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Mathematical analysis of morphogen transport models.

Phd Dissertation

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Author's declaration:

Aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Supervisors' declaration:

The dissertation is ready to be reviewed.

date supervisors' signatures

To my loving mother and my teacher Zdzisław Matuski.

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Abstract

In this dissertation two models of morphogen transport are studied analytically. Both models are based on the concept of positional signalling as introduced by Wolpert in the late sixties. Positional signalling explains the mechanism of cell differentiation and pattern formation in developing embryo using the process of morphogen gradient formation. Both models couple semilinear parabolic equations with a system of ordinary differential equations. The first model by Hufnagel et al. describes the space distribution of chemicals in a rectangle reflecting part of the tissue. The main mathematical difficulty in the analysis of this model stems from the presence of a Dirac Delta in the boundary flux of one of the diffusing components. Besides obtaining well-posedness of the evolutionary problem and proving the existence of the unique equilibrium we perform the dimension reduction. This justifies rigorously that the simplified one dimensional model is a reduced version of the model posed on a rectangle. For the second model by Lander et al. we generalise results obtained previously by Krzyżanowski et al. to domains of arbitrary dimension. Moreover the topology of convergence of the solution to the steady state is improved. We use tools of functional analysis such as theory of analytic semigroups, interpolation of Banach spaces and fixed point theorems.

Keywords: morphogen transport, ligands, receptors, reaction-diffusion systems, analytic semigroups, Radon measures, asymptotic analysis, dimension reduction, interpolation of Banach spaces, uniqueness of solutions, Lyapunov functionals

Contents

Chapter 1

Introduction

1.1. Modelling of morphogenesis

Explaining the process of tissue formation in embryo is one of the most challenging problems of developmental biology. Understanding the process of cell differentiation leads to the study of mechanism of localised gene expression and transformation of initially identical cells into specialised tissues. Besides extensive experimental studies many mathematical models were constructed starting from the seminal work of Turing from 1952 (see [32]) where the concept of morphogen was introduced and then developed by Gierer and Meinhard (see [9]) and many other authors. Gierer and Meinhard introduced two hypothetical diffusive substances - activator and inhibitor whose local concentration determines the fate of cells. The effect of pattern formation is a consequence of the so called diffusive instability in a system of two reaction-diffusion equations with linear diffusion and complicated inhibitor-activator nonlinear reaction kinetics. The main drawback of this approach is related to the experimental identification of chemicals playing the role of activator and inhibitor. Although this approach leads to solutions which resemble many patterns observed in nature, it is unclear whether the activator-inhibitor mechanism is indeed responsible for appearance of real complicated patterns.

An alternative approach was based on the work by Lewis Wolpert from the late sixties (see [33]), who introduced the concept of morphogen gradient formation. His famous French Flag Model is described in the next section. In this approach the role of morphogen is played by particular proteins called ligands. Many of them were already identified (Decapentaplegic, Wingless, Hedgehog). Most experimental studies on morphogenesis are performed on fruit fly (Drosophila Melanogaster) - a species which is very convenient to cultivate in the laboratory environment.

1.1.1. Positional signalling

According to the French Flag Model (see Figure) morphogens are molecules which due to mechanism of positional signalling govern the fate of cells in living organisms. It has been observed that certain proteins called ligands after being secreted from a source, typically a group of cells, spread through the tissue and after a certain amount of time form a stable gradient of concentration. Next receptors located on the surfaces of the cells detect levels of morphogen concentration and transmit these information to the nucleus. This leads to the activation of appropriate genes, synthesis of proteins and finally differentiation of cells.

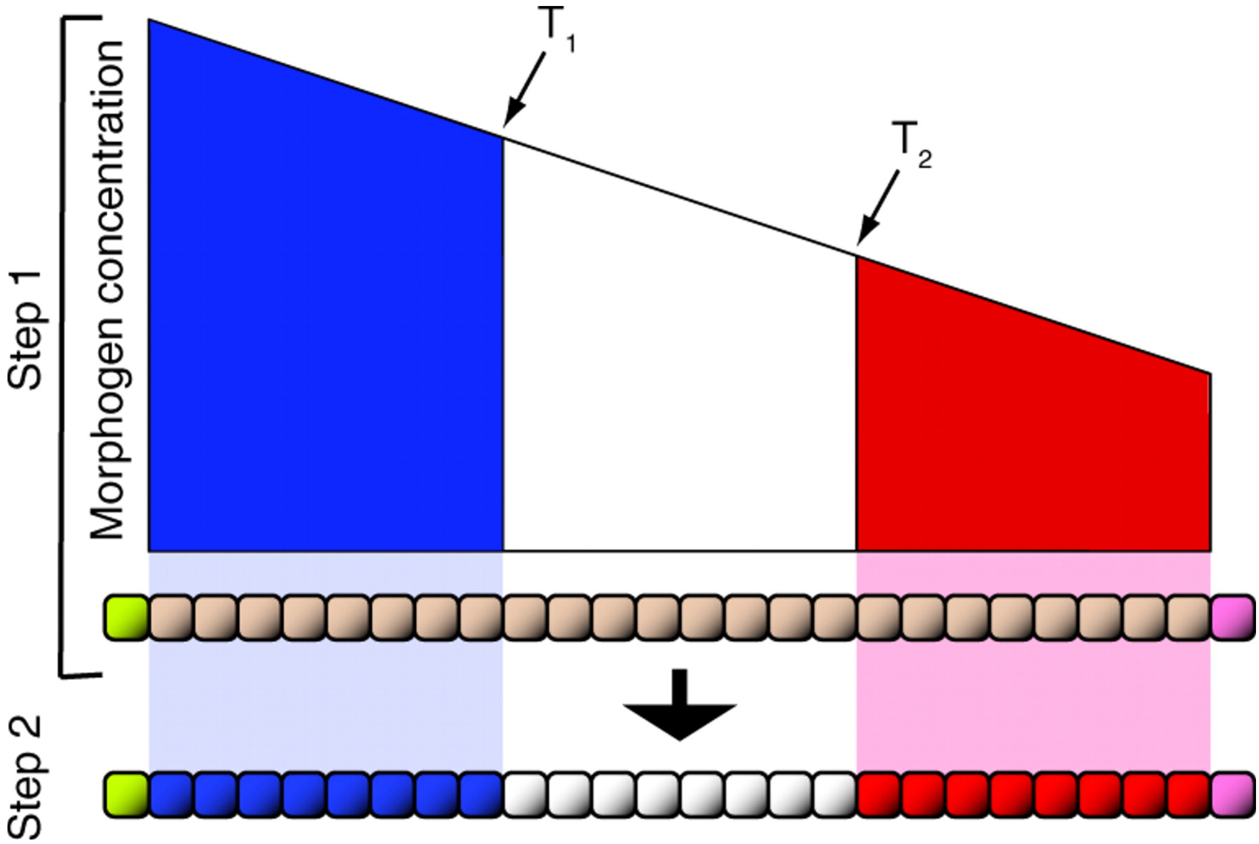

Figure 1.1: The French Flag Model of positional signalling. In Step 1 the morphogen substance is being secreted from the green cell and spreads above the tissue. In Step 2 receptors detect concentration of the morphogens and transmit this information to the nuclei. The colors represent thresholds of distinct gene activation.

Although the role of morphogen in cell differentiation, as described above, is commonly accepted there is still discussion regarding the exact kinetic mechanism of the movement of morphogen molecules and the role of reactions of morphogen with receptors in forming the gradient of concentration (see [11],[17],[18]). To determine the mechanism of morphogen transport, several mathematical models consisting of systems of semilinear parabolic PDEs of reaction diffusion-type coupled with ODEs were recently proposed and analysed (see [15],[16],[30],[23],[29]).

1.2. Presentation of the mathematical models

We present two models ([**LNW**].**B** and [**HKCS**]) of the transport of two distinct morphogens (Dpp and Wg) in the imaginary wing disc of the fruit fly. Both models take into account diffusion of morphogen molecules and their reactions with receptors distributed on the cell surface. Model [**HKCS**] additionally accounts for reactions of morphogens with glypicans - special type of receptors which have an active role in the transport. Another feature which distinguish the models is that in [**HKCS**] the transport of morphogens takes place in the extracellular space as well as on the cell surfaces while in [**LNW**].**B** only the latter mechanism is present. Details are presented in the following sections.

1.2.1. The [HKCS] model

The model [**HKCS**] introduced by Hufnagel et al. in [13] describes the formation of the gradient of morphogen Wingless (Wg) in the imaginal wing disc of the Drosophila Melanogaster individual. Model [**HKCS**] has two counterparts - one and two dimensional, depending on the dimensionality of the domain representing the imaginal wing disc. We denote these models [**HKCS**]**.1D** and [**HKCS**]**.2D** respectively. In mathematical terms [**HKCS**]**.1D** is a system of two semilinear parabolic PDEs of reaction diffusion type coupled with three nonlinear ODEs posed on the interval $I^L = (-L, L)$, while [**HKCS**]**.2D** consists of a linear parabolic PDE posed on rectangle $\Omega^{L,H} = (-L, L) \times (0, H)$ which is coupled via nonlinear boundary condition on $\partial_1\Omega^{L,H} = (-L, L) \times \{0\}$ with a semilinear parabolic PDE and three ODEs.

The [HKCS].2D model.

For $L, H > 0$, and $\infty \geq T > 0$ denote

$$\begin{aligned}
I^L &= (-L, L),\ x_1 \in I^L,\ I^1 = I,\\
\Omega^{L,H} &= (-L, L) \times (0, H),\ x = (x_1, x_2) \in \Omega^{L,H},\ \Omega = \Omega^{1,1},\\
\partial_0\Omega^{L,H} &= \{-L, L\} \times [0, H] \cup (-L, L) \times \{H\},\ \partial_1\Omega^{L,H} = (-L, L) \times \{0\},\ \partial\Omega^{L,H} = \partial_1\Omega^{L,H} \cup \partial_0\Omega^{L,H},\\
\Omega_T^{L,H} &= (0, T) \times \Omega^{L,H},\ (\partial\Omega^{L,H})_T = (0, T) \times \partial\Omega^{L,H}.
\end{aligned}$$

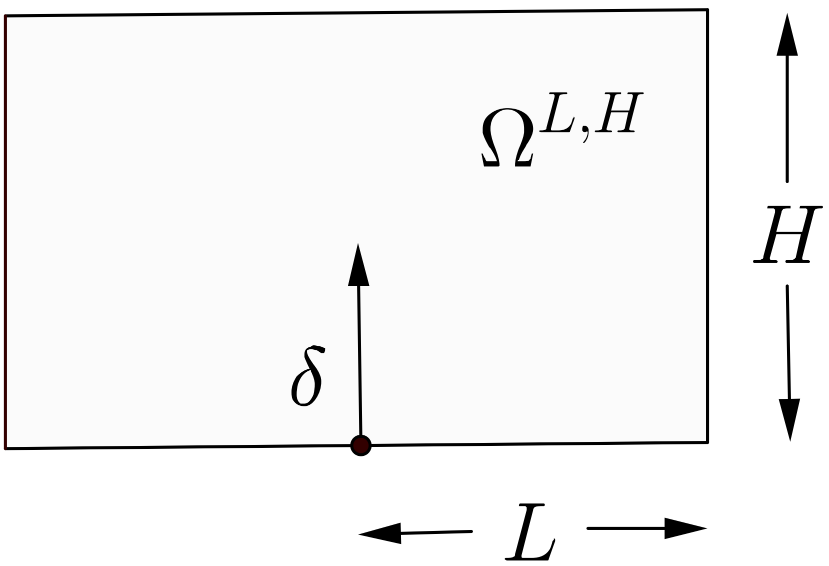

Figure 1.2: Graph of the domain $\Omega^{L,H}$. The arrow pointing towards the rectangle represents a point source of the morphogen (a Dirac Delta) on the boundary.

The domain $\Omega^{L,H}$ (see Figure 1.2) represents the imaginal wing disc of the Drosophila Melanogaster individual and the x_2 direction corresponds to the thickness of the disc, so that in practice H is much smaller than L.

Let ν denote a unit outer normal vector to $\partial\Omega^{L,H}$ and let δ be a one dimensional Dirac Delta. Moreover denote by $\nabla = (\partial_{x_1}, \partial_{x_2})$ the gradient and by $\Delta = \partial^2_{x_1} + \partial^2_{x_2}$ the Laplace operator.

The model **[HKCS].2D** is a system which consists of one evolutionary PDE posed on $\Omega^{L,H}$, one evolutionary PDE and 3 ODEs posed on $\partial_1\Omega^{L,H}$:

[HKCS].2D

$$
\begin{aligned}
\partial_t W - D\Delta W &= -\gamma W, && (t,x) \in \Omega^{L,H}_T && (1.1a)\\
\partial_t W^* - D^*\partial^2_{x_1} W^* &= -\gamma^* W^* + \Xi_1 - \Xi_2, && (t,x) \in (\partial_1\Omega^{L,H})_T && (1.1b)\\
\partial_t R &= -\Xi_2 - \Xi_3 - \alpha R + \Gamma, && (t,x) \in (\partial_1\Omega^{L,H})_T && (1.1c)\\
\partial_t R^* &= \Xi_2 - \alpha^* R^*, && (t,x) \in (\partial_1\Omega^{L,H})_T && (1.1d)\\
\partial_t R^*_g &= \Xi_3 - \alpha^* R^*_g, && (t,x) \in (\partial_1\Omega^{L,H})_T && (1.1e)
\end{aligned}
$$

supplemented by the boundary conditions:

$$
\begin{aligned}
D\nabla W\nu &= 0, && (t,x) \in (\partial_0\Omega^{L,H})_T && (1.2a)\\
D\nabla W\nu &= -\Xi_1 - \Xi_2 + s\delta, && (t,x) \in (\partial_1\Omega^{L,H})_T && (1.2b)\\
\partial_{x_1} W^* &= 0, && (t,x) \in (\partial\partial_1\Omega^{L,H})_T && (1.2c)
\end{aligned}
$$

and initial conditions:

$$
\begin{aligned}
&W(0) = W_0, && x \in \Omega^{L,H} && (1.3a)\\
&W^*(0) = W^*_0,\ R(0) = R_0,\ R^*(0) = R^*_0,\ R^*_g(0) = R^*_{g0}, && x \in \partial_1\Omega^{L,H} && (1.3b)
\end{aligned}
$$

where

$$
\begin{aligned}
\Xi_1 &= \Xi_1(G, W, W^*) = kGW - k'W^*,\\
\Xi_2 &= \Xi_2(R, W, R^*) = k_R RW - k'_R R^*,\\
\Xi_3 &= \Xi_3(R, W^*, R^*_g) = k_{Rg} RW^* - k'_{Rg} R^*_g.
\end{aligned}
$$

In (1.1),(1.2) and (1.3) W (resp. G, R, W^*, R^* and R^*_g) denotes concentration of free morphogens Wg (resp. free glypicans Dlp, free receptors, morphogen-glypican complexes, morphogen-receptor complexes and morphogen-glypican-receptor complexes). It is assumed that W is located on $\Omega^{L,H}$ and is thus a function of (t, x_1, x_2), while other substances are present only on $\partial_1\Omega^{L,H}$ and depend only on (t, x_1). Substances R, R^* and R^*_g may be internalised from the cell surface to its interior. The model takes into account association-dissociation mechanism of

- W and G with rates $k, k' : \Xi_1$,
- W and R with rates $k_R, k'_R : \Xi_2$,
- W^* and R with rates $k_{Rg}, k'_{Rg} : \Xi_3$.

Other terms of the system account for

- diffusion of W in $\Omega^{L,H}$ (resp. W^* on $\partial_1\Omega^{L,H}$) with rate D (resp D^*): $-D\Delta W$ (resp. $-D^*\partial_{x_1}^2 W^*$),
- degradation of W in $\Omega^{L,H}$ (resp. W^* on $\partial_1\Omega^{L,H}$) with rate γ (resp. γ^*): $-\gamma W$ (resp. $-\gamma^* W^*$),
- internalisation of R (resp. R^*, R_g^*) with rate α (resp. α^*, α^*) : $-\alpha R$ (resp. $-\alpha^* R^*, -\alpha^* R_g^*$),
- secretion of W with rate s from the source localised at the boundary point $x = 0 \in \partial_1\Omega^{L,H}$: $s\delta$,
- production of R: Γ.

For simplicity we assume that G and Γ are given and constant (in time and space).

In order to analyse the reduction of the dimension of the domain we introduce for $\epsilon > 0$ the **[HKCS].(2D,ϵ)** model, which is obtained from **[HKCS].2D** by changing $\Omega^{L,H}$ into $\Omega^{L,\epsilon H}$ and rescaling the source term for W in the boundary conditions (1.2):

[HKCS].(2D,ϵ)

$$\begin{aligned}
\partial_t W^\epsilon - D\Delta W^\epsilon &= -\gamma W^\epsilon, & (t,x) &\in \Omega_T^{L,\epsilon H} & \text{(1.4a)}\\
\partial_t W^{*,\epsilon} - D^*\partial_{x_1}^2 W^{*,\epsilon} &= -\gamma^* W^{*,\epsilon} + \Xi_1^\epsilon - \Xi_2^\epsilon, & (t,x) &\in (\partial_1\Omega^{L,\epsilon H})_T & \text{(1.4b)}\\
\partial_t R^\epsilon &= -\Xi_2^\epsilon - \Xi_3^\epsilon - \alpha R^\epsilon + \Gamma, & (t,x) &\in (\partial_1\Omega^{L,\epsilon H})_T & \text{(1.4c)}\\
\partial_t R^{*,\epsilon} &= \Xi_2^\epsilon - \alpha^* R^{*,\epsilon}, & (t,x) &\in (\partial_1\Omega^{L,\epsilon H})_T & \text{(1.4d)}\\
\partial_t R_g^{*,\epsilon} &= \Xi_3^\epsilon - \alpha^* R_g^{*,\epsilon}, & (t,x) &\in (\partial_1\Omega^{L,\epsilon H})_T & \text{(1.4e)}
\end{aligned}$$

with boundary conditions

$$\begin{aligned}
\epsilon^{-1} D\nabla W^\epsilon \nu &= 0, & (t,x) &\in (\partial_0\Omega^{L,\epsilon H})_T & \text{(1.5a)}\\
\epsilon^{-1} D\nabla W^\epsilon \nu &= -\Xi_1^\epsilon - \Xi_2^\epsilon + s\delta, & (t,x) &\in (\partial_1\Omega^{L,\epsilon H})_T & \text{(1.5b)}\\
\partial_{x_1} W^{*,\epsilon} &= 0, & (t,x) &\in (\partial\partial_1\Omega^{L,\epsilon H})_T & \text{(1.5c)}
\end{aligned}$$

and initial conditions

$$\begin{aligned}
&W^\epsilon(0) = W_0^\epsilon, & x &\in \Omega^{L,\epsilon H}\\
&W^{*,\epsilon}(0) = W_0^*,\ R^\epsilon(0) = R_0,\ R^{*,\epsilon}(0) = R_0^*,\ R_g^{*,\epsilon}(0) = R_{g0}^*, & x &\in \partial_1\Omega^{L,\epsilon H},
\end{aligned}$$

where

$$\begin{aligned}
\Xi_1^\epsilon &= \Xi_1^\epsilon(G, W^\epsilon, W^{*,^\epsilon}) = kGW^\epsilon - k'W^{*,\epsilon},\\
\Xi_2^\epsilon &= \Xi_2^\epsilon(R^\epsilon, W^\epsilon, R^{*,\epsilon}) = k_R R^\epsilon W^\epsilon - k_R' R^{*,\epsilon},\\
\Xi_3^\epsilon &= \Xi_3^\epsilon(R^\epsilon, W^{*,\epsilon}, R_g^{*,\epsilon}) = k_{Rg} R^\epsilon W^{*,\epsilon} - k_{Rg}' R_g^{*,\epsilon},\\
W_0^\epsilon(x_1,x_2) &= W_0(x_1, x_2/\epsilon).
\end{aligned}$$

Observe that **[HKCS].(2D,1)**=**[HKCS].2D**. Roughly speaking besides the well-posedness of **[HKCS]**, our main result is that

$$\lim_{\epsilon\to 0^+} \textbf{[HKCS].(2D,}\epsilon\textbf{)} = \textbf{[HKCS].1D}, \tag{1.6}$$

where **[HKCS].1D** is a simplified model analysed in section 2.4. The precise meaning of the limit (1.6) is given in Theorem 2.3.

1.2.2. The [LNW].B model

For the case of morphogen Decapentaplegic (Dpp) acting in the wing disc of the Drosophila Melanogaster individuals, several models have been proposed in [20]. In this dissertation we are concerned with model **[LNW].B** (Model B [20] p.786). In mathematical terms the model is a system of two differential equations (PDE+ODE equipped with initial and boundary conditions), posed on an annular shaped domain $\Omega' \subset \mathbb{R}^n$, which represents a fragment of the wing tissue. The boundary of Ω' consists of two disjoint sets Γ'_N and Γ'_D. An example of a two dimensional domain Ω' is provided on Figure 1.3.

In the model movement of morphogen molecules (A) occurs by passive diffusion while being affected by reactions of reversible binding with receptors (C) and degradation of morphogen-receptor complexes (B). It is assumed that the total concentration of free and bounded receptors $B + C$ is constant and equal to R_{tot}. Morphogen is being delivered to the system by secretion from a source localised on Γ'_N.

[LNW].B

$$\partial_t A - D'\Delta A = k_{off}B - k_{on}A(R_{tot} - B), \qquad (t,x) \in (0,T) \times \Omega' \tag{1.7a}$$

$$\partial_t B = k_{on}A(R_{tot} - B) - (k_{off} + k_{deg})B, \qquad (t,x) \in (0,T) \times \Omega' \tag{1.7b}$$

$$D'\nabla A\nu = g', \qquad (t,x) \in (0,T) \times \Gamma'_N \tag{1.7c}$$

$$A = 0, \qquad (t,x) \in (0,T) \times \Gamma'_D \tag{1.7d}$$

$$A(0) = A_0, \; B(0) = B_0, \qquad x \in \Omega' \tag{1.7e}$$

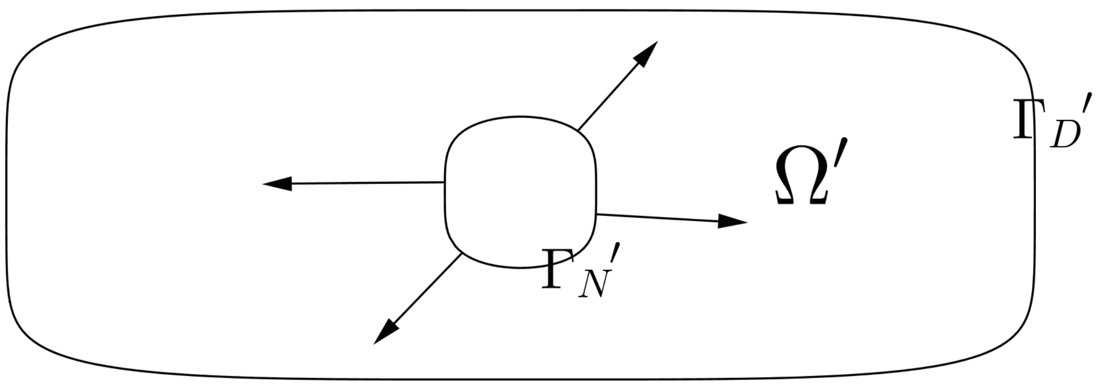

Figure 1.3: Graph of a two dimensional domain Ω'. The arrows pointing towards Ω' represent the secretion of morphogen from Γ'_N.

In case of one dimensional domains a detailed mathematical analysis of this model was performed in [15] and [30].

In [15] the case $\Omega' = (0, L)$ is analysed. Finding Lyapunov functional allowed to prove well-posedness and $L_2(\Omega)$ exponential convergence to the unique equilibrium, with rate χ expressed explicitly by the parameters of the model.

In [30] the case $\Omega' = (0, \infty)$, with a nonlinear dynamic boundary condition at $x = 0$ and vanishing boundary condition at $x \to \infty$ is considered. Well-posedness and $L_p(\Omega')$ convergence of the solution to the unique steady state were proved.

1.3. The main results of the dissertation

1.3.1. The [HKCS] model

In section 2.2 we analyse the evolution part of the **[HKCS].2D** model. Using analytic semigroup theory we prove its global well-posedness (Theorem 2.2) in appropriately chosen function setting and justify rigourously that model **[HKCS].1D** can be obtained from **[HKCS].2D** through "ironing of the wing disc" - i.e. dimension reduction of the domain in the direction perpendicular to the surface of the wing disc (Theorem 2.3). The main analytic problem which we have to overcome stems from two factors: the lack of smoothing effect in the ODEs and the presence of a point source term (a Dirac Delta) in the boundary condition for the equation posed on $(-L, L) \times (0, H)$, which causes the solution to be unbounded for every $t > 0$.
Stationary problem for the **[HKCS].2D** is analysed in section 2.3. We prove that there is a unique steady state (Theorem 2.4) which converges to the equilibrium of **[HKCS].1D** as $h \to 0$ (Theorem 2.5). We illustrate our result by performing numerical computations which show that the graph of the stationary solution to **[HKCS].2D** becomes homogeneous in the x_2 direction as $h \to 0$ (Figure 2.1). It is worth underlining that all our results are proved without imposing any artificial conditions on the parameters which are present in the system.
Well-posedness (Theorem 2.7) and the existence of the unique stationary solution (Theorem 2.6) to model **[HKCS].1D** are established in Section 2.4.

1.3.2. The [LNW].B model

In chapter 3 we examine model **[LNW].B** in the [15] setting for bounded domains of arbitrary dimension n. Although $n \in \{1, 2, 3\}$ is, from the biological point of view, the only relevant case, we do not impose this restriction on n (methods that we use do not depend on the dimension). Using fixed point theorem and monotonicity of the nonlinearity we prove that our model has a unique non-negative steady state (Theorem 3.1). Using theory of analytic semigroups and comparison principle arguments we show existence of classical global solutions (Theorem 3.2). We check that the Lyapunov functional, obtained in [15], also works for arbitrary n and thanks to appropriate semigroup estimates and bootstrap arguments we improve the topology of the convergence to the equilibrium from $L_2 \times L_2$ to $C^{1,\alpha} \times C^{0,\alpha}$ without losing the exponential rate χ (Theorem 3.3).

1.4. Preliminaries

1.4.1. Notation

We introduce the following notation

- δ_{ij} for $i,j \in \mathbb{N}$ - the Kronecker symbol $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$
- $x \vee y = \max\{x,y\},\ x \wedge y = \min\{x,y\}$ for $x,y \in \mathbb{R}$
- $x_+ = x \vee 0,\ x_- = (-x) \vee 0$ for $x,y \in \mathbb{R}$
- $sgn(x) = \begin{cases} |x|/x & ,x \neq 0 \\ 0 & ,x = 0 \end{cases}$ for $x \in \mathbb{R}$
- $\overline{\boldsymbol{x}} = \max\{x_i \colon 1 \leq i \leq n\},\ \underline{\boldsymbol{x}} = \min\{x_i \colon 1 \leq i \leq n\}$ for $\boldsymbol{x} \in \mathbb{R}^n$

If X is a vector space we denote by

- X^n - the n-th product power of X
- $X_+ = \{x \in X : \ x \geq 0\}$ - the positive cone of $(X, \geq)$ when $\geq$ is a partial ordering
- $lin(Y)$ - the linear subspace of X spanned by a subset $Y \subset X$
- $Id : X \to X$ - the identity map $Id(x) = x$

If X, Y are normed vector spaces we denote by

- $cl_X(U)$ - the closure of $U \subset X$ in X
- $\|\cdot\|_X$ - the norm in X
- X^* - the topological dual of X
- $\left\langle x^*, x \right\rangle_{(X^*,X)} = x^*(x)$ for $x^* \in X^*, x \in X$ - the natural duality pairing between X and X^*
- $\mathcal{L}(X,Y)$ - the space of bounded linear operators between X and Y, $\mathcal{L}(X) = \mathcal{L}(X,X)$
- $A' \in \mathcal{L}(Y^*, X^*)$ - the transpose of $A \in \mathcal{L}(X,Y)$
- $A : X \supset D(A) \to X$ - the unbounded linear operator with domain $D(A)$
- $G(A)$ - the graph of operator A
- $\rho(A)$ - the resolvent set of A
- $\sigma(A)$ - the spectrum of A
- $\sigma_p(A)$ - the point spectrum (the set of eigenvalues) of A
- $R(\lambda, A)$ for $\lambda \in \rho(A)$ - the resolvent operator of A

If X, Y are Hilbert spaces we denote by

- $(\cdot|\cdot)_X$ - the scalar product in X
- $X \otimes Y$, $x \otimes y$ - the tensor products

- $A^* : Y \supset D(A^*) \to X$ - the adjoint of operator $A : X \supset D(A) \to Y$

If U is a subset of $\mathbb{R}^n$

- $\overline{U} = cl_{\mathbb{R}^n}(U)$ - the closure of U
- ∂U - the boundary of U
- $W^s_p(U)$ for $1 \le p \le \infty,\ s \in \mathbb{R}$ - the fractional Sobolev (Sobolev-Slobodecki) spaces,
- $C^{k,\alpha}(\overline{U})$ for $k \in \mathbb{N},\ \alpha \in [0,1]$ - the Hölder spaces,
- $\mathcal{M}(\overline{U}) = (C(\overline{U}))^*$ - the space of finite, signed Radon measures.

Moreover we denote by

- $\|\cdot\|_\infty$ - the supremum norm
- $[\cdot,\cdot]_\theta$ - the complex interpolation functor

For a comprehensive treatment on

- normed, Hilbert and Banach spaces we refer to [[28], Chap. I-III]
- bounded and unbounded operators as well as their spectral theory we refer to [[28], Chap. VI-VIII]
- spaces $W^s_p(U), C^{k,\alpha}(\overline{U}), \mathcal{M}(\overline{U})$ and functor $[\cdot,\cdot]_\theta$ we refer to [1] and [31].

1.4.2. Inequalities

In Lemma 1.1 we collect three elementary estimates which are used in the following chapters. For completeness of the reasoning we provide short proofs.

Lemma 1.1. *The following inequalities hold*

$$\sup\{t^\alpha e^{-rt} : t \ge t_0\} \le C(r^{-\alpha} + t_0^\alpha)e^{-rt_0}, \qquad t_0 \ge 0, \alpha \ge 0, r > 0, \tag{1.8a}$$

$$\int_0^t \frac{d\tau}{\tau^\alpha (t-\tau)^\beta} \le C t^{1-\alpha-\beta}, \qquad t > 0, \alpha, \beta \ge 0, \alpha + \beta < 1, \tag{1.8b}$$

$$\int_0^t e^{-r\tau} \frac{d\tau}{\tau^\alpha (t-\tau)^\beta} \le C\Big(\frac{t^{\alpha+\beta}}{r}\Big)^{\frac{1-(\alpha+\beta)}{1+\alpha+\beta}}, \qquad t > 0, \alpha, \beta \ge 0, \alpha + \beta < 1, r > 0, \tag{1.8c}$$

where constant C depends only on α and β.

Proof. To prove (1.8a) define for $t \ge 0$ function $f(t) = t^\alpha e^{-rt}$. Then $f'(t) = \alpha t^{\alpha-1} e^{-rt} - r t^\alpha e^{-rt} = t^{\alpha-1} e^{-rt}(\alpha - rt)$. Analysing the sign of f' we obtain that function f is increasing on $[0, \alpha/r]$ and decreasing on $[\alpha/r, \infty)$. It follows that

$$\sup\{t^\alpha e^{-rt} : t \ge t_0\} = \begin{cases} f(\alpha/r) & \text{if } t_0 \le \alpha/r \\ f(t_0) & \text{if } t_0 > \alpha/r \end{cases} \le C(r^{-\alpha} + t_0^\alpha)e^{-rt_0},$$

where one can take $C = \max\{\alpha^\alpha, 1\}$. To prove inequality (1.8b) we change variables $\tau = ty$. Then we have

$$\int_0^t \frac{d\tau}{\tau^\alpha (t-\tau)^\beta} = t^{1-\alpha-\beta} \int_0^1 \frac{dy}{y^\alpha (1-y)^\beta} \le C t^{1-\alpha-\beta}.$$

Finally we prove (1.8c). Set $q = \frac{\alpha+\beta+1}{2(\alpha+\beta)}$. It is easy to check that $1 < q$ and $(\alpha+\beta)q < 1$. Let $p = \frac{q}{q-1} = \frac{1+(\alpha+\beta)}{1-(\alpha+\beta)}$ be q's Hölder conjugate exponent. Using Hölder inequality we obtain

$$\int_0^t e^{-r\tau}\frac{1}{\tau^\alpha(t-\tau)^\beta}d\tau = t^{1-(\alpha+\beta)}\int_0^1 e^{-rty}\frac{1}{y^\alpha(1-y)^\beta}dy \le t^{1-(\alpha+\beta)}\Big(\int_0^1 e^{-prty}dy\Big)^{1/p}\Big(\int_0^1 \frac{dy}{y^{q\alpha}(1-y)^{q\beta}}\Big)^{1/q}$$
$$\le Ct^{1-(\alpha+\beta)}\Big(\frac{1-e^{-rtp}}{rtp}\Big)^{1/p} \le C\frac{t^{1-(\alpha+\beta)-1/p}}{r^{1/p}} = C\frac{t^{(\alpha+\beta)\frac{1-(\alpha+\beta)}{1+\alpha+\beta}}}{r^{\frac{1-(\alpha+\beta)}{1+\alpha+\beta}}} = C\Big(\frac{t^{\alpha+\beta}}{r}\Big)^{\frac{1-(\alpha+\beta)}{1+\alpha+\beta}}.$$

□

Lemma 1.2 is an extension of the well known Gronwall inequality in integral form. Although several results of similar type can be found in the literature (for instance in [27]), we were not able to find a reference to the one which would cover the full range of parameters. Our method of proof is taken from [27].

Lemma 1.2. *Let $0 \le \alpha, \beta,\ \alpha+\beta < 1,\ 0 \le a,\ 0 < b,\ 0 < T < \infty$. Assume that $f \in L_\infty(0,T')$ for every $T' < T$ and that for a.e. $t \in (0,T)$ the following inequality holds*

$$0 \le f(t) \le a + b\int_0^t \frac{f(\tau)}{\tau^\alpha(t-\tau)^\beta}d\tau,$$

then $f \in L_\infty(0,T)$ and

$$\|f\|_{L_\infty(0,T)} \le Ca\exp\Big(Cb^{\frac{1+\alpha+\beta}{1-\alpha-\beta}}T^{1+\alpha+\beta}\Big),$$

where C depends only on α and β. Moreover $C = 1$ when $\alpha = \beta = 0$.

