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# Resolutions of quotient singularities and their Cox rings

*PhD dissertation*

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Author's declaration:

I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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**Abstract**

The aim of this dissertation is to investigate the geometry of resolutions of quotient singularities $\mathbb{C}^n/G$ for $G \subset \mathrm{SL}_n(\mathbb{C})$ with use of an associated algebraic object – the Cox ring. We are interested in the construction of all crepant resolutions and a combinatorial description of birational relations among them. This information can be read off from the structure of the Cox ring.

We give a method to construct certain finitely generated subalgebras of the Cox ring $\mathcal{R}(X)$ of a crepant resolution $X \to \mathbb{C}^n/G$ and present three methods to verify when such a subring is actually the whole Cox ring. The construction relies on the embedding, investigated by Donten-Bury and Wiśniewski, of the Cox ring into the Laurent polynomial ring over an invariant ring of the commutator subgroup $[G,G] \subset G$.

The first method to verify whether a constructed subalgebra is equal to $\mathcal{R}(X)$ relies on a criterion involving valuations of crepant divisors over the singularity $\mathbb{C}^n/G$. Such valuations can be computed using the McKay correspondence of Ito and Reid as restrictions of certain monomial valuations on the field of rational functions $\mathbb{C}(x_1, \ldots, x_n)$. We apply this method to the family of three-dimensional quotient singularities given by groups acting via reducible representations on $\mathbb{C}^3$. We obtain a presentation of $\mathcal{R}(X)$ in terms of generators and relations, which is then used to study the geometry of resolutions for several examples of quotient singularities. One example is the infinite series of quotient singularities given by dihedral groups, investigated previously by Nolla de Cellis and Sekiya with different methods. We provide an alternative treatment for such quotients. We also investigate the geometry of crepant resolutions in the simplest examples of the quotient $\mathbb{C}^3/G$ when such a resolution contracts a divisor to a point. Due to limitations of methods used prior to our work examples exhibiting such a phenomenon were not studied earlier even though they are typical among resolutions of three-dimensional quotient singularities. Two examples we present belong to the family of reducible representations and one belongs to the family of irreducible representations which is substantially harder to analyze.

The second method is based on the characterization theorem for the Cox ring in terms of Geometric Invariant Theory. We present an example of its use when we give another proof in the case of dihedral groups.

We also give a general method to bound degrees of generators of the Cox ring by use of the Kawamata-Viehweg vanishing and multigraded Castelnuovo-Mumford regularity. We use this method in the study of three examples of symplectic quotient singularities in dimension four. In this study we use another technique to verify that a constructed subalgebra is equal to the Cox ring, based on the algebraic torus action on the resolution. The action allows us, via the Lefschetz-Riemann-Roch theorem, to compute the important part of the Hilbert series of the Cox ring. We expect that this idea may be generalized and used to study other examples.

**Keywords:** finite group action, quotient singularity, resolution of singularities, crepant resolution, symplectic resolution, Cox ring, algebraic torus action.

**AMS MSC 2010 classification:** 14E15, 14E30, 14E16, 14L30, 14L24, 14C35, 14C40, 14C20, 14Q15

## Streszczenie


Celem niniejszej rozprawy jest zbadanie geometrii rozwiązań osobliwości ilorazowych $\mathbb{C}^n/G$ dla $G \subset \mathrm{SL}_n(\mathbb{C})$ przy użyciu stowarzyszonego obiektu algebraicznego – pierścienia Coxa. Interesuje nas skonstruowanie wszystkich rozwiązań krepantnych i kombinatoryczny opis relacji biwymiernych pomiędzy nimi. Opis taki można odczytać ze struktury pierścienia Coxa.

Podajemy metodę konstrukcji pewnych skończenie generowanych podalgebr pierścienia Coxa $\mathcal{R}(X)$ rozwiązania krepantnego $X \to \mathbb{C}^n/G$ i trzy metody pozwalające rozstrzygać, kiedy taki podpierścień jest całym pierścieniem Coxa. Konstrukcję przeprowadzamy w oparciu o badane przez Donten-Bury i Wiśniewskiego zanurzenie pierścienia Coxa w pierścień wielomianów Laurenta nad pierścieniem niezmienników komutanta $[G, G] \subset G$.

Pierwszy sposób sprawdzenia, czy skonstruowana algebra jest równa $\mathcal{R}(X)$, opiera się na kryterium związanym z waluacjami dywizorów krepantnych nad osobliwością $\mathbb{C}^n/G$. Waluacje te można obliczyć przy użyciu odpowiedniości McKaya w sensie Ito i Reida jako zawężenia pewnych waluacji jednomianowych na pierścieniu funkcji wymiernych $\mathbb{C}(x_1, \ldots, x_n)$. Stosujemy to kryterium do rodziny trójwymiarowych osobliwości ilorazowych związanych z reprezentacjami rozkładalnymi. Otrzymujemy prezentację $\mathcal{R}(X)$ poprzez generatory i relacje, której następnie używamy, aby zbadać geometrię rozwiązań kilku przykładów osobliwości ilorazowych. Jednym z nich jest nieskończona seria osobliwości ilorazowych zadanych przez grupy dihedralne, badana wcześniej przez Nollę de Cellis i Sekiyę przy użyciu innych metod. Analizujemy również geometrię rozwiązań krepantnych w najprostszych przypadkach osobliwości $\mathbb{C}^3/G$ o rozwiązaniu ściągającym dywizor do punktu. Ze względu na ograniczenia stosowanych uprzednio metod przykłady takich rozwiązań nie były dotychczas badane pomimo ich powszechności wśród rozwiązań trójwymiarowych osobliwości ilorazowych. Dwa przykłady należą do rodziny reprezentacji rozkładalnych a jeden do rodziny reprezentacji nieprzywiedlnych, które są istotnie trudniejsze do przeanalizowania.

Drugi sposób jest oparty o twierdzenie charakteryzujące pierścienie Coxa w terminach geometrycznej teorii niezmienników (GIT). Przedstawiamy jego przykładowe zastosowanie, przeprowadzając alternatywny dowód w przypadku grup dihedralnych.

Podajemy również ogólną metodę ograniczania stopni generatorów pierścienia Coxa przy użyciu twierdzenia Kawamaty-Viehwega o znikaniu i wielogradowanej wersji regularności Castelnuovo-Mumforda. Stosujemy tę metodę do zbadania trzech przykładów symplektycznych osobliwości ilorazowych w wymiarze cztery. Stosujemy tutaj inną technikę do rozstrzygnięcia, czy skonstruowana podalgebra jest równa całemu pierścieniowi Coxa, opierając się na działaniu algebraicznego torusa na rozwiązaniu. To działanie pozwala nam, dzięki twierdzeniu Lefschetza-Riemanna-Rocha, obliczyć ważną część funkcji Hilberta pierścienia Coxa. Spodziewamy się, że przedstawione w tej części pracy metody związane z działaniem torusa mogą zostać uogólnione i wykorzystane do badania innych przykładów.



**Słowa kluczowe:** działanie grupy skończonej, osobliwość ilorazowa, rozwiązanie osobliwości, rozwiązanie krepantne, rozwiązanie symplektyczne, pierścień Coxa, działanie torusa algebraicznego

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# Contents

CHAPTER 1

# Introduction

The resolution of singularities allows one to modify an algebraic variety to obtain a nonsingular variety sharing many properties with the original, singular one. The study of nonsingular varieties is an easier task, due to their good geometric and topological properties. By the theorem of Hironaka [**53**] it is always possible to resolve singularities of a complex algebraic variety.

Quotient singularities are the singularities occurring in quotients of algebraic varieties by finite group actions. A good local model of such a singularity is a quotient $\mathbb{C}^n/G$ of an affine space by a linear action of a finite group. The study of quotient singularities dates back to the work of Hilbert and Noether [**52**], [**78**], who proved finite generation of the ring of invariants of a linear finite group action on a ring of polynomials. Another important classical result is due to Chevalley, Shephard and Todd [**19**], [**86**], who characterized groups $G$ that yield nonsingular quotients. Nowadays, quotient singularities form an important class of singularities, since they are in some sense typical well-behaved (*log terminal*) singularities in dimensions two and three (see [**66**, Sect. 3.2]). Methods of construction of algebraic varieties via quotients, such as the classical Kummer construction, are being developed and used in various contexts, including higher dimensions, see [**4**] and [**36**] for a recent example of application.

In the well-studied case of surfaces every quotient singularity admits a unique minimal resolution. The geometry of such a resolution can be described by the incidence diagram of the exceptional curves together with their self-intersection numbers. Among surface quotient singularities one distinguishes the class of Du Val singularities (see e.g. [**39**]) associated with Dynkin diagrams, which appear also in representation theory.

The modern notion of a crepant resolution generalizes the notion of the minimal resolution of a Du Val singularity. A remarkable difference is that in higher dimension a crepant resolution is rarely unique and it may even not exist at all. Nevertheless, the minimality property makes crepant resolutions highly desired objects. Moreover, the problem of non-existence can be avoided by introducing a further generalization, a minimal model, which is a partial resolution admitting a certain minimality property and which for a quotient singularity always exists. The study of crepant resolutions and minimal models of quotient singularities shows an interesting interplay of many techniques from various fields. Among them are geometry, representation theory, topology and even mathematical physics (see [**29, 28**]). One may observe such connections in the research on the McKay correspondence, which describes the relation between the structure of a group and the geometry of minimal models of a quotient (see [**80**], [**84**] for a survey). This research field appeared at the end of the previous century and is still developing.

The aim of this work is to develop general methods for construction and study of crepant resolutions of a quotient singularity via their Cox ring and apply them to three- and four-dimensional examples exhibiting interesting phenomena.

The Cox ring of a normal complex algebraic variety $X$ with a finitely generated class group $\mathrm{Cl}(X)$ is a $\mathrm{Cl}(X)$-graded ring:

$$\mathcal{R}(X) = \bigoplus_{D \in \mathrm{Cl}(X)} H^0(X, D)$$

(see chapter 3 for the precise definition). According to ideas presented in the seminal paper by Hu and Keel [**54**] if $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra it gives a powerful tool to study the geometry of $X$ and its small modifications. In particular one may recover $X$ and all its codimension two modifications as quotients of open subsets of $\operatorname{Spec} \mathcal{R}(X)$ by the action of the characteristic quasitorus $\mathbb{T} = \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$. This observation is the central theme of our work, the basis of all our research that contributes to this dissertation. As explained in section 3.4, the finite generation assumption is satisfied whenever $X$ is a minimal model of a quotient singularity. In this case one may recover all the minimal models of the singularity together with birational relations between them (*flops*, see [**83**]) by using the geometric invariant theory (GIT) to construct quotients corresponding to chambers in a certain subdivision of the movable cone of $X$. Knowing the Cox ring one may find the movable cone of $X$, its subdivision and GIT quotients corresponding to chambers.

## 1.1. Motivation and state of the art

The existence and geometry of crepant resolutions of quotient singularities is an active field of research. Here we outline problems that motivated our work and summarize what was known before and what developed while we were conducting the research presented in the thesis.

**Existence and construction of symplectic resolutions.** The open problem that originally motivated our work is the problem of existence and construction of crepant resolutions of symplectic quotient singularities. A crepant resolution of such a singularity preserves the symplectic structure and thus is often called a symplectic resolution (the converse also holds – a resolution preserving the symplectic structure is crepant). Such resolutions may find applications in constructing Hyperkähler manifolds via the generalized Kummer construction [**4**]. Finding new Hyperkähler manifolds is an important and difficult problem in complex geometry.

A theorem of Verbitsky [**90**] gives a necessary (but insufficient) criterion for the existence of a crepant resolution of a quotient $\mathbb{C}^{2n}/G$, where $G$ is a finite group of linear transformations preserving the symplectic form on $\mathbb{C}^{2n}$. The condition says that the group $G$ has to be generated by symplectic reflections, i.e. elements which fix a subspace of codimension 2 (see section 2.3).

Finite groups generated by symplectic reflections were classified by Cohen in [**21**]. The problem of existence of symplectic resolutions was investigated in [**43, 9, 11, 12, 94**]. According to [**9**] and [**12**, 4.1], for the following groups it is known that a symplectic resolution exists:

1. $S_{n+1}$ acting on $\mathbb{C}^{2n}$ via direct sum of two copies of the standard $n$-dimensional representation of $S_{n+1}$. Here a resolution can be constructed via the Hilbert scheme $\mathrm{Hilb}^{n+1}(\mathbb{C}^2)$. The example of $S_3$ from section 7.3 is the only four-dimensional representative of his family.
2. $H \wr S_n = H^n \rtimes S_n$, where $H$ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, so that $\mathbb{C}^2/H$ is a Du Val singularity, and $S_n$ acts on $H^n$ by permutations of coordinates. The natural product representation of $H^n$ on $\mathbb{C}^{2n}$ extends naturally to the action of $H \wr S_n$.

One of the resolutions is $\mathrm{Hilb}^n(S)$ where $S$ is the minimal resolution of the Du Val singularity $\mathbb{C}^2/H$. The example of $\mathbb{Z}_2 \wr S_2$ analyzed in section 7.4 belongs to this family. Note that for $n = 1$ we obtain Du Val singularities and their minimal resolutions.

(3) A certain four-dimensional representation of the binary tetrahedral group. The existence of crepant resolutions was first proved by Bellamy in [**9**]; they were constructed later by Lehn and Sorger, see [**68**]. We consider this case in section 7.5.

(4) A group of order 32 acting on $\mathbb{C}^4$, for which the existence of crepant resolutions was proved in [**11**]. Their construction and the Cox ring were investigated in [**37**] and [**47**].

Apart from this list, [**9**], [**12**] and [**94**] give also negative results. The non-existence of crepant resolutions is shown for certain infinite families of representations from the Cohen's classification [**21**]. Notably, the results of the paper [**94**], that appeared when this work was in progress, rely on the techniques developed in [**37**] and [**32**] (the second paper contributed partially to this thesis), using Cox rings. On the other hand, even in dimension 4 there still are infinitely many groups on the Cohen's list for which the question of existence of a crepant resolution remains unanswered.

There are also other recent results on the birational geometry of symplectic quotient singularities, related to our work. For example in [**10**], Bellamy used the work of Namikawa [**77**] to find the number of minimal models of a given symplectic quotient singularity. And as we were finishing this thesis, there appeared paper [**8**] by Bellamy and Craw in which they study an interpretation of resolutions for wreath products $H \wr S_n$ as in (2) above as certain moduli spaces, constructing them with the use of GIT.

**Geometry of three-dimensional crepant resolutions.** Three-dimensional quotient singularities $\mathbb{C}^3/G$ for $G \subset \mathrm{SL}_3(\mathbb{C})$ have also been studied extensively. In this case it is known, see [**55, 56, 71, 85**], that a crepant resolution exists. It is usually nonunique, but each two crepant resolutions differ by a modification in codimension two, and the set of all such resolutions for a given quotient singularity is finite. This set together with birational relations between its elements form a natural object of study.

As shown in [**17**], a crepant resolution of $\mathbb{C}^3/G$ can be constructed as the equivariant Hilbert scheme $G$-Hilb, which was conjectured by Nakamura in [**75**] (see also [**25**] for the case of $G$ abelian). In [**24**] all small $\mathbb{Q}$-factorial modifications of $G$-Hilb for $G$ abelian are analyzed and constructed as certain moduli spaces. It was also conjectured that such a construction might be the possible for arbitrary finite subgroup of $\mathrm{SL}_3(\mathbb{C})$ – this is the celebrated Craw-Ishii conjecture. Another successful approach to investigating such resolutions is based on homological methods and noncommutative algebra [**16, 91, 79**]. However, up to now, significant results have been obtained only for groups not containing any elements of age 2. Geometrically, this condition is equivalent to saying that the resolution does not contract a divisor to a point (see corollary 2.2.27, or the original source [**57**]), which apparently makes these resolutions easier to deal with. See e.g. [**79**] for an application of this method to finite subgroups $G \subset \mathrm{SO}_3$, in particular to representations of dihedral groups, which we study in 5.3.

**Cox rings of crepant resolutions.** The case of surface quotient singularities was analyzed by Donten-Bury in [**31**]. In [**37**] Donten-Bury and Wiśniewski studied the example of the symplectic quotient singularity of dimension 4 by a certain group of order 32 (see (4) above, in the part concerning symplectic quotient singularities). These two papers laid the groundwork for the study of crepant resolutions $X$ of quotient singularity $\mathbb{C}^n/G$ via Cox ring $\mathcal{R}(X)$ by use of the natural embedding $\mathcal{R}(X) \subset \mathbb{C}[x_1, \ldots, x_n]^{[G,G]}[\mathrm{Cl}(X)]$. Here

$[G, G] \subset G$ is the commutator subgroup and the rank of the finitely generated free abelian group $\mathrm{Cl}(X)$ may be computed via the McKay correspondence of Ito and Reid [**57**]. In other words, it was known from these works that the Cox ring of a resolution of a quotient singularity given by $G$ embeds into the Laurent polynomial ring over the ring of invariants of the commutator subgroup of $G$. This embedding is the starting point for the research presented here.

Prior to our studies there were also known computational methods by Hausen, Keicher and Laface [**49**], [**47**] of finding Cox rings of birational modifications of a variety with known Cox ring. The main difficulty to apply them in our context is describing the crepant resolution in terms of the blow-up operation (but see [**47**] for an application to singularity studied in [**37**]).

There are also methods specific to the complexity one case, i.e. for varieties with an effective action of an algebraic torus of the rank one smaller that the dimension of the variety [**50**]. These methods were generalized from complete to not necessarily complete case by Hausen and Wrobel in [**48**] when this work was in progress. As our ultimate goal is to understand the more general case we do not use these results.

As this work was in progress there appeared two more articles developing computational methods related to resolving singularities via Cox rings. The first is paper [**35**] by Donten-Bury and Keicher in which the methods of [**49**] were combined with tropical geometry to give an algorithm that in some cases computes the Cox ring of a resolution of a quotient singularity. This algorithm also relies heavily on the computational power, and we do not use it in this work. The second was the work of Yamagishi [**94**] providing an algorithm to check the slightly generalized valuative criterion given in [**32**] (we presented the criterion here in even more general form in chapter 4, see theorem 4.1.15). We use a part of this algorithm, with some improvements, once, in section 5.5, to study the geometry of crepant resolutions in the case of a three-dimensional quotient singularity given by a trihedral group of order 21. The quick growth of computational complexity of this algorithm with the growth of the number of crepant divisors as well as the fact that it does not always produce the minimal set of generators suggests to seek for other methods.

As we mentioned before, there is an ongoing research on the explicit minimal model program with use of homological methods by Wemyss and others [**91**], and the research on the symplectic varieties with use of noncommutative algebra by Bellamy, Namikawa, Schedler and others [**9**], [**11**], [**77**],[**10**]. We believe that as these studies develop – the one via Cox rings presented here, and the ones via homological methods and noncommutative algebra – each of them may benefit from the existence of the other ones.

## 1.2. Results and content of the dissertation

Chapters 2 and 3 give the general background for the main objects of our interest in this work. These are quotient singularities and resolutions in chapter 2 and Cox rings, together with the concepts of affine GIT of algebraic quasitorus actions, indispensable for retrieving geometric information from them, in chapter 3.

The first result that to our best knowledge is not explicitly written elsewhere is in section 3.4.2, where we generalize directly the well-known result by Hu and Keel [**54**, Proposition 2.9] characterizing complete Mori Dream Spaces to the relative setting (theorem 3.4.7). Then we use this fact to show that crepant resolutions or, more generally, minimal models of quotient singularities defined by subgroups of $\mathrm{SL}_n(\mathbb{C})$, are relative Mori Dream Spaces. In other words, their Cox rings are finitely generated and knowing the Cox ring one may reconstruct all such resolutions or minimal models together with their flops via GIT of the characteristic quasitorus action on the spectrum of the Cox ring. In particular the information such as the number of minimal models, or graph of flops are encoded by the subdivision of the cone of movable divisors. The structure of this cone is described in terms of the generators of the Cox ring by proposition 7.5.7, which is a straightforward generalization of the analogous result for the complete Mori Dream Spaces given in [**5**, Sect. 4.3.3].

The main results of this work are presented in chapters 4, 5 and 7.

In chapter 4, given a projective birational map $X \to Y$ we study the natural embedding of $\mathcal{R}(X)$ into a Laurent polynomial ring over $\mathcal{R}(Y)$ generalizing the setup from [**37**]. We propose a natural way of constructing a finite 'candidate' for a set of generators of $\mathcal{R}(X)$ from a given set of generators of $\mathcal{R}(Y)$ and we give a valuative criterion to verify whether such a 'candidate' is an actual set of generators. Then, specializing to the case of quotient singularities $\mathbb{C}^n/G$ for $G \subset \mathrm{SL}_n(\mathbb{C})$, we make this construction and valuative criterion effective in terms of monomial valuations of junior conjugacy classes of $G$, using the McKay correspondence of Ito and Reid [**57**]. We also present another criterion, based directly on the characterization of the Cox ring in terms of GIT from [**5**, Sect. 1.6.4]. Finally, we show how one may use multigraded Castelnuovo-Mumford regularity from [**70**] together with the Kawamata-Viehweg vanishing to bound the degrees of generators of a Cox ring.

Chapter 5 consists of the systematic application of results of the first part of the previous chapter to study the three-dimensional case. Here, we give the general treatment of quotient singularities $\mathbb{C}^3/G$, where $\mathbb{C}^3$ is a reducible $G$-representation and $G$ is a nonabelian group (so that the quotient is not a toric variety). Then, we analyze the case of dihedral groups – we reobtain the results of Nolla de Cellis and Sekiya on geometry of crepant resolutions using methods of Cox rings, where they used theory of noncommutative resolutions. We give two proofs for the presentation of the Cox ring in terms of generators and relations – one proof is based on the general theorem using valuative criterion of 4.1 and the other employs the criterion using characterization theorem for Cox rings from section 4.3. Finally, we study some examples of groups with elements of age two. We study two such examples that belong to the general family of reducible representations studied in the first part of the chapter. Then, in the last section, we study the simplest possible case of the quotient singularity $\mathbb{C}^3/G$ where $\mathbb{C}^3$ is an irreducible $G$-representation and $G$ contains elements of age two. Here we again use the construction of a 'candidate' for a set of generators together with a part of the algorithm of Yamagishi [**94**, Sect. 4] to verify that the appropriate 'candidate' is an actual set of generators.

Chapter 6 is an interlude dedicated to various facts on algebraic torus actions such as lifting the action to a crepant resolution, local structure of the action on a smooth variety, the Białynicki-Birula decomposition and the equivariant Euler characteristic. Unfortunately, for the last concept we did not find a satisfactory exposition including the Lefschetz-Riemann-Roch theorem at the level of generality adequate to our purpose. Therefore we decided to recall the basic definitions of the algebraic equivariant $K_0$ for an algebraic torus action and sketch the proof using the localization formula in 6.4.

In the final chapter 7 we exemplify multigraded Castelnuovo-Mumford bounding techniques of section 4.4 in the study of three symplectic quotient singularities in dimension four. In each case we find the Cox ring and describe the codimension two maximal cycles (the components of the fibre over 0) as well as the flops. The arguments here are based on the presence of a two-dimensional algebraic torus action, which comes from the fact that each of the three groups gives a reducible representation. The line of the argument here is more subtle than in previous chapters as we are simultaneously studying the geometry of a resolution and moving toward the proof that the given 'candidate set' is an actual set of generators of the Cox ring. The final proof of the latter part is based on the Lefschetz-Riemann-Roch formula from section 6.4 and the bounding technique from 4.4. Along the way we use computer calculations several times. First, to study the central fibre of a candidate for a resolution. Then, to prove that the GIT quotient of the ring generated by the appropriate 'candidate set' is a crepant resolution. Finally, we use a computer algebra system to find the Hilbert series of this ring and compare it with results of the Lefschetz-Riemann-Roch formula to prove that this ring is the whole Cox ring.

## Notation and conventions

We consider algebraic varieties over the field $\mathbb{C}$ of complex numbers and make the usual assumption that they are irreducible.

Let $X$ be an algebraic variety. By $\mathbb{C}(X)$ we denote its field of rational functions. By $X_{\mathrm{sm}}$ we mean the smooth locus of $X$. If $Z \subset X$ is a closed subvariety, then by $\mathcal{O}_{X,Z}$ we denote the local ring of $X$ at $Z$, i.e. the stalk of the structure sheaf $\mathcal{O}_X$ at the generic point of $Z$.

Let $R$ be a commutative ring with unity. By $R^*$ we denote the group of units of $R$. If $r_1, r_2 \in R$ then by $r_1 \mid r_2$ we denote the fact that $r_1$ divides $r_2$, i.e. that there exists $r'$ such that $r_1 r' = r_2$.

All algebraic group actions on algebraic varieties will be algebraic.

Assume that an algebraic group $H$ acts on algebraic variety $X$. For $x \in X$ we denote by $H_x$ the isotropy group of $x$, i.e. $H_x = \{h \in H : hx = x\}$. By $X^H$ we denote the fixed point locus of the action and for $h \in H$ by $X^h$ we denote the fixed point locus of the element $h$.

By a torus we mean an algebraic group isomorphic to $(\mathbb{C}^*)^m$ for some $m$. By a quasitorus we mean an algebraic group isomorphic to $(\mathbb{C}^*)^m \times H$, where $H$ is a finite abelian group.

If $\sigma$ is a cone in a rational or real vector space then by $\sigma^\circ$ we denote the relative interior of $\sigma$ and by $\sigma^\vee$ we denote the dual cone $\{w : \langle w, v\rangle \geq 0\}$ in the dual vector space.

$\zeta_r$ is a fixed complex primitive root of unity of order $r$.

$\mathbb{Z}_{\geq 0}$ is the set of nonnegative integers $\{0, 1, 2, \ldots\}$.

CHAPTER 2

# Singularities and their resolutions

## 2.1. General results from birational geometry

Here we collect a number of notions from birational geometry and related properties that we will use to describe objects of our studies and to formulate our results. First, in 2.1.1 we give the definition of a (discrete) valuation and relate it to divisors over an algebraic variety. These concepts are essential for later parts of the work as general results in chapter 4 will be formulated in terms of valuations (see in particular theorem 4.1.15) and the subsequent chapters will depend on these general results. In 2.1.2 we define the discrepancy of a divisor over an algebraic variety and we introduce various types of singularities. The last two sections 2.1.3 and 2.1.4 introduce concepts of a resolution of singularities and of a minimal model of an algebraic variety. The general reference for birational geometry is [**64**], especially Chapter 2 for the content of this part of the work.

**2.1.1. Valuations and divisors.** Let $k$ be a field and let $K/k$ be a field extension.

DEFINITION 2.1.1 (Discrete valuation). A *discrete valuation over $k$ on $K$* is a function $\nu : K \to \mathbb{Z} \cup \{\infty\}$ satisfying:

(1) $\nu(K^*) \subset \mathbb{Z},\ \nu(0) = \infty,\ \nu(k^*) = 0,$
(2) $\nu(xy) = \nu(x) + \nu(y),$
(3) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}.$

The ring $A_\nu = \{f \in K : \nu(f) \geq 0\}$ is called a *valuation ring* of $\nu$ and its quotient $k_\nu$ by its unique maximal ideal $\mathfrak{m}_\nu := \{f \in K : \nu(f) \geq 0\}$ is called the *residual field* of valuation $\nu$. The ring of the form $A_\nu$ is called a *discrete valuation ring* or in short *DVR*.

