $K$, $\delta : C_{|R|} \to K$ is a graph homomorphism. It is easily checked from definitions that $r_{\gamma(W)}$ is the endpoint of $\overline{\mu(W)}$ for any walk $W$ from $f_0$ to $f$, and thus $\delta(\gamma(f)) = \mu(f)$ for $f \in V(F)$. □

We are now ready to conclude the main theorem, in a slightly stronger form. It gives $G \to K$, $H \to K$, or homomorphisms $G \times H \to C_n$ and $C_n \to K$. By multiplicativity of cycles, the latter implies $G \to C_n \to K$ or $H \to C_n \to K$, concluding the proof of multiplicativity of square-free $K$.

**5.7 Theorem.** *Let $\mu : G \times H \to K$ for $K$ square-free, $G$ and $H$ connected and non-bipartite. Then there are graph homomorphisms $\mu^* : G \times H \to K$, $\gamma : G \times H \to I$ and $\delta : I \to K$ such that $\mu^* = \delta \circ \gamma$ and $\mu^*$ is reachable from $\mu$ by recoloring, where $I$ is either $G$, $H$ or $C_n$ for some $n \in \mathbb{N}$.*

*Proof.* Let $\{(g_0, h_0), (g_1, h_1)\}$ be an edge of $G \times H$. By Theorem 3.7, one of the following holds:

- $\overline{\mu(C)} = \varepsilon$ for every closed walk $C$ from $(g_0, h_0)$ in $G \times h_0h_1$. Then, by repeatedly applying Lemma 5.2, we can recolor $\mu$ to eventually reach a $K$-coloring $\mu^*$ with no $H$-extremal sets. Since it is reached by recoloring, Lemma 5.1 guarantees that it still has the same property: $\overline{\mu^*(C)} = \varepsilon$ for every closed walk $C$ from $(g_0, h_0)$ in $G \times h_0h_1$.

  Therefore $\mu^*|_{V(G\times H')} : G \times H' \to K$ satisfies the conditions of Lemma 5.3 for $R = \varepsilon$, $G$ and $H' = h_0h_1$ (a graph isomorphic to $K_2$). The second outcome of the lemma cannot hold, because the reduced image of a one-edge walk $\overline{\mu^*((g_0, h_0)(g_1, h_1))}$ has odd length and thus cannot be a prefix of $\varepsilon^{\cdot i}$ (the empty walk) for any $i \in \mathbb{Z}$. Hence the first outcome is true, that is, $\mu^*$ is constant on $V(G) \times \{h\}$ for some $h \in \{h_0, h_1\}$. The claim then follows for $I = H$ from Lemma 5.4.

- $\overline{\mu(D)} = \varepsilon$ for every closed walk $D$ from $(g_0, h_0)$ in $g_0g_1 \times H$. This case is entirely symmetric with the previous one, swapping the roles of $G$ and $H$. The claim then follows for $I = G$.

- There is an $R \in \boldsymbol{\pi}(K)$ such that for every closed walk $C$ from $(g_0, h_0)$ in $G \times H$, $\overline{\mu(C)} = R^{\cdot i}$ for some $i \in \mathbb{Z}$. Then, again by repeatedly applying Lemma 5.2, we can recolor $\mu$ to eventually reach a $K$-coloring $\mu^*$ with no $H$-extremal sets. Since it is reached by recoloring, Lemma 5.1 guarantees that there is an $R' \in \boldsymbol{\pi}(K)$ such that for every closed walk $C$ from $(g_0, h_0)$ in $G \times H$, $\overline{\mu^*(C)} = R'^{\cdot i}$ for some $i \in \mathbb{Z}$.

  Hence $\mu^* : G \times H \to K$ and $R'$ satisfy the conditions of Lemma 5.3 directly. If the first outcome of the lemma holds, then the claim follows for $I = G$ from Lemma 5.4. Otherwise the second outcome is true, that is, for every walk $W$ in $G \times H$ starting from $(g_0, h_0)$, $\overline{\mu(W)}$ is a prefix of $R'^{i}$ for some $i \in \mathbb{Z}$. By Lemma 5.5, $R'$ is cyclically reduced. Then the claim follows for $I = C_{|R'|}$ from Lemma 5.6. □

## 6. Conclusions

Some further conclusions can be drawn from Theorem 5.7. For one example, let $\mu : G \times H \to K$ for a square-free graph $K$, and suppose that $G \not\to K$, $G \not\to K_3$, $H \not\to K_3$, and that $H$ has only one $K$-coloring $\gamma$, up to automorphisms of $K$. Then $\mu$ is the only $K$-coloring of $G \times H$ (up to automorphisms of $K$). Indeed, the only possible outcome of Theorem 5.7 (again up to automorphisms) is that $\mu$ can be recolored to $\mu^* = \delta \circ \gamma$, where $\delta$ is the projection to $H$. That is, $\mu^*$ is constant on $V(G) \times \{h\}$ for each $h \in V(H)$. If $\mu^* \neq \mu$, then it was obtained by recoloring; let the last recoloring step change the color of $(g, h) \in V(G \times H)$ from $a \in V(K)$ to $b := \mu^*(g, h)$. Before this step, the $K$-coloring was still constant on $V(G) \times \{h'\}$ for $h' \neq h \in V(H)$. Hence replacing all values of $\mu^*(\cdot, h) = b$ with $a$ gives a different $K$-coloring of $G \times H$, which is a composition of the projection to $H$ with a different $K$-coloring of $H$ (but different only on $g$). But this is impossible, thus in fact $\mu = \mu^*$ is the only $K$-coloring of $G \times H$ (up to automorphisms of $K$).

Second, multiplicativity of square-free graphs $K$ can be strengthened to the following statement: for graphs $G, H$ and odd cycles $G'$ in $G$ and $H'$ in $H$, if $G \times H' \cup G' \times H$ (an induced subgraph of $G \times H$) has a $K$-coloring, then $G \to K$ or $H \to K$. This follows by adapting Theorem 3.7 so that depending on the types of $G'$ and $H'$ we have one of the same three conclusions, with the first two limited to closed walks in $G' \times h_0h_1$ and $g_0g_1 \times H'$ respectively (instead of $G \times h_0h_1$ and $g_0g_1 \times H$). The third case is without change. In the first two, we then consider $G'$ instead of $G$ (or $H'$ instead of $H$, respectively) and continue the proof without change (applying Lemma 5.3 to $G' \times h_0h_1$ only) to eventually get $H \to K$ (or $G \to K$, respectively). In other words, if $\overline{\mu(C \otimes h_0h_1)} = \varepsilon$ for one odd cycle $C$ of $G$, then this already implies $H \to K$.

Unfortunately, this means our methods have the same limitations as previous ones: Tardif and Zhu [TZ02a] showed that an analogous extension is false for $K = K_n$ with $n \geq 4$. Namely, for any $m > n \geq 4$ there exists $m$-chromatic graphs $G, H$ with $n$-chromatic subgraphs $G', H'$ such that $G \times H' \cup G' \times H$ is $n$-chromatic. On the other hand, our approach yields results for high-chromatic graphs while still only relying on cycles, essentially. Nevertheless, all the author's attempts to get even partial results for $K_4$, the next case of Hedetniemi's conjecture, fizzled. The underlying reason is that the topology of $K_4$ is a two-dimensional sphere, and extensions of the topological arguments to that case remain elusive, regardless of any combinatorics. We discuss this obstacle in the next chapter, formalizing a natural topological conjecture which turns out to be implied by Hedetniemi's conjecture.

Let us also mention that the proofs here are constructive, in the sense that given a $K$-coloring of $G \times H$, a $K$-coloring of $G$ or $H$ can be found in polynomial time. This is straightforward for circular cliques, while for square-free graphs this follows from the fact that a $K$-coloring of $G \times H$ can be $H$-improved only polynomially many times. More explicit colorings, for example describing colors of nodes of the exponential graphs $K^G$ in time polynomial in $G$, remain an interesting open problem, see [Tar06].

# Chapter IV

# Inverse powers of graphs and topological implications of Hedetniemi's conjecture

## 1. Introduction

**Graph functors** In this chapter we show that the 'inverse power' operation $\Omega_k$ on graphs, defined next, has surprising topological properties. It preserves the topology (the $\mathbb{Z}_2$-homotopy type) of the box complex and refines its geometry. This means that $\mathbb{Z}_2$-maps (equivariant, continuous maps) between box complexes of graphs $G, H$ can be approximated by homomorphisms from the refined graph $\Omega_k(G)$ to $H$, for high enough $k$. This allows to generalize some theorems on coloring powers of graphs and to characterize topological properties in combinatorial terms. The most interesting corollary is that Hedetniemi's conjecture implies an analogous conjecture in topology, as independently proved by Matsushita [Mat17a]. We discuss this and other implications, arguing the importance of the topological conjecture.

We consider three interrelated families of graph operations $\Lambda_k$, $\Gamma_k$, $\Omega_k$, parameterized by an odd integer $k$. The left operation, the *graph $k$-subdivision* $\Lambda_k(G)$ of a graph $G$, is obtained by replacing every edge with a path on $k$ edges (this is sometimes denoted $G^{\frac{1}{k}}$). The central operation, the $k$-th *power* $\Gamma_k(G)$ of $G$ is the graph on the same vertex set $V(G)$, with two vertices joined by an edge if they were connected by a walk of length exactly $k$ in $G$ (equivalently, the adjacency matrix is taken to the $k$-th power; this is sometimes denoted $G^k$, note however this is *not* the same as joining vertices at distance at most $k$). Our results concern the right operation, $\Omega_k$, which is a certain inverse to the powering operation $\Gamma_k$, as we shall now make precise.

Each operation in the above families is a *functor* in the (thin) category of graphs, which means simply that $G \to H$ implies $\Pi(G) \to \Pi(H)$, for any graphs $G, H$ (for $\Pi = \Lambda_k, \Gamma_k, \Omega_k$ with $k$ odd).[1] More importantly, $\Gamma_k$ is a *right adjoint* to $\Lambda_k$, meaning that $\Lambda_k(G) \to H$ holds if and only if $G \to \Gamma_k(H)$ does. Similarly (but less trivially), $\Omega_k$ is a right adjoint to $\Gamma_k$, that is, $\Gamma_k(G) \to H$ iff $G \to \Omega_k(H)$. This characterizes $\Omega_k$ up to homomorphic equivalence, but we give the explicit construction with other definitions in Section 2. For example, the third power of a graph $G$ admits an $n$-coloring (a homomorphism into the clique $K_n$) if and only if $G \to \Omega_3(K_n)$.

Adjointness of various graph constructions is the principal tool behind Hell and Nešetřil's celebrated theorem (characterizing the complexity of deciding $G \to H$, for a fixed $H$) [HN90], in particular the adjointness of $\Gamma_k$ to $\Lambda_k$ is used in the first of many steps of the proof. The construction $\Omega_k$ was used implicitly by Gyárfás et al. [GJS04], to answer a question on $n$-chromatic graphs with "strongly independent colour classes": they showed that $\Omega_3(K_n)$ gives an example of an $n$-chromatic graph whose third power is still $n$-chromatic. The construction has also been used in a homomorphism duality theorem by Häggkvist and Hell [HH93]. Tardif [Tar05] used

[1] We are only concerned with the existence of homomorphisms and maps, not with their identity (compositions, uniqueness). Thus we only consider the *thin* category of graphs (where all homomorphisms $G \to H$ are identified as one arrow), or equivalently, the poset of graphs (with $G \leq H$ when $G \to H$). In the language of posets, functors are just order-preserving maps, while adjoint functors are known as Galois connections. Additional properties required of adjoint functors in the usual (non-thin) categories are not necessarily met, see [FT18].

iterations of $\Omega_3$ and $\Gamma_3$ to extend results on Hedetniemi's conjecture to the circular chromatic number, showing the multiplicativity of circular cliques. Iterating $\Omega_3$ $k$ times is equivalent to applying $\Omega_{3^k}$; the functors $\Omega_k$ for general odd $k$ can thus be considered just a smoother way to express such iterations. They were first considered by Hajiabolhassan and Taherkhani [HT10], who proved, among other results, a characterization of the circular chromatic number in terms of the chromatic number (or even just 3-colorability) of powers of graph subdivisions.

The operations $\Lambda_k$, $\Gamma_k$, $\Omega_k$ are also the simplest example of so called left, central, and right Pultr functors, a more general construction of adjoint graph functors [Pul70]. Another simple example of a (central) Pultr functor is the so called *arc graph construction* (see eg. [Ror+16]), also crucial in applications to Hedetniemi's conjecture [PR81; Tar08]. Graph products and exponential graphs can also be seen as applying Pultr functors. See [FT18] for a survey on graph functors focused around Hedetniemi's conjecture and [FT15] for the question of when both left and right adjoints to a common functor exist.

**The topology of graphs** Recall from Chapter I that the box complex is a way to assign a $\mathbb{Z}_2$-space (a topological space with a symmetry in the form of a $\mathbb{Z}_2$-action) to a graph, with the property that a homomorphism $G \to H$ induces a $\mathbb{Z}_2$-map $|\mathrm{Box}(G)| \to_{\mathbb{Z}_2} |\mathrm{Box}(H)|$ (it is a functor from the category of graphs to the category of $\mathbb{Z}_2$-spaces with $\mathbb{Z}_2$-maps). Knowing that eg. the box complex of a clique is a sphere, $|\mathrm{Box}(K_n)| \simeq_{\mathbb{Z}_2} \mathcal{S}^{n-2}$, we can deduce topological obstructions to homomorphisms using the Borsuk-Ulam theorem, which states that there is no $\mathbb{Z}_2$-map from a higher-dimensional sphere to a lower-dimensional one, $\mathcal{S}^m \not\to_{\mathbb{Z}_2} \mathcal{S}^n$ for $m > n$.