Proof. When $\alpha = \beta = 0$ the result is the well known Gronwall inequality in integral form. Otherwise we proceed similarly as in the proof of inequality (1.8c). Fix $q > 1$ such that $q(\alpha+\beta) < 1$ and let $p = \frac{q}{q-1}$ be q's Hölder conjugate exponent. Using Hölder inequality we obtain

$$\begin{aligned}\int_0^t \frac{f(\tau)}{\tau^\alpha(t-\tau)^\beta}d\tau &\le \Big(\int_0^t f(\tau)^p d\tau\Big)^{1/p}\Big(\int_0^t \frac{d\tau}{\tau^{\alpha q}(t-\tau)^{\beta q}}\Big)^{1/q}\\ &= \Big(\int_0^t f(\tau)^p d\tau\Big)^{1/p} t^{1/q-(\alpha+\beta)}\Big(\int_0^1 \frac{d\tau}{\tau^{\alpha q}(1-\tau)^{\beta q}}\Big)^{1/q}\\ &\le C_0 T^{1/q-(\alpha+\beta)}\Big(\int_0^t f(\tau)^p d\tau\Big)^{1/p},\end{aligned}$$

where $C_0 = \int_0^1 \frac{d\tau}{\tau^{\alpha q}(1-\tau)^{\beta q}}$. Thus

$$f(t)^p \le \Big(a + bC_0T^{1/q-(\alpha+\beta)}\Big(\int_0^t f(\tau)^p d\tau\Big)^{1/p}\Big)^p \le 2^{p-1}a^p + 2^{p-1}b^pC_0^pT^{p/q-p(\alpha+\beta)}\int_0^t f(\tau)^p d\tau.$$

Using Lemma 1.2 with $\alpha = \beta = 0$ we obtain

$$\begin{aligned} f(t)^p &\le 2^{p-1}a^p\exp\Big(2^{p-1}b^pC_0^pT^{p/q-p(\alpha+\beta)}t\Big)\\ f(t) &\le 2^{1/q}a\exp\Big(p^{-1}2^{p-1}b^pC_0^pT^{p/q-p(\alpha+\beta)+1}\Big) = 2^{1/q}a\exp\Big(p^{-1}2^{p-1}b^pC_0^pT^{p(1-\alpha-\beta)}\Big)\\ &\le Ca\exp(Cb^pT^{p(1-\alpha-\beta)}),\end{aligned}$$

with $C = \max\{2^{1/q}, p^{-1}2^{p-1}C_0^p\}$. To finish the proof observe that for $q = \frac{1}{2}\Big(1+\frac{1}{\alpha+\beta}\Big)$ one has $p = \frac{1+\alpha+\beta}{1-\alpha-\beta}$. □

1.4.3. Existence result for a system of abstract ODE's

For $i = 1, \dots, n$ let $(\mathcal{X}_i, \mathcal{X}_i^1)$ be a densely injected Banach couple (i.e. $\mathcal{X}_i^1$ is a dense subspace of $\mathcal{X}_i$ in the topology of $\mathcal{X}_i$). For $\alpha_i \in (0,1)$ denote $\mathcal{X}_i^{\alpha_i} = [\mathcal{X}_i, \mathcal{X}_i^1]_{\alpha_i}$ (where $[.,.]_{\alpha_i}$ is the complex interpolation functor). Finally note

$$\alpha = (\alpha_1, \dots, \alpha_n),\ \mathcal{X} = \mathcal{X}_1 \times \dots \times \mathcal{X}_n,\ \mathcal{X}^1 = \mathcal{X}_1^1 \times \dots \times \mathcal{X}_n^1,\ \mathcal{X}^\alpha = \mathcal{X}_1^{\alpha_1} \times \dots \times \mathcal{X}_n^{\alpha_n}. \tag{1.9}$$

Lemma 1.3. *Assume that for $i = 1, \dots, n$ the following three conditions are satisfied*

1. *The operator $\mathcal{A}_i : \mathcal{X}_i \supset \mathcal{X}_i^1 \to \mathcal{X}_i$ generates an analytic strongly continuous semigroup $e^{t\mathcal{A}_i}$.*
2. *The map $\mathcal{F}_i : \mathcal{X}^\alpha \to \mathcal{X}_i$ is Lipschitz continuous on bounded sets i.e.*

$$\forall_{R>0} \exists_{C_R} \left[\|\boldsymbol{u}\|_{\mathcal{X}^\alpha}, \|\boldsymbol{w}\|_{\mathcal{X}^\alpha} \le R \Longrightarrow \|\mathcal{F}_i(\boldsymbol{u}) - \mathcal{F}_i(\boldsymbol{w})\|_{\mathcal{X}_i} \le C_R \|\boldsymbol{u} - \boldsymbol{w}\|_{\mathcal{X}^\alpha} \right]$$

3. *$u_{0i} \in \mathcal{X}_i^{\alpha_i}$.*

Then the following system of abstract ODE's

$$\frac{d}{dt} u_i - \mathcal{A}_i u_i = \mathcal{F}_i(\boldsymbol{u}),\ t > 0 \tag{1.10}$$

$$u_i(0) = u_{0i} \tag{1.11}$$

has a unique maximal $\mathcal{X}^\alpha$ solution $\boldsymbol{u} = (u_1, \dots, u_n)$ i.e. there exists a unique

$$\boldsymbol{u} \in C([0, T_{\max}); \mathcal{X}^\alpha) \cap C^1((0, T_{\max}); \mathcal{X}) \cap C((0, T_{\max}); \mathcal{X}^1),$$

which satisfies system (1.10)-(1.11) *in the classical sense. For $t \in (0, T_{\max})$ the following Duhamel formulas hold:*

$$u_i(t) = e^{t\mathcal{A}_i} u_{0i} + \int_0^t e^{(t-\tau)\mathcal{A}_i} \mathcal{F}_i(\boldsymbol{u}(\tau)) d\tau,\ 1 \le i \le n,$$

and $T_{\max}$ satisfies the blow-up condition:

$$\text{if } T_{\max} < \infty \text{ then } \limsup_{t \to T_{\max}^-} \|\boldsymbol{u}(t)\|_{\mathcal{X}^\alpha} = \infty. \tag{1.12}$$

In particular if there exists C such that

$$\sum_{i=1}^n \|\mathcal{F}_i(\boldsymbol{u}(t))\|_{\mathcal{X}_i} \le C(\|\boldsymbol{u}(t)\|_{\mathcal{X}^\alpha} + 1) \text{ for } t \in [0, T_{\max}) \tag{1.13}$$

then $T_{\max} = \infty$.

Proof. If $n = 1$ the result is well-known and can be proved using contraction mapping principle (see for instance [[21], Theorem 6.3.2]) . For $n > 1$ one can adapt the same method with obvious modifications. $\square$

Chapter 2

Well-posedness and dimension reduction in the [HKCS] model

In the present chapter we investigate evolution and stationary problems for the **[HKCS]** model. We begin with nondimensionalisation of system (1.4).

2.1. Nondimensionalisation and basic assumptions

Introduce the following nondimensional parameters:

$$T = L^2/D,\ K_1 = k_R T,\ K_2 = k_R T/H,\ h = \epsilon H/L,\ d = D^*/D,$$
$$\boldsymbol{b} = (b_1, b_2, b_3, b_4, b_5) = (T\gamma, T\gamma^*, T\alpha, T\alpha^*, T\alpha^*),$$
$$\boldsymbol{c} = (c_1, c_2, c_3, c_4, c_5) = (TkG/H, Tk', Hk_{Rg}/k_R, Tk'_R, Tk'_{Rg}),$$
$$\boldsymbol{p} = (p_1, p_2, p_3, p_4, p_5) = (K_2 Ts, 0, K_2 T\Gamma, 0, 0).$$

For $(t, x) = (t, x_1, x_2) \in \Omega_T = (0, T) \times (-1, 1) \times (0, 1)$ we define functions

$$u_1^h(t, x_1, x_2) = K_1 W^\epsilon(Tt, Lx_1, \epsilon H x_2),\ u_2^h(t, x_1) = K_2 W^{*,\epsilon}(Tt, Lx_1),\ u_3^h(t, x_1) = K_2 R^\epsilon(Tt, Lx_1)$$
$$u_4^h(t, x_1) = K_2 R^{*,\epsilon}(Tt, Lx_1),\ u_5^h(t, x_1) = K_2 R_g^{*,\epsilon}(Tt, Lx_1)$$
$$\boldsymbol{u}^h = (u_1^h, u_2^h, u_3^h, u_4^h, u_5^h)$$
$$u_{01}(x_1, x_2) = K_1 W_0^\epsilon(Lx_1, \epsilon H x_2) = K_1 W_0(Lx_1, Hx_2),\ u_{02}(x_1) = K_2 W_0^*(Lx_1),\ u_{03}(x_1) = K_2 R_0(Lx_1)$$
$$u_{04}(x_1) = K_2 R_0^*(Lx_1),\ u_{05}(x_1) = K_2 R_{g0}^*(Lx_1)$$
$$\boldsymbol{u}_0 = (u_{01}, u_{02}, u_{03}, u_{04}, u_{05}),$$

then system **[HKCS].(2D,ϵ)** rewritten in the nondimensional form reads

$$\partial_t u_1^h + div(J_h(u_1^h)) = -b_1 u_1^h, \qquad (t, x) \in \Omega_T \tag{2.1a}$$
$$\partial_t u_2^h - d\partial_{x_1}^2 u_2^h = c_1 u_1^h - (b_2 + c_2 + c_3 u_3^h) u_2^h + c_5 u_5^h, \qquad (t, x) \in (\partial_1 \Omega)_T \tag{2.1b}$$
$$\partial_t u_3^h = -(b_3 + u_1^h + c_3 u_2^h) u_3^h + c_4 u_4^h + c_5 u_5^h + p_3, \qquad (t, x) \in (\partial_1 \Omega)_T \tag{2.1c}$$
$$\partial_t u_4^h = u_1^h u_3^h - (b_4 + c_4) u_4^h, \qquad (t, x) \in (\partial_1 \Omega)_T \tag{2.1d}$$
$$\partial_t u_5^h = c_3 u_2^h u_3^h - (b_5 + c_5) u_5^h, \qquad (t, x) \in (\partial_1 \Omega)_T \tag{2.1e}$$

with boundary and initial conditions

$$
\begin{aligned}
-J_h(u_1^h)\nu &= 0, && (t,x) \in (\partial_0\Omega)_T \\
-J_h(u_1^h)\nu &= -(c_1 + u_3^h)u_1^h + c_2u_2^h + c_4u_4^h + p_1\delta, && (t,x) \in (\partial_1\Omega)_T \\
\partial_{x_1} u_2^h &= 0, && (t,x) \in (\partial\partial_1\Omega)_T \\
\boldsymbol{u}^h(0,\cdot) &= \boldsymbol{u}_0
\end{aligned}
$$

where

- $J_h(u) = -(\partial_{x_1}u, h^{-2}\partial_{x_2}u)$ denotes the flux of u_1^h,
- ν denotes the outer normal unit vector to $\partial\Omega$,
- δ denotes a one dimensional Dirac Delta i.e $\delta(\phi) = \phi(0)$ for any $\phi \in C([-1,1])$.

From now on we impose the following natural assumptions on the signs of the constant parameters and (possibly nonconstant) initial conditions

$$
d, \boldsymbol{b} > 0,\ \ \boldsymbol{c}, \boldsymbol{p}, \boldsymbol{u_0} \geq 0. \tag{2.2}
$$

where $\boldsymbol{b} = (b_1, \ldots, b_5)$ and similarly for $\boldsymbol{c}, \boldsymbol{p}$.

In the whole chapter $I_+ = (0,1),\ I = (-1,1)$ and $\Omega = (-1,1) \times (0,1)$ are fixed domains.

2.2. Evolution problem

In this section we study well-posedness and dimension reduction in the system (2.1). We begin by introducing a functional analytic framework which will be used to put system (2.1) in the form of a system of abstract ODEs.

2.2.1. Operators, semigroups, estimates

The X^s spaces, operators A_0, A_h.

Let us recall that $I_+ = (0,1),\ I = (-1,1),\ \Omega = I \times I_+$. For $U \in \{I_+, I, \Omega\}$ we denote

$$
X(U) = L_2(U),\ (\cdot|\cdot)_{X(U)} = (\cdot|\cdot)_{L_2(U)}.
$$

For $i, j \in \mathbb{N}$ we define functions u_i, v_i, w_{ij}

$$
u_i(x_1) = c_{1i}\cos(i\pi(x_1+1)/2),\ x_1 \in I, \quad v_i(x_2) = c_{2i}\cos(i\pi x_2),\ x_2 \in I_+ \tag{2.3}
$$

$$
w_{ij}(x_1,x_2) = u_i(x_1)v_j(x_2),\ (x_1,x_2) \in \Omega, \tag{2.4}
$$

where constants c_{1i}, c_{2i} are such that $\|u_i\|_{X(I)} = \|v_i\|_{X(I_+)} = 1$ i.e.

$$
c_{1i} = \begin{cases} 1/\sqrt{2} & \text{if } i = 0 \\ 1 & \text{if } i > 0 \end{cases}, \quad c_{2i} = \begin{cases} 1 & \text{if } i = 0 \\ \sqrt{2} & \text{if } i > 0 \end{cases}. \tag{2.5}
$$

The reason we introduce functions u_i, v_i, w_{ij} is given in the following

Lemma 2.1. *The set $\{v_i : i \in \mathbb{N}\}$ (resp. $\{u_i : i \in \mathbb{N}\}$ and $\{w_{ij} : i, j \in \mathbb{N}\}$) is a complete orthonormal system in $X(I_+)$ (resp. $X(I)$ and $X(\Omega)$).*

Proof. The fact that $\{v_i : i \in \mathbb{N}\}$ is a complete orthonormal system in $X(I_+)$ is well known. Since $u_i(x) = (c_{1i}/c_{2i})v_i((x + 1)/2)$ the thesis for the set $\{u_i : i \in \mathbb{N}\}$ follows. Finally observe that since $w_{ij} = u_i \otimes v_j$ and $X(\Omega) = X(I) \otimes X(I_+)$ then the claim for the set $\{w_{ij} : i, j \in \mathbb{N}\}$ follows from [[28], Chap. II.4, Prop. 2].

□

Denote

$$X_{fin}(I) = lin(\{u_i : \; i \in \mathbb{N}\}), \quad X_{fin}(\Omega) = lin(\{w_{ij} : \; i, j \in \mathbb{N}\}).$$

Define sequences

$$\begin{aligned}
\lambda^{I_+} &= (\lambda_i^{I_+})_{i\in\mathbb{N}}, \; \lambda_i^{I_+} = -(i\pi)^2, \; i \in \mathbb{N} \\
\lambda^{I} &= (\lambda_i^{I})_{i\in\mathbb{N}}, \; \lambda_i^{I} = -(i\pi/2)^2, \; i \in \mathbb{N} \\
\lambda^{\Omega}_{,h} &= (\lambda^{\Omega}_{ij,h})_{i,j\in\mathbb{N}}, \; \lambda^{\Omega}_{ij,h} = \lambda_i^I + h^{-2}\lambda_j^{I_+} = -(i\pi/2)^2 - (j\pi/h)^2, \; i, j \in \mathbb{N}, \; h \in (0, 1]
\end{aligned}$$

and denote $\lambda^{\Omega} = \lambda^{\Omega}_{,1}$, $\lambda^{\Omega}_{ij} = \lambda^{\Omega}_{ij,1}$. Next we define $X(I)$ and $X(\Omega)$ realisations of the perturbed Laplace operator witn Neumann boundary condition. Define $\tilde{A}_0$ and $\tilde{A}_h$ for $h \in (0, 1]$ to be the unique unbounded linear operator such that

$$\begin{aligned}
&\tilde{A}_0 : X(I) \supset X_{fin}(I) \to X(I), \; \tilde{A}_0 u_i = \partial^2_{x_1 x_1} u_i = \lambda_i^I u_i, \\
&\tilde{A}_h : X(\Omega) \supset X_{fin}(\Omega) \to X(\Omega), \; \tilde{A}_h w_{ij} = -div J_h(w_{ij}) = (\partial^2_{x_1} + h^{-2}\partial^2_{x_2})w_{ij} = \lambda^{\Omega}_{ij,h} w_{ij},
\end{aligned}$$

and denote $\tilde{A} = \tilde{A}_1$.
Define the unbounded linear operators A_0 and A_h for $h \in (0, 1]$:

$$A_0 : X(I) \supset D(A_0) \to X(I), \; D(A_0) = \{u \in X(I) : \sum_{i\in\mathbb{N}} (1 - \lambda_i^I)^2 (u|u_i)^2_{X(I)} < \infty\},$$

$$A_0 u = \sum_{i\in\mathbb{N}} \lambda_i^I (u|u_i)_{X(I)} u_i,$$

$$A_h : X(\Omega) \supset D(A_h) \to X(\Omega), \; D(A_h) = \{w \in X(\Omega) : \sum_{i,j\in\mathbb{N}} (1 - \lambda^{\Omega}_{ij,h})^2 (w|w_{ij})^2_{X(\Omega)} < \infty\},$$

$$A_h w = \sum_{i,j\in\mathbb{N}} \lambda^{\Omega}_{ij,h} (w|w_{ij})_{X(\Omega)} w_{ij},$$

and denote $A = A_1$. Observe that the domain $D(A_h)$ does not depend on h since $\lambda^{\Omega}_{ij} \leq \lambda^{\Omega}_{ij,h} \leq h^{-2}\lambda^{\Omega}_{ij}$, i.e. $D(A_h) = D(A)$ for any $h \in (0, 1]$.

Next we collect spectral properties of operators A_0, A_h.
Lemma 2.2.

1. *The operator A_0 (resp. A_h) is the closure of the operator $\tilde{A}_0$ (resp. $\tilde{A}_h$).*
2. *Operators A_0 and A_h are self-adjoint and nonpositive.*

3. The spectra of operators A_0, A_h consist entirely of eigenvalues:

$$\sigma(A_0) = \sigma_p(A_0) = \lambda^I, \quad \sigma(A_h) = \sigma_p(A_h) = \lambda^\Omega_{,h}. \tag{2.6}$$

4. Resolvent operators $R(\lambda, A_0)$ and $R(\lambda, A_h)$ satisfy

$$R(\lambda, A_0)u = \sum_{i\in\mathbb{N}} (\lambda - \lambda^I_i)^{-1}(u|u_i)_{X(I)}u_i, \text{ for } \lambda \in \rho(A_0),\ u \in X(I), \tag{2.7}$$

$$R(\lambda, A_h)w = \sum_{i,j\in\mathbb{N}} (\lambda - \lambda^\Omega_{ij,h})^{-1}(w|w_{ij})_{X(\Omega)}w_{ij}, \text{ for } \lambda \in \rho(A_h),\ w \in X(\Omega). \tag{2.8}$$

Proof. We give the proof only for A_h (for A_0 it is similar). Moreover it is clear that it is enough to consider the case $h = 1$.

Step 1 It is readily seen that the operator A is an extension of the operator $\tilde{A}$. To show that the operator A is closed let us consider an arbitrary sequence $(w_n)_{n=1}^\infty \subset D(A)$ such that

$$w_n \to w, \text{ in } X(\Omega), \quad Aw_n \to v, \text{ in } X(\Omega),$$

for certain $w, v \in X(\Omega)$. It follows that $(w_n)_{n=1}^\infty$ is a Cauchy sequence in the Hilbert space $X^1(\Omega) = (D(A), (\cdot|\cdot)_{X^1(\Omega)})$, where

$$(w|w')_{X^1(\Omega)} = \sum_{i,j\in\mathbb{N}} (1 - \lambda^\Omega_{ij})^2 (w|w_{ij})_{X(\Omega)}(w'|w_{ij})_{X(\Omega)}, \text{ for } w, w' \in D(A).$$

Thus $w \in D(A)$ and $Aw = v$ which proves that A is closed. It is left to prove that $G(A) \subset cl_{X(\Omega)\times X(\Omega)}G(\tilde{A})$. To achieve this goal choose an arbitrary $(w, Aw) \in G(A)$. Then the sequence $(w_n)_{n=1}^\infty$ defined by $w_n = \sum_{i+j\le n}(w|w_{ij})_{X(\Omega)}w_{ij}$ satisfies

$$w_n \in D(\tilde{A}), \text{ for } n \ge 1,$$
$$(w_n, Aw_n) \to (w, Aw), \text{ in } X(\Omega) \times X(\Omega),$$

which completes the proof of 1.

Step 2 To prove that the operator A is symmetric and nonpositive let us observe that for $w, w' \in D(A)$ we have

$$\begin{aligned}(Aw|w')_{X(\Omega)} &= \sum_{i,j\in\mathbb{N}} (Aw|w_{ij})_{X(\Omega)}(w'|w_{ij})_{X(\Omega)} = \sum_{i,j\in\mathbb{N}} \lambda^\Omega_{ij}(w|w_{ij})_{X(\Omega)}(w'|w_{ij})_{X(\Omega)} = \\ &= \sum_{i,j\in\mathbb{N}} (w|w_{ij})_{X(\Omega)}(Aw'|w_{ij})_{X(\Omega)} = (w|Aw')_{X(\Omega)}, \\ (Aw|w)_{X(\Omega)} &= \sum_{i,j\in\mathbb{N}} \lambda^\Omega_{ij}(w|w_{ij})^2_{X(\Omega)} \le 0.\end{aligned}$$

Moreover A is densely defined since $X_{fin}(\Omega) \subset D(A)$ and $X_{fin}(\Omega)$ is dense in $X(\Omega)$ by Lemma 2.1, thus it is possible to define the adjoint operator A^*. To prove that A is self-adjoint it is left to prove that $D(A^*) = D(A)$, which is equivalent to $D(A^*) \subset D(A)$ since the opposite inclusion always holds. Choose arbitrary $w' \in D(A^*)$. By definition of $D(A^*)$ there exists unique $v \in X(\Omega)$ such that for every $w \in D(A)$ one has $(w'|Aw)_{X(\Omega)} = (v|w)_{X(\Omega)}$. Choosing $w = w_{ij}$ we obtain $\lambda^\Omega_{ij}(w'|w_{ij})_{X(\Omega)} = (v|w_{ij})_{X(\Omega)}$. Finally $w' \in D(A)$ since $\sum_{i,j}(1 - \lambda^\Omega_{ij})^2(w'|w_{ij})^2_{X(\Omega)} \le 2\sum_{i,j}(1 + |\lambda^\Omega_{ij}|^2)(w'|w_{ij})^2_{X(\Omega)} =$

$2(\|w'\|^2_{X(\Omega)} + \|v\|^2_{X(\Omega)}) < \infty$.

Step 3 Since $Aw_{ij} = \lambda^{\Omega}_{ij} w_{ij}$ we obtain that $\lambda^{\Omega} \subset \sigma_p(A)$. For $\lambda \notin \lambda^{\Omega}$ define the operator

$$B(\lambda, A)w = \sum_{i,j} (\lambda - \lambda^{\Omega}_{ij})^{-1} (w|w_{ij})_{X(\Omega)} w_{ij}.$$

One checks easily that $B(\lambda, A) \in \mathcal{L}(X(\Omega)),\ B(\lambda, A)w \in D(A)$ for $w \in X(\Omega)$. Moreover $B(\lambda, A)(\lambda - A)w = w$ for $w \in D(A)$ and $(\lambda - A)B(\lambda, A)w = w$ for $w \in X(\Omega)$ which is easily seen for $w \in X_{fin}(\Omega)$ and by the density argument can be extended to $D(A)$ and $X(\Omega)$. Thus $\rho(A) = \mathbb{C} \setminus \lambda^{\Omega},\ R(\lambda, A) = B(\lambda, A)$ for $\lambda \in \rho(A)$ and $\sigma(A) = \sigma_p(A) = \lambda^{\Omega}$. □

Since operators A_0, A_h are self-adjoint and nonpositive they generate strongly continuous analytic semigroups e^{tA_0} and e^{tA_h} :

$$e^{tA_0} u = \sum_{i \in \mathbb{N}} e^{t\lambda^I_i} (u|u_i)_{X(I)} u_i, \text{ for } u \in X(I), \tag{2.9}$$

$$e^{tA_h} w = \sum_{i,j \in \mathbb{N}} e^{t\lambda^{\Omega}_{ij,h}} (w|w_{ij})_{X(\Omega)} w_{ij}, \text{ for } w \in X(\Omega). \tag{2.10}$$

Since operators $I - A_0$ and $I - A$ are self-adjoint and positive one can define their fractional powers $(I - A_0)^s$ and $(I - A)^s$ for $s \geq 0$. Their domains $D((I - A_0)^s)$ and $D((I - A)^s)$ become Hilbert spaces (which we denote $X^s(I)$ and $X^s(\Omega)$) when equipped with appropriate scalar products. For $s \geq 0$ spaces $X^s(I)$ and $X^s(\Omega)$ are defined as follows

$$X^s(I) = \{u \in X(I) : \sum_{i \in \mathbb{N}} (1 - \lambda^I_i)^{2s} (u|u_i)^2_{X(I)} < \infty\},$$

$$(u|u')_{X^s(I)} = \sum_{i \in \mathbb{N}} (1 - \lambda^I_i)^{2s} (u|u_i)_{X(I)} (u'|u_i)_{X(I)}, \text{ for } u, u' \in X^s(I),$$

$$X^s(\Omega) = \{w \in X(\Omega) : \sum_{i,j \in \mathbb{N}} (1 - \lambda^{\Omega}_{ij})^{2s} (w|w_{ij})^2_{X(\Omega)} < \infty\},$$

$$(w|w')_{X^s(\Omega)} = \sum_{i,j \in \mathbb{N}} (1 - \lambda^{\Omega}_{ij})^{2s} (w|w_{ij})_{X(\Omega)} (w'|w_{ij})_{X(\Omega)}, \text{ for } w, w' \in X^s(\Omega).$$

In the next lemma we give correspondence between scalar products in X^s spaces.

Lemma 2.3.

1. *For $s_1 > s_2 \geq 0$ the following equalities hold*

$$(u|u_i)_{X^{s_1}(I)} = (1 - \lambda^I_i)^{2(s_1 - s_2)} (u|u_i)_{X^{s_2}(I)}, \text{ for } u \in X^{s_1}(I), \tag{2.11}$$

$$(w|w_{ij})_{X^{s_1}(\Omega)} = (1 - \lambda^{\Omega}_{ij})^{2(s_1 - s_2)} (w|w_{ij})_{X^{s_2}(\Omega)}, \text{ for } w \in X^{s_1}(\Omega). \tag{2.12}$$

2. *The set $\{u_i : i \in \mathbb{N}\}$ (resp. $\{w_{ij} : i, j \in \mathbb{N}\}$) is a complete orthogonal system in $X^s(I)$ (resp. $X^s(\Omega)$) for any $s \geq 0$. In particular if $s_1 > s_2 \geq 0$ then $X^{s_1}(I)$ (resp. $X^{s_1}(\Omega)$) is a dense subspace of $X^{s_2}(I)$ (resp. $X^{s_2}(\Omega)$).*

3. *For $U \in \{I, \Omega\},\ s_1 > s_2 \geq 0$ the space $X^{s_1}(U)$ imbeds compactly into $X^{s_2}(U)$:*

$$X^{s_1}(U) \subset\subset X^{s_2}(U). \tag{2.13}$$

Proof.
Step 1 We give the proof for $U = I$ only as the one for $U = \Omega$ is analogous. For $s \geq 0$ and $u \in X^s(I)$ we have

$$(u|u_i)_{X^s(I)} = \sum_{k\in\mathbb{N}}(1-\lambda_k^I)^{2s}(u|u_k)_{X(I)}(u_i|u_k)_{X(I)} = (1-\lambda_i^I)^{2s}(u|u_i)_{X(I)},$$

from which (2.11) follows. The proof of (2.12) is similar.
Step 2 Orthogonality in X^s follows from (2.11),(2.12) and Lemma 2.1 . For the proof of completeness of $\{u_i : i \in \mathbb{N}\}$ notice that if $u \in X^s(I)$ then denoting $u_n = \sum_{i=0}^n (u|u_i)_{X(I)} u_i$ one has $\lim_{n\to\infty}\|u - u_n\|^2_{X^s(I)} = \lim_{n\to\infty}\sum_{k\geq n+1}(1-\lambda_i^I)^{2s}(u|u_k)^2_{X(I)} = 0$. Similarly one prooves completeness of $\{w_{ij} : i,j \in \mathbb{N}\}$.
Step 3 Let $(u^n)_{n=1}^\infty$ be a bounded sequence in $X^{s_1}(I)$. Denote $M = \sup\{\|u^n\|_{X^{s_1}(I)} :\ n \in \mathbb{N}\}$. Since $X^{s_1}(I)$ is a Hilbert space we can choose a subsequence $(u^{n_k})_{k=1}$ weakly convergent in $X^{s_1}(I)$ to certain $u^\infty \in X^{s_1}(I)$. In particular

$$\lim_{k\to\infty}(u^{n_k}|u_i)_{X(I)} = (u^\infty|u_i)_{X(I)},\ \text{for } i \in \mathbb{N},$$
$$\|u^\infty\|_{X^{s_1}(I)} \leq M.$$

For any $i_0 \in \mathbb{N}_+$ we estimate

$$\|u^{n_k} - u^\infty\|^2_{X^{s_2}(I)} \leq \sum_{i=0}^{i_0-1}(1-\lambda_i^I)^{2s_2}|(u^{n_k}-u^\infty|u_i)_{X(I)}|^2 + (1-\lambda_{i_0}^I)^{2s_2-2s_1}\sum_{i=i_0}^{\infty}(1-\lambda_i^I)^{2s_1}|(u^{n_k}-u^\infty|u_i)_{X(I)}|^2$$
$$\leq \sum_{i=0}^{i_0-1}(1-\lambda_i^I)^{2s_2}|(u^{n_k}-u^\infty|u_i)_{X(I)}|^2 + (1-\lambda_{i_0}^I)^{2s_2-2s_1}\|u^{n_k}-u^\infty\|^2_{X^{s_1}(I)}$$
$$\leq \sum_{i=0}^{i_0-1}(1-\lambda_i^I)^{2s_2}|(u^{n_k}-u^\infty|u_i)_{X(I)}|^2 + 4M^2(1-\lambda_{i_0}^I)^{2s_2-2s_1}.$$

Fix $\epsilon > 0$. Choose $i_0 \in \mathbb{N}_+$ such that $4M^2(1-\lambda_{i_0}^I)^{2s_2-2s_1} \leq \epsilon^2$. Then $\limsup_{k\to\infty}\|u^{n_k} - u^\infty\|_{X^{s_2}(I)} \leq \epsilon$ and consequently $\lim_{k\to\infty}\|u^{n_k} - u^\infty\|_{X^{s_2}(I)} = 0$. □

Next we extend the scale of Hilbert spaces $X^s(I), X^s(\Omega)$ to $s \in [-1, 0)$ by duality. More precisely for any $s \in [-1, 0)$ we define $X^s(I) = (X^{-s}(I))^*, X^s(\Omega) = (X^{-s}(\Omega))^*$. Then for $s \in [-1, 0)$ Banach spaces X^s become Hilbert spaces when equipped with the following scalar products

$$(u|u')_{X^s(I)} = \sum_{i\in\mathbb{N}}(1-\lambda_i^I)^{2s}\Big\langle u, u_i\Big\rangle_{(X^s(I),X^{-s}(I))}\Big\langle u', u_i\Big\rangle_{(X^s(I),X^{-s}(I))},\ \text{for } u, u' \in X^s(I), \tag{2.14}$$

$$(w|w')_{X^s(\Omega)} = \sum_{i,j\in\mathbb{N}}(1-\lambda_{ij}^\Omega)^{2s}\Big\langle w, w_{ij}\Big\rangle_{(X^s(\Omega),X^{-s}(\Omega))}\Big\langle w', w_{ij}\Big\rangle_{(X^s(\Omega),X^{-s}(\Omega))},\ \text{for } w, w' \in X^s(\Omega). \tag{2.15}$$

Observe that assertions of Lemma 2.3 are still valid without assumming that s, s_1, s_2 are nonnegative.

In the following lemma we give relation between X^s spaces and complex interpolation.

Lemma 2.4. $[X^{s_1}(U), X^{s_2}(U)]_\theta = X^{s_1(1-\theta)+s_2\theta}(U)$ *for* $s_1, s_2 \geq -1,\ \theta \in [0,1],\ U \in \{I, \Omega\}$.

Proof. We provide the proof for $U = \Omega$ as the one for $U = I$ can be carried out similarly. For $s \geq -1$ and $i, j \in \mathbb{N}$ define Hilbert spaces $Z^s_{ij} = (\mathbb{R}, (\cdot|\cdot)_{Z^s_{ij}})$, where $(a|a')_{Z^s_{ij}} = (1 - \lambda^\Omega_{ij})^{2s} aa'$ for $a, a' \in \mathbb{R}$ and $l_2(Z^s_{ij}) = (\{\mathbf{a} = (a_{ij})_{i,j\in\mathbb{N}} : a_{ij} \in \mathbb{R},\ \sum_{i,j\in\mathbb{N}} \|a_{ij}\|^2_{Z^s_{ij}} < \infty\}, (\cdot|\cdot)_{l_2(Z^s_{ij})})$, where $(\mathbf{a}|\mathbf{a}')_{l_2(Z^s_{ij})} = \sum_{i,j\in\mathbb{N}} (a_{ij}|a'_{ij})_{Z^s_{ij}}$ for $\mathbf{a}, \mathbf{a}' \in l_2(Z^s_{ij})$. Define map

$$\Phi(w) = \left(\left\langle w, w_{ij} \right\rangle_{(X^{-1}(\Omega), X^1(\Omega))} \right)_{i,j\in\mathbb{N}}, \text{ for } w \in X^{-1}(\Omega).$$

Observe that Φ is an isometric isomorphism between $X^s(\Omega)$ and $l_2(Z^s_{ij})$ for any $s \geq -1$. This fact allows as to justify the first and the fourth equality in

$$[X^{s_1}(\Omega), X^{s_2}(\Omega)]_\theta = [l_2(Z^{s_1}_{ij}), l_2(Z^{s_2}_{ij})]_\theta = l_2([Z^{s_1}_{ij}, Z^{s_2}_{ij}]_\theta) = l_2(Z^{s_1(1-\theta)+s_2\theta}_{ij}) = X^{s_1(1-\theta)+s_2\theta}(\Omega),$$

while the second equality follows from [[31], Chap. 1.18.1, Theorem]. $\square$

In the next lemma we characterise X^s spaces as Sobolev-Slobodecki spaces with Neumann boundary condition.

Lemma 2.5. *For* $s \in [0,1],\ U \in \{I, \Omega\}$ *we have the following characterisation of the spaces* $X^s(U)$*:*

$$X^s(U) = \begin{cases} W^{2s}_2(U) \text{ if } 0 \leq s < 3/4 \\ W^{2s}_{2,N}(U) = \{u \in W^{2s}_2(U) : \nabla u \cdot \nu = 0 \text{ on } \partial U\} \text{ if } 3/4 < s \leq 1 \end{cases}, X^{3/4}(U) \subset W^{2s}_2(U). \tag{2.16}$$

Proof. The case when U is an open bounded domain of $\mathbb{R}^n$ with a smooth boudary (in particular $U = I$) or U is a half space - $U = \mathbb{R}_+ \times \mathbb{R}^{n-1}$ was treated in [[8], Theorem 2]. The case when $U = \Omega$ we divide in several steps.

Step 1 We will show that

$$X^1(\Omega) = W^2_{2,N}(\Omega). \tag{2.17}$$

Denote

$$\overline{u}_i(x_1) = c_{1i} \sin(i\pi(x_1 + 1)/2),\ x_1 \in I, \quad \overline{v}_i(x_2) = c_{2i} \sin(i\pi x_2),\ x_2 \in I_+.$$

Reasoning similarly as in the proof of Lemma 2.1 we obtain that the set $\{\overline{u}_i : i \in \mathbb{N}_+\}$ (resp. $\{\overline{v}_i : i \in \mathbb{N}_+\}$, $\{\overline{u}_i \otimes \overline{v}_j : i, j \in \mathbb{N}_+\}, \{\overline{u}_i \otimes v_j : i, \in \mathbb{N}_+, j \in \mathbb{N}\}, \{u_i \otimes \overline{v}_j : i \in \mathbb{N}_+, j \in \mathbb{N}\}$) is a complete orthonormal system in $X(I)$ (resp. $X(I_+), X(\Omega), X(\Omega), X(\Omega)$). Compute

$$\begin{aligned} &\partial_{x_1} w_{ij} = -(i\pi/2)\overline{u}_i \otimes v_j = -\sqrt{|\lambda^I_i|}\overline{u}_i \otimes v_j, \quad \partial_{x_2} w_{ij} = -(j\pi) u_i \otimes \overline{v}_j = -\sqrt{|\lambda^{I_+}_j|} u_i \otimes \overline{v}_j \\ &\partial^2_{x_1x_1} w_{ij} = -(i\pi/2)^2 w_{ij} = \lambda^I_i w_{ij}, \quad \partial^2_{x_2x_2} w_{ij} = -(j\pi)^2 w_{ij} = \lambda^{I_+}_j w_{ij} \\ &\partial^2_{x_1x_2} w_{ij} = \partial^2_{x_2x_1} w_{ij} = (i\pi/2)(j\pi)\overline{u}_i \otimes \overline{v}_j = \sqrt{|\lambda^I_i||\lambda^{I_+}_j|}\overline{u}_i \otimes \overline{v}_j. \end{aligned}$$

Observe that $X_{fin}(\Omega) \subset W^2_{2,N}(\Omega)$. Let $w \in X_{fin}(\Omega)$. Using the triangle inequality and $(a + b)^2 \leq 2(a^2 + b^2)$ we estimate

$$\|w\|^2_{X^1(\Omega)} = \|(I - \Delta)w\|^2_{L_2(\Omega)} \leq 2(\|w\|^2_{L_2(\Omega)} + 2(\|\partial^2_{x_1x_1} w\|^2_{L_2(\Omega)} + \|\partial^2_{x_2x_2} w\|^2_{L_2(\Omega)})) \leq 4\|w\|^2_{W^2_2(\Omega)}.$$

On the other hand

$$\begin{aligned}\|w\|^2_{W^2_2(\Omega)} &= \|w\|^2_{L_2(\Omega)} + \sum_{i=1}^{2}\|\partial_{x_i} w\|^2_{L_2(\Omega)} + \sum_{i=1}^{2}\sum_{j=1}^{2}\|\partial^2_{x_i x_j} w\|^2_{L_2(\Omega)} \\ &= \sum_{i,j}(1+|\lambda^I_i|+|\lambda^{I_+}_j|+|\lambda^I_i|^2+|\lambda^{I_+}_j|^2+2|\lambda^I_i||\lambda^{I_+}_j|)(w|w_{ij})^2_{X(\Omega)} \\ &\le 2\sum_{i,j}(1-\lambda^I_i-\lambda^{I_+}_j)^2(w|w_{ij})^2_{X(\Omega)} \\ &= 2\sum_{i,j}(1-\lambda^\Omega_{ij})^2(w|w_{ij})^2_{X(\Omega)} = 2\|w\|^2_{X^1(\Omega)}.\end{aligned}$$

Thus norms $\|\cdot\|_{W^2_2(\Omega)}$ and $\|\cdot\|_{X^1(\Omega)}$ are equivalent on $X_{fin}(\Omega)$.
In particular $X^1(\Omega) = cl_{X^1}(X_{fin}(\Omega)) = cl_{W^2_2}(X_{fin}(\Omega)) \subset W^2_{2,N}(\Omega)$. It is left to prove that $W^2_{2,N}(\Omega) \subset X^1(\Omega)$. Choose arbitrary $u \in W^2_{2,N}(\Omega)$ and let $f = u - \Delta u$. Then $f \in X(\Omega)$. Let $w = R(1,A)f$. Since $X^1(\Omega) \subset W^2_{2,N}(\Omega)$ thus $w \in W^2_{2,N}(\Omega)$ and $f = w - \Delta w$. We have

$$0 = \int_\Omega (w-u)(f-f) = \int_\Omega (w-u)^2 - \int_\Omega (w-u)\Delta(w-u) = \int_\Omega (w-u)^2 + \int_\Omega |\nabla(w-u)|^2,$$

since $\nabla(w-u)\cdot\nu = 0$ on $\partial\Omega$. Finally we obtain that $u = w \in X^1(\Omega)$.
Step 2 We will show that for $U \in \{(\mathbb{R}_+)^2, \Omega\}$:

$$[L_2(U), W^2_{2,N}(U)]_s = \begin{cases} W^{2s}_2(U) \text{ if } 0 \le s < 3/4 \\ W^{2s}_{2,N}(U) \text{ if } 3/4 < s \le 1. \end{cases} \tag{2.18}$$

To prove (2.18) for $U = (\mathbb{R}_+)^2$ we proceed as in the proof of [[8], Theorem 2] for the case $U = \mathbb{R}_+ \times \mathbb{R}$ substituting functions π and ν from that proof by

$$\begin{aligned}&\nu' : L_2((\mathbb{R}_+)^2) \to L_2(\mathbb{R}^2),\ \nu' u(x_1,x_2) = u(|x_1|,|x_2|), \\ &\pi' : L_2(\mathbb{R}^2) \to L_2((\mathbb{R}_+)^2),\ \pi' v(x_1,x_2) = \frac{1}{4}\sum_{\epsilon_1,\epsilon_2\in\{-1,1\}} v(\epsilon_1 x_1, \epsilon_2 x_2).\end{aligned}$$

Observe that the only nonsmooth points of rectangle Ω are the corners. We choose the covering of Ω by four open subsets $\{\Omega_i\}$ such that each of them contains exactly one corner. Then a standard argument involving partition of unity inscribed in the covering $\{\Omega_i\}$ allows us to adapt (2.18) from $U = (\mathbb{R}_+)^2$ to $U = \Omega$.