Throughout this text we use only discrete valuations over $\mathbb{C}$, so unless we specify otherwise, whenever we write *valuation* we mean *discrete valuation over* $\mathbb{C}$. We will be using the concept of valuation in the geometric setting that we now present. Let $Y$ be a normal algebraic variety.

EXAMPLE 2.1.2. Let $E$ be an irreducible divisor on $Y$. Then we have a valuation $\nu_E$ on the field of fractions $\mathbb{C}(Y)$ such that $\nu_E(f)$ is the order of vanishing of $f$ at $E$. Formally $\nu_E$ is defined first on the local ring $(\mathcal{O}_{Y,E}, \mathfrak{m}_{Y,E})$ as $\nu_E(f) = \min\{n : f \in \mathfrak{m}_{Y,E}^n\}$, and then extended uniquely to $\mathbb{C}(Y)$ by $\nu_E(f/g) = \nu_E(f) - \nu_E(g)$.

This *order of pole or order of vanishing of a divisor* type of valuation can be generalized using divisors on birational models of $Y$.

DEFINITION 2.1.3 (Divisors over $Y$ and algebraic valuations). Now let $E$ be a divisor on some normal algebraic variety equipped with birational morphism $\varphi\colon X \to Y$. We call divisors of this form for various $X$ and $\varphi$ *divisors over* $Y$. Then, as $\varphi$ induces an isomorphism $\mathbb{C}(X) \cong \mathbb{C}(Y)$, we have a valuation $\nu_E$ on $\mathbb{C}(Y)$ even though $\varphi(E)$ might not be a divisor on $Y$. The valuation of this form is called an *algebraic valuation* on $\mathbb{C}(Y)$ and the image of $E$ in $Y$ is called the *center* of valuation $\nu_E$.

The residual fields of algebraic valuations are of transcendence degree over $\mathbb{C}$ equal to $\dim Y - 1$. In fact, by theorem of Zariski, every valuation of $\mathbb{C}(Y)$ over $\mathbb{C}$ having this two properties is algebraic, see [**64**, Lemma 2.45].

**2.1.2. Discrepancies and related types of singularity.** Here we introduce the notion of the discrepancy of a divisor over an algebraic variety and related types of singularity. These are important technical notions in birational geometry. In this work they will be used in two subsequent sections to define special classes of (partial) resolutions of singularities – crepant resolution and minimal model – the main objects of our interest.

Let $Y$ be a normal algebraic variety.

Definition 2.1.4. (Discrepancy of a divisor over $X$) Assume that $mK_Y$ is a Cartier divisor for a positive integer $m$. Let $\varphi\colon X \to Y$ be a birational map from another normal algebraic variety $X$. Denote by $E_i$ the irreducible exceptional divisors of $\varphi$. Then we have an equivalence of divisors:

$$mK_X \sim \varphi^*(mK_Y) + \sum_i m \cdot a(E_i, Y)E_i$$

for some rational numbers $a(E_i, Y)$ such that $m \cdot a(E_i, Y) \in \mathbb{Z}$. The number $a(E_i, Y)$ is called the *discrepancy* of $E_i$ with respect to $Y$.

As the notation suggests, the discrepancy of a divisor over $Y$ depend only on $Y$ and the valuation $\nu_E$ on the function field $\mathbb{C}(Y)$, but not on the choice of the particular variety $X$ with a birational map $\varphi\colon X \to Y$ such that $\nu_E$ is a valuation of a divisor on $X$ (see [**64**, Remark 2.23]).

Definition 2.1.5 (Crepant divisors). An irreducible divisor $E$ over $Y$ is called *crepant* if its discrepancy vanishes, i.e. if $a(E, Y) = 0$.

If $Y$ is smooth then $a(E, Y) \geq 1$ for every divisor $E$ over $Y$ [**64**, Lemma 2.29]. In fact, discrepancies of divisors over $Y$ allow one to introduce the various types of singularity.

Definition 2.1.6 (Singularities related to discrepancies). We will say that $Y$ has:

(1) *terminal singularities* if $a(E, Y) > 0$ for every divisor $E$ over $Y$,
(2) *canonical singularities* if $a(E, Y) \geq 0$ for every divisor $E$ over $Y$,
(3) *log terminal singularities* if $a(E, Y) > -1$ for every divisor $E$ over $Y$.

Remark 2.1.7. The discrepancy of a divisor over $Y$ is a special case of the more general concept of the discrepancy of a pair $(Y, \Delta)$, where $\Delta$ is an effective divisor on $Y$. For this generalization and related notions of singularities of pairs we refer the reader to [**64**, Sect. 2.3]. We will not use them except for one step in the proof of theorem 3.4.10. The notion of discrepancy from definition 2.1.4 coincides with the notion of discrepancy of pair $(Y, \Delta)$ for $\Delta = 0$.

**2.1.3. Resolution of singularities.** Let $Y$ be an algebraic variety. Due to many desired properties of smooth varieties it is interesting to find a way of altering a singular variety to obtain a smooth one. One of such ways – resolving singularities – is a theme of this work. Informally speaking, to resolve singularities of $Y$ means to cut out the singular locus of $Y$ and replace it with a Zariski closed subset to obtain a smooth variety. This intuition is formalized in the following definition.

Definition 2.1.8 (Resolution of singularities). A *resolution of singularities* of $Y$ is a proper birational morphism $\varphi\colon X \to Y$ from a smooth variety $X$, which is an isomorphism outside the singular locus of $Y$, i.e. $\varphi|_{X \setminus \varphi^{-1}(\operatorname{Sing} Y)}\colon X \setminus \varphi^{-1}(\operatorname{Sing} Y) \to Y \setminus \operatorname{Sing} Y$ is an isomorphism. A resolution is called a *projective resolution* when it is a projective morphism.

In this work we consider only projective resolutions. By abuse of notation, whenever we say a *resolution* we mean a *projective resolution.* In particular, the domain of a resolution will always be a quasiprojective variety, whenever we will be resolving singularities of an affine variety.

In characteristic zero, for every algebraic variety over algebraically closed field there exists a projective resolution of singularities – this is the celebrated result of Hironaka on resolution of singularities [**53**, Main theorem I].

Given the existence of a resolution one may would like to ask what is the minimal possible way of resolving singularities of $Y$. For example, if $Y$ is a curve, such a resolution is given by the normalization. If $Y$ is a surface, then there always exists a *minimal resolution*, i.e. a resolution on which one cannot contract any curve to obtain another resolution. In higher dimensions there are also concepts that express such a minimality postulate, at least under certain mild assumptions on the singularities. We will present two of them, one – a crepant resolution – in this section, and another – a minimal model – in the next one.

Assume that the variety $Y$ is normal. Then, there is a well-defined Weil canonical divisor class $K_Y$ on $Y$ which is the unique class of a divisor restricting to the canonical divisor on the smooth locus of $Y$. Assume that $K_Y$ is $\mathbb{Q}$-Cartier.

Definition 2.1.9 (Crepant resolution of singularities). A *crepant resolution of singularities* of $Y$ is a resolution of singularities $\varphi\colon X \to Y$ such that every exceptional divisor on $X$ is a crepant divisor over $Y$. In other words, a resolution $\varphi\colon X \to Y$ is crepant if and only if $\varphi^* K_Y = K_X$.

We will see in theorem 2.2.20 that in the case of canonical quotient surface singularities the concept of crepant resolution coincides with the concept of minimal resolution. Unfortunately, as we also will see in examples 2.2.25 and 2.3.6 respectively, a crepant resolution of a canonical quotient singularity does not have to be unique, nor has it to exist at all. The problem of non-existence of crepant resolution will be discussed further in the next section, where we introduce minimal models. As for non-uniqueness, for our purposes it will be sufficient to note that a crepant resolution is unique in codimension one, since the components of its exceptional locus are precisely the crepant divisors over the resolved singular space.

**2.1.4. Minimal models.** In this section we present a concept of a minimal model in the relative setting. As we noted at the end of the previous section there are canonical quotient singularities for which a crepant resolution does not always exists. On the other hand a minimal model always exists (by nontrivial results from the Minimal Model Program), but in general it does not have to be smooth – it gives only a partial resolution of singularities. Nevertheless, in our setting a smooth minimal model is precisely a crepant resolution.

Although in concrete examples studied in detail in the later parts of this work the crepant resolution will exist, introducing minimal models will help us to spell out the results of chapter 4 in a more general setting.

Let $\varphi\colon X \to Y$ be a projective morphism from a normal $\mathbb{Q}$-factorial variety $X$ with terminal singularities to a normal $\mathbb{Q}$-factorial variety $Y$. The following is the relative version of the notion of minimal model from Mori theory (see [**64**, Definition 2.13, Example 2.16]).

Definition 2.1.10 (Minimal models in the relative setting).

(1) A *minimal model of* $\varphi$ is a normal $\mathbb{Q}$-factorial variety $X'$ with terminal singularities together with a projective morphism $\varphi' \colon X' \to Y$ and a projective birational morphism $\psi : X \to X'$ such that:
   (a) $K_{X'/Y}$ is $\varphi'$-nef.
   (b) The following diagram commutes:

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & X' \arrow[dl, "\varphi'"] \\ & Y & \end{tikzcd}$$

(2) A *minimal model of* $Y$ is a projective morphism $\varphi \colon X \to Y$ such that $X$ is terminal and $K_{X/Y}$ is $\varphi$-nef.

In later chapters we use the following characterization of minimal models for varieties with canonical singularities and trivial canonical divisor:

Proposition 2.1.11. *Assume that $K_Y = 0$ and $Y$ has canonical singularities. Then $\varphi \colon X \to Y$ is a minimal model of $Y$ if and only if $X$ is terminal and $K_X = 0$. In particular, in this situation every crepant resolution of $Y$ is a minimal model of $Y$ and every smooth minimal model of $Y$ is a crepant resolution of $Y$.*

Proof. One direction is immediate since a linearly trivial divisor is obviously relatively nef. On the other hand if $\varphi \colon X \to Y$ is a minimal model of $Y$ then $K_X = \sum_i a_i E_i$ for exceptional divisors $E_i$ and coefficients $a_i \geq 0$. Since $K_X$ is nef one has $a_i = 0$ by negativity lemma [**64**, Lemma 3.39]. □

Theorem 2.1.12. *Let $Y$ be a normal $\mathbb{Q}$-factorial variety with canonical singularities and assume that $K_Y = 0$. Then there exists a minimal model $\varphi : X \to Y$.*

Proof. This follows from [**15**, Corollary 1.4.3] applied to valuations of crepant divisors on any resolution of singularities of $Y$ and combined with proposition 2.1.11. □

## 2.2. Quotient singularities and their crepant resolutions

In this section we introduce the singular spaces whose resolutions are main objects of study in this work – quotient singularities, and we outline their basic properties. In 2.2.1 we summarize the results on their birational geometry and then, in 2.2.2 and 2.2.3 we present the known results on two- and three-dimensional cases. The general reference for algebraic properties of quotient singularities is [**13**].

Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite subgroup.

Theorem 2.2.1 (Hilbert-Noether [**13**, Theorem 1.3.1]). *The ring of invariants $\mathbb{C}[x_1, \ldots, x_n]^G$ is a finitely generated $\mathbb{C}$-algebra.*

Definition 2.2.2 (Quotient variety, quotient singularity). We denote the corresponding affine variety $\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]^G$ by $\mathbb{C}^n/G$ and call it a *quotient variety* of the action of $G$ on $\mathbb{C}^n$. We call a quotient variety $\mathbb{C}^n/G$ a *quotient singularity* if it is singular.

The celebrated Chevalley-Shephard-Todd theorem tells us which quotient spaces $\mathbb{C}^n/G$ are singular in terms of the group $G$.

Definition 2.2.3 (Complex reflection, complex reflection group, small group). An element $g \in \mathrm{GL}_n(\mathbb{C})$ is called a *complex reflection* if $\dim(\mathbb{C}^n)^g = n - 1$. A subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ is called a *complex reflection group* if $G$ is generated by complex reflections. A subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ is called a *small group* if $G$ does not contain complex reflections.

Example 2.2.4. The matrix $\left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$ is a complex reflection. On the other hand every subgroup $G \subset \mathrm{SL}_n(\mathbb{C})$ is a small group.

Theorem 2.2.5 (Chevalley-Shephard-Todd [**13**, Theorem 7.2.1]). *The quotient algebraic variety $\mathbb{C}^n/G$ is smooth if and only if $G$ is a complex reflection group. In such a case $\mathbb{C}^n/G \cong \mathbb{C}^n$.*

Note that the subgroup generated by complex reflections is normal as a matrix conjugate of a complex reflection is a complex reflection. As a consequence of this simple observation and of the theorem above, in the study of quotient varieties $\mathbb{C}^n/G$ for $G \subset \mathrm{GL}_n(\mathbb{C})$ one may restrict attention to quotient singularities for small groups $G$.

Proposition 2.2.6 (Properties of quotient singularities). *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a small finite group. Then*

*(i)* $\mathbb{C}^n/G$ *is a normal variety.*
*(ii)* $\mathrm{Pic}(\mathbb{C}^n/G) = 0$.
*(iii) There is a natural isomorphism* $\mathrm{Cl}(\mathbb{C}^n/G) \cong \mathrm{Ab}(G)^\vee := \mathrm{Hom}(G, \mathbb{C}^*)$. *In particular* $\mathbb{C}^n/G$ *is* $\mathbb{Q}$*-factorial.*
*(iv) The canonical Weil divisor class* $K_{\mathbb{C}^n/G}$ *is trivial if and only if* $G \subset \mathrm{SL}_n(\mathbb{C})$.

References. See [**13**, Proposition 1.1.1] for (i), [**13**, Theorem 3.6.1] for (ii), [**13**, Theorem 3.9.2] for (iii) and [**13**, Theorem 4.6.2] for (iv). ☐

The following result will be used to analyze the structure of quotient singularities in corollary 2.2.8 and in proposition 2.3.12.

Proposition 2.2.7. *Let $H \subset G$ and denote by $p_H \colon \mathbb{C}^n \to \mathbb{C}^n/H$ the quotient map, and by $p_{G,H} \colon \mathbb{C}^n/H \to \mathbb{C}^n/G$ be the map induced by inclusion of invariants. Let $x \in \mathbb{C}^n$. Then $p_{G,H}$ is étale at $p_H(x) \in \mathbb{C}^n/H$ if and only if $G_x \subset H$. In particular the quotient map $p\colon \mathbb{C}^n \to \mathbb{C}^n/G$ is étale at $x \in \mathbb{C}^n$ if and only if $G_x$ is trivial.*

Proof. It may be checked topologically (using standard Euclidean topology over $\mathbb{C}$) that the map $p_H$ is unramified. Under our assumptions this is equivalent to $p_H$ being étale. For the algebraic version of the proposition, which works more generally over any algebraically closed field of characterstic 0 see [**69**, Sect. II, Lemme 2]. ☐

Now we may use Chevalley-Shephard-Todd theorem to specify which points on the quotient variety $\mathbb{C}^n/G$ are singular.

Corollary 2.2.8. *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite group, not necessarily small. Then $\mathrm{Sing}(\mathbb{C}^n/G)$ is the image of the points in $\mathbb{C}^n$ for which the isotropy group $G_x$ is not generated by complex reflections.*

Proof. Let $x \in \mathbb{C}^n$. By proposition 2.2.7 the natural map induced by inclusion of invariants $\mathbb{C}^n/G_x \to \mathbb{C}^n/G$ is étale. In particular the image of $x$ in $\mathbb{C}^n/G$ is singular if and only if the image of $x$ in $\mathbb{C}^n/G_x$ is singular. By theorem 2.2.5 the latter holds if and only if $G_x$ is not generated by complex reflections. ☐

**2.2.1. Discrepancies and the McKay correspondence.** Here we collect the general results concerning the birational geometry of quotient singularities. For the later chapters the most important result of this part is the McKay correspondence of Ito and Reid (theorem 2.2.18). The main references for this section are [**66**, Sect. 3.2] and papers on the McKay correspondence [**57**], [**84**].

For each $r$ fix a complex primitive root $\zeta_r$ of order $r$.

NOTATION 2.2.9. Let $g \in \mathrm{GL}_n(\mathbb{C})$ be an element of finite order $r$. Then $g$ has a diagonal form $\mathrm{diag}(\zeta_r^{a_1}, \dots, \zeta_r^{a_n})$ for some integers $a_i \in [0, r)$. In such situation we will write $g \sim \frac{1}{r}(a_1, \dots, a_n)$. Note that this does not depend on the representative of the conjugacy class of $g$.

DEFINITION 2.2.10 (Age). The age of the conjugacy class of $g \sim \frac{1}{r}(a_1, \dots, a_n)$ is the number $\mathrm{age}(g) = \frac{1}{r}\sum_{i=1}^n a_i$.

By definition $\mathrm{age}(\mathrm{id}) = 0$, the age of an element $g \neq \mathrm{id}$ is always positive and if $g \in \mathrm{SL}_n(\mathbb{C})$ then $\mathrm{age}(g)$ is an integer.

DEFINITION 2.2.11 (Monomial valuation).

(1) Let $(a_1, \dots, a_n)$ be a tuple of integers such that $\gcd(a_1, \dots, a_n) = 1$. The corresponding *monomial valuation* on the field $\mathbb{C}(x_1, \dots, x_n)$ of rational functions is the unique valuation $\nu$ over $\mathbb{C}$ satisfying

$$\nu\left(\sum_\alpha c_\alpha x^\alpha\right) = \min\left\{\sum_{i=1}^n \alpha_i a_i \colon c_\alpha \neq 0\right\}.$$

(2) If $g \sim \frac{1}{r}(a_1, \dots, a_n)$ has a diagonal form in coordinates $x_1, \dots, x_n$, then we define the monomial valuation $\nu_g$ of $g$ as the monomial valuation of the tuple $(a_1, \dots, a_n)$.

The concept of monomial valuation has the following geometric meaning:

PROPOSITION 2.2.12 ([**57**, Sect. 2.4]). *If $(a_1, \dots, a_n)$ are positive integers with $\gcd(a_1, \dots, a_n) = 1$ and $W \to \mathbb{C}^n$ is the corresponding weighted blow-up with exceptional divisor denoted by $E$ then $\nu_{E,\mathbb{C}^n}$ is the monomial valuation defined by $(a_1, \dots, a_n)$.*

The following technical result describes in detail divisors over quotient singularities in terms of the age and monomial valuations of elements of $G$. To make statement more concise we use the following convention: whenever we write that $E \subset X \xrightarrow{\varphi} Y$ is a divisor over $Y$ we mean that $X$ is a normal variety, $\varphi\colon X \to Y$ is a birational morphism and $E \subset X$ is a divisor.

THEOREM 2.2.13 (Divisors over a quotient). *Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a small group and denote $Y = \mathbb{C}^n/G$.*

*(i) If $F \subset X' \xrightarrow{\varphi} \mathbb{C}^n$ is a divisor over $\mathbb{C}^n$ then $X'$ admits a rational action of $G$ such that $\varphi$ is an equivariant map and the group $\mathrm{Fix}(F) := \{g \in G\colon g \text{ fixes a general point of } F\}$ is cyclic.*

*(ii) Every divisor $E$ over $Y$ arises as a divisor over $\mathbb{C}^n$. That is, if $E \subset X \xrightarrow{\varphi} Y$ is a divisor over $Y$ then there is a birational morphism $\varphi' : X' \to \mathbb{C}^n$, a morphism $\psi : X' \to X$ and a divisor $F \subset X'$ such that $\psi(F) = E$ and the following diagram commutes:*

$$\begin{tikzcd} X' \arrow[r, "\varphi'"] \arrow[d, "\psi"'] & \mathbb{C}^n \arrow[d] \\ X \arrow[r, "\varphi"] & Y \end{tikzcd}$$

*Moreover $\varphi, \varphi', \psi$ can be chosen as proper morphisms and then $a(E, Y) = \frac{a(F,\mathbb{C}^n)+1}{\#\mathrm{Fix}(F)} - 1$.*

*(iii)* *Let* $g \in G$, $g \sim \frac{1}{r}(a_1, \dots, a_n)$, $\gcd(a_1, \dots, a_n) = 1$. *Assume that there is no element* $g' \in G$ *with* $g' \sim \frac{1}{r'}(a_1, \dots, a_n)$, $r \mid r'$ *diagonalizable in the same basis. Consider the weighted blow-up* $W \to \mathbb{C}^n$ *with weights* $(a_1, \dots, a_n)$ *in coordinates in which* $g$ *has the diagonal form, its exceptional divisor* $F$. *Let* $E$ *be image of* $F$ *over* $Y$, *i.e. the divisor* $E \subset X \xrightarrow{\varphi} Y$ *such that there is a rational map* $\psi \colon W \dashrightarrow X$ *with* $\psi|_F : F \dashrightarrow E$ *well-defined and dominating and such that the diagram below commutes:*

$$\begin{tikzcd} W \arrow[r, "\varphi'"] \arrow[d, dashed, "\psi"'] & \mathbb{C}^n \arrow[d] \\ X \arrow[r, "\varphi"] & Y. \end{tikzcd}$$

*Then* $\mathrm{Fix}(F) = \langle g \rangle$, $a(E, Y) = \mathrm{age}(g) - 1$ *and* $\nu_E = \frac{1}{r}\nu_g|_{\mathbb{C}(Y)}$. *Moreover, if* $\mathrm{age}(g) < 2$, *then every divisor* $E$ *over* $Y$ *with* $\mathrm{Fix}(E) = \langle g \rangle$ *and* $a(E, Y) = \mathrm{age}(g) - 1$ *has this form.*

*(iv)* *Let* $E$ *be a divisor over* $Y$ *and denote the corresponding divisor over* $\mathbb{C}^n$ *as in (ii) by* $F$. *Assume* $\mathrm{Fix}(F) = \langle g \rangle \neq \{\mathrm{id}\}$ *and* $g \sim \frac{1}{r}(a_1, \dots, a_n)$ *with* $\gcd(a_1, \dots, a_n) = 1$. *Then* $a(E, Y) \geq \mathrm{age}(g) - 1$.

References. See [**57**, Sect. 2.3-4, 2.6-7], [**84**, Theorem 2.1] or [**66**, Theorem 3.21] and its proof. The map $\psi$ in (iii) exists by a general lemma of Abhyankar and Zariski [**66**, Lemma 2.22]. See also [**61**, Lemma 2.2] for a result related to (iv). □

We now draw some corollaries of theorem 2.2.13. Among them are the Reid-Tai criterion and the McKay correspondence of Ito and Reid. But first, by (ii) and the smoothness of $\mathbb{C}^n$ we obtain immediately that:

Corollary 2.2.14. $\mathbb{C}^n/G$ *has log terminal singularities.*

Theorem 2.2.15 (Reid-Tai criterion [**82**, 4.11]). $\mathbb{C}^n/G$ *has canonical (terminal) singularities if and only if age*$(g) \geq 1 (> 1)$ *for every* $g \in G \setminus \{\mathrm{id}\}$. *In particular if* $G \subset \mathrm{SL}_n(\mathbb{C})$ *then the corresponding quotient singularity is canonical.*

Proof. We use theorem 2.2.13. As in the theorem denote $Y = \mathbb{C}^n/G$. Let $E$ be a divisor over $Y$ and let $F, \varphi, \varphi', \psi$ be as in (ii) with $\varphi, \varphi', \psi$ proper. By (i) we have $\mathrm{Fix}(F) = \langle g \rangle$ for some $g \in G$. By (ii) if $\mathrm{Fix}(F) = \{\mathrm{id}\}$ then $a(E, Y) = a(F, \mathbb{C}^n) \geq 1$. The claim follows now by (iii) and (iv). □

By Reid-Tai criterion, proposition 2.2.6(iv) and theorem 2.1.12 we obtain the existence of minimal models for quotient singularities for $G \subset \mathrm{SL}_n(\mathbb{C})$.

Corollary 2.2.16. *If* $G \subset \mathrm{SL}_n(\mathbb{C})$ *then there exists a minimal model of the corresponding quotient singularity* $\mathbb{C}^n/G$.

Definition 2.2.17. If $G \subset \mathrm{SL}_n(\mathbb{C})$ the classes of age one in $G$ are called *junior conjugacy classes* of $G$.

The following theorem is the most important part of this section from the point of view of our applications. It belongs to the field of the McKay correspondence, which studies relations between properties of geometric objects associated to a quotient singularity $\mathbb{C}^n/G$ and algebraic objects associated to $G$ and its representations. This instance relates codimension one geometry of a minimal model of a quotient to the group and its action on the field of rational functions.

THEOREM 2.2.18 (McKay correspondence of Ito and Reid [**57**]). *Let $G \subset \mathrm{SL}_n(\mathbb{C})$ and denote $Y = \mathbb{C}^n/G$. There is a one-to-one correspondence between junior conjugacy classes of $G$ and irreducible crepant divisors over $Y$, i.e. the irreducible exceptional divisors on a minimal model $\varphi\colon X \to Y$. More precisely, if $E$ is a divisor corresponding to conjugacy class of element $g$ of order $r$, then the divisorial valuation $\nu_E$ is equal to $\frac{1}{r}\nu_g|_{\mathbb{C}(Y)}$, where we use identification $\mathbb{C}(X) = \mathbb{C}(Y) = \mathbb{C}(x_1, \ldots, x_n)^G$.*

SKETCH OF THE PROOF, CF. [**57**, Sect. 2.8]. We identify crepant divisors over $Y$ with their valuations on $\mathbb{C}(Y)$. To junior conjugacy class $[g]$ of element $g$ of order $r$ we assign the valuation $\frac{1}{r}\nu_g|_{\mathbb{C}(Y)}$. By theorem 2.2.13(iii) this is the valuation of a crepant divisor over $Y$ which arise as in theorem 2.2.13(ii) from the weighted blow-up of $\mathbb{C}^n$ with weights $(a_1, \ldots, a_n)$ in coordinates diagonalizing $g$. The assumptions on $g$ needed in 2.2.13(iii) follow from the fact that $G \subset \mathrm{SL}_n(\mathbb{C})$ and $[g]$ is a junior conjugacy class. The assignment $[g] \mapsto \frac{1}{r}\nu_g|_{\mathbb{C}(Y)}$ is well-defined, since $\mathbb{C}(Y) = \mathbb{C}(x_1, \ldots, x_n)^G$. It is injective, because the field extension $\mathbb{C}(x_1, \ldots, x_n)/\mathbb{C}(Y)$ is Galois. The surjectivity follows from the last part of theorem 2.2.13(iii), since by points (ii) and (i) of the theorem $\mathrm{Fix}(E)$ is cyclic for every divisor $E$ over $Y$ and by point (iv) if $E$ is crepant, then $\mathrm{Fix}(E)$ is generated by a junior element. □

We find this form of the McKay correspondence particularly useful for our needs. Not only it helps to find the class group of a minimal model or a crepant resolution, but it also gives us a precise recipe how to calculate divisorial valuations of crepant divisors over the quotient singularities for $G \subset \mathrm{SL}_n(\mathbb{C})$ in terms of elements of the group $G$.