Our main technical result is that $\Omega_k$ functors behave much like subdivision (in the topological sense) on the box complex. That is, they preserve the homotopy type and they refine the geometric structure, so that any continuous maps between box complexes can be approximated with graph homomorphisms from refinements $\Omega_k(G)$ of $G$. See Figure IV.1 for a particularly simple example. Formally (here $p_k$ is a certain natural homomorphism $\Omega_k(G) \to G$, see Section 2 for definitions):

**1.1 Theorem.** (Equivalence) *$|\mathrm{Box}(G)|$ and $|\mathrm{Box}(\Omega_k(G))|$ are $\mathbb{Z}_2$-homotopy equivalent, for all odd integers $k$. Moreover, $p_k$ induces a $\mathbb{Z}_2$-homotopy equivalence.*

**1.2 Theorem.** (Approximation) *There exists a $\mathbb{Z}_2$-map from $|\mathrm{Box}(G)|$ to $|\mathrm{Box}(H)|$ if and only if for some odd $k$, $\Omega_k(G)$ has a homomorphism to $H$.*

*Moreover, for every $\mathbb{Z}_2$-map $f : |\mathrm{Box}(G)| \to_{\mathbb{Z}_2} |\mathrm{Box}(H)|$, there is an odd $k$ and a homomorphism $\Omega_k(G) \to H$ that induces a map $\mathbb{Z}_2$-homotopic to $p_k \circ f$.*

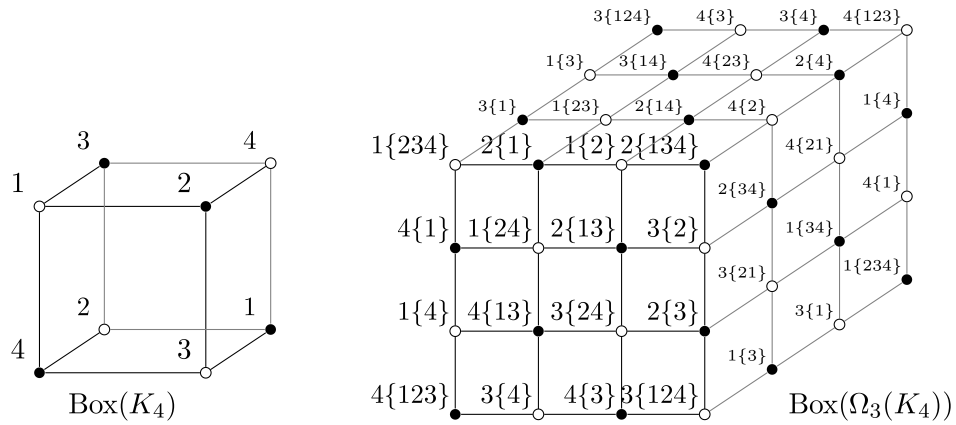



**Figure IV.1** The box complex of the clique graph $K_4$ and of $\Omega_3(K_4)$. As $G$ becomes $G \times K_2$, for each vertex $v$ of the graph there are two vertices $v_\circ$ and $v_\bullet$ in the complex. Faces are glued to each 4-cycle, making both complexes equivalent to the (hollow) sphere. The vertex $(\{v\}, \{v_1, v_2, \dots\})$ of $\Omega_3(K_4)$ is labeled $v\{v_1 v_2 \dots\}$ for short. (A careful reader may note that the definition of Box() includes also faces corresponding to each vertex neighborhood, such as the tetrahedron $\{1_\circ, 2_\bullet, 3_\bullet, 4_\bullet\}$, but it can be shown that these can be collapsed).

Csorba [Cso08] gave a construction showing that any simplicial complex is equivalent to some box complex (see also a generalization to actions of groups other than $\mathbb{Z}_2$ and to complexes of homomorphisms in [DS12]).

**1.3 Theorem.** (Universality, [Cso08]) *For every $\mathbb{Z}_2$-complex $X$, there is a graph $G$ such that $|X|$ and $\mathrm{Box}(G)$ are $\mathbb{Z}_2$-homotopy equivalent.*

Together, these three theorems show that the homotopy theory of $\mathbb{Z}_2$-spaces is largely reflected in graphs, with $\Omega_k$ functors as the connection. (Equivalently, in all of our results, 'for some odd $k$' can be replaced by 'for large enough odd $k$' and $\Omega_k$ by iterations $\Omega_3(\dots(\Omega_3(G))\dots)$ of $\Omega_3$).

The existence of some sequence of functors which satisfy the above Equivalence and Approximation Theorems already follows from the work of Dochtermann and Schultz [DS12, Proposition 4.7]. Essentially, the idea is to go to the box complex, apply barycentric subdivision iteratively, and return to graphs with the construction from the Universality Theorem. The construction is however ad-hoc and tedious to describe directly, it cannot be described as iterating a single functor, and it is not clear whether the resulting graph functors admit left adjoints.

One application of the Equivalence Theorem, for $\Omega_k$ functors specifically, is that it immediately implies the result of Gyárfás et al. [GJS04]: since $\Omega_k(K_n)$ has the same homotopy type as $K_n$, it is not $(n-1)$-colorable (by the Borsuk-Ulam theorem, as explained above). It is then easy to check that it is in fact an $n$-chromatic graph whose $k$-th power is still only $n$-chromatic (in particular it has no loops, so $\Omega_k(K_n)$ has no odd cycle of length $\leq k$). The chromatic number of $\Omega_k(K_n)$ (as a "universal graph for wide colorings") has also been shown in [ST06] and [BS05]. More generally, for any graph $K$ without loops, $\Omega_k(K)$ gives a graph with a $\mathbb{Z}_2$-homotopy equivalent box complex, but with arbitrarily high odd girth (no odd cycles of length $\leq k$).

## Hedetniemi's conjecture and multiplicative graphs

The strong connection between graphs and topology, together with the fact that the functors $\Omega_k$ commute with the product (which follows from them being right adjoints, see Lemma 2.2), allow us to show that Hedetniemi's conjecture implies an analogous statement in topology (recall that $\mathcal{S}^d$ denotes the $d$-dimensional sphere with antipodal $\mathbb{Z}_2$-action). This has recently been shown independently by Matsushita [Mat17a]:

**1.4 Theorem.** *Suppose Hedetniemi's conjecture is true. Then $|X| \times |Y| \to_{\mathbb{Z}_2} \mathcal{S}^d$ implies $|X| \to_{\mathbb{Z}_2} \mathcal{S}^d$ or $|Y| \to_{\mathbb{Z}_2} \mathcal{S}^d$, for any $\mathbb{Z}_2$-complexes $X, Y$ and any integer $d$.*

Matsushita in fact adapts the box complex construction, the functors of Dochtermann and Schultz [DS12], and the construction of Csorba [Cso08], to give a particularly elegant connection between the category of graphs and the category of $\mathbb{Z}_2$-spaces in the form of adjoint functors preserving finite limits, from which the statement readily follows. While the approach in this thesis does not give such a graceful connection, the author finds it surprising that the most important topological conclusions can also be made using more natural graph functors $\Omega_k$, which have already proven to be useful for purely combinatorial theorems. Our methods do not give here any stronger results than Matsushita's (except maybe for Theorem 1.7, where the appearance of $\Omega_k$ will make the statement more meaningful as a combinatorial characterization), but we comment more on the implications on Hedetniemi's conjecture and further argue on the importance of topological approaches.

Recall that a graph $K$ is *multiplicative* if $G \times H \to K$ implies $G \to K$ or $H \to K$, for all graphs $G, H$. Hedetniemi's conjecture is then that all clique graphs $K_n$ are multiplicative. We can analogously define a $\mathbb{Z}_2$-space $Z$ to be *multiplicative* when $|X| \times |Y| \to_{\mathbb{Z}_2} Z$ implies $|X| \to_{\mathbb{Z}_2} Z$ or $|Y| \to_{\mathbb{Z}_2} Z$, for all $\mathbb{Z}_2$-complexes $X, Y$ (we do not care here about $\mathbb{Z}_2$-spaces not coming from (finite) $\mathbb{Z}_2$-complexes). Since the box complex of the clique $|\mathrm{Box}(K_n)|$ is ($\mathbb{Z}_2$-homotopy equivalent to) the $(n-2)$-dimensional sphere $\mathcal{S}^{n-2}$, Theorem 1.4 is a special case of the following: multipl. $\mathbb{Z}_2$-space

**1.5 Theorem.** *Let $K$ be a multiplicative graph. Then $|\mathrm{Box}(K)|$ is a multiplicative $\mathbb{Z}_2$-space.*

In other words, this means Hedetniemi's conjecture implies the following:

**1.6 Conjecture.** *All spheres $\mathcal{S}^d$ are multiplicative.*

We do not know if the converse implication is true. However, from the multiplicativity of a $\mathbb{Z}_2$-space we can deduce a weaker statement, which can be seen as a relaxation of graph multiplicativity, and a combinatorial characterization of multiplicative spaces:

**1.7 Theorem.** *Let $Z$ be a $\mathbb{Z}_2$-space and let $K$ be a graph such that $|\mathrm{Box}(K)| \simeq_{\mathbb{Z}_2} Z$. Then $Z$ is multiplicative if and only if: for all graphs $G, H$, $G \times H \to K$ implies that for some odd $k$, $\Omega_k(G) \to K$ or $\Omega_k(H) \to K$.*

Thus Conjecture 1.6 can be stated as a purely combinatorial statement, relaxing Hedetniemi's conjecture. However, we note that the conclusion that $\Omega_k(G) \to K$ is much weaker than the desired $G \to K$. For example, circular cliques $K_{p/q}$ with $3 < p/q < 4$ do not admit a homomorphism into $K_3$, but $\Omega_k(K_{p/q})$ does (for high enough odd $k$ depending on $p/q$), since the box complex of $K_{p/q}$ is a circle (up to homotopy, in this range of $p/q$). More strikingly, when $G$ has high girth, then $G$ can have high chromatic number, but $\Omega_k(G)$ coincides with the graph $k$-subdivision of $G$ (Lemma 2.3.(vii)), which is always 3-colorable.

Nevertheless, quite surprisingly, known proofs of multiplicativity for graphs largely follow topological ideas. In Section 6 we give direct, elementary proofs of the multiplicativity of the circle $\mathcal{S}^1$ and discuss the few additional steps needed to conclude the multiplicativity of $K_3$, cycles, and circular cliques, as done in Chapter III. (We note that Matsushita [Mat17a] gives a different, though in essence somewhat similar, direct proof of the multiplicativity of $\mathcal{S}^1$, using the theory of covering spaces).

This strongly suggests that Conjecture 1.6 is crucial to resolving Hedetniemi's conjecture: any counter-example immediately implies a counter-example to Hedetniemi's conjecture, while a proof could be an important first step to a strengthening for graphs (and at least implies a weaker graph-theoretical statement). Furthemore, a proof of Conjecture 1.6 should be in principle easier than any proof of Hedetniemi's conjecture, while obstacles to proving Conjecture 1.6 are also obstacles for certain approaches to Hedetniemi's conjecture. We discuss these in Section 6.

To complement this, we show new multiplicative graphs: powers of graphs of high girth.

**1.8 Theorem.** *If $K$ has girth $> 12$ (ie. no cycles of length $\leq 12$), then $\Gamma_3(K)$ is multiplicative.*

The proof is combinatorial and follows the ideas of Tardif's [Tar05] original proof of the multiplicativity of circular cliques, deduced from the multiplicativity of cycles using $\Gamma_k$ and $\Omega_k$ functors. On the other hand, the requirement that the girth be larger than 12 is precisely the case where $K$ and $\Gamma_k(K)$ can be shown to have the same topology (ie. $\mathbb{Z}_2$-homotopy equivalent box complexes). It thus seems that these methods are also limited by topology, though whether this can be formally proved remains an open question: for example, is there a general connection between whether a graph $K$ admits a homomorphism $\Omega_3(\Gamma_3(K)) \to K$ and its topology?

### Organization

Section 2 gives basic definitions and lists some properties of $\Lambda_k, \Gamma_k, \Omega_k$ functors. The Equivalence Theorem 1.1 is proved in Section 3 using Discrete Morse Theory, which is also introduced there. The Approximation Theorem 1.2 is proved in Section 4, by considering the geometry of $|\mathrm{Box}(\Omega_k(G))|$ and then a fairly standard use of the simplicial approximation technique. In Section 5 we show Theorem 1.8 with a short, combinatorial proof. Finally Section 6 gives the proofs of Theorems 1.4, 1.5 and 1.7, which are straighforward applications of the main technical theorems. We then consider in more detail the implications on Hedetniemi's conjecture and multiplicative graphs; we also comment more on obstacles to generalizations, on Conjecture 1.6, and on open questions that arise from these results.

## 2. Preliminaries

### Graphs

For a graph $G$ and two vertex subsets $A, B \subseteq V(G)$, we write $A \boxtimes B$ ($A$ is *joined* to $B$) if all vertices of $A$ are adjacent to all vertices of $B$ (and $A \not\boxtimes B$ otherwise). We denote the *common neighborhood* of $A \subseteq V(G)$ as $\mathrm{CN}(A) := \bigcap_{v \in A} N(v)$ ($\mathrm{CN}(\emptyset) = V(G)$). Observe that $A \boxtimes B$ iff $A \subseteq \mathrm{CN}(B)$ iff $B \subseteq \mathrm{CN}(A)$. Note that $A \boxtimes B$ implies that $A$ and $B$ are disjoint, if $G$ has no loops. Recall that a graph without loops is *square-free* if it contains no $C_4$ as a subgraph. More generally, a graph $G$ (with loops allowed) is square-free if it has no quadruple $v_1, v_2, v_3, v_4 \in V(G)$ such that $v_1 \neq v_3$, $v_2 \neq v_4$ and $v_1v_2, v_2v_3, v_3v_4, v_4v_1 \in G$. Equivalently $A \boxtimes B$ implies $|A| \leq 1$ or $|B| \leq 1$ for $A, B \subseteq V(G)$. $A \boxtimes B$ $\mathrm{CN}(A)$

In this chapter a *walk* of length $n$ is a sequence of vertices $v_0, \ldots, v_n$ such that $v_i$ is adjacent to $v_{i+1}$ ($i = 0 \ldots n-1$); that is, vertices and edges may repeat, and the length is the number of edges. A *path* is a walk with no vertex (nor edge) repetitions. walk path

### Box complex

The definitions in equivariant topology were given in Chapter I: $\mathbb{Z}_2$-spaces and their products, the spheres $\mathcal{S}^d$, a $\mathbb{Z}_2$-complex $X$ and its geometric realization $|X|$, a $\mathbb{Z}_2$-map $X \to_{\mathbb{Z}_2} Y$, and $\mathbb{Z}_2$-homotopy equivalence $X \simeq_{\mathbb{Z}_2} Y$. Note that thus is much stronger than just requiring $X \to_{\mathbb{Z}_2} Y$ and $Y \to_{\mathbb{Z}_2} X$.