Step 3 Using Lemma 2.4 and (2.17) we obtain $X^s(\Omega) = [X(\Omega), X^1(\Omega)]_s = [L_2(\Omega), W^2_{2,N}(\Omega)]_s$ from which Lemma 2.5 for $U = \Omega$ follows due to (2.18). □

In the next lemma we collect imbeddings of X^s spaces into Lebesgue spaces L_p and the space of continuous functions.

Lemma 2.6. *We have the following imbeddings*

$$X^s(I) \subset\subset \begin{cases} C(\overline{I}) \text{ if } 1/4 < s \\ L_p(I) \text{ if } 0 \le s < 1/4,\ 1 \le p < 2/(1-4s) \end{cases},\ X^s(I) \subset L_{2/(1-4s)}(I), \tag{2.19}$$

$$X^s(\Omega) \subset\subset \begin{cases} C(\overline{\Omega}) \text{ if } 1/2 < s \\ L_p(\Omega) \text{ if } 0 \le s < 1/2, \; 1 \le p < 2/(1-2s) \end{cases}, X^s(\Omega) \subset L_{2/(1-2s)}(\Omega). \tag{2.20}$$

Proof. Imbeddings (2.19) and (2.20) are straigthforward consequences of the well-known continuous imbeddings of fractional Sobolev spaces W_p^s (see for instance [[1], Theorem 7.27]), characterisation of X^s spaces given in 2.5 and compact imbeddings of X^s spaces given in (2.13). □

For $s \ge -1$ define operator $A_{0,s}$ (resp. $A_{h,s}$) as $X^s(I)$ (resp. $X^s(\Omega)$) realisation of operator A_0 (resp. A_h) i.e.

$$A_{0,s} : X^s(I) \supset X^{s+1}(I) \to X^s(I), \; A_{0,s}u = \sum_{i\in\mathbb{N}} \lambda_i^I (u|u_i)_{X(I)} u_i, \text{ for } u \in X^{s+1}(I),$$
$$A_{h,s} : X^s(\Omega) \supset X^{s+1}(\Omega) \to X^s(\Omega), \; A_{h,s}w = \sum_{i,j\in\mathbb{N}} \lambda_{ij,h}^\Omega (w|w_{ij})_{X(\Omega)} w_{ij}, \text{ for } w \in X^{s+1}(\Omega).$$

Operators $A_{0,s}, A_{h,s}$ are self-adjoint and nonpositive and thus generate strongly continuous, analytic semigroups of contractions $e^{tA_{0,s}} \in \mathcal{L}(X^s(I)), \; e^{tA_{h,s}} \in \mathcal{L}(X^s(\Omega))$.

If $s_1 \ge s_2 \ge -1$ then operators $A_{0,s_1}, R(\lambda, A_{0,s_1}), e^{tA_{0,s_1}}$ are restrictions of operators $A_{0,s_2}, R(\lambda, A_{0,s_2}), e^{tA_{0,s_2}}$ and operators $A_{h,s_1}, R(\lambda, A_{h,s_1}), e^{tA_{h,s_1}}$ are restrictions of operators $A_{h,s_2}, R(\lambda, A_{h,s_2}), e^{tA_{h,s_2}}$ i.e.

$$\begin{aligned} A_{0,s_1}u &= A_{0,s_2}u, \text{ for } u \in X^{s_1+1}(I), \\ R(\lambda, A_{0,s_1})u &= R(\lambda, A_{0,s_2})u, \; e^{tA_{0,s_1}}u = e^{tA_{0,s_2}}u, \text{ for } u \in X^{s_1}(I), \\ A_{h,s_1}w &= A_{h,s_2}w, \text{ for } w \in X^{s_1+1}(\Omega), \\ R(\lambda, A_{h,s_1})w &= R(\lambda, A_{h,s_2})w, \; e^{tA_{h,s_1}}w = e^{tA_{h,s_2}}w, \text{ for } w \in X^{s_1}(\Omega). \end{aligned}$$

From now on we will loose s-dependence in notation and write $A_0, A_h, R(\lambda, A_0), R(\lambda, A_h), e^{tA_0}, e^{tA_h}$ instead of $A_{0,s}, A_{h,s}, R(\lambda, A_{0,s}), R(\lambda, A_{h,s}), e^{tA_{0,s}}, e^{tA_{h,s}}$.

In the next lemma we collect basic estimates for the resolvents $R(\lambda, A_0), R(\lambda, A_h)$ and semigroups e^{tA_0}, e^{tA_h}.

Lemma 2.7. *For $h \in (0,1], \; \lambda > 0, \; t > 0$ the following estimates hold*

$$\|R(\lambda, A_0)\|_{\mathcal{L}(X^s(I),X^{s'}(I))} + \|R(\lambda, A_h)\|_{\mathcal{L}(X^s(\Omega),X^{s'}(\Omega))} \le C\frac{1}{\lambda}(1 + \lambda^{s'-s}), \; -1 \le s \le s' \le s+1, \tag{2.21}$$
$$\|e^{tA_0}\|_{\mathcal{L}(X^s(I),X^{s'}(I))} + \|e^{tA_h}\|_{\mathcal{L}(X^s(\Omega),X^{s'}(\Omega))} \le C\Big(1 + \frac{1}{t^{s'-s}}\Big), \; -1 \le s \le s', \tag{2.22}$$

where C depends only on s, s'.

Proof. The proof may be obtained with the use of spectral decomposition. For details we refer to the proof of the Lemma 2.11 where we use the same technique. □

Operators E, P and Tr

Define operators

$$E \in \mathcal{L}(X(I), X(\Omega)), \; [Eu](x_1, x_2) = u(x_1), \text{ for } u \in X(I), \tag{2.23}$$
$$P \in \mathcal{L}(X(\Omega), X(I)), \; [Pw](x_1) = \int_{I_+} w(x_1, x_2)dx_2, \text{ for } w \in X(\Omega). \tag{2.24}$$

Basic properties of operators E and P are collected in the following

Lemma 2.8. *Operators E and P are mutually adjoint i.e. $E^* = P$. Moreover*

$$E \in \mathcal{L}(X^s(I), X^s(\Omega)), \ P \in \mathcal{L}(X^s(\Omega), X^s(I)), \text{ for } s \geq 0. \tag{2.25}$$

Proof. To prove that $E^* = P$ we need to show that

$$(Eu|w)_{X(\Omega)} = (u|Pw)_{X(I)}, \text{ for } u \in X(I), w \in X(\Omega). \tag{2.26}$$

Observe that for $i, j, k \in \mathbb{N}$

$$Eu_k = w_{k0} \text{ and } Pw_{ij} = u_i \delta_{0j}. \tag{2.27}$$

Thus

$$(Eu_k|w_{ij})_{X(\Omega)} = (w_{k0}|w_{ij})_{X(\Omega)} = \delta_{ki}\delta_{0j} = (u_k|u_i\delta_{0j})_{X(I)} = (u_k|Pw_{ij})_{X(I)}.$$

Owing to bilinearity of scalar products we obtain (2.26) for $u \in X_{fin}(I), w \in X_{fin}(\Omega)$ and finally by density of $X_{fin}(I)$ (resp. $X_{fin}(\Omega)$) in $X(I)$ (resp. $X(\Omega)$) and continuity of scalar products and operators E, P we obtain (2.26) for arbitrary $u \in X(I), w \in X(\Omega)$.

For $u \in X^s(I)$ we obtain

$$\begin{aligned}\|Eu\|^2_{X^s(\Omega)} &= \sum_{i,j}(1-\lambda^{\Omega}_{ij})^{2s}(Eu|w_{ij})^2_{X(\Omega)} = \sum_{i,j}(1-\lambda^{\Omega}_{ij})^{2s}(u|Pw_{ij})^2_{X(\Omega)} = \sum_{i,j}(1-\lambda^{\Omega}_{ij})^{2s}(u|u_i)^2_{X(\Omega)}\delta_{0j} \\ &= \sum_{i}(1-\lambda^{\Omega}_{i0})^{2s}(u|u_i)^2_{X(I)} = \|u\|^2_{X^s(I)},\end{aligned}$$

since $\lambda^{\Omega}_{i0} = \lambda^I_i$. Similarly

$$\begin{aligned}\|Pw\|_{X^s(I)} &= \sum_k (1-\lambda^I_k)^{2s}(Pw|u_k)^2_{X(I)} = \sum_k (1-\lambda^I_k)^{2s}(w|Eu_k)^2_{X(I)} \\ &= \sum_k (1-\lambda^{\Omega}_{k0})^{2s}(w|w_{k0})^2_{X(I)} \leq \|w\|^2_{X^s(\Omega)}.\end{aligned}$$

□

Define operator $P_{-1} = E'$ and operator $E_{-1} = P'$. Using Lemma 2.8 we obtain that P_{-1} and E_{-1} satisfy

$$P_{-1} \in \mathcal{L}(X^{-s}(\Omega), X^{-s}(I)), \ E_{-1} \in \mathcal{L}(X^{-s}(I), X^{-s}(\Omega)), \ s \in [0,1].$$

Moreover for $u \in X(I), w \in X(\Omega)$

$$P_{-1}w = E'w = E^*w = Pw, \quad E_{-1}u = P'u = P^*u = Eu.$$

From now on we will write E, P instead of E_{-1}, P_{-1}.

For $w \in X_{fin}(\Omega)$ denote by $Tr(w)$ the trace operator i.e. restriction of w to $I \times \{0\}$:

$$Tr(w)(x_1) = w(x_1, 0), \text{ for } x_1 \in I.$$

Lemma 2.9. *For any $s > 1/4$ there exists C depending only on s such that for any $w \in X_{fin}(\Omega)$*

$$\|Tr(w)\|_{X^{s-1/4}(I)} \le C\|w\|_{X^s(\Omega)}. \tag{2.28}$$

The operator Tr can be uniquely extended to an operator $\tilde{Tr} \in \mathcal{L}(X^s(\Omega), X^{s-1/4}(I))$.

Proof. For $w = \sum_{i,j\ge 0} a_{ij} w_{ij}$ where only finitely many a_{ij} are nonzero we have

$$Tr(w) = \sum_{i,j\ge 0} a_{ij} u_i v_j(0) = \sum_{i\ge 0}(\sum_{j\ge 0} a_{ij} c_{2j}) u_i.$$

Using Lemma 2.3 we get that the system $\{u_i\}$ is orthogonal in $X^{s-1/4}(I)$ and $\|u_i\|^2_{X^{s-1/4}(I)} = (1 - \lambda_i^I)^{2s-1/2} = (1 + (i\pi/2)^2)^{2s-1/2}$. Since $0 < c_{2j} \le \sqrt{2}$ (see (2.5)) we thus obtain

$$\begin{aligned}\|Tr(w)\|^2_{X^{s-1/4}(I)} &= \sum_{i\ge 0}(\sum_{j\ge 0} a_{ij} c_{2j})^2 \|u_i\|^2_{X^{s-1/4}(I)} \le 2\sum_{i\ge 0}(\sum_{j\ge 0} |a_{ij}|)^2 (1 + (i\pi/2)^2)^{2s-1/2} \\ &\le 2\Big(\frac{\pi}{2}\Big)^{4s-1} \sum_{i\ge 0}(\sum_{j\ge 0} |a_{ij}|)^2 (1+i)^{4s-1}.\end{aligned}$$

Using Cauchy-Schwarz inequality to estimate the inner sum we further obtain that

$$\begin{aligned}\|Tr(w)\|^2_{X^{s-1/4}(I)} &\le 2\Big(\frac{\pi}{2}\Big)^{4s-1} \sum_{i\ge 0}(\sum_{j\ge 0} |a_{ij}|^2 (1+i+j)^{4s})(\sum_{j\ge 0} \frac{1}{(1+i+j)^{4s}})(1+i)^{4s-1} \\ &\le 2\Big(\frac{\pi}{2}\Big)^{4s-1} \frac{4s}{4s-1} \sum_{i\ge 0}(\sum_{j\ge 0} |a_{ij}|^2 (1+i+j)^{4s}),\end{aligned} \tag{2.29}$$

where the last inequality is a consequence of the following estimate

$$\begin{aligned}\sum_{j\ge 0} \frac{(1+i)^{4s-1}}{(1+i+j)^{4s}} &= \frac{1}{1+i} + \sum_{j\ge 1} \frac{(1+i)^{4s-1}}{(1+i+j)^{4s}} \le 1 + (1+i)^{4s-1} \int_{1+i}^{\infty} \frac{dt}{t^{4s}} \\ &= 1 + (1+i)^{4s-1} \frac{1}{4s-1} (1+i)^{1-4s} = \frac{4s}{4s-1}.\end{aligned}$$

On the other hand since the system $\{w_{ij}\}$ is orthogonal in $X^s(\Omega)$ and

$$\|w_{ij}\|^2_{X^s(\Omega)} = (1 - \lambda_{ij}^{\Omega})^{2s} = (1 + (i\pi/2)^2 + (j\pi)^2)^{2s} \ge 3^{-2s}(1+i+j)^{4s}$$

we have

$$\begin{aligned}\|w\|^2_{X^s(\Omega)} &= \sum_{i,j\ge 0} |a_{ij}|^2 \|w_{ij}\|^2_{X^s(\Omega)} = \sum_{i\ge 0}(\sum_{j\ge 0} |a_{ij}|^2 (1 + (i\pi/2)^2 + (j\pi)^2)^{2s}) \\ &\ge 3^{-2s} \sum_{i\ge 0}(\sum_{j\ge 0} |a_{ij}|^2 (1+i+j)^{4s}).\end{aligned} \tag{2.30}$$

Combining (2.29) and (2.30) we obtain (2.28) with $C^2 = 3^{2s}(\pi/2)^{4s-1} 8s/(4s-1)$. Since $X_{fin}(\Omega)$ is dense in $X^s(\Omega)$ (see part 2 of Lemma 2.3) the latter part of the Lemma 2.9 follows. □

From now on we write Tr instead of $\tilde{Tr}$.
Next we collect several identities involving operators $P, E, Tr, R(\lambda, A_h), R(\lambda, A_0), e^{tA_h}$ and e^{tA_0}.

Lemma 2.10. *The following identities hold*

$$PTr'u = TrEu = PEu = u, \text{ for } u \in X^{-s}(I), \ 1 \geq s > 0 \tag{2.31a}$$

$$R(\lambda, A_h)E = ER(\lambda, A_0), \text{ for } h > 0 \tag{2.31b}$$

$$e^{tA_h}E = Ee^{tA_0}, \text{ for } h > 0. \tag{2.31c}$$

Proof. Identities $TrEu = PEu = u$ are obvious for $u \in X(I)$ and can be extended to the case when $u \in X^{-s}(I)$ by a density argument. Then

$$PTr'u = E'Tr'u = (TrE)'u = u,$$

from which (2.31a) follows.
Since $Eu_i = w_{i0}$ for any $i \geq 0$ hence

$$R(\lambda, A_h)Eu_i = R(\lambda, A_h)w_{i0} = \frac{1}{\lambda - \lambda^{\Omega}_{i0}} w_{i0} = E\frac{1}{\lambda - \lambda^{I}_{i}} u_i = ER(\lambda, A_0)u_i.$$

Since $\{u_i\}_{i\geq 0}$ is a Schauder basis in every $X^s(I)$ (see part 2 of Lemma 2.3) we obtain (2.31b). Similarly one proves (2.31c). □

Resolvent and semigroup estimates used in the dimension reduction

Estimates for semigroup e^{tA_h} and resolvent operator $R(\lambda, A_h)$ which are presented in the next lemma are of fundamental importance in the dimension reduction carried out in Section 2.2.5.

Lemma 2.11. *For $h \in (0, 1]$, $s, s' \geq -1$, $t, \lambda > 0$, $w \in X^s(\Omega)$ the following estimates hold*

$$\|R(\lambda, A_h)(I - EP)w\|_{X^{s'}(\Omega)} \leq \frac{1}{\lambda - \lambda^{\Omega}_{01,h}}(1 + (\lambda - \lambda^{\Omega}_{01,h})^{s'-s})\|(I - EP)w\|_{X^s(\Omega)}, \ 0 \leq s' - s \leq 1, \tag{2.32}$$

$$\|e^{tA_h}(I - EP)w\|_{X^{s'}(\Omega)} \leq C\Big(1 + \frac{1}{t^{s'-s}}\Big)e^{t\lambda^{\Omega}_{01,h}}\|(I - EP)w\|_{X^s(\Omega)}, \ 0 \leq s' - s. \tag{2.33}$$

where C depends only on s, s'.

Proof. Since $X_{fin}(\Omega)$ is dense in $X^s(\Omega)$ (see part 2 of Lemma 2.3) one can assume that $w \in X_{fin}(\Omega)$ i.e. $w = \sum_{i,j\geq 0} a_{ij}w_{ij}$ where only finitely many a_{ij} are nonzero. Define

$$M_1 = \sup\Big\{\frac{(1 - \lambda^{\Omega}_{ij})^{s'-s}}{\lambda - \lambda^{\Omega}_{ij,h}} \ : \ i \geq 0, j \geq 1\Big\}.$$

Observe that

$$(I - EP)w_{ij} = w_{ij} - E(u_i\delta_{0j}) = w_{ij} - w_{i0}\delta_{0j} = w_{ij}(1 - \delta_{0j}), \tag{2.34}$$

$$(w_{ij}|w_{kl})_{X^{s'}(\Omega)} = (1 - \lambda^{\Omega}_{ij})^{2s'}\delta_{ij}\delta_{jl}, \tag{2.35}$$

$$\|w_{ij}\|_{X^{s'}(\Omega)} = (1 - \lambda^{\Omega}_{ij})^{s'-s}\|w_{ij}\|_{X^s(\Omega)}. \tag{2.36}$$

Indeed (2.34) is a simple consequence of the definitions of operators E and P (see (2.23),(2.24)) and (2.4), while (2.35) and (2.36) follow from Lemma 2.3. Using (2.34), (2.35) and (2.36) we estimate

$$\|R(\lambda, A_h)(I - EP)w\|^2_{X^{s'}(\Omega)} = \|\sum_{i\geq 0, j\geq 1} \frac{1}{\lambda - \lambda^{\Omega}_{ij,h}} a_{ij} w_{ij}\|^2_{X^{s'}(\Omega)} = \sum_{i\geq 0, j\geq 1} \frac{1}{(\lambda - \lambda^{\Omega}_{ij,h})^2} a^2_{ij} \|w_{ij}\|^2_{X^{s'}(\Omega)}$$
$$= \sum_{i\geq 0, j\geq 1} \frac{1}{(\lambda - \lambda^{\Omega}_{ij,h})^2} a^2_{ij} (1 - \lambda^{\Omega}_{ij})^{2(s'-s)} \|w_{ij}\|^2_{X^{s}(\Omega)} \leq M^2_1 \sum_{i\geq 0, j\geq 1} a^2_{ij} \|w_{ij}\|^2_{X^{s}(\Omega)}$$
$$= M^2_1 \|(I - EP)w\|^2_{X^{s}(\Omega)},$$

To finish the proof of (2.32) it is left to show that

$$M_1 \leq \frac{1}{\lambda - \lambda^{\Omega}_{01,h}} (1 + (\lambda - \lambda^{\Omega}_{01,h})^{s'-s}). \tag{2.37}$$

Using condition $0 \leq s' - s \leq 1$ and the following inequality

$$(1 + x)^{\alpha} \leq 1 + x^{\alpha} \text{ for } x > 0,\ 0 \leq \alpha \leq 1$$

we estimate

$$\frac{(1 - \lambda^{\Omega}_{ij})^{s'-s}}{\lambda - \lambda^{\Omega}_{ij,h}} \leq \frac{(1 - \lambda^{\Omega}_{ij,h})^{s'-s}}{\lambda - \lambda^{\Omega}_{ij,h}} \leq \frac{1 + (-\lambda^{\Omega}_{ij,h})^{s'-s}}{\lambda - \lambda^{\Omega}_{ij,h}} \leq \frac{1 + (\lambda - \lambda^{\Omega}_{ij,h})^{s'-s}}{\lambda - \lambda^{\Omega}_{ij,h}}$$
$$= \frac{1}{(\lambda - \lambda^{\Omega}_{ij,h})^{1-(s'-s)}} + \frac{1}{\lambda - \lambda^{\Omega}_{ij,h}} \leq \frac{1}{(\lambda - \lambda^{\Omega}_{01,h})^{1-(s'-s)}} + \frac{1}{\lambda - \lambda^{\Omega}_{01,h}} = \frac{1}{\lambda - \lambda^{\Omega}_{01,h}} (1 + (\lambda - \lambda^{\Omega}_{01,h})^{s'-s})$$

from which (2.37) and consequently (2.32) follows. We move to the proof of (2.33). Reasoning as in the proof of (2.32) we obtain that for $w \in X^s(\Omega)$

$$\|e^{tA_h}(w - EPw)\|_{X^{s'}(\Omega)} \leq M_2 \|w - EPw\|_{X^s(\Omega)},$$

where

$$M_2 = \sup\{(1 - \lambda^{\Omega}_{ij})^{s'-s} \exp(t\lambda^{\Omega}_{ij,h}) : i \geq 0, j \geq 1\}.$$

Using inequality (1.8a) from Lemma 1.1 we estimate for $i \geq 0, j \geq 1$

$$(1 - \lambda^{\Omega}_{ij})^{s'-s} \exp(t\lambda^{\Omega}_{ij,h}) = (1 + (i\pi/2)^2 + (j\pi)^2)^{s'-s} \exp(-t((i\pi/2)^2 + (j\pi/h)^2))$$
$$= (1 + (i\pi/2)^2 + (j\pi)^2)^{s'-s} \exp(-\frac{t}{h^2}(1 + (i\pi/2)^2 + (j\pi)^2)) \exp(-t(i\pi/2)^2) \exp(\frac{t}{h^2}(1 + (i\pi/2)^2))$$
$$\leq \sup\{x^{s'-s} \exp(-\frac{t}{h^2}x) : x \geq 1 + (i\pi/2)^2 + \pi^2\} \exp(-t(i\pi/2)^2) \exp(\frac{t}{h^2}(1 + (i\pi/2)^2))$$
$$\leq C((\frac{h^2}{t})^{s'-s} + (1 + (i\pi/2)^2 + \pi^2)^{s'-s}) \exp(-\frac{t}{h^2}(1 + (i\pi/2)^2 + \pi^2)) \exp(-t(i\pi/2)^2) \exp(\frac{t}{h^2}(1 + (i\pi/2)^2))$$
$$= C((\frac{h^2}{t})^{s'-s} + (1 + (i\pi/2)^2 + \pi^2)^{s'-s}) \exp(-t(i\pi/2)^2) \exp(-\frac{t\pi^2}{h^2})$$
$$\leq C(\frac{1}{t^{s'-s}} + 1 + \frac{(t(i\pi/2)^2)^{s'-s}}{t^{s'-s}}) \exp(-t(i\pi/2)^2) \exp(-\frac{t\pi^2}{h^2})$$
$$\leq C(1 + \frac{1}{t^{s'-s}} + \frac{1}{t^{s'-s}} \sup\{x^{s'-s} \exp(-x) : x \geq 0\}) \exp(-\frac{t\pi^2}{h^2}) \leq C(1 + \frac{1}{t^{s'-s}}) \exp(-\frac{t\pi^2}{h^2}).$$

□

The multiplication operator

For $1 \le p < \infty$ and $0 \ge f \in L_p(I)$ we define the multiplication operator M_f

$$M_f : L_\infty(I) \supset D(M_f) \to L_\infty(I),\ M_f u = fu, \tag{2.38}$$

where $D(M_f) = \{u \in L_\infty(I) : fu \in L_\infty(I)\}$. Observe that if $u \in L_\infty(I)$ and $Re(\lambda) > 0$ then $R(\lambda, M_f)u = \frac{u}{\lambda - f} \in L_\infty(I)$ and $\|R(\lambda, M_f)\|_{\mathcal{L}(L_\infty(I))} \le 1/|\lambda|$, which proves that M_f is sectorial and thus generates an analytic semigroup e^{tM_f}:

$$e^{tM_f} u = e^{tf} u,\ u \in L_\infty(I).$$

Basic estimates concerning e^{tM_f} are collected in the following

Lemma 2.12. *Assume that* $0 \ge f, f_1, f_2 \in L_p(I)$*. Then for* $t, t' \ge 0$

$$\|e^{tM_f}\|_{\mathcal{L}(L_\infty(I))} \le 1, \tag{2.39}$$

$$\|e^{t'M_f} - e^{tM_f}\|_{\mathcal{L}(L_\infty(I), L_p(I))} \le |t' - t| \|f\|_{L_p(I)}, \tag{2.40}$$

$$\|e^{tM_{f_1}} - e^{tM_{f_2}}\|_{\mathcal{L}(L_\infty(I), L_p(I))} \le t \|f_1 - f_2\|_{L_p(I)}. \tag{2.41}$$

Proof. Using inequalities

$$0 < e^x \le 1,\ |e^x - e^y| \le |x - y|,\ x, y < 0,$$

we get for $u \in L_\infty(I), t, t' \ge 0$

$$\|e^{tM_f} u\|_\infty = \|e^{tf} u\|_{L_\infty(I)} \le \|e^{tf}\|_\infty \|u\|_{L_\infty(I)} \le \|u\|_\infty,$$
$$\|(e^{t'M_f} - e^{tM_f})u\|_{L_p(I)} = \|(e^{t'f} - e^{tf})u\|_{L_p(I)} \le \|e^{t'f} - e^{tf}\|_{L_p(I)} \|u\|_\infty \le |t' - t| \|f\|_{L_p(I)} \|u\|_\infty,$$
$$\|(e^{tM_{f_1}} - e^{tM_{f_2}})u\|_{L_p(I)} = \|(e^{tf_1} - e^{tf_2})u\|_{L_p(I)} \le \|e^{tf_1} - e^{tf_2}\|_{L_p(I)} \|u\|_\infty \le t \|f_1 - f_2\|_{L_p(I)} \|u\|_\infty,$$

from which (2.39), (2.40) and (2.41) follow. □

2.2.2. The case of a regular source

Denote

$$\boldsymbol{u} = (u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^5,\ f_1, f_2, f_3, f_4, f_5 : \mathbb{R}^5 \to \mathbb{R},$$
$$f_1(\boldsymbol{u}) = -(c_1 + u_3)u_1 + c_2 u_2 + c_4 u_4,$$
$$f_2(\boldsymbol{u}) = c_1 u_1 - (b_2 + c_2 + c_3 u_3)u_2 + c_5 u_5,$$
$$f_3(\boldsymbol{u}) = -(b_3 + u_1 + c_3 u_2)u_3 + c_4 u_4 + c_5 u_5 + p_3,$$
$$f_4(\boldsymbol{u}) = u_1 u_3 - (b_4 + c_4)u_4,$$
$$f_5(\boldsymbol{u}) = c_3 u_2 u_3 - (b_5 + c_5)u_5.$$

In this section we study system (2.1) with δ substituted by a regular function ω:

$$\partial_t u_1 + div(J_h(u_1)) + b_1 u_1 = 0, \quad (t, x) \in \Omega_T \tag{2.42a}$$
$$\partial_t u_2 - d\partial_{x_1}^2 u_2 = f_2(\boldsymbol{u}), \quad (t, x) \in (\partial_1\Omega)_T \tag{2.42b}$$
$$\partial_t u_3 = f_3(\boldsymbol{u}), \quad (t, x) \in (\partial_1\Omega)_T \tag{2.42c}$$
$$\partial_t u_4 = f_4(\boldsymbol{u}), \quad (t, x) \in (\partial_1\Omega)_T \tag{2.42d}$$
$$\partial_t u_5 = f_5(\boldsymbol{u}), \quad (t, x) \in (\partial_1\Omega)_T \tag{2.42e}$$

with boundary and initial conditions

$$
\begin{aligned}
-J_h(u_1)\nu &= 0, && (t,x) \in (\partial_0\Omega)_T \\
-J_h(u_1)\nu &= f_1(\boldsymbol{u}) + \omega, && (t,x) \in (\partial_1\Omega)_T \\
\partial_{x_1} u_2 &= 0, && (t,x) \in (\partial\partial_1\Omega)_T \\
\boldsymbol{u}(0,\cdot) &= \boldsymbol{u_0}
\end{aligned}
$$

To obtain well-posedness of system (2.42) we interpret it as a system of abstract ODE's (1.10)-(1.11).

Assume that

$$
d, \boldsymbol{b} > 0,\ \boldsymbol{c}, \boldsymbol{p} \geq 0, \tag{2.43a}
$$

$$
1/2 < s < s' < 3/4, \tag{2.43b}
$$

$$
0 \leq \omega \in L_\infty(I). \tag{2.43c}
$$

Define spaces

$$
\mathcal{X}_1 = X^{-1+s'}(\Omega), \mathcal{X}_2 = X^0(I), \mathcal{X}_3 = \mathcal{X}_4 = \mathcal{X}_5 = L_\infty(I)
$$

$$
\mathcal{X}_1^1 = X^{s'}(\Omega), \mathcal{X}_2^1 = X^1(I), \mathcal{X}_3^1 = \mathcal{X}_4^1 = \mathcal{X}_5^1 = L_\infty(I)
$$

Set $\alpha = (1 + s - s', 1/2, 1/2, 1/2, 1/2)$ and observe that due to Lemma 2.4 we have

$$
\mathcal{X}^\alpha = X^s(\Omega) \times X^{1/2}(I) \times (L_\infty(I))^3.
$$

Define operators

$$
\begin{aligned}
\mathcal{A}_1 u &= (A_h - b_1)u,\ u \in \mathcal{X}_1^1, \\
\mathcal{A}_2 u &= dA_0 u,\ u \in \mathcal{X}_2^1 \\
\mathcal{A}_i &= 0,\ i = 3,4,5
\end{aligned}
$$

and for $\boldsymbol{u} \in \mathcal{X}^\alpha$ set

$$
\begin{aligned}
\mathcal{F}_1(\boldsymbol{u}) &= Tr'[f_1(Tr(u_1), u_2, \ldots, u_5) + \omega] \\
\mathcal{F}_i(\boldsymbol{u}) &= f_i(Tr(u_1), u_2, \ldots, u_5),\ i = 2,3,4,5.
\end{aligned}
$$

The main result of the present section is the following

Theorem 2.1. *Assume* (2.43). *Then for every* $0 \leq \boldsymbol{u}_0 \in \mathcal{X}^\alpha$ *system* (2.42) *has a unique globally in time defined* $\mathcal{X}^\alpha$ *solution* $\boldsymbol{u}$. *The solution* $\boldsymbol{u}$ *is nonnegative and satisfies for all times the following estimate*

$$
\|u_3(t)\|_\infty + \|u_4(t)\|_\infty + \|u_5(t)\|_\infty \leq C, \tag{2.44}
$$

where C *depends only on* $\|u_{30}\|_\infty + \|u_{40}\|_\infty + \|u_{50}\|_\infty, b_3, b_4, b_5, p_3$.

Proof.
Step 1 - local existence of solution.

Using assumption (2.43b), Lemma 2.6 and Lemma 2.9 we get

$$\begin{aligned}
&\mathcal{X}^\alpha = X^s(\Omega) \times X^{1/2}(I) \times (L_\infty(I))^3 \subset C(\overline{\Omega}) \times C(\overline{I}) \times (L_\infty(I))^3, \\
&Tr' \in \mathcal{L}(L_\infty(I), \mathcal{X}_1), \\
&Tr \in \mathcal{L}(\mathcal{X}_1^{\alpha_1}, C(\overline{I}))
\end{aligned}$$

from where we deduce that for $\boldsymbol{u}, \boldsymbol{w} \in \mathcal{X}^\alpha$ the following estimates hold

$$\sum_{i=1}^{5} \|\mathcal{F}_i(\boldsymbol{u})\|_{\mathcal{X}_i} \leq C\Big\{(1 + \sum_{i=1}^{2} \|u_i\|_{\mathcal{X}_i^{\alpha_i}})(1 + \|u_3\|_{\mathcal{X}_3^{\alpha_3}}) + \sum_{i=4}^{5} \|u_i\|_{\mathcal{X}_i^{\alpha_i}} + \|\omega\|_\infty\Big\} \tag{2.45}$$

$$\begin{aligned}
&\sum_{i=1}^{5} \|\mathcal{F}_i(\boldsymbol{u}) - \mathcal{F}_i(\boldsymbol{u}')\|_{\mathcal{X}_i^{\alpha_i}} \leq C\Big\{\sum_{i=1}^{2} \|u_i - u_i'\|_{\mathcal{X}_i^{\alpha_i}}(1 + \|u_3\|_{\mathcal{X}_3^{\alpha_3}} + \|u_3'\|_{\mathcal{X}_3^{\alpha_3}}) \\
&+ \|u_3 - u_3'\|_{\mathcal{X}_3^{\alpha_3}}(1 + \sum_{i=1}^{2} (\|u_i\|_{\mathcal{X}_i^{\alpha_i}} + \|u_i'\|_{\mathcal{X}_i^{\alpha_i}})) + \sum_{i=4}^{5} \|u_i - u_i'\|_{\mathcal{X}_i^{\alpha_i}}\Big\}
\end{aligned}$$

Using above estimates we conclude that assumptions of Lemma 1.3 are satisfied which results in the existence of a unique maximally defined $\mathcal{X}^\alpha$ solution to (2.42).

Step 2 - nonnegativity of solution.

Reasoning as in Step 1 we obtain that system

$$\partial_t v_1 + div(J_h(v_1)) + b_1 v_1 = 0, \qquad (t,x) \in \Omega_T \tag{2.46a}$$

$$\partial_t v_2 - d\partial_{x_1}^2 v_2 = f_{2+}(\boldsymbol{v}), \qquad (t,x) \in (\partial_1\Omega)_T \tag{2.46b}$$

$$\partial_t v_3 = f_{3+}(\boldsymbol{v}), \qquad (t,x) \in (\partial_1\Omega)_T \tag{2.46c}$$

$$\partial_t v_4 = f_{4+}(\boldsymbol{v}), \qquad (t,x) \in (\partial_1\Omega)_T \tag{2.46d}$$

$$\partial_t v_5 = f_{5+}(\boldsymbol{v}), \qquad (t,x) \in (\partial_1\Omega)_T \tag{2.46e}$$

with boundary and initial conditions

$$\begin{aligned}
-J_h(v_1)\nu &= 0, && (t,x) \in (\partial_0\Omega)_T \\
-J_h(v_1)\nu &= f_{1+}(\boldsymbol{v}) + \omega, && (t,x) \in (\partial_1\Omega)_T \\
\partial_{x_1} v_2 &= 0, && (t,x) \in (\partial\partial_1\Omega)_T \\
\boldsymbol{v}(0,\cdot) &= \boldsymbol{u_0}
\end{aligned}$$

where for $i = 1, \ldots, 5$ and $\boldsymbol{v} \in \mathbb{R}^5$

$$f_{i+}(\boldsymbol{v}) = f_i((v_1)_+, \ldots, (v_5)_+)$$

has a unique maximal $\mathcal{X}^\alpha$ solution $\boldsymbol{v}(t)$. Note by $T'_{\max}$ its time of existence.
Testing (2.46a),…,(2.46e) by $(v_1)_-, \ldots, (v_5)_-$ we obtain

$$\begin{aligned}
&-\frac{1}{2}\frac{d}{dt}\|(v_1)_-\|_{X(\Omega)}^2 - \|\partial_{x_1}(v_1)_-\|_{X(\Omega)}^2 - h^{-2}\|\partial_{x_2}(v_1)_-\|_{X(\Omega)}^2 - b_1\|(v_1)_-\|_{X(\Omega)}^2 = \\
&\int_I (f_{1+}(v_1(x_1,0), v_2(x_1), \ldots, v_5(x_1)) + \omega(x_1))(v_1(x_1,0))_- dx_1 \\
&-\frac{1}{2}\frac{d}{dt}\|(v_2)_-\|_{X(I)}^2 - d\|\partial_{x_1}(v_2)_-\|_{X(I)}^2 = \int_I f_{2+}(v_1(x_1,0), v_2(x_1), \ldots, v_5(x_1))(v_2(x_1))_- dx_1 \\
&-\frac{1}{2}\frac{d}{dt}\|(v_i)_-\|_{X(I)}^2 = \int_I f_{i+}(v_1(x_1,0), v_2(x_1), \ldots, v_5(x_1))(v_i(x_1))_- dx_1, \ i = 3,4,5.
\end{aligned}$$

Since right hand sides of above equalities are nonnegative we obtain that

$$\frac{d}{dt}[\|(v_1)_-\|^2_{X(\Omega)} + \sum_{i=2}^{5}\|(v_i)_-\|^2_{X(I)}] \leq 0$$
$$\|(v_1(t))_-\|^2_{X(\Omega)} + \sum_{i=2}^{5}\|(v_i(t))_-\|^2_{X(I)} \leq \|(v_{01})_-\|^2_{X(\Omega)} + \sum_{i=2}^{5}\|(v_{0i})_-\|^2_{X(I)} = 0.$$

Which proves that the only solution of system (2.46) is nonnegative. Since for $\boldsymbol{v} \geq 0$ there is $f_{i+}(\boldsymbol{v}) = f_i(\boldsymbol{v})$ we see that $T_{\max} \geq T'_{\max}$ and $\boldsymbol{u}(t) = \boldsymbol{v}(t)$ for $t \in [0, T'_{max})$. Finally observe that if $T'_{\max} < \infty$ then owing to the blow-up condition (1.12)

$$\limsup_{t\to {T'_{\max}}^-} \|\boldsymbol{u}(t)\|_{\mathcal{X}^\alpha} = \limsup_{t\to {T'_{\max}}^-} \|\boldsymbol{v}(t)\|_{\mathcal{X}^\alpha} = \infty$$

whence $T_{\max} = T'_{\max}$ and finally $\boldsymbol{u}(t) \geq 0$ for $t \in [0, T_{\max})$.