**2.2.2. Dimension two case.** Here we collect the results about subgroups $G \subset \mathrm{GL}_2(\mathbb{C})$ and corresponding quotients two-dimensional quotients $\mathbb{C}^2/G$. They serve as motivating examples to study quotient singularities and they will also find use in later parts of the work, especially in chapter 5. We begin with the subgroups of $\mathrm{SL}_2(\mathbb{C})$.

THEOREM 2.2.19 (Finite subgroups of $\mathrm{SL}_2(\mathbb{C})$, [**73**], [**22**, Sect. 3]). *Every finite subgroup $G \subset \mathrm{SL}_2(\mathbb{C})$ is conjugate to one of the following groups*

- $(A_n)$ *Cyclic group $C_n$ of order $n$ generated by* $\begin{pmatrix} \zeta_n & 0 \\ 0 & \zeta_n^{-1} \end{pmatrix}$.
- $(D_n)$ *Binary dihedral group $\mathrm{BD}_{4n}$ of order $4n$, $n \geq 2$ generated by $C_{2n}$ and* $\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$.
- $(E_6)$ *Binary tetrahedral group $\mathrm{BT}$ of order 24 generated by $\mathrm{BD}_8$ and a matrix* $\frac{1}{\sqrt{2}}\begin{pmatrix} \zeta_8 & \zeta_8^3 \\ \zeta_8 & \zeta_8^7 \end{pmatrix}$.
- $(E_7)$ *Binary octahedral group $\mathrm{BO}$ of order 48 generated by $BT$ and a matrix* $\begin{pmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8^5 \end{pmatrix}$.
- $(E_8)$ *Binary icosahedral group of order 120 generated by matrices* $\frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5^4-\zeta_5 & \zeta_5^2-\zeta_5^3 \\ \zeta_5^2-\zeta_5^3 & \zeta_5-\zeta_5^4 \end{pmatrix}$ *and* $\frac{1}{\sqrt{5}}\begin{pmatrix} \zeta_5^2-\zeta_5^4 & \zeta_5^4-1 \\ 1-\zeta_5 & \zeta_5^3-\zeta_5 \end{pmatrix}$.

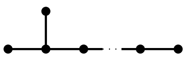
*The corresponding invariant rings $\mathbb{C}[x,y]^G$ are generated by three invariant polynomials $p_1, p_2, p_3$ with the following relations:*

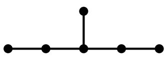
| *Group G* | *generators* $p_1, p_2, p_3$ | *relation* |
|---|---|---|
| $C_n$ | $x^n,\ y^n,\ xy$ | $Z_1Z_2 - Z_3^2$ |
| $\mathrm{BD}_{4q}$ | $x^q + y^q,\ x^q - y^q,\ xy$ | $Z_1^2 - Z_2^2 - 4Z_3^q$ |
| BT | $x^4 + y^4 + \sqrt{-12}x^2y^2,\ x^4 + y^4 - \sqrt{-12}x^2y^2,\ x^5y - xy^5$ | $Z_1^3 - Z_2^3 - 12(\zeta_3 - \zeta_3^2)Z_3^2$ |
| BO | $x^5y - xy^5,\ x^8 + 14x^4y^4 + y^8,\ x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$ | $108Z_1^4 - Z_2^3 + Z_3^2$ |
| BI | $x^{11}y + 11x^6y^6 - xy^{11}$, $x^{20} - 228x^{15}y^5 + 494x^{10}y^{10} + 228x^5y^{15} + y^{20}$, $x^{30} + 522x^{25}y^5 - 10005x^{20}y^{10} - 10005x^{10}y^{20} - 522x^5y^{25} + y^{30}$ | $1728Z_1^5 + Z_2^3 - Z_3^2$ |

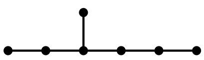
The corresponding quotient singularities $\mathbb{C}^2/G$ for $G \subset \mathrm{SL}_2(\mathbb{C})$ are called *ADE-singularities* or *Du Val singularities.*

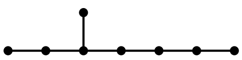
Theorem 2.2.20 (Minimal resolutions of Du Val singularities, [**64**, Theorem 4.20, Theorem 4.22]). *There is a unique crepant resolution of a Du Val singularity and it coincides with the minimal resolution. The exceptional divisor of such a resolution consists of smooth rational curves with self-intersection* $(-2)$ *and the dual of its incidence graph is a Dynkin diagram of type corresponding to the type of the singularity:*

$(A_n)$ 

$(D_n)$ 

$(E_6)$ 

$(E_7)$ 

$(E_8)$ 

Definition 2.2.21 (Cartan matrices). We define the *Cartan matrices* of types $A_n, D_n, E_6, E_7, E_8$ as the intersection matrices $(C_i.C_j)_{i,j}$ for the components $C_i$ of the exceptional divisor on the crepant resolution of the corresponding Du Val singularity.

By Reid-Tai criterion (see theorem 2.2.15) Du Val singularities are two-dimensional canonical singularities. It turns out that the converse is also true, see [**64**, Theorem 4.20].

Now we will recall the classification of all finite subgroups of $\mathrm{GL}_2(\mathbb{C})$. To do so, we need to introduce some notation first.

Notation 2.2.22. Let $G_1, G_2 \subset \mathrm{GL}_n(\mathbb{C})$ be matrix groups with normal subgroups $H_1, H_2$, and $\varphi$ be the isomorphism of $G_1/H_1$ and $G_2/H_2$. Set $G_1 \times_\varphi G_2 = \{(g_1, g_2) \in G_1 \times G_2 \mid \varphi(g_1H_1) = g_2H_2\}$. Then by $(G_1 \mid H_1,\ G_2 \mid H_2)$ we understand $\psi(G_1 \times_\varphi G_2)$, where $\psi$ is the multiplication map. By $\mu_d$ we denote the group of complex $d$-th roots of unity. We use the embedding $\mu_d \cdot I_n \subset \mathrm{GL}_n(\mathbb{C})$.

Theorem 2.2.23 ([**22**, Sect. 3]). *Every finite subgroup* $G \subset \mathrm{GL}_2(\mathbb{C})$ *is conjugate to one of the following groups:*

(0) *Cyclic subgroup* $C_{m,q}$ *of order* $m$ *generated by the matrix* $\mathrm{diag}(\zeta_m, \zeta_m^q)$.
(1) $(\mu_{4m} \mid \mu_{2m},\ \mathrm{BD}_{4q} \mid C_{2q})$ *of order* $4mq$.
(2) $\mu_{2m} \cdot \mathrm{BT}$ *of order* $24m$.

*(3)* $(\mu_{4m} \mid \mu_{2m},\ \mathrm{BD}_{8q} \mid \mathrm{BD}_{4q})$ *of order* $8mq$.
*(4)* $(\mu_{6m} \mid \mu_{2m},\ \mathrm{BT} \mid \mathrm{BD}_8)$ *of order* $8mq$.
*(5)* $\mu_{2m} \cdot \mathrm{BD}_{4q}$ *of order* $4mq$.
*(6)* $\mu_{2m} \cdot \mathrm{BO}$ *of order* $48m$.
*(7)* $(\mu_{4m} \mid \mu_m,\ \mathrm{BD}_{4q} \mid C_q)$ *for* $q$ *odd, of order* $2mq$ *if* $m$ *is odd and of order* $4mq$ *if* $m$ *is even.*
*(8)* $(\mu_{4m} \mid \mu_{2m},\ \mathrm{BO} \mid \mathrm{BT})$ *of order* $48m$.
*(9)* $\mu_{2m} \cdot BI$ *of order* $120m$.

Just as in the case of Du Val singularities, general quotient singularities in the surface case also coincide with a class of singularities defined in terms of discrepancies – the log terminal singularities (see [**64**, Proposition 4.18]).

**2.2.3. Dimension three case.** In chapter 5 we will be interested in finite subgroups of the special linear group in dimension three. Let $G \subset \mathrm{SL}_3(\mathbb{C})$. Like in the 2-dimensional case one can find a crepant resolution of the corresponding quotient singularity.

THEOREM 2.2.24 ([**17**],[**85**]). *There exists a crepant resolution of* $\mathbb{C}^3/G$.

The proof by Ito [**55**], [**56**], Markushevich [**71**] and Roan [**85**] follows the classification of finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ which we recall in the theorem below. The proof by Bridgeland, King and Reid in [**17**] is more conceptual and constructs a crepant resolution uniformly as so-called $G$-Hilbert scheme of $\mathbb{C}^3$.

On the other hand, the uniqueness property for crepant resolutions in 2-dimensional case does not extend to dimension three.

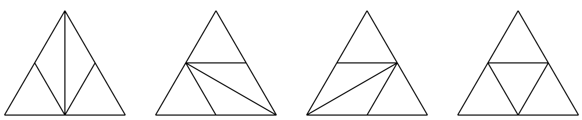
EXAMPLE 2.2.25 (Non-uniqueness of a crepant resolution in dimension three). Consider the subgroup $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ of $\mathrm{SL}_3(\mathbb{C})$ generated by matrices:

$$g_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then $Y = \mathbb{C}^3/G$ is a toric variety of the cone $\sigma = cone(e_1, e_2, e_3) \subset N'_{\mathbb{R}}$ for the torus with one-parameter group lattice $N' = \mathbb{Z} \cdot \frac{e_1+e_2}{2} + \mathbb{Z} \cdot \frac{e_2+e_3}{2} + \mathbb{Z} \cdot \frac{e_1+e_3}{2}$. Using toric geometry one may prove that:

(1) Blowing-up subsequently in any order strict transforms of images of loci $S_1, S_2, S_3$ of the points in $\mathbb{C}^3$ with isotropy groups $\langle g_1 \rangle, \langle g_2 \rangle, \langle g_1 g_2 \rangle$ one obtains a crepant resolution of $Y$.
(2) Blowing-up in (1) strict transforms of the images of $S_1, S_2, S_3$ in various orders yield 3 resolutions non-isomorphic as varieties over $Y$.

In fact there are four crepant resolutions of $Y$ which correspond to the subdivisions of the cone $\sigma$ defining $Y$. Slices of these subdivisions are pictured below:



The first three subdivisions correspond to the resolutions described above. Note that they give isomorphic toric varieties, but the point is that these isomorphisms are not over $Y$. The fourth subdivision yields a toric variety which is not even isomorphic as a variety over $\mathbb{C}$ with any of the first three resolutions.

As a consequence the crepant resolution of the quotient singularity $Y$ is not unique.

The classification of finite subgroups of $\mathrm{SL}_3(\mathbb{C})$ up to conjugacy is well-understood:

THEOREM 2.2.26 ([**95**, Sect. 1]). *Every finite subgroup $G \subset \mathrm{SL}_3(\mathbb{C})$ is conjugate to one of the following groups*

(A) *Finite diagonal abelian groups, i.e. finite groups consisting of matrices of the form* $\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$.

(B) *Nonabelian groups corresponding to nonabelian finite subgroups of* $\mathrm{GL}_2(\mathbb{C})$, *i.e. groups of the form*
$$G = \left\{ \begin{pmatrix} M & 0 \\ 0 & \det(M)^{-1} \end{pmatrix} : M \in \overline{G} \right\}, \ \textit{where } \overline{G} \subset \mathrm{GL}_2(\mathbb{C}), \ \textit{nonabelian.}$$

(C) *Groups generated by finite diagonal abelian group as in (A) and the matrix* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

(D) *Groups generated by a group of the form (C) and a matrix* $\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & b \\ 0 & c & 0 \end{pmatrix}$, *where* $abc = -1$.

(E)-(L) 8 *exceptional groups.*

The description of exceptional groups is omitted as we will not be dealing with them in this work. We end this section with the following corollary of the McKay correspondence (theorem 2.2.18).

COROLLARY 2.2.27. *If $n = 3$, then we can compose the bijection from the McKay correspondence with the involution sending $g$ to $g^{-1}$ to obtain that irreducible exceptional divisors of a crepant resolution $\varphi \colon X \to \mathbb{C}^3/G$ which are contracted to the point $[0] \in \mathbb{C}^3/G$ are in one-to-one correspondence with conjugacy classes of age 2 in $G$.*

PROOF. Note that the exceptional divisor corresponding to the class of $g$ is contracted to the point $[0]$ if and only if all exponents $a_i$ in the expression $g \sim \frac{1}{r}(a_1, a_2, a_3)$ are positive. Then apply theorem 2.2.18 and observe that the involution $g \mapsto g^{-1}$ sends the classes of age 1 with all exponents $a_i$ positive precisely onto the classes of age 2. $\square$

## 2.3. Symplectic quotient singularities and their symplectic resolutions

Here we define and give basic properties of the special class of quotient singularities, generalizing Du Val singularities in even dimensions, namely symplectic quotient singularities. For crepant resolutions of such singularities we have a stronger version of the McKay correspondence proven by Kaledin, see theorem 2.3.11. We will also need a result of Andreatta and Wiśniewski presented in theorem 2.3.13. The content of this section will be used in chapter 7, where we will analyze the geometry and Cox rings of crepant resolutions of examples of symplectic quotient singularities. For a survey on symplectic singularities (not necessarily of the quotient nature) see [**40**].

DEFINITION 2.3.1 (Symplectic structure on a smooth variety). Let $X$ be a smooth algebraic variety. A *symplectic form* on $X$ is a global section $\omega \in H^0(X, \Omega^2_X)$ which is closed, i.e. $d\omega = 0$ and nowhere degenerate, i.e. it restricts to a nondegenerate bilinear form on the tangent space $T_xX$ for every $x \in X$. A *symplectic variety* is a smooth variety with chosen symplectic form.

Let $\mathrm{Sp}_{2n}(\mathbb{C})$ be a group of linear transformations of the $\mathbb{C}^{2n}$ preserving a standard symplectic form $\omega_{\mathbb{C}^{2n}} = \sum_{i=1}^{2n} dx_i \wedge dy_i$ on $\mathbb{C}^{2n}$.

DEFINITION 2.3.2 (Symplectic quotient singularity). A *symplectic quotient singularity* is a quotient singularity $\mathbb{C}^{2n}/G$ where $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$ is a finite subgroup.

EXAMPLE 2.3.3. Since $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$ every Du Val singularity is a symplectic quotient singularity.

Let $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$ be a finite subgroup. Denote by $p\colon \mathbb{C}^{2n} \to \mathbb{C}^{2n}/G$ the quotient map and let $(\mathbb{C}^{2n}/G)_{\mathrm{sm}}$ be the smooth locus of $\mathbb{C}^{2n}/G$. Since the action of $G$ preserves $\omega_{\mathbb{C}^{2n}}$ and $p|_{p^{-1}((\mathbb{C}^{2n}/G)_{\mathrm{sm}})}$ is étale by proposition 2.2.7 and corollary 2.2.8, the restriction of $\omega_{\mathbb{C}^{2n}}$ to the preimage of the smooth locus of $\mathbb{C}^{2n}/G$ induces a symplectic structure on the smooth locus of $\mathbb{C}^{2n}/G$. This allows one to give a characterization of crepant resolutions of $\mathbb{C}^{2n}/G$ in terms of the symplectic form.

Definition 2.3.4 (Symplectic resolution). A resolution of the symplectic quotient singularity $X \to \mathbb{C}^{2n}/G$ is called a *symplectic resolution* if the pullback of the symplectic structure on the smooth locus of $\mathbb{C}^{2n}/G$ extends to a symplectic structure on $X$.

Proposition 2.3.5 ([**90**, Theorem 2.5]). *A resolution of the symplectic quotient singularity $X \to \mathbb{C}^{2n}/G$ is crepant if and only if it is symplectic.*

The symplectic resolution does not always exists.

Example 2.3.6. Let $G = \langle -\mathrm{id}_{\mathbb{C}^4} \rangle \subset \mathrm{Sp}_4(\mathbb{C})$. Then, by Reid-Tai criterion (see theorem 2.2.15) the singularity $Y = \mathbb{C}^4/G$ is terminal. In particular, every crepant divisor over $Y$ is present on $Y$. Since $Y$ is $\mathbb{Q}$-factorial, it follows that a resolution of singularities of $Y$ has to contract a divisor (see [**64**, Lemma 2.62], the proof given there generalizes verbatim to a projective birational morphism from a quasiprojective variety). In particular no resolution of singularities of $Y$ is crepant. By virtue of proposition 2.3.5 there is no symplectic resolution of $Y$.

We proceed to give a necessary criterion for existence of a symplectic resolution.

Definition 2.3.7 (Symplectic reflection). A *symplectic reflection* is an element $g \in \mathrm{Sp}_{2n}(\mathbb{C})$ such that $\dim(\mathbb{C}^{2n})^g = 2n-2$.

Note that $g \in \mathrm{Sp}_{2n}(\mathbb{C})$ is a symplectic reflection if and only if $\mathrm{age}(g) = 1$. As in the example, employing Reid-Tai criterion we may show that the necessary condition for the existence of the symplectic resolution of $\mathbb{C}^{2n}/G$ is the existence of symplectic reflection in $G$. The following result of Verbitsky gives a stronger criterion.

Theorem 2.3.8 ([**90**, Theorem 1.2],[**60**, Theorem 1.7]). *Let $X \to \mathbb{C}^{2n}/G$ be a symplectic resolution. Then $G$ is generated by symplectic reflections.*

Groups generated by symplectic reflections are classified in [**21**].

In the case of symplectic quotient singularities there is a stronger version of the McKay correspondence, assigning to each conjugacy class of $X$ a subvariety of $X$ with a certain special property. Let us make this precise.

Theorem 2.3.9 ([**59**, Proposition 4.4],[**90**, Theorem 2.8]). *Let $\varphi\colon X \to \mathbb{C}^{2n}/G$ be a symplectic resolution. Then $\varphi$ is semismall, i.e.* $\operatorname{codim}\varphi(Z) \leq 2\operatorname{codim} Z$ *for every algebraic subvariety $Z \subset X$.*

Definition 2.3.10 (Maximal cycle). Let $\varphi\colon X \to \mathbb{C}^{2n}/G$ be a symplectic resolution. A *maximal cycle* on $X$ is an algebraic subvariety $Z \subset X$ such that $\operatorname{codim}\varphi(Z) = 2\operatorname{codim} Z$.

Theorem 2.3.11 (Symplectic McKay correspondence [**61**]). *Let $\varphi\colon X \to \mathbb{C}^{2n}/G$ be a symplectic resolution. There is a bijective correspondence between conjugacy classes in $G$ and maximal cycles on $X$ given as follows. A conjugacy class $[g]$ of $G$ correspond to the center $Z_g$ of monomial valuation $v_g$. Moreover* $\operatorname{codim} Z_g = \mathrm{age}(g)$.

The simplest examples of symplectic resolutions are given by minimal resolutions of Du Val singularities 2.2.20. It turns out that every symplectic resolution of a symplectic quotient singularity is a minimal resolution of transversal Du Val singularity in codimension one in the sense of the following proposition (cf. [**92**, Theorem 1.4]).

Proposition 2.3.12. *Let $G \subset \mathrm{Sp}_{2n}(\mathbb{C})$ be a finite group. Assume that there exists a symplectic resolution $\varphi\colon X \to \mathbb{C}^{2n}/G$.*

(i) *Each component $\Sigma_0$ of the singular locus $\mathrm{Sing}(\mathbb{C}^{2n}/G)$ is of dimension $2n-2$ and for each such component there is an open subset $U \subset \mathbb{C}^{2n}/G$ such that $\Sigma_0 \cap U \neq \varnothing$ and a surjective étale rational map $(\mathbb{C}^2/H) \times \mathbb{C}^{2n-2} \dashrightarrow U$ for some finite group $H \subset \mathrm{SL}_2(\mathbb{C})$.*

(ii) *Assume that $\Sigma_1, \dots, \Sigma_k$ are the components of the singular locus of $\mathbb{C}^{2n}/G$. Let $E_1, \dots, E_m \subset X$ be the irreducible components of the exceptional divisor of $\varphi$. If $C_i$ is the general fibre of $\varphi|_{E_i}\colon E_i \to \varphi(E_i)$ then the intersection matrix $(E_i.C_j)_{i,j}$ is the direct sum of Cartan matrices (see definition 2.2.21) corresponding to types of the Du Val singularities $\mathbb{C}^2/H_i$, where $H_i \subset \mathrm{SL}_2(\mathbb{C})$ is obtained as in (i) for the component $\Sigma_i$.*

Proof. To prove (i), let $y \in \mathbb{C}^{2n}/G$ be a singular point and let $x \in \mathbb{C}^{2n}$ be a point in its preimage. The stabilizer group $G_x$ is nontrivial by corollary 2.2.8. By [**59**, Theorem 1.6] the quotient $V'/G_x$ admits a symplectic resolution, where $\mathbb{C}^{2n} = V' \oplus (\mathbb{C}^{2n})^{G_x}$ as $G_x$-representations. One may check that $(\mathbb{C}^{2n})^{G_x}$ and $V'$ are symplectic vector subspaces of $\mathbb{C}^{2n}$. In particular, by theorem 2.3.8 it follows that $G_x \subset \mathrm{Sp}(V')$ is generated by symplectic reflections. As the action of $G_x$ on $(\mathbb{C}^{2n})^{G_x}$ is trivial it means that $G_x \subset \mathrm{Sp}_{2n}(\mathbb{C})$ is generated by symplectic reflections. Now if $g \in G_x$ is any symplectic reflection, then $(\mathbb{C}^{2n})^g \subset \mathbb{C}^{2n}$ is a subspace of codimension $2n-2$ and by corollary 2.2.8 it corresponds to a $2n-2$ dimensional component of the singular locus of $\mathbb{C}^{2n}/G$ containing $y$. Moreover, for a general point $x'$ of $(\mathbb{C}^{2n})^g$ we have $G_{x'} = \{g' \in G\colon g'|_{(\mathbb{C}^{2n})^g} = \mathrm{id}\} =: H$. If $\mathbb{C}^{2n} = (\mathbb{C}^{2n})^g \oplus V''$ as $H$-representations then again both subspaces $(\mathbb{C}^{2n})^g$ and $V''$ are symplectic and $H \subset \mathrm{Sp}(V'') \cong \mathrm{SL}_2(\mathbb{C})$ because $\dim V'' = 2$. The claim with $U = (\mathbb{C}^{2n}/G) \setminus \overline{(\mathrm{Sing}(X) \setminus \Sigma_0)}$ follows by proposition 2.2.7.

The second part follows directly by the first one and the uniqueness of the crepant resolution in codimension one. □

**Dimension four case.** In our studies on symplectic resolutions we concentrate on the dimension four case. The properties of four-dimensional symplectic resolutions were studied in a more general setting than the one given here in [**93**] and in [**3**]. We recall only a single result that we will need in chapter 7. Assume that $G \subset \mathrm{Sp}_4(\mathbb{C})$ is a finite subgroup. The theorem below is a special case of [**3**, Theorem 3.5] (see also section 3.1 for a discussion of cones of divisors).

Theorem 2.3.13. *Let $\varphi\colon X \to \mathbb{C}^4/G$ be a symplectic resolution. Then the cone $\mathrm{Mov}(X) \subset N^1(X)$ is dual to the cone in $N_1(X)$ spanned by the classes of proper curves in $X \setminus \varphi^{-1}([0])$.*

CHAPTER 3

# Cox rings and relative Mori Dream Spaces

This chapter is dedicated to the notion of a Cox ring and related ideas in birational geometry. We follow the fruitful insight of Hu and Keel [**54**] that one can generalize the notion of total coordinate ring of a toric variety to the broader context of algebraic varieties and when this ring turns out to be finitely generated it is a powerful tool to study the birational geometry of a variety. This is reflected in the very name of the projective variety with finitely generated Cox ring – such varieties are called Mori Dream Spaces. We start with introducing several cones of divisors related to birational geometry in 3.1. Then we define and study basic properties of the Cox ring of a variety in 3.2, in particular in section 3.2.2 we describe cones of divisors from 3.1 in terms of degrees of generators of the Cox ring. Section 3.3 is an interlude containing the necessary notions from geometric invariant theory that will be used in 3.4, where we present a generalization of the notion of Mori Dream Space to the relative situation. The results of this final part of the chapter will be used in chapters 5 and 7 to represent all the crepant resolutions of a given quotient singularity as GIT quotients of the spectrum of its Cox ring.

## 3.1. Cones of divisors

Here we introduce the vector space spanned by the classes of divisors on an algebraic variety and various cones which will allow us to study certain properties of divisors (and in consequence, the birational geometry of a variety) via convex geometry. The references are [**26**, Sect. 1.6], [**67**, Sect. 1.7] and [**5**, Sect. 3.3.2].

Let $X$ be a normal variety with finitely generated class group $\mathrm{Cl}(X)$. Denote by $\mathrm{WDiv}(X)$ the group of *Weil divisors* on $X$, i.e. the free abelian group generated by irreducible codimension one subvarieties of $X$. To each $D \in \mathrm{WDiv}(X)$ we have an associated rank one reflexive sheaf $\mathcal{O}_X(D)$ of $\mathcal{O}_X$-modules:

$$H^0(U, \mathcal{O}_X(D)) = \{f \in \mathbb{C}(X)^* \colon (\operatorname{div} f + D)|_U \geq 0\} \cup \{0\},$$

which is a subsheaf of the constant sheaf $\mathbb{C}(X)$ of rational functions on $X$. See [**81**, Sect. 1, Appendix] and references given therein. In this section we recollect a few facts about various cones of divisors in the rational vector space $N^1(X) = \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $D \in \mathrm{WDiv}(X)$.

For $D \in \mathrm{WDiv}(X)$ and $f \in H^0(X, \mathcal{O}_X(D))$ we denote $\operatorname{div}_D f := \operatorname{div} f + D$ $(\geq 0)$. In other words $\operatorname{div}_D f$ is the effective (Weil) divisor, which is the divisor of zeroes of $f$ viewed as the global section of the reflexive sheaf $\mathcal{O}_X(D)$.

Definition 3.1.1 (Base locus and stable base locus). The *base locus* and *stable base locus* of $D$ are the following subsets of $X$:

$$\mathrm{Bs}(D) := \bigcap_{f \in H^0(X, \mathcal{O}_X(D))} \mathrm{Supp}(\operatorname{div}_D f), \qquad \mathbf{B}(D) := \bigcap_{m=1}^{\infty} \mathrm{Bs}(mD)$$

Definition 3.1.2 (Movable, semiample and (absolutely) ample divisors). The divisor $D$ is called *movable* if $\mathbf{B}(D)$ is of codimension at least two in $X$. The divisor $D$ is called *semiample* if $\mathbf{B}(D) = \varnothing$. The divisor $D$ is called *absolutely ample* if the sets of the form $X \setminus \operatorname{Supp}(\operatorname{div}_{nD} f)$ for $f \in H^0(X, \mathcal{O}_X(nD))$ and $n > 0$ give a basis for Zariski topology on $X$.