Let us recall the definition of the *box complex* $\mathrm{Box}(G)$ of a graph $G$ with new notation. If $G$ has isolated vertices (vertices with no neighbors), first remove all of them from $G$. Let the vertex set of $\mathrm{Box}(G)$ be $V(G) \times \{\circ, \bullet\}$; that is, for every (non-isolated) vertex $v \in V(G)$, the simplicial complex has two vertices, which we denote $v_\circ$ and $v_\bullet$. We will also write $v_?$ when $? \in \{\circ, \bullet\}$ is clear from the context. For a set $\sigma \subseteq V(G) \times \{\circ, \bullet\}$, we write $\sigma_\circ, \sigma_\bullet$ for $\sigma \cap (V(G) \times \{\circ\})$ and $\sigma \cap (V(G) \times \{\bullet\})$, respectively. For a set $A \subseteq V(G)$, we write $A_\circ$ and $A_\bullet$ for $\{v_\circ \mid v \in A\}$ and $\{v_\bullet \mid v \in A\}$ (to avoid confusion, we denote faces with small greek letters and vertex subsets with capital latin letters). The faces of $\mathrm{Box}(G)$ are exactly those sets $\sigma \subseteq V(G) \times \{\circ, \bullet\}$ such that $\sigma_\circ \boxtimes \sigma_\bullet$ and both $\mathrm{CN}(\sigma_\circ)$ and $\mathrm{CN}(\sigma_\bullet)$ are non-empty (in other words, the non-trivial complete bipartite subgraphs of $G$ and their subsets). Note that if $\sigma_\circ \neq \emptyset$, then $\sigma_\circ \cup \mathrm{CN}(\sigma_\circ) = \sigma \cup \mathrm{CN}(\sigma_\circ)$ is again a face; similarly for $\sigma_\bullet$; hence all maximal faces $\sigma$ have both $\sigma_\circ$ and $\sigma_\bullet$ non-empty. The $\mathbb{Z}_2$-action $-$ on $\mathrm{Box}(G)$ is defined as $-v_\circ = v_\bullet$ and $-v_\bullet = v_\circ$ for each $v \in V(G)$. As mentioned in the introduction, a homomorphism $G \to H$ induces a $\mathbb{Z}_2$-map $|\mathrm{Box}(G)| \to_{\mathbb{Z}_2} |\mathrm{Box}(H)|$. $\mathrm{Box}(G)$ $v_\circ, v_\bullet$ $v_?$ $\sigma_\circ, \sigma_\bullet$ $A_\circ, A_\bullet$

### Graph functors

An operation $\Gamma$ on graphs is a (thin) *functor* if $G \to H$ implies $\Gamma(G) \to \Gamma(H)$, for all graphs $G, H$. Two functors $\Gamma, \Omega$ are called a (thin) *adjoint pair* when $\Gamma(G) \to H$ holds if and only if $G \to \Omega(H)$ does. In this case $\Gamma, \Omega$ are called *left* and *right* adjoints, respectively. Note that a right adjoint functor may be a left adjoint in another pair, as is the case for the $\Gamma_k$ functor. The graph subdivision functor $\Lambda_k$ and powering functor $\Gamma_k$ were defined in the introduction. functor, adjoint

For a graph $G$ and an integer $\ell$, the graph $\Omega_{2\ell+1}(G)$ is defined as follows. Its vertices are tuples $\bar{A} = (A_0, \ldots, A_\ell)$ of vertex subsets $A_i \subseteq V(G)$ such that $A_0$ is a singleton (contains exactly one vertex) and $A_{i-1} \boxtimes A_i$ (for $i = 1 \ldots \ell$). Its edges are pairs $\{\bar{A}, \bar{B}\}$ such that $A_{i-1} \subseteq B_i$, $B_{i-1} \subseteq A_i$, and $A_i \boxtimes B_i$ (for $i = 1 \ldots \ell$). We define the homomorphism $p_{2\ell+1} \colon \Omega_{2\ell+1}(G) \to G$ as $p_{2\ell+1}((\{v\}, A_1, \ldots, A_\ell)) := v$. We do not define $\Omega_k(G)$ for even integers $k$ (see [FT18] for a functor $\Omega_2$ that shares some properties). $\Omega_k$ $p_k$

We now list a few basic properties of these functors. For the box complex, an important property is that $\mathrm{Box}()$ commutes with products:

**2.1 Lemma.** ([Cso08], see [SZ10]) $|\mathrm{Box}(G)| \times |\mathrm{Box}(H)| \simeq_{\mathbb{Z}_2} |\mathrm{Box}(G \times H)|$.

Similarly, any right adjoint graph functor commutes with the tensor product. We state this together with a few other simple properties. The proofs are straightforward. The applications to multiplicativity (Lemma 2.2.(vi) and 2.3.(x)) have first been shown and used by Tardif [Tar05]; we do not use them except for the proof of Theorem 1.8.

**2.2 Lemma.** *Let $G, G_1, G_2$ be any graphs. Then:*

*(i)* $G_1 \times G_2 \to G_i$, *for* $i = 1, 2$*;*

*(ii)* $G \to G_1 \times G_2$ *if and only if* $G \to G_1$ *and* $G \to G_2$*;*

*(iii) if* $\Gamma$ *is a functor, then* $\Gamma(G_1 \times G_2) \to \Gamma(G_1) \times \Gamma(G_2)$*;*

*(iv) if* $(\Gamma, \Omega)$ *is an adjoint pair of functors, then* $\Gamma(\Omega(G)) \to G \to \Omega(\Gamma(G))$*;*

*(v) if* $\Omega$ *is a right adjoint, then* $\Omega(G_1 \times G_2) \leftrightarrow \Omega(G_1) \times \Omega(G_2)$*;*

*(vi) if* $\Omega$ *is a right adjoint to a functor that is a right adjoint itself, and if* $K$ *is a multiplicative graph, then* $\Omega(K)$ *is multiplicative too.*

We follow with a few properties more specific to $\Lambda_k$, $\Gamma_k$, and $\Omega_k$. Most of these have been shown by Tardif [Tar05] or by Hajiabolhassan and Taherkhani [Haj09; HT10], who also proved many properties of other compositions of these functors (which can be interpreted as "fractional powers"). As far as we know, (vi) and (vii) are folklore, but have not appeared earlier in literature.

**2.3 Lemma.** *Let $G, H, K$ be graphs and let $k, k'$ be odd integers. Then:*

*(i)* $\Lambda_k(G) \to H$ *if and only if* $G \to \Gamma_k(H)$ *(that is,* $(\Lambda_k, \Gamma_k)$ *is an adjoint pair);*

*(ii)* $\Gamma_k(G) \to H$ *if and only if* $G \to \Omega_k(H)$ *(that is,* $(\Gamma_k, \Omega_k)$ *is an adjoint pair);*

*(iii)* $\Lambda_k(G) \to \Lambda_{k-2}(G) \to \cdots \to \Lambda_1(G) = G = \Gamma_1(G) \to \cdots \to \Gamma_{k-2}(G) \to \Gamma_k(G)$*;*

*(iv)* $\Omega_k(G) \to \Omega_{k-2}(G) \to \cdots \to \Omega_1(G) = G$*;*

*(v)* $\Lambda_k(\Lambda_{k'}(G)) \leftrightarrow \Lambda_{k\cdot k'}(G)$, $\Gamma_k(\Gamma_{k'}(G)) \leftrightarrow \Gamma_{k\cdot k'}(G)$, *and* $\Omega_k(\Omega_{k'}(G)) \leftrightarrow \Omega_{k\cdot k'}(G)$*;*

*(vi)* $\Lambda_k(G) \subseteq \Omega_k(G)$, *in particular* $\Lambda_k(G) \to \Omega_k(G)$*;*

*(vii) if* $G$ *is square-free, then* $\Lambda_k(G) \leftrightarrow \Omega_k(G)$*;*

*(viii)* $\Gamma_k(\Omega_k(G)) \leftrightarrow G \leftrightarrow \Gamma_k(\Lambda_k(G))$*;*

*(ix)* $G \to H$ *if and only if* $\Omega_k(G) \to \Omega_k(H)$*;*

*(x)* $K$ *is multiplicative if and only if* $\Omega_k(K)$ *is.*

*Proof.* Let $k = 2\ell + 1$. (i) and (iii) follow straight from definitions. For one direction of (ii), let $f : \Gamma_k(G) \to H$; then a homomorphism $G \to \Omega_k(H)$ is given by $v \mapsto (N^0(f(v)), \dots, N^\ell(f(v)))$ where $N^i(f(v))$ is the set of vertices reachable from $f(v)$ by walks of length $i$, as one can easily check. For the other direction, let $f : G \to \Omega_k(H)$; then a homomorphism $\Gamma_k(G) \to H$ is given by mapping $v$ to the only vertex in the first, singleton set of $f(v) = (A_0, \dots, A_\ell)$.

For (iv), $(A_0, \dots, A_{\ell-1}, A_\ell) \mapsto (A_0, \dots, A_{\ell-1})$ gives the homomorphism (where $k = 2\ell+1$). For (v) observe that $\Lambda_k(\Lambda_{k'}(G)) \leftrightarrow \Lambda_{k\cdot k'}(G)$ follows from the definition. Then since $\Gamma_{k\cdot k'}$ is a right adjoint to $\Lambda_{k\cdot k'}$, which is homomorphically equivalent (when applied to any graph) to $\Lambda_k(\Lambda_{k'}(\cdot))$, which in turn is a left adjoint to $\Gamma_{k'}(\Gamma_k(\cdot))$, it follows that $\Gamma_{k\cdot k'}(G) \leftrightarrow \Gamma_{k'}(\Gamma_k(G))$ (for all $G, k, k'$). Similarly the same follows for $\Omega_k$.

For (vi), let us define the following injective homomorphism $\Lambda_k(G) \to \Omega_k(G)$. For $a, b \in V(G)$, the path of length $k$ between $a$ and $b$ in the graph $k$-subdivision of $G$ is mapped to the following path in $\Omega_k(G)$:

$$
\begin{array}{llllllll}
(\{a\}, & N(a), & \{a\}, & N(a), & \{a\}, & \ldots), & \\
(\{b\}, & \{a\}, & N(a), & \{a\}, & N(a), & \ldots), & \\
(\{a\}, & \{b\}, & \{a\}, & N(a), & \{a\}, & \ldots), & \\
\vdots & & & & & & \\
(\{b\}, & \{a\}, & \{b\}, & \{a\}, & \{b\}, & \ldots), & (\leftarrow) \\
(\{a\}, & \{b\}, & \{a\}, & \{b\}, & \{a\}, & \ldots), & (\leftarrow \text{ or vice-versa}) \\
\vdots & & & & & & \\
(\{b\}, & \{a\}, & \{b\}, & N(b), & \{b\}, & \ldots), & \\
(\{a\}, & \{b\}, & N(b), & \{b\}, & N(b), & \ldots), & \\
(\{b\}, & N(b), & \{b\}, & N(b), & \{b\}, & \ldots). &
\end{array}
$$

(The two vertices in the middle should be swapped when $\lfloor \frac{k}{2} \rfloor$ is even). It is straightforward to check this defines an injective homomorphism in a consistent way.

To show (vii), let $k = 2\ell + 1$. We construct $f : \Omega_{2\ell+1}(G) \to \Lambda_{2\ell+1}(G)$ as follows. For $\bar{A} = (A_0, \ldots, A_\ell) \in V(\Omega_{2\ell+1}(G))$ with $A_0 = \{a\}$, let $j_{\bar{A}}$ be the maximum index such that $A_i$ are singletons for $i \leq j_{\bar{A}}$. If $j_{\bar{A}} = 0$, we set $f(\bar{A}) = a$, otherwise let $A_1 = \{b\}$ and we set $f(\bar{A})$ to be the $i$-th vertex on the path between $a$ and $b$ (counting $a$ as the 0-th vertex), where $A_1 = \{b\}$ and $i = j_{\bar{A}}$ if $j_{\bar{A}}$ is even, while $i = 2\ell + 1 - j_{\bar{A}}$ if $j_{\bar{A}}$ is odd.

Let $\bar{A}, \bar{B}$ be adjacent in $\Omega_{2\ell+1}(G)$. Since $A_\ell \boxtimes B_\ell$ and $G$ is square-free, one of $A_\ell, B_\ell$ must be of size at most 1. Assume without loss of generality that $|A_\ell| \leq 1$ (otherwise swap $\bar{A}$ and $\bar{B}$). Since $A_\ell \supseteq B_{\ell-1} \supseteq A_{\ell-2} \supseteq \ldots$ is a sequence of containments ending in a singleton $A_0$ or $B_0$, all these containments are equalities. Let us also assume that $\ell$ is odd (the proof is the same with $\ell$ even). That is, the sequence ends in $B_0$ and $A_\ell = B_{\ell-1} = A_{\ell-2} = \cdots = B_0$ is a singleton. Let $B_0 = \{b\}$ and $A_0 = \{a\}$. Consider the sequence $A_0 \subseteq B_1 \subseteq A_2 \subseteq \cdots \subseteq B_\ell$ and let $j$ be the maximum index such that the $j$-th set of this sequence is a singleton, and hence equal to $A_0 = \{a\}$, as well as to all the sets in between. Then, since the next sets in the sequence (if there are any) are not singletons, we have $j_{\bar{A}} = j$ and $j_{\bar{B}} = j + 1$ or vice versa (depending on the parity of $j$), unless $j = \ell$, in which case $j_{\bar{A}} = j_{\bar{B}} = \ell$. It each case, it is easily checked that $f(\bar{A})$ and $f(\bar{B})$ are adjacent in $\Lambda_{2\ell+1}(G)$.

Observe that (vi) implies $\Lambda_k(G) \to \Omega_k(G)$, which by adjointness is equivalent to $G \to \Gamma_k(\Omega_k(G))$ and to $\Gamma_k(\Lambda_k(G)) \to G$. This, together with Lemma 2.2.(iv), implies (viii). Applying $\Gamma_k$ to both sides of the assumption $\Omega_k(G) \to \Omega_k(H)$ thus yields the non-trivial direction of (ix).

For (x), one direction follows from Lemma 2.2.(vi). For the other, suppose $\Omega_k(K)$ is multiplicative. Let $G \times H \to K$. Then $\Omega_k(G) \times \Omega_k(H) \to \Omega_k(G \times H) \to \Omega_k(K)$, hence $\Omega_k(G) \to \Omega_k(K)$ or $\Omega_k(H) \to \Omega_k(K)$, which by (ix) implies $G \to K$ or $H \to K$. $\square$

## 3. Proof of the Equivalence Theorem 1.1 – collapses and expansions

The goal of this section is to show Theorem 1.1, in particular that $|\mathrm{Box}(G)|$ and $|\mathrm{Box}(\Omega_{2k+1}(G))|$ are $\mathbb{Z}_2$-homotopy equivalent, for all $k$. Following ideas of Csorba [Cso08], we use basics of Discrete Morse Theory, a framework introduced by Forman [For98] which allows to show homotopy equivalence in a very combinatorial way. We refer to [For02] for an introduction and [Koz08b] for an in depth coverage.

Let us introduce the required notions. We will construct homotopy equivalences by composing a sequence of small steps. If $K$ is a simplicial complex with a face $\tau$ such that there is a unique face $\sigma \neq \tau$ in $K$ containing $\tau$, then it is not hard to show that $K \setminus \{\tau, \sigma\}$ is homotopy equivalent to $K$; this is called an *elementary collapse*. If $K'$ can be obtained from $K$ by a sequence of elementary collapses, we say that $K$ *collapses* to $K'$. If $K'$ can be obtained from $K$ by a sequence

collapse
simple homot. eq.

of elementary collapses and expansions (operations inverse to elementary collapses), we say that $K'$ is *simple homotopy equivalent* to $K$ (Whitehead showed that this notion is slightly stronger than just homotopy equivalence, see [Coh73]). The definitions are naturally extended to free $\mathbb{Z}_2$-simplicial complexes (where elementary collapses have to be performed in pair: $\tau, \sigma$ are removed together with their $\mathbb{Z}_2$-image $\tau', \sigma'$).