Step 3 - global solvability: $T_{\max} = \infty$.

Adding equations (2.42c),(2.42d),(2.42e) and using nonnegativity of $\boldsymbol{u}$ we get

$$\partial_t(u_3 + u_4 + u_5) + \min\{b_3, b_4, b_5\}(u_3 + u_4 + u_5) \leq p_3$$

from which we conclude that there exists C depending only on $\|u_{30}\|_\infty + \|u_{40}\|_\infty + \|u_{50}\|_\infty, b_3, b_4, b_5, p_3$ such that

$$\|u_3(t)\|_\infty + \|u_4(t)\|_\infty + \|u_5(t)\|_\infty \leq C,\ t \in [0, T_{\max}). \tag{2.47}$$

Using (2.47) and (2.45) we get that condition (1.13) is satisfied which gives $T_{\max} = \infty$.

□

The case of a singular source and dimension reduction

We begin by introducing auxiliary functions which are used in the definition of M-mild solution presented in section 2.2.4.

2.2.3. Auxiliary functions

Let us recall the definition of the standard one dimensional mollifier

$$\eta(x_1) = \begin{cases} C\exp\left(\frac{1}{|x_1|^2-1}\right), & |x_1| < 1 \\ 0, & |x_1| \geq 1 \end{cases} \quad , \quad \eta^\epsilon(x_1) = \eta(x_1/\epsilon)/\epsilon,\ \epsilon > 0$$

where C is such that $\int_{\mathbb{R}} \eta = 1$.

The next lemma concerns convergence of η^ϵ as $\epsilon \to 0$.

Lemma 2.13. *For any $0 < s$ the following convergence holds*

$$\lim_{\epsilon\to 0^+} \|\eta^\epsilon - \delta\|_{X^{-1/4-s}(I)} = 0. \tag{2.48}$$

Proof. Without loss of generality assume that $s < 1/8$. It is enough to show that every sequence $(\epsilon_n)_{n=1}^{\infty}$ of positive numbers which converges to 0 has a subsequence $(\epsilon_{n_k})_{k=1}^{\infty}$ such that

$$\eta^{\epsilon_{n_k}} \to \delta \text{ in } X^{-1/4-s}(I). \tag{2.49}$$

Since $X^{1/4+s}(I) \subset\subset C(\overline{I})$ (see Lemma 2.6) thus $\mathcal{M}(\overline{I}) = C(\overline{I})^* \subset\subset X^{-1/4-s}(I)$. Fix any sequence $(\epsilon_n)_{n=1}^{\infty}$ of positive numbers which converges to 0. Since $(\eta^{\epsilon_n})_{n=1}^{\infty}$ is a bounded sequence in $\mathcal{M}(\overline{I})$ then, by the previous observation, one can choose a subsequence $(\epsilon_{n_k})_{k=1}^{\infty}$ such that

$$\eta^{\epsilon_{n_k}} \to u \text{ in } X^{-1/4-s}(I),$$

for certain $u \in X^{-1/4-s}(I)$. Finally observe that for any $v \in X^{1/4+s}(I)$ one has

$$\left\langle u, v \right\rangle_{(X^{-1/4-s}(I), X^{1/4+s}(I))} = \lim_{k\to\infty} \left\langle \eta^{\epsilon_{n_k}}, v \right\rangle_{(X^{-1/4-s}(I), X^{1/4+s}(I))} = \lim_{k\to\infty} \int_I \eta^{\epsilon_{n_k}} v = v(0),$$

where the first equality is a consequence of the fact that strong convergence in $X^{-1/4-s}(I)$ implies convergence in the weak star topology of $X^{-1/4-s}(I)$ while the third equality follows from a well known fact that η^{ϵ} converges to δ in the weak star topology of $\mathcal{M}(\overline{I})$. Thus $u = \delta$ and (2.49) follows. □

From now on we denote

$$\eta^0 = \delta, \ \mu = (h, \epsilon) \in (0, 1] \times [0, 1] \text{ and } \mu_0 = (h, 0). \tag{2.50}$$

Next we define auxiliary functions m^{μ} and m^0 which play a fundamental role in the definition of M-mild solution which is given in section 2.2.4:

$$m^{\mu} = R(b_1, A_h)(p_1 Tr' \eta^{\epsilon}), \ m^0 = R(b_1, A_0)(p_1 \delta). \tag{2.51}$$

From (2.51) we get that m^{μ} for $\epsilon > 0$ and m^0 are W^1_2 weak solutions of the following boundary value problems

$$b_1 m^{\mu} + div(J_h(m^{\mu})) = 0, \qquad x \in \Omega \tag{2.52a}$$
$$-J_h(m^{\mu})\nu = 0, \qquad x \in \partial_0\Omega \tag{2.52b}$$
$$-J_h(m^{\mu})\nu = p_1 \eta^{\epsilon}, \qquad x \in \partial_1\Omega, \tag{2.52c}$$

$$b_1 m^0 - d\partial^2_{x_1} m^0 = p_1 \delta, \qquad x_1 \in I \tag{2.53a}$$
$$\partial_{x_1} m^0 = 0, \qquad x_1 \in \partial I. \tag{2.53b}$$

Concerning regularity of m^0 and m^{μ} we have the following

Lemma 2.14. *Let m^0 and m^{μ} be given by (2.51) then*

$$m^0 \in W^1_{\infty}(I), \tag{2.54}$$
$$m^{\mu} \in W^1_p(\Omega) \text{ for any } 1 \le p < 2, \tag{2.55}$$
$$\|m^{\mu}\|_{X^{1/2-s}(\Omega)} \le C, \ 0 < s \le 3/2 \tag{2.56}$$
$$\|m^{\mu} - m^{\mu_0}\|_{X^{1/2-s}(\Omega)} \le C\|\eta^{\epsilon} - \delta\|_{X^{-1/4-s}(I)}, \ 0 < s \le 3/4 \tag{2.57}$$
$$\|m^{\mu_0} - Em^0\|_{X^{1/2-s}(\Omega)} \le C\frac{1}{|\lambda^{\Omega}_{01,h}|^{s/2}}, \ 0 < s \le 3/2, \tag{2.58}$$

where C does not depend on μ. Moreover $m^{\mu}, m^0 \ge 0$.

Proof. To prove (2.54) define $u(x_1) = m^0(x_1) + \frac{p_1}{2d}|x_1|$. Then using (2.53) we obtain that $b_1 u - d\partial_{x_1}^2 u = \frac{b_1 p_1}{2d}|x_1|$ for $x_1 \in I$. We conclude that $u \in C^2(\overline{I})$ from where (2.54) follows. The claim (2.55) is a consequence of Lemma 2.17.
Using (2.21), (2.28), (2.48) we estimate

$$\|m^\mu\|_{X^{1/2-s}(\Omega)} \leq p_1 \|R(b_1, A_h)\|_{\mathcal{L}(X^{-1/2-s}(\Omega), X^{1/2-s}(\Omega))} \|Tr'\|_{\mathcal{L}(X^{-1/4-s}(I), X^{-1/2-s}(\Omega))} \|\eta^\epsilon\|_{X^{-1/4-s}(I)} \leq C,$$

from which (2.56) follows. To prove (2.57) we proceed in a similar manner

$$\|m^\mu - m^{\mu_0}\|_{X^{1/2-s}(\Omega)} \leq p_1 \|R(b_1, A_h)\|_{\mathcal{L}(X^{-1/2-s}(\Omega), X^{1/2-s}(\Omega))} \|Tr'\|_{\mathcal{L}(X^{-1/4-s}(I), X^{-1/2-s}(\Omega))} \|\eta^\epsilon - \delta\|_{X^{-1/4-s}(I)}.$$

Using (2.31a) we get that $\delta = PTr'\delta$ hence using (2.31b) and (2.32) we obtain

$$\begin{aligned}
&\|m^{\mu_0} - Em^0\|_{X^{1/2-s}(\Omega)} = p_1 \|R(b_1, A_h) Tr'\delta - ER(b_1, A_0) PTr'\delta\|_{X^{1/2-s}(\Omega)} \\
&= p_1 \|R(b_1, A_h)(I - EP)Tr'\delta)\|_{X^{1/2-s}(\Omega)} \leq C\Big(\frac{1}{(b_1 - \lambda_{01,h}^\Omega)^{s/2}} + \frac{1}{b_1 - \lambda_{01,h}^\Omega}\Big) \|(I - EP)Tr'\delta\|_{X^{-1/2-s/2}(\Omega)}.
\end{aligned}$$

Moreover using (2.28) and (2.48) we have

$$\|(I - EP)Tr'\delta\|_{X^{-1/2-s/2}(\Omega)} \leq \|I - EP\|_{\mathcal{L}(X^{-1/2-s/2}(\Omega))} \|Tr'\|_{\mathcal{L}(X^{-1/4-s/2}(I), X^{-1/2-s/2}(\Omega))} \|\delta\|_{X^{-1/4-s/2}(I)} \leq C.$$

Finally to finish the proof of (2.58) observe that

$$\frac{1}{(b_1 - \lambda_{01,h}^\Omega)^{s/2}} + \frac{1}{b_1 - \lambda_{01,h}^\Omega} \leq C \frac{1}{|\lambda_{01,h}^\Omega|^{s/2}}.$$

Using maximum principle for elliptic boundary value problem (2.52) we get that $m^\mu \geq 0$ for $\epsilon > 0$. Then (2.57) implies that $m^{\mu_0} \geq 0$ while $m^0 \geq 0$ follows from (2.58). □

Recall that $\mu = (h, \epsilon) \in (0, 1] \times [0, 1]$. Substituting δ by η^ϵ in (2.1) we get

$$\begin{aligned}
\partial_t u_1^\mu + div(J_h(u_1^\mu)) + b_1 u_1^\mu &= 0, && (t, x) \in \Omega_T && (2.59a) \\
\partial_t u_2^\mu - d\partial_{x_1}^2 u_2^\mu &= f_2(\boldsymbol{u}^\mu), && (t, x) \in (\partial_1\Omega)_T && (2.59b) \\
\partial_t u_3^\mu &= f_3(\boldsymbol{u}^\mu), && (t, x) \in (\partial_1\Omega)_T && (2.59c) \\
\partial_t u_4^\mu &= f_4(\boldsymbol{u}^\mu), && (t, x) \in (\partial_1\Omega)_T && (2.59d) \\
\partial_t u_5^\mu &= f_5(\boldsymbol{u}^\mu), && (t, x) \in (\partial_1\Omega)_T && (2.59e)
\end{aligned}$$

with boundary and initial conditions

$$\begin{aligned}
-J_h(u_1^\mu)\nu &= 0, && (t, x) \in (\partial_0\Omega)_T \\
-J_h(u_1^\mu)\nu &= f_1(\boldsymbol{u}^\mu) + p_1\eta^\epsilon, && (t, x) \in (\partial_1\Omega)_T \qquad (2.60) \\
\partial_{x_1} u_2^\mu &= 0, && (t, x) \in (\partial\partial_1\Omega)_T \\
\boldsymbol{u}^\mu(0, \cdot) &= \boldsymbol{u}_0,
\end{aligned}$$

where

$$\boldsymbol{u}^\mu = (u_1^\mu, u_2^\mu, u_3^\mu, u_4^\mu, u_5^\mu).$$

2.2.4. Definition of M-mild solution

Using Theorem 2.1 we obtain that for $\epsilon \in (0,1]$ system (2.59) has a unique globally defined $\mathcal{X}^\alpha$ solution. Unfortunately due to regularity issues the notion of $\mathcal{X}^\alpha$ solution is insufficient for the case $\epsilon = 0$. Due to the presence of a singular source term any potential solution $u_1^{\mu 0}$ has to be unbounded function of x for any positive time which causes problems in the ODE part of the system. This motivates us to generalize the notion of solution. We rewrite our problem in the new variables so that system (2.59) with singular source term is transformed into system (2.63) with regular sources and low regularity initial data.

Observe that putting

$$\boldsymbol{z}^\mu = (z_1^\mu, z_2^\mu, z_3^\mu, z_4^\mu, z_5^\mu) = M(u_1^\mu - m^\mu, u_2^\mu, u_3^\mu, u_4^\mu, u_5^\mu), \tag{2.61}$$

$$\boldsymbol{z}_0^\mu = (z_{01}^\mu, z_{02}, z_{03}, z_{04}, z_{05}) = M(u_{01} - m^\mu, u_{02}, u_{03}, u_{04}, u_{05}), \tag{2.62}$$

where m^μ was defined in (2.51) and M denotes the following matrix

$$M = \begin{bmatrix} 1,0,0,0,0 \\ 0,1,0,0,0 \\ 0,0,1,0,0 \\ 0,0,1,1,0 \\ 0,0,1,1,1 \end{bmatrix},$$

system (2.59) can be rewritten as

$$\partial_t z_1^\mu + div(J_h(z_1^\mu)) + b_1 z_1^\mu = 0, \qquad (t,x) \in \Omega_T \tag{2.63a}$$

$$\partial_t z_2^\mu - d\partial_{x_1}^2 z_2^\mu = g_2^\mu(\boldsymbol{z}^\mu), \qquad (t,x_1) \in (\partial_1\Omega)_T \tag{2.63b}$$

$$\partial_t z_3^\mu + Tr(m^\mu) z_3^\mu = g_3(\boldsymbol{z}^\mu), \qquad (t,x_1) \in (\partial_1\Omega)_T \tag{2.63c}$$

$$\partial_t z_4^\mu = g_4(\boldsymbol{z}^\mu), \qquad (t,x_1) \in (\partial_1\Omega)_T \tag{2.63d}$$

$$\partial_t z_5^\mu = g_5(\boldsymbol{z}^\mu), \qquad (t,x_1) \in (\partial_1\Omega)_T \tag{2.63e}$$

with boundary and initial conditions

$$\begin{aligned} -J_h(z_1^\mu)\nu &= 0, && (t,x) \in (\partial_0\Omega)_T \\ -J_h(z_1^\mu)\nu &= g_1^\mu(\boldsymbol{z}^\mu), && (t,x_1) \in (\partial_1\Omega)_T \\ \partial_{x_1} z_2^\mu &= 0, && (t,x_1) \in (\partial\partial_1\Omega)_T \\ \boldsymbol{z}^\mu(0,\cdot) &= \boldsymbol{z}_0^\mu, \end{aligned}$$

where

$$\begin{aligned} &g_1^\mu, g_2^\mu : I \times \mathbb{R}^5 \to \mathbb{R}, \ g_3, g_4, g_5 : \mathbb{R}^5 \to \mathbb{R}, \\ &g_1^\mu(\boldsymbol{z}) = -c_1 z_1 + c_2 z_2 - z_1 z_3 + c_4(z_4 - z_3) - (c_1 + z_3) Tr(m^\mu), \\ &g_2^\mu(\boldsymbol{z}) = -b_2 z_2 + c_1 z_1 - c_2 z_2 - c_3 z_2 z_3 + c_5(z_5 - z_4) + c_1 Tr(m^\mu), \\ &g_3(\boldsymbol{z}) = -b_3 z_3 - z_1 z_3 - c_3 z_2 z_3 + c_4(z_4 - z_3) + c_5(z_5 - z_4) + p_3, \\ &g_4(\boldsymbol{z}) = -b_3 z_3 - b_4(z_4 - z_3) - c_3 z_2 z_3 + c_5(z_5 - z_4) + p_3, \\ &g_5(\boldsymbol{z}) = -b_3 z_3 - b_4(z_4 - z_3) - b_5(z_5 - z_4) + p_3. \end{aligned}$$

Assume that:

$$d, \boldsymbol{b} > 0,\ \boldsymbol{c}, \boldsymbol{p} \geq 0, \tag{2.64a}$$

$$2 < p < \infty,\ 0 < \theta < \min\left\{\frac{1}{16}, \frac{1}{2p}\right\}, \tag{2.64b}$$

$$0 \leq \boldsymbol{u}_0 = (u_{01}, \ldots, u_{05}) \in X^{1/2+\theta}(\Omega) \times X^{1/2}(I) \times \{L_\infty(I)\}^3. \tag{2.64c}$$

Define Banach spaces

$$\begin{aligned}
\boldsymbol{Z}_- &= Z_{1-} \times Z_{2-} \times Z_{3-} \times Z_{4-} \times Z_{5-} = X^{-1/4-\theta}(\Omega) \times X(I) \times L_p(I) \times L_p(I) \times L_p(I),\\
\boldsymbol{Z} &= Z_1 \times Z_2 \times Z_3 \times Z_4 \times Z_5 = X^{1/2-\theta}(\Omega) \times X^{1/2}(I) \times L_p(I) \times L_p(I) \times L_p(I),\\
\boldsymbol{Z}_+ &= Z_{1+} \times Z_{2+} \times Z_{3+} \times Z_{4+} \times Z_{5+} = X^{1/2+\theta}(\Omega) \times X^{1/2}(I) \times L_\infty(I) \times L_\infty(I) \times L_\infty(I).
\end{aligned}$$

For $\boldsymbol{z} \in \boldsymbol{Z}_+$ put

$$\begin{aligned}
G_1^\mu(\boldsymbol{z}) &= Tr'(g_1^\mu(Tr(z_1), z_2, z_3, z_4, z_5)),\\
G_2^\mu(\boldsymbol{z}) &= g_2^\mu(Tr(z_1), z_2, z_3, z_4, z_5),\\
G_i(\boldsymbol{z}) &= g_i(Tr(z_1), z_2, z_3, z_4, z_5),\ i \in \{3, 4, 5\}.
\end{aligned}$$

Definition 1. *Fix $\mu = (h, \epsilon) \in (0, 1] \times [0, 1]$ and let $\boldsymbol{z}^\mu, \boldsymbol{z}_0^\mu$ be related with $\boldsymbol{u}^\mu, \boldsymbol{u}_0$ by equations* (2.61) *and* (2.62)*. We define $\boldsymbol{u}^\mu$ as a **M-mild** solution of system* (2.59) *on $[0, T)$ if the following three conditions are satisfied*

1. *Assumptions* (2.64) *hold.*
2. *The function $\boldsymbol{z}^\mu$ has the following regularity*

$$z_1^\mu \in C([0, T), Z_1),\ t^{2\theta} z_1^\mu \in L_\infty(0, T'; Z_{1+}) \text{ for } T' < T, \tag{2.65a}$$

$$z_2^\mu \in C([0, T), Z_2), \tag{2.65b}$$

$$z_3^\mu \in C([0, T), Z_3) \cap L_\infty(0, T; Z_{3+}), \tag{2.65c}$$

$$z_i^\mu \in C([0, T), Z_{i+}),\ i \in \{4, 5\}. \tag{2.65d}$$

3. *For every $t \in [0, T)$ the following Duhamel formulas hold*

$$z_1^\mu(t) = e^{t(A_h - b_1)} z_{01}^\mu + \int_0^t e^{(t-\tau)(A_h - b_1)} G_1^\mu(\boldsymbol{z}^\mu(\tau)) d\tau, \tag{2.66a}$$

$$z_2^\mu(t) = e^{tdA_0} z_{02} + \int_0^t e^{(t-\tau)dA_0} G_2^\mu(\boldsymbol{z}^\mu(\tau)) d\tau, \tag{2.66b}$$

$$z_3^\mu(t) = e^{-tTr(m^\mu)} z_{03} + \int_0^t e^{-(t-\tau)Tr(m^\mu)} G_3(\boldsymbol{z}^\mu(\tau)) d\tau, \tag{2.66c}$$

$$z_i^\mu(t) = z_{0i} + \int_0^t G_i(\boldsymbol{z}^\mu(\tau)) d\tau,\ i \in \{4, 5\}. \tag{2.66d}$$

Concerning regularity of M-mild solutions we have the following

Remark 1. *If $\boldsymbol{u}^\mu$ is a M-mild solution of system* (2.59) *then*

$$u_1^\mu \in C([0, T), W_2^{1-2\theta}(\Omega)),\ t^{2\theta} u_1^\mu \in L_\infty(0, T'; W_p^1(\Omega)) \text{ for } 1 \leq p < 2, \tag{2.67a}$$

$$u_2^\mu \in C([0, T), W_2^1(I)), \tag{2.67b}$$

$$u_i \in C([0, T), L_p(I)) \cap L_\infty(0, T'; L_\infty(I)) \text{ for } i \in \{3, 4, 5\}. \tag{2.67c}$$

Proof. Using Lemma 2.5 we obtain that $Z_1 = W_2^{1-2\theta}(\Omega), Z_{1+} = W_2^{1+2\theta}(\Omega), Z_2 = W_2^1(I)$. Using Lemma 2.14 $m^\mu \in W_p^1(\Omega) \cap W_2^{1-2\theta}(\Omega)$ for $1 \le p < 2$. Thus using (2.61) and (2.65) we obtain that

$$\begin{aligned} &u_1^\mu = z_1^\mu + m^\mu \subset C([0,T), W_2^{1-2\theta}(\Omega)), \\ &t^{2\theta} u_1^\mu = t^{2\theta} z_1^\mu + t^{2\theta} m^\mu \subset L_\infty(0,T'; W_2^1(\Omega)) + L_\infty(0,T'; W_p^1(\Omega)) \subset L_\infty(0,T'; W_p^1(\Omega)). \end{aligned}$$

Similarly one shows (2.67b) and (2.67c). □

2.2.5. The main results of Section 2.2

We first prove that for $\epsilon > 0$ system (2.59) has a unique M-mild solution and study its convergence as $\epsilon \to 0$.

Theorem 2.2. *Assume* (2.64). *Then*

1. *For every $\mu = (h,\epsilon) \in (0,1] \times (0,1]$, $0 < T \le \infty$ system* (2.59) *has a unique M-mild solution $\boldsymbol{u}^\mu$ defined on $[0,T)$. This solution is nonnegative and is also $\mathcal{X}^\alpha$ solution.*
2. *For every $h \in (0,1]$, $\epsilon = 0$, $0 < T \le \infty$ system* (2.59) *has a unique M-mild solution $\boldsymbol{u}^{\mu_0}$ defined on $[0,T)$. The solution is nonnegative. Moreover if $T = \infty$ then for every $0 < T' < \infty$ the following convergence holds*

$$\lim_{\epsilon \to 0^+} \left\{ \sum_{i=1}^{5} \|u_i^\mu - u_i^{\mu_0}\|_{L_\infty(0,T';Z_i)} \right\} = 0. \tag{2.68}$$

Next we consider the dimension reduction problem. We show that for $\epsilon = 0$ the solution of system (2.59) converges to the solution of an appropriate one dimensional problem when $h \to 0$.

Theorem 2.3. *Let $\boldsymbol{u}^{\mu_0}$ be the unique, global in time M-mild solution of system* (2.59) *for $h \in (0,1]$ and $\epsilon = 0$. Then for every $0 < T < \infty$*

$$\lim_{h \to 0^+} \left\{ \|t^{2\theta}(z_1^{\mu_0} - z_1^0)\|_{L_\infty(0,T;Z_{1+})} + \sum_{i=2}^{5} \|u_i^{\mu_0} - u_i^0\|_{L_\infty(0,T;Z_i)} \right\} = 0, \tag{2.69}$$

where $z_1^{\mu_0} = u_1^{\mu_0} - m^{\mu_0}$, $z_1^0 = E(u_1^0 - m^0)$ and $\boldsymbol{u}^0 = (u_1^0, \dots, u_5^0)$ is the unique classical solution of

$$\begin{aligned} \partial_t u_1 - \partial_{x_1}^2 u_1 + b_1 u_1 &= f_1(\boldsymbol{u}) + p_1 \delta & (t,x_1) &\in I_\infty & \text{(2.70a)} \\ \partial_t u_2 - d\partial_{x_1}^2 u_2 &= f_2(\boldsymbol{u}), & (t,x_1) &\in I_\infty & \text{(2.70b)} \\ \partial_t u_3 &= f_3(\boldsymbol{u}), & (t,x_1) &\in I_\infty & \text{(2.70c)} \\ \partial_t u_4 &= f_4(\boldsymbol{u}), & (t,x_1) &\in I_\infty & \text{(2.70d)} \\ \partial_t u_5 &= f_5(\boldsymbol{u}), & (t,x_1) &\in I_\infty & \text{(2.70e)} \end{aligned}$$

with boundary and initial conditions

$$\begin{aligned} &\partial_{x_1} u_1 = \partial_{x_1} u_2 = 0, && (t,x_1) \in (\partial I)_T \\ &\boldsymbol{u}(0,.) = \boldsymbol{u}_0^0 = [Pu_{01}, u_{02}, u_{03}, u_{04}, u_{05}]. \end{aligned}$$

Remark 2. *Global well-posedness of system* (2.70) *is established in Section 2.4.*

2.2.6. Proof of Theorem 2.2

Step 1 - estimates for G_i's.

Lemma 2.15. *For $\boldsymbol{z}, \boldsymbol{z}' \in \boldsymbol{Z}_+$, $\mu \in \mathbb{Z}_+ \times [0,1]$ the following estimates hold*

$$\sum_{i=1}^{2} \|G_i^\mu(\boldsymbol{z})\|_{Z_{i-}} + \sum_{i=3}^{5} \|G_i(\boldsymbol{z})\|_{Z_{i+}} \leq C\Big((1 + \sum_{i=1}^{2} \|z_i\|_{Z_{i+}})(1 + \|z_3\|_{Z_{3+}}) + \sum_{i=4}^{5} \|z_i\|_{Z_{i+}}\Big),$$

$$\sum_{i=1}^{2} \|G_i^\mu(\boldsymbol{z}) - G_i^\mu(\boldsymbol{z}')\|_{Z_{i-}} + \sum_{i=3}^{5} \|G_i(\boldsymbol{z}) - G_i(\boldsymbol{z}')\|_{Z_i} \leq C\Big((1 + \|z_3\|_{Z_{3+}} + \|z_3'\|_{Z_{3+}}) \sum_{i=1}^{2} \|z_i - z_i'\|_{Z_i}$$
$$+ (1 + \sum_{i=1}^{2} (\|z_i\|_{Z_{i+}} + \|z_i'\|_{Z_{i+}}))\|z_3 - z_3'\|_{Z_3} + \sum_{i=4}^{5} \|z_i - z_i'\|_{Z_i}\Big),$$

$$\sum_{i=1}^{2} \|G_i^\mu(\boldsymbol{z}) - G_i^\mu(\boldsymbol{z}')\|_{Z_{i-}} + \sum_{i=3}^{5} \|G_i(\boldsymbol{z}) - G_i(\boldsymbol{z}')\|_{Z_{i+}} \leq C\Big((1 + \|z_3\|_{Z_{3+}} + \|z_3'\|_{Z_{3+}}) \sum_{i=1}^{2} \|z_i - z_i'\|_{Z_{i+}}$$
$$+ (1 + \sum_{i=1}^{2} (\|z_i\|_{Z_{i+}} + \|z_i'\|_{Z_{i+}}))\|z_3 - z_3'\|_{Z_{3+}} + \sum_{i=4}^{5} \|z_i - z_i'\|_{Z_{i+}}\Big),$$

$$\sum_{i=1}^{2} \|G_i^\mu(\boldsymbol{z}) - G_i^{\mu_0}(\boldsymbol{z})\|_{Z_{i-}} \leq C(1 + \|z_3\|_{Z_{3+}})\|\eta^\epsilon - \delta\|_{X^{-1/4-\theta}(I)},$$

where C does not depend on μ.

Proof. We will prove inequalities involving G_1^μ and G_3. Inequalities involving G_2^μ, G_4 and G_5 can be derived analogously. Using condition (2.64b) and Lemma 2.6 we get that $X^{1/4-\theta}(I) \subset L_p(I)$, from which $Tr \in \mathcal{L}(Z_1, L_p(I))$ by Lemma 2.9. Using the above observation and Hölder's inequality we estimate

$$\|G_1^\mu(\boldsymbol{z})\|_{Z_{1-}} = \|Tr' g_1^\mu(Tr(z_1), z_2, z_3, z_4, z_5)\|_{X^{-1/4-\theta}(\Omega)} \leq C\|g_1^\mu(Tr(z_1), z_2, z_3, z_4, z_5)\|_{L_2(I)}$$
$$\leq C\Big(\|Tr(z_1)\|_{L_2(I)} + \|z_2\|_{L_2(I)} + \|Tr(z_1)\|_{L_2(I)}\|z_3\|_\infty + \|z_4\|_{L_2(I)} + \|z_3\|_{L_2(I)}$$
$$+ (1 + \|z_3\|_\infty)\|Tr(m^\mu)\|_{L_2(I)}\Big) \leq C\Big((1 + \sum_{i=1}^{2} \|z_i\|_{Z_{i+}})(1 + \|z_3\|_{Z_{3+}}) + \sum_{i=4}^{5} \|z_i\|_{Z_{i+}}\Big),$$

$$\|G_3(\boldsymbol{z})\|_{Z_{3+}} \leq C\Big(\|z_3\|_\infty + \|Tr(z_1)\|_\infty\|z_3\|_\infty + \|z_2\|_\infty\|z_3\|_\infty + \sum_{i=3}^{5} \|z_i\|_\infty + 1\Big)$$
$$\leq C\Big((1 + \sum_{i=1}^{2} \|z_i\|_{Z_{i+}})(1 + \|z_3\|_{Z_{3+}}) + \sum_{i=4}^{5} \|z_i\|_{Z_{i+}}\Big),$$

$$\|G_1^\mu(\boldsymbol{z}) - G_1^\mu(\boldsymbol{z}')\|_{Z_{1-}} \leq C\|g_1^\mu(Tr(z_1), z_2, z_3, z_4, z_5) - g_1^\mu(Tr(z_1'), z_2', z_3', z_4', z_5')\|_{L_2(I)}$$
$$\leq C\Big(\|Tr(z_1 - z_1')\|_{L_2(I)} + \|z_2 - z_2'\|_{L_2(I)} + \|Tr(z_1 - z_1')\|_{L_2(I)}\|z_3\|_\infty + \|z_3 - z_3'\|_{L_2(I)}\|Tr(z_1')\|_\infty$$
$$+ \|z_4 - z_4'\|_{L_2(I)} + \|z_3 - z_3'\|_{L_2(I)} + \|Tr(m^\mu)\|_{L_{\frac{2p}{p-2}}(I)}\|z_3 - z_3'\|_{L_p(I)}\Big) \leq C\Big((1 + \|z_3\|_{Z_{3+}}$$
$$+ \|z_3'\|_{Z_{3+}}) \sum_{i=1}^{2} \|z_i - z_i'\|_{Z_i} + (1 + \sum_{i=1}^{2} (\|z_i\|_{Z_{i+}} + \|z_i'\|_{Z_{i+}}))\|z_3 - z_3'\|_{Z_3} + \sum_{i=4}^{5} \|z_i - z_i'\|_{Z_i}\Big),$$

$$\|G_3(\boldsymbol{z})-G_3(\boldsymbol{z}')\|_{Z_3}\le C\Big(\|z_3-z_3'\|_{L_p(I)}+\|Tr(z_1-z_1')\|_{L_p(I)}\|z_3\|_\infty+\|z_3-z_3'\|_{L_p(I)}\|Tr(z_1')\|_\infty$$
$$+\|z_2-z_2'\|_{L_p(I)}\|z_3\|_\infty+\|z_3-z_3'\|_{L_p(I)}\|z_2'\|_\infty+\sum_{i=4}^5\|z_i-z_i'\|_{L_p(I)}\Big)\le C\Big((1+\|z_3\|_{Z_{3+}}$$
$$+\|z_3'\|_{Z_{3+}})\sum_{i=1}^2\|z_i-z_i'\|_{Z_i}+(1+\sum_{i=1}^2(\|z_i\|_{Z_{i+}}+\|z_i'\|_{Z_{i+}}))\|z_3-z_3'\|_{Z_3}+\sum_{i=4}^5\|z_i-z_i'\|_{Z_i}\Big),$$
$$\|G_3(\boldsymbol{z})-G_3(\boldsymbol{z}')\|_{Z_{3+}}\le C\Big(\|z_3-z_3'\|_\infty+\|Tr(z_1-z_1')\|_\infty\|z_3\|_\infty+\|z_3-z_3'\|_\infty\|Tr(z_1')\|_\infty$$
$$+\|z_2-z_2'\|_\infty\|z_3\|_\infty+\|z_3-z_3'\|_\infty\|z_2'\|_\infty+\sum_{i=4}^5\|z_i-z_i'\|_\infty\Big)\le C\Big((1+\|z_3\|_{Z_{3+}}$$
$$+\|z_3'\|_{Z_{3+}})\sum_{i=1}^2\|z_i-z_i'\|_{Z_{i+}}+(1+\sum_{i=1}^2(\|z_i\|_{Z_{i+}}+\|z_i'\|_{Z_{i+}}))\|z_3-z_3'\|_{Z_{3+}}+\sum_{i=4}^5\|z_i-z_i'\|_{Z_{i+}}\Big),$$
$$\|G_1^\mu(\boldsymbol{z})-G_1^{\mu_0}(\boldsymbol{z})\|_{Z_{1-}}\le C\|(c_1+z_3)Tr(m^\mu-m^{\mu_0})\|_{L_2(I)}\le C(1+\|z_3\|_\infty)\|Tr(m^\mu-m^{\mu_0})\|_{L_2(I)}$$
$$\le C(1+\|z_3\|_{Z_{3+}})\|\eta^\epsilon-\delta\|_{X^{-1/4-\theta}(I)}.$$

□

Step 2 - uniqueness of M-mild solution to system (2.59)**.**

Assume that $\boldsymbol{u},\boldsymbol{u}'$ are two M-mild solutions of system (2.59) on $[0,T), 0<T\le\infty$, with the same initial condition. Let $\boldsymbol{z},\boldsymbol{z}'$ be related with $\boldsymbol{u},\boldsymbol{u}'$ by (2.61), (2.62). Fix $T'<T$.

For $t\in(0,T')$ denote $f(t)=\sum_{i=1}^5\|z_i(t)-z_i'(t)\|_{Z_i}$. Put

$$K_1(T')=\|t^{2\theta}z_1\|_{L_\infty(0,T';Z_{1+})}+\|t^{2\theta}z_1'\|_{L_\infty(0,T';Z_{1+})}$$
$$,K_i(T')=\|z_i\|_{L_\infty(0,T';Z_{i+})}+\|z_i'\|_{L_\infty(0,T';Z_{i+})},\ i=2,3$$
$$\overline{K}(T')=\max\{K_1(T'),K_2(T'),K_3(T')\}$$

Using condition (2.65) we get that $f\in L_\infty(0,T')$ and $\overline{K}(T')<\infty$. Owing to Lemma 2.15 we obtain that for $t\in(0,T')$ there is

$$\sum_{i=1}^2\|G_i^\mu(\boldsymbol{z}(t))-G_i^\mu(\boldsymbol{z}'(t))\|_{Z_{i-}}+\sum_{i=3}^5\|G_i(\boldsymbol{z}(t))-G_i(\boldsymbol{z}'(t))\|_{Z_i}\le C\Big\{\Big(1+\|z_3(t)\|_{Z_{3+}}$$
$$+\|z_3'(t)\|_{Z_{3+}}\Big)\sum_{i=1}^2\|z_i(t)-z_i'(t)\|_{Z_i}+\Big(1+\sum_{i=1}^2(\|z_i(t)\|_{Z_{i+}}+\|z_i'(t)\|_{Z_{i+}})\Big)\|z_3(t)-z_3'(t)\|_{Z_3}$$
$$+\sum_{i=4}^5\|z_i(t)-z_i'(t)\|_{Z_i}\Big\}\le C\Big\{\Big(1+K_3(T')\Big)\sum_{i=1}^2\|z_i(t)-z_i'(t)\|_{Z_i}+\Big(1+\frac{1}{t^{2\theta}}K_1(T')$$
$$+K_2(T')\Big)\|z_3(t)-z_3'(t)\|_{Z_3}+\sum_{i=4}^5\|z_i(t)-z_i'(t)\|_{Z_i}\Big\}\le C(1+\overline{K}(T'))\Big(1+\frac{1}{t^{2\theta}}\Big)f(t).$$

Using Lemma 2.7 and owing to the fact that $\boldsymbol{z}, \boldsymbol{z}'$ satisfy (2.66) we obtain for $t \in (0, T')$

$$f(t) \le \int_0^t \Big\{\|e^{(t-\tau)A_h}\|_{\mathcal{L}(Z_{1-},Z_1)}\|G_1^\mu(\boldsymbol{z}(\tau)) - G_1^\mu(\boldsymbol{z}'(\tau))\|_{Z_{1-}}$$
$$+ \|e^{(t-\tau)dA_0}\|_{\mathcal{L}(Z_{2-},Z_2)}\|G_2^\mu(\boldsymbol{z}(\tau)) - G_2^\mu(\boldsymbol{z}'(\tau))\|_{Z_{2-}} + \|e^{-(t-\tau)Tr(m^\mu)}\|_{Z_{3+}}\|G_3(\boldsymbol{z}(\tau)) - G_3(\boldsymbol{z}'(\tau))\|_{Z_3}$$
$$+ \sum_{i=4}^{5}\|G_i(\boldsymbol{z}(\tau)) - G_i(\boldsymbol{z}'(\tau))\|_{Z_i}\Big\}d\tau \le C(1+\overline{K}(T'))\int_0^t \Big(1 + \frac{1}{(t-\tau)^{3/4}} + \frac{1}{(t-\tau)^{1/2}}\Big)\Big(1+\frac{1}{\tau^{2\theta}}\Big)f(\tau)d\tau$$
$$\le C(1+\overline{K}(T'))(1+(T')^{3/4+2\theta})\int_0^t \frac{f(\tau)}{\tau^{2\theta}(t-\tau)^{3/4}}d\tau.$$

Finally using Lemma (1.2) (see (2.64b)) we conclude that $f \equiv 0$ on $(0, T')$ hence $\boldsymbol{u} \equiv \boldsymbol{u}'$.