We define the cones $\operatorname{Eff}(X)$, $\operatorname{Mov}(X)$, $\operatorname{SAmp}(X), \operatorname{Amp}(X)$ in the $\mathbb{Q}$-vector space $N^1(X)$, called respectively: *effective cone*, *movable cone*, *semiample cone* and *(absolutely) ample cone*, as the convex cones spanned by classes of, respectively: effective, movable and semiample and absolutely ample Weil divisors on $X$. One has $\operatorname{Amp}(X) \subset \operatorname{SAmp}(X) \subset \operatorname{Mov}(X) \subset \operatorname{Eff}(X)$.

If $\pi\colon X \to Y$ is a projective morphism then we may consider also the cones $\operatorname{Amp}(X/Y)$ and $\operatorname{Nef}(X/Y)$ of, respectively, relatively ample and relatively nef divisors in $N^1(X/Y) = (\operatorname{Cl}(X)/\pi^* \operatorname{Pic}(Y)) \otimes_{\mathbb{Z}} \mathbb{Q}$. By definition $\operatorname{Amp}(X/Y)$ is spanned by classes of relatively ample divisors and $\operatorname{Nef}(X/Y)$ is spanned by relatively nef Cartier divisors, i.e. the Cartier divisors with nonnegative intersection with all effective curves contracted by $\pi$. The cone $\operatorname{Amp}(X/Y)$ is the relative interior of $\operatorname{Nef}(X/Y)$. $\operatorname{Amp}(X/Y)$ is open if $X$ is $\mathbb{Q}$-factorial. If $\operatorname{Cl}(Y)$ is a torsion group then $N^1(X) = N^1(X/Y)$ and $\operatorname{SAmp}(X) \subset \operatorname{Nef}(X/Y) \subset \overline{\operatorname{Eff}(X)}$. If moreover $Y$ is affine, then $\operatorname{Amp}(X) = \operatorname{Amp}(X/Y)$.

## 3.2. Cox rings

In this section we introduce the Cox ring of an algebraic variety, which is the main tool as well as the primary object of study in our work. First, we give the definition and list the most important algebraic properties in 3.2.1. Then, in 3.2.2 we show, how to obtain descriptions of various cones of divisors introduced in 3.1 in terms of degrees of generators of the Cox ring. The general reference here is [**5**, Chapter 1].

### 3.2.1. Definition and algebraic properties.

Let $X$ be a normal variety.

Definition 3.2.1 (Cox ring – the free class group case). Assume that the class group $\operatorname{Cl}(X)$ of $X$ is finitely generated and free. Let $K \subset \operatorname{WDiv}(X)$ be a subgroup projecting isomorphically onto $\operatorname{Cl}(X)$. Then the *Cox ring* of $X$ is the ring:

$$\mathcal{R}(X) = \bigoplus_{D \in K} H^0(X, \mathcal{O}_X(D)),$$

with multiplication induced from $\mathbb{C}(X)$. We introduce a $\operatorname{Cl}(X)$-grading on $\mathcal{R}(X)$ using the isomorphism $K \to \operatorname{Cl}(X)$ induced by projection.

Up to the isomorphism preserving $\operatorname{Cl}(X)$-grading, the Cox ring does not depend on the choice of the subgroup $K \subset \operatorname{Cl}(X)$ projecting isomorphically onto $\operatorname{Cl}(X)$ ([**5**, Construction 1.4.1.1]).

The construction of the Cox ring of $X$ can be generalized to the case when $\operatorname{Cl}(X)$ is an arbitrary finitely generated abelian group under an additional assumption on the global invertible functions on $X$.

Definition 3.2.2 (Cox ring – general case). Assume that the class group $\operatorname{Cl}(X)$ of $X$ is finitely generated and that $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$. Fix a finitely generated subgroup $K \subset \operatorname{WDiv}(X)$ projecting onto $\operatorname{Cl}(X)$. Let $K^0$ be the kernel of the natural epimorphism $K \to \operatorname{Cl}(X)$. Fix a homomorphism $\chi\colon K^0 \to \mathbb{C}(X)^*$ satisfying $\operatorname{div}(\chi(E)) = E$. Define the *Cox ring* $\mathcal{R}(X)$ of $X$ as

$$\mathcal{R}(X) = \mathcal{S}(X)/\mathcal{I}(X).$$

Here $\mathcal{S}(X) = \bigoplus_{D \in K} H^0(X, \mathcal{O}_X(D))$ with multiplication induced by the one in $\mathbb{C}(X)$ and $\mathcal{I}(X)$ is the ideal generated by the elements of the form $1 - \chi(E)$ where $E \in K^0$

and $\chi(E) \in \mathbb{C}(X)^*$ is viewed as homogeneous element of degree $-E$ in $\mathcal{S}(X)$. Using the canonical epimorphism $\pi\colon \mathcal{S}(X) \to \mathcal{R}(X)$ we introduce a $\mathrm{Cl}(X)$-grading on $\mathcal{R}(X)$ as follows:

$$\mathcal{R}(X) = \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}(X)_{[D]}, \qquad \mathcal{R}(X)_{[D]} = \pi\left(\bigoplus_{\substack{D'\in K\\ [D']=[D]}} \mathcal{S}(X)_{D'}\right)$$

Again, up to the isomorphism preserving $\mathrm{Cl}(X)$-grading the Cox ring of $X$ does not depend on the choices of $K$ and $\chi$ ([**5**, Sect. 1.4.3.]) – here the assumptions on $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$, which at the first glance may look unmotivated plays an important role. In particular definition 3.2.2 is consistent with definition 3.2.1. By [**5**, Lemma. 1.4.3.5] the above definition coincides with [**5**, Construction 1.4.2.1].

The canonical epimorphism $\pi\colon \mathcal{S}(X) \to \mathcal{R}(X)$ preserves graded pieces:

Proposition 3.2.3 ([**5**, Lemma 1.4.3.4]). *In the situation of definition 3.2.2 the restriction* $\pi|_{\mathcal{S}_D(X)}\colon \mathcal{S}(X)_D \to \mathcal{R}(X)_{[D]}$ *is an isomorphism of* $H^0(X, \mathcal{O}_X)$*-modules for each* $D \in K$.

The next proposition summarizes the algebraic properties of Cox rings, but first, we need a definition.

Definition 3.2.4 (Factorial grading). Let $A$ be an integral domain, graded by finitely generated abelian group $M$. A homogeneous element $f \in A$ is called *$M$-prime* if for every homogeneous elements $g, h \in A$ if $f \mid gh$ then $f \mid g$ or $f \mid h$. $A$ is called *$M$-factorial* (or *factorially $M$-graded*) if every homogeneous element of $A$ which is not unit, can be expressed as a product of $M$-prime homogeneous elements.

The concept of $M$-factoriality will be motivated by theorem 3.3.13. This theorem in turn will be used in section 3.4.2. It will allow us to obtain all small modifications of a variety projective over affine base via variation of GIT (see section 3.3.2), enabling us to study geometry of resolutions via the combinatorial methods of convex geometry in chapters 5 and 7.

Proposition 3.2.5. *Let $X$ be a normal variety with finitely generated class group. Assume that either* $\mathrm{Cl}(X)$ *is free or* $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$. *Then*

*(i) The Cox ring* $\mathcal{R}(X)$ *is a normal* $\mathrm{Cl}(X)$*-graded integral* $H^0(X, \mathcal{O}_X)$*-algebra.*
*(ii) If* $\mathrm{Cl}(X)$ *is free, then* $\mathcal{R}(X)^* = H^0(X, \mathcal{O}_X^*)$.
*(iii) If* $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$, *then every homogeneous invertible element of* $\mathcal{R}(X)$ *is constant.*
*(iv) If* $\mathrm{Cl}(X)$ *is free, then* $\mathcal{R}(X)$ *is an unique factorization domain.*
*(v) If* $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$, *then* $\mathcal{R}(X)$ *is* $\mathrm{Cl}(X)$*-factorial (see definition 3.2.4).*

References. For normality and integrality in (i) see [**5**, Thm. 1.5.1.1], (ii) and (iv) are [**5**, Prop. 1.4.1.5], (iii) follows from [**5**, Prop. 1.5.2.5(i)] and (v) is the content of [**5**, Thm. 1.5.3.7]. □

**3.2.2. Cones of divisors via generators of the Cox ring.** To work with concrete examples, as well as to present the characterization of relative Mori Dream Spaces in section 3.4.2 we need to describe cones of divisors from section 3.1 in terms of generators of the Cox ring. Here we summarize related results from [**5**]. We also generalize [**5**, Prop. 3.2.2.3] to the non-complete case in proposition 3.2.9.

For varieties with finitely generated class group we have the following characterization of $\mathrm{Eff}(X)$ in terms of generators of Cox ring.

PROPOSITION 3.2.6 ([**5**, Prop. 3.3.2.1]). *Let $X$ be a normal variety with finitely generated class group and $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$. Let $(f_i)_{i\in I}$ be any system of nonzero homogeneous generators of the Cox ring $\mathcal{R}(X)$. Then*

$$\mathrm{Eff}(X) = cone(\deg(f_i) \colon i \in I).$$

*In particular, if $\mathcal{R}(X)$ is finitely generated, then* $\mathrm{Eff}(X) = \overline{\mathrm{Eff}(X)}$.

We also have a description of the movable cone in the similar vein. To present it we introduce an additional piece of notation.

NOTATION 3.2.7. Let $X$ and $K \subset \mathrm{WDiv}(X)$ be as in definition 3.2.2. Let $D \in K$. By proposition 3.2.3 it follows that we have a canonical bijection $H^0(X, \mathcal{O}_X(D)) \to \mathcal{R}(X)_{[D]}$. Now take $f \in \mathcal{R}(X)_{[D]}$ and let $\widetilde{f} \in H^0(X, \mathcal{O}_X(D))$ be an element corresponding to $f$. We define $\mathrm{div}_{[D]} f = \mathrm{div}_D \widetilde{f} \in \mathrm{WDiv}(X)$.

By [**5**, Proposition 1.5.2.2] the Weil divisor $\mathrm{div}_{[D]} f$ is well-defined and by [**5**, Proposition 1.5.3.5.(ii)] $D \mapsto f_{[D]}$ gives a one-to one correspondence between effective divisors on $X$ and homogeneous elements of $\mathcal{R}(X)$ up to multiplication by scalars.

PROPOSITION 3.2.8 ([**5**, Proposition 1.5.3.5(i)]). *Let $X$ be as in definition 3.2.2. For $f \in \mathcal{R}(X)_{[D]}$ and $g \in \mathcal{R}(X)_{[E]}$ one has $f \mid g$ if and only if $\mathrm{div}_{[D]} f \leq \mathrm{div}_{[E]} g$.*

We are ready to give the description of the movable cone in terms of degrees of the generators of the Cox ring. The following theorem, together with the lemma below is a direct generalization of [**5**, Prop. 3.3.2.3] and [**5**, Lem. 3.3.2.4] to the non-complete setting. The only technical difference is that instead of the condition: $\dim_{\mathbb{C}} \mathcal{R}(X)_w = 1$ one should use the condition: $\mathcal{R}(X)_w \cong H^0(X, \mathcal{O}_X)$ as $H^0(X, \mathcal{O}_X)$-modules.

PROPOSITION 3.2.9. *Let $X$ be a normal variety with finitely generated class group and $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$. Let $(f_i)_{i\in I}$ be any system of pairwise nonassociated* $\mathrm{Cl}(X)$*-prime generators of the Cox ring $\mathcal{R}(X)$. Then*

$$\mathrm{Mov}(X) = \bigcap_{i\in I} cone(\deg(f_j) \ : \ j \neq i).$$

PROOF. Denote $w_i := \deg f_i$ and let $I_0 \subset I$ be the set of indices $i$ such that $\mathcal{R}(X)_{nw_i} \cong H^0(X, \mathcal{O}_X)$ as $H^0(X, \mathcal{O}_X)$-modules for all $n \geq 1$. If $w \in \mathrm{Mov}(X)$ then lemma 3.2.10 shows that for each $i \in I_0$ there is $m \geq 1$ so that $\mathcal{R}(X)_{mw}$ contains a monomial in generators which is not divisible by $f_i$. For $i \notin I_0$ the same is true since $\mathcal{R}(X)_{nw_i} \supsetneq H^0(X, \mathcal{O}_X) f_i^n$ for some $n \geq 1$. Hence $w$ belongs to the right-hand side

Conversely, let $w$ be contained in the right-hand side. Then for each $i \in I$ we can express $w$ as a nonnegative combination of $w_j$ with $j \neq i$. This means that for each $i \in I$ there is $m \geq 1$ such that $\mathcal{R}(X)_{mw}$ contains a monomial in generators which is not divisible by $f_i$. Specializing to the case $i \in I_0$ and using lemma 3.2.10 we deduce that $w \in \mathrm{Mov}(X)$. ☐

LEMMA 3.2.10 (cf. [**5**, Lem. 3.3.2.4]). *Let $X$ be a normal variety with finitely generated class group and $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$. Let $w \in \mathrm{Cl}(X)$ be an effective class. The following conditions are equivalent:*

*(a) The stable base locus of the class $w$ contains a divisor.*
*(b) There exists $w_0 \in \mathrm{Cl}(X)$ with $\mathcal{R}(X)_{nw_0} \cong H^0(X, \mathcal{O}_X)$ as $H^0(X, \mathcal{O}_X)$-modules for every $n \geq 0$ and $f_0 \in \mathcal{R}(X)_{w_0}$ such that for every $m \geq 1$ and $f \in \mathcal{R}(X)_{mw}$ one has $f = f_0 f'$ for some $f' \in \mathcal{R}(X)_{mw-w_0}$.*

PROOF. The implication $(b) \implies (a)$ follows by the equality $\operatorname{div}_w f = \operatorname{div}_{w_0} f_0 + \operatorname{div}_{mw-w_0} f'$.

Assume $(a)$. Let $D$ be an effective divisor representing class $w$. Let $D_0$ be a prime divisor contained in the stable base locus of $D$ and let $w_0 \in \mathrm{Cl}(X)$ be its class. We take as $f_0$ the element of $\mathcal{R}(X)_{w_0}$ which is the image of a constant nonzero function in $H^0(X, \mathcal{O}_X(D_0))$. Since $D_0$ is a fixed component of any effective divisor linearly equivalent to $mD$ then $f_0$ divides any element of $\mathcal{R}(X)_{mw}$ for any $m \geq 1$. By the definition of $D_0$ we have $\mathcal{R}(X)_{nw_0} \cong H^0(X, \mathcal{O}_X)$ for any $n \geq 0$. Indeed, suppose that $n_0$ is minimal such that $\mathcal{R}(X)_{n_0 w_0} \supsetneq H^0(X, \mathcal{O}_X) f_0^{n_0}$. By minimality of $n_0$ there exists $g \in \mathcal{R}(X)_{n_0 w_0}$ such that $f_0 \nmid g$. By $\mathrm{Cl}(X)$-factoriality of $\mathcal{R}(X)$ we also have $f_0 \nmid g^{a_0}$, where $a_0$ is the multiplicity of $D_0$ in $D$. Then $\operatorname{Supp}(\operatorname{div}_{[na_0 D_0]} g^{a_0}) \not\supset D_0$, and so $nD - na_0 D_0 + \operatorname{div}_{[na_0 D_0]} g^{a_0}$ is an effective divisor equivalent to $D$ and not containing $D_0$ in its support – a contradiction. □

If the Cox ring is finitely generated, then the cones of semiample and (absolutely) ample divisors have the description in terms of the orbit cones of characteristic quasitorus $\mathbb{T}_X = \operatorname{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ action on $\operatorname{Spec} \mathcal{R}(X)$ (see definition 3.3.4).

PROPOSITION 3.2.11 ([**5**, Proposition 3.3.2.6]). *Let $X$ be a normal variety with finitely generated class group, $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$ and such that the Cox ring $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra. Let $f_1, \ldots, f_s$ be a system of homogeneous generators of $\mathcal{R}(X)$. Consider the homomorphism $Q\colon \mathbb{Z}^s \to \mathrm{Cl}(X)$ defined on the canonical basis of $\mathbb{Z}^s$ as $Q(e_i) = \deg f_i$ (cf. section 3.3.3). Let $Z = \operatorname{Spec} \mathcal{R}(X)$ with the natural action of $\mathbb{T}_X$ and let $W$ be an open subset of $Z$ with a good quotient $p\colon W \to X$ as in theorem 3.4.5. Then*

$$\operatorname{SAmp}(X) = \bigcap_{x \in W} Q(\omega_x), \qquad \operatorname{Amp}(X) = \bigcap_{x \in W} Q(\omega_x^\circ).$$

## 3.3. Results from GIT

In this section we collect important notions from geometric invariant theory (in short: GIT). The techniques of GIT lie at the very heart of our work, giving us methods to study birational geometry of algebraic varieties via their Cox rings, following ideas of Hu and Keel from [**54**]. First, we recall the notion of good quotient in 3.3.1. Then, in 3.3.2, we list the results on GIT for quasitorus actions in the affine case. In the last section 3.3.3 we present tools to work effectively with concrete examples. The main references here are [**74**], [**30**] and [**5**, Sect. 1.2, 3.1].

### 3.3.1. Good quotients.
Here we introduce the notions of a good quotient and a geometric quotient by the action of an algebraic group, which are fundamental notions in geometric invariant theory.

Let $G$ be an algebraic group acting on an algebraic variety $X$.

DEFINITION 3.3.1 (Good quotient and geometric quotient). We call a morphism $p\colon X \to Y$ a *good quotient* of the action of $G$ on $X$ if the following conditions hold:

(i) $p$ is a $G$-invariant, affine surjective morphism.
(ii) The induced map $\mathcal{O}_Y \to p_*\mathcal{O}_X$ is an isomorphism between $\mathcal{O}_Y$ and $(p_*\mathcal{O}_X)^G$.
(iii) For every closed, $G$-invariant subset $W \subset X$ the image $p(W) \subset Y$ is closed.
(iv) For every pair $W_1, W_2$ of disjoint closed, $G$-invariant subsets of $X$ their images $p(W_1), p(W_2)$ are disjoint.

We call a good quotient $p\colon X \to Y$ a *geometric quotient* if every fibre of $p$ consist of a single orbit of the $G$-action.

The following property partially motivates the above definition.

PROPOSITION 3.3.2 ([**5**, Corollary 1.2.3.8.(ii)]). *Let $p\colon X \to Y$ be a good quotient of the action of $G$ on $X$. Then, for every $G$-invariant morphism of varieties $q\colon X \to Z$ there exists a unique morphism $f\colon Y \to Z$ such that $q = f \circ p$. In particular a good quotient, if it exists, is unique up to isomorphism.*

In this work we will consider mostly the actions of finite groups and algebraic tori.

EXAMPLE 3.3.3. Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a finite group. The quotient map $\mathbb{C}^n \to \mathbb{C}^n/G$ is a good quotient.

**3.3.2. Affine GIT for quasitorus actions.** From the point of view of our applications the most important are quotients of an action of algebraic (quasi)torus on an affine variety. In this section we summarize results concerning such quotients needed in the further parts of the work – the (semi)stability conditions given by a linearization of a line bundle and the variation of GIT related to the change of linearization. We introduce the notion of an orbit cone of a point of a variety and the GIT cone of a weight of the action. For each weight we have the corresponding good quotient of the semistable locus of this weight. Altogether GIT cones form a (quasi)fan which encodes the totality of such quotients – each cone corresponds to a different quotient, and the quotient corresponding to a face of a GIT cone admits a projective morphism to the quotient given by this cone. Finally, it turns out that every quotient satisfying certain mild assumptions is given by a GIT cone.

Let $T \cong (\mathbb{C}^*)^r \times H$, where $H$ is a finite abelian group and let $X$ be an affine variety with a $T$-action. Then $X = \operatorname{Spec} A$ for a finitely generated $M$-graded algebra $A = \bigoplus_{m\in M} A_m$, where $M = \operatorname{Hom}(T, \mathbb{C}^*) \cong \mathbb{Z}^r \times H$ is the group of characters of $T$.

We introduce notation:

DEFINITION 3.3.4. The *orbit cone* of a point $x \in X$ is the cone:

$$\omega_x = \mathit{cone}(m \in M\colon \exists_{f\in A_m} f(x) \neq 0) \subset M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The *weight cone* of $X$ is the cone:

$$\omega_X = \mathit{cone}(m \in M\colon A_m \neq 0) \subset M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The *GIT cone* of weight $m \in \omega_X$ is the cone:

$$\lambda(m) = \bigcap_{\substack{m\in\omega_x \\ x\in X}} \omega_x \subset M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The relative interior of a top-dimensional GIT cone is called a *GIT chamber.*

REMARK 3.3.5. The weight cone of $X$ and every orbit cone are closed polyhedral cones [**5**, Rem. 3.1.1.2]. There are only finitely many pairwise distinct orbit cones [**5**, Prop. 3.1.1.10], in particular every GIT cone is also a closed polyhedral cone.

To characterize good and geometric quotients of the action of $T$ on $X$ we have to consider the following notion of (semi)stability, which coincides with the classical notion of (semi)stability with respect to a linearization of the trivial line bundle [**30**, Sect. 8.1].

DEFINITION 3.3.6 (Semistable and stable points). The *set of semistable points* associated with an element $m \in M \otimes_{\mathbb{Z}} \mathbb{Q}$ is the set:

$$X^{ss}(m) = \{x \in X\colon \exists_{k>0}\, \exists_{f\in A_{km}}\ f(x) \neq 0\}.$$

The *set of stable points* associated with an element $m \in M \otimes_{\mathbb{Z}} \mathbb{Q}$ is the set of points in $X^{ss}(m)$ with finite isotropy group and closed orbit in $X^{ss}(m)$. The complement of the set of semistable points is called the *set of unstable points.*

REMARK 3.3.7 ([**74**, Sect. 1.4(1)]). $X^s(m) \subset X^{ss}(m) \subset X$ are open and $T$-invariant.

The set of semistable points always admits a good quotient and its restriction to the set of stable points is geometric.

PROPOSITION 3.3.8 ([**5**, Proposition 3.1.2.2]). *For every $m \in \omega_X$ there is a good quotient $p\colon X^{ss}(m) \to Y(m)$ of the $T$-action, with $Y(m)$ projective over $Y(0) = \operatorname{Spec} A_0$. Moreover the restriction $p|_{X^s(m)}\colon X^s(m) \to p(X^s(m))$ is a geometric quotient.*

Relation between various good quotients thus obtained is described in the next proposition.

PROPOSITION 3.3.9 ([**5**, Proposition 3.1.2.3]). *For any $m_1, m_2 \in \omega_X$ if $X^{ss}(m_1) \subset X^{ss}(m_2)$ then there exists a projective and surjective morphism of quotients $Y(m_1) \to Y(m_2)$ making the following diagram commutative*

$$\begin{tikzcd} X^{ss}(m_1) \arrow[d, "//T"'] \arrow[r, phantom, "\subset"] & X^{ss}(m_2) \arrow[d, "//T"] \\ Y(m_1) \arrow[r] & Y(m_2). \end{tikzcd}$$

The GIT cones provide the description of semistability in terms of convex geometry.

REMARK 3.3.10. For $m \in \omega_X$ we have $X^{ss}(m) = \{x \in X\colon m \in \omega_x\} = \{x \in X\colon \lambda(m) \subset \omega_x\}$.

THEOREM 3.3.11 ([**5**, Theorem 3.1.2.8]). *The collection $\Lambda(X) := \{\lambda(m)\colon m \in \omega_X\}$ of all GIT cones is a quasifan (*quasi- *means that it does not necessarily consist of strongly convex cones) in $M \otimes_{\mathbb{Z}} \mathbb{Q}$. The support of $\Lambda(X)$ (i.e. the set theoretic sum of its elements) is equal to the weight cone $\omega_X$. Moreover for $m_1, m_2 \in \omega_X$*

$$\lambda(m_1) \subset \lambda(m_2) \iff X^{ss}(m_1) \supset X^{ss}(m_2).$$

*In particular $m_1, m_2$ are contained in the relative interior of the same GIT cone if and only if the equality $X^{ss}(m_1) = X^{ss}(m_2)$ holds, i.e. if and only if they give the same good quotients.*

DEFINITION 3.3.12. We call $\Lambda(X)$ the *GIT (quasi)fan* of $T$-variety $X$.

Theorem 3.3.11 allows us to introduce the notation $X^{ss}(\lambda) = X^{ss}(m)$, for GIT cone $\lambda = \lambda(m)$, as the set of semistable points does not depend on the choice of weight used to define $\lambda$.

If $A$ is factorial, or more generally: $M$-factorial (see definition 3.2.4), then the open sets of the form $X^{ss}(w)$ are the all open and $T$-saturated subsets of $X = \operatorname{Spec} A$ from which one can form good quotients that are varieties projective over $Y(0)$ ($T$-saturated means here that for each $T$-orbit contained in the subset its closure is also contained).

THEOREM 3.3.13. *Assume that $A$ is $M$-factorial. The assignment $\lambda \mapsto X^{ss}(\lambda)$ gives a bijection between $\Lambda(X)$ and $T$-saturated open subsets of $X = \operatorname{Spec} A$ admitting a good quotient which is a variety projective over $Y(0) = \operatorname{Spec} A^T$.*

PROOF. By [**5**, Theorem 3.1.4.3] the above assignment gives a bijection between $\Lambda(X)$ and maximal $T$-saturated open subsets of $X = \operatorname{Spec} A$ admitting a good quotient which is a quasiprojective variety. Since every quotient $Y(\lambda) = X^{ss}(\lambda)//T$ is projective over $Y(0)$ and open subset of $Y(\lambda)$ is not projective over $Y(0)$ unless it is the whole $Y(\lambda)$, the theorem follows. $\square$

**3.3.3. Orbits and $I$-faces.** To work with concrete examples of affine GIT quotients by a (quasi)torus action we need effective methods to describe orbit cones. In this section we introduce the notion of $I$-face, which is used to relate orbit cones with generators of the coordinate ring of an affine variety with a quasitorus action.

Assume that an algebraic quasitorus $T \cong (\mathbb{C}^*)^r \times H$ with character group $M \cong \mathbb{Z}^r \times H$ act on affine variety $X = \operatorname{Spec} A$. Consider the $M$-grading on $A$ induced by this action. Let $f_1, \ldots, f_s$ be a system of homogeneous generators of $A$. Consider the homomorphism $Q\colon \mathbb{Z}^s \to M$ defined on the canonical basis of $\mathbb{Z}^s$ as $Q(e_i) = \deg f_i$. Let $I$ be the ideal of relations between generators $f_1, \ldots, f_s$, i.e. $I$ is a kernel of the surjective ring homomorphism $\mathbb{C}[T_1, \ldots, T_s] \ni T_i \mapsto f_i \in A$ corresponding to a closed embedding $X \subset \mathbb{C}^s$ (in other words $I = I(X)$ for this affine embedding).

Definition 3.3.14 ($I$-face). An $I$-face is a face $\sigma$ of $cone(e_1, \ldots, e_s)$ such that the product of $f_i$ with $e_i \in \sigma$ does not belong to the radical of the ideal $(f_i\colon e_i \notin \sigma)$.

The next proposition unravels the geometric meaning of the concept of an $I$-face:

Proposition 3.3.15 ([**5**, Proposition 3.1.1.9]). *The face $\sigma$ of $cone(e_1, \ldots, e_s)$ is an $I$-face if and only if the corresponding orbit $\{(z_1, \ldots, z_s)\colon z_i \neq 0 \iff i \in \sigma\}$ of the big torus $(\mathbb{C}^*)^s$ coordinatewise action on $\mathbb{C}^s$ intersects the embedded variety $X$.*

Using $I$-faces we can describe effectively orbit cones of the $T$-action on $X$.