A sequence of elementary collapses can be described more concisely using matchings. For a simplicial complex $K$ and a subcomplex $K'$, a *matching* is a bijective function $\mu$ on the set of faces matching $K \setminus K'$ such that $\mu \circ \mu = \mathrm{id}$ and for each $\sigma \in K \setminus K'$, $\mu(\sigma)$ contains or is contained in $\sigma$. We also require that $\dim \mu(\sigma) = \dim \sigma \pm 1$. Since all of the faces of $K \setminus K'$ are matched into pairs, we can try to order them into a sequence of elementary collapses. The sufficient and necessary condition turns out to be the following. A matching is *acyclic* if there is no sequence of containments of the following form (for $n \geq 2$ pair-wise different $\sigma_i$ in $K \setminus K'$): acyclic

$$\sigma_1 \subsetneq \mu(\sigma_1) \supsetneq \sigma_2 \subsetneq \mu(\sigma_2) \supsetneq \cdots \subsetneq \mu(\sigma_n) \supsetneq \sigma_1 \tag{$*$}$$

With those definitions, we can state the basic theorem of Discrete Morse Theory (we note this is only the simplest version of the statement, but we will not need anything more):

**3.1 Theorem.** ([For98]) *Let $K$ be a $\mathbb{Z}_2$-simplicial complex and $K'$ a $\mathbb{Z}_2$-subcomplex. If there is an acyclic $\mathbb{Z}_2$-matching $M$ on the set of faces $K \setminus K'$, then $K'$ is $\mathbb{Z}_2$-homotopy equivalent to $K$ and the inclusion map $K' \hookrightarrow K$ is a $\mathbb{Z}_2$-homotopy equivalence.*

We will show that $\mathrm{Box}(\Omega_{2k+1}(G))$ and $\mathrm{Box}(\Omega_{2k-1}(G))$ are (simple) homotopy equivalent by defining an intermediate complex that collapses to both. For $\bar{A} \in V(\Omega_{2k+1}(G))$, let $\phi(\bar{A})$

$$\phi(\bar{A}) := (A_0, \ldots, A_{k-1}, \mathrm{CN}(A_{k-1})) \in V(\Omega_{2k+1}(G)).$$

Define the graph $\Omega'_{2k+1}(G)$ by adding the following edges to $\Omega_{2k+1}(G)$: for each existing edge $\Omega'_{2k+1}$ $\{\bar{A}, \bar{B}\}$, add new edges $\{\bar{A}, \phi(\bar{B})\}$, $\{\phi(\bar{A}), \bar{B}\}$, and $\{\phi(\bar{A}), \phi(\bar{B})\}$. Observe that $\phi(\bar{A})$ is adjacent to $\phi(\bar{B})$ if and only if $(A_0, \ldots, A_{k-1})$ and $(B_0, \ldots, B_{k-1})$ are adjacent in $\Omega_{2k-1}(G)$. In particular the subgraph of $\Omega'_{2k+1}(G)$ induced on vertices of $\mathrm{im}\,\phi$ is isomorphic to $\Omega_{2k-1}(G)$. We show that it induces a homotopy equivalent subcomplex. (We write $\sigma \triangle \{v\}$ for the symmetric difference, that $\sigma \triangle \{v\}$ is, $\sigma \cup \{v\}$ if $v \notin \sigma$ and $\sigma \setminus \{v\}$ otherwise).

**3.2 Lemma.** $\mathrm{Box}(\Omega'_{2k+1}(G))$ *$\mathbb{Z}_2$-collapses to the subcomplex induced by* $\mathrm{im}\,\phi$ *(isomorphic to* $\mathrm{Box}(\Omega_{2k-1}(G))$*).*

*Proof.* The faces not in the subcomplex are exactly those that contain $\bar{A}_\circ$ (or $\bar{A}_\bullet$) for some vertex $\bar{A}$ from outside $\mathrm{im}\,\phi$. We define a matching $\mu$ by matching every such face $\sigma$ with $\sigma \triangle \{\phi(\bar{A})_\circ\}$ (or $\sigma \triangle \{\phi(\bar{A})_\bullet\}$), where $\bar{A}$ is chosen to be the smallest vertex in $\sigma \setminus \mathrm{im}\,\phi$, according to an arbitrary, fixed ordering on $V(\Omega'_{2k+1}(G))$. Note that exactly one of $\bar{A}_\circ$, $\bar{A}_\bullet$ is in $\sigma$, so this is a well defined $\mathbb{Z}_2$-matching. The fact that $\sigma \triangle \{\phi(\bar{A})_?\}$ is a face of $\mathrm{Box}(\Omega'_{2k+1}(G))$ follows from the definition of $\phi$ and $\Omega'_{2k+1}$.

To show that the matching is acyclic, suppose $\sigma_1, \ldots, \sigma_n$ $(n \geq 2)$ forms a cycle as in $(*)$. When going up the matching, from $\sigma_i$ to $\mu(\sigma_i)$, we always add a vertex in $\mathrm{im}\,\phi$. Therefore, since the sequence forms a cycle, when going down from $\mu(\sigma_i)$ to $\sigma_{i+1}$ we can only remove vertices in $\mathrm{im}\,\phi$; the set of vertices of $\sigma$ not in $\mathrm{im}\,\phi$ remains constant. But then the vertex $\phi(\bar{A})_?$ added when going up the matching from $\sigma_1$ to $\mu(\sigma_1)$ is also the vertex in $\sigma_2 \triangle \mu(\sigma_2)$, by definition of the matching $\mu$. This vertex is not removed when going down from $\mu(\sigma_1)$ to $\sigma_2$, since $\sigma_1 \neq \sigma_2$ $(n \geq 2)$. Hence $\sigma_2$ contains this vertex and $\mu(\sigma_2) = \sigma_2 \setminus \{\phi(\bar{A})_?\}$, contradicting that the sequence should go up the matching $(\sigma_2 \subseteq \mu(\sigma_2))$. $\square$

The collapse onto $\mathrm{Box}(\Omega_{2k+1}(G))$ is less easy to describe. Let us first characterize minimal faces that have to be collapsed.

**3.3 Lemma.** *A face $\sigma \in \mathrm{Box}(\Omega'_{2k+1}(G))$ is not in the subcomplex $\mathrm{Box}(\Omega_{2k+1}(G))$ if and only if*

*(i) $\sigma$ contains $\bar{A}_\circ, \bar{B}_\bullet$ such that $A_k \not\supseteq\!\!\!\subseteq B_k$, or*

*(ii) $\sigma$ contains $\bar{A}_\circ, \bar{C}_\circ$ (or $\bar{A}_\bullet, \bar{C}_\bullet$) such that $A_k \not\supseteq\!\!\!\subseteq C_{k-1}$.*

*Proof.* Let us first show one direction. If $\sigma \in \mathrm{Box}(\Omega'_{2k+1}(G))$ contains $\bar{A}_\circ, \bar{B}_\bullet$ such that $A_k \not\supseteq\!\!\!\subseteq B_k$, then these are clearly not adjacent in $\Omega_{2k+1}(G)$, so $\sigma$ is not in the subcomplex. If $\sigma$ contains $\bar{A}_\circ, \bar{C}_\circ$ (or $\bar{A}_\bullet, \bar{C}_\bullet$) such that $A_k \not\supseteq\!\!\!\subseteq C_{k-1}$, then suppose $\sigma$ is in the subcomplex. By definition this implies $\mathrm{CN}(\sigma_\circ)$ (meaning the common neighborhood in $\Omega_{2k+1}(G)$) is non-empty, so let $\bar{B}$ be a common neighbor of $\bar{A}$ and $\bar{C}$ in $\Omega_{2k+1}(G)$. Then $\bar{A}$ is adjacent to $\bar{B}$ in $\Omega_{2k+1}(G)$, which implies $A_k \supseteq\!\!\!\subseteq B_k$, and $\bar{B}$ is adjacent to $\bar{C}$, which implies $B_k \supseteq C_{k-1}$, contradicting $A_k \not\supseteq\!\!\!\subseteq C_{k-1}$. Hence $\sigma$ cannot be in the subcomplex.

For the other direction, consider a face $\sigma \in \mathrm{Box}(\Omega'_{2k+1}(G))$ that is not in the subcomplex. That is, there are $\bar{A}_\circ, \bar{B}_\bullet \in \sigma$ such that $\bar{A}$ and $\bar{B}$ are not adjacent in $\Omega_{2k+1}(G)$, or it must be that $\mathrm{CN}(\sigma_\circ)$ or $\mathrm{CN}(\sigma_\bullet)$ is empty. In the former case, since $\bar{A}$ and $\bar{B}$ are adjacent in $\Omega'_{2k+1}(G)$, we conclude that $A_k \not\supseteq\!\!\!\subseteq B_k$. In the latter case, say $\mathrm{CN}(\sigma_\circ)$ is empty. That is, the vertices of $\sigma_\circ$ do not have a common neighbor in $\Omega_{2k+1}(G)$, although they do have some common neighbor $\bar{B}$ in $\Omega'_{2k+1}(G)$. Let $\bar{B}' := (B_0, \dots, B_{k-1}, B'_k)$ where $B'_k := \bigcup_{\bar{C} \in \sigma_\circ} C_{k-1}$. Since $\bar{B}'$ in particular is not a common neighbor of $\sigma_\circ$ in $\Omega_{2k+1}(G)$, it must be that $B'_k \not\supseteq\!\!\!\subseteq A_k$ for some $\bar{A} \in \sigma_\circ$. By definition of $B'_k$, this means that $C_{k-1} \not\supseteq\!\!\!\subseteq A_k$ for some $\bar{A}, \bar{C} \in \sigma_\circ$ □

We can now show the necessary collapse, in phases corresponding to the points in Lemma 3.3. The reader is warned that the proof is not very illuminating, it is just trying the simplest collapses that come to mind, carefully adapted into a few phases until all cases are covered, and checking that all the technical conditions are satisfied.

**3.4 Lemma.** $\mathrm{Box}(\Omega'_{2k+1}(G))$ *$\mathbb{Z}_2$-collapses to the subcomplex* $\mathrm{Box}(\Omega_{2k+1}(G))$.

*Proof.* We first collapse faces containing some vertices $\bar{A}_\circ, \bar{C}_\circ$ (or $\bar{A}_\bullet, \bar{C}_\bullet$) such that $A_k \not\supseteq\!\!\!\subseteq C_{k-1}$. Among those, we first collapse faces where $\bar{A}$ can be chosen from outside $\mathrm{im}\,\phi$.

For any such face $\sigma$, choose $\bar{A}_? \in \sigma \setminus \mathrm{im}\,\phi$, $\bar{C}_? \in \sigma$ such that $A_k \not\supseteq\!\!\!\subseteq C_{k-1}$ and $(\bar{A}, \bar{C})$ is lexicographically minimum (according to some arbitrary fixed ordering of vertices of $\Omega_{2k+1}(G)$). Without loss of generality assume $? = \circ$ for this minimum pair. Let

$$\bar{A}^* := (A_0, \dots, A_{k-1}, \mathrm{CN}(S)) \quad \text{where} \quad S := \bigcup_{\bar{A}'_\circ \in \sigma} A'_{k-1} \ \cup \bigcup_{\bar{B}'_\bullet \in \sigma \setminus \mathrm{im}\,\phi} B'_k$$

We define a matching $\mu(\sigma) := \sigma \vartriangle \{\bar{A}^*_\circ\}$. We need to check a series of technical conditions:

(i) the vertex $\bar{A}^*$ is well defined;

(ii) $\sigma \vartriangle \{\bar{A}^*_\circ\}$ is a face of $\mathrm{Box}(\Omega'_{2k+1}(G))$; equivalently, that $\bar{A}^*$ is adjacent to vertices in $\sigma_\bullet$ and has a common neighbor together with all the vertices in $\sigma_\circ$;

(iii) $\sigma \vartriangle \{\bar{A}^*_\circ\}$ still contains $\bar{A}_\circ$ and $\bar{C}_\circ$ (so it is not in the subcomplex we collapse to);

(iv) $\mu(\sigma \vartriangle \{\bar{A}^*_\circ\}) = \sigma$ (so that $\mu$ is indeed a matching);

(v) $\mu$ is acyclic.

For (i), observe that $S$ contains $A_{k-1}$, which implies $A_{k-1} \supseteq\!\!\!\subseteq \mathrm{CN}(S)$, as required for a vertex.

For (ii), let us first show that $\bar{A}^*$ is adjacent to each vertex in $\sigma_\bullet$. Let $\bar{B} \in \sigma_\bullet$. One condition for adjacency is that $B_{k-1} \subseteq \mathrm{CN}(S)$, or equivalently, that $B_{k-1} \boxtimes S$. This holds, because $B_{k-1} \boxtimes A'_{k-1}$ for each $\bar{A}'_\circ \in \sigma$ (because $\bar{B}_\bullet, \bar{A}'_\circ$ are adjacent, as they are contained in $\sigma$). Furthermore, $B_{k-1} \boxtimes B'_k$ for each $\bar{B}'_\bullet \in \sigma \setminus \operatorname{im}\phi$, because $B_{k-1} \subseteq A_k$ and $A_k \boxtimes B'_k$ (because $\bar{B}_\bullet$ and $\bar{A}_\circ$ are adjacent, while $\bar{A}_\circ$ and $\bar{B}'_\bullet$ are adjacent and not in $\operatorname{im}\phi$). Thus $B_{k-1} \boxtimes S$, that is, $B_{k-1} \subseteq \mathrm{CN}(S)$.

If $\bar{B} \in \operatorname{im}\phi$, then $\bar{B} = (B_0, \ldots, B_{k-1}, \mathrm{CN}(B_{k-1})) = \phi((B_0, \ldots, B_{k-1}, A_{k-1}))$. But $\bar{A}^*$ is adjacent to $(B_0, \ldots, B_{k-1}, A_{k-1})$, because $\bar{A}$ was adjacent to $\bar{B}$, $B_{k-1} \subseteq \mathrm{CN}(S)$ and $A_{k-1} \boxtimes \mathrm{CN}(S)$. Hence $\bar{A}^*$ is (by definition of $\Omega'_{2k+1}$) also adjacent to $\phi((B_0, \ldots, B_{k-1}, A_{k-1}))$, which is $\bar{B}$.

If on the other hand $\bar{B} \notin \operatorname{im}\phi$, then $\bar{A}^*$ is again adjacent to it, because $\bar{A}$ was, $B_{k-1} \subseteq \mathrm{CN}(S)$ (as shown above), and $B_k \boxtimes \mathrm{CN}(S)$ (because $B_k \subseteq S$).

To conclude (ii), it remains to show that $\bar{A}^*$ has a common neighbor together with all vertices in $\sigma_\circ$. If $\sigma_\bullet$ is non-empty, then any vertex in it is such a common neighbor. If however $\sigma_\bullet$ is empty, then there must exists a vertex $\bar{B} \in \mathrm{CN}(\sigma_\circ)$, so $\sigma \cup \{\bar{B}_\bullet\}$ is a face and the same analysis as above shows that $\bar{A}^*$ is also adjacent to $\bar{B}$, proving that $\bar{B}$ is a common neighbor of $\sigma_\circ \triangle \{\bar{A}^*\}$.