Step 3 - existence of global solutions for $\epsilon > 0$ and μ-independence of bounds.

Using Theorem 2.1 with $s = 1/2 + \theta,\ s' = 1/2 + 2\theta,\ \omega = \eta^\epsilon$ we obtain that system (2.59) has for $\mu \in (0,1] \times (0,1]$ a unique global $\mathcal{X}^\alpha$ solution $\boldsymbol{u}^\mu$ which is nonnegative. Let $\boldsymbol{z}^\mu, \boldsymbol{z}_0^\mu$ be related with $\boldsymbol{u}^\mu, \boldsymbol{u}_0$ by (2.61) and (2.62). It is easy to see that $\boldsymbol{z}^\mu$ satisfies formulas (2.66) from which one concludes that $\boldsymbol{u}^\mu$ is also a M-mild solution of system (2.59). Using estimate (2.44) from Theorem 2.1 we get that

$$M_3 = \sup_{\mu\in(0,1]\times(0,1]} \sum_{i=3}^{5}\|z_i^\mu\|_{L_\infty(0,\infty;Z_{i+})} \tag{2.71}$$

is finite. Fix $T < \infty$ and for $0 < t < T$ denote $g(t) = 1 + t^{2\theta}\|z_1^\mu(t)\|_{Z_{1+}} + \|z_2^\mu(t)\|_{Z_{2+}}$. Owing to Lemma 2.15 we obtain

$$\sum_{i=1}^{2}\|G_i^\mu(\boldsymbol{z}^\mu(t))\|_{Z_{i-}} \le C\Big((1 + \|z_1^\mu(t)\|_{Z_{1+}} + \|z_2^\mu(t)\|_{Z_{2+}})(1 + \|z_3^\mu(t)\|_{Z_{3+}}) + \|z_4^\mu(t)\|_{Z_{4+}} + \|z_5^\mu(t)\|_{Z_{5+}}\Big)$$
$$\le C(1+M_3)(1 + \|z_1^\mu(t)\|_{Z_{1+}} + \|z_2^\mu(t)\|_{Z_{2+}}) \le C(1+M_3)\Big(1+\frac{1}{t^{2\theta}}\Big)g(t).$$

Using (2.66) and Lemma 2.7 we estimate (recall that $Z_2 = Z_{2+}$)

$$g(t) \le 1 + t^{2\theta}\|e^{tA_h}\|_{\mathcal{L}(Z_1,Z_{1+})}\|z_{01}^\mu\|_{Z_1} + \|e^{tdA_0}\|_{\mathcal{L}(Z_2)}\|z_{02}\|_{Z_2}$$
$$+ \int_0^t \Big(t^{2\theta}\|e^{(t-\tau)A_h}\|_{\mathcal{L}(Z_{1-},Z_{1+})}\|G_1^\mu(\boldsymbol{z}^\mu(\tau))\|_{Z_{1-}} + \|e^{(t-\tau)dA_0}\|_{\mathcal{L}(Z_{2-},Z_2)}\|G_2^\mu(\boldsymbol{z}^\mu(\tau))\|_{Z_{2-}}\Big)d\tau$$
$$\le C(1+t^{2\theta})\Big\{1 + (1+M_3)\int_0^t \Big(1 + \frac{1}{(t-\tau)^{3/4+2\theta}} + \frac{1}{(t-\tau)^{1/2}}\Big)\Big(1+\frac{1}{\tau^{2\theta}}\Big)g(\tau)d\tau\Big\}$$
$$\le C(1+T^{2\theta})\Big\{1 + (1+M_3)(1+T^{3/4+4\theta})\int_0^t \frac{g(\tau)}{(t-\tau)^{3/4+2\theta}\tau^{2\theta}}d\tau\Big\}$$
$$\le C(1+T^{2\theta}) + C(1+M_3)(1+T^{3/4+6\theta})\int_0^t \frac{g(\tau)}{(t-\tau)^{3/4+2\theta}\tau^{2\theta}}d\tau.$$

Thus using Lemma 1.2 (see (2.64b)) we get that for every $T > 0$

$$M_1(T) = \sup_{\mu\in(0,1]\times(0,1]}\|t^{2\theta}z_1^\mu\|_{L_\infty(0,T;Z_{1+})} \text{ and } M_2(T) = \sup_{\mu\in(0,1]\times(0,1]}\|z_2^\mu\|_{L_\infty(0,T;Z_{2+})} \tag{2.72}$$

are finite.

Step 4 - existence of local M-mild solutions for $\epsilon = 0$.

To prove existence of local M-mild solutions we use the contraction mapping principle in appropriate weighted in time spaces. For $R, T > 0$ define

$$
\begin{aligned}
&\mathcal{Z}_1 = \{z_1 \in C([0,T], Z_1) : \ \|z_1\|_{L_\infty(0,T;Z_1)} + \|t^{2\theta} z_1\|_{L_\infty(0,T;Z_{1+})} \le R\},\\
&d_{\mathcal{Z}_1}(z_1, z_1') = \|z_1 - z_1'\|_{L_\infty(0,T;Z_1)} + \|t^{2\theta}(z_1 - z_1')\|_{L_\infty(0,T;Z_{1+})},\\
&\mathcal{Z}_2 = \{z_2 \in C([0,T], Z_2) : \|z_2\|_{L_\infty(0,T;Z_2)} \le R\}, \ d_{\mathcal{Z}_2}(z_2, z_2') = \|z_2 - z_2'\|_{L_\infty(0,T;Z_2)},\\
&\mathcal{Z}_i = \{z_i \in C([0,T], Z_i) : \|z_i\|_{L_\infty(0,T;Z_{i+})} \le R\}, \ d_{\mathcal{Z}_i}(z_i, z_i') = \|z_i - z_i'\|_{L_\infty(0,T;Z_{i+})}, \ i = 3,4,5\\
&\boldsymbol{\mathcal{Z}} = \mathcal{Z}_1 \times \ldots \times \mathcal{Z}_5, \ d_{\boldsymbol{\mathcal{Z}}}(\boldsymbol{z}, \boldsymbol{z}') = \sum_{i=1}^{5} d_{\mathcal{Z}_i}(z_i, z_i').
\end{aligned}
$$

Observe that $\mathcal{Z}_i$ and $\boldsymbol{\mathcal{Z}}$ are complete metric spaces.

For $\boldsymbol{z} \in \boldsymbol{\mathcal{Z}}, \ \mu = (h, \epsilon) \in (0,1] \times [0,1]$ define

$$
\begin{aligned}
[\Phi_1^\mu(\boldsymbol{z})](t) &= e^{t(A_h - b_1)} z_{01}^\mu + \int_0^t e^{(t-\tau)(A_h - b_1)} G_1^\mu(\boldsymbol{z}(\tau)) d\tau,\\
[\Phi_2^\mu(\boldsymbol{z})](t) &= e^{tdA_0} z_{02} + \int_0^t e^{(t-\tau)dA_0} G_2^\mu(\boldsymbol{z}(\tau)) d\tau,\\
[\Phi_3^\mu(\boldsymbol{z})](t) &= e^{-tTr(m^\mu)} z_{03} + \int_0^t e^{-(t-\tau)Tr(m^\mu)} G_3(\boldsymbol{z}(\tau)) d\tau,\\
[\Phi_i(\boldsymbol{z})](t) &= z_{0i} + \int_0^t G_i(\boldsymbol{z}(\tau)) d\tau, \ i = 4,5\\
\boldsymbol{\Phi}^\mu &= (\Phi_1^\mu, \Phi_2^\mu, \Phi_3^\mu, \Phi_4, \Phi_5).
\end{aligned}
$$

Lemma 2.16. *There exist $R, T > 0$ such that for every $\mu \in (0,1] \times [0,1]$ the map $\boldsymbol{\Phi}^\mu$ maps $\boldsymbol{\mathcal{Z}}$ into itself and satisfies for every $\boldsymbol{z}, \boldsymbol{z}' \in \boldsymbol{\mathcal{Z}}$ the following condition*

$$
d_{\boldsymbol{\mathcal{Z}}}(\boldsymbol{\Phi}^\mu(\boldsymbol{z}), \boldsymbol{\Phi}^\mu(\boldsymbol{z}')) \le (1/2) d_{\boldsymbol{\mathcal{Z}}}(\boldsymbol{z}, \boldsymbol{z}'). \tag{2.73}
$$

Proof. Fix $R \ge 1 \ge T > 0$. Using Lemma 2.15 we have for $t \in [0,T]$ and $\boldsymbol{z}, \boldsymbol{z}' \in \boldsymbol{\mathcal{Z}}$

$$
\sum_{i=1}^{2} \|G_i^\mu(\boldsymbol{z}(t))\|_{Z_{i-}} + \sum_{i=3}^{5} \|G_i(\boldsymbol{z}(t))\|_{Z_{i+}} \le CR^2 \Big(1 + \frac{1}{t^{2\theta}}\Big) \tag{2.74}
$$

$$
\sum_{i=1}^{2} \|G_i^\mu(\boldsymbol{z}(t)) - G_i^\mu(\boldsymbol{z}'(t))\|_{Z_{i-}} + \sum_{i=3}^{5} \|G_i(\boldsymbol{z}(t)) - G_i(\boldsymbol{z}'(t))\|_{Z_{i+}} \le CR \Big(1 + \frac{1}{t^{2\theta}}\Big) d_{\boldsymbol{\mathcal{Z}}}(\boldsymbol{z}, \boldsymbol{z}'). \tag{2.75}
$$

Using (2.74) and Lemma 2.7 we estimate

$$t^{2\theta}\|[\Phi_1^\mu(\boldsymbol{z})](t)\|_{Z_{1+}} + \|[\Phi_1^\mu(\boldsymbol{z})](t)\|_{Z_1} + \sum_{i=2}^5 \|[\Phi_i^\mu(\boldsymbol{z})](t)\|_{Z_{i+}} \le (t^{2\theta}\|e^{tA_h}\|_{\mathcal{L}(Z_1,Z_{1+})} + \|e^{tA_h}\|_{\mathcal{L}(Z_1)})\|z_{01}^\mu\|_{Z_1}$$

$$+ \|e^{tdA_0}\|_{\mathcal{L}(Z_2)}\|z_{02}\|_{Z_2} + \|e^{-tTr(m^\mu)}\|_{Z_{3+}}\|z_{03}\|_{Z_{3+}} + \sum_{i=4}^5 \|z_{0i}\|_{Z_{i+}} + \int_0^t \Big\{\Big(t^{2\theta}\|e^{(t-\tau)A_h}\|_{\mathcal{L}(Z_{1-},Z_{1+})}$$

$$+ \|e^{(t-\tau)A_h}\|_{\mathcal{L}(Z_{1-},Z_1)}\Big)\|G_1^\mu(\boldsymbol{z}(\tau))\|_{Z_{1-}} + \|e^{(t-\tau)dA_0}\|_{\mathcal{L}(Z_{2-},Z_2)}\|G_2^\mu(\boldsymbol{z}(\tau))\|_{Z_{2-}}$$

$$+ \|e^{-(t-\tau)Tr(m^\mu)}\|_{Z_{3+}}\|G_3(\boldsymbol{z}(\tau))\|_{Z_{3+}} + \sum_{i=4}^5 \|G_i(\boldsymbol{z}(\tau))\|_{Z_{i+}}\Big\}d\tau \le C(t^{2\theta}+1)\Big\{\|z_{01}^\mu\|_{Z_1} + \|z_{02}\|_{Z_2}$$

$$+ \sum_{i=3}^5 \|z_{0i}\|_{Z_{i+}} + R^2\int_0^t \Big(1 + \frac{1}{(t-\tau)^{3/4+2\theta}} + \frac{1}{(t-\tau)^{3/4}} + \frac{1}{(t-\tau)^{1/2}}\Big)\Big(1+\frac{1}{\tau^{2\theta}}\Big)d\tau\Big\}$$

$$\le C\Big\{\|z_{01}^\mu\|_{Z_1} + \|z_{02}\|_{Z_2} + \sum_{i=3}^5 \|z_{0i}\|_{Z_{i+}} + R^2\int_0^t \frac{1}{(t-\tau)^{3/4+2\theta}\tau^{2\theta}}d\tau\Big\} \le C(\|z_{01}^\mu\|_{Z_1} + \|z_{02}\|_{Z_2}$$

$$+ \sum_{i=3}^5 \|z_{0i}\|_{Z_{i+}}) + CR^2T^{1/4-4\theta}.$$

Taking R, T such that $R \ge \max\{1, 2C(\|z_{01}^\mu\|_{Z_1} + \|z_{02}\|_{Z_2} + \sum_{i=3}^5 \|z_{0i}\|_{Z_{i+}})\}$ and $T \le \min\{1, (2CR)^{4/(16\theta-1)}\}$ we obtain

$$t^{2\theta}\|[\Phi_1^\mu(\boldsymbol{z})](t)\|_{Z_{1+}} + \|[\Phi_1^\mu(\boldsymbol{z})](t)\|_{Z_1} + \sum_{i=2}^5 \|[\Phi_i^\mu(\boldsymbol{z})](t)\|_{Z_{i+}} \le R/2 + R/2 = R$$

which proves that $\boldsymbol{\Phi}^\mu$ maps $\boldsymbol{\mathcal{Z}}$ into itself. Using (2.75) we prove analogously that condition (2.73) holds after making T smaller if needed. □

We obtain from Lemma 2.16 that the map $\boldsymbol{\Phi}^\mu : \boldsymbol{\mathcal{Z}} \to \boldsymbol{\mathcal{Z}}$ satisfies, for certain R, T which are independent of μ, the assumptions of the contraction mapping principle. We conclude that system (2.59) has for $\epsilon = 0$ a unique maximally defined M-mild solution $\boldsymbol{u}^{\mu_0}$ defined on $[0, T_{\max}^h)$, where $T^* := \inf\{T_{\max}^h : h \in (0,1]\} > 0$.

Step 5 - For any fixed $h \in (0,1]$: $\boldsymbol{u}^\mu$ converges to $\boldsymbol{u}^{\mu_0}$ as $\epsilon \to 0$. Moreover $T_{\max}^h = \infty$.

Fix $T < T_{\max}^h$ and for $0 < t < T$ denote: $f^\mu(t) = \sum_{i=1}^5 \|z_i^\mu(t) - z_i^{\mu_0}(t)\|_{Z_i}$. Put

$$K_1^h(T) = \sup_{\epsilon\in[0,1]} \|t^{2\theta} z_1^\mu\|_{L_\infty(0,T;Z_{1+})},\ K_i^h(T) = \sup_{\epsilon\in[0,1]} \|z_i^\mu\|_{L_\infty(0,T;Z_{i+})},\ i = 2,3$$

Observe that due to (2.71),(2.72) $K_i^h(T)$ are finite. Denote $\overline{K}^h(T) = \max\{K_1^h(T), K_2^h(T), K_3^h(T)\}$. Using Lemma 2.15 we have for $0 < t < T$

$$\sum_{i=1}^2 \|G_i^\mu(\boldsymbol{z}^\mu(t)) - G_i^\mu(\boldsymbol{z}^{\mu_0}(t))\|_{Z_{i-}} + \sum_{i=3}^5 \|G_i(\boldsymbol{z}^\mu(t)) - G_i(\boldsymbol{z}^{\mu_0}(t))\|_{Z_i} \le C(1+\overline{K}^h(T))\Big(1+\frac{1}{t^{2\theta}}\Big)f^\mu(t)$$

$$\sum_{i=1}^2 \|G_i^\mu(\boldsymbol{z}^{\mu_0}(t)) - G_i^{\mu_0}(\boldsymbol{z}^{\mu_0}(t))\|_{Z_{i-}} \le C(1+\overline{K}^h(T))\|\eta^\epsilon - \delta\|_{X^{-1/4-\theta}(I)}$$

$$\|G_3(\boldsymbol{z}^{\mu_0}(t))\|_{Z_{3+}} \le C\Big(1+\frac{1}{t^{2\theta}}\Big)\Big(1+(\overline{K}^h(T))^2\Big)$$

Thus owing to (2.66) we estimate

$$f^\mu(t) \le \|e^{tA_h}\|_{\mathcal{L}(Z_1)}\|z_{01}^\mu - z_{01}^{\mu_0}\|_{Z_1} + \|e^{-tTr(m^\mu)} - e^{-tTr(m^{\mu_0})}\|_{Z_3}\|z_{03}\|_{Z_{3+}}$$
$$+ \int_0^t \Big\{\|e^{(t-\tau)A_h}\|_{\mathcal{L}(Z_{1-},Z_1)}\Big(\|G_1^\mu(\boldsymbol{z}^\mu(\tau)) - G_1^\mu(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_{1-}} + \|G_1^\mu(\boldsymbol{z}^{\mu_0}(\tau)) - G_1^{\mu_0}(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_{1-}}\Big)$$
$$+ \|e^{(t-\tau)dA_0}\|_{\mathcal{L}(Z_{2-},Z_2)}\Big(\|G_2^\mu(\boldsymbol{z}^\mu(\tau)) - G_2^\mu(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_{2-}} + \|G_2^\mu(\boldsymbol{z}^{\mu_0}(\tau)) - G_2^{\mu_0}(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_{2-}}\Big)$$
$$+ \Big(\|e^{-(t-\tau)Tr(m^\mu)}\|_{Z_{3+}}\|G_3(\boldsymbol{z}^\mu(\tau)) - G_3(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_3}$$
$$+ \|e^{-(t-\tau)Tr(m^\mu)} - e^{-(t-\tau)Tr(m^{\mu_0})}\|_{Z_3}\|G_3(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_{3+}}\Big) + \sum_{i=4}^{5}\|G_i(\boldsymbol{z}^\mu(\tau)) - G_i(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_i}\Big\}d\tau.$$

Using Lemma 2.7 and Lemma (2.14) we obtain

$$f^\mu(t) \le C\|m^\mu - m^{\mu_0}\|_{Z_1} + t\|Tr(m^\mu - m^{\mu_0})\|_{Z_3}\|z_{03}\|_{Z_{3+}}$$
$$+ \int_0^t \Big\{\Big(\|e^{(t-\tau)A_h}\|_{\mathcal{L}(Z_{1-},Z_1)} + \|e^{(t-\tau)dA_0}\|_{\mathcal{L}(Z_{2-},Z_2)} + \|e^{-(t-\tau)Tr(m^\mu)}\|_{Z_{3+}} + 1\Big)\Big(\sum_{i=1}^{2}\|G_i^\mu(\boldsymbol{z}^\mu(\tau))$$
$$- G_i^\mu(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_{i-}} + \sum_{i=3}^{5}\|G_i(\boldsymbol{z}^\mu(\tau)) - G_i(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_i}\Big)\Big\}d\tau$$
$$+ \int_0^t \Big\{\Big(\|e^{(t-\tau)(A_h-b_1)}\|_{\mathcal{L}(Z_{1-},Z_1)} + \|e^{(t-\tau)dA_0}\|_{\mathcal{L}(Z_{2-},Z_2)}\Big)\Big(\sum_{i=1}^{2}\|G_i^\mu(\boldsymbol{z}^{\mu_0}(\tau)) - G_i^{\mu_0}(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_{i-}}\Big)\Big\}d\tau$$
$$+ \int_0^t \|e^{-(t-\tau)Tr(m^\mu)} - e^{-(t-\tau)Tr(m^{\mu_0})}\|_{Z_3}\|G_3(\boldsymbol{z}^{\mu_0}(\tau))\|_{Z_{3+}}d\tau$$
$$\le C(1+t)\|\eta^\epsilon - \delta\|_{X^{-1/4-\theta}(I)} + C(1+\overline{K}^h(T))\int_0^t \Big(1 + \frac{1}{(t-\tau)^{3/4}} + \frac{1}{(t-\tau)^{1/2}}\Big)\Big(1 + \frac{1}{\tau^{2\theta}}\Big)f^\mu(\tau)d\tau$$
$$+ C(1+\overline{K}^h(T))\|\eta^\epsilon - \delta\|_{X^{-1/4-\theta}(I)}\int_0^t \Big(\frac{1}{(t-\tau)^{3/4}} + \frac{1}{(t-\tau)^{1/2}}\Big)d\tau$$
$$+ C\Big(1 + (\overline{K}^h(T))^2\Big)\|\eta^\epsilon - \delta\|_{X^{-1/4-\theta}(I)}\int_0^t (t-\tau)\Big(1 + \frac{1}{\tau^{2\theta}}\Big)d\tau$$
$$\le a_h(T)\|\eta^\epsilon - \delta\|_{X^{-1/4-\theta}(I)} + b_h(T)\int_0^t \frac{f^\mu(\tau)}{(t-\tau)^{3/4}\tau^{2\theta}}d\tau.$$

Using Lemma 1.2 (see (2.64b)) we get that

$$\|f^\mu\|_{L_\infty(0,T)} \le a_h(T)\|\eta^\epsilon - \delta\|_{X^{-1/4-\theta}(I)}C\exp\Big(b_h(T)^{\frac{7/4+2\theta}{1/4-2\theta}}CT^{7/4+2\theta}\Big),$$

from which we conclude that $\lim_{\epsilon\to 0^+}\|f^\mu\|_{L_\infty(0,T)} = 0$ for every $h \in (0,1]$, $T < T_{max}^h$by Lemma 2.13. In particular $\boldsymbol{u}^{\mu_0}$ is nonnegative on $[0, T_{max}^h)$ and for every $T < T_{max}^h$

$$\|t^{2\theta}z_1^{\mu_0}\|_{L_\infty(0,T;Z_{1+})} \le M_1(T),$$
$$\|z_2^{\mu_0}\|_{L_\infty(0,T;Z_{2+})} \le M_2(T),$$
$$\sum_{i=3}^{5}\|z_i^{\mu_0}\|_{L_\infty(0,T;Z_{i+})} \le M_3,$$

where $M_1(T), M_2(T)$ are defined in (2.72) while M_3 is defined in (2.71). Hence we observe that $\boldsymbol{z}^{\mu_0}$ does not blow-up in finite time in $\|\cdot\|_{\boldsymbol{Z}_+}$. Using standard continuation argument we conclude that $T_{max}^h = \infty$ for any $h \in (0,1]$.

2.2.7. Proof of Theorem 2.3

Recall that m^0 is defined in (2.51) while u_{01} in (2.64c). Denote

$$\begin{aligned} z_{01}^0 &= Pu_{01} - m^0, \\ g_1^0, g_2^0 &: I \times \mathbb{R}^5 \to \mathbb{R}, \\ g_1^0(\boldsymbol{z}) &= -c_1 z_1 + c_2 z_2 - z_1 z_3 + c_4(z_4 - z_3) - (c_1 + z_3)m^0, \\ g_2^0(\boldsymbol{z}) &= -b_2 z_2 + c_1 z_1 - c_2 z_2 - c_3 z_2 z_3 + c_5(z_5 - z_4) + c_1 m^0. \end{aligned}$$

For $\boldsymbol{z} \in \boldsymbol{Z}_+$ define

$$\begin{aligned} G_1^0(\boldsymbol{z}) &= Tr'(g_1^0(Tr(z_1), z_2, z_3, z_4, z_5)), \\ G_2^0(\boldsymbol{z}) &= g_2^0(Tr(z_1), z_2, z_3, z_4, z_5). \end{aligned}$$

Observe that since $\boldsymbol{u}^0 = (u_1^0, \ldots, u_5^0)$ solves (2.70a), $\boldsymbol{z}^0 = (z_1^0, \ldots, z_5^0) = M(E(u_1^0 - m^0), u_2^0, \ldots, u_5^0)$ satisfies the following Duhamel formulas:

$$z_1^0(t) = E\Big\{e^{t(A_0 - b_1)} z_{01}^0 + \int_0^t e^{(t-\tau)(A_0 - b_1)} PG_1^0(\boldsymbol{z}^0(\tau))d\tau\Big\}, \tag{2.76a}$$

$$z_2^0(t) = e^{tdA_0} z_{02} + \int_0^t e^{(t-\tau)dA_0} G_2^0(\boldsymbol{z}^0(\tau))d\tau, \tag{2.76b}$$

$$z_3^0(t) = e^{-tm^0} z_{03} + \int_0^t e^{-(t-\tau)m^0} G_3(\boldsymbol{z}^0(\tau))d\tau, \tag{2.76c}$$

$$z_i^0(t) = z_{0i} + \int_0^t G_i(\boldsymbol{z}^0(\tau))d\tau, \ i \in \{4, 5\}. \tag{2.76d}$$

For $t < T < \infty$ denote

$$N(T) = \sup_{h \in (0,1]} \Big(\|t^{2\theta} z_1^{\mu_0}\|_{L_\infty(0,T;Z_{1+})} + \|t^{2\theta} z_1^0\|_{L_\infty(0,T;Z_{1+})} + \sum_{i=2}^{3} (\|z_i^{\mu_0}\|_{L_\infty(0,T;Z_{i+})} + \|z_i^0\|_{L_\infty(0,T;Z_{i+})}) \Big),$$

$$f^{\mu_0}(t) = t^{2\theta} \|z_1^{\mu_0}(t) - z_1^0(t)\|_{Z_{1+}} + \sum_{i=2}^{5} \|z_i^{\mu_0}(t) - z_i^0(t)\|_{Z_i}.$$

Observe that $N(T) \le M_1(T) + M_2(T) + M_3 < \infty$ as was proved in Step 3 of Theorem 2.2. Owing to Lemma 2.15 and Lemma 2.14 we have

$$\sum_{i=1}^{2} \|G_i^{\mu_0}(\boldsymbol{z}^{\mu_0}(t)) - G_i^{\mu_0}(\boldsymbol{z}^0(t))\|_{Z_{i-}} + \sum_{i=3}^{5} \|G_i(\boldsymbol{z}^{\mu_0}(t)) - G_i(\boldsymbol{z}^0(t))\|_{Z_i} \le C(1 + N(T))\Big(1 + \frac{1}{t^{2\theta}}\Big) f^{\mu_0}(t),$$

$$\|G_1^0(\boldsymbol{z}^0(t))\|_{Z_{1-}} + \|G_3(\boldsymbol{z}^0(t))\|_{Z_{3+}} \le C(1 + (N(T))^2)\Big(1 + \frac{1}{t^{2\theta}}\Big),$$

$$\sum_{i=1}^{2} \|G_i^{\mu_0}(\boldsymbol{z}^0(t)) - G_i^0(\boldsymbol{z}^0(t))\|_{Z_{i-}} \le C(1 + N(T)) \frac{1}{|\lambda_{01,h}^{\Omega}|^{\theta/2}}.$$

Since $\boldsymbol{z}^{\mu_0}$ (resp. $\boldsymbol{z}^0$) satisfies (2.66) (resp. (2.76)) thus using (2.31) and Lemma 2.12 we obtain

$$
\begin{aligned}
&f^{\mu_0}(t) \le t^{2\theta}\|e^{tA_h}(z_{01}^{\mu_0} - Ez_{01}^0)\|_{Z_{1+}} + \|e^{-tTr(m^{\mu_0})} - e^{-tm^0}\|_{Z_3}\|z_{03}\|_{Z_{3+}} \\
&+ \int_0^t \Big\{t^{2\theta}\|e^{(t-\tau)A_h}\Big(G_1^{\mu_0}(\boldsymbol{z}^{\mu_0}(\tau)) - EPG_1^0(\boldsymbol{z}^0(\tau))\Big)\|_{Z_{1+}}\Big\}d\tau + \int_0^t \Big\{\|e^{(t-\tau)dA_0}\Big(G_2^{\mu_0}(\boldsymbol{z}^{\mu_0}(\tau)) \\
&- G_2^0(\boldsymbol{z}^0(\tau))\Big)\|_{Z_2}\Big\}d\tau + \int_0^t \Big\{\|e^{-(t-\tau)Tr(m^{\mu_0})}G_3(\boldsymbol{z}^{\mu_0}(\tau)) - e^{-(t-\tau)m^0}G_3(\boldsymbol{z}^0(\tau))\|_{Z_3}\Big\}d\tau \\
&+ \sum_{i=4}^5 \int_0^t \|G_i(\boldsymbol{z}^{\mu_0}(\tau)) - G_i(\boldsymbol{z}^0(\tau))\|_{Z_i}d\tau \le t^{2\theta}\|e^{tA_h}(I-EP)u_{01}\|_{Z_{1+}} + t^{2\theta}\|e^{tA_h}\|_{\mathcal{L}(Z_1,Z_{1+})}\|m^{\mu_0} - Em^0\|_{Z_1} \\
&+ t\|Tr(m^{\mu_0}) - m^0\|_{Z_3}\|z_{03}\|_{Z_{3+}} + T^{2\theta}\int_0^t \Big\{\|e^{(t-\tau)A_h}\|_{\mathcal{L}(Z_{1-},Z_{1+})}\Big(\|G_1^{\mu_0}(\boldsymbol{z}^{\mu_0}(\tau)) - G_1^{\mu_0}(\boldsymbol{z}^0(\tau))\|_{Z_{1-}} \\
&+ \|G_1^{\mu_0}(\boldsymbol{z}^0(\tau)) - G_1^0(\boldsymbol{z}^0(\tau))\|_{Z_{1-}}\Big)\Big\}d\tau + T^{2\theta}\int_0^t \Big\{\|e^{(t-\tau)A_h}(I-EP)G_1^0(\boldsymbol{z}^0(\tau))\|_{Z_{1+}}\Big\}d\tau \\
&+ \int_0^t \Big\{\|e^{(t-\tau)dA_0}\|_{\mathcal{L}(Z_{2-},Z_2)}\Big(\|G_2^{\mu_0}(\boldsymbol{z}^{\mu_0}(\tau)) - G_2^{\mu_0}(\boldsymbol{z}^0(\tau))\|_{Z_{2-}} + \|G_2^{\mu_0}(\boldsymbol{z}^0(\tau)) - G_2^0(\boldsymbol{z}^0(\tau))\|_{Z_{2-}}\Big)\Big\}d\tau \\
&+ \int_0^t \Big\{\|e^{-(t-\tau)Tr(m^{\mu_0})}\|_{Z_{3+}}\|G_3(\boldsymbol{z}^{\mu_0}(\tau)) - G_3(\boldsymbol{z}^0(\tau))\|_{Z_3} \\
&+ \|e^{-(t-\tau)Tr(m^{\mu_0})} - e^{-(t-\tau)m^0}\|_{Z_3}\|G_3(\boldsymbol{z}^0(\tau))\|_{Z_{3+}}\Big\}d\tau + \sum_{i=4}^5 \int_0^t \|G_i(\boldsymbol{z}^{\mu_0}(\tau)) - G_i(\boldsymbol{z}^0(\tau))\|_{Z_i}d\tau.
\end{aligned}
$$

Using Lemma 2.7, Lemma 2.33, Lemma 2.14, Lemma 2.11 and Lemma 1.1 we have

$$
\begin{aligned}
&f^{\mu_0}(t) \le Ct^{2\theta}e^{t\lambda_{01,h}}\|u_{01}\|_{Z_{1+}} + C(1+t^{2\theta})\frac{1}{|\lambda_{01,h}^\Omega|^{\theta/2}} + Ct\frac{1}{|\lambda_{01,h}^\Omega|^{\theta/2}}\|z_{03}\|_{Z_{3+}} \\
&+ CT^{2\theta}(1+N(T))\int_0^t \Big(1+\frac{1}{(t-\tau)^{3/4+2\theta}}\Big)\Big(\Big(1+\frac{1}{\tau^{2\theta}}\Big)f^{\mu_0}(\tau) + \frac{1}{|\lambda_{01,h}^\Omega|^{\theta/2}}\Big)d\tau \\
&+ CT^{2\theta}(1+(N(T))^2)\int_0^t \Big(1+\frac{1}{(t-\tau)^{3/4+2\theta}}\Big)\Big(1+\frac{1}{\tau^{2\theta}}\Big)e^{(t-\tau)\lambda_{01,h}^\Omega}d\tau \\
&+ C(1+N(T))\int_0^t \Big(1+\frac{1}{(t-\tau)^{1/2}}\Big)\Big(\Big(1+\frac{1}{\tau^{2\theta}}\Big)f^{\mu_0}(\tau) + \frac{1}{|\lambda_{01,h}^\Omega|^{\theta/2}}\Big)d\tau \\
&+ C(1+N(T))\int_0^t \Big(1+\frac{1}{\tau^{2\theta}}\Big)f^{\mu_0}(\tau) + C(1+(N(T))^2)\frac{1}{|\lambda_{01,h}^\Omega|^{\theta/2}}\int_0^t (t-\tau)\Big(1+\frac{1}{\tau^{2\theta}}\Big)d\tau \\
&\le C(1+T)\Big(\frac{1}{|\lambda_{01,h}^\Omega|^{2\theta}} + \frac{1}{|\lambda_{01,h}^\Omega|^{\theta/2}}\Big) + C(1+T^{2\theta})(1+(N(T))^2)\Big\{\frac{1}{|\lambda_{01,h}^\Omega|^{\theta/2}}\int_0^t \Big(1+\frac{1}{(t-\tau)^{3/4+2\theta}} \\
&+ \frac{1}{(t-\tau)^{1/2}} + (t-\tau)\Big(1+\frac{1}{\tau^{2\theta}}\Big)\Big)d\tau + \int_0^t \Big(1+\frac{1}{\tau^{3/4+2\theta}}\Big)\Big(1+\frac{1}{(t-\tau)^{2\theta}}\Big)e^{\tau\lambda_{01,h}^\Omega}d\tau \\
&+ \int_0^t \Big(1+\frac{1}{(t-\tau)^{3/4+2\theta}} + \frac{1}{(t-\tau)^{1/2}}\Big)\Big(1+\frac{1}{\tau^{2\theta}}\Big)f^{\mu_0}(\tau)d\tau\Big\} \\
&\le a(T)\Big(\frac{1}{|\lambda_{01,h}^\Omega|^{\theta/2}} + \frac{1}{|\lambda_{01,h}^\Omega|^{\frac{1/4-4\theta}{7/4+4\theta}}}\Big) + b(T)\int_0^t \frac{f^{\mu_0}(\tau)}{(t-\tau)^{3/4+2\theta}\tau^{2\theta}}d\tau.
\end{aligned}
$$

Using Lemma (1.2) (see (2.64b)) we get that

$$\|f^{\mu_0}\|_{L_\infty(0,T)} \le a(T)\Big(\frac{1}{|\lambda^\Omega_{01,h}|^{\theta/2}} + \frac{1}{|\lambda^\Omega_{01,h}|^{\frac{1/4-4\theta}{7/4+4\theta}}}\Big) C \exp\Big(b(T)^{\frac{7/4+4\theta}{1/4-4\theta}} C T^{7/4+4\theta}\Big),$$

from which we conclude that $\lim_{h\to 0^+}\|f^{\mu_0}\|_{L_\infty(0,T)} = 0$ since $|\lambda^\Omega_{01,h}| = (\pi/h)^2 \to \infty$ as $h \to 0$.

2.3. Stationary problem

In this section we show that system (2.1) has a unique equilibrium. Moreover we study the problem of the dimension reduction for the stationary case.

2.3.1. The results of Section 2.3

Let us observe that due to the presence of three ODE's in the system (2.1), the stationary problem may be reduced to a system of two elliptic equations:

$$div(J_h(u_1)) + b_1 u_1 = 0, \qquad x \in \Omega \tag{2.77a}$$

$$-d\partial^2_{x_1} u_2 - c_1 u_1 + (b_2 + c_2 + k_2 H(u_1,u_2))u_2 = 0, \qquad x \in \partial_1\Omega \tag{2.77b}$$

with boundary conditions

$$-J_h(u_1)\nu = 0, \qquad x \in \partial_0\Omega \tag{2.78a}$$

$$-J_h(u_1)\nu = -(c_1 + k_1 H(u_1,u_2))u_1 + c_2 u_2 + p_1\delta, \qquad x \in \partial_1\Omega \tag{2.78b}$$

$$\partial_{x_1} u_2 = 0, \qquad x \in \partial\partial_1\Omega, \tag{2.78c}$$

where

$$k_1 = b_4/(b_4 + c_4),\ k_2 = c_3 b_5/(b_5 + c_5),\ H(u_1,u_2) = p_3/(k_1 u_1 + k_2 u_2 + b_3) \tag{2.79}$$

and

$$u_3 = H(u_1,u_2),\ u_4 = \frac{k_1}{b_4} u_1 H(u_1,u_2),\ u_5 = \frac{k_2}{b_5} u_2 H(u_1,u_2).$$

We will prove the following two theorems.