Proposition 3.3.16 ([**5**, Proposition 3.1.10]). *Let $\sigma$ be an $I$-face. Then, for every $x \in X \cap \{(z_1, \ldots, z_s)\colon z_i \neq 0 \iff i \in \sigma\}$ we have $\omega_x = Q(\sigma)$.*

## 3.4. Cox rings via GIT

Using the results from invariant theory that we introduced in preceeding sections we can now present the relation of a finitely generated Cox ring to the birational geometry of a variety, as described in the absolutely projective setting by Hu and Keel in [**54**]. First, in section 3.4.1 we state a theorem that characterizes the Cox ring of a variety $X$ as the coordinate ring of an affine variety with the action of the characteristic quasitorus, dividing by which we may obtain $X$ as a good quotient. In section 3.4.2 we introduce and study the notion of a relative Mori Dream Space. In particular, theorem 3.4.10 based on nontrivial results from birational geometry [**15**] shows that minimal models of quotient singularities defined by finite subgroups of $\mathrm{SL}_n(\mathbb{C})$ are relative Mori Dream Spaces.

**3.4.1. Characterization theorem.** In this section we recall the characterization of the Cox ring in terms of GIT from [**5**]. We apply it in the next section to present a generalization of ideas of Hu and Keel to relative setting. It is also used in section 4.3 to give a method of confirming that a certain ring is the Cox ring of a crepant resolution, which is then illustrated in section 5.3. The reference for this section is [**5**, Sect. 1.6.]

Let $X$ be as in definition 3.2.2. Assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. We consider a quasitorus action on $\operatorname{Spec} \mathcal{R}(X)$ induced by the $\mathrm{Cl}(X)$-grading on $\mathcal{R}(X)$.

Definition 3.4.1 (Characteristic quasitorus / Picard torus). The quasitorus $\mathbb{T}_X = \operatorname{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ is called the *characteristic quasitorus* of $X$. If $\mathbb{T}_X$ is a torus (i.e. if $\mathrm{Cl}(X)$ is free) and $X$ is smooth (in particular $\mathrm{Cl}(X) = \mathrm{Pic}(X)$) we call $\mathbb{T}_X$ the *Picard torus* of $X$.

We proceed to give a characterization of the variety $\operatorname{Spec} \mathcal{R}(X)$ in terms of the action of the characteristic quasitorus. To do so, we will need to introduce two notions. The notion of the $\mathbb{T}_X$-factoriality is the geometric counterpart of the algebraic notion of $\mathrm{Cl}(X)$-factoriality from definition 3.2.4.

Definition 3.4.2 ($T$-factorial variety). Let $X$ be a normal variety with the action of a quasitorus $T$. We say that $X$ is *$T$-factorial* if every $T$-invariant divisor on $X$ is a divisor of a rational function which is homogeneous with respect to the grading on $\mathcal{O}_X$ induced by the action of $T$.

Proposition 3.4.3 ([**5**, Prop. 1.5.3.3]). *Let $M$ be a finitely generated abelian group and denote by $T = \operatorname{Hom}(M, \mathbb{C}^*)$ the associated quasitorus. Let $X = \operatorname{Spec} A$ be a normal affine variety with a $T$-action. The following are equivalent:*

*(a) $X$ is $T$-factorial.*
*(b) $A$ is $M$-factorial.*

We will also need the notion of strongly stable action, which encompasses the fact that there are sufficiently many closed orbits with trivial isotropy to cover an open subset with complement of codimension at least two.

Definition 3.4.4 (Strongly stable action). Let $G$ be a linear algebraic group acting on the variety $W$. We say that the action of $G$ on $W$ is *strongly stable* if there exists a open subset $W' \subset W$ invariant with respect to the action of $G$ satisfying:

(1) Complement $W \setminus W'$ is of codimension at least two.
(2) The action of $G$ on $W'$ is free, i.e. there are no nontrivial stabilizers.
(3) Each point $x \in W'$ has a closed orbit in $W$.

Let $X$ be as in definition 3.2.2 and assume that the Cox ring $\mathcal{R}(X)$ is finitely generated. The following theorem gives a characterization of the Cox ring of $X$ in terms of its spectrum.

Theorem 3.4.5 ([**5**, Construction 1.6.3.1, Corollary 1.6.4.4]). *Up to an isomorphism $Z = \operatorname{Spec} \mathcal{R}(X)$ is the unique normal affine variety with an action of a quasitorus $\mathbb{T}_X$ satisfying the following conditions:*

*(1) $Z$ has only constant invertible global homogeneous functions,*
*(2) there exists an open $\mathbb{T}_X$-invariant subset $W \subset Z$ with complement of codimension at least two, such that $\mathbb{T}_X$-action on $W$ is strongly stable and that $W$ admits a good quotient $q\colon W \to X$,*
*(3) $Z$ is $\mathbb{T}_X$-factorial.*

**3.4.2. Relative Mori Dream Spaces.** In this section we relate the finite generation of the Cox ring to the birational geometry while studying the notion of a relative Mori Dream Space – a generalization of the notion from [**54**].

Let $Y$ be a normal affine variety with torsion class group and let $\varphi\colon X \to Y$ be a projective morphism. Assume moreover that $X$ is normal and $\mathbb{Q}$-factorial.

Definition 3.4.6 (Relative Mori Dream Space). $X$ is called a *Mori Dream Space* (in short a *MDS*) *over* $Y$ if:

(1) $\operatorname{Cl}(X)$ is a finitely generated group,
(2) the cone $\operatorname{Nef}(X/Y) \subset N^1(X/Y) = N^1(X)$ is polyhedral and generated by semi-ample line bundles, and
(3) there exist finitely many $\mathbb{Q}$-factorial varieties $\varphi_i\colon X_i \to Y$ projective over $Y$ satisfying (1) and (2), and there exist isomorphisms $\psi_i\colon X \dashrightarrow X_i$ in codimension 1 over $Y$ and such that strict transforms $(\psi_i^{-1})_*(\operatorname{Nef}(X_i/Y))$ form a subdivision of $\operatorname{Mov}(X)$.

The following theorem relates the notion of a relative Mori Dream Space with the finite generation of the Cox ring.

THEOREM 3.4.7. *Let $Y$ be a normal affine variety with torsion class group and let $\varphi\colon X \to Y$ be a projective morphism. Assume moreover that $X$ is normal and $\mathbb{Q}$-factorial and that the assumptions of the definition of the Cox ring are satisfied, i.e.* $\mathrm{Cl}(X)$ *is finitely generated and* $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$. *Then the two following conditions are equivalent:*

*(a) $X$ is a Mori Dream Space over $Y$.*
*(b) The Cox ring $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra.*

To prove that the Cox ring of a relative Mori Dream Space is finitely generated we will use the following theorem, which goes back to the paper of Zariski [**96**].

THEOREM 3.4.8 ([**26**, Proposition 1.8.20]). *Let $Y$ be a normal affine variety and let $\varphi\colon X \to Y$ be a projective morphism. Assume moreover that $X$ is normal and $\mathbb{Q}$-factorial. If $D_1, \ldots, D_r \in$* $\mathrm{WDiv}(X)$ *are semiample then the ring* $\bigoplus_{n_1,\ldots,n_r \geq 0} H^0(X, n_1L_1 + \ldots n_rL_r)$ *with multiplication induced from the function field $\mathbb{C}(X)$ is finitely generated $\mathbb{C}$-algebra.*

We will use also the following lemma.

PROPOSITION 3.4.9. *Let $X$ be a normal variety with finitely generated class group and* $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^*$. *If* $\mathrm{Mov}(X)$ *is polyhedral then* $\mathrm{Eff}(X)$ *is polyhedral.*

PROOF. This is [**5**, Lemma 4.3.3.3], except that here we do not assume that $X$ is complete. Nevertheless, the proof goes analogously, using the generalization of the description of the movable cone from the complete case to the general case given in proposition 3.2.9. □

PROOF OF THE IMPLICATION $(a) \implies (b)$ IN THEOREM 3.4.7. We are following the proof in the complete case given in [**5**, Sect. 4.3.3]. Let $X \to Y$ be a relative MDS. Then the movable cone $\mathrm{Mov}(X)$ is polyhedral. By proposition 3.4.9 the effective cone $\mathrm{Eff}(X)$ is also polyhedral. Denote by $w_1, \ldots, w_d$ the primitive generators of rays of $\mathrm{Eff}(X)$ such that given $n \geq 0$ we have either $\mathcal{R}(X)_{nw_i} = 0$ or $\mathcal{R}(X)_{nw_i} \cong H^0(X, \mathcal{O}_X)$ as $H^0(X, \mathcal{O}_X)$-module. Note that $\mathcal{R}(X)_{nw_i} \neq 0$ for some $n \geq 0$ as $w_i$ is effective. Let $n_i > 0$ be the minimal positive integer satisfying $\mathcal{R}(X)_{n_iw_i} \neq 0$ and take the generator $f_i$ of the $H^0(X, \mathcal{O}_X)$-module $\mathcal{R}(X)_{n_iw_i}$. It follows that $f_i$ is $\mathrm{Cl}(X)$-prime and $H^0(X, \mathcal{O}_X)[f_i] = \bigoplus_{n=0}^{\infty} \mathcal{R}(X)_{nw_i}$ for each $i$. Let $\lambda_i := (\varphi_i^{-1})_*(\mathrm{Nef}(X_i/Y))$. By Gordan's lemma [**23**, Prop. 1.2.7] the semigroup $\mathrm{Cl}(X) \cap \lambda_i$ is finitely generated. Therefore by theorem 3.4.8 the algebra $\bigoplus_{w \in \mathrm{Cl}(X) \cap \lambda_i} \mathcal{R}(X)_w$ is finitely generated for every $i$. It follows that the algebra

$$\mathcal{A}(X) := \bigoplus_{w \in \mathrm{Cl}(X) \cap \mathrm{Mov}(X)} \mathcal{R}(X)_w = \sum_{i=1}^{r} \left( \bigoplus_{w \in \mathrm{Cl}(X) \cap \lambda_i} \mathcal{R}(X)_w \right),$$

is finitely generated. We claim that elements $f_i \in \mathcal{R}(X)_{n_iw_i}$ together with a choice of finitely many generators for $\mathcal{A}(X)$ generate the Cox ring $\mathcal{R}(X)$. Take $f \in \mathcal{R}(X)_w \setminus 0$. If $w$ is not movable, then by lemma 3.2.10 we have $f = f_if^{(1)}$ for some $i$ and $f^{(1)} \in \mathcal{R}(X)_{w - n_iw_i} \setminus 0$. Repeating the same procedure we get the sequence of elements $f^{(m)} \in \mathcal{R}(X)_{w - n_{i_1}w_{i_1} - \ldots - n_{i_m}w_{i_m}}$. Since for sufficiently large $m$ the element of the form $w - n_{i_1}w_{i_1} - \ldots - n_{i_m}w_{i_m}$ is not in $\mathrm{Eff}(X)$ the sequence must be finite and hence the last term has degree from $\mathrm{Mov}(X)$. In other words $f = f_{i_1} \cdot \ldots \cdot f_{i_M} \cdot f^{(M)}$ where $\deg f^{(M)} \in \mathrm{Mov}(X)$ and the proof is finished. □

PROOF OF THE IMPLICATION $(b) \implies (a)$ IN THEOREM 3.4.7. By description of the ample cone in proposition 3.2.11 we see that $\mathrm{Amp}(X)$ is polyhedral. Since $Y$ is affine and $\mathrm{Cl}(Y)$ is a torsion group $\mathrm{Amp}(X) = \mathrm{Amp}(X/Y)$ and, as a consequence, $\mathrm{Nef}(X/Y)$ is polyhedral. Moreover, by the same proposition 3.2.11, we know that $\mathrm{SAmp}(X) =$

$\overline{\mathrm{Amp}(X)}(= \overline{\mathrm{Amp}(X/Y)} = \mathrm{Nef}(X/Y))$ is a GIT cone of $\mathbb{T}_X$-action on $Z = \operatorname{Spec}\mathcal{R}(X)$, contained in $\mathrm{Mov}(X)$.

By theorem 3.3.13 every variety which is projective over $Y$ and is a quotient of an invariant open subset of $Z$ is of the form $X_\lambda = Z^{ss}(\lambda)/\!/\mathbb{T}_X$ for some GIT cone $\lambda$. Note that varieties isomorphic in codimension one have the same Cox ring. Hence, by theorem 3.4.5 every variety isomorphic to $X$ in codimension 1 is a quotient of an invariant open subset of $Z$. Similarly as before, by proposition 3.2.11 we see that $\mathrm{SAmp}(X) = \lambda$. Therefore if $X_\lambda$ is isomorphic in codimension 1 to $X$ over $Y$ then $\lambda \subset \mathrm{Mov}(X)$, since its semiample cone has to be contained in $\mathrm{Mov}(X)$.

On the other hand, by theorem 3.4.5 and the definition of strongly stable action (definition 3.4.4) it follows, that if $\lambda \subset \mathrm{Mov}(X)$, then $X_\lambda$ is isomorphic in codimension 1 to $X$ over $Y$. Moreover $X_\lambda$ is $\mathbb{Q}$-factorial as $\mathrm{Amp}(X_\lambda) = \lambda^\circ$ is of full dimension. □

The next theorem shows that the objects of our study in this work – crepant resolutions, or more generally, minimal models of quotient singularities given by subgroups of $\mathrm{SL}_n(\mathbb{C})$ are relative Mori Dream Spaces.

THEOREM 3.4.10. *Let $G \subset \mathrm{SL}_n(\mathbb{C})$ be a finite subgroup and let $\varphi\colon X \to \mathbb{C}^n/G$ be a minimal model. Then $X$ is a relative Mori Dream Space over $\mathbb{C}^n/G$.*

REMARK 3.4.11. If $G \subset \mathrm{Sp}_4(\mathbb{C})$ and $X$ smooth this is a particular case of [**3**, Theorem 3.2]. In case of higher-dimensional symplectic singularities the theorem follows from [**77**, Main Theorem].

PROOF OF THEOREM 3.4.10. By proposition 2.1.11 for arbitrary $G \subset \mathrm{SL}_n(\mathbb{C})$ a minimal model $X$ of $\mathbb{C}^n/G$ is a $\mathbb{Q}$-factorial variety with terminal singularities and $K_X = 0$. By argument as in [**64**, Lemma 2.62] there is a relatively ample divisor which is the negative of a divisor $B$ supported on the exceptional locus of $\varphi$.

Now observe that for every effective divisor $D$ supported on the exceptional locus of $\varphi$ the pair $(X, \varepsilon D)$ is klt for sufficiently small $\varepsilon > 0$ (see [**64**, Definition 2.34]). Indeed, since $X$ has terminal singularities, for any log resolution $\psi : W \to X$ of $(X, \varepsilon D)$ with irreducible components of exceptional divisor $E_i$ we have $K_W = \psi^* K_X + \sum_i a_i E_i$, where all $a_i$ are positive. Then for sufficiently small $\varepsilon > 0$ we get $a_i + \varepsilon b_i > 0$, where $\psi^* D + \sum_j b_j E_j$ is the strict transform of $D$. Cf. [**92**, Proposition 4.5(i)].

Since every klt pair is dlt ([**64**, Proposition 2.41]) the pair $(X, \varepsilon B)$ is dlt for sufficiently small $\varepsilon > 0$. Hence the pair $(X, \varepsilon B)$ satisfies the assumptions of [**15**, Corollary 1.3.2] and so the Cox ring of $X$ is a finitely generated $\mathbb{C}$-algebra. □

CHAPTER 4

# Cox rings of resolutions – general results

In this chapter we present general results concerning computing Cox rings that are illustrated by the analysis of concrete examples in chapters 5 and 7. Section 4.1 presents the main idea – for a projective birational morphism $\varphi\colon X \to Y$ of varieties satisfying general assumptions under which the Cox ring is defined we embed the Cox ring of $X$ into a ring of Laurent polynomials over the Cox ring of $Y$. We give a certain *valuation compatibility condition* (theorem 4.1.15) to decide whether a natural candidate for a set of generators indeed generates the image of the embedding. In section 4.2, specializing to the case of a crepant resolution of a quotient singularity we describe explicitly the Cox ring of $Y$ and the valuations involved. These results are crucial in chapter 5, where we analyze three-dimensional quotient singularities and their crepant resolutions. Then, in section 4.3, based on the characterization of the Cox ring given in section 3.4.1, we give another criterion to check whether the image of the embedding is generated by a candidate set. We apply this theorem in section 5.3 to present an alternative treatment of Cox rings of crepant resolutions of three-dimensional quotient singularities given by dihedral groups. Finally, section 4.4 outlines a direct approach to bounding the degrees of generators of the Cox ring of a relative Mori Dream Space, by use of multigraded Castelnuovo-Mumford regularity. This method is important in the study of symplectic examples in chapter 7.

Part of the results from this chapter (especially in sections 4.1 and 4.2) form a generalization of results of the joint work with Maria Donten-Bury [**32**].

## 4.1. Embedding of the Cox ring and the compatibility criterion

In this section we work under the following general assumptions.

SITUATION 4.1.1. Let $\varphi\colon X \to Y$ be a projective birational morphism of normal varieties with finitely generated class groups and with $H^0(X, \mathcal{O}_X^*) = \mathbb{C}^* = H^0(Y, \mathcal{O}_Y^*)$. Assume moreover that $Y$ is $\mathbb{Q}$-factorial and that $E_1, \ldots, E_m$ are the irreducible components of the exceptional divisor of $\varphi$.

The following result by Hausen, Keicher and Laface describes the change of the Cox ring under the birational morphisms.

PROPOSITION 4.1.2 ([**49**, Proposition 2.2]). *Under the assumptions in situation 4.1.1 there exist:*

*(i) A natural pushforward homomorphism $\varphi_*\colon \mathrm{WDiv}(X) \to \mathrm{WDiv}(Y)$ such that for every prime divisor $D$ either $\varphi_*(D) = \varphi(D)$ if $\varphi(D)$ is a divisor on $Y$ or $\varphi_*(D) = 0$ if $\operatorname{codim} \varphi(D) > 1$. This homomorphism induce a homomorphism of class groups $\varphi_*\colon \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$.*

*(ii) A pushforward map $H^0(X, \mathcal{O}_X(D)) \to H^0(Y, \mathcal{O}_Y(\varphi_* D))$ defined as $f \mapsto f$, via isomorphism $\mathbb{C}(X) \cong \mathbb{C}(Y)$ induced by $\varphi$.*

*Moreover the maps from (ii) give a surjective ring homomorphism $\varphi_*\colon \mathcal{R}(X) \to \mathcal{R}(Y)$ compatible with the pushforward map $\varphi_*\colon \mathrm{Cl}(X) \to \mathrm{Cl}(Y)$. If the ring $\mathcal{R}(X)$ is finitely* *generated then the kernel of* $\varphi_*\colon \mathcal{R}(X) \to \mathcal{R}(Y)$ *is generated by elements of the form* $1 - f_{E_i}$*, for some* $f_{E_i}$ *with* $\operatorname{div}_{[E_i]} f_{E_i} = E_i$*.*

The proposition above gives in particular a recipe how to compute $\mathcal{R}(Y)$ in terms of $\mathcal{R}(X)$, assuming that $\mathcal{R}(X)$ is finitely generated. What follows is motivated by an attempt to give an inverse construction. More precisely, our ultimate goal is to find $\mathcal{R}(X)$ assuming that we know $\mathcal{R}(Y)$, at least when $\varphi$ is a crepant resolution of a quotient singularity. In this section we present a general construction for a natural 'candidate set' for a set of generators and a condition when such set actually generates $\mathcal{R}(X)$. For crepant resolutions of a quotient singularity we describe objects appearing in these constructions in 4.2. This allows us to use them effectively in chapters 5 and 7. The construction is a direct generalization of the constructions given in [**31**], [**37**], [**32**] and [**94**] (the order of references here corresponds to the growing level of generality).

Denote $\operatorname{Cl}(X/Y) = \operatorname{Cl}(X)/\varphi^* \operatorname{Pic}(Y)$ and by $\operatorname{Cl}(X/Y)_{\mathrm{free}}$ the free part of $\operatorname{Cl}(X/Y)$.

Proposition 4.1.3. *Under the assumptions in situation 4.1.1 we have exact sequences:*

$$0 \to \bigoplus_{i=1}^{m} \mathbb{Z}[E_i] \to \operatorname{Cl}(X) \xrightarrow{\varphi_*} \operatorname{Cl}(Y) \to 0$$

*and*

$$0 \to \bigoplus_{i=1}^{m} \mathbb{Z}[E_i] \to \operatorname{Cl}(X/Y) \xrightarrow{\varphi_*} \operatorname{Cl}(Y)/\operatorname{Pic}(Y) \to 0.$$

Proof. By [**46**, Proposition II.6.5(d)] we have an exact sequence:

$$\bigoplus_{i=1}^{m} \mathbb{Z}[E_i] \to \operatorname{Cl}(X) \xrightarrow{\varphi_*} \operatorname{Cl}(Y) \to 0.$$

By definition of $\operatorname{Cl}(X/Y)$ this yields an exact sequence

$$\bigoplus_{i=1}^{m} \mathbb{Z}[E_i] \to \operatorname{Cl}(X/Y) \xrightarrow{\varphi_*} \operatorname{Cl}(Y)/\operatorname{Pic}(Y) \to 0.$$

To finish the proof it suffices to prove that the map $\bigoplus_{i=1}^{m} \mathbb{Z}[E_i] \to \operatorname{Cl}(X/Y)$ is injective. We first note that there exists a $\varphi$-ample divisor on $X$ of the form $A = -\sum_{i=1}^{m} a_i E_i$, where $a_i > 0$ for every $i$. Indeed, take any $\varphi$-ample divisor $A_0$, then define $A = A_0 - \varphi^*\varphi_* A_0$ and note that $A \sim_\varphi A_0$ is $\varphi$-ample and $A = -\sum_{i=1}^{m} a_i E_i$ with $a_i \geq 0$.

Now note that for each $i$ there exists $y_i \in Y$ such that $\varphi^{-1}(y_i) \cap E_i \setminus \bigcup_{j\neq i} E_j \neq \varnothing$. Since $E_i$ is contracted by $\varphi$ there exists a curve $C_i \subset \varphi^{-1}(y_i)$ which has nonempty intersection with $E_i \setminus \bigcup_{j\neq i} E_j$. As $A.C_i > 0$ and $C_i.E_j \geq 0$ for $i \neq j$ because $C_i \not\subset E_j$, we must have $E_i.C_i < 0$ and $a_i > 0$.

To finish the argument assume that a linear combination $L$ of $E_i$ is $\varphi$-trivial. In particular it is nef. By negativity lemma [**64**, Lemma 3.39] at least one of coefficients in $L$ is negative. Then for some combination $pA - qL$ of $A$ and $L$ with $p, q > 0$ we have $E_i \not\subset \operatorname{Supp}(pA - qL)$ for some $i$. But then $(pA - qL).C_i < 0$ even though $pA - qL \sim_\varphi pA$ is ample as $p > 0$ – a contradiction. $\square$

Lemma 4.1.4. *Under the assumptions in situation 4.1.1 the homomorphism* $\operatorname{Cl}(X) \ni \alpha \mapsto (\varphi_*\alpha, [\alpha]) \in \operatorname{Cl}(Y) \oplus \operatorname{Cl}(X/Y)_{\mathit{free}}$ *is injective.*

Proof. Assume that $\alpha$ is send to zero by this homomorphism. Then $\varphi_*\alpha = 0$ in $\operatorname{Cl}(Y)$. By proposition 4.1.3 we have $\alpha \in \bigoplus_{i=1}^{m} \mathbb{Z}[E_i] \subset \operatorname{Cl}(X)$. But $[\alpha] = 0$ in $\operatorname{Cl}(X/Y)_{\mathrm{free}}$ and $\bigoplus_{i=1}^{m} \mathbb{Z}[E_i]$ embeds into $\operatorname{Cl}(X/Y)_{\mathrm{free}}$, hence $\alpha = 0$. $\square$

PROPOSITION 4.1.5. *Under the assumptions in situation 4.1.1 the ring homomorphism* $\Theta\colon \mathcal{R}(X) \to \mathcal{R}(Y) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X/Y)_{free}]$*:*

$$\mathcal{R}(X)_\alpha \ni f \overset{\Theta}{\mapsto} \varphi_* f \otimes_{\mathbb{C}} t^\alpha$$

*is injective, where* $t^\alpha$ *is the character of the torus* $\mathrm{Hom}(\mathrm{Cl}(X/Y)_{free}, \mathbb{C}^*)$ *corresponding to the class of* $\alpha$ *in* $\mathrm{Cl}(X/Y)_{free}$*.*

PROOF. Assume that $\Theta(f) = 0$. By lemma 4.1.4 we may assume that $f$ is homogeneous, but then the claim follows as the map $\varphi_*\colon \mathcal{R}(X)_\alpha \to \mathcal{R}(Y)_{\varphi_*\alpha}$ is injective. $\square$

The proposition that we just have proven gives us an embedding of the ring $\mathcal{R}(X)$, that we would like to understand, into the ring $\mathcal{R}(Y) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X/Y)_{\mathrm{free}}]$, which is just a Laurent polynomial ring over the ring $\mathcal{R}(Y)$. As we noted before, in cases of our interest the structure of the ring $\mathcal{R}(Y)$ will be known (more precisely, we will know the generators of $\mathcal{R}(Y)$ and relations among them). In particular, the structure of the ring $\mathcal{R}(Y) \otimes \mathbb{C}[\mathrm{Cl}(X/Y)_{\mathrm{free}}]$ will be easier to understand than the structure of $\mathcal{R}(X)$. Our hope is to find a finite set of elements of $\mathcal{R}(Y) \otimes \mathbb{C}[\mathrm{Cl}(X/Y)_{\mathrm{free}}]$ which generate $\Theta(\mathcal{R}(X))$. We proceed to describe a 'candidate' for such a set of elements and give a criterion when it actually generates $\Theta(\mathcal{R}(X))$. Both description of the 'candidate set' and the criterion will rely on the following notion.

DEFINITION 4.1.6. Under the assumptions in situation 4.1.1:

(i) Consider an effective Weil divisor $D$ on $Y$. Let $\overline{D} = \varphi_*^{-1} D$ be the strict transform of $D$ via $\varphi$. Then $\varphi^* D = \overline{D} + \sum_i \nu_i(D) E_i$. We call the number $\nu_i(D)$ the *valuation of* $D$ *at* $E_i$.

(ii) If $D$ is any Weil divisor on $Y$, then for every $f \in H^0(Y, \mathcal{O}(D)) = \{h \in \mathbb{C}(Y)^*\colon \operatorname{div} h + D \geq 0\}$ we define *the valuation of* $f$ *at* $E_i$ *as* $\nu_i(f) = \nu_i(\operatorname{div} f + D)$.