For (iii), we need to show that $\bar{A}^*_\circ \neq \bar{A}_\circ$ and $\bar{A}^*_\circ \neq \bar{C}_\circ$. The former follows from the fact that $S \supseteq C_{k-1}$, so $A^*_k = \mathrm{CN}(S) \boxtimes C_{k-1}$, while $A_k \not\boxtimes C_{k-1}$, thus $A^*_k \neq A_k$. The latter follows from the fact that $C_{k-1} \not\boxtimes A_k$, but $A^*_{k-1} = A_{k-1} \boxtimes A_k$, so $C_{k-1} \neq A^*_{k-1}$.

For (iv), we need to show that the initial choice of a pair $\bar{A}_?, \bar{C}_?$ for $\sigma \cup \{\bar{A}^*_\circ\}$ will be the same as for $\sigma \setminus \{\bar{A}^*_\circ\}$. Recall that valid choices are pairs $\bar{A}_?, \bar{C}_?$ of vertices in the face such that $\bar{A} \notin \operatorname{im}\phi$ and $A_k \not\boxtimes C_{k-1}$, and we select the lexicographically minimum valid choice. Without loss of generality assume $\sigma \triangle \{\bar{A}^*_\circ\} = \sigma \cup \{\bar{A}^*_\circ\}$ and suppose to the contrary that the choice for $\sigma \cup \{\bar{A}^*_\circ\}$ is $(\bar{A}^\dagger, \bar{C}^\dagger)$, different from the choice $(\bar{A}, \bar{C})$ for $\sigma$. Since $(\bar{A}, \bar{C})$ is a valid choice for $\sigma \cup \{\bar{A}^*_\circ\}$ as well, $(\bar{A}^\dagger, \bar{C}^\dagger)$ must be lexicographically smaller. That is, either $\bar{A}^\dagger < \bar{A}$, or $\bar{A}^\dagger = \bar{A}$ and $\bar{C}^\dagger < \bar{C}$. Since $(\bar{A}^\dagger, \bar{C}^\dagger)$ was not a valid choice for $\sigma$, we have $\bar{A}^\dagger = \bar{A}^*$ or $\bar{C}^\dagger = \bar{A}^*$. If $\bar{A}^\dagger = \bar{A}^*$, then $A^\dagger_k = A^*_k = \mathrm{CN}(S) \boxtimes C^\dagger_{k-1}$ (because $S \supseteq C^\dagger_{k-1}$), contradicting that $(\bar{A}^\dagger, \bar{C}^\dagger)$ was a valid choice. If on the other hand $\bar{C}^\dagger = \bar{A}^*$, then validity of the choice implies $A^\dagger_k \not\boxtimes C^\dagger_{k-1} = A^*_{k-1} = A_{k-1}$. In particular $A^\dagger \neq A$, so $A^\dagger < A$. But then the pair $(A^\dagger, A)$ would have been a valid, lexicographically smaller choice for $\sigma$, a contradiction.

Finally we show (v), that is, the matching $\mu$ is acyclic. Suppose to the contrary that $\sigma_1, \ldots, \sigma_n$ ($n \geq 2$) forms a cycle as in ($*$). When going up the matching, from $\sigma_i$ to $\mu(\sigma_i)$, the initial choice of $\bar{A}_?, \bar{C}_?$ remains the same, as shown in (iv). When going down from $\mu(\sigma_i)$ to $\sigma_{i+1}$ contained in it, the initial choice can only stay the same or increase lexicographically (since it is also available for $\mu(\sigma_i)$). Hence the choice of $\bar{A}_?, \bar{C}_?$ must in fact remain unchanged throughout the cycle, say it is $\bar{A}_\circ, \bar{C}_\circ$ for all $\sigma_i$ and $\mu(\sigma_i)$. Therefore, the vertices we add (and hence also those we remove) in the cycle are all of the form $(A_0, \ldots, A_{k-1}, X)_\circ$ for some vertex subsets $X$. This implies that the set $S$, as defined above, and hence also $\bar{A}^*$, is always the same when defining the face $\mu(\sigma_i)$ matched to $\sigma_i$. But then $\bar{A}^*$ is always the vertex added (the vertex in $\mu(\sigma_i) \setminus \sigma_i$) and hence also the only vertex removed (the one in $\mu(\sigma_i) \setminus \sigma_{i+1}$), which implies $\sigma_1 = \sigma_2$, a contradiction.

We now collapse the remaining faces $\sigma$ that contain some vertices $\bar{A}_\circ, \bar{C}_\circ$ (or $\bar{A}_\bullet, \bar{C}_\bullet$) such that $A_k \not\boxtimes C_{k-1}$. By the previous collapsing phase, we know that $\bar{A} \in \operatorname{im}\phi$ and symmetrically:

$$\text{For any } \bar{B}'_\bullet, \bar{B}_\bullet \in \sigma \text{ with } B'_k \not\boxtimes B_{k-1}, \text{ we know that } \bar{B}' \in \operatorname{im}\phi. \tag{1}$$

For any such face $\sigma$, choose $\bar{A}_?, \bar{C}_? \in \sigma$ such that $A_k \not\boxtimes C_{k-1}$ and $(\bar{A}, \bar{C})$ is lexicographically minimum (according to some arbitrary fixed ordering of vertices of $\Omega_{2k+1}(G)$). Without loss of generality assume $? = \circ$ for this minimum pair. Just as before, let

$$\bar{A}^* := (A_0, \ldots, A_{k-1}, \mathrm{CN}(S)) \quad \text{where} \quad S := \bigcup_{\bar{A}'_\circ \in \sigma} A'_{k-1} \quad \cup \quad \bigcup_{\bar{B}'_\bullet \in \sigma \setminus \operatorname{im}\phi} B'_k$$

We define a matching $\mu(\sigma) := \sigma \triangle \{\bar{A}^*_\circ\}$. Similarly as before, we need to show (i)–(v). The proof of (i) is unchanged: $S$ contains $A_{k-1}$, which implies $A_{k-1} \boxtimes \mathrm{CN}(S)$, as required for a vertex.

For (ii), let us first show that $\bar{A}^*$ is adjacent to each vertex in $\sigma_\bullet$. Let $\bar{B} \in \sigma_\bullet$. Observe that $B_{k-1} \boxtimes A'_{k-1}$ for all $\bar{A}'_\circ \in \sigma$ and by (1), $B_{k-1} \boxtimes B'_k$ for $\bar{B}'_\bullet \in \sigma \setminus \operatorname{im}\phi$, hence $B_{k-1} \boxtimes S$, which means $B_{k-1} \subseteq \mathrm{CN}(S)$. The remaining proof proceeds just as before (with two cases depending on $\bar{B} \in \operatorname{im}\phi$ or $\bar{B} \notin \operatorname{im}\phi$), concluding (ii). The proofs of (iii)–(v) also proceed without change, since they never used the fact that $\bar{A} \notin \operatorname{im}\phi$.

Finally, we collapse all faces $\sigma$ containing $\bar{A}_\circ, \bar{B}_\bullet$ such that $A_k \not\boxtimes B_k$. Fortunately this is considerably simpler, since $\sigma_\bullet$ is non-empty, and by the previous collapses, we known that

$$B'_k \boxtimes B''_{k-1} \text{ for any } \bar{B}', \bar{B}'' \in \sigma_\bullet \tag{2}$$

For any such face $\sigma$, choose a lexicographically minimum pair $\bar{A}_\circ, \bar{B}_\bullet$ or $\bar{A}_\bullet, \bar{B}_\circ$ such that $A_k \not\boxtimes B_k$. Without loss of generality assume it is $\bar{A}_\circ, \bar{B}_\bullet$. Let $\bar{A}^* := (A_0, \ldots, A_{k-1}, \bigcup_{\bar{B}' \in \sigma} B'_{k-1})$. We define a matching $\mu(\sigma) := \sigma \triangle \{\bar{A}^*_\circ\}$ and check (i)–(v). It is now easy to check (using (2)) that (i) and (ii) are satisfied.

For (iii), we need to show that $\bar{A}^*_\circ \neq \bar{A}_\circ$ (and trivially $\bar{A}^*_\circ \neq \bar{B}_\bullet$). This follows from the fact that $A_k \not\boxtimes B_k$, but $A^*_k = \bigcup_{\bar{B}' \in \sigma} B'_{k-1} \boxtimes B_k$ (by (2)).

For (iv), we need to show that the initial choice of a pair $\bar{A}_\circ, \bar{B}_\bullet$ for $\sigma \cup \{\bar{A}^*_\circ\}$ will be the same as for $\sigma \setminus \{\bar{A}^*_\circ\}$. This follows from the fact that $A^*_k \boxtimes B'_k$ for all $\bar{B}' \in \sigma_\bullet$, so $\bar{A}^*$ does not contribute in any way to this choice (since it only consider vertices such that $A_k \not\boxtimes B_k$).

Finally to show (v), suppose to the contrary that $\sigma_1, \ldots, \sigma_n$ ($n \geq 2$) forms a cycle as in ($*$). When going up the matching, from $\sigma_i$ to $\mu(\sigma_i)$, the initial choice of $\bar{A}_\circ, \bar{B}_\bullet$ remains the same, as shown in (iv). When going down from $\mu(\sigma_i)$ to the face $\sigma_{i+1}$ contained in it, the initial choice can only stay the same or increase lexicographically (since it is also available for $\mu(\sigma_i)$). Hence the choice of $\bar{A}_\circ, \bar{B}_\bullet$ must in fact remain unchanged throughout the cycle. This implies that when going up the matching, we only add vertices to $\sigma_\circ$, so when going through the cycle we also only remove vertices from $\sigma_\circ$, and $\sigma_\bullet$ is unchanged. But then the vertex $\bar{A}^*$ added in the matching (in $\mu(\sigma_i) \setminus \sigma_i$) is always the same, so the only possible vertex in $\mu(\sigma_i) \setminus \sigma_{i+1}$ is also $\bar{A}^*$, implying that $\sigma_1 = \sigma_2$, a contradiction. □

Theorem 3.1 with Lemma 3.2 and 3.4 already imply the $\mathbb{Z}_2$-simple homotopy equivalence of $\mathrm{Box}(\Omega_{2k+1}(G))$ and $\mathrm{Box}(\Omega_{2k-1}(G))$. To describe an explicit homotopy equivalence, Theorem 3.1 is insufficient, as it only guarantees a map in one direction of a collapse (the containment map) to be a homotopy equivalence. We hence replace the use of Lemma 3.2 with an explicit homotopy to conclude our theorem. The Equivalence Theorem 1.1 follows by applying it repeatedly.

**3.5 Theorem.** *For any graph $G$ and $k \in \mathbb{N}$, $\mathrm{Box}(\Omega_{2k+1}(G))$ and $\mathrm{Box}(\Omega_{2k-1}(G))$ are $\mathbb{Z}_2$-simple homotopy equivalent. Moreover, the homomorphism $(A_0, \ldots, A_{k-1}, A_k) \mapsto (A_0, \ldots, A_{k-1}) : \Omega_{2k+1}(G) \to \Omega_{2k-1}(G)$ induces a $\mathbb{Z}_2$-homotopy equivalence.*

*Proof.* By Theorem 3.1 and Lemma 3.4 the containment map of $\mathrm{Box}(\Omega_{2k+1}(G))$ in $\mathrm{Box}(\Omega'_{2k+1}(G))$ is a $\mathbb{Z}_2$-homotopy equivalence. It remains to show that the following map $q$ from $\mathrm{Box}(\Omega'_{2k+1}(G))$ to $\mathrm{Box}(\Omega_{2k-1}(G))$ is a $\mathbb{Z}_2$-homotopy equivalence:

$$q\colon (A_0, \ldots, A_{k-1}, A_k)_? \mapsto (A_0, \ldots, A_{k-1})_? \quad (\text{for } ? \in \{\circ, \bullet\})$$

Consider the containment map $\iota : (A_0, \ldots, A_{k-1})_? \mapsto (A_0, \ldots, A_{k-1}, \mathrm{CN}(A_k))_?$ of $\mathrm{Box}(\Omega_{2k-1}(G))$ in $\mathrm{Box}(\Omega'_{2k+1}(G))$. One composition, $\iota \circ q : \mathrm{Box}(\Omega_{2k-1}(G)) \to \mathrm{Box}(\Omega_{2k-1}(G))$, is just the identity.

The other composition is $q \circ \iota : (A_0, \ldots, A_{k-1}, A_k)_? \mapsto (A_0, \ldots, A_{k-1}, \mathrm{CN}(A_{k-1}))_?$. For $t \in [0,1]$, define $q_t(\bar{A}) = (1-t) \cdot \bar{A} + t \cdot \iota(q(\bar{A})) \in |\mathrm{Box}(\Omega'_{2k+1}(G))|$ and extend it linearly from vertices to each face. For this to be well defined, we need to show that for any $\sigma \in \mathrm{Box}(\Omega'_{2k+1}(G))$, the set $\sigma \cup \{\iota(q(\bar{A}_?)) \mid \bar{A}_? \in \sigma\}$ is again a face. This follows from the definition of $\Omega'_{2k+1}$ and the fact that $q \circ \iota$ coincides with the map $\phi$ used in this definition. Thus $q_t$ defines a $\mathbb{Z}_2$-homotopy from $q_1 = q \circ \iota$ to $q_0$, the identity map. Therefore $q$ and $\iota$ are $\mathbb{Z}_2$-homotopy equivalences, and hence $q$ composed with the containment map of $\mathrm{Box}(\Omega_{2k+1}(G))$ into $\mathrm{Box}(\Omega'_{2k+1}(G))$ is a $\mathbb{Z}_2$-homotopy equivalence. □

# 4. Proof of the Approximation Theorem 1.2 – simplicial approximation

We first show that $|\mathrm{Box}(\Omega_{2k+1}(G))|$ refines $|\mathrm{Box}(G)|$. This is the part where using $\Omega_{2k+1}$ instead of iterations of $\Omega_3$ makes the proof considerably simpler, because when iteratively applying $\Omega_3$, one would have to deal with the fact that degrees in the graph, and hence dimensions of simplicies, grow exponentially.