Theorem 2.4. *For every $h \in (0,1]$ system (2.77)-(2.78) has a unique nonnegative W^1_1 solution (u^h_1, u^h_2) i.e. there exists a unique nonnegative $(u^h_1,u^h_2) \in W^1_1(\Omega) \times W^1_1(\partial_1\Omega)$ such that for every $(v_1,v_2) \in W^1_\infty(\Omega) \times W^1_\infty(\partial_1\Omega)$*

$$-\int_\Omega [J_h(u^h_1)\nabla v_1 + b_1 u^h_1 v_1] = p_1 v_1(0) + \int_{\partial_1\Omega} [-(c_1 + k_1 H(u^h_1,u^h_2))u^h_1 + c_2 u^h_2] v_1, \tag{2.80a}$$

$$\int_{\partial_1\Omega} d\partial_{x_1} u^h_2 \partial_{x_1} v_2 = \int_{\partial_1\Omega} [c_1 u^h_1 - (b_2 + c_2 + k_2 H(u^h_1,u^h_2))u^h_2] v_2. \tag{2.80b}$$

Moreover $(u^h_1,u^h_2) \in W^1_p(\Omega) \times W^2_q(\partial_1\Omega)$ for every $1 \le p < 2, 1 \le q < \infty$ and

$$\|u^h_1\|_{W^1_p(\Omega)} + h^{-1}\|\partial_{x_2} u^h_1\|_{L_p(\Omega)} + \|u^h_2\|_{W^2_q(\partial_1\Omega)} \le C, \tag{2.81}$$

where C does not depend on h.

The following theorem concerns dimension reduction in the stationary problem.

Theorem 2.5. *Let* (u_1^h, u_2^h) *be the unique solution of system* (2.77)-(2.78)*. Then for every* $1 \le p < 2, 1 \le q < \infty$ *we have the following weak convergence as* $h \to 0^+$

$$u_1^h \rightharpoonup u_1^0 \quad \text{in} \quad W_p^1(\Omega), \tag{2.82a}$$

$$u_2^h \rightharpoonup u_2^0 \quad \text{in} \quad W_q^2(\partial_1\Omega). \tag{2.82b}$$

Moreover $\partial_{x_2} u_1^0 = 0$ *(so that* u_1^0 *depends only on* x_1*) and* $(u_1^0, u_2^0) \in W_\infty^1(I) \times C^2(\overline{I})$ *is the unique solution of*

$$-u_1'' + (b_1 + c_1 + k_1 H(u_1, u_2))u_1 - c_2 u_2 = p_1 \delta, \qquad x \in I \tag{2.83a}$$

$$-du_2'' - c_1 u_1 + (b_2 + c_2 + k_2 H(u_1, u_2))u_2 = 0, \qquad x \in I \tag{2.83b}$$

$$u_1' = u_2' = 0, \qquad x \in \partial I. \tag{2.83c}$$

Proofs of Theorem 2.4 and 2.5 are given in sections 2.3.3 and 2.3.4.

Remark 3. *Notice that* (2.83) *is the stationary problem associated with model* ***[HKCS].1D*** *(analysed in Section 2.4). Thus Theorem 2.5 is the rigourous formulation of the dimension reduction of the model* ***[HKCS].2D*** *in the stationary case.*

In Figure 2.1 placed at the end of this chapter we present graphs of u_1^h for several values of h. Notice that as h becomes smaller the graph of u_1^h becomes homogeneous in the x_2 direction.

2.3.2. Solvability of certain linear system with measure valued sources

To prove Theorem 2.4 we will use two lemmas concerning solvability of linear elliptic boundary value problems with low regularity data.

Lemma 2.17. *Assume that* $0 \le a_0 \in L_\infty(\Omega),\ 0 \le a_{11} \in L_\infty(\partial_1\Omega)$*. Then for every* $h \in (0,1],\ \lambda > 0$ *and* $\mu_\Omega \in \mathcal{M}(\Omega),\ \mu_I \in \mathcal{M}(I)$ *the following boundary value problem*

$$div(J_h(u)) + (\lambda + a_0)u = \mu_\Omega, \qquad x \in \Omega \tag{2.84a}$$

$$-J_h(u)\nu = 0, \qquad x \in \partial_0\Omega \tag{2.84b}$$

$$-J_h(u)\nu + a_{11}u = \mu_I, \qquad x \in \partial_1\Omega \tag{2.84c}$$

has a unique W_1^1 *solution i.e. there exists a unique* $u \in W_1^1(\Omega)$ *such that for every* $v \in W_\infty^1(\Omega)$

$$\int_\Omega [-J_h(u)\nabla v + (\lambda + a_0)uv] + \int_{\partial_1\Omega} a_{11}uv = \int_\Omega v d\mu_\Omega + \int_{\partial_1\Omega} v d\mu_I. \tag{2.85}$$

Moreover $u \in W_p^1(\Omega)$ *for every* $p < 2$ *and*

$$\|u\|_{W_p^1(\Omega)} + h^{-1}\|\partial_{x_2} u\|_{L_p(\Omega)} \le C(\|\mu_\Omega\|_{TV} + \|\mu_I\|_{TV}), \tag{2.86}$$

where C *depends only on* $p, \lambda, \|a_0\|_{L_\infty(\Omega)}, \|a_{11}\|_{L_\infty(\partial_1\Omega)}$*. If* $\mu_\Omega, \mu_I \ge 0$ *then* $u \ge 0$*.*

Proof. We divide the proof into two parts. In the first part we employ the technique from [3] to prove existence of the solution which additionally satisfies (2.86). Notice that one has to use a slight modification due to the Robin boundary condition instead of the Dirichlet condition which is treated in [3]. In the second part of the proof, using duality technique from [5], we show that the solution is unique in the W^1_1 class.

Step 1 - existence of solutions.
Observe that due to linearity of the problem (2.84) one can assume without loss of generality that

$$\|\mu_\Omega\|_{TV} + \|\mu_I\|_{TV} \leq 1.$$

First let us consider $\mu_\Omega \in L_\infty(\Omega), \mu_I \in L_\infty(\partial_1\Omega)$. Using the Lax-Milgram lemma we obtain that the problem (2.84) has a unique solution $u \in W^1_2(\Omega)$. We will now prove that this solution satisfies (2.86). Observe that if $\phi \in W^1_\infty(\mathbb{R})$ is such that

$$\|\phi\|_{L_\infty(\mathbb{R})} \leq 1,\ \ y\phi(y) \geq 0,\ \ \phi'(y) \geq 0, \tag{2.87}$$

then testing (2.85) by $v = \phi(u) \in W^1_2(\Omega)$ we obtain

$$\lambda \int_\Omega u\phi(u) \leq 1, \tag{2.88a}$$

$$0 \leq \int_\Omega -\phi'(u)J_h(u)\nabla u \leq 1. \tag{2.88b}$$

For $n \geq 1$ define

$$\varphi_n(y) = \begin{cases} ny & \text{if } |y| < 1/n \\ sgn(y) & \text{if } |y| \geq 1/n \end{cases}. \tag{2.89}$$

Choosing in (2.88a) $\phi = \varphi_n$ and taking $n \to \infty$ we obtain that

$$\|u\|_{L_1(\Omega)} \leq 1/\lambda \leq C. \tag{2.90}$$

For $n \geq 0$ define $B_n = \{x : n \leq |u(x)| \leq n+1\}$ and

$$\psi_n(y) = \begin{cases} 0 & \text{if } |y| < n \\ y - sgn(y)\cdot n & \text{if } n \leq |y| \leq n+1 \\ sgn(y) & \text{if } |y| > n+1 \end{cases}.$$

Choosing in (2.88b) $\phi = \psi_n$ we obtain that

$$\|m_h(u)\mathbf{1}_{B_n}\|^2_{L_2(\Omega)} = \int_{B_n} -J_h(u)\nabla u \leq 1, \tag{2.91}$$

where $m_h(u) = \sqrt{|\partial_{x_1}u|^2 + h^{-2}|\partial_{x_2}u|^2}$. Using Hölder's inequality with $1 = p/2 + p/p^*$ we have

$$\|m_h(u)\mathbf{1}_{B_n}\|^p_{L_p(\Omega)} \leq \|m_h(u)\mathbf{1}_{B_n}\|^p_{L_2(\Omega)}|B_n|^{p/p^*} \leq |B_n|^{p/p^*} \leq C. \tag{2.92}$$

Using Sobolev's inequality and (2.90) we have

$$\|u\|_{L_{p^*}(\Omega)} \le C(\|m_1(u)\|_{L_p(\Omega)} + \|u\|_{L_1(\Omega)}) \le C(\|m_h(u)\|_{L_p(\Omega)} + 1). \tag{2.93}$$

From (2.92), Hölder's inequality (for series) and (2.93) we have

$$\begin{aligned}\|m_h(u)\|^p_{L_p(\Omega)} &= \sum_{n=0}^{N}\|m_h(u)\mathbf{1}_{B_n}\|^p_{L_p(\Omega)} + \sum_{n=N+1}^{\infty}\|m_h(u)\mathbf{1}_{B_n}\|^p_{L_p(\Omega)} \le C(N+1) + \sum_{n=N+1}^{\infty}|B_n|^{p/p^*}\\ &\le C(N+1) + \sum_{n=N+1}^{\infty} n^{-p}\|u\mathbf{1}_{B_n}\|^p_{L_{p^*}(\Omega)} \le C(N+1) + \Big(\sum_{n=N+1}^{\infty} n^{-2}\Big)^{p/2}\|u\|^p_{L_{p^*}(\Omega)}\\ &= C(N+1) + A(N)^p\|u\|^p_{L_{p^*}(\Omega)} \le C(N+1) + CA(N)^p(\|m_h(u)\|^p_{L_p(\Omega)} + 1).\end{aligned}$$

Taking N sufficiently large we obtain

$$\|m_h(u)\|_{L_p(\Omega)} \le C.$$

Finally from (2.93) it follows that $\|u\|_{L_p(\Omega)} \le C\|u\|_{L_{p^*}(\Omega)} \le C(\|m_h(u)\|_{L_p(\Omega)}+1) \le C$ which completes the proof of (2.86).

The case of arbitrary Radon measures μ_Ω, μ_I follows by standard approximation, see [3] for instance.

Step 2- uniqueness of solution.
We shall use duality technique. Let u be a W_1^1 solution to

$$div(J_h(u)) + (\lambda + a_0)u = 0, \qquad x \in \Omega \tag{2.94a}$$
$$-J_h(u)\nu = 0, \qquad x \in \partial_0\Omega \tag{2.94b}$$
$$-J_h(u)\nu + a_{11}u = 0, \qquad x \in \partial_1\Omega. \tag{2.94c}$$

We intend to prove that $u \equiv 0$. First we assume additionally that $a_{11} \equiv 0$. Using [7] we get that for every $f \in L_q(\Omega), q > 2$, problem

$$div(J_h(v)) + (\lambda + a_0)v = f, \qquad x \in \Omega \tag{2.95a}$$
$$-J_h(v)\nu = 0, \qquad x \in \partial\Omega \tag{2.95b}$$

has a unique solution $v \in W_q^2(\Omega)$. Since $q > 2$ we have $W_q^2(\Omega) \subset W_\infty^1(\Omega)$, so that for every $w \in W_1^1(\Omega)$ we have

$$\int_\Omega [-J_h(v)\nabla w + (\lambda + a_0)vw] = \int_\Omega fw.$$

Taking $w = u$ we thus get $\int_\Omega fu = 0$ and since f was arbitrary - $u \equiv 0$ follows.
Now let us take $0 \le a_{11} \in L_\infty(\partial_1\Omega)$. Denote $g = -a_{11}u$. Observe that u is a W_1^1 solution of

$$div(J_h(u)) + (\lambda + a_0)u = 0, \qquad x \in \Omega \tag{2.96a}$$
$$-J_h(u)\nu = 0, \qquad x \in \partial_0\Omega \tag{2.96b}$$
$$-J_h(u)\nu = g, \qquad x \in \partial_1\Omega. \tag{2.96c}$$

As we already showed (2.96) has a unique W_1^1 solution and, thus $u \in W_p^1(\Omega)$ for every $p < 2$. In particular $g \in L_q(\partial_1\Omega)$ for every $q < \infty$. We can now use Lax-Milgram theorem to prove that (2.96) has a unique W_2^1 solution and thus conclude that $u \in W_2^1(\Omega)$. It follows that, u is also a W_2^1 solution of (2.94), whence $u \equiv 0$.

□

Lemma 2.18. *Assume that $d > 0,\ 0 \leq a_0 \in L_\infty(\Omega)$ and*

$$a_{ij} \in L_\infty(\partial_1\Omega),\ a_{11} \geq |a_{21}|, a_{22} \geq |a_{12}|. \tag{2.97}$$

Then for every $h \in (0,1], \lambda > 0, \mu_\Omega \in \mathcal{M}(\Omega), \mu_I \in \mathcal{M}(I)$ the following system

$$div(J_h(u_1)) + (\lambda + a_0)u_1 = \mu_\Omega, \qquad x \in \Omega \tag{2.98a}$$

$$-d\partial_{x_1}^2 u_2 - a_{21}u_1 + (\lambda + a_{22})u_2 = 0, \qquad x \in \partial_1\Omega \tag{2.98b}$$

with boundary conditions

$$-J_h(u_1)\nu = 0, \qquad x \in \partial_0\Omega \tag{2.99a}$$

$$-J_h(u_1)\nu + a_{11}u_1 - a_{12}u_2 = \mu_I, \qquad x \in \partial_1\Omega \tag{2.99b}$$

$$\partial_{x_1} u_2 = 0, \qquad x \in \partial\partial_1\Omega, \tag{2.99c}$$

has a unique W_1^1 solution i.e. there exists a unique $(u_1, u_2) \in W_1^1(\Omega) \times W_1^1(\partial_1\Omega)$ such that for every $(v_1, v_2) \in W_\infty^1(\Omega) \times W_\infty^1(\partial_1\Omega)$:

$$\int_\Omega [-J_h(u_1)\nabla v_1 + (\lambda + a_0)u_1 v_1] + \int_{\partial_1\Omega} \Big[d\partial_{x_1}u_2\partial_{x_1}v_2 + \lambda u_2 v_2 - \Big(M(u_1,u_2)\Big|(v_1,v_2)\Big)_{\mathbb{R}^2}\Big]$$
$$= \int_\Omega v_1 d\mu_\Omega + \int_{\partial_1\Omega} v_1 d\mu_I,$$

where $M(u_1, u_2) = (-a_{11}u_1 + a_{12}u_2, a_{21}u_1 - a_{22}u_2)$.

Moreover $(u_1, u_2) \in W_p^1(\Omega) \times W_q^2(\partial_1\Omega)$ for every $1 \leq p < 2, 1 \leq q < \infty$ and

$$\|u_1\|_{W_p^1(\Omega)} + h^{-1}\|\partial_{x_2}u_1\|_{L_p(\Omega)} + \|u_2\|_{W_q^2(\partial_1\Omega)} \leq C(\|\mu_\Omega\|_{TV} + \|\mu_I\|_{TV}, \tag{2.100}$$

where C depends only on $p, \lambda, d, \|a_0\|_{L_\infty(\Omega)}, \|a_{ij}\|_{L_\infty(\partial_1\Omega)}$. If $\mu_\Omega, \mu_I, a_{12}, a_{21} \geq 0$ then $u_1, u_2 \geq 0$.

Proof.
Step 1 - existence of solution.
Let us define the Hilbert spaces $X_{1/2} = W_2^1(\Omega) \times W_2^1(\partial_1\Omega),\ X_{-1/2} = X_{1/2}^*$ and an unbounded operator $A : X_{-1/2} \supset X_{1/2} \to X_{-1/2}$ by

$$\Big\langle A(u_1,u_2),(v_1,v_2)\Big\rangle_{(X_{-1/2},X_{1/2})} = \int_\Omega [J_h(u_1)\nabla v_1 - a_0 u_1 v_1]$$
$$+ \int_{\partial_1\Omega} \Big[-d\partial_{x_1}u_2\partial_{x_1}v_2 + \Big(M(u_1,u_2)\Big|(v_1,v_2)\Big)_{\mathbb{R}^2}\Big].$$

Due to boundedness of a_0 and a_{ij} operator $\lambda - A$ is coercive for λ large enough and the Lax-Milgram lemma guarantees that there is $\lambda_0 > 0$ such that $[\lambda_0, \infty) \subset \rho(A)$ ($\rho(A)$ denotes the resolvent set of A) . Because $X_{1/2}$ is compactly embedded into $X_{-1/2}$ we get that for $\lambda \in \rho(A)$ the resolvent operator $(\lambda - A)^{-1}$ is compact and thus the spectrum $\sigma(A)$ consists entirely of eigenvalues. Choose any $\lambda \in \mathbb{R},\ \theta \in X_{-1/2}$ and $u = (u_1, u_2) \in X_{1/2}$ such that $(\lambda - A)u = \theta$. Let φ_n be the function defined in (2.89). Then

$$\Big\langle\theta,(\varphi_n(u_1),\varphi_n(u_2))\Big\rangle_{(X_{-1/2},X_{1/2})} = \Big\langle(\lambda - A)(u_1,u_2),(\varphi_n(u_1),\varphi_n(u_2))\Big\rangle_{(X_{-1/2},X_{1/2})}$$
$$= \int_\Omega [-\varphi_n'(u_1)J_h(u_1)\nabla u_1 + (\lambda + a_0)u_1\varphi_n(u_1)]$$
$$+ \int_{\partial_1\Omega} \Big[d\varphi_n'(u_2)|\partial_{x_1}u_2|^2 - \Big(M(u_1,u_2)\Big|(\varphi_n(u_1),\varphi_n(u_2))\Big)_{\mathbb{R}^2} + \lambda u_2\varphi_n(u_2)\Big]$$
$$\geq \lambda\Big(\int_\Omega u_1\varphi_n(u_1) + \int_{\partial_1\Omega} u_2\varphi_n(u_2)\Big) - \int_\Omega \Big(M(u_1,u_2)\Big|(\varphi_n(u_1),\varphi_n(u_2))\Big)_{\mathbb{R}^2}.$$

Thus taking $n \to \infty$ and using (2.97) we get

$$\liminf_{n\to\infty} \Big\langle \theta, (\varphi_n(u_1), \varphi_n(u_2)) \Big\rangle_{(X_{-1/2}, X_{1/2})} \geq \lambda(\|u_1\|_{L_1(\Omega)} + \|u_2\|_{L_1(\partial_1\Omega)}). \tag{2.101}$$

In particular it follows from (2.101) that for $\lambda > 0$ equation $(\lambda - A)u = 0$ does not have nontrivial solutions, whence $(0, \infty) \subset \rho(A)$.
Observe that when μ_Ω, μ_I are bounded functions then the distribution θ defined by

$$\Big\langle \theta, (v_1, v_2) \Big\rangle = \int_\Omega v_1 d\mu_\Omega + \int_I v_1 d\mu_I = \int_\Omega v_1 \mu_\Omega dx + \int_I v_1(\cdot, 0)\mu_I dx_1 \tag{2.102}$$

belongs to $X^*_{1/2}$ thus equation $(\lambda - A)u = \theta$ has a unique solution $u = (u_1, u_2) \in X_{1/2}$ which is a solution to problem (2.98)-(2.99). We will now prove that u satisfies (2.100). Due to linearity of (2.98), (2.99) we can assume, without loss of generality, that

$$\|\mu_\Omega\|_{TV} + \|\mu_I\|_{TV} \leq 1.$$

Next we prove respectively that

$$\lambda(\|u_1\|_{L_1(\Omega)} + \|u_2\|_{L_1(\partial_1\Omega)}) \leq C, \tag{2.103}$$

$$\|u_1\|_{W^1_p(\Omega)} + h^{-1}\|\partial_{x_2} u_1\|_{L_p(\partial_1\Omega)} \leq C, \tag{2.104}$$

$$\|u_2\|_{W^2_q(\partial_1\Omega)} \leq C. \tag{2.105}$$

To get (2.103) observe that from (2.101) with θ given by (2.102) one has

$$\begin{aligned}\lambda(\|u_1\|_{L_1(\Omega)} + \|u_2\|_{L_1(\partial_1\Omega)}) &\leq \liminf_{n\to\infty} \Big\langle \theta, (\varphi_n(u_1), \varphi_n(u_2)) \Big\rangle_{(X_{-1/2}, X_{1/2})} \\ &\leq \|\mu_\Omega\|_{L_1(\Omega)} + \|\mu_I\|_{L_1(I)} \leq 1,\end{aligned}$$

since $|\varphi_n(y)| \leq 1$ for $y \in \mathbb{R}$. Then (2.104) follows from (2.103) and Lemma 2.17, while (2.105) follows from (2.98b), (2.104) and the fact that for every $1 \leq q < \infty$ there exists $1 \leq p < 2$ such that the trace operator maps $W^1_p(\Omega)$ into $L_q(\partial_1\Omega)$. To prove existence of solutions to (2.98), (2.99) for the case when μ_Ω and μ_I are finite Radon measures one proceeds by the standard approximation technique with the use of (2.100).

Step 2 - uniqueness of solution.
Let (u_1, u_2) be a W^1_1 solution of problem (2.98), (2.99) with $\lambda > 0, \mu_\Omega = 0, \mu_I = 0$.

Denoting $g_1 = a_{12}u_2 \in L_\infty(I), g_2 = a_{21}u_1 \in L_1(I)$ we see that u_1 is a W^1_1 solution of

$$div(J_h(u)) + (\lambda + a_0)u = 0, \qquad x \in \Omega \tag{2.106a}$$

$$-J_h(u)\nu = 0, \qquad x \in \partial_0\Omega \tag{2.106b}$$

$$-J_h(u)\nu + a_{11}u = g_1, \qquad x \in \partial_1\Omega \tag{2.106c}$$

and u_2 is a W^1_1 solution of

$$-d\partial^2_{x_1} u + (\lambda + a_{22})u = g_2, \qquad x \in I \tag{2.107a}$$

$$\partial_{x_1} u = 0, \qquad x \in \partial I. \tag{2.107b}$$

Since $g_1 \in L_\infty(\partial_1\Omega)$ then by Lax-Milgram lemma problem (2.106) has a W^1_2 solution which by Lemma 2.17 is unique in W^1_1 class. Thus u_1 is a W^1_2 solution of (2.106) and $g_2 \in L_2(I)$. From Lax-Milgram lemma we obtain that (2.107) has a W^1_2 solution which due to duality technique is unique in W^1_1 class. Thus $u_2 \in W^1_2$. Finally we observe that $(u_1, u_2) \in X_{1/2}$ is in the kernel of the operator $(\lambda - A)$ and thus $(u_1, u_2) \equiv 0$. □

2.3.3. Proof of Theorem 2.4

Step 1 - existence of solution.
Fix $1 > s > 1/p,\ \infty > q > 1$ and for $R > 0$ define

$$K_R = \{(v_1, v_2) \in W_p^s(\Omega) \times L_q(\partial_1\Omega) : v_1, v_2 \geq 0, \|v_1\|_{W_p^s(\Omega)} + \|v_2\|_{L_q(\partial_1\Omega)} \leq R\}.$$

K_R is a bounded, convex and closed subset of the Banach space $B = W_p^s(\Omega) \times L_q(\partial_1\Omega)$. For $(v_1, v_2) \in K_R$ consider problem (2.77)-(2.78) with $H(u_1, u_2)$ replaced by $H(v_1, v_2)$ (notice that $v_1(0, \cdot)$ is well defined as $s > 1/p$) i.e.

$$div(J_h(u_1)) + b_1 u_1 = 0, \qquad x \in \Omega \tag{2.108a}$$
$$-d\partial_{x_1}^2 u_2 - c_1 u_1 + (b_2 + c_2 + k_2 H(v_1, v_2))u_2 = 0, \qquad x \in \partial_1\Omega \tag{2.108b}$$

with boundary conditions

$$-J_h(u_1)\nu = 0, \qquad x \in \partial_0\Omega \tag{2.109a}$$
$$-J_h(u_1)\nu = -(c_1 + k_1 H(v_1, v_2))u_1 + c_2 u_2 + p_1\delta, \qquad x \in \partial_1\Omega \tag{2.109b}$$
$$\partial_{x_1} u_2 = 0, \qquad x \in \partial\partial_1\Omega. \tag{2.109c}$$

Using Lemma 2.18 with

$$\begin{aligned}
&\lambda = \min\{b_1, b_2\},\ a_0 = b_1 - \lambda,\ \mu_\Omega = 0,\ \mu_I = p_1\delta, \\
&a_{11} = c_1 + k_1 H(v_1, v_2), \qquad a_{12} = c_2, \\
&a_{21} = c_1, \qquad a_{22} = b_2 - \lambda + c_2 + k_2 H(v_1, v_2),
\end{aligned}$$

we obtain that problem (2.108) has the unique solution $(u_1, u_2) = T(v_1, v_2)$ satisfying (2.100) with C independent of R (since H is bounded on $\mathbb{R}_+^2$). Thus for large R the nonlinear operator T maps K_R into itself. Since $W_p^1(\Omega) \times W_q^2(\partial_1\Omega)$ embeds compactly into $W_p^s(\Omega) \times L_q(\partial_1\Omega)$ the nonlinear operator T is compact. Since H is globally Lipchitz we conclude that T is continuous in the topology of B. Thus, using Schauder fixed point theorem, T has a fixed point, which additionally satisfies (2.81).

Step 2 - uniqueness of solution.
Assume that $(u_1, u_2), (v_1, v_2)$ are two W_1^1 solutions of (2.77)-(2.78). Denoting $z_i = u_i - v_i$ for $i = 1, 2$ we have:

$$div(J_h(z_1)) + b_1 z_1 = 0, \qquad x \in \Omega$$
$$-d\partial_{x_1}^2 z_2 - c_1 z_1 + (b_2 + c_2)z_2 + k_2(H(u_1, u_2)u_2 - H(v_1, v_2)v_2) = 0, \qquad x \in \partial_1\Omega$$

with boundary conditions

$$-J_h(z_1)\nu = 0, \qquad x \in \partial_0\Omega$$
$$-J_h(z_1)\nu = -c_1 z_1 - k_1(H(u_1, u_2)u_1 - H(v_1, v_2)v_1) + c_2 z_2, \qquad x \in \partial_1\Omega$$
$$\partial_{x_1} z_2 = 0, \qquad x \in \partial\partial_1\Omega.$$

Define

$$\begin{aligned}
D &= (k_1 u_1 + k_2 u_2 + b_3)(k_1 v_1 + k_2 v_2 + b_3), \\
w_i &= (u_i + v_i)/2,\ i = 1, 2.
\end{aligned}$$

Notice that

$$
\begin{aligned}
u_1v_2 - u_2v_1 &= z_1(u_2+v_2)/2 - z_2(u_1+v_1)/2 = z_1w_2 - z_2w_1,\\
H(u_1,u_2)u_1 - H(v_1,v_2)v_1 &= p_3\Big(\frac{u_1}{k_1u_1+k_2u_2+b_3} - \frac{v_1}{k_1v_1+k_2v_2+b_3}\Big)\\
&= \frac{p_3}{D}(k_2(u_1v_2-u_2v_1)+b_3z_1)\\
&= \frac{p_3}{D}((k_2w_2+b_3)z_1 - k_2w_1z_2),\\
H(u_1,u_2)u_2 - H(v_1,v_2)v_2 &= p_3\Big(\frac{u_2}{k_1u_1+k_2u_2+b_3} - \frac{v_2}{k_1v_1+k_2v_2+b_3}\Big)\\
&= \frac{p_3}{D}(-k_1(u_1v_2-u_2v_1)+b_3z_2)\\
&= \frac{p_3}{D}(-k_1w_2z_1 + (k_1w_1+b_3)z_2).
\end{aligned}
$$

Thus

$$
\begin{aligned}
div(J_h(z_1)) + b_1z_1 &= 0, && x\in\Omega\\
-d\partial_{x_1}^2 z_2 - (c_1 + \frac{k_1k_2p_3w_2}{D})z_1 + (b_2 + \frac{k_2p_3b_3}{D} + c_2 + \frac{k_1k_2p_3w_1}{D})z_2 &= 0, && x\in\partial_1\Omega
\end{aligned}
$$

with boundary conditions

$$
\begin{aligned}
-J_h(z_1)\nu &= 0, && x\in\partial_0\Omega\\
-J_h(z_1)\nu + (\frac{k_1p_3b_3}{D} + c_1 + \frac{k_1k_2p_3w_2}{D})z_1 - (c_2 + \frac{k_1k_2p_3w_1}{D})z_2 &= 0, && x\in\partial_1\Omega\\
\partial_{x_1}z_2 &= 0, && x\in\partial\partial_1\Omega.
\end{aligned}
$$

Hence, using the notation introduced in Lemma 2.18, (z_1,z_2) is a W_1^1 solution of (2.98),(2.99) with

$$
\begin{aligned}
&\lambda = \min\{b_1,b_2\},\ a_0 = b_1-\lambda,\ \mu_\Omega = 0,\ \mu_I = 0\\
&a_{11} = \frac{k_1p_3b_3}{D} + c_1 + \frac{k_1k_2p_3w_2}{D}, && a_{12} = c_2 + \frac{k_1k_2p_3w_1}{D},\\
&a_{21} = c_1 + \frac{k_1k_2p_3w_2}{D}, && a_{22} = b_2 - \lambda + \frac{k_2p_3b_3}{D} + c_2 + \frac{k_1k_2p_3w_1}{D}.
\end{aligned}
$$

Since the nonnegativity of w_1, w_2 ensures that assumption (2.97) is fulfilled we infer that $z_1 = z_2 = 0$.

2.3.4. Proof of Theorem 2.5

Since the spaces $W_p^1(\Omega)$ and $W_q^2(\partial_1\Omega)$ are reflexive for $1<p<2,\ 1<q<\infty$ thus, owing to (2.81), there exists a sequence $(h_k)_{k=1}^\infty \subset (0,1]$ such that $\lim_{k\to\infty} h_k = 0$ and

$$u_1^{h_k} \rightharpoonup w_1 \quad \text{in} \quad W_p^1(\Omega), \tag{2.110a}$$

$$u_2^{h_k} \rightharpoonup w_2 \quad \text{in} \quad W_q^2(\partial_1\Omega). \tag{2.110b}$$

Now we claim that

$$\partial_{x_2} w_1 \equiv 0, \tag{2.111a}$$

$$u_1^{h_k}(0,\cdot) \to w_1(0,\cdot) \quad \text{in} \quad L_q(\partial_1\Omega), \tag{2.111b}$$

$$u_2^{h_k} \to w_2 \quad \text{in} \quad X. \tag{2.111c}$$

Indeed (2.111a) comes from (2.81). To prove (2.111b) fix any $1 < q < \infty$, then choose s, p such that $1 < p < 2, 1/p < s < 1, s-2/p \geq -1/q$. Then $W^1_p(\Omega)$ embeds compactly into $W^s_p(\Omega)$, the trace operator maps $W^s_p(\Omega)$ into $W^{s-1/p}_p(\partial_1\Omega)$ and the latter space embeds continuously into $L_q(\partial_1\Omega)$. Finally (2.111c) follows from compact embedding of $W^2_q(\partial_1\Omega)$ into $C(\overline{\partial_1\Omega})$. Choose $v_1 \in C^1(\overline{\Omega}),\ v_2 \in C^1(\overline{\partial_1\Omega})$, then by (2.80)

$$\int_\Omega [\partial_{x_1} u_1^{h_k} \partial_{x_1} v_1 + b_1 u_1^{h_k} v_1] + \int_{\partial_1\Omega} [d\partial_{x_1} u_2^{h_k} \partial_{x_1} v_2 - c_1 u_1^{h_k} v_2] = p_1 v_1(0),$$
$$\int_{\partial_1\Omega} [c_1 H(u_1^{h_k}, u_2^{h_k}) u_1^{h_k} v_1 - c_2 u_2^{h_k} v_1 + (b_2 + c_2 H(u_1^{h_k}, u_2^{h_k}) v_2)] = 0.$$

Using (2.110) and (2.111) we can pass to the limit with $k \to \infty$ and identify that $(w_1, w_2) = (u_1^0, u_2^0)$ is a solution of (2.83). Finally notice that (2.82) follows from (2.110) and the fact that (2.83) has a unique solution, as proved in Section 2.4.

2.4. Limit problem

After nondimensionalisation the **[HKCS].1D** model reads:

$$\begin{aligned}
\partial_t u_1 - \partial^2_{xx} u_1 &= -(b_1 + c_1 + u_3)u_1 + c_2 u_2 + c_4 u_4 + p_1 \delta, && (t,x) \in I_T && (2.112a)\\
\partial_t u_2 - d\partial^2_{xx} u_2 &= -(b_2 + c_2 + c_3 u_3)u_2 + c_1 u_1 + c_5 u_5, && (t,x) \in I_T && (2.112b)\\
\partial_t u_3 &= -(b_3 + u_1 + c_3 u_2)u_3 + c_4 u_4 + c_5 u_5 + p_3, && (t,x) \in I_T && (2.112c)\\
\partial_t u_4 &= -(b_4 + c_4)u_4 + u_1 u_3, && (t,x) \in I_T && (2.112d)\\
\partial_t u_5 &= -(b_5 + c_5)u_5 + c_3 u_2 u_3, && (t,x) \in I_T && (2.112e)
\end{aligned}$$

with boundary and initial conditions

$$\begin{aligned}
\partial_x u_1 = \partial_x u_2 &= 0, && (t,x) \in (\partial I)_T\\
\boldsymbol{u}(0,\cdot) &= \boldsymbol{u_0}, && x \in I
\end{aligned}$$

The aim of this section is to establish well-posedness of (2.112) and to prove existence of a unique steady state.

During the analysis we encounter the following difficulties:

- absence of diffusion in equations (2.112c),(2.112d),(2.112e) so that there is no smoothing effect for u_3, u_4, u_5,
- singular source term in (2.112a),
- nonsymmetric zero order part of the operator which appears in the stationary problem.

We first solve the stationary problem for (2.112) by using Schauder's fixed point theorem. The key observation is that the linear operator which appears in the definition of $T_n(v)$ (see proof of Theorem 2.6), has a diagonally dominant structure. This leads us to analyse the problem in an L_1 setting rather than L_2. To prove uniqueness we consider the system which is satisfied by the difference of two possible solutions and after algebraic manipulations show that it also has a diagonally dominant structure.
To remove the singularity $p_1\delta$ from (2.112a) we change variables $\boldsymbol{z} = \boldsymbol{u} - \boldsymbol{u}^*$, where $\boldsymbol{u}^*$ is the steady

state to (2.112). Then local well-posedness in the space of continuous functions of the system for $\mathbf{z}$ follows from the classical perturbation theory for sectorial operators. To prove global existence we notice that the quasipositivity of the vector field which appears on the right hand side of (2.112) guarantees that the semiflow generated by (2.112) preserves the positive cone. Then using compensation effects it is easy to show that $u_3, u_4, u_5 \in L_\infty(0, T_{max}; C(I))$ and $u_1, u_2 \in L_\infty(0, T_{max}; L_1(I))$. Finally using smoothing effects of the heat semigroup we prove that $u_1, u_2 \in L_\infty(0, T_{max}; C(I))$, from which we finally conclude that the system (2.112) is globally well posed and has bounded trajectories.

Before stating the results precisely we introduce the notation and function spaces which we will use to analyse the system (2.112).

2.4.1. Function spaces

To analyse the problem we will use the following Banach spaces

$$X = C(\overline{I}),\ X_1 = C^2_N(\overline{I}) = \{u\colon u \in C^2(\overline{I}),\ u'(-1) = u'(1) = 0\},\ X_{1/2} = W^1_\infty(I),$$
$$Y = L_1(I),\ Y_1 = W^2_{1,N}(I) = \{u\colon u \in W^2_1(I), u'(-1) = u'(1) = 0\}.$$

Notice that due to the imbedding $W^2_1(I) \subset C^1(\overline{I})$ the boundary conditions in the definition of Y_1 are meaningful.

2.4.2. The results of Section 2.4

From now on we assume that

$$d, \boldsymbol{b} > 0,\ \boldsymbol{c}, \boldsymbol{p} \geq 0,\ \boldsymbol{u_0} \in X^5_+.$$

We start with the analysis of the stationary problem and prove that there exists unique nonnegative steady state. Observe that due to the absence of diffusion in (2.112c),(2.112d),(2.112e) the stationary problem reduces to the system (2.115) (see below) of two semilinear elliptic equations for u_1^* and u_2^*.

Theorem 2.6. *System* (2.112) *possesses a unique nonnegative steady state* $\boldsymbol{u^*} \in X_{1/2} \times X_1 \times X^3_{1/2}$ *such that*

$$u_3^* = H(u_1^*, u_2^*),\ b_4 u_4^* = k_1 u_1^* H(u_1^*, u_2^*),\ b_5 u_5^* = k_2 u_2^* H(u_1^*, u_2^*), \tag{2.113a}$$

where

$$k_1 = b_4/(b_4 + c_4),\ k_2 = c_3 b_5/(b_5 + c_5),\ H(x_1, x_2) = p_3/(k_1 x_1 + k_2 x_2 + b_3) \tag{2.114}$$

and (u_1^*, u_2^*) *is a solution of the following boundary value problem*

$$-u_1^{*\prime\prime} + (b_1 + c_1 + k_1 H(u_1^*, u_2^*))u_1^* - c_2 u_2^* = p_1 \delta, \qquad x \in I \tag{2.115a}$$
$$-d u_2^{*\prime\prime} - c_1 u_1^* + (b_2 + c_2 + k_2 H(u_1^*, u_2^*))u_2^* = 0, \qquad x \in I \tag{2.115b}$$
$$u_1^{*\prime} = u_2^{*\prime} = 0, \qquad x \in \partial I \tag{2.115c}$$

i.e. for every $\varphi \in X_{1/2}$

$$\int_I [u_1^{*\prime}\varphi' + ((b_1 + c_1 + k_1 H(u_1^*, u_2^*))u_1^* - c_2 u_2^*)\varphi] = p_1 \varphi(0)$$

and (2.115b) *is satisfied in the classical sense. Moreover*

$$u_1^* + p_1|x|/2 \in C^2(\overline{I}). \tag{2.116}$$

A typical shape of the steady state is to be found in Figure 2.2 present at the end of this chapter. Numerical scheme based on the standard finite difference method was implemented in Octave. To approximate δ we have used Gaussian with small variance.