REMARK 4.1.7. In the situation of definition 4.1.6 (i) if $\varphi^* K_Y = K_X$ then $\nu_i(D) = -a(Y, E_i, D)$, where $a(Y, E_i, D)$ is the discrepancy of $(Y, D)$ at $E_i$ (see [**64**, Definition 2.25]).

REMARK 4.1.8. In the situation of definition 4.1.6 (ii) if $rD$ is a Cartier divisor, then $f^r$ is a global section of line bundle $\mathcal{O}_Y(rD)$. Take a trivialization of $\mathcal{O}_Y(rD)$ along open subset $U \subset Y$ such that $E_i \cap U \neq \varnothing$ and let $f_Y$ be a section of $\mathcal{O}_Y$ along $U$ which corresponds to $f^r$. Then $r\nu_i(f) = \nu_{E_i}(f_Y)$, where $\nu_{E_i}\colon \mathbb{C}(Y)^* \to \mathbb{Z}$ is a divisorial valuation of $E_i$.

Observe that the notion of a valuation of a divisor or of a nonzero section of $H^0(Y, \mathcal{O}_Y(D))$ at $E_i$ is not precisely a valuation defined on the field extension of $\mathbb{C}$ in the sense of section 2.1.1. Nevertheless, the remark 4.1.8 relates the two notions and in the next section we will see an even closer relation in the case of crepant resolution of quotient singularity.

PROPOSITION 4.1.9. *Under the assumptions in situation 4.1.1 assume further that* $D$ *is an effective Weil divisor on* $Y$ *and* $f \in \mathcal{R}(Y)_{[D]}$ *such that* $\mathrm{div}_{[D]} f = D$*. Then in* $\mathrm{Cl}(X/Y)_{free}$ *we have the equality of classes* $\overline{D} = -\sum_i \nu_i(f) E_i$*.*

PROOF. This follows by definition of $\nu_i$ – since $Y$ is $\mathbb{Q}$-factorial for some positive integer $r$ we have $\varphi^*(rD) = r\overline{D} + \sum_i r\nu_i(f)E_i$, where $rD$ is a Cartier divisor. Therefore $r\overline{D} = -r\sum_i \nu_i(f)E$ in $\mathrm{Cl}(X/Y)$ and we may cancel $r$ in the free abelian group $\mathrm{Cl}(X/Y)_{\mathrm{free}}$. $\square$

We are now ready to describe the promised 'candidate' for generating set of $\Theta(\mathcal{R}(X))$ and to give a valuative criterion when it is indeed a generating set.

SITUATION 4.1.10. Under the assumptions in situation 4.1.1 assume further that $\mathcal{R}(Y)$ is finitely generated and let $\phi_1, \ldots, \phi_s$ be a system of homogeneous generators of $\mathcal{R}(Y)$. Consider the subring $\mathcal{R} \subset \Theta(\mathcal{R}(X))$ defined by generators as:

$$\mathcal{R} = \mathbb{C}[\phi_i t^{\overline{D}_{\phi_i}},\ t^{E_j} \text{ for } i = 1, \ldots, s \text{ and } j = 1, \ldots, m],$$

where $\overline{D}_{\phi_i}$ is the strict transform of a divisor $D_{\phi_i} \in \mathrm{WDiv}(Y)$ corresponding to $\phi_i$, i.e. $D_{\phi_i} = \mathrm{div}_{\deg \phi_i} \phi_i$.

REMARK 4.1.11. Note that in the above situation $\mathcal{R}$ is generated by images of homogeneous elements in $\mathcal{R}(X)$, and so it has a natural $\mathrm{Cl}(X)$-grading. Moreover, this grading is compatible with the $\mathrm{Cl}(Y) \oplus \mathrm{Cl}(X/Y)_{\mathrm{free}}$-grading on $\mathcal{R}(Y) \otimes \mathbb{C}[\mathrm{Cl}(X/Y)_{\mathrm{free}}]$ via the map from lemma 4.1.4.

REMARK 4.1.12. Also note that in the above situation if $\mathcal{R}(X)$ is finitely generated then there always exists the system of generators of $\mathcal{R}(X)$ the form as in the definition of $\mathcal{R}$ – indeed, one may take any system of homogeneous generators, and take as $\phi_i$ pushforwards of generators (cf. the proof of lemma 4.1.18).

The choice of generators of $\mathcal{R}$ in situation 4.1.10 has the following simple geometric interpretation.

PROPOSITION 4.1.13. $\mathcal{R} = \Theta(\mathcal{R}(X))$ *if and only if for every effective divisor* $D \in \mathrm{WDiv}(Y)$ *one has* $\overline{D} = \sum_{i=1}^{s} a_i \overline{D}_{\phi_i} + \sum_{j=1}^{m} b_j E_j$ *for some nonnegative integers* $a_i, b_j$*.*

PROOF. Every effective divisor on $X$ can be expressed as the sum $\overline{D} + E$, where $D$ is an effective divisor on $Y$ and $E$ is an effective divisor supported on the components of the exceptional divisor. Now the assertion follows from the correspondence between homogeneous elements of $\mathcal{R}(X)$ (up to scalar multiplication) and effective divisors on $X$. □

We now restate this observation in the language of valuations to make it more effective for crepant resolutions of quotient singularities in the next section. Let $\kappa\colon \mathbb{C}[Z_1, \ldots, Z_s] \to \mathcal{R}(Y)$ be a surjective map defined by $Z_i \mapsto \phi_i$. Define a monomial valuation $\widetilde{\nu}_i$ on $\mathbb{C}(Z_1, \ldots, Z_s)$ by setting $\widetilde{\nu}_i(Z_j) = \nu_i(\phi_j)$.

ASSUMPTION 4.1.14 (Valuation compatibility condition). Assume that for every $\mathrm{Cl}(Y)$-homogeneous element $f \in \mathcal{R}(Y)_\alpha$ there exists an element $\widetilde{f} \in \mathbb{C}[Z_1, \ldots, Z_s]$ with $\kappa(\widetilde{f}) = f$ and such that for every $i$ we have $\nu_i(f) = \widetilde{\nu}_i(\widetilde{f})$. (Equivalently: $f$ can be expressed as a sum of monomials $f_j$ in $\phi_1, \ldots, \phi_s$ such that $\nu_i(f) \leq \nu_i(f_j)$ for every $i, j$.)

THEOREM 4.1.15 (Valuation compatibility criterion). *In situation 4.1.10 the following conditions are equivalent:*

*(i) Assumption 4.1.14.*
*(ii)* $\mathcal{R} = \Theta(\mathcal{R}(X))$*, i.e. elements* $\phi_i t^{\overline{D}_{\phi_i}}$ *and* $t^{E_j}$*,* $i = 1, \ldots, s$*,* $j = 1, \ldots, m$ *generate the embedded Cox ring* $\Theta(\mathcal{R}(X))$*.*

PROOF. The proof of $(i) \implies (ii)$ is the combination of the three lemmas below. For the converse note that if $\mathcal{R} = \Theta(\mathcal{R}(X))$ and $f \in \mathcal{R}(Y)_\alpha$ then we can find $h \in \mathbb{C}[Z_1, \ldots, Z_s, Z_{s+1}, \ldots, Z_{s+m}]$ with minimal number of monomials, satisfying $ft^{\overline{D}_f} = h(\phi_1 t^{\overline{D}_{\phi_1}}, \ldots, \phi_s t^{\overline{D}_{\phi_s}}, t^{E_1}, \ldots, t^{E_m})$ and such that every monomial occurring in polynomial $h$ after substituting $\phi_i t^{\overline{D}_{\phi_i}},\ t^{E_j}$ is of the same degree in $\mathcal{R}$ as $ft^{\overline{D}_f}$. We may take $\widetilde{f} = h(Z_1, \ldots, Z_s, 1, \ldots, 1)$ for such $h$ to get assumption 4.1.14. □

LEMMA 4.1.16. *If $f \in \mathcal{R}(Y)_\alpha$ is a monomial in generators $\phi_1, \ldots, \phi_s$, then $ft^{\overline{D}_f} \in \mathcal{R}$.*

PROOF. It is immediate that $ft^{\overline{D}_f}$ is the same monomial evaluated at $\phi_i t^{\overline{D}_{\phi_i}}$ instead of $\phi_i$. □

LEMMA 4.1.17. *Assume that $f \in \mathcal{R}(Y)_\alpha$ satisfies the condition from the assumption 4.1.14 then $ft^{\overline{D}_f} \in \mathcal{R}$.*

PROOF. By the assumption we may write $f = \sum_i f_i$ where $f_i$ is a monomial in generators $\phi_1, \ldots, \phi_s$ and $\nu_i(f) \le \nu_i(f_j)$ for every $i, j$. By proposition 4.1.9 we have $\overline{D}_f = -\sum_i \nu_i(f) E_i$ and $\overline{D}_{f_j} = -\sum_i \nu_i(f_j) E_i$ in $\mathrm{Cl}(X/Y)_{\mathrm{free}}$. Therefore $\overline{D}_f = \overline{D}_{f_j} + \sum_j n_{i,j} E_i$, with $n_{i,j} > 0$. Since $f$ and $f_j$ are homogeneous we may assume that they have the same degree, and therefore $n_{i,j} \in \mathbb{Z}$ by proposition 4.1.3. Then $ft^{\overline{D}_f} = \sum_k f_j t^{\overline{D}_{f_j}} \cdot \prod_i (t^{E_i})^{n_{i,j}} \in \mathcal{R}$ by the previous lemma. □

LEMMA 4.1.18. *Assume that $f \in \mathcal{R}(Y)_\alpha$. If $ft^{\overline{D}_f} \in \mathcal{R}$ and $ft^D \in \Theta(\mathcal{R}(X))$ for some $D \in \mathrm{WDiv}(X)$, then $ft^D \in \mathcal{R}$.*

PROOF. Let $ft^D \in \Theta(\mathcal{R}(X))$. Then $D = \overline{D}_f + \sum_i n_i E_i$ in $\mathrm{Cl}(X/Y)_{\mathrm{free}}$ for some nonnegative integers $n_i$. Therefore $ft^D = ft^{\overline{D}_f} \cdot \prod_i (t^{E_i})^{n_i} \in \mathcal{R}$ by the previous lemma. □

We finish with a description of the movable cone of $X$ in terms of degrees of generators of the form considered in situation 4.1.10 and the geometric information encoded by its GIT subdivision under assumptions dictated by applications in chapters 5 and 7.
Denote

$$\mathrm{Mov}(\mathcal{R}) = \bigcap_{k=1}^{s} cone(\{\overline{D}_{\phi_i} : i \neq k\} \cup \{E_1, \ldots, E_m\}) \cap \bigcap_{k=1}^{m} cone(\{\overline{D}_{\phi_1}, \ldots, \overline{D}_{\phi_s}\} \cup \{E_j : j \neq k\}) \subset N^1(X). \tag{4.1.19}$$

Note that the characteristic quasitorus $\mathbb{T} = \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ of $X$ acts on $\mathrm{Spec}\,\mathcal{R}$ as $\mathcal{R}$ is $\mathrm{Cl}(X)$-graded by remark 4.1.11.

PROPOSITION 4.1.20. *Under the assumptions at the beginning of the section if $\mathcal{R} = \Theta(\mathcal{R}(X))$ then $\mathrm{Mov}(X) = \mathrm{Mov}(\mathcal{R})$. If moreover $Y$ is affine with $K_Y = 0$, canonical singularities and torsion class group, and $\varphi\colon X \to Y$ is a minimal model of $Y$ then there is a one-to-one correspondence between minimal models of $Y$ and GIT chambers of the action of $\mathbb{T}$ on $\mathrm{Spec}\,\mathcal{R}$ contained in $\mathrm{Mov}(\mathcal{R})$. Namely, taking GIT quotients corresponding to GIT chambers of this action we obtain all, pairwise nonisomorphic minimal models of $Y$.*

PROOF. The equality $\mathrm{Mov}(X) = \mathrm{Mov}(\mathcal{R})$ is a particular case of proposition 3.2.9. The second part follows as $X$ is a Mori Dream Space over $Y$ by theorem 3.4.7 and every two minimal models are isomorphic in codimension one (for the last fact note that the exceptional divisors on a minimal model of $Y$ are precisely the crepant divisors over $Y$ by proposition 2.1.11). □

## 4.2. Case of a resolution of a quotient singularity

Here we describe objects appearing in the previous section specializing to crepant resolutions of quotient singularities. In this case, we describe more effectively the Cox ring $\mathcal{R}(Y)$, the valuation of a divisor and, more generally, the valuation of a section of a reflexive sheaf of a divisor at an exceptional divisor.

Let $G \subset \mathrm{GL}_n(\mathbb{C})$ be a small group. Recall from proposition 2.2.6 (iii) that there is a natural isomorphism $\mathrm{Cl}(\mathbb{C}^n/G) \cong \mathrm{Ab}(G)^\vee = \mathrm{Hom}(G, \mathbb{C}^*)$. Note that we have an $\mathrm{Ab}(G)$-action on $\mathbb{C}^n/[G,G]$ and the corresponding $\mathrm{Ab}(G)^\vee$-grading on $\mathbb{C}[x_1, \ldots, x_n]^{[G,G]}$, where $[G,G] \subset G$ is the commutator subgroup and $\mathrm{Ab}(G) = G/[G,G]$.

Proposition 4.2.1 ([**6**, Theorem 3.1]). *There is a natural isomorphism of* $\mathrm{Ab}(G)^\vee$*-graded algebras* $\mathcal{R}(\mathbb{C}^n/G) \cong \mathbb{C}[x_1, \ldots, x_n]^{[G,G]}$.

Since $\mathrm{Pic}(\mathbb{C}^n/G) = 0$ by proposition 2.2.6 (ii), proposition 4.1.3 takes the following form.

Corollary 4.2.2. *If* $E_1, \ldots, E_m$ *are the components of the exceptional divisor of a resolution* $\varphi\colon X \to \mathbb{C}^n/G$ *then we have the exact sequence:*

$$0 \to \bigoplus_{i=1}^{m} \mathbb{Z}[E_i] \to \mathrm{Cl}(X) \xrightarrow{\varphi_*} \mathrm{Cl}(\mathbb{C}^n/G) \to 0.$$

The next proposition implies that in our setting $\mathrm{Cl}(X) = \mathrm{Cl}(X/Y)_{\mathrm{free}}$.

Proposition 4.2.3 ([**37**, Lemma 2.13]). *Let* $\varphi\colon X \to \mathbb{C}^n/G$ *be a resolution of singularities. Then the class group* $\mathrm{Cl}(X)$ *is free.*

By propositions 4.2.1 and 4.2.3 proposition 4.1.5 simplifies to the following.

Corollary 4.2.4. *Let* $\varphi\colon X \to \mathbb{C}^n/G$ *be a resolution of singularities. The ring homomorphism* $\Theta\colon \mathcal{R}(X) \to \mathcal{R}(\mathbb{C}^n/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)]$*:*

$$\mathcal{R}(X)_\alpha \ni f \overset{\Theta}{\mapsto} \varphi_* f \otimes_{\mathbb{C}} t^\alpha \in \mathbb{C}[x_1, \ldots, x_n]^{[G,G]}_\alpha \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)]$$

*is injective, where* $t^\alpha$ *is the character of the Picard torus* $\mathbb{T} = \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ *corresponding to the class of* $\alpha$ *in* $\mathrm{Cl}(X)$*.*

We are now going to describe valuations from the previous section in the setting of crepant resolutions of quotient singularities. Assume that $G \subset \mathrm{SL}_n(\mathbb{C})$ and that $\varphi\colon X \to \mathbb{C}^n/G$ is a crepant resolution. Let $g_1, \ldots, g_m$ be representatives of junior conjugacy classes corresponding to the components $E_1, \ldots, E_m$ of the exceptional divisor via McKay correspondence (theorem 2.2.18). Denote by $r_i$ the order of $g_i$ and by $\nu_{g_i}\colon \mathbb{C}(x_1, \ldots, x_n)^* \to \mathbb{Z}$ the monomial valuation of $g_i$. Note that $\nu_{g_i}$ restricts to a well-defined function on the quotient field of the Cox ring $\mathcal{R}(\mathbb{C}^n/G) = \mathbb{C}[x_1, \ldots, x_n]^{[G,G]}$ which is equal to $\mathbb{C}(x_1, \ldots, x_n)^{[G,G]} \subset \mathbb{C}(x_1, \ldots, x_n)$. By the McKay correspondence we have:

Corollary 4.2.5. *Under the assumptions above let* $D$ *be an effective Weil divisor on* $\mathbb{C}^n/G$ *and* $f \in \mathbb{C}[x_1, \ldots, x_n]^{[G,G]}_{[D]}$ *such that* $\mathrm{div}_{[D]} f = D$*. Then* $\nu_i(f) = \nu_i(D) = \frac{1}{r_i}\nu_{g_i}(f)$*.*

Proof. Note that $f^r \in \mathcal{R}(\mathbb{C}^n/G)_0 = \mathbb{C}[x_1, \ldots, x_n]^G$ for some $r$. By remark 4.1.8 we have $r\nu_i(f) = \nu_{E_i}(f^r)$ and by theorem 2.2.18 we know that $\nu_{E_i} = \frac{1}{r_i}\nu_{g_i}|_{\mathbb{C}(x_1,\ldots,x_n)^G}$. □

It is often convenient to work with a free abelian group $\Lambda_X$ of which $\mathrm{Cl}(X)$ is a subgroup of finite index. This group is defined as the subgroup of $\mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ with a basis $\frac{1}{r_1}E_1, \ldots, \frac{1}{r_m}E_m$. It allows us to express the embedding $\Theta$ a bit more clearly in terms of monomial valuations $\nu_{g_i}$ and orders $r_i$ of $g_i$. By identifying $\mathbb{C}[\Lambda_X]$ with $\mathbb{C}[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$, where $t_i^{-r_i} = t^{E_i}$ we obtain another form of the embedding $\Theta$ (denoted by the same letter by abuse of notation).

Corollary 4.2.6. *Under the assumptions above if $D$ is an effective divisor on $Y$ and a homogeneous element $f \in \mathcal{R}(X)$ corresponds to $\overline{D}$ then $\Theta(f) = \varphi_*(f)\prod_{i=1}^m t_i^{\nu_{g_i}(\varphi_*(f))} \in \mathbb{C}[x_1,\dots,x_n]^{[G,G]}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$. Moreover $\Theta(t^{E_i}) = t_i^{-r_i}$.*

## 4.3. Approach via characterization theorem

Assume that we are in situation 4.1.10, assume moreover that $\varphi\colon X \to Y$ is a crepant resolution of singularities, $\mathrm{Cl}(X)$ is free abelian group and that the Cox ring $\mathcal{R}(X)$ is finitely generated. This setup is of most interest for us as it includes the case of quotient singularities by proposition 4.2.3 and theorem 3.4.10. It is natural to try to use the characterization theorem for Cox rings (theorem 3.4.5) to give a criterion when the inclusion $\mathcal{R} \subset \Theta(\mathcal{R}(X))$ is the equality. We will now give such a criterion. Unfortunately, the criterion may not be easy to apply efficiently in concrete examples, as one of the conditions to check is whether of some elements of $\mathcal{R}$ are prime. Nevertheless, we will give an example how to use it in section 5.3, in the study of the three-dimensional quotients by dihedral groups.

We need two general lemmas relating $\mathcal{R}$ to $\mathcal{R}(X)$ and $Y$. They are interesting on their own and the first one finds also another use in chapter 7. It shows that the GIT quotient of the spectrum of $\mathcal{R}$ with respect to the Picard torus $\mathbb{T}$ action linearized by trivial character is the same as the analogous quotient for $\mathcal{R}(X)$.

Lemma 4.3.1. *Under the assumptions at the beginning of the section we have $\mathcal{R}^{\mathbb{T}} = \mathcal{R}(Y)_0 \otimes 1$ ($\cong H^0(Y, \mathcal{O}_Y)$). In particular, if we assume moreover that $Y$ is affine, then $(\operatorname{Spec}\mathcal{R})/\mathbb{T} \cong Y$.*

Proof. To see that $\mathcal{R}^{\mathbb{T}} \subset \mathcal{R}(Y)_0 \otimes 1$ note that every element of $\mathcal{R}^{\mathbb{T}}$ is of the form $\widetilde{f} = \varphi_*(f) \otimes 1$ for some $f \in \mathcal{R}(X)$. Then $\deg \varphi_*(f) = 0$ since $\deg \widetilde{f} = 0$ and $\varphi_*\colon \mathcal{R}(X) \to \mathcal{R}(Y)$ is a map of graded rings.

For the opposite inclusion let $\psi_1,\dots,\psi_k$ be generators of $\mathcal{R}(Y)_0$. By the construction of the generators of $\mathcal{R}$ we have $\psi_i t^{\overline{D}_i} \in \mathcal{R}$ for some effective divisors $D_i$ on $Y$. Moreover, by proposition 4.1.9 we have $\overline{D}_i = -\sum_j \nu_j(D_i)E_j$ for nonnegative rational numbers $\nu_j(D_i)$ which are in fact nonnegative integers as $\deg \psi_i = 0 \in \mathrm{Cl}(Y)$ imply $\overline{D}_i \in \bigoplus_j \mathbb{Z}E_j$ in $\mathrm{Cl}(X)$ (see proposition 4.1.3). Then $\psi_i = \psi_i t^{\overline{D}_i} \cdot \prod_j (t^{E_j})^{\nu_j(D_j)} \in \mathcal{R}$. □

Let $S$ be a multiplicative system generated by elements $t^{E_1},\dots,t^{E_m} \in \mathcal{R} \subset \Theta(\mathcal{R}(X))$.

Lemma 4.3.2. *Under the assumptions at the beginning of the section $S^{-1}\mathcal{R} = S^{-1}\Theta(\mathcal{R}(X))$.*

Proof. The inclusion $\subset$ is trivial and the inclusion $\supset$ can be deduced from proposition 4.1.3 as follows. First, note that if $f \in \mathcal{R}(Y)$ is $\mathrm{Cl}(Y)$-homogeneous, $D, D' \in \mathrm{Cl}(X/Y)_{\mathrm{free}}$ and $ft^D, ft^{D'} \in \Theta(\mathcal{R}(X))$ then $D - D' \in \bigoplus_{i=1}^n \mathbb{Z}[E_i]$, because $\varphi_*(ft^D) = \varphi_*(ft^{D'}) = f$ and so $\varphi_* D = \varphi_* D'$. Now since elements $t^{E_i}$ are $\mathrm{Cl}(X)$-homogeneous we have a natural $\mathrm{Cl}(X)$-grading on $S^{-1}\Theta(\mathcal{R}(X))$. As every $\mathrm{Cl}(X)$-homogeneous element of $\Theta(\mathcal{R}(X))$ is of the form $ft^D$ (see proposition 4.1.5) we deduce that homogeneous elements of $\Theta(\mathcal{R}(X))$ are contained in $S^{-1}\mathcal{R}$ and this suffice to conclude that $S^{-1}\mathcal{R} \supset S^{-1}\Theta(\mathcal{R}(X))$. □

Now we are ready to state and prove the main result of this section. In the notation of proposition 4.1.20 let $\sigma \subset \mathrm{Mov}(\mathcal{R})$ be a GIT cone of the Picard torus $\mathbb{T}$ action on $\operatorname{Spec}\mathcal{R}$.

Theorem 4.3.3. *Under the assumptions given at the beginning of the section assume moreover that $Y$ is affine. The following conditions are equivalent:*

*(i)* *Each element* $t^{E_i}$ *is prime in* $\mathcal{R}$*, there exists* $\lambda \in \sigma^\circ$ *such that the good quotient* $(\operatorname{Spec}\mathcal{R})^{ss}(\lambda)/\!/\mathbb{T}$ *is smooth and the set of points of* $\operatorname{Spec}\mathcal{R}$ *that are unstable with respect to* $\lambda$ *or have nontrivial isotropy group has the codimension greater than one.*

*(ii)* *The natural map* $(\operatorname{Spec}\mathcal{R})^{ss}(\lambda)/\!/\mathbb{T} \to (\operatorname{Spec}\mathcal{R})^{ss}(0)/\!/\mathbb{T} = Y$ *is a resolution which is isomorphic to* $X$ *in codimension one and* $\mathcal{R} = \Theta(\mathcal{R}(X))$.

Proof. To prove $(i) \implies (ii)$ we use two previous lemmas.
By lemma 4.3.2 we have $S^{-1}\mathcal{R} = S^{-1}\Theta(\mathcal{R}(X))$. Therefore, as $\mathrm{Cl}(X)$ is free, we may use [**5**, Proposition 3.4.1.8] and [**5**, Theorem 3.4.1.11] to conclude that the ring $\mathcal{R}$ is factorial, in particular it is factorially graded and normal. Moreover, $\operatorname{Spec} S^{-1}\mathcal{R} \cong \operatorname{Spec} S^{-1}\mathcal{R}(X)$ as $\mathbb{T}$-varieties.

By the assumption on the codimension of unstable point locus and the locus of points with nontrivial isotropy group the $\mathbb{T}$-action on $\operatorname{Spec}\mathcal{R}$ is strongly stable and we may use theorem 3.4.5 to get $\mathcal{R} \cong \mathcal{R}(X')$, where $X' = (\operatorname{Spec}\mathcal{R})^{ss}(\lambda)/\!/\mathbb{T}$. By lemma 4.3.1 we have $(\operatorname{Spec}\mathcal{R})^{ss}(0)/\!/\mathbb{T} \cong Y$.

Now the map $X' \to (\operatorname{Spec}\mathcal{R})^{ss}(0)/\!/\mathbb{T} \cong Y$ is an isomorphism outside the set $\bigcup_{i=1}^m \{t^{E_i} = 0\}$ and $\{t^{E_i} = 0\}$ are irreducible divisors on $(\operatorname{Spec}\mathcal{R})^{ss}(\lambda)/\!/\mathbb{T}$. Since all crepant divisors over $Y$ have to be divisors on every nonsingular variety admitting a projective birational map to $Y$ it means that $\{t^{E_i} = 0\}$ are crepant divisors over $Y$. Hence $X'$ is a crepant resolution and as such it is isomorphic in codimension one to $X$. Thus we get $(ii)$.

The implication $(ii) \implies (i)$ follows from theorem 3.4.5. □

## 4.4. Bounding degrees of generators with multigraded regularity

In this section we give a direct bound on the degrees of generators of the Cox rings using the results in the sheaf cohomology – the Kawamata Viehweg vanishing (as in [**62**, Theorem 1-2-3]) and the properties of multigraded Castelnuovo-Mumford regularity. We start with a brief introduction to multigraded regularity.

**4.4.1. Multigraded Castelnuovo-Mumford regularity.** The following generalization of the Castelnuovo-Mumford regularity is due to Maclagan and Smith [**70**]. Let $X$ be an algebraic variety, let $B_1, \ldots, B_\ell$ be globally generated line bundles on $X$ and let $L$ be a line bundle on $X$.

Definition 4.4.1 (See [**51**, Sect. 2.]). A coherent sheaf $\mathcal{F}$ is called *$L$-regular* (with respect to $B_1, \ldots, B_\ell$) if $H^i(X, \mathcal{F} \otimes L \otimes B^{-\mathbf{u}}) = 0$ for all $i > 0$ and $|\mathbf{u}| = i$, where $\mathbf{u} := (u_1, \ldots, u_\ell)$ is a tuple of nonnegative integers, $|\mathbf{u}| := u_1 + \ldots + u_\ell$ and $B^{\mathbf{u}} := B_1^{u_1} \otimes \ldots \otimes B_\ell^{u_\ell}$.