**4.1 Theorem.** *There is a $\mathbb{Z}_2$-map $g$ from $|\mathrm{Box}(\Omega_{2k+1}(G))|$ to $|\mathrm{Box}(G)|$ such that:*

- *$g$ is homotopic to the map induced by the homomorphism $p_k : \Omega_{2k+1}(G) \to G$,*
- *in particular, $g$ is a $\mathbb{Z}_2$-homotopy equivalence (by Theorem 1.1),*
- *$g$ maps every face of $\mathrm{Box}(\Omega_{2k+1}(G))$ into a subset of $|\mathrm{Box}(G)|$ of diameter less than $\frac{6D}{k}$, where $D$ is the maximum degree of $G$.*

*Proof.* For a set of points $S = \{s_1, \dots, s_n\} \subseteq |\mathrm{Box}(G)|$, define $\mathrm{avg}(S) := \frac{1}{n}(s_1 + \dots + s_n)$. For $\bar{A}_\circ \in V(\mathrm{Box}(\Omega_{2k+1}(G)))$, let

$$g(\bar{A}_\circ) := \mathrm{avg}(\{\mathrm{avg}(A_{0\circ}), \mathrm{avg}(A_{1\bullet}), \mathrm{avg}(A_{2\circ}), \dots, \mathrm{avg}(A_{k?})\})$$

(with $\circ$ and $\bullet$ alternating). Since $\bar{A}$ is not an isolated vertex of $G$ (by definition of the box complex), we have $A_i \subseteq A_{i+2}$ for $i = 0 \dots k-2$ and $A_i \boxtimes A_{i+1}$ for $i = 0 \dots k-1$. Hence the set $\tau_{\bar{A}_\circ} := \bigcup_{i \text{ even}} A_{i\circ} \cup \bigcup_{i \text{ odd}} A_{i\bullet}$ is a face of $\mathrm{Box}(G)$. As $g(\bar{A}_\circ)$ is a convex combination of vertices in this face, it defines a point in the geometric realization $|\mathrm{Box}(G)|$. Define $g(\bar{A}_\bullet) \in |\mathrm{Box}(G)|$ symmetrically and extend this map linearly to all of $|\mathrm{Box}(\Omega_{2k+1}(G))|$. This is well defined, since the set $\tau_\sigma := \bigcup_{\bar{A}_? \in \sigma} \tau_{\bar{A}_?}$ is easily checked to be a face of $\mathrm{Box}(G)$.

For $t \in [0,1]$ and $\bar{A}_\circ \in V(\mathrm{Box}(\Omega_{2k+1}(G)))$ with $A_0 = \{v\}$, define $g_t(\bar{A}_\circ) := (1-t) \cdot g(\bar{A}_\circ) + t \cdot v$. This is again a convex combination of vertices in $\tau_{\bar{A}}$ that extends linearly, hence $g_t$ defines a homotopy from $g_0 = g$ to $g_1 = p_k$.

It remains to bound the diameter of images of faces. Let $\sigma$ be a maximal face of $\mathrm{Box}(\Omega_{2k+1}(G))$. Let $\bar{A}_\circ, \bar{B}_\bullet \in \sigma$. Then $A_0 \subseteq B_1 \subseteq A_2 \subseteq \dots$ and $B_0 \subseteq A_1 \subseteq B_2 \subseteq \dots$ are two subset chains of length $k+1$. Moreover, since $A_k \boxtimes B_k$, all sets have size at most $|A_k|, |B_k| \leq D$, where $D$ is the maximum degree in $G$. Since $|A_0| = |B_0| = 1$, in both chains at most $D-1$ containments $\subseteq$ are strict, all the other ones are equalities. In particular, among the pairs $(A_0, B_1), (A_1, B_0), (A_2, B_3), (A_3, B_2), \dots$ at most $2D-2$ are pairs of different sets. Therefore $g(\bar{A}_\circ)$ and $g(\bar{B}_\bullet)$ can be written as $\frac{2D-2}{k} p + \frac{k-(2D-2)}{k} q$ and $\frac{2D-2}{k} p' + \frac{k-(2D-2)}{k} q$ for some points $p, p', q$ in $\tau_\sigma$ (defined as averages of subsets of $\{\mathrm{avg}(A_{0\circ}), \mathrm{avg}(A_{1\bullet}), \dots\}$ and $\{\mathrm{avg}(B_{0\bullet}), \mathrm{avg}(B_{1\circ}), \dots\}$). Hence the distance between $g(\bar{A}_\circ)$ and $g(\bar{B}_\bullet)$ is the same as the distance between $\frac{2D-2}{k} p$ and $\frac{2D-2}{k} p'$, which is at most $\frac{2D-2}{k}\sqrt{2}$, since the distance between any two point $p, p'$ in a simplex (namely in the face $\tau_\sigma$) of a geometric realization is at most $\sqrt{2}$. The distance between $g(\bar{A}_\circ)$ and $g(\bar{A}'_\circ)$ for any $\bar{A}_\circ, \bar{A}'_\circ \in \sigma$ is at most the sum of their distances to $g(\bar{B}_\bullet)$, hence at most $\frac{4\sqrt{2}(D-1)}{k}$. Since the image of $\sigma$ is the convex hull of the images of its vertices, its diameter is also bounded by $\frac{4\sqrt{2}(D-1)}{k}$, which we bound by $\frac{6D}{k}$ for conciseness. □

We are now ready to show the Approximation Theorem 1.2. This closely follows the standard technique of simplicial approximation (see eg. Theorem 2C.1. in [Hat01]). The main difference is that we consider $\mathbb{Z}_2$-maps instead of just continuous maps, and that finding a $\mathbb{Z}_2$-map between box complexes that is simplicial (ie., a linear extension of a map on vertices of the complex) is not enough to find a graph homomorphism (because a simplicial map can map two adjacent vertices into a single vertex). To avoid these problems, we use a variant of the fact that the box complex is equivalent to the complex $\mathrm{Hom}(K_2, G)$ [Cso08], allowing us to avoid certain extremal points of the box complex.

**1.2 Theorem.** (restated) *There exists a $\mathbb{Z}_2$-map from $|\mathrm{Box}(G)|$ to $|\mathrm{Box}(H)|$ if and only if for some $k \in \mathbb{N}$, $\Omega_{2k+1}(G)$ has a homomorphism to $H$.*

*Moreover, for any $\mathbb{Z}_2$-map $f : |\mathrm{Box}(G)| \to_{\mathbb{Z}_2} |\mathrm{Box}(H)|$ there is an integer $k$ and a homomorphism $\Omega_{2k+1}(G) \to H$ that induces a map $\mathbb{Z}_2$-homotopic to $p_k \circ f$.*

*Proof.* For one direction, suppose $\Omega_{2k+1}(G)$ has a homomorphism to $H$, for some $k \in \mathbb{N}$. This induces a $\mathbb{Z}_2$-map from $|\mathrm{Box}(\Omega_{2k+1}(G))|$ to $|\mathrm{Box}(H)|$. By Theorem 1.1, there is a $\mathbb{Z}_2$-map from $|\mathrm{Box}(G)|$ to $|\mathrm{Box}(\Omega_{2k+1}(G))|$. Composition then gives a $\mathbb{Z}_2$-map from $|\mathrm{Box}(G)|$ to $|\mathrm{Box}(H)|$.

For the other direction, let $f : |\mathrm{Box}(G)| \to_{\mathbb{Z}_2} |\mathrm{Box}(H)|$ be a $\mathbb{Z}_2$-map. Let $X \subseteq |\mathrm{Box}(H)|$ be the set of points $x \in |\mathrm{Box}(H)|$ that, when written as a convex combination $x = \sum_{v \in V(H)} \lambda_v v_\circ + \sum_{v \in V(H)} \mu_v v_\bullet$, satisfy $\sum_v \lambda_v = \sum_v \mu_v = \frac{1}{2}$. Simonyi et al. [STV09] observed that the equivalences of various versions of the box complex imply that $|\mathrm{Box}(H)|$ is $\mathbb{Z}_2$-homotopy equivalent to the subspace $X$. Therefore, up to $\mathbb{Z}_2$-homotopy, we can assume that the image of $f$ is contained in $X$.

Define the *star* of a face $\sigma$ in a simplicial complex $K$ to be the subcomplex made of all faces containing $\sigma$ (and their subsets), that is: $\{\tau \mid \tau \cup \sigma \in K\}$. Define the *closed star* $\mathrm{St}\,\sigma \subseteq |K|$ as the geometric realization of the star of $\sigma$ and the *open star* $\mathrm{st}\,\sigma \subseteq |K|$ as the sum of interiors of geometric realizations of faces in the star of $\sigma$. Thus $\mathrm{St}\,\sigma$ is the closure of the open set $\mathrm{st}\,\sigma$. For a vertex $v$ of the complex, we write $\mathrm{St}\,v$ for short instead of $\mathrm{St}\{v\}$.

Observe that the sets $\mathrm{st}\,v_\circ$ for $v \in V(H)$ cover (a superset of) $X$ in $|\mathrm{Box}(H)|$ (to cover all of $|\mathrm{Box}(H)|$ we would need $\mathrm{st}\,v_\bullet$ as well). Consider the family of sets $\mathcal{C}_\circ := \{f^{-1}(\mathrm{st}\,v_\circ) \mid v \in V(H)\}$. This is a family of open sets covering $|\mathrm{Box}(G)|$, a compact space (as a closed and bounded subset of $\mathbb{R}^n$). Therefore, we can let $\varepsilon > 0$ be the Lebesgue number of $\mathcal{C}_\circ$, that is, a number such that any set $X \subseteq |\mathrm{Box}(G)|$ of diameter less than $\varepsilon$ is contained in some set of $\mathcal{C}_\circ$.

Let $D$ be the maximum degree of $G$ and let $k := 12D \cdot \frac{1}{\varepsilon}$. Let $g$ be the $\mathbb{Z}_2$-map from $|\mathrm{Box}(\Omega_{2k+1}(G))|$ to $|\mathrm{Box}(G)|$ given by Theorem 4.1. For every face $\sigma$ of $|\mathrm{Box}(\Omega_{2k+1}(G))|$, $g(\sigma)$ has diameter at most $\frac{6D}{k}$ in $|\mathrm{Box}(G)|$. Thus, for every vertex $\bar{A}$ of $\Omega_{2k+1}(G)$, $g(\mathrm{St}\,\bar{A}_\circ)$ has diameter less than $\frac{12D}{k} = \varepsilon$ in $|\mathrm{Box}(G)|$, hence there is a vertex $h(\bar{A}) \in V(H)$ such that $g(\mathrm{St}\,\bar{A}_\circ) \subseteq f^{-1}(\mathrm{st}\,h(\bar{A})_\circ)$, that is, $f(g(\mathrm{St}\,\bar{A}_\circ)) \subseteq \mathrm{st}\,h(\bar{A})_\circ$. We claim that $h : V(\Omega_{2k+1}(G)) \to V(H)$ is a graph homomorphism.

Indeed, let $\bar{A}, \bar{B}$ be two adjacent vertices of $\Omega_{2k+1}(G)$. Then $f(g(\mathrm{St}\,\bar{A}_\circ)) \subseteq \mathrm{st}\,h(\bar{A})_\circ$ and $f(g(\mathrm{St}\,\bar{B}_\circ)) \subseteq \mathrm{st}\,h(\bar{B})_\circ$, or equivalently (since $f, g$ are $\mathbb{Z}_2$-maps), $f(g(\mathrm{St}\,\bar{B}_\bullet)) \subseteq \mathrm{st}\,h(\bar{B})_\bullet$. Since $\bar{A}_\circ$ is contained in both $\mathrm{St}\,\bar{A}_\circ$ and $\mathrm{St}\,\bar{B}_\bullet$, $f(g(\mathrm{St}\,\bar{A}_\circ))$ is contained in both $\mathrm{st}\,h(\bar{A})_\circ$ and $\mathrm{st}\,h(\bar{B})_\bullet$. Hence $\mathrm{st}\,h(\bar{A})_\circ \cap \mathrm{st}\,h(\bar{B})_\bullet \neq \emptyset$, which implies that $h(\bar{A})$ and $h(\bar{B})$ must be adjacent graph vertices.

Finally we show that the $\mathbb{Z}_2$-map induced by $h$ is homotopic to $g \circ f$, which in turn is homotopic to $p_k \circ f$, as guaranteed by Theorem 4.1. Indeed, let $x$ be a point in a face $|\sigma| = |\{\bar{A}^1_\circ, \dots, \bar{A}^n_\circ, \bar{B}^1_\bullet, \dots, \bar{B}^m_\bullet\}|$ of $|\mathrm{Box}(\Omega_{2k+1}(G))|$. Then $h(x)$ is a point in the face $|h(\sigma)| := |\{h(\bar{A}^1)_\circ, \dots, h(\bar{B}^1)_\bullet, \dots\}|$ (since the $\mathbb{Z}_2$-map induced by the graph homomorphism is defined as a linear extension of the map on vertices). On the other hand $x$ is in $\mathrm{St}\,\bar{A}^1_\circ \cap \dots \cap \mathrm{St}\,\bar{B}^1_\bullet \cap \dots$, hence $f(g(x))$ is in $\mathrm{st}\,h(\bar{A}^1)_\circ \cap \dots \mathrm{st}\,h(\bar{B}^1)_\bullet \cap \dots$, which is equal to $\mathrm{st}\{h(\bar{A}^1)_\circ, \dots, h(\bar{B}^1)_\bullet, \dots\} = \mathrm{st}\,h(\sigma)$ (see Lemma 2C.2. in [Hat01]). Therefore $h(x)$ and $f(g(x))$ are both contained in a common face (a face in the star of $h(\sigma)$). We can thus define a homotopy $t \cdot f(g(x)) + (1-t) \cdot h(x)$ (this is clearly continuous for $x$ varying on any face $|\sigma|$, hence everywhere) from $h$ to $g \circ f$. □

## 5. Powers of graphs of high girth are multiplicative

In this section we show that 3rd powers of a graphs of girth $> 12$ are multiplicative. For this, we use $\Omega_3$ and $\Gamma_3$ functors in a similar way as Tardif [Tar05] did to prove the multiplicativity of circular cliques. There is always a homomorphism $K \to \Omega_k(\Gamma_k(K))$, namely $v \mapsto (\{v\}, N(v), \ldots, N^\ell(v))$, as is easy to check (here $N^d(v)$ is the set of vertices reachable by walks of length exactly $d$ from $v$). If for some $K$ we can show a homomorphism in the other direction, then $K$ is homomorphically equivalent to $\Omega_k(\Gamma_k(K))$. By Lemma 2.3 (x), this is multiplicative if and only if $\Gamma_k(K)$ is. We use this to infer the multiplicativity of $\Gamma_k(K)$ from that of a square-free graph $K$.

The proof here is combinatorial, but an informal intuition behind it is that for graphs of girth $> 4k$, taking the $k$-th power does not change the topology (one can show that the box complex remains $\mathbb{Z}_2$-homotopy equivalent); one may thus hope that applying the 'inverse' $\Omega_k$ will get us back to the original graph. Whether the proof can really be extended to $k$-th powers of graphs of girth $> 4k$ for all $k \in \mathbb{N}$ remains an open question. It would also be interesting to find more graphs for which $\Omega_3(\Gamma_3(K)) \to K$.

**1.8 Theorem.** (restated) *Let $K$ be a graph of girth $> 3 \cdot 4$ (no cycles of length $\leq 12$). Then $\Omega_3(\Gamma_3(K)) \to K$, hence $\Omega_3(\Gamma_3(K)) \leftrightarrow K$ and $\Gamma_3(K)$ is multiplicative.*

*Proof.* Throughout the proof all neighborhoods and walks are always meant in $K$, not in $\Gamma_3(K)$ (only $A \boxtimes B$ will mean joined sets in $\Gamma_3(K)$). We define a homomorphism $f\colon \Omega_3(\Gamma_3(K)) \to K$ as follows. For a vertex $\bar{A} = (\{x\}, A_1)$, let $f(\bar{A})$ be any vertex in $\mathrm{CN}(A_1)$ closest to $x$ (possibly $x$ itself) if $\mathrm{CN}(A_1) \neq \emptyset$, otherwise let $f(\bar{A}) := x$.