In the following remark we analyse the behavior of the stationary solution near the source of morphogen.

Remark 4 ($\boldsymbol{u}^*$ near $x = 0$)**.** *Observe that as $\boldsymbol{u}^*$ is unique it must be even. Indeed otherwise $\boldsymbol{u}^*(-x)$ would be a second solution as the system* (2.115) *is invariant under the transformation $x \to -x$. Thus using* (2.116)

$$(u_1^*)'(0^+) = -p_1/2 < 0, \tag{2.117a}$$
$$(u_2^*)'(0) = 0, \tag{2.117b}$$

where $(u_1^*)'(0^+)$ denotes the right-sided derivative of u_1^* at $x = 0$.

Using (2.113) *and* (2.117) *we compute directly*

$$(u_3^*)'(0^+) = \frac{p_1 k_1}{2p_3}[H(u_1^*(0), u_2^*(0))]^2 > 0, \tag{2.118a}$$
$$(u_4^*)'(0^+) = -\frac{p_1 k_1 k_2}{2p_3 b_4}[H(u_1^*(0), u_2^*(0))]^2(u_2^*(0) + b_3/k_2) < 0, \tag{2.118b}$$
$$(u_5^*)'(0^+) = \frac{p_1 k_1 k_2}{2p_3 b_5}[H(u_1^*(0), u_2^*(0))]^2 u_2^*(0) > 0. \tag{2.118c}$$

In particular from (2.118b), (2.118c) *and the fact that $\boldsymbol{u}^*$ is even we infer that $u_4^*(0)$ (resp. $u_5^*(0)$) is a strict local maximum (resp. minimum), which explains the difference near $x = 0$ in u_4^* (steep spike) and u_5^* (depletion effect) as observed in Figure 2.2.*

Next we turn our attention to the evolution problem and establish its well-posedness and the uniform boundedness of trajectories in X^5.

Theorem 2.7. *System* (2.112) *possesses a unique, global in time, nonnegative solution*

$$u_1 \in C([0,\infty); X) \cap C^1((0,\infty); X) \cap C((0,\infty); X_{1/2}) \tag{2.119a}$$
$$u_2 \in C([0,\infty); X) \cap C^1((0,\infty); X) \cap C((0,\infty); X_1) \tag{2.119b}$$
$$u_3, u_4, u_5 \in C^1([0,\infty); X) \tag{2.119c}$$

such that for every $\varphi \in X_{1/2}, t \in (0,\infty)$

$$\int_I \partial_t u_1 \varphi + D\int_I \partial_x u_1 \partial_x \varphi = \int_I [-(b_1 + c_1 + u_3)u_1 + c_2 u_2 + c_4 u_4]\varphi + p_1\varphi(0)$$

and other equations are satisfied in the sense of X. Moreover $\boldsymbol{u} \in L_\infty(0,\infty; X^5)$ and the following estimates hold

$$\sum_{i=3}^{5} u_i(t) \le e^{-\underline{\boldsymbol{b}}t}\sum_{i=3}^{5} u_{i0} + p_3(1 - e^{-\underline{\boldsymbol{b}}t})/\underline{\boldsymbol{b}}, \tag{2.120a}$$
$$\sum_{i\in\{1,2,4,5\}} \|u_i(t)\|_Y \le e^{-\underline{\boldsymbol{b}}t}\sum_{i\in\{1,2,4,5\}} \|u_{i0}\|_Y + p_1(1 - e^{-\underline{\boldsymbol{b}}t})/\underline{\boldsymbol{b}}. \tag{2.120b}$$

We conclude with a remark concerning the discussion about the asymptotic behavior.

Remark 5 (Asymptotics)**.** *For the case of morphogen Dpp acting in the imaginal wing disc of the fruit fly without the presence of glypicans, it is proved in [15] that the morphogen gradient (i.e. steady state of the appropriate evolution system) is globally exponentially stable. It is expected that an analogous result should hold for* ***[HKCS].1D****, though we are not able to prove even the local stability of the steady state. However it may also be the case that the presence of glypicans has a destabilising effect on the equilibrium within a certain range of parameters.*

2.4.3. Lemmas

In this section we collect lemmas which are used in the proofs of the results. The first lemma states that realisations of one dimensional Laplace operator in the chosen Banach spaces are sectorial. Since this result is well known we state it only to make the dissertation more self-contained.

For $Z \in \{X, Y\}$ we define the Z-realisation of the Laplace operator with Neumann boundary condition:

$$A_Z : Z \supset Z_1 \to Z,\ A_Z u = u'',\ u \in Z_1.$$

Proof. For the proof we refer the interested reader to [[22], Chapter 3.1]. □

Lemma 2.19. *A_Z is a sectorial, densely defined operator with compact resolvent. It generates an analytic, strongly continuous semigroup e^{tA_Z} and for $t > 0$ the following estimates hold*

$$\|e^{tA_Z}\|_{\mathcal{L}(Z)} \le 1, \quad \|e^{tA_Y}\|_{\mathcal{L}(Y,X)} \le C(1 \wedge t)^{-1/2}.$$

Moreover (A_X, e^{tA_X}) is a restriction of (A_Y, e^{tA_Y}) to X i.e.

$$A_X u = A_Y u,\ u \in X_1, \quad e^{tA_X} u = e^{tA_Y} u,\ (t, u) \in [0, \infty) \times X.$$

The second lemma concerns solvability of linear elliptic systems with diagonally dominant zero order term. It is crucial in the proofs of existence and uniqueness of the steady state of the system (2.112).

Lemma 2.20. *Assume that for $i, j = 1, 2,\ d_i > 0, a_{ij} \in X_+$ and*

$$a_{11} \ge a_{21}, \quad a_{22} \ge a_{12}. \tag{2.121a}$$

Define operators

$$\begin{aligned} &M : Y^2 \to Y^2,\ Mu = (-a_{11}u_1 + a_{12}u_2, a_{21}u_1 - a_{22}u_2), \\ &G : Y^2 \supset Y_1^2 \to Y^2,\ G = (d_1 A_Y) \times (d_2 A_Y) + M. \end{aligned}$$

Then G is a sectorial, densely defined operator with a compact resolvent $R(\lambda, G) = (\lambda - G)^{-1}$ and the following hold

$$(0, \infty) \subset \rho(G) \text{ and } \|R(\lambda, G)\|_{\mathcal{L}(Y^2)} \le 1/\lambda, \tag{2.122a}$$

$$\|R(\lambda, G)\|_{\mathcal{L}(Y^2, Y_1^2)} \le C(1 + 1/\lambda), \tag{2.122b}$$

$$R(\lambda, G) \text{ preserves } Y_+^2, \tag{2.122c}$$

where $\lambda > 0$ and C depends only on $d_i, \|a_{ij}\|_X$.

Proof. To prove that G is sectorial and has a compact resolvent notice that it is a perturbation of the operator $(d_1 A_Y) \times (d_2 A_Y)$ having these two properties by a bounded operator $M \in \mathcal{L}(Y^2)$. From the compactness of the resolvent of G we get that the spectrum $\sigma(G)$ only contains eigenvalues (see [[14], Theorem 6.29]).
In the rest of the proof we will use the following observation. For $a \leq 0 \leq b$ define

$$\gamma(x) = \begin{cases} a, & x < 0 \\ 0, & x = 0 \\ b, & x > 0. \end{cases}$$

Then for every x, y one has $x\gamma(x) \geq 0$ and $x\gamma(x) \geq x\gamma(y)$. Using (2.121) we obtain the following pointwise inequality

$$\Big(Mu, (\gamma(u_1)|\gamma(u_2))\Big)_{\mathbb{R}^2} = -a_{11}u_1\gamma(u_1) + a_{12}u_2\gamma(u_1) + a_{21}u_1\gamma(u_2) - a_{22}u_2\gamma(u_2) \tag{2.123a}$$

$$\leq -u_1\gamma(u_1)(a_{11} - a_{21}) - u_2\gamma(u_2)(a_{22} - a_{12}) \leq 0. \tag{2.123b}$$

Choose $\lambda > 0,\ f \in Y^2, u \in Y_1^2$ such that

$$f = (\lambda - G)u. \tag{2.124}$$

To proove (2.122a) we estimate

$$\|f\|_{Y^2} \geq \int_I \Big(f, (sgn(u_1)|sgn(u_2))\Big)_{\mathbb{R}^2} = \lambda\|u\|_{Y^2} - \sum_{i=1}^{2} d_i \int_I u_i'' sgn(u_i)$$
$$- \int_I \Big(Mu, (sgn(u_1), sgn(u_2))\Big)_{\mathbb{R}^2} \geq \lambda\|u\|_{Y^2},$$

where we used (2.123) with $\gamma = sgn$ and the following Kato's inequality (see Lemma 2 in [5])

$$-\int_I v'' sgn(v) \geq 0, \quad v \in Y_1. \tag{2.125}$$

To prove (2.122b) observe that from (2.124),(2.122a) we have

$$\|Gu\|_{Y^2} \leq \|f\|_{Y^2} + \lambda\|u\|_{Y^2} \leq 2\|f\|_{Y^2},$$

whence

$$\begin{aligned}\|u\|_{Y_1^2} &\leq C(\|(A_Y u_1, A_Y u_2)\|_{Y^2} + \|u\|_{Y^2}) \leq C[(d_1 \wedge d_2)^{-1}\|Gu - Mu\|_{Y^2} + \|f\|_{Y^2}/\lambda] \\ &\leq C[(d_1 \wedge d_2)^{-1}(\|Gu\|_{(Y)^2} + \|M\|_{\mathcal{L}((Y)^2)}\|u\|_{Y^2}) + \|f\|_{Y^2}/\lambda] \\ &\leq C[(d_1 \wedge d_2)^{-1}(2 + \|M\|_{\mathcal{L}((Y)^2)}/\lambda) + 1/\lambda]\|f\|_{Y^2} \\ &\leq C(1 + 1/\lambda)\|f\|_{Y^2}.\end{aligned}$$

Finally to prove (2.122c) fix $f \in Y_+^2$. Using (2.123) with

$$\gamma(x) = \begin{cases} x_-/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

we obtain

$$0 \geq \int_I \Big(f, (\gamma(u_1)|\gamma(u_2))\Big)_{\mathbb{R}^2} = \lambda \sum_{i=1}^{2} \int_I u_i\gamma(u_i) - \frac{1}{2}\sum_{i=1}^{2} d_i \int_I u_i'' sgn(u_i) - \int_I \Big(Mu|(\gamma(u_1), \gamma(u_2))\Big)_{\mathbb{R}^2}$$
$$\geq \lambda\|u_-\|_{Y^2},$$

whence $u \geq 0$. ∎

2.4.4. Proof of Theorem 2.6

We divide the proof into two steps. To prove existence of a solution of (2.115) we first approximate the singular source term $p_1\delta$ by more regular functions $h_n \in Y_+$. Using Schauder's fixed point theorem we prove solvability of the approximated problem. Finally using compactness methods we show that the approximated solutions converge to a solution of (2.115). In the proof of uniqueness we show that the difference of any two possible steady states belongs to the kernel of a certain operator $\lambda - G$, where $\lambda > 0$ and G satisfies assumptions of Lemma 2.20.

Step 1 - Existence of solutions

Choose a sequence $h_n \in Y_+$ such that $h_n \rightharpoonup^* \delta$ in $\mathcal{M}([-1,1])$ - the space of signed Radon measures. For $\mathbf{v} \in (X_+)^2$ consider the following problem

$$-u_1'' + (b_1 + c_1 + k_1 H(v_1, v_2))u_1 - c_2 u_2 = p_1 h_n, \qquad x \in I \tag{2.126a}$$
$$-du_2'' - c_1 u_1 + (b_2 + c_2 + k_2 H(v_1, v_2))u_2 = 0, \qquad x \in I, \tag{2.126b}$$
$$u_1' = u_2' = 0, \qquad x \in \partial I, \tag{2.126c}$$

where H is defined in (2.114). Using notation introduced in Lemma 2.20 system (2.126) is equivalent to

$$(\lambda - G)(u_1, u_2) = (p_1 h_n, 0),$$

where

$$\begin{aligned}
&\lambda = \underline{\boldsymbol{b}}, \\
&d_1 = 1, d_2 = d, \\
&a_{11} = b_1 - \underline{\boldsymbol{b}} + c_1 + k_1 H(v_1, v_2), \qquad && a_{12} = c_2, \\
&a_{21} = c_1, && a_{22} = b_2 - \underline{\boldsymbol{b}} + c_2 + k_2 H(v_1, v_2).
\end{aligned}$$

Observe that condition (2.121) holds, thus using Lemma 2.20 we obtain that (2.126) has a unique solution $(u_1, u_2) \in Y_{1,+}^2$ and there exists C_1 which does not depend on $(v_1, v_2), (u_1, u_2), h_n$ such that

$$\|(u_1, u_2)\|_{Y_1^2} \leq C_1 \|h_n\|_Y. \tag{2.127}$$

Using the compact imbedding

$$Y_1 \subset\subset X \tag{2.128}$$

and (2.127) we obtain that there exists C_2 such that

$$\|(u_1, u_2)\|_{(X)^2} \leq C_2 \|(u_1, u_2)\|_{Y_1^2} \leq C_1 C_2 \|h_n\|_Y. \tag{2.129}$$

Define

$$V_n = \{(v_1, v_2) \in (X_+)^2 \colon \|(v_1, v_2)\|_{(X)^2} \leq C_1 C_2 \|h_n\|_Y\},$$
$$T_n \colon V_n \to V_n, \; T_n(v_1, v_2) = (u_1, u_2),$$

where (u_1, u_2) is the solution of (2.126). Observe that V_n is a closed and convex subset of a Banach space $(X)^2$ and T_n is well defined, and continuous (because H is a globally Lipschitz continuous function on $\mathbb{R}_+^2$). Moreover due to (2.128) and (2.129) $T_n(V_n)$ is precompact. Hence by Schauder's

theorem T_n has a fixed point $(u^*_{n,1}, u^*_{n,2}) \in V_n$.
Since $(h_n)_{n=1}^\infty$ is bounded in Y we get, by (2.129), that $(u^*_{n,1}, u^*_{n,2})_{n=1}^\infty$ is bounded in Y_1^2. From the imbeddings $Y_1 \subset X_{1/2} \subset\subset X$ there exist $(u_1^*, u_2^*) \in X_{1/2}$ and a subsequence $(u^*_{n_k,1}, u^*_{n_k,2})_{k=1}^\infty$ such that for $i = 1, 2$

$$(u^*_{n_k,i})' \rightharpoonup^* (u_i^*)', \text{ in } L_\infty(I)$$
$$u^*_{n_k,i} \to u_i^*, \text{ in } X.$$

Fix $\varphi \in X_{1/2}$, then since $T_{n_k}(u^*_{n_k,1}, u^*_{n_k,2}) = (u^*_{n_k,1}, u^*_{n_k,2})$ we have:

$$\int_I (u^*_{n_k,1})'\varphi' + [(b_1 + c_1 + k_1 H(u^*_{n_k,1}, u^*_{n_k,2}))u^*_{n_k,1} - c_2 u^*_{n_k,2}]\varphi = p_1 \int_I h_{n_k}\varphi, \tag{2.130a}$$

$$d\int_I (u^*_{n_k,2})'\varphi' + [-c_1 u^*_{n_k,1} + (b_2 + c_2 + k_2 H(u^*_{n_k,1}, u^*_{n_k,2}))u^*_{n_k,2}]\varphi = 0, \tag{2.130b}$$

Using again the fact that H is globally Lipschitz continuous on $\mathbb{R}^2_+$ we can pass in (2.130) with $n_k \to \infty$ and obtain that (u_1^*, u_2^*) is a solution of (2.126).

Step 2 - Uniqueness

Assume that $(u_1, u_2), (v_1, v_2)$ are two solutions of (2.115). Noting $z_i = u_i - v_i$ for $i = 1, 2$ we have:

$$-z_1'' + (b_1 + c_1)z_1 - c_2 z_2 + k_1(H(u_1, u_2)u_1 - H(v_1, v_2)v_1) = 0$$
$$-dz_2'' - c_1 z_1 + (b_2 + c_2)z_2 + k_2(H(u_1, u_2)u_2 - H(v_1, v_2)v_2) = 0.$$

Define

$$D = (k_1 u_1 + k_2 u_2 + b_3)(k_1 v_1 + k_2 v_2 + b_3)$$
$$w_i = (u_i + v_i)/2, \ i = 1, 2$$

and compute

$$u_1 v_2 - u_2 v_1 = z_1(u_2 + v_2)/2 - z_2(u_1 + v_1)/2 = z_1 w_2 - z_2 w_1$$
$$H(u_1, u_2)u_1 - H(v_1, v_2)v_1 = p_3\Big(\frac{u_1}{k_1 u_1 + k_2 u_2 + b_3} - \frac{v_1}{k_1 v_1 + k_2 v_2 + b_3}\Big)$$
$$= \frac{p_3}{D}(k_2(u_1 v_2 - u_2 v_1) + b_3 z_1) = \frac{p_3}{D}((k_2 w_2 + b_3)z_1 - k_2 w_1 z_2)$$
$$H(u_1, u_2)u_2 - H(v_1, v_2)v_2 = p_3\Big(\frac{u_2}{k_1 u_1 + k_2 u_2 + b_3} - \frac{v_2}{k_1 v_1 + k_2 v_2 + b_3}\Big)$$
$$= \frac{p_3}{D}(-k_1(u_1 v_2 - u_2 v_1) + b_3 z_2) = \frac{p_3}{D}(-k_1 w_2 z_1 + (k_1 w_1 + b_3)z_2).$$

Thus

$$-z_1'' + (b_1 + \frac{k_1 p_3 b_3}{D} + c_1 + \frac{k_1 k_2 p_3 w_2}{D})z_1 - (c_2 + \frac{k_1 k_2 p_3 w_1}{D})z_2 = 0$$
$$-dz_2'' - (c_1 + \frac{k_1 k_2 p_3 w_2}{D})z_1 + (b_2 + \frac{k_2 p_3 b_3}{D} + c_2 + \frac{k_1 k_2 p_3 w_1}{D})z_2 = 0.$$

Hence, using the notation introduced in Lemma 3, (z_1, z_2) belongs to the kernel of the operator $\underline{\mathbf{b}} - G$ where

$$d_1 = 1, d_2 = d$$
$$a_{11} = b_1 - \underline{\boldsymbol{b}} + \frac{k_1 p_3 b_3}{D} + c_1 + \frac{k_1 k_2 p_3 w_2}{D}, \qquad a_{12} = c_2 + \frac{k_1 k_2 p_3 w_1}{D},$$
$$a_{21} = c_1 + \frac{k_1 k_2 p_3 w_2}{D}, \qquad a_{22} = b_2 - \underline{\boldsymbol{b}} + \frac{k_2 p_3 b_3}{D} + c_2 + \frac{k_1 k_2 p_3 w_1}{D}.$$

Since nonnegativity of w_1, w_2 ensures that assumption (2.121) is fulfilled we infer that $z_1 = z_2 = 0$ which finishes the proof.

Higher regularity of u_1^* outside $x = 0$

Observe that $E = -p_1|x|/2$ satisfies $-E'' = p_1\delta$ in the sense of distributions. Owing to (2.115a) $v = u_1^* - E$ solves the following boundary value problem

$$\begin{aligned} -v'' &= f, && x \in I \\ v' &= -E', && x \in \partial I \end{aligned}$$

with $f = c_2 u_2^* - (b_1 + c_1 + k_1 H(u_1^*, u_2^*))u_1^*$. Since $f \in X$ then (2.116) follows.

2.4.5. Proof of Theorem 2.7

Using the theory of analytic semigroups we first establish the local well-posedness of (2.112). Using quasipositivity of the right hand side of (2.112) we next prove that the generated semiflow preserves nonnegativity of initial conditions. Then using a compensation effect we derive $L_\infty(0,\infty,X)$ estimate for u_3, u_4, u_5 and $L_\infty(0,\infty,Y)$ estimate for u_1, u_2. Finally thanks to the regularising properties of the semigroup $e^{A_Y t}$ we bootstrap the estimate to $\boldsymbol{u} \in L_\infty(0,\infty,X^5)$.

Step 1 - local existence

We rewrite system (2.112) in the new variables $\boldsymbol{z} = \boldsymbol{u} - \boldsymbol{u}^*$, where $\mathbf{u}^*$ is the unique steady state of (2.112), and put it into the semigroup framework:

$$\boldsymbol{z}' - \boldsymbol{A_X z} = \boldsymbol{f}(\boldsymbol{z}), \qquad t > 0 \tag{2.131a}$$

$$\boldsymbol{z}(0) = \mathbf{z_0} = \boldsymbol{u_0} - \boldsymbol{u}^*, \tag{2.131b}$$

where

$$\begin{aligned} \boldsymbol{A} &= A_X \times (dA_X) \times 0^3 \\ \boldsymbol{f} &= (f_1, f_2, f_3, f_4, f_5) : (X)^5 \to (X)^5 \\ f_1(\boldsymbol{z}) &= -(b_1 + c_1)z_1 - (z_1 z_3 + u_1^* z_3 + u_3^* z_1) + c_2 z_2 + c_4 z_4 \\ f_2(\boldsymbol{z}) &= -(b_2 + c_2)z_2 - c_3(z_2 z_3 + u_3^* z_2 + u_2^* z_3) + c_1 z_1 + c_5 z_5 \\ f_3(\boldsymbol{z}) &= -b_3 z_3 - (z_1 z_3 + u_1^* z_3 + u_3^* z_1) - c_3(z_2 z_3 + u_3^* z_2 + u_2^* z_3) + c_4 z_4 + c_5 z_5 \\ f_4(\boldsymbol{z}) &= -(b_4 + c_4)z_4 + (z_1 z_3 + u_1^* z_3 + u_3^* z_1) \\ f_5(\boldsymbol{z}) &= -(b_5 + c_5)z_5 + c_3(z_2 z_3 + u_3^* z_2 + u_2^* z_3). \end{aligned}$$

Observe that $\boldsymbol{A}$ generates an analytic, strongly continuous semigroup in X^5 :
$e^{t\boldsymbol{A}} = e^{tA_X} \times e^{tdA_X} \times (Id)^3$. Moreover $\boldsymbol{f}$ is Lipschitz continuous on bounded subsets of X^5. Using Lemma 1.3 we obtain that (2.131) possesses a unique solution defined on a maximal time interval $[0, T_{max})$ with the following regularity:

$$\begin{aligned} z_1, z_2 &\in C([0,T_{max}); X) \cap C^1((0,T_{max}); X) \cap C((0,T_{max}); X_1) \\ z_3, z_4, z_5 &\in C^1([0,T_{max}); X). \end{aligned}$$

Setting $\boldsymbol{u} = \boldsymbol{z} + \boldsymbol{u^*}$ it is obvious that $\boldsymbol{u}$ is the unique solution to (2.112).

Step 2 - nonnegativity of solutions

Consider the following system

$$\partial_t v_1 - \partial^2_{xx} v_1 = -(b_1 + c_1 + (v_3)_+)v_1 + c_2(v_2)_+ + c_4(v_4)_+ + p_1\delta, \qquad (t,x) \in I_\infty \tag{2.132a}$$

$$\partial_t v_2 - d\partial^2_{xx} v_2 = -(b_2 + c_2 + c_3(v_3)_+)v_2 + c_1(v_1)_+ + c_5(v_5)_+, \qquad (t,x) \in I_\infty \tag{2.132b}$$

$$\partial_t v_3 = -(b_3 + (v_1)_+ + c_3(v_2)_+)v_3 + c_4(v_4)_+ + c_5(v_5)_+ + p_3, \qquad (t,x) \in I_\infty \tag{2.132c}$$

$$\partial_t v_4 = -(b_4 + c_4)v_4 + (v_1)_+(v_3)_+, \qquad (t,x) \in I_\infty \tag{2.132d}$$

$$\partial_t v_5 = -(b_5 + c_5)v_5 + c_3(v_2)_+(v_3)_+, \qquad (t,x) \in I_\infty \tag{2.132e}$$

with boundary and initial conditions

$$\partial_x v_1 = \partial_x v_2 = 0, \qquad (t,x) \in (\partial I)_\infty$$
$$\boldsymbol{v}(0,\cdot) = \boldsymbol{u_0}, \qquad x \in I.$$

Reasoning as in the previous section, the system (2.132) possesses a unique maximally defined solution $\mathbf{v}(t)$ on $[0, T'_{max})$ in $(X)^5$. We will now prove that $\mathbf{v(t)} \geq 0$ for $t \in [0, T'_{max})$. Multiplying (2.132) by $-(\mathbf{v})_-$ and adding equations we obtain:

$$\frac{d}{dt}\sum_{i=1}^{5} \|(v_i)_-\|_2^2 + \|\partial_x(v_1)_-\|_2^2 + 2d\|\partial_x(v_2)_-\|_2^2 \leq 0$$

$$\sum_{i=1}^{5} \|(v_i(0))_-\|_2^2 = 0.$$

Thus $\boldsymbol{v}(t) \geq 0$ for $t \in [0, T'_{max})$. Then $\boldsymbol{v}_+ = \mathbf{v}$ and it readily follows from (2.132) that $\boldsymbol{v}$ solves (2.112) on $[0, T'_{max})$. Consequently, $\boldsymbol{v} = \boldsymbol{u}$ on $[0, T'_{max})$ and $T'_{max} \leq T_{max}$. Finally observe that if $T'_{max} < \infty$ then, by (1.12), $\limsup_{t \to T'_{max}} \|\boldsymbol{u}(t)\|_{(X)^5} = \limsup_{t \to T'_{max}} \|\boldsymbol{v}(t)\|_{(X)^5} = \infty$ thus $T'_{max} = T_{max}$ and $\boldsymbol{u(t)} \geq 0$ on $[0, T_{max})$.

Step 3 - $u_3, u_4, u_5 \in L_\infty(0, T_{max}; X)$

Adding equations (2.112c),(2.112d),(2.112e) and using nonnegativity of $\boldsymbol{u}$, we obtain

$$\partial_t \sum_{i=3}^{5} u_i + \underline{\boldsymbol{b}} \sum_{i=3}^{5} u_i \leq p_3, \quad (t,x) \in [0, T_{max}) \times I.$$

Thus

$$0 \leq \sum_{i=3}^{5} u_i \leq e^{-\underline{\boldsymbol{b}}t} \sum_{i=3}^{5} u_{i0} + p_3(1 - e^{-\underline{\boldsymbol{b}}t})/\underline{\boldsymbol{b}}, \quad (t,x) \in [0, T_{max}) \times I. \tag{2.133}$$

Step 4 - $T_{max} = \infty$

Observe that due to (2.133)

$$\|z_i(t) z_3(t)\|_X \leq C\|z_i(t)\|_X, \; i = 1, 2.$$

Thus $\boldsymbol{f(z(t))}$ satisfies (1.13), whence $T_{max} = \infty$ by Lemma 1.3.

Step 5 - $u_1, u_2 \in L_\infty(0, \infty; Y)$

After integrating equations (2.112a), (2.112b), (2.112d), (2.112e) over the set I and adding them together we obtain

$$\frac{d}{dt}(\sum_{i\in\{1,2,4,5\}} \|u_i\|_Y) + \underline{\boldsymbol{b}} \sum_{i\in\{1,2,4,5\}} \|u_i\|_Y \leq p_1.$$

Thus

$$\sum_{i\in\{1,2,4,5\}} \|u_i(t)\|_Y \leq e^{-\underline{\boldsymbol{b}}t} \sum_{i\in\{1,2,4,5\}} \|u_{i0}\|_Y + p_1(1 - e^{-\underline{\boldsymbol{b}}t})/\underline{\boldsymbol{b}}. \tag{2.134}$$

Step 6 - $u_1, u_2 \in L_\infty(0, \infty; X)$

From (2.133), (2.134) we obtain that $f_1(\boldsymbol{z}) + z_1 \in L_\infty(0, \infty; Y)$. Using the Duhamel formula and estimates from Lemma 2.19 we get

$$\begin{aligned}\|z_1(t)\|_X &\leq e^{-t}\|e^{tA_X}\|_{\mathcal{L}(X)}\|z_{10}\|_X + \int_0^t e^{-s}\|e^{sA_Y}\|_{\mathcal{L}(Y,X)}\|f_1(\boldsymbol{z}(t-s)) + z_1(t-s)\|_Y ds \\ &\leq \|z_{10}\|_X + C\|f_1(\boldsymbol{z}) + z_1\|_{L_\infty(Y)} \int_0^\infty (1 \wedge s)^{-1/2} e^{-s} ds \leq \|z_{10}\|_X + C\|f_1(\boldsymbol{z}) + z_1\|_{L_\infty(Y)},\end{aligned}$$

whence $u_1 \in L_\infty(0, \infty; X)$. A similar argument gives $u_2 \in L_\infty(0, \infty; X)$ and completes the proof.

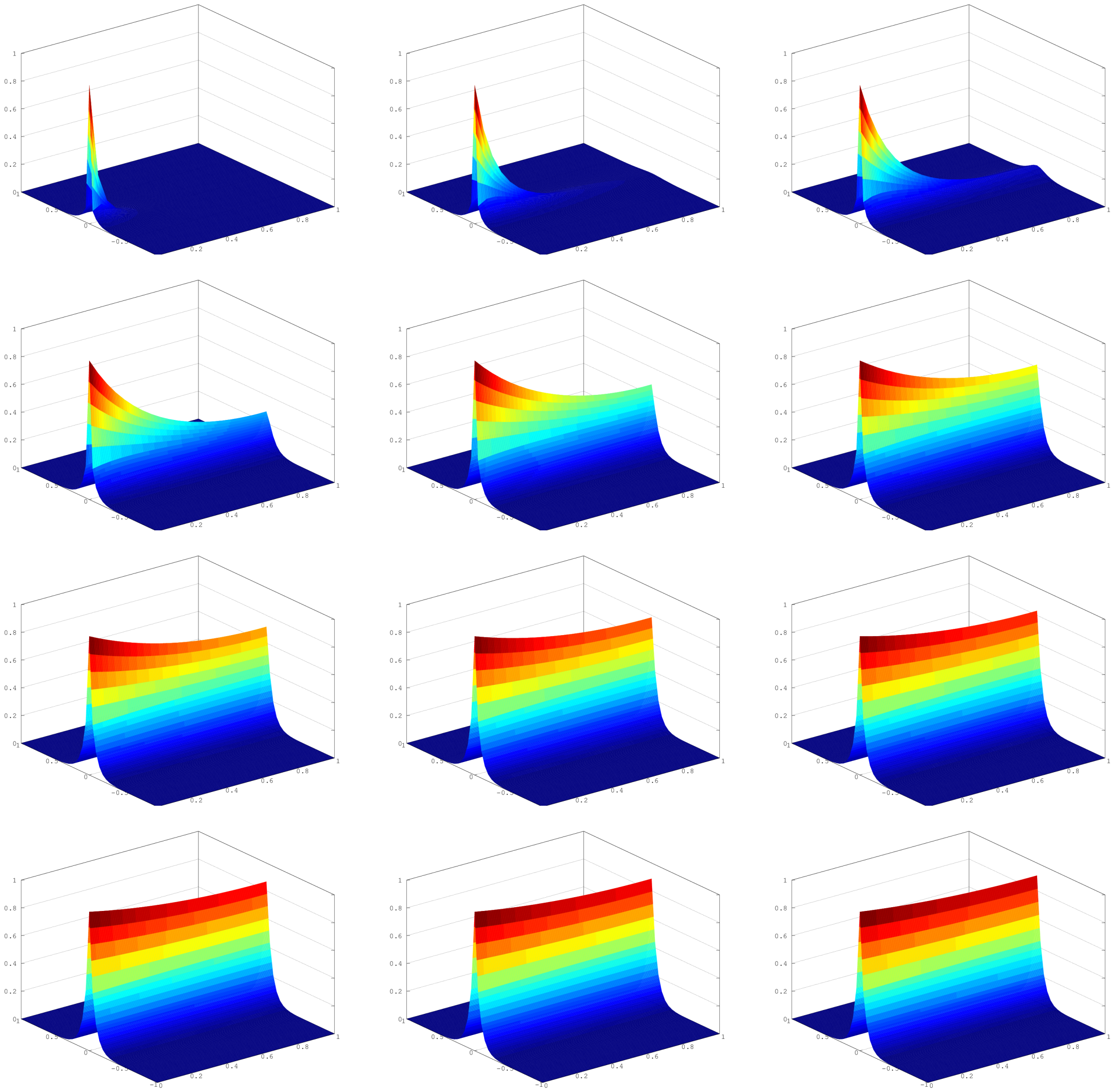

Figure 2.1: Graph of u_1^h - the stationary solution to problem (2.1) (normalised to 1) computed for the following values of parameters: $\mathbf{b} = [100, 10, 10, 10, 10]$, $\mathbf{c} = [10, 10, 1, 10, 10]$, $\mathbf{p} = [100, 0, 100, 0, 0]$, $d = 1/10$. First row - $h \in \{1, 1/3, 1/5\}$, second row - $h \in \{1/10, 1/15, 1/20\}$, third row - $h \in \{1/25, 1/30, 1/35\}$, fourth row - $h \in \{1/40, 1/45, 1/50\}$. A numerical scheme based on the finite difference method was implemented using the software Octave.

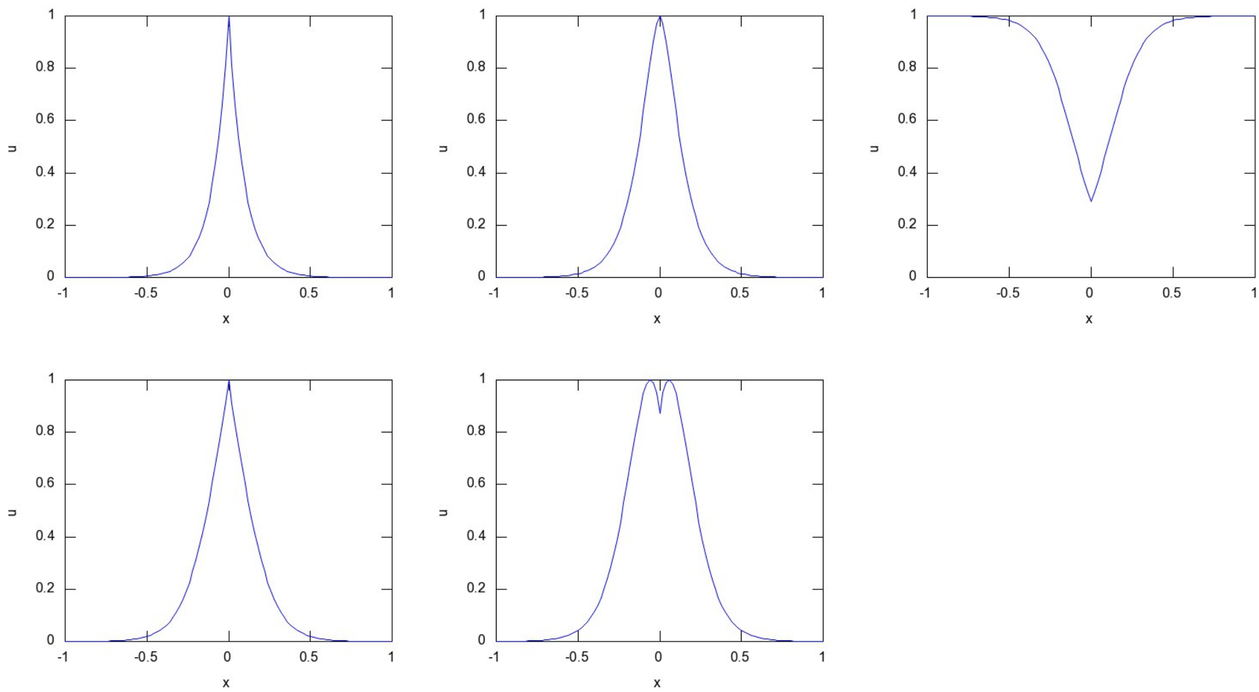

Figure 2.2: Graph of $\mathbf{u}^*$ - the stationary solution to problem (2.112) (normalised to 1) computed for the following values of parameters: $\mathbf{b} = [100, 10, 10, 10, 10]$, $\mathbf{c} = [10, 10, 1, 10, 10]$, $\mathbf{p} = [100, 0, 100, 0, 0]$, $d = 1/10$. First row - $u_1^*/\|u_1^*\|_X$,$u_2^*/\|u_2^*\|_X$,$u_3^*/\|u_3^*\|_X$, second row - $u_4^*/\|u_4^*\|_X$,$u_5^*/\|u_5^*\|_X$. Notice the surprising difference in behavior of u_4^* and u_5^* near $x = 0$ (see Remark 4 for explanation).

Chapter 3

Well-posedness and asymptotic behaviour in the [LNW].B model

3.1. Nondimensionalisation and basic assumptions

After nondimensionalisation the **[LNW].B** model reads

[LNW].B

$$
\begin{aligned}
\partial_t l - d\Delta l &= cs - l(1-s), && (t,x) \in (0,\infty)\times\Omega \\
\partial_t s &= -(c+b)s + l(1-s), && (t,x) \in (0,\infty)\times\Omega \\
d\frac{\partial l}{\partial \nu} &= g, && (t,x) \in (0,\infty)\times\Gamma_N \\
l &= 0, && (t,x) \in (0,\infty)\times\Gamma_D \\
l(0) &= l_0,\ s(0) = s_0, && x \in \Omega
\end{aligned}
$$

where by $\frac{\partial}{\partial\nu}$ we denote the derivative in the direction of the outer normal vector to Γ_N.