The concept of multigraded regularity is interesting for us due to the following property.

Theorem 4.4.2 ([**51**, Theorem 2.1(2)]). *Let* $\mathcal{F}$ *be a* $L$*-regular coherent sheaf on* $X$*. Then the multiplication map:*

$$H^0(X, \mathcal{F} \otimes L \otimes B^{\mathbf{u}}) \otimes H^0(X, B^{\mathbf{v}}) \to H^0(X, \mathcal{F} \otimes L \otimes B^{\mathbf{u}+\mathbf{v}})$$

*is surjective for all* $\ell$*-tuples of nonnegative integers* $\mathbf{u}, \mathbf{v}$.

We will be using this theorem via the following corollary.

Corollary 4.4.3. *Assume that* $X$ *is a smooth algebraic variety with* $K_X = 0$*. Let* $\varphi\colon X \to Y$ *be a projective morphism onto an affine variety* $Y$*. Let* $B_1, \ldots, B_\ell$ *be globally generated line bundles on* $X$*. Assume that the fibres of* $\varphi$ *are of dimension at most* $r$*. Let* $A$ *be a* *line bundle on $X$ such that $A \otimes B^{-\mathbf{u}}$ is $\varphi$-nef and $\varphi$-big for every tuple $\mathbf{u}$ of nonnegative integers with $|\mathbf{u}| \leq r$. Then the multiplication map:*

$$H^0(X, A \otimes B^{\mathbf{v}}) \otimes H^0(X, B^{\mathbf{w}}) \to H^0(X, A \otimes B^{\mathbf{v}+\mathbf{w}})$$

*is surjective for all $\ell$-tuples of nonnegative integers $\mathbf{v}, \mathbf{w}$.*

Proof. This is a direct consequence of theorem 4.4.2 for $L = \mathcal{O}_X$ and $\mathcal{F} = A$ since the defining property of $\mathcal{O}_X$-regular sheaf follows then by Kawamata-Viehweg vanishing theorem [**62**, Theorem 1-2-3]. The vanishing of cohomologies $H^i(X, A \otimes B^{-\mathbf{u}})$ with $i > r$ follows from the assumption on the dimension of fibres of $\varphi$ as $Y$ is affine, see [**46**, Corollary III.11.2] $\square$

**4.4.2. Degrees of generators of $\mathcal{R}(X)$.** Assume that we are in situation 4.1.10. Assume moreover that $\varphi$ is a relative Mori Dream Space, $Y$ is affine and $K_X = 0$.

Let $\Sigma$ be a GIT subdivision of the movable cone $\mathrm{Mov}(X)$ and let $\sigma \in \Sigma$ be a cone of maximal dimension. Then $\sigma = \mathrm{SAmp}(X')$ for some $\varphi' \colon X' \to Y$ isomorphic to $X$ over $Y$ in codimension one. Let $\rho_1, \ldots, \rho_s$ be rays of $\sigma$ and let $D_1, \ldots, D_s \in \mathrm{Cl}(X')$ be ray generators. Assume that $m_1, \ldots, m_s > 0$ are such that $m_1 D_1, \ldots, m_s D_s$ are divisors with base point free linear systems. Let $\mathcal{R}(X)_\sigma = \bigoplus_{\alpha \in \mathrm{Cl}(X) \cap \sigma} \mathcal{R}(X)_\alpha$.

Proposition 4.4.4. *Assume that $X'$ is smooth and the dimensions of fibres of $\varphi'$ are at most $r$. Then the $\mathbb{C}$-algebra $\mathcal{R}(X)_\sigma$ is generated by the elements corresponding to global sections of all line bundles of the form $\sum_{i=1}^{s} k_i D_i$ with $0 \leq k_i < (r+1)m_i$ for every $i$.*

Proof. We apply corollary 4.4.3 multiple times for globally generated line bundles $\{B_1, \ldots, B_\ell\} = \{m_i D_i \colon i \in I\}$ and $A = \sum_{i \in I}(rm_i + k_i)D_i$, for all $I \subset \{1, \ldots, s\}$ and all $0 \leq k_i < m_i$. $\square$

Proposition 4.4.5. *(cf. the proof of implication (a) $\implies$ (b) in theorem 3.4.7) Under the assumptions above the Cox ring $\mathcal{R}(X)$ is generated by generators of $\mathcal{R}(X)_\sigma$ for $\sigma \in \Sigma$ together with the elements corresponding to the components of the exceptional divisor.*

Proof. We argue as in the proof of proposition 4.1.13. Every effective Weil divisor on $X$ is of the form $\overline{D} + E$ where $\overline{D}$ is the strict transform of a Weil divisor on $Y$ and $E$ is an effective divisor supported on the components of the exceptional divisor.

Then $\overline{D}$ is movable as $D$ is movable on $Y$ because $Y$ is affine. Indeed, assume that $D \sim D'$ on $Y$ for an effective divisor $D'$ and $D'$ does not contain a certain prime component $D_0$ of $D$ in its support. Then $\overline{D} + F_1 \sim \overline{D'} + F_2$, where $F_1, F_2$ are effective divisors supported on the components of the exceptional divisor. Let $A$ be a relatively ample divisor. Let $A' = A - \varphi^*\varphi_* A$. Then $A'$ is relatively ample and $-A'$ is an effective divisor supported on the exceptional divisor. By replacing $A'$ with some positive multiple we may assume that $-(A' + F_1)$ is effective and $A' + F_2$ is relatively ample. Let $A'' \sim A' + F_2$ be an effective divisor which does not contain $D_0$ in its support. Now $\overline{D} \sim \overline{D'} + F_2 - F_1 = \overline{D'} - A' - F_1 + A + F_2 \sim \overline{D'} - A' - F_1 + A''$ and the latter is an effective divisor which does not contain $\overline{D_0}$ in its support. As we may argue in the same manner for each prime component $D_0$, we conclude that $\overline{D}$ is movable.

As homogeneous elements of $\mathcal{R}(X)$ up to multiplication by a constant correspond to effective divisors on $X$ we see that every homogeneous element of $\mathcal{R}(X)$ belongs to the $\mathbb{C}$-algebra generated by $\mathcal{R}(X)_{\mathrm{Mov}(X)}$ and the elements corresponding to the components of exceptional divisor. $\square$

Note that propositions 4.4.4 and 4.4.5 combined allow one to bound degrees of generators of $\mathcal{R}(X)$ under the assumption that all the codimension two modifications $X' \to Y$ of

$X \to Y$ corresponding to the chambers in movable cone are smooth. This assumption is satisfied for example in the case of three-dimensional quotient singularities and in the case of symplectic quotient singularities considered in chapters 5 and 7 (see proposition 5.1.1 and proposition 7.1.4 respectively). Combining this with proposition 4.4.5 and McKay correspondence, which helps to bound the degree of exceptional divisors (see section 4.2), we obtain the method to bound degrees of such quotient singularities. This method will be used in chapter 7.

CHAPTER 5

# Resolutions of three-dimensional quotient singularities

In this chapter we apply the general results from the previous chapter to crepant resolutions of quotient singularities in dimension three. In section 5.1 we introduce briefly the setup in which we work and we classify all the crepant resolutions of a given quotient singularity as GIT quotients of the spectrum of the Cox ring of a single resolution. In section 5.2 we compute Cox rings of quotient singularities corresponding to faithful reducible representations of nonabelian groups. Then in sections 5.3 and 5.4 we specialize to concrete examples – representations of dihedral groups and two quotients for which the resolution contracts a divisor to a point. The last section presents an example of a different nature – the analysis of the quotient corresponding to the simplest irreducible representation with a crepant resolution contracting a divisor to a point.

We refer to section 2.2.3 for general facts such that the existence of crepant resolutions in dimension three and the classification of finite subgroups of $\mathrm{SL}_3(\mathbb{C})$.

Most of the content of this section was originally published in the joint paper with Maria Donten-Bury [**33**].

## 5.1. Setting

Let $G \subset \mathrm{SL}_3(\mathbb{C})$ be a finite group. Let $\varphi\colon X \to \mathbb{C}^3/G$ be a crepant resolution with components $E_1, \dots, E_m$ of the exceptional divisor of $\varphi$. Then we are in situation 4.1.10. By results of section 4.2 we have an embedding

$$\Theta\colon \mathcal{R}(X) \to \mathcal{R}(\mathbb{C}^3/G) \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)] \subset \mathcal{R}(\mathbb{C}^3/G) \otimes_{\mathbb{C}} \mathbb{C}[\Lambda_X] = \mathbb{C}[x,y,z]^{[G,G]}[t_1^{\pm 1}, \dots, t_m^{\pm 1}].$$

Given the system of $\mathrm{Ab}(G)^\vee$-homogeneous generators $\phi_1, \dots, \phi_s$ of $\mathbb{C}[x,y,z]^{[G,G]}$ we consider the subalgebra $\mathcal{R} \subset \Theta(\mathcal{R}(X))$ generated by elements of the form $\phi_i t^{\overline{D}_{\phi_i}}$ and $t^{E_j}$, where $\overline{D}_{\phi_i}$ is the strict transform of the divisor $D_{\phi_i} \in \mathrm{WDiv}(\mathbb{C}^3/G)$ corresponding to $\phi_i$. By corollary 4.2.5 $\nu_{E_i} = \frac{1}{r_i}\nu_{g_i}$, where $g_i \in G$ is an element in the conjugacy class corresponding to $E_i$ via McKay correspondence and $r_i$ is the order of $g_i$. Moreover by corollary 4.2.6 we have $t^{\overline{D}_{\phi_i}} = \prod_{j=1}^m t^{\nu_{g_j}(\phi_i)}$ and $t^{E_j} = t^{-r_j}$.

Recall the cone $\mathrm{Mov}(\mathcal{R})$ introduced in (4.1.19). The next proposition gives the geometric motivation for seeking the generators of $\mathcal{R}(X)$.

Proposition 5.1.1. *If $\mathcal{R} = \Theta(\mathcal{R}(X))$ then $\mathrm{Mov}(X) = \mathrm{Mov}(\mathcal{R})$. Moreover, there is a one-to-one correspondence between the crepant resolutions of $\mathbb{C}^3/G$ and GIT chambers of the action of $\mathbb{T}$ on $\mathrm{Spec}\,\mathcal{R}$ contained in $\mathrm{Mov}(\mathcal{R})$. Namely, taking GIT quotients corresponding to GIT chambers of this action we obtain all pairwise nonisomorphic crepant resolutions of $\mathbb{C}^3/G$.*

Proof. It is a corollary from proposition 4.1.20 if we show that every minimal model of $X$ is smooth. In the three-dimensional case the smoothness of minimal models follows from [**65**, Corollary 4.11] – this result implies that in dimension three if one minimal model is smooth then the others are smooth as well. By theorem 2.2.24 we know that a crepant resolution exists, by definition it is smooth and by proposition 2.1.11 it is a minimal model. □

## 5.2. Case of a reducible representation

In this section we give a description of $\mathcal{R}(X)$ using the compatibility criterion (theorem 4.1.15) under the additional assumption that $G$ is nonabelian and $\mathbb{C}^3 = V_1 \oplus V_2$ as $G$-representations, with $\dim V_i = i$ (case (B) of theorem 2.2.26).

In the case of abelian groups (case (A) of theorem 2.2.26) the quotient and crepant resolutions are toric and hence the structure of the Cox ring is well-known – it is then isomorphic to the polynomial ring, with number of variables and grading described by the combinatorial data related to the toric structure. See [**23**, Chapter 5] for discussion of Cox rings of toric varieties.

In the case considered in this section the quotient space $\mathbb{C}^3/G$ is not toric, but it inherits a natural 2-dimensional torus action (where each $\mathbb{C}^*$ factor comes from a homothety on a component), which lifts to a resolution (see section 6.1). This means that we obtain a (non-complete) $T$-variety of complexity one. The spectrum of the Cox ring for such varieties is always defined by trinomial relations; in the projective case it is shown in [**50**], and the case of non-complete rational varieties is treated in [**48**]. Since the structure of $[G, G]$-invariants for reducible representations is quite simple (there are always just four invariants) we obtain that there is always a single trinomial relation in the Cox ring, see remark 5.2.7. But in general the structure of the Cox ring of a (crepant) resolution of a 3-dimensional quotient singularity can be more complex, see the case of $A_4$ in [**35**, Thm 4.5] and section 5.5 of this work.

We start from the 2-dimensional representation, that is $\overline{G} \subset \mathrm{GL}_2(\mathbb{C})$, and we construct $G$ as the set of $3 \times 3$ matrices of the following form, for all $M$ in $\overline{G}$:

$$\begin{pmatrix} M & 0 \\ 0 & \det(M)^{-1} \end{pmatrix}$$

Let $X \to \mathbb{C}^3/G$ be a projective crepant resolution. We are going to find the presentation of the Cox ring $\mathcal{R}(X)$ in terms of generators and relations using theorem 4.1.15. By a classification of finite subgroups of $\mathrm{GL}_2(\mathbb{C})$ (theorem 2.2.23) we obtain the following corollaries.

Corollary 5.2.1. *A finite group $\overline{G} \subset \mathrm{GL}_2(\mathbb{C})$ is of the form $(\mu_{wd} \mid \mu_d,\ H \mid K)$ for some integers $w, d$ and a normal subgroup $K \subset H = (\mathbb{C}^* \cdot \overline{G}) \cap \mathrm{SL}_2(\mathbb{C})$.*

Corollary 5.2.2. *For a finite subgroup $\overline{G} \subset \mathrm{GL}_2(\mathbb{C})$ we have the equality $[\overline{G}, \overline{G}] = [H, H]$, hence also $\mathbb{C}[x, y]^{[\overline{G},\overline{G}]} = \mathbb{C}[x, y]^{[H,H]}$. Moreover, an element $p \in \mathbb{C}[x, y]^{[\overline{G},\overline{G}]}$ homogeneous with respect to the degree grading is homogeneous with respect to the $\mathrm{Ab}(\overline{G})$-action if and only if it is homogeneous with respect to the $\mathrm{Ab}(H)$-action.*

Consider the ring of invariants $\mathbb{C}[x, y, z]^{[G,G]} = \mathbb{C}[x, y]^{[\overline{G},\overline{G}]}[z]$. Recall that we listed finite subgroups of $\mathrm{SL}_2(\mathbb{C})$ up to conjugacy in theorem 2.2.19. In particular the generators of invariant rings for $H \subset \mathrm{SL}_2(\mathbb{C})$, are given in the table in theorem 2.2.19. It can be verified directly that they are $\mathrm{Ab}(H)^\vee$-homogeneous.

Consider the ring homomorphism $\kappa\colon \mathbb{C}[Z_1, Z_2, Z_3, Z_4] \to \mathbb{C}[x, y, z]^{[G,G]}$ defined by $Z_j \mapsto p_j$ for $j = 1, 2, 3$, and $Z_4 \mapsto z$. Note that we have an $\mathrm{Ab}(H)$-action on $\mathbb{C}[Z_1, Z_2, Z_3, Z_4]$ induced from $\mathbb{C}[x, y, z]^{[G,G]} = \mathbb{C}[x, y, z]^{[H,H]}$, where $H = (\mathbb{C}^* \cdot \overline{G}) \cap \mathrm{SL}_2(\mathbb{C})$ acts on $x, y$ linearly and trivially on $z$.

Let $g_1, \dots, g_m$ be representatives of all junior conjugacy classes of $G$, $r_1, \dots, r_m$ their orders and $\nu_1, \dots, \nu_m$ corresponding monomial valuations on $\mathbb{C}(x, y, z)$. We will also use monomial valuations $\widetilde{\nu}_1, \dots, \widetilde{\nu}_m$ on $\mathbb{C}[Z_1, Z_2, Z_3, Z_4]$ defined by setting $\widetilde{\nu}_i(Z_j) = \nu_i(\kappa(Z_j))$.

We can now state the main result of this section.

THEOREM 5.2.3. *The image $\Theta(\mathcal{R}(X))$ of the Cox ring of a crepant resolution $X \to Y$ via the embedding $\Theta$ from corollary 4.2.4 is generated by $m+4$ elements*

$$p_j \prod_{i=1}^{m} t_i^{\nu_i(p_j)} \text{ for } j = 1, 2, 3, \qquad z \prod_{i=1}^{m} t_i^{\nu_i(z)}, \qquad t_i^{-r_i} \text{ for } i = 1, \dots, m.$$

The proof uses theorem 4.1.15 and corollary 4.2.5 by showing the compatibility condition for valuations $\nu_i$ and $\widetilde{\nu}_i$. We begin with introducing some more notation and proving a lemma, which actually implies that homogeneous elements can be lifted from $\mathbb{C}(x, y, z)$ to $\mathbb{C}(Z_1, Z_2, Z_3, Z_4)$ via $\kappa$ compatibly with respect to a single valuation $\nu_i$ (in the sense of Theorem 4.1.15). Then we finish with the main part of the argument, which is to show that homogeneous elements can be lifted correctly with respect to multiple valuations simultaneously.

If we choose a monomial valuation $\nu$ then we may write any polynomial $F$ as $F_0 + F_1$, where all monomials in $F_0$ have valuation equal to $\nu(F)$ and $\nu(F_1) > \nu(F)$.

Denote by $R$ the trinomial relation between $p_1, p_2, p_3$, that is the generator of $\ker \kappa$. Choose a valuation $\nu := \nu_i$, and if necessary, make a linear change of coordinates such that the corresponding matrix $g_i$ acts diagonally. In these coordinates we may take minimal parts of generators with respect to $\nu_i$: $p_{1,0}, p_{2,0}, p_{3,0}$. We also decompose $R$ with respect to $\widetilde{\nu} := \widetilde{\nu}_i$: $R = R_0 + R_1$. Note that $R_0$ is a relation between $p_{1,0}, p_{2,0}, p_{3,0}$. In this setting we state the following result.

LEMMA 5.2.4. *Every relation between $p_{1,0}, p_{2,0}, p_{3,0}$ homogeneous with respect to* $\mathrm{Ab}(H)$-*action lies in the ideal generated by $R_0$ in $\mathbb{C}[Z_1, Z_2, Z_3]$.*

PROOF. First notice that $p_{1,0}, p_{2,0}, p_{3,0}$ depend only on the inequality between values of $\nu$ on coordinates $u, v$ diagonalizing $g_i$, that is whether $\nu(u) > \nu(v)$, $\nu(u) = \nu(v)$ or $\nu(u) < \nu(v)$. If $g_i = \zeta_d h$ for some $h \in H$ then the inequality between $\nu(u)$ and $\nu(v)$ is the same as the inequality between values on $u, v$ either of the valuation corresponding to $h$ or the one corresponding to $h^{-1}$. Thus without loss of generality we may assume that $p_{1,0}, p_{2,0}, p_{3,0}$ are determined by monomial valuation $\nu_h$ corresponding to $h \in H$.

Now we reduce to the case of Cox rings of minimal resolutions of $\mathbb{C}^2/H$ for $H \subset \mathrm{SL}(2, \mathbb{C})$. By [**31**, Theorem 6.12] we find that $p_1, p_2, p_3$ satisfy valuation compatibility property. Here we use implication $(ii) \implies (i)$ theorem 4.1.15: if $\phi_1, \dots, \phi_s$ can be used for producing a generating set of the Cox ring then they satisfy the valuation compatibility condition (assumption 4.1.14).

Hence it suffices to show that for $\nu_h$ the valuation compatibility property implies the statement of the lemma. Let $Q \in \mathbb{C}[Z_1, Z_2, Z_3]$ be any $\mathrm{Ab}(H)$-homogeneous relation between $p_{1,0}, p_{2,0}, p_{3,0}$. We can change it by adding an element $W(Z_1, Z_2, Z_3)R$, to obtain a compatible lift $\widetilde{q} = Q + WR$ of $q = Q(p_1, p_2, p_3)$. That is, $\widetilde{\nu}_h(Q + WR) \geq \nu_h(q)$. But $\widetilde{\nu}_h(Q) < \nu_h(q)$ because $Q$ is a relation between initial forms of $p_1, p_2, p_3$ with respect to $\nu_h$. Thus the initial forms of $Q$ and $WR$ with respect to $\widetilde{\nu}_h$ must cancel: $-Q_0 = (WR)_0 = W_0 R_0$. Since $Q_0$ is a relation between $p_{1,0}, p_{2,0}, p_{3,0}$, $Q_1$ also is, and if $Q_1 \neq 0$ we proceed by induction, repeating the argument for $Q_1$, to show that $Q$ is a multiple of $R_0$. $\square$

PROOF OF THEOREM 5.2.3. We show that the valuation compatibility condition from Theorem 4.1.15 is satisfied. Note that by definition of a monomial valuation it suffices to check only these elements of $\mathbb{C}[x,y,z]^{[G,G]}$ which are homogeneous with respect to the standard degree grading and with respect to the $\mathrm{Ab}(G)$-action, i.e., by corollary 5.2.2, precisely the elements homogeneous with respect to the standard grading and $\mathrm{Ab}(H)$-action.

Let $f \in \mathbb{C}[x,y,z]^{[G,G]}$ be any element homogeneous with respect to both the standard degree and the $\mathrm{Ab}(H)$-action. Take any $F \in \kappa^{-1}(f) \subset \mathbb{C}(Z_1,Z_2,Z_3,Z_4)$. Set $N = \max_{i=1,\ldots,m}(\nu_i(f) - \widetilde{\nu}_i(F))$. We may assume $N > 0$, since $N \geq 0$ by definition of $\widetilde{\nu}_i$, and the valuation compatibility condition is equivalent to $N = 0$. We proceed by induction: we find $F'$ such that

(1) $\kappa(F') = f$,
(2) $\widetilde{\nu}_i(F') \geq \widetilde{\nu}_i(F)$ for all $i$,
(3) $\widetilde{\nu}_{i_0}(F') > \widetilde{\nu}_{i_0}(F)$ for some $i_0$.

Let $j$ be any index such that $\widetilde{\nu}_j(F) < \nu_j(f)$. We decompose $F = F_0 + F_1$ with respect to $\widetilde{\nu}_j$. Changing coordinates if necessary, we may assume that the corresponding matrix $g_j$ acts diagonally and decompose $p_i = p_{i,0} + p_{i,1}$ with respect to $\nu_j$.

Since $\widetilde{\nu}_j(F) < \nu_j(f)$, the part of $F$ with the smallest $j$-th valuation must annihilate parts of generators with smallest $j$-th valuation in order to increase valuation when passing through $\kappa$. Formally, we have $F_0(p_{1,0}, p_{2,0}, p_{3,0}, z) = 0$, but since $z$ is algebraically independent of $p_{1,0}, p_{2,0}, p_{3,0}$, in fact $F_0 \in \mathbb{C}[Z_1,Z_2,Z_3]$ is a relation between $p_{1,0}, p_{2,0}, p_{3,0}$. Moreover, as a sum of monomials in an element of $\mathbb{C}[Z_1,Z_2,Z_3,Z_4]$ homogeneous with respect to $\mathrm{Ab}(H)$-action, $F_0$ is homogeneous with respect to $\mathrm{Ab}(H)$-action.

By Lemma 5.2.4 applied to $\nu_j$ and $\widetilde{\nu}_j$ we have $F_0 = PR_0$ for some $P \in \mathbb{C}[Z_1,Z_2,Z_3]$. We claim that $F' = F - PR$ satisfies conditions (1)-(3) with $i_0 = j$. Condition (1) is immediate since $\kappa(R) = 0$. Condition (3) holds because $F' = F - PR = F_1 - R_1P$ and $\widetilde{\nu}_j(R_1) > \widetilde{\nu}_j(R_0)$, implying $\widetilde{\nu}_j(R_1P) > \widetilde{\nu}_j(R_0P) = \widetilde{\nu}_j(F)$. To prove condition (2) we use again $F' = F_1 - R_1P$ and the fact that $R$ is a trinomial. We have to show that $\widetilde{\nu}_i(R_1) \geq \widetilde{\nu}_i(R_0)$. Since $R_0$ consists of at least two monomials (as a relation between leading forms), $R_1$ is either 0 or a monomial. Assuming the latter and repeating the same argument for $\widetilde{\nu}_i$ for $i \neq j$ we see that there are at least two monomials in $R$ with valuation $\widetilde{\nu}_i(R)$. Thus at least one of them is in $R_0$, which implies $\widetilde{\nu}_i(R_1) \geq \widetilde{\nu}_i(R) = \widetilde{\nu}_i(R_0)$. □

Remark 5.2.5. The argument generalizes immediately to the analogous description of the Cox ring of a minimal model of a quotient singularity $\mathbb{C}^{n+2}/G$ for a finite nonabelian subgroup $G \subset \mathrm{SL}_{n+2}(\mathbb{C})$, acting on $\mathbb{C}^{n+2}$ via a representation which splits into one 2-dimensional component and $n$ components of dimension 1.

Remark 5.2.6. Some ideas in the above proof are related to the algorithm presented in [**94**, Sect. 4]. In the notation therein, $F_0 = \min_j(F)$, $p_{i,0} = \min_j(p_i)$ and $\min_j(I) = (R_0) = \min_j(J)$. One may also check that all steps of the algorithm in [**94**, Sect. 4] end without introducing additional generators which gives a different proof of Theorem 5.2.3.

Remark 5.2.7. Note that to get the generator of the ideal of relations between elements generating the Cox ring $\mathcal{R}(X)$ listed in Theorem 5.2.3 it suffices to take the trinomial generator of relations between $p_1, p_2, p_3$ and homogenize it with respect to $\mathrm{Ab}(G)$-action using variables mapped into generators of the form $t_i^{-r_i}$.

The first application of Theorem 5.2.3 is the case of dihedral groups, obtained in the case of even $n$ as $(\mu_4 \mid \langle 1 \rangle,\ \mathrm{BD}_{2n} \mid C_n)$ and in the odd $n$ case as $(\mu_4 \mid \langle 1 \rangle,\ \mathrm{BD}_{4n} \mid C_n)$, studied in section 5.3. In section 5.4 we provide two more examples with an interesting feature, not appearing in the series of dihedral groups: the groups contain elements of age 2.

## 5.3. Example – Dihedral groups

Let $G = D_{2n}$, $n \geq 3$, be a dihedral group of order $2n$, i.e. the group of isometries of the regular $n$-gon in the plane. In terms of generators and relations $G$ can be presented as $\langle \rho, \varepsilon \mid \rho^n,\ \varepsilon^2,\ (\varepsilon\rho)^2 \rangle$, and its elements are: the identity, rotations $\rho, \rho^2, \dots, \rho^{n-1}$ and reflections $\varepsilon, \varepsilon\rho, \dots, \varepsilon\rho^{n-1}$. The structure of the sets of conjugacy classes of $G$, which are very important for the properties of the resolution because of the McKay correspondence, differs depending on the parity of $n$ – this is why we describe these cases separately.