To show that $f$ is indeed a homomorphism, consider two adjacent vertices $(\{x\}, A), (\{y\}, B)$ in $\Omega_3(\Gamma_3(K))$. That is, $x \in B$, $y \in A$ and $A \boxtimes B$ in $\Gamma_3(K)$. We first show that

$$\mathrm{CN}(A) \neq \emptyset \text{ or } \mathrm{CN}(B) \neq \emptyset \qquad (*)$$

Suppose that to the contrary $\mathrm{CN}(A) = \mathrm{CN}(B) = \emptyset$. Then there are $a, a' \in A$ with no common neighbor in $K$ (since otherwise there would be distinct $a, a', a'' \in A$ with pairwise common neighbors, contradicting that the girth is $> 6$) and similarly $b, b' \in B$ with no common neighbor in $K$. Since $A \boxtimes B$ in $\Gamma_3(K)$, there are walks of length 3 between $a, a'$ and $b, b'$ in $K$. Concatenated together, these four walks give one closed walk of length 12 going through $a, a', b, b'$ in order. Since $K$ has girth $> 12$, this closed walk must be a walk in a tree subgraph $T$ of $K$. Hence there are walks of length 3 between $a, a'$ and $b, b'$ in $T \subseteq K$.

Let $P$ be the shortest path between $a$ and $a'$ in $T$: since there is a walk of length exactly 6 between them (going though $b$ or $b'$), the shortest path has even length $\leq 6$ and $> 2$, since $a, a'$ have no common neighbor. Hence $P$ has length 4 or 6. If $P$ has length 6, then $b$ and $b'$, which are accessible via walks of length 3 from both endpoints of $P$ in $T$, must both be equal to middle vertex of $P$, a contradiction. If $P$ has length 4, then similarly $b$ and $b'$ must be adjacent to the middle vertex of $P$ in $T$, hence they have a common neighbor in $T \subseteq K$, a contradiction. This proves (*).

If $\mathrm{CN}(A) \neq \emptyset$ and $f(\{y\}, B) = y$, then $y \in A$ is adjacent by definition to any vertex in $\mathrm{CN}(A)$, so $f(\{y\}, B)$ is adjacent to $f(\{x\}, A) \in \mathrm{CN}(A)$. In particular if $\mathrm{CN}(B) = \emptyset$, then $f(\{y\}, B) = y$ and $\mathrm{CN}(A) \neq \emptyset$ by (*), hence the mapping is correct. Symmetrically if $\mathrm{CN}(B) \neq \emptyset$ and $f(\{x\}, A) = x$, then the mapping is correct. In particular if $\mathrm{CN}(A) = \emptyset$, then we are done.

It thus remains to consider the case where $f(\{x\}, A) \neq x$ and $f(\{y\}, B) \neq y$. In this case, we have that $\mathrm{CN}(A) \neq \emptyset$ but $x \notin \mathrm{CN}(A)$ and similarly for $y, B$. Let $a$ be a vertex in $A$ not adjacent to $x$ in $K$. Since $x \in B \boxtimes A \ni a$ in $\Gamma_3(K)$, there is a (unique, since $K$ has girth $> 6$) walk of length 3 between $a$ and $x$ in $K$; let $x, p, q, a$ be the vertices on it. If there is any vertex $a' \in A$ other than $a$, there is a walk of length 2 between $a$ and $a'$ (as they have a common neighbor), a walk of length 3 from $a'$ to $x$, and the walk of length 3 $x, p, q, a$; together they form a closed walk of length 8, so they must map to a tree in $K$ and it is easy to see that the common neighbor of $a$ and $a'$ must be $q$. Hence $q \in \mathrm{CN}(A)$. If $|A| > 1$ there can be only one vertex in $\mathrm{CN}(A)$ (since $K$ has no $C_4$), while if $A = \{a\}$, then vertex in $\mathrm{CN}(A) = N(a)$ closest to $x$ is $q$. Hence $f(\{x\}, A) = q$.

Since $f(\{y\}, B) \neq y$, we have $\mathrm{CN}(B) \neq \emptyset$ but $y \notin \mathrm{CN}(B)$. Then $y \in A$ is adjacent to $q \in \mathrm{CN}(A)$. If $y$ is not adjacent to $x$, then $y, q, p, a$ is a (unique) walk of length 3 and just as above we conclude that $f(\{y\}, B)$ is the second vertex on this walk, namely $p$; thus $f(\{x\}, A) = q$ and $f(\{y\}, B)$ are adjacent. Otherwise, if $y$ is adjacent to $x$, then it is a common neighbor of $x$ and $q$, hence $y = p$ (since $K$ has no $C_4$). For any vertex $b \in B$ other than $x$, similarly as before there is a walk of length 2 between $x$ and $b$ (since $\mathrm{CN}(B) \neq \emptyset$), a walk of length 3 between $b$ and $a$, an the walk $a, q, p, x$; together, they form a closed walk of length 8, so they must map to a tree in $K$ and it is easy to see that the common neighbor of $x$ and $b$ must be $y$. Hence $y \in CN(B)$, which we assumed was not the case. Therefore in all cases $f(\{x\}, A)$ is adjacent to $f(\{y\}, B)$ in $K$, concluding the proof. $\square$

One can check that the same proof works with walks in $K \times K_2$ used in place of $K$, so in fact it suffices to assume that $K \times K_2$ has girth $> 12$, that is, cycles of length $7, 9, 11$ are allowed in $K$.

## 6. Conclusions on the multiplicativity of graphs and spaces

### Proof of Theorem 1.5 and Theorem 1.7

We show how the following theorem easily follows from the Equivalence, Approximation and Universality Theorems. (Recall that a $\mathbb{Z}_2$-space $Z$ is *multiplicative* if $X \times Y \to_{\mathbb{Z}_2} Z$ implies $X \to_{\mathbb{Z}_2} Z$ or $Y \to_{\mathbb{Z}_2} Z$ for any $\mathbb{Z}_2$-spaces $X, Y$.)

**1.5 Theorem.** (restated) *Let $K$ be a multiplicative graph. Then $|\mathrm{Box}(K)|$ is multiplicative.*

*Proof.* Let $X, Y$, be finite simplicial $\mathbb{Z}_2$-spaces such that $X \times Y \to_{\mathbb{Z}_2} |\mathrm{Box}(K)|$. By the Universality Theorem 1.3, there are graphs $G, H$ such that $X \simeq_{\mathbb{Z}_2} |\mathrm{Box}(G)|$ and $Y \simeq_{\mathbb{Z}_2} |\mathrm{Box}(H)|$. Thus:

$$|\mathrm{Box}(G \times H)| \overset{\text{Lem 2.1}}{\simeq_{\mathbb{Z}_2}} |\mathrm{Box}(G)| \times |\mathrm{Box}(H)| \simeq_{\mathbb{Z}_2} X \times Y \to_{\mathbb{Z}_2} |\mathrm{Box}(K)|.$$

By the Approximation Theorem 1.2, there is an odd integer $k$ such that $\Omega_k(G) \times \Omega_k(H) \overset{\text{Lem 2.2}}{\leftrightarrow} \Omega_k(G \times H) \to K$. By definition of multiplicativity of $K$, we have $\Omega_k(G) \to K$ or $\Omega_k(H) \to K$, hence by the other direction of the Approximation Theorem 1.2, $X \simeq_{\mathbb{Z}_2} |\mathrm{Box}(G)| \to_{\mathbb{Z}_2} |\mathrm{Box}(K)|$ or $Y \simeq_{\mathbb{Z}_2} |\mathrm{Box}(H)| \to_{\mathbb{Z}_2} |\mathrm{Box}(K)|$. $\square$

The proof of Theorem 1.7 is similarly straightforward.

**1.7 Theorem.** (restated) *Let $Z$ be a $\mathbb{Z}_2$-space and let $K$ be a graph such that $Z \simeq_{\mathbb{Z}_2} |\mathrm{Box}(K)|$. Then $Z$ is multiplicative if and only if for all graphs $G, H$ the following holds: if $G \times H \to K$, then for some odd $k$, $\Omega_k(G) \to K$ or $\Omega_k(H) \to K$.*

*Proof.* For one direction, let $K$ be a graph such that $|\mathrm{Box}(K)|$ is a multiplicative $\mathbb{Z}_2$-space. Let $G, H$ be graphs and suppose that $G \times H \to K$. Then

$$|\mathrm{Box}(G)| \times |\mathrm{Box}(H)| \overset{\text{Lem 2.1}}{\simeq_{\mathbb{Z}_2}} |\mathrm{Box}(G \times H)| \to_{\mathbb{Z}_2} |\mathrm{Box}(K)|$$

By multiplicativity of $|\mathrm{Box}(K)|$, we have $|\mathrm{Box}(G)| \to_{\mathbb{Z}_2} |\mathrm{Box}(K)|$ (or the same for $H$). By the Approximation Theorem 1.2, this implies that $\Omega_k(G) \to K$ for some odd integer $k$.

For the other direction, suppose $K$ has the property that for all graphs $G, H$, $G \times H \to K$ implies $\Omega_k(G) \to K$ or $\Omega_k(H) \to K$ for some odd $k$. Let $X, Y$ be any $\mathbb{Z}_2$-spaces such that $X \times Y \to_{\mathbb{Z}_2} |\mathrm{Box}(K)|$. By the Universality Theorem 1.3, there are graphs $G, H$ such that $|\mathrm{Box}(G)| \simeq_{\mathbb{Z}_2} X$ and $|\mathrm{Box}(H)| \simeq_{\mathbb{Z}_2} Y$.

Then

$$|\mathrm{Box}(G\times H)| \xrightarrow[\simeq_{\mathbb{Z}_2}]{\text{Lem 2.1}} |\mathrm{Box}(G)|\times|\mathrm{Box}(H)| \to_{\mathbb{Z}_2} |\mathrm{Box}(K)|$$

Hence by the Approximation Theorem 1.2, there is an odd integer $k$ such that

$$\Omega_k(G)\times\Omega_k(H) \overset{\text{Lem 2.2}}{\leftrightarrow} \Omega_k(G\times H)\to K$$

By the property of $K$, there is an odd integer $k'$ such that $\Omega_{k'}(\Omega_k(G)) \to K$ (or the same for $H$). By the Approximation Theorem 1.2 and the Equivalence Theorem 1.1, this implies $X \simeq_{\mathbb{Z}_2} |\mathrm{Box}(G)| \simeq_{\mathbb{Z}_2} |\mathrm{Box}(\Omega_k(G))| \to_{\mathbb{Z}_2} |\mathrm{Box}(K)| \simeq_{\mathbb{Z}_2} Z$ (or $Y \to_{\mathbb{Z}_2} Z$). □

## Known cases of multiplicativity

Let us give as a warm-up an elementary proof of the multiplicativity of the 0-dimensional sphere $\mathcal{S}^0$ (two points $-1, 1$ on the real line, with the $\mathbb{Z}_2$-action swapping them), in the following two lemmas. (The first will be crucial to the multiplicativity of $\mathcal{S}^1$ as well).

**6.1 Lemma.** *Let $X$ be a $\mathbb{Z}_2$-space. Then $\mathcal{S}^1 \not\to_{\mathbb{Z}_2} X$ if and only if $X \to_{\mathbb{Z}_2} \mathcal{S}^0$.*

*Proof.* Let $p\colon \mathcal{S}^1 \to_{\mathbb{Z}_2} X$ be a $\mathbb{Z}_2$-map. Then $p$ on one half of $\mathcal{S}^1$ gives a path $p'\colon [0,1]\to X$ from some point $p(0) = x \in X$ to $p(1) = -x$. If there was a map $f\colon X \to_{\mathbb{Z}_2} \mathcal{S}^0$, then each connected (path-)component of $X$ would have to map all into $-1$ or all into $1 \in \mathcal{S}^0$, in particular $f(x) = f(-x)$, a contradiction.

For the other direction, assume $\mathcal{S}^1 \not\to_{\mathbb{Z}_2} X$. Then there is no path $p\colon [0,1]\to X$ from a point $x\in X$ to $-x$, since concatenating such a path $t\mapsto p(t)$ with $t \mapsto -p(t)$ gives a $\mathbb{Z}_2$-map $\mathcal{S}^1 \to_{\mathbb{Z}_2} X$. Therefore, the $\mathbb{Z}_2$-action $-$ matches the connected (path-)components of $X$ into pairs. We can choose a map that maps one component of each pair into $-1$ and the other into 1, giving a $\mathbb{Z}_2$-map $X \to_{\mathbb{Z}_2} \mathcal{S}^0$. □

To translate the above proof to graphs, recall that the antipode of $v_\circ$ in the box complex of a graph $G$ (for a vertex $v$ of $G$) is $v_\bullet$ and observe that there is a path from $v_\circ$ to $v_\bullet$ in the box complex if and only if there is a walk of odd length in the graph $G$ from $v$ to $v$ itself. That is, 'equivariant' circles in the box complex, represented as $\mathbb{Z}_2$-maps $\mathcal{S}^1 \to_{\mathbb{Z}_2} X$, correspond to odd closed walks in the graph. The above lemma then corresponds to the fact that a graph has no odd closed walks (equivalently, no odd cycles) if and only if it has a homomorphism to $K_2$ (equivalently, it is bipartite). The proof can also be made entirely analogous, by considering connected components of $G\times K_2$. We proceed with a proof of multiplicativity.

**6.2 Lemma.** *$\mathcal{S}^0$ is multiplicative. That is, for any $\mathbb{Z}_2$-spaces $X, Y$, if $X\times Y \to_{\mathbb{Z}_2} \mathcal{S}^0$, then $X \to_{\mathbb{Z}_2} \mathcal{S}^0$ or $Y \to_{\mathbb{Z}_2} \mathcal{S}^0$.*

*Proof.* Suppose that $X \not\to_{\mathbb{Z}_2} \mathcal{S}^0$ and $Y \not\to_{\mathbb{Z}_2} \mathcal{S}^0$. By the above lemma, there are $\mathbb{Z}_2$-maps $p\colon \mathcal{S}^1 \to_{\mathbb{Z}_2} X$ and $q\colon \mathcal{S}^1 \to_{\mathbb{Z}_2} Y$. But then $t\mapsto (p(t), q(t))$ is a $\mathbb{Z}_2$-map $\mathcal{S}^1 \to_{\mathbb{Z}_2} X\times Y$ (since by definition of the product of $\mathbb{Z}_2$-spaces, $-(p(t),q(t)) = (-p(t), -q(t)) = (p(-t), q(-t))$). Therefore $X\times Y \not\to_{\mathbb{Z}_2} \mathcal{S}^0$. □

The multiplicativity of $K_2$ is a simple translation of this proof: if $G \not\to K_2$ and $H \not\to K_2$, then there are odd closed walks $P = (P_1,\dots,P_n)$ $(P_i \in V(G))$ in $G$ and $Q = (Q_1,\dots,Q_m)$ in $H$. We can turn them into odd closed walks of equal length, say $P' = P$ and $Q' = (Q_1,\dots,Q_m,Q_{m-1},Q_m,\dots,Q_{m-1},Q_m)$ if $n\le m$. Thus $((P'_1,Q'_1),(P'_2,Q'_2),\dots)$ is an odd closed walk in $G\times H$, hence $G\times H \not\to K_2$.