In the whole chapter we assume that

A1 $n \in \mathbb{N},\ p > n \geq 1.$

A2 $\Omega \subset \mathbb{R}^n$ is a bounded domain (open, connected) with ($C^{1,1}$) boundary which consists of two disjoint parts: $\partial\Omega = \Gamma_D \sqcup \Gamma_N$.

A3 $0 \leq g \in W_p^{1-1/p}(\Gamma_N)$.

A4 $l_0, s_0 \in W_p^1(\Omega)$; $\quad 0 \leq l_0(x), \quad 0 \leq s_0(x) < 1$, for $x \in \Omega$; $\quad l_0(x) = s_0(x) = 0$, for $x \in \Gamma_D$.

3.2. Notation and preliminaries

In this chapter C denotes a positive constant which may depend on a subset of $\{l_0, s_0, g, c, b, d, \Omega, p\}$ and may change its value from line to line.

For $1 < q < \infty, \alpha \in \{1,2\}$ we introduce the spaces $W^{\alpha}_{q,\mathcal{B}^\alpha}(\Omega)$:

$$W^1_{q,\mathcal{B}^1}(\Omega) = \{u \in W^1_q(\Omega) : u|_{\Gamma_D} = 0\},$$
$$W^2_{q,\mathcal{B}^2}(\Omega) = \{u \in W^2_q(\Omega) : u|_{\Gamma_D} = 0, \frac{\partial u}{\partial \nu}|_{\Gamma_N} = 0\},$$

with standard Sobolev norms $\|.\|_{\alpha,q}$.
We will often use the following version of Poincaré's inequality

$$\|u\|_q \le C\|\nabla u\|_q, \text{ for } u \in W^1_{q,\mathcal{B}^1}(\Omega), \tag{3.1}$$

where C depends only on on q and Ω.

In what follows we denote by A_q the $L_q(\Omega)$ realisation of the Laplace operator with mixed boundary condition.

$$A_q : L_q(\Omega) \supset D(A_q) \to L_q(\Omega),\ D(A_q) = W^2_{q,\mathcal{B}^2}(\Omega)\ A_q u = \Delta u \text{ for } u \in D(A_q).$$

The properties of operator A_q are collected in the following

Lemma 3.1.

1. *The operator A_q generates an analytic, strongly continuous semigroup e^{tA_q} for $1 < q < \infty$. Moreover $e^{tA_{q_1}}u = e^{tA_{q_2}}u$ for $1 < q_1 < q_2 < \infty$ and $u \in L_{q_2}(\Omega)$.*
2. *The spectrum of A_q does not depend on q and consists entirely of negative eigenvalues*

$$\sigma(A_q) = \sigma_p(A_q) = \{\lambda_i :\ i \in \mathbb{N}_+\},\ 0 > \lambda_1 \ge \lambda_2 \ge \dots.$$

3. *For $\alpha, \beta \in \{0,1,2\},\ \alpha \le \beta,\ 1 < q_1 \le q_2 < \infty$ and $t > 0$ the following estimates hold*

$$\|e^{tA_q}u\|_{\beta,q} \le C(t\wedge 1)^{(\alpha-\beta)/2}e^{\lambda_1 t}\|u\|_{\alpha,q} \le Ct^{(\alpha-\beta)/2}\|u\|_{\alpha,q}, \qquad u \in W^{\alpha}_{q,\mathcal{B}^\alpha}(\Omega) \tag{3.2a}$$
$$\|e^{tA_q}u\|_{q_2} \le C(t\wedge 1)^{-n/2(1/q_1-1/q_2)}e^{\lambda_1 t}\|u\|_{q_1} \le Ct^{-n/2(1/q_1-1/q_2)}\|u\|_{q_1}, \quad u \in L_{q_1}(\Omega) \tag{3.2b}$$

where $\lambda_1 < 0$ is the first eigenvalue of A_2 and C depends only on q, q_1, q_2, Ω.

Proof.
Step 1 Observe that Part 1 follows from [[2], Theorem 4.1].
Step 2 As a straightforward consequence of Part 1 we obtain that for any $1 < q < \infty$ the resolvent set $\rho(A_q)$ is not empty. Moreover using the compact imbedding

$$W^2_{q,\mathcal{B}^2}(\Omega) \subset\subset L_q(\Omega),$$

we get that $R(\lambda, A_q)$ is compact for $\lambda \in \rho(A_q)$ and thus $\sigma(A_q) = \sigma_p(A_q)$. We will now prove that

$$0 \in \rho(A_q) \text{ for } 1 < q < \infty. \tag{3.3}$$

Assume that $u \in ker(A_q)$. To prove (3.3) it is enough to show that $u = 0$. If $q \ge 2$ we have

$$0 = \int_\Omega A_q u u = -\int_\Omega |\nabla u|^2,$$

whence $u = 0$ due to the boundary condition . If $1 < q \leq 2$ denote $p = q/(q-1) > 2$, choose $f \in L_p(\Omega)$ and let $w = R(0, A_p)(-f)$. Then

$$0 = \int_\Omega A_q u w = -\int_\Omega \nabla u \nabla w = \int_\Omega u f. \tag{3.4}$$

Since (3.4) holds for any $f \in L_p(\Omega)$ we conclude that $u = 0$. We will now show that

$$\sigma(A_q) = \sigma(A_{q'}), \text{ for } 1 < q < q' < \infty. \tag{3.5}$$

It is clear that $\sigma(A_{q'}) \subset \sigma(A_q)$ as $D(A_{q'}) \subset D(A_q)$. To prove the opposite inclusion choose $\lambda \in \sigma(A_q)$ and $0 \neq \phi \in D(A_q)$ such that $A_q\phi = \lambda\phi$. To finish the proof of (3.5) it suffices to show that $\phi \in D(A_{q'})$. Define sequence $(q_i)_{i=1}^m$ such that $q = q_1 < q_2 < \ldots < q_m = q'$ and $W^2_{q_i}(\Omega) \subset L_{q_{i+1}}(\Omega)$. We will prove inductively that $\phi \in D(A_{q_i})$ for $i = 1, \ldots, m$. The base of induction follows from the definition of ϕ. Assume that $\phi \in D(A_{q_i})$ for certain $i \in 1, \ldots, m-1$. Then $\phi \in L_{q+1}(\Omega)$. Denote $\psi = R(0, A_{q+1})(-\lambda\phi)$. Then $\psi \in D(A_{q+1})$ and $A_q(\psi - \phi) = 0$ hence $\phi = \psi \in D(A_{q+1})$.

Finally since operator A_2 is self-adjoint, negative and has a compact resolvent we get from the spectral theorem for unbounded operators on Hilbert spaces that $\sigma(A_q) = \sigma(A_2) = \sigma_p(A_2) = \{\lambda_i : \ i \in \mathbb{N}_+\}$ and $0 > \lambda_1 \geq \lambda_2 \geq \ldots$.
Step 3 Observe that $\lambda_1 = \sup Re(\sigma(A_q))$ for any $1 < q < \infty$. Thus using [[22], Corollary 2.3.5.] we obtain the following estimates

$$\|e^{tA_q}u\|_q \leq M_0 e^{\lambda_1 t}\|u\|_q, \tag{3.6}$$

$$\|t(A_q + \lambda_1 I)e^{tA_q}u\|_q \leq M_1 e^{\lambda_1 t}\|u\|_q. \tag{3.7}$$

From (3.6) and (3.7) we obtain

$$\begin{aligned} \|e^{tA_q}u\|_{2,q} &\leq C\|A_q e^{tA_q}u\|_q \leq C\|(A_q + \lambda_1 I)e^{tA_q}u\|_q + C\lambda_1\|e^{tA_q}u\|_q \\ &\leq C(M_1/t + M_0\lambda_1)e^{\lambda_1 t}\|u\|_q \leq C(t \wedge 1)^{-1}e^{\lambda_1 t}\|u\|_q. \end{aligned} \tag{3.8}$$

From [[2], Theorem 5.2] we obtain

$$[L_q(\Omega), W^2_{q,B^2}(\Omega)]_{\alpha/2} = W^\alpha_{q,B^\alpha}(\Omega), \text{ for } \alpha \in \{0, 1, 2\}, \tag{3.9}$$

$$[L_{q_1}(\Omega), W^2_{q_1,B^2}(\Omega)]_\theta \subset L_{q_2}(\Omega), \text{ for } \theta \geq n/2(1/q_1 - 1/q_2). \tag{3.10}$$

Finally estimates (3.2a) and (3.2b) follow from (3.6), (3.8), (3.9) and (3.10). □

From now on we will omit subscript q and write A instead of A_q.

3.3. Stationary problem

In this section we prove the following

Theorem 3.1. ***[LNW].B*** *has a unique nonnegative steady state* (l_∞, s_∞), *where* $0 \leq l_\infty \in W^2_p(\Omega)$ *is the unique solution to*

$$-d\Delta l_\infty = -\frac{b l_\infty}{c + b + l_\infty}, \qquad x \in \Omega \tag{3.11a}$$

$$d\frac{\partial l_\infty}{\partial \nu} = g, \qquad x \in \Gamma_N \tag{3.11b}$$

$$l_\infty = 0, \qquad x \in \Gamma_D. \tag{3.11c}$$

and $s_\infty = l_\infty/(b + c + l_\infty)$.

The proof of existence is based on maximal regularity for uniformly elliptic operators in Sobolev spaces, compact embedding, comparison principle and Schauder fixed point theorem. Uniqueness follows from monotonicity of the nonlinear part in (3.11a).

Proof of Theorem 3.1. For $x \geq 0$ let $f(x) = \frac{b}{c+b+x}$. For $u \in L_p(\Omega)_+$ define operator

$$A(u) : L_p(\Omega) \supset W^2_{p,\mathcal{B}^2}(\Omega) \to L_p(\Omega), \;\; A(u)w = dAw - f(u)w.$$

Using [[2], Theorem 4.1] we see that $A(u)$ generates an analytic, strongly continuous semigroup. In particular there exists $0 < \lambda(u) \in \rho(A(u))$. From compact imbedding $W^2_{p,\mathcal{B}^2}(\Omega) \subset\subset L_p(\Omega)$ we get that the resolvent operator $R(\lambda(u), A(u))$ is compact and thus $\sigma(A(u)) = \sigma_p(A(u))$ (i.e. $A(u)$ consists of eigenvalues only). Finally since $ker(A(u)) = \{0\}$ we obtain that $0 \in \rho(A(u))$ hence operator $A(u)$ is invertible. Let $G \in W^2_p(\Omega)$ be such that $G = 0$ on Γ_D and $d\partial G/\partial\nu = g$ on Γ_N. Consider the operator $T : L_2(\Omega)_+ \to L_2(\Omega)$, defined by

$$T(u) = (-A(u))^{-1}(d\Delta G - f(u)G) + G.$$

Observe that $T(u) \in W^2_p(\Omega)$ satisfies

$$-d\Delta T(u) + f(u)T(u) = 0, \qquad x \in \Omega \tag{3.12a}$$

$$d\frac{\partial T(u)}{\partial \nu} = g, \qquad x \in \Gamma_N \tag{3.12b}$$

$$T(u) = 0, \qquad x \in \Gamma_D \tag{3.12c}$$

We will show that T has a bounded image in $L_2(\Omega)_+$, is compact and continuous (this via the Schauder fixed point theorem will imply existence of a solution of (3.11) in $W^2_p(\Omega)$). Multiplying (3.12a) by $T(u)$, integrating by parts and using positivity of f and Poincaré's inequality we obtain the following energy estimate

$$\|T(u)\|_{W^1_2(\Omega)} \leq C\|g\|_{L_2(\Gamma_N)}. \tag{3.13}$$

where C does not depend on u. From (3.13) we obtain that the range of T is bounded in $W^1_2(\Omega)$ and therefore in $L_2(\Omega)$. Compactness of T follows from the compact imbedding $W^1_2(\Omega) \subset\subset L_2(\Omega)$. To show that $T(u) \geq 0$ we multiply (3.12a) by $T(u)_-$ and integrate by parts

$$-d\int_\Omega |\nabla T(u)_-|^2 - \int_{\Gamma_N} gT(u)_- - \int_\Omega f(u)(T(u)_-)^2 = 0.$$

Thus $T(u)_-$ is constant in Ω and since $T(u) = 0$ on Γ_D therefore $T(u) \geq 0$ in Ω.
Assume that $u_n \to u$ in $L_2(\Omega)$. Let $w = T(u), w_n = T(u_n)$, then

$$-d\Delta(w_n - w) + f(u_n)(w_n - w) + w(f(u_n) - f(u)) = 0, \qquad x \in \Omega$$

$$d\frac{\partial(w_n - w)}{\partial\nu} = 0, \qquad x \in \Gamma_N$$

$$w_n - w = 0, \qquad x \in \Gamma_D$$

therefore

$$\|w_n - w\|_{L_2(\Omega)} \leq C\|w(f(u_n) - f(u))\|_{L_2(\Omega)} \leq C\|w\|_{L_\infty(\Omega)}\|f'\|_{L_\infty(0,\infty)}\|u_n - u\|_{L_2(\Omega)}$$

which proves that T is continuous. Using Schauder fixed point theorem we obtain existence of $l_\infty \in W^2_p(\Omega)$ which solves (3.11).

To prove uniqueness, assume that l^1_∞, l^2_∞ are solutions of (3.11). Subtracting equations (3.11a) for l^1_∞, l^2_∞, multiplying by $l^1_\infty - l^2_\infty$, integrating by parts and using the monotonicity of function $\mathbb{R}_+ \ni x \to xf(x)$ we get

$$-d\int_\Omega |\nabla(l^1_\infty - l^2_\infty)|^2 = \int_\Omega (f(l^1_\infty)l^1_\infty - f(l^2_\infty)l^2_\infty)(l^1_\infty - l^2_\infty) \geq 0,$$

which by (3.11c) implies $l^1_\infty \equiv l^2_\infty$. □

3.4. Evolution problem

We next turn to the evolution problem and establish its well-posedness.

Theorem 3.2. ***[LNW].B*** *has unique solution* (l, s) *such that*

$$l - l_\infty \in C([0,\infty); W^1_p(\Omega)) \cap C^1((0,\infty); L_p(\Omega)) \cap C((0,\infty); W^2_p(\Omega)), \tag{3.14a}$$

$$s \in C^1([0,\infty); W^1_p(\Omega)). \tag{3.14b}$$

Moreover for $(t,x) \in [0,\infty) \times \Omega$

$$0 \leq l(t,x), \quad 0 \leq s(t,x) < 1. \tag{3.14c}$$

Local existence and uniqueness are obtained by putting system **[LNW].B** into the semigroup framework and using general theory for abstract parabolic semilinear problems. Comparison principle allows us to deduce that (3.14c) is satisfied from which we get that our solution is global.

Proof of Theorem 3.2. To deal with nonhomogeneous boundary condition on Γ_N we subtract from (l,s) the stationary state (l_∞, s_∞). Setting $(z_1, z_2) = (l - l_\infty, s - s_\infty)$ we arrive at

$$\partial_t z_1 - d\Delta z_1 = cz_2 - z_1(1 - z_2) + s_\infty z_1 + l_\infty z_2, \qquad (t,x) \in (0,\infty) \times \Omega \tag{3.15a}$$

$$\partial_t z_2 = -(c+b)z_2 + z_1(1 - z_2) - s_\infty z_1 - l_\infty z_2, \qquad (t,x) \in (0,\infty) \times \Omega \tag{3.15b}$$

$$d\frac{\partial z_1}{\partial \nu} = 0 \qquad , (t,x) \in (0,\infty) \times \Gamma_N \tag{3.15c}$$

$$z_1 = 0, \qquad (t,x) \in (0,\infty) \times \Gamma_D \tag{3.15d}$$

$$z_1(0) = z_{10} = l_0 - l_\infty, \qquad x \in \Omega \tag{3.15e}$$

$$z_2(0) = z_{20} = s_0 - s_\infty, \qquad x \in \Omega \tag{3.15f}$$

We interpret system (3.15) as a differential equation in a Banach space specified below

$$\frac{dz}{dt} - \mathcal{A}z = H(z), \qquad t \in (0,\infty) \tag{3.16a}$$

$$z(0) = z_0 = (z_{10}, z_{20}) \tag{3.16b}$$

where $z = (z_1, z_2)$, $\mathcal{A}z = (dAz_1, 0)$, $H = (H^1, H^2)$,

$$H^1(z) = cz_2 - z_1(1 - z_2) + s_\infty z_1 + l_\infty z_2, \tag{3.17a}$$

$$H^2(z) = -(c+b)z_2 + z_1(1 - z_2) - s_\infty z_1 - l_\infty z_2. \tag{3.17b}$$

In the following lemma we prove local existence for (3.16).

Lemma 3.2. *For $\alpha \in \{0,1,2\}$ denote $Z_{\alpha,p} = W^{\alpha}_{p,\mathcal{B}^\alpha} \times W^1_{p,\mathcal{B}^1}$. For every $z_0 \in Z_{1,p}$ the Cauchy problem (3.16) possess a unique maximal local solution*

$$z \in C([0,T_{\max}); Z_{1,p}) \cap C^1((0,T_{\max}); Z_{0,p}) \cap C((0,T_{\max}); Z_{2,p}).$$

which satisfies for $t \in [0,T_{\max})$ the following Duhamel formula:

$$z_1(t) = e^{tdA} z_{10} + \int_0^t e^{(t-s)dA} H^1(z(s))ds, \tag{3.18a}$$

$$z_2(t) = z_{20} + \int_0^t H^2(z(s))ds. \tag{3.18b}$$

Moreover if $T_{\max} < \infty$ then $\limsup_{t\to T_{\max}^-} \|z(t)\|_{1,p} = \infty$.

Proof of Lemma 3.2. The operator $\mathcal{A} : Z_p \supset Z_{2,p} \to Z_p$ is a generator of an analytic strongly continuous semigroup $e^{t\mathcal{A}} = e^{tdA} \times Id$ (as a product of two generators). Moreover since $Z_{1,p}$ is a Banach algebra ($p > n$) we observe that $H : Z_{1,p} \to Z_{1,p}$ is locally Lipschitz on bounded sets. The claim follows from Lemma1.3 . □

We next turn to the proof of (3.14c).

To prove that for $t \in [0,T_{\max})$ $l(t), s(t) \geq 0$ we consider the system

$$\partial_t l' - d\Delta l' = cs'_+ - l'_+(1-s'_+), \quad (t,x) \in (0,\infty)\times\Omega \tag{3.19a}$$

$$\partial_t s' = -(c+b)s'_+ + l'_+(1-s'_+), \quad (t,x) \in (0,\infty)\times\Omega \tag{3.19b}$$

$$d\frac{\partial l'}{\partial \nu} = g, \quad (t,x) \in (0,\infty)\times\Gamma_N \tag{3.19c}$$

$$l' = 0, \quad (t,x) \in (0,\infty)\times\Gamma_D \tag{3.19d}$$

$$l'(0) = l_0, \quad x \in \Omega \tag{3.19e}$$

$$s'(0) = s_0, \quad x \in \Omega \tag{3.19f}$$

As before one can show that (3.19) possess unique classical local solution (l', s'). After multiplying (3.19a) by l_- and integrating by parts we obtain

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |l'_-|^2 dx - d\int_\Omega |\nabla l'_-|^2 dx - d\int_{\Gamma_N} l'_- g dS = c\int_\Omega s'_+ l'_- dx \geq 0.$$

Similarly multiplying (3.19b) by s_- yields

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |s'_-|^2 dx = \int_\Omega l'_+ s'_- dx \geq 0.$$

Therefore for $t \in [0,T_{\max})$

$$\|l'(t)_-\|_2^2 + \|s'(t)_-\|_2^2 \leq \|l'(0)_-\|_2^2 + \|s'(0)_-\|_2^2 = 0$$

and consequently $l'(t) \geq 0, s'(t) \geq 0$. We observe now that (l', s') is a solution of **[LNW].B** and using uniqueness we finally get that $l(t) = l'(t) \geq 0, s(t) = s'(t) \geq 0$ for $t \in [0,T_{\max})$.

To show that $s(t,x) < 1$ for $(t,x) \in [0, T_{\max}) \times \overline{\Omega}$ we get from Lemma 3.2, that for every fixed $x \in \overline{\Omega}$ the function $m = 1 - s = 1 - z_2 - s_\infty \in C^1([0; T_{\max}), \mathbb{R})$ satisfies for $t > 0$ the following ODE

$$\frac{dm}{dt} + (c + b + l)m = c + b.$$

Therefore

$$m(t) = e^{-(c+b)t - \int_0^t l(\tau)d\tau}(1 - s_0) + (c + b)\int_0^t e^{-(c+b)(t-t') - \int_0^{t-t'} l(\tau)d\tau} dt' > 0.$$

We finally show that $T_{\max} = \infty$. Reasoning by contradiction assume that $T_{\max} < \infty$. Using uniform L_∞ boundedness of s (and therefore of z_2) we obtain for $t \in (0, T_{\max})$:

$$\|H^1(z(t))\|_p \le C(1 + \|z_1(t)\|_p) \le C(1 + \|z_1(t)\|_{1,p}). \tag{3.20}$$

Using (3.18a),(3.2a),(3.20) we obtain

$$\begin{aligned}
\|z_1(t)\|_{1,p} &\le \|e^{tdA} z_{10}\|_{1,p} + \int_0^t \|e^{(t-\tau)dA} H^1(z(\tau))\|_{1,p} d\tau \\
&\le C\|z_{10}\|_{1,p} + C\int_0^t (t-\tau)^{-1/2} \|H^1(z(t))\|_p d\tau \\
&\le C\|z_{10}\|_{1,p} + C\int_0^t (t-\tau)^{-1/2}(1 + \|z_1(\tau)\|_{1,p}) d\tau \\
&\le C(\|z_{10}\|_{1,p} + 1) + C\int_0^t (t-\tau)^{-1/2} \|z_1(\tau)\|_{1,p} d\tau.
\end{aligned}$$

Using Lemma 1.2 we get that $\|z_1(t)\|_{1,p} \le C$ and therefore

$$\|H^2(z(t))\|_{1,p} \le C(1 + \|z_2(t)\|_{1,p}). \tag{3.21}$$

Using (3.18b) and (3.21) we obtain

$$\begin{aligned}
\|z_2(t)\|_{1,p} &\le \|z_{20}\|_{1,p} + \int_0^t \|H^2(z(\tau))\|_{1,p} d\tau \le \|z_{20}\|_{1,p} + C\int_0^t (1 + \|z_2(\tau)\|_{1,p}) d\tau \\
&\le C(\|z_{20}\|_{1,p} + 1) + C\int_0^t \|z_2(\tau)\|_{1,p} d\tau.
\end{aligned}$$

Another application of Lemma 1.2 gives the desired contradiction from which we deduce that $T_{\max} = \infty$.

□

We finally study the stability of the steady state and show that it attracts all trajectories with the uniform exponential rate.

Theorem 3.3. *There exists a positive constant C depending on $l_0, s_0, g, c, b, d, \Omega, p$ such that for every $t > 0$*

$$\|l(t) - l_\infty\|_{1,p} + \|s(t) - s_\infty\|_{1,p} \le Ce^{-(\chi/2)t}, \tag{3.22a}$$

$$\|l(t) - l_\infty\|_{2,p} \le C\max\{1/\sqrt{t}, 1\} e^{-(\chi/2)t}, \tag{3.22b}$$

where

$$\chi = \min\Big\{ - d\lambda_1, \frac{-d\lambda_1(c+b)}{2(-d\lambda_1+2)} + \frac{b}{2}\Big\} \tag{3.22c}$$

and λ_1 *is defined in Lemma 3.1.*

By extending Lyapunov functional (derived in [15] for one dimensional interval) to the case of arbitrary dimension we obtain estimates on the distance between solution and steady state in $L_2 \times L_2$ topology. Using regularising properties of the heat semigroup we next bootstrap the topology of convergence to $W_p^2 \times W_p^1$.

Remark
Using embedding $W_p^2(\Omega) \times W_p^1(\Omega) \subset C^{1,\alpha}(\Omega) \times C^{0,\alpha}(\Omega)$ valid for $p > n,\ 0 \le \alpha \le 1 - n/p$ we obtain topology of convergence as claimed in the introduction.

Proof of Theorem 3.3. The proof of Theorem 3.3 is based on L_2 estimates obtained for $n = 1$ in [15] and bootstrap method to improve convergence from X_i-topology to X_{i+1}-topology, where $X_{i+1} \subset X_i$ are appropriately chosen Banach spaces. We use (as long as the regularity of our solution permits) the following two step

Bootstrap scheme

1. $\|z_1(t)\|_{X_i} + \|z_2(t)\|_{X_i} \le Ce^{-(\chi/2)t}$ gives $\|z_1(t)\|_{X_{i+1}} \le Ce^{-(\chi/2)t}$.
2. $\|z_1(t)\|_{X_{i+1}} \le Ce^{-(\chi/2)t}$ gives $\|z_2(t)\|_{X_{i+1}} \le Ce^{-(\chi/2)t}$.

Part 1. is a consequence of the Duhamel formula (3.18a) and semigroup estimates (3.2).
Part 2. follows from the fact that we can solve equation (3.15b) explicitly for z_2 in terms of z_1.

Step 1 - L_2 estimate

We first show that, as in the one dimensional case **[LNW].B** has a Lyapunov functional from which exponential convergence to the equlibrium (l_∞, s_∞) follows.

Lemma 3.3. *For* $x \in [0,1), u, v \in W^1_{p,\mathcal{B}^1}(\Omega), 0 \le v < 1,$ *define*

$$\begin{aligned}
\Sigma_I(x) &= -\ln(1-x),\\
\Lambda_0(v) &= \int_\Omega (1-s_\infty)(l_\infty + c + 2b)\Big[\Sigma_I(v) - \Sigma_I(s_\infty) - \frac{v - s_\infty}{1 - s_\infty}\Big]dx,\\
\Lambda(u,v) &= \frac{1}{2}\|u - l_\infty\|_2^2 + \Lambda_0(v),\\
\mathcal{D}_\Lambda(u,v) &= d\|\nabla(u - l_\infty)\|_2^2 + \int_\Omega \frac{[u(1-v) - (c+b)v]^2 + b(l_\infty + c + b)(v - s_\infty)^2}{1-v}dx.
\end{aligned}$$

Then for $t \ge 0$

$$\begin{aligned}
\Lambda(l(t), s(t)) + \int_0^t \mathcal{D}_\Lambda(l(\tau), s(\tau))d\tau &= \Lambda(l_0, s_0),\\
\chi\Lambda(l(t), s(t)) &\le \mathcal{D}_\Lambda(l(t), s(t)),\\
(c+b)\|s(t) - s_\infty\|_2^2 &\le 2\Lambda_0(s(t))
\end{aligned}$$

and

$$\|l(t) - l_\infty\|_2^2 + (c+b)\|s(t) - s_\infty\|_2^2 \le 2\Lambda(l_0, s_0)e^{-\chi t}, \tag{3.23}$$

where χ *satisfies* (3.22c).

Proof of Lemma 3.3. The proof can be obtained exactly as in [15] (part of Theorem 8 and Proposition 9 pp 1740-1744). For the case $n = 1, p \in (1,2)$, to justify integration by parts and Poincaré's inequality, we observe that for $t > 0 : l(t) \in W_p^2(\Omega) \subset W_2^1(\Omega)$. □

Step 2 - L_p estimate

In this subsection we will prove that for $t \ge 0$

$$\|z_1(t)\|_p + \|z_2(t)\|_p \le Ce^{-(\chi/2)t}, \tag{3.24}$$

the parameter p being defined in **A1**.

Notice that if $p \in (1,2]$ (which can only happen if $n = 1$), the inequality (3.24) follows from (3.23).

Otherwise we have $p > (2 \vee n)$. We choose an increasing sequence $(p_i)_{i=1}^m$ such that

$$p_1 = 2, p_m = p,$$
$$n/2(1/p_i - 1/p_{i+1}) < 1 \text{ for } 1 \le i \le m-1.$$

Notice that for $n \in \{1,2,3,4\}$ one can take $m = 2$. Inductively we will prove that

$$\|z_1(t)\|_{p_i} + \|z_2(t)\|_{p_i} \le Ce^{-(\chi/2)t},\ 1 \le i \le m. \tag{3.25}$$

For $i = 1$ (3.25) follows from (3.23). Assume that (3.25) is true for some $1 \le i \le m-1$. Then

$$\|H^1(z(t))\|_{p_i} \le \|z_1\|_{p_i}\|1 - z_2 + s_\infty\|_\infty + \|z_2\|_{p_i}\|c + b + l_\infty\|_\infty \le Ce^{-(\chi/2)t}. \tag{3.26}$$

Using (3.18a), (3.2b), (3.26) and $\chi/2 < d\lambda_1$ we obtain

$$\begin{aligned}
\|z_1(t)\|_{p_{i+1}} &\le \|e^{tdA}z_{10}\|_{p_{i+1}} + \int_0^t \|e^{\tau dA}H^1(z(t-\tau))\|_{p_{i+1}}d\tau \\
&\le Ce^{d\lambda_1 t} + C\int_0^t (\tau d \wedge 1)^{-n/2(1/p_i - 1/p_{i+1})}e^{d\lambda_1\tau}\|H^1(z(t-\tau))\|_{p_i}d\tau \\
&\le Ce^{d\lambda_1 t} + C\int_0^t (\tau d \wedge 1)^{-n/2(1/p_i - 1/p_{i+1})}e^{d\lambda_1\tau}e^{-(\chi/2)(t-\tau)}d\tau \\
&\le Ce^{d\lambda_1 t} + Ce^{-(\chi/2)t}\int_0^t (\tau d \wedge 1)^{-n/2(1/p_i - 1/p_{i+1})}e^{(d\lambda_1 + \chi/2)\tau}d\tau \\
&\le Ce^{-(\chi/2)t}.
\end{aligned}$$

We now show that $\|z_2(t)\|_{p_{i+1}} \le Ce^{-(\chi/2)t}$ for $t > 0$. Indeed from Theorem 3.2 we obtain that for each fixed $x \in \overline{\Omega}$ the function $z_2 \in C^1([0,\infty);\mathbb{R})$ satisfies the ODE

$$\frac{dz_2}{dt} + (c + b + l_\infty + z_1)z_2 = (1 - s_\infty)z_1,$$

hence

$$z_2(t) = A(t)z_{20} + (1 - s_\infty) \int_0^t A(\tau)z_1(t-\tau)d\tau, \tag{3.27}$$

where

$$A(t) = \exp\Big(- \int_0^t (c + b + l_\infty + z_1(\tau))d\tau\Big). \tag{3.28}$$

From $l_\infty + z_1 = l \geq 0$ we get $\|A(t)\|_\infty \leq e^{-(c+b)t}$. Using $\chi/2 < c + b$ we obtain

$$\begin{aligned}\|z_2(t)\|_{p_{i+1}} &\leq \|A(t)\|_\infty \|z_{20}\|_{p_{i+1}} + \|1 - s_\infty\|_\infty \int_0^t \|A(\tau)\|_\infty \|z_1(t-\tau)\|_{p_{i+1}} d\tau \\ &\leq Ce^{-(c+b)t} + Ce^{-(\chi/2)t} \int_0^t e^{-(c+b-\chi/2)\tau} d\tau \leq Ce^{-(\chi/2)t},\end{aligned}$$

thus finishing the proof of (3.25), whence that of (3.24).

In the next two sections we use the smoothing properties of e^{tA} to extend convergence to the first and second derivatives.

Step 3 - W_p^1 estimate

Using (3.18a), (3.2a), (3.24) and $\chi/2 < -d\lambda_1$ we obtain

$$\begin{aligned}\|z_1(t)\|_{1,p} &\leq \|e^{tdA}z_{10}\|_{1,p} + \int_0^t \|e^{\tau dA}H^1(z(t-\tau))\|_{1,p} d\tau \\ &\leq Ce^{d\lambda_1 t} + C\int_0^t (\tau d \wedge 1)^{-1/2} e^{\lambda_1 d\tau} \|H^1(z(t-\tau))\|_p d\tau \\ &\leq Ce^{d\lambda_1 t} + C\int_0^t (\tau d \wedge 1)^{-1/2} e^{\lambda_1 d\tau} e^{-(\chi/2)(t-\tau)} d\tau \\ &\leq Ce^{d\lambda_1 t} + Ce^{-(\chi/2)t} \int_0^t (\tau d \wedge 1)^{-1/2} e^{(d\lambda_1 + \chi/2)\tau} d\tau \\ &\leq Ce^{-(\chi/2)t}.\end{aligned}$$

Using the above estimate for z_1 we obtain that $A(t)$ given by (3.28) satisfies

$$\begin{aligned}\|A(t)\|_p &\leq C\|A(t)\|_\infty \leq Ce^{-(c+b)t} \\ \|\nabla A(t)\|_p &= \|-A(t) \int_0^t (\nabla l_\infty + \nabla z_1(\tau))d\tau\|_p \leq \|A(t)\|_\infty \int_0^t (\|\nabla l_\infty\|_p + \|\nabla z_1(\tau)\|_p) d\tau \\ &\leq Ce^{-(c+b)t} \int_0^t (1 + e^{-(\chi/2)\tau})d\tau \leq Cte^{-(c+b)t}.\end{aligned}$$

Thus using (3.27) we have

$$\begin{aligned}
\|z_2(t)\|_{1,p} &\le \|A(t)\|_{1,p}\|z_{20}\|_{1,p} + C\|1-s_\infty\|_{1,p}\int_0^t \|A(\tau)\|_{1,p}\|z_1(t-\tau)\|_{1,p}d\tau \\
&\le C(t+1)e^{-(c+b)t} + C\int_0^t (\tau+1)e^{-(c+b)\tau}e^{-(\chi/2)(t-\tau)}d\tau \\
&\le C(t+1)e^{-(c+b)t} + Ce^{-(\chi/2)t}\int_0^t (\tau+1)e^{-(c+b-\chi/2)\tau}d\tau \\
&\le Ce^{-(\chi/2)t}
\end{aligned}$$

which finishes the proof of (3.22a).

Step 4 - W_p^2 estimate for z_1

Using (3.18a), (3.2a), (3.22a) and $\chi/2 < d\lambda_1$ we obtain

$$\begin{aligned}
\|z_1(t)\|_{2,p} &\le \|e^{tdA}z_{10}\|_{2,p} + \int_0^t \|e^{\tau dA}H^1(z(t-\tau))\|_{2,p}d\tau \\
&\le C(td\wedge 1)^{-1/2}e^{d\lambda_1 t} + C\int_0^t (\tau d\wedge 1)^{-1/2}e^{\lambda_1 d\tau}e^{-(\chi/2)(t-\tau)}d\tau \\
&\le C(t\wedge 1)^{-1/2}e^{d\lambda_1 t} + Ce^{-(\chi/2)t}\int_0^t (\tau\wedge 1)^{-1/2}e^{(d\lambda_1+\chi/2)\tau}d\tau \\
&\le C\max\{1/\sqrt{t},1\}e^{-(\chi/2)t},
\end{aligned}$$

which finishes the proof of (3.22b). □

Chapter 4

Conclusions and final remarks

We summarise the content of the dissertation as well as provide few additional remarks and state one open problem.
The subject of the dissertation is a rigorous mathematical analysis of two models that were recently proposed to describe morphogen transport - a biological process governing cell differentiation in living organisms. The models that are taken under consideration describe the movement of two distinct morphogens (Wg and Dpp) in the imaginal wing disc of the fruit fly.
In Chapter 2 we have analysed the **[HKCS]** model of morphogen Wg transport introduced by Hufnagel et al. in [13]. We have shown that the model is well-posed in appropriately chosen function setting (Theorem 2.2 and Theorem 2.7) and has a unique stationary solution (Theorem 2.4 and Theorem 2.6). The most significant result of the dissertation is a mathematically rigorous justification of the fact that the one dimensional version of the **[HKCS]** model can be obtained from the full two dimensional version by an appropriate limiting process in the evolutionary (Theorem 2.3) as well as in the stationary (Theorem 2.5) case. This process can be interpreted either as dimension reduction (the two dimensional domain of the full model is "ironed" to the interval) or as sending to infinity the flux of morphogen molecules in the direction perpendicular to the wing disc. Above result may be seen as an argument to justify that the one dimensional domain is sufficient to model the process. However the topology of convergence is to weak to exclude one qualitative difference in the behaviour of solutions at the source point $x = 0$. Namely the concentration of morphogen in the **[HKCS].2D** model blows up at $x \to 0$ while it stays bounded in the case of one dimensional domain. Roughly speaking this phenomenon is a consequence of the fact that the Dirac Delta which is used to represent the point source of morphogen in both models, is a more singular distribution in the second dimension. Another interesting phenomenon observed during the analysis is the surprising behaviour of the concentration of the triple morphogen-glypican-receptor complexes near the source point (see Figure 2.2). Although we are able to justify that analytically (see Remark 4) we believe that there should also be a biological explanation. What remains open in the analysis of the **[HKCS]** model is the asymptotic behaviour of solutions as $t \to \infty$.
In Chapter 3 we have analysed the **[LNW].B** model of morphogen Dpp transport introduced by Lander et al. in [20]. We have shown that all results obtained before by Krzyżanowski et al. in [15] for the case of a one dimensional domain (i.e. well-posedness and existence of a unique equilibrium which is globally exponentially stable in the L_2 topology) hold in the domains of arbitrary dimension (Theorem 3.1, Theorem 3.2 and Lemma 3.3). Moreover we have shown that the topology of convergence of time dependent solution to the equilibrium is at least $C^{1,\alpha} \times C^{0,\alpha}$ (Theorem 3.3) which improves the result from [15] and shows that the gradient of morphogen is being formed in a more regular manner.

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