After this warm-up, let us turn to the circle $\mathcal{S}^1$.

**6.3 Lemma.** *$\mathcal{S}^1$ is multiplicative. That is, for any $\mathbb{Z}_2$-spaces $X, Y$, if $X \times Y \to_{\mathbb{Z}_2} \mathcal{S}^1$, then $X \to_{\mathbb{Z}_2} \mathcal{S}^1$ or $Y \to_{\mathbb{Z}_2} \mathcal{S}^1$.*

*Proof.* Let $f\colon X \times Y \to_{\mathbb{Z}_2} \mathcal{S}^1$ be a $\mathbb{Z}_2$ map (that is, $-f(x,y) = f(-x,-y)$ for all $x \in X, y \in Y$). Without loss of generality assume $X, Y$ are (path-)connected (otherwise we can consider each pair of connected components separately). Fix arbitrary points $x_0 \in X, y_0 \in Y$.

Consider any $\mathbb{Z}_2$-maps $p\colon \mathcal{S}^1 \to_{\mathbb{Z}_2} X$ and $q\colon \mathcal{S}^1 \to_{\mathbb{Z}_2} Y$ starting (and ending) at $x_0$ and $y_0$, respectively (if there are none, the claim already follows from Lemma 6.1). For $t \in \mathcal{S}^1$, let $p'(t) := f(p(t), y_0)$ and $q'(t) := f(x_0, q(t))$. The functions $p', q'$ are continuous maps from $\mathcal{S}^1$ to $\mathcal{S}^1$ (not necessarily $\mathbb{Z}_2$-maps). Since the concatenation of the path $p(t)$ with the constant path $t \mapsto x_0$ is homotopic to $p(t)$ (and similarly for $q(t)$ and $y_0$), the concatenation of paths $t \mapsto (p(t), y_0)$ and $t \mapsto (x_0, q(t))$ in $X \times Y$ is homotopic to $t \mapsto (p(t), q(t))$. Therefore the concatenation of $p'(t) = f(p(t), y_0)$ and $q'(t) = f(x_0, q(t))$ is homotopic to $t \mapsto f(p(t), q(t))$. Thus the winding numbers of $p'$ and $q'$ sum to the winding number of $t \mapsto f(p(t), q(t))$. The latter is a $\mathbb{Z}_2$-map (because $f, p, q$ are) and hence has odd winding number. Therefore exactly one of the winding numbers of $p'$ and $q'$ is odd. Without loss of generality suppose the winding number of $p'$ is odd and the winding number of $q'$ is even. Then the winding number of $p'$ is odd for any choice of $p\colon \mathcal{S}^1 \to_{\mathbb{Z}_2} X$ (starting and ending at $x_0$), as we can keep the choice of $q, q'$ unchanged (with even winding number). Moreover, for any $p\colon \mathcal{S}^1 \to_{\mathbb{Z}_2} X$, even if $p$ does not necessarily start and end at $x_0$, then $p$ is still homotopic to a $\mathbb{Z}_2$-map that does, so $p'$ has odd winding number in this case as well.

For any $p\colon \mathcal{S}^1 \to_{\mathbb{Z}_2} X$, since the winding number of $p'$ is odd, there is a point $t \in \mathcal{S}^1$ such that $p'(-t) = -p'(t)$. That is $f(p(-t), y_0) = -f(p(t), y_0)$. Let us call a point $x \in X$ a *coincidence point* if $f(x, y_0) = -f(-x, y_0)$ (equivalently, $f(x, y_0) = f(x, -y_0)$). Let $X' \subseteq X$ be the set of coincidence points (observe that if $x \in X'$, then $-x \in X'$ as well). Then we know that there is no $\mathbb{Z}_2$-map $p\colon \mathcal{S}^1 \to_{\mathbb{Z}_2} X \setminus X'$. Therefore, there is a $\mathbb{Z}_2$-map $h\colon X \setminus X' \to \mathcal{S}^0$. We can then define a $\mathbb{Z}_2$-map from $X$ to $\mathcal{S}^1$ as follows: if $x \in X \setminus X'$, we map $x$ to $f(x, -y_0)$ or to $f(x, y_0)$ depending on $h(x) \in \{-1, 1\}$; otherwise, if $x \in X'$, we map $x$ to $f(x, y_0) = f(x, -y_0)$. This is easily checked to give a $\mathbb{Z}_2$-map from $X$ to $\mathcal{S}^1$. □

The proof of the multiplicativity of $K_3$ by El-Zahar and Sauer [ES85], its generalization to odd cycles by Häggkvist et al. [Häg+88], and especially its reformulation and generalization to circular cliques $K_{p/q}$ (with $2 < p/q < 4$) given in Chapter III, largely follows the steps of the above proof of Lemma 6.3. An invariant on odd cycles is considered, which turns out to be exactly the winding number assigned as above to the corresponding map $\mathcal{S}^1 \to_{\mathbb{Z}_2} |\mathrm{Box}(G)|$. One then proves that all odd cycles on one side of the product must have an odd invariant, which implies that certain coincidence points must exists on every such cycle (this part can be done just as above, purely topologically, in the box complex). If those coincidence points occur on vertices of the box complex (corresponding to vertices of the graph), as opposed to some general position on edges or larger faces, then they can be temporarily removed to conclude a homomorphism just as above. The only additional step is thus showing that the coincidence points can be assumed to lie on vertices, which is indeed true and not hard to show for $K_3$ and odd cycles. However, for circular cliques a certain relaxation of this notion is necessary (but still possible, see Chapter III), while for other graphs $G$ with $|\mathrm{Box}(G)| \simeq_{\mathbb{Z}_2} \mathcal{S}^1$ we do not know whether this approach can work at all, indeed we do not know whether all such graphs are multiplicative.

The multiplicativity of square-free graphs, also shown in Chapter III, corresponds to the multiplicativity of 1-dimensional $\mathbb{Z}_2$-spaces, that is, those coming from simplicial complexes with no faces of size larger than 2. However, all such spaces can be shown to admit a $\mathbb{Z}_2$-map to $\mathcal{S}^1$, and then their multiplicativity easily follows from that of $\mathcal{S}^1$. This reasoning does not extend to the combinatorial setting, unfortunately.

### Obstacles and non-tidy spaces

When attempting to generalize the above proofs to higher dimensional spheres, even just to $\mathcal{S}^2$, while some steps do extend (the arguments on the parity of winding numbers, in particular), there are nevertheless substantial obstacles. Perhaps the most important is the fact that Lemma 6.1 becomes false: there are $\mathbb{Z}_2$-spaces $X$ such that $\mathcal{S}^2 \not\to_{\mathbb{Z}_2} X$, but $X \not\to_{\mathbb{Z}_2} \mathcal{S}^1$.

This gap can in fact get much worse. Consider the following two parameters of a $\mathbb{Z}_2$-space $X$. The *coindex* $\operatorname{coind}(X)$ is the largest $n$ such that $\mathcal{S}^n \to_{\mathbb{Z}_2} X$. The *index* is the least $n$ such that ind
$X \to_{\mathbb{Z}_2} \mathcal{S}^n$. The Borsuk-Ulam theorem thus states that $\operatorname{coind}(X) \leq \operatorname{ind}(X)$. These parameters are coind
analogous to the clique number $\omega(G)$ (the size of the largest clique subgraph) and the chromatic number $\chi(G)$ of a graph $G$. In fact:

$$\omega(G) \;\leq\; \operatorname{coind}(|\mathrm{Box}(G)|) + 2 \;\leq\; \operatorname{ind}(|\mathrm{Box}(G)|) + 2 \;\leq\; \chi(G).$$

Spaces where the coindex is strictly smaller than the index are called *non-tidy* (see [Mat08], p. 100). Lemma 6.1 states that the coindex is 0 if and only if the index is 0, so non-tidy spaces are counter-examples to its generalization, and thus a significant problem when attempting to extend known cases of Hedetniemi's conjecture. Moreover, Conjecture 1.6 is equivalent to the statement that, for all $\mathbb{Z}_2$-spaces $X, Y$:

$$\operatorname{ind}(X \times Y) = \min(\operatorname{ind}(X), \operatorname{ind}(Y)).$$

Since the inequality $\operatorname{ind}(X \times Y) \leq \min(\operatorname{ind}(X), \operatorname{ind}(Y))$ is trivial and since $X \to_{\mathbb{Z}_2} Y$ easily implies the other direction, any counter-example to Conjecture 1.6 must satisfy $X \not\to_{\mathbb{Z}_2} Y$ and $Y \not\to_{\mathbb{Z}_2} X$. Since $\operatorname{coind}(X) \geq \operatorname{ind}(Y)$ implies $Y \to_{\mathbb{Z}_2} \mathcal{S}^{\operatorname{ind}(Y)} \to_{\mathbb{Z}_2} X$, this means any counter-example to Conjecture 1.6 must involve a non-tidy space.

Non-tidy spaces are not so easy to come by, at least for a combinatorialist, but a few examples are known. The following (and others: Stiefel manifolds, constructions using the Hopf map) are discussed in more detail in Matoušek's book [Mat08] and in a chapter of Csorba's thesis [Cso05] devoted to the topic. The simplest is perhaps the torus with two holes (that is, the 2-dimensional orientable surface of genus 2, with $\mathbb{Z}_2$-action $x \mapsto -x$ in a symmetric embedding in $\mathbb{R}^3$, ie., swapping the holes) which has coindex 1 and index 2, that is, $\mathcal{S}^2 \not\to_{\mathbb{Z}_2} X$, but $X \not\to_{\mathbb{Z}_2} \mathcal{S}^1$. Real projective spaces (with an appropriate $\mathbb{Z}_2$-action) provide examples with the worst possible gap: they have coindex 1 and arbitrarily high index, that is, $\mathcal{S}^2 \not\to_{\mathbb{Z}_2} X$, but $X \not\to_{\mathbb{Z}_2} \mathcal{S}^n$, for an arbitrarily high $n$ (the index has been computed by Stolz [Sto89], see also an exposition in [Pfi95]). Matsushita [Mat17b] proved an even stronger example where not only the index is arbitrarily high, but so is a cohomological lower bound of it; his proof also uses considerably fewer tools of algebraic topology.

The dual to Conjecture 1.6, namely $\operatorname{coind}(X \times Y) = \min(\operatorname{coind}(X), \operatorname{coind}(Y))$, has been considered by Simonyi and Zsbán [SZ10]. This statement is trivial in topology, that is, $\mathcal{S}^n \to_{\mathbb{Z}_2} X \times Y$ if and only if $\mathcal{S}^n \to_{\mathbb{Z}_2} X$ and $\mathcal{S}^n \to_{\mathbb{Z}_2} Y$. However, they showed that $\operatorname{coind}|\mathrm{Box}(G \times H)| = \min(\operatorname{coind}|\mathrm{Box}(G)|, \operatorname{coind}|\mathrm{Box}(H)|)$ (without resorting to $|\mathrm{Box}(G)| \times |\mathrm{Box}(H)| \simeq_{\mathbb{Z}_2} |\mathrm{Box}(G \times H)|$), which allowed them to conclude that Hedetniemi's conjecture is true on all graphs for which the topological bound on the chromatic number $\operatorname{coind}(|\mathrm{Box}(G)|) + 2 \leq \chi(G)$ is tight. Conjecture 1.6 would imply that tightness of the stronger bound $\operatorname{ind}(|\mathrm{Box}(G)|) + 2$ would suffice.

In topological literature on the index (see eg. [Yan54; Yan55; CF60; CF62; Ucc72; Tan03]), $\mathbb{Z}_2$-maps are usually called equivariant maps. The names 'coindex' and 'index' are usually swapped with respect to their usage in (topological) combinatorics. The index has also been called the B-index, level, genus. Nevertheless, the only mention of the index of products of spaces seems to be [Kau13].

We note that the index has important applications in algebra, see [DLP80; DL84]; Dai and Lam proved a crucial connection and stated a question [DL84, (11.2)] about tensor products of commutative $\mathbb{R}$-algebras that is closely related, via this connection, to Conjecture 1.6. The *level* $s(A)$ of an algebra $A$ is the least $n$ such that $-1$ can be represented as the sum of $n$ squares: $-1 = a_1^2 + \cdots + a_n^2$ for some $a_i \in A$. The question is whether $s(A \otimes_{\mathbb{R}} B) = \min(s(A), s(B))$, for all commutative $\mathbb{R}$-affine algebras $A, B$. As far as we know, this question has not been explored further, regrettably.

## Open questions

For a topologist, the main question stemming from this chapter and the whole of this thesis is Conjecture 1.6. Even though we state it as a conjecture, we have no serious reason to believe it to be true. In fact so little is known that any partial result would be interesting. In particular, is $\mathcal{S}^2$, or really any non-1-dimensional $\mathbb{Z}_2$-space, multiplicative? An example of a $\mathbb{Z}_2$-space that is *not* multiplicative is $X \times Y$ for any $X, Y$ such that $X \not\to_{\mathbb{Z}_2} Y$ and $Y \not\to_{\mathbb{Z}_2} X$; can any other examples be given? As far as we know, it could even turn out that known non-tidy spaces provide relatively simple counter-examples to Conjecture 1.6 and hence to Hedetniemi's conjecture. Can one compute the index of some non-trivial products involving non-tidy spaces? How about some subspaces of the space of maps from $\mathcal{S}^3$ to $\mathcal{S}^2$?

Closer to combinatorics, we ask how close can the connection with topology be. Is every graph $K$ with $|\mathrm{Box}(K)| \simeq_{\mathbb{Z}_2} \mathcal{S}^1$ multiplicative? All known examples suggest so, but very little is known on graphs that are not multiplicative, so any new method for disproving multiplicativity would be interesting. Beside taking $K = G \times H$ for graphs such that $G \not\to H$ and $H \not\to G$, the only construction known to the author comes from Kneser graphs, see [TZ02b].

Finally, do other functors have similar properties to $\Omega_k$, in particular do all "adjoint fractional powers" of the form $\Gamma_\ell(\Omega_k(\cdot))$ with $l < k$ preserve the homotopy type (as in the Equivalence Theorem 1.1)? How about right adjoints to the arc graph construction? Can the properties be derived from more general principles? Similarly, aiming to generalize the proof that powers of graphs of high girth are multiplicative, for which graphs $K$ can we show a homomorphism $\Omega_k(\Gamma_k(K)) \to K$?

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