We consider the following 3-dimensional representation of $G$:

$$\rho \mapsto \begin{pmatrix} \zeta & 0 & 0 \\ 0 & \zeta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \varepsilon \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{5.3.1}$$

where $\zeta$ denotes the primitive $n$-th root of unity. It is easy to check that the image of any other faithful representation of $G$ in $\mathrm{SL}_3(\mathbb{C})$ is the same subgroup up to conjugacy. By abuse of notation, from now on we will denote by $G$ the image of the given representation and by $\rho, \varepsilon$ their images.

**5.3.1. The odd case: $n = 2k+1$.** The commutator subgroup consists of all rotations: $[G, G] = \langle \rho \rangle$. All reflections are conjugate and pairs of rotations are conjugate, so there are $k + 2$ conjugacy classes:

$$\{1\},\ \{\rho, \rho^{-1}\}, \dots, \{\rho^k, \rho^{-k}\},\ \{\varepsilon, \varepsilon\rho, \varepsilon\rho^2, \dots, \varepsilon\rho^{2k}\}.$$

The set of points in $\mathbb{C}^3$ with nontrivial isotropy group consists of the line $x_1 = x_2 = 0$ fixed by $\langle \rho \rangle$ and $n$ lines, each fixed by a reflection, e.g. $x_1 - x_2 = x_3 = 0$ fixed by $\langle \varepsilon \rangle$. In the quotient $\mathbb{C}^3/G$ these lines are mapped to two components of the singular points set: the first one to a component $L_\rho$ with transversal $A_{2k}$ singularity and lines fixed by reflections to a component $L_\varepsilon$ with transversal $A_1$ singularity (away from 0). The image of 0, as the intersection point of these components, has a worse singularity.

From the McKay correspondence we obtain that each nontrivial conjugacy class in $G$ correspond to an exceptional divisor on the resolution which is mapped to a line of singular points in $\mathbb{C}^3/G$. This is because there are no elements of age 2, which would give an exceptional divisor in the fibre over the origin. Thus we have $k$ exceptional divisors $E_1, \dots, E_k$ mapped to $L_\rho$ and $E_\varepsilon$ mapped to $L_\varepsilon$.

*The Cox ring.* The Cox ring $\mathcal{R}(\mathbb{C}^3/G)$ is the ring of invariants $\mathbb{C}[x, y, z]^{[G,G]}$ of the commutator subgroup $[G, G] = \langle \rho \rangle$. To find the Cox ring of a crepant resolution we also have to determine the eigenspaces of the action of the abelianization $\mathrm{Ab}(G) \simeq \mathbb{Z}_2$, generated by the class of $\varepsilon$, on $\mathcal{R}(\mathbb{C}^3/G)$.

LEMMA 5.3.2. *$\mathcal{R}(\mathbb{C}^3/G) = \mathbb{C}[x, y, z]^{[G,G]}$ is generated by $x^n + y^n$, $x^n - y^n$, $xy$, $z$, where $x^n + y^n$ and $xy$ are also $\varepsilon$-invariant, but on $x^n - y^n$ and $z$ the reflection $\varepsilon$ acts by multiplying by $-1$.*

Next, we need the values of monomial valuations corresponding to all conjugacy classes (of age 1) in $G$.

LEMMA 5.3.3. *The values of monomial valuations on given generators of $\mathcal{R}(\mathbb{C}^3/G)$ are as follows:*

| $val \backslash gen$ | $x^n + y^n$ | $x^n - y^n$ | $xy$ | $z$ |
|---|---|---|---|---|
| $\nu_{\rho^i}$ | $\frac{in}{\gcd(n,i)}$ | $\frac{in}{\gcd(n,i)}$ | $\frac{n}{\gcd(n,i)}$ | $0$ |
| $\nu_\varepsilon$ | $0$ | $1$ | $0$ | $1$ |

Let $\varphi : X \to \mathbb{C}^3/G$ be a crepant resolution.

THEOREM 5.3.4. *The Cox ring $\mathcal{R}(X)$ as a $\mathbb{C}$-subalgebra of $\mathbb{C}[x,y,z]^{[G,G]}[t_\varepsilon^{\pm 1}, t_i^{\pm 1} \colon i \in \{1,\dots,k\}]$ is generated by*

$$(x^n+y^n)\prod_{i=1}^k t_i^{\frac{in}{\gcd(n,i)}},\ (x^n-y^n)t_\varepsilon\prod_{i=1}^k t_i^{\frac{in}{\gcd(n,i)}},\ xy\prod_{i=1}^k t_i^{\frac{n}{\gcd(n,i)}},\ zt_\varepsilon,$$
$$t_\varepsilon^{-2},\ \{t_i^{-\frac{n}{\gcd(n,i)}} \colon i\in\{1,\dots,k\}\}.$$

PROOF. It follows from a more general result, theorem 5.2.3. One can also give an alternative proof based on theorem 4.3.3, which we sketch below. □

We define the surjective map

$$\kappa\colon \mathbb{C}[Z_1,Z_2,Z_3,Z_4,T_\varepsilon,T_i\colon i\in\{1,\dots,k\}] \longrightarrow \mathcal{R}(X) \tag{5.3.5}$$

which sends each variable to the respective generator of $\mathcal{R}(X)$.

COROLLARY 5.3.6. *The ideal $I_k = \ker\kappa$, corresponding to the embedding of* $\operatorname{Spec}\mathcal{R}(X)$ *in $\mathbb{C}^{k+5}$, is generated by a single trinomial*

$$Z_1^2 - Z_2^2 T_\varepsilon - 4Z_3^n \prod_{i=1}^k T_i^{n-2i}. \tag{5.3.7}$$

AN ALTERNATIVE PROOF OF THEOREM 5.3.4. We will use theorem 4.3.3.
Denote the $\mathbb{C}$-algebra generated by elements from the statement by $\mathcal{R}$. We prove first that $\mathcal{R} = \mathbb{C}[Z_1,Z_2,Z_3,Z_4,T_\varepsilon,T_i\colon i\in\{1,\dots,k\}]/I_k$. That is we give an alternative proof of corollary 5.3.6. One verifies directly that $I_k \subset \ker\kappa$. On the other hand the polynomial from the statement generates the prime ideal (e.g. by the Eisenstein criterion for irreducibility [**38**, Proposition 9.4.13]). Corollary 5.3.6 follows now by comparing Krull dimensions of $\mathcal{R}(X)$ and $\mathbb{C}[Z_1,Z_2,Z_3,Z_4,T_1,\dots,T_k,T_\varepsilon]/I_k$.
Using the Eisenstein criterion again we can check that $(T_i)+I_k$ and $(T_\varepsilon)+I_k$ are prime ideals, dividing first by the variable, and then by the image of the generator of $I_k$ in the quotient polynomial ring. In particular the elements $\kappa(T_i)$ and $\kappa(T_\varepsilon)$ are prime.
Next we will show that the set of stable points of $\operatorname{Spec}\mathcal{R}$ with respect to the Picard torus action induced by grading on $\mathcal{R}$ is contained in the smooth locus of $\operatorname{Spec}\mathcal{R}$. We have an explicit description of the singular locus:

$$\operatorname{Sing}(\operatorname{Spec}\mathcal{R}) = \{Z_1=Z_2=Z_3=0\} \cup \bigcup_{i=1}^{k-1}\{Z_1=Z_2=T_i=0\},$$

which follows from Jacobian criterion for smoothness. By the description of the cone $\operatorname{Mov}(\mathcal{R})$ in proposition 5.3.10 and its GIT subdivision in proposition 5.3.11 it follows that $\operatorname{Sing}(\operatorname{Spec}\mathcal{R}) \cap (\operatorname{Spec}\mathcal{R})^{ss}(\lambda) = \varnothing$ for any $\lambda$ inside a GIT chamber inside $\operatorname{Mov}(\mathcal{R})^\circ$. Moreover, all the points in $(\operatorname{Spec}\mathcal{R})^{ss}(\lambda)$ have trivial isotropy groups, in particular the quotient $(\operatorname{Spec}\mathcal{R})^{ss}(\lambda)/\!/\mathbb{T}$ is smooth by the Luna slice theorem [**69**]. The set of points of $\operatorname{Spec}\mathcal{R}$ unstable with respect to such $\lambda$ has codimension at least two. Therefore we may conclude by theorem 4.3.3. □

*The Mov cone and the set of all crepant resolutions.* The monomial valuations from lemma 5.3.3 yield a description of the action of the Picard torus $\mathbb{T} = \operatorname{Hom}(\operatorname{Cl}(X),\mathbb{C}^*)$ on $\operatorname{Spec}\mathcal{R}(X)$ in the following lemma:

LEMMA 5.3.8. *The class group* $\mathrm{Cl}(X)$ *as a subgroup of* $\Lambda_X$ *is generated by* $E_1, \ldots, E_k$ *and* $\frac{1}{2}E_\varepsilon$.

PROOF. The class group contains the group $M$ generated by degrees of generators of $\Theta(\mathcal{R}(X))$. This group is equal to the group generated by $E_1, \ldots, E_k$ and $\frac{1}{2}E_\varepsilon$, in particular it contains the group generated by the components of the exceptional divisor as a subgroup of index 2. By corollary 4.2.2 the group $\mathrm{Cl}(X)$ also contains group generated by the components of the exceptional divisor as a subgroup of index 2 as $\mathrm{Cl}(\mathbb{C}^3/G) \cong \mathrm{Ab}(G)^\vee$ is of order 2. Hence $\mathrm{Cl}(X) = M$. $\square$

In particular the matrix of the $\mathbb{T}$-action on $\operatorname{Spec}\mathcal{R}(X)$ is:

$$U_k := \left(\begin{array}{c|cc|cccccc|c} 1 & 1 & 0 & 1 & -1 & 0 & & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & -1 & & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ k & k & 0 & 1 & 0 & 0 & & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & & 0 & 1 \end{array}\right)$$

The first three groups of columns correspond to sets of variables in monomials in the trinomial relation defining $\operatorname{Spec}\mathcal{R}(X)$ in corollary 5.3.7, and the last one corresponds to $Z_4$ not involved in the relation.

We are ready to describe the cone $\mathrm{Mov}(\mathcal{R})$ in $N^1(X) = \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^{k+1}$. We denote the standard basis by $e_1, \ldots, e_{k+1}$. Note that in these coordinates the exceptional divisors are $E_i = -e_i$ for $i \in \{1, \ldots, k\}$ and $E_\varepsilon = -2e_{k+1}$. We also use the following vectors:

$$\begin{aligned} q &= e_{k+1} = (0, \ldots, 0, 1), \\ v_i &= (1, 2, \ldots, i-1, i, \ldots, i, 0) \quad \text{for } i \in \{1, \ldots, k\}, \\ w_i &= (1, 2, \ldots, i, \ldots, i, 1) = v_i + q \quad \text{for } i \in \{1, \ldots, k\}. \end{aligned} \tag{5.3.9}$$

PROPOSITION 5.3.10. *The cone* $\mathrm{Mov}(X) = \mathrm{Mov}(\mathcal{R})$ *is spanned by rays* $q, v_1, \ldots, v_k$ *and defined by the inequalities*

$$\{(a_1, \ldots, a_{k+1}) \ : \ a_{k+1} \geq 0,\ 2a_1 \geq a_2,\ a_k \geq a_{k-1},\ 2a_i \geq a_{i-1} + a_{i+1},\ \ 1 < i < k\}.$$

PROOF. First, it is easy to check that the cones defined by given rays and by given inequalities are equal: obviously all the rays satisfy the inequalities, and one can construct inductively a positive combination of the rays equal to a vector satisfying the inequalities. We denote this cone by $\tau_k$.

Then by proposition 4.1.20 we have to show that $\tau_k$ is equal to $\mathrm{Mov}(\mathcal{R})$ defined as the intersection of the images of facets of the positive orthant $\gamma \subset \mathbb{Q}^{k+5}$ under $U_k$. That is, we describe $\mathrm{Mov}(\mathcal{R})$ as the intersection of cones spanned by any set of $k+4$ columns of $U_k$. To prove $\tau_k \subset \mathrm{Mov}(\mathcal{R})$ we look at the rays of $\tau_k$ and show that they belong to the image of any facet. And for $\tau_k \supset \mathrm{Mov}(\mathcal{R})$ we use the inequalities for $\tau_k$: for each of them one can find a facet whose image satisfies the considered inequality. We skip the details of this purely combinatorial argument. $\square$

PROPOSITION 5.3.11. *There are* $k+1$ *GIT chambers of* $\mathrm{Mov}(\mathcal{R}) = \mathrm{Mov}(X)$*, which are relative interiors of the cones:*

$$\sigma_0 = cone(q, w_1, \ldots, w_k),\ \sigma_i = cone(v_1, v_2, \ldots, v_i, w_i, w_{i+1}, \ldots, w_k) \text{ for } i \in \{1, \ldots, k\}$$

*where* $q$, $v_i$ *and* $w_i$ *are as in* (5.3.9).

To give the proof we need to describe projected $I_k$-faces of the positive orthant $\gamma \subset \mathbb{Q}^{k+5}$ under $U_k$ which are big enough to cut out a full-dimensional cone.

LEMMA 5.3.12. *The projected $I_k$-faces of dimension $k+1$ are spanned by the sets of rays given below. Note that indices $i \leq j$ (possibly equal!) are always from $\{1,\dots,k\}$.*

$$\{v_1,-e_1,\dots,-e_k,\pm e_{k+1}\},\{v_1,-e_1,\dots,-e_k,e_{k+1}\},\{v_1,-e_1,\dots,-e_{k+1}\}, \tag{5.3.13}$$

$$\begin{aligned}&\{q,v_1,w_k,-e_1,\dots,-e_{k+1}\}\setminus\{-e_i,-e_j\},\ \{q,w_k,-e_1,\dots,-e_{k+1}\}\setminus\{-e_i\},\\ &\{v_1,w_k,-e_1,\dots,-e_{k+1}\}\setminus\{-e_i,-e_j\},\quad \{w_k,-e_1,\dots,-e_{k+1}\}\setminus\{-e_i\},\end{aligned} \tag{5.3.14}$$

$$\{-e_1,\dots,-e_k,\pm e_{k+1}\},\quad \{-e_1,\dots,-e_k,-e_{k+1}\},\quad \{-e_1,\dots,-e_k,e_{k+1}\}, \tag{5.3.15}$$

$$\{q,v_1,w_k,-e_1,\dots,-e_k\}\setminus\{-e_i,-e_j\},\quad \{q,w_k,-e_1,\dots,-e_k\}\setminus\{-e_i\}, \tag{5.3.16}$$

$$\{v_1,-e_1,\dots,-e_{k+1}\}\setminus\{-e_i\},\quad \{q,v_1,-e_1,\dots,-e_{k+1}\}\setminus\{-e_i\}. \tag{5.3.17}$$

PROOF. By definition 3.3.14 we need to choose all sets of columns of $U_k$ which span $\mathbb{Q}^{k+1}$ such that if we set all variables corresponding to *not* chosen columns to 0 in the trinomial (5.3.7) then we still can find nonzero values for the remaining variables to make the trinomial vanish. This last condition means that we may set to 0 precisely 3, 1 or 0 monomials in the trinomial (by choosing columns corresponding to any sets of their variables), but not 2 of them. In this way we make the list of all possible sets given above, checking if they span $\mathbb{Q}^k$, simplifying them and avoiding redundancies.

If the third monomial $Z_3\prod_i T_i$ is nonzero, then we can get only the whole space or a halfspace, (5.3.13). Four sets in (5.3.14) correspond to setting only the third monomial to 0: note that we may remove at most 2 columns from the third group to have a set spanning $\mathbb{Q}^{k+1}$. The last three cases describe the situation with all three monomials set to 0. In (5.3.15) we have $Z_1=Z_2=Z_3=0$, in (5.3.16) $Z_1=T_\varepsilon=Z_3\prod_i T_i=0$ (note that here we need the last column to span the whole $\mathbb{Q}^{k+1}$) and in (5.3.17) $Z_1=Z_2=\prod_i T_i=0$. □

PROOF OF PROPOSITION 5.3.11. First, note that $\mathrm{Mov}(X)$ is indeed a sum of given cones. All the cones are contained in $\mathrm{Mov}(X)$ since all their rays lie in this cone. For the other inclusion, take $v=aq+a_1v_1+\ldots+a_kv_k\in\mathrm{Mov}(X)$. If $a\geq a_1+\ldots+a_k$ then $v$ is in $\sigma_0$:

$$v=(a-a_1-\ldots-a_k)q+a_1w_1+\ldots+a_kw_k.$$

Otherwise, let $j$ be maximal such that $a<a_j+\ldots+a_k$. Let $S_j=a_j+\ldots+a_k$. Then $v$ can be written as an element of $\sigma_j$:

$$v=a_1v_1+\ldots+a_{j-1}v_{j-1}+(S_j-a)v_j+(a-S_{j+1})w_j+a_{j+1}w_{j+1}+\ldots+a_kw_k.$$

Next, we prove that projected $I_k$-faces do not subdivide given chambers. That is, any projected $I_k$-face of maximal dimension either contains a whole chamber, or does not contain any of its points. It can be checked directly, by showing for each chamber and each type of $I_k$-face of maximal dimension, that either all rays of the chamber lie within the $I_k$-face or that there is a hyperplane separating interiors of these cones. An easy way for finding such a hyperplane is to determine it by computation for small $k$ and then deduce the equation in the general case. Since the results are similar for all types of $I_k$-faces, we present here just a single case, one of the least trivial.

Consider $I_k$-faces of the first type from (5.3.16). If $j=i$ or $j=i+1$ then such an $I_k$-face contains chambers $\sigma_0,\dots,\sigma_i$, because one can construct $v_1,\dots,v_i$ from its rays as combinations of $mv_1$ and negatives of basis vectors, and $w_1,\dots,w_k$ either as $v_i+q$ or as combinations of $w_k$ and negatives of basis vectors. The hyperplane $a_{i+1}-a_i=a_{k+1}$ separates interior of such an $I_k$-face and chambers $\sigma_{i+1},\dots,\sigma_k$. Now assume that $j-i>1$.

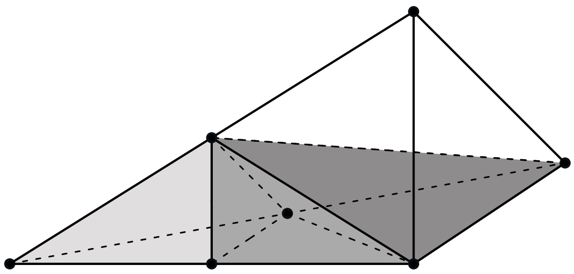
Then the hyperplane $a_i + a_j = a_{i+1} + a_{j-1}$ separates the interior of the $I_k$-face and any of the chambers (i.e. the whole $\mathrm{Mov}(X)$).

Finally, we have to show that any chamber can be separated from any other using a projected $I_k$-face. But this follows from the previous paragraph: $I_k$-faces of the first type from (5.3.16) separate $\sigma_i$ from $\sigma_{i+1}, \dots, \sigma_k$ for $i = 0, \dots, k-1$. □

EXAMPLE 5.3.18. For $D_{14}$, that is $k = 3$, we give the equation for the Cox ring of crepant resolutions and the matrix of the Picard torus action. The picture shows a 3-dimensional slice of the Mov cone subdivided into chambers. Three chambers, sharing an edge on the back face of the tetrahedron, are shown in grey, the last one is white. One sees that there is only one way of walking through chambers, starting from the leftmost or the topmost, i.e. crepant resolutions and flops form a sequence.

$$\left(\begin{array}{c|cc|cccc|c} 1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 1 & 0 & -1 & 0 & 0 \\ 3 & 3 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$Z_1^2 - Z_2^2 T_\varepsilon - 4 Z_3^7 T_1^5 T_2^3 T_3$$



REMARK 5.3.19. In fact, it follows from the proof of the Proposition 5.3.11 that for any $k$ flops of crepant resolutions form a sequence, as in the example above.

*Structure of the central fibre.* We describe the structure of the central fibre of the resolution, check how it changes under a flop and find the chamber corresponding to the $G$-Hilb resolution.

PROPOSITION 5.3.20. *If $G$ is a representation of a dihedral group defined as in* (5.3.1) *then* $\mathbb{C}[x,y,z]^G = \mathbb{C}[x^n + y^n, (x^n - y^n)z, xy, z^2]$.

PROOF. We take a generating set of the invariants of the first (diagonal) generator $\rho$ and modify it to obtain eigenvectors of the action of the second generator $\varepsilon$. (In the case of $n$ odd we have already seen this, since $\rho$ generates $[G,G]$ and $\varepsilon$ determines the action of $\mathrm{Ab}(G)$.) Thus we have an induced diagonal action of $\varepsilon$ on the invariant ring of $\rho$, so we just need to determine invariant monomials in generators of this ring. □

The ideal of $\mathbb{C}[Z_1, Z_2, Z_3, Z_4, T_\varepsilon, T_i \colon i \in \{1, \dots, k\}]$ of the closure of the subset of $\operatorname{Spec} \mathcal{R}(X)$ which is mapped to the central fibre of the resolution by the GIT quotient morphism is the inverse image under $\kappa$, see (5.3.5), of the ideal generated by the nonconstant elements of $\mathbb{C}[x,y,z]^G$ in $\mathcal{R}(X)$. This is the same as the ideal generated by $I_k$ and the Picard torus invariants.

COROLLARY 5.3.21. *The ideal of the closure of the subset of* $\operatorname{Spec} \mathcal{R}(X) \subset \mathbb{C}^{k+5}$ *mapped to the central fibre of $X$ is*

$$J_k = I_k + (Z_1 T_1 T_2^2 \cdots T_k^k,\ Z_2 Z_4 T_\varepsilon T_1 T_2^2 \cdots T_k^k,\ Z_3 T_1 T_2 \cdots T_k,\ Z_4^2 T_\varepsilon).$$

LEMMA 5.3.22. *The subset of* $\operatorname{Spec} \mathcal{R}(X) \subset \mathbb{C}^{k+5}$ *mapped to the central fibre of the resolution $X$ decomposes into the following irreducible components:*

$$W_i = \{Z_4 = T_i = Z_1^2 - Z_2^2 T_\varepsilon = 0\} \text{ and } W_i' = \{Z_1 = T_\varepsilon = T_i = 0\} \text{ for } i \in \{1, \dots, k\},$$
$$W_0 = \{Z_1 = Z_3 = T_\varepsilon = 0\}, \quad W_u = \{Z_1 = Z_2 = Z_3 = Z_4 = 0\}.$$

PROOF. One can prove directly that the radical of $J_k$ is equal to the intersection of all ideals listed above; the key observation is that $Z_1^2 - Z_2^2 T_\varepsilon$ is in the radical of $J_k$. $\square$

Let $\varphi_i \colon X_i \to \mathbb{C}^3/G$ be the resolution corresponding to the choice of the linearisation of the Picard torus action given by a character from the chamber inside the cone $\sigma_i$.

PROPOSITION 5.3.23. *For $X_i$ the stable components of the set mapped to the central fibre are $W_1, \ldots, W_i, W_i', \ldots, W_k'$. For $X_0$ the stable components are $W_0, W_1', \ldots, W_k'$.*

PROOF. We show that $W_j'$ for $j = 1, \ldots, i-1$ are not stable on $X_i$; the remaining assertions are proved similarly. To fix a linearisation corresponding to $X_i$ we pick a vector $s_i$ from $\sigma_i$ which is the sum of its rays. We need to show that it does not belong to the interior of the orbit cone $\omega_j'$ corresponding to an open subset of $W_j'$. The cone $\omega_j'$ is spanned by all columns of $U_k$ except the ones corresponding to $Z_1$, $T_\varepsilon$ and $T_j$.

Take a positive combination of rays of $\omega_j'$ with coefficient $\alpha$ at $(1, 2, \ldots, k, 1)$ and $\beta$ at $(1, \ldots, 1, 0)$, and assume it gives $s_i$. From a direct computation of coordinates of $s_i$ ($j$, $j+1$ and $k+1$ respectively) and the negativity of certain rays of $\omega_j'$ we get three conditions:

$$\begin{aligned} j\alpha + \beta &= (k(k+1) - (k-j)(k-j+1))/2 + j, \\ (j+1)\alpha + \beta &\geq (k(k+1) - (k-j-1)(k-j))/2 + j + 1, \\ \alpha &\leq k - j + 1. \end{aligned}$$

This leads to $k - i + 1 \geq \alpha \geq k - j + 1$, which contradicts the assumption $j < i$.

The investigation of stability of $W_i$ relies on the fact that its open subset corresponds to an orbit cone spanned by all columns of $U_k$ except these corresponding to $Z_4$ and $T_i$. Finally, note that in a similar way one could list all orbit cones corresponding to stable points of $\operatorname{Spec} \mathcal{R}(X)$, which we skip for the sake of brevity. $\square$

Using similar methods to investigate the stability of the intersections of components $W_0, W_1, \ldots, W_k, W_1', \ldots, W_k'$ one also checks that the components of the central fibre form a chain of smooth rational curves.

COROLLARY 5.3.24. *By analysing the proof of Theorem 5.1 in* [**79**] *one checks that the* $G$-Hilb *resolution corresponds to the chamber inside the cone $\sigma_0$.*

**5.3.2. The even case: $n = 2k$.** Since we have presented the details in the case of $D_{2n}$ for odd $n$, here we skip some details and arguments if they are the same as in the previous case.

The commutator subgroup consists of all even rotations: $[G, G] = \langle \rho^2 \rangle$. Then $\mathrm{Ab}(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ and we use classes of $\varepsilon$ and $\varepsilon\rho$ as its generators. Pairs of mutually inverse rotations are conjugate and reflections make two conjugacy classes, so there are $k + 3$ conjugacy classes:

$$\{1\},\ \{\rho, \rho^{-1}\}, \ldots, \{\rho^{k-1}, \rho^{-k+1}\},\ \{\rho^k\},\ \{\varepsilon, \varepsilon\rho^2, \ldots, \varepsilon\rho^{2k-2}\},\ \{\varepsilon\rho, \varepsilon\rho^3, \ldots, \varepsilon\rho^{2k-1}\}.$$

The set of points in $\mathbb{C}^3$ with nontrivial isotropy group consists of the line $x_1 = x_2 = 0$ fixed by $\langle \rho \rangle$ and $n$ lines, each fixed by a reflection, e.g. $x_1 - x_2 = x_3 = 0$ fixed by $\langle \varepsilon \rangle$ and $x_1 - \zeta x_2 = x_3 = 0$ fixed by $\langle \varepsilon\rho \rangle$. In the quotient $\mathbb{C}^3/G$ these lines are mapped to three components of the singular points set: a component $L_\rho$ with transversal $A_{2k-1}$ singularity and two components $L_\varepsilon$ and $L_{\varepsilon\rho}$ with transversal $A_1$ singularity (away from 0). The image of 0 has a worse singularity.

By the McKay correspondence we have $k$ exceptional divisors $E_1, \ldots, E_k$ mapped to $L_\rho$ and $E_\varepsilon, E_{\varepsilon\rho}$ mapped to $L_\varepsilon, L_{\varepsilon\rho}$ respectively.

*The Cox ring.* We compute the ring of $[G,G]$-invariants, give a generating set of eigenvectors of the $\mathrm{Ab}(G)$ action and provide values of all monomial valuations corresponding to conjugacy classes (of age 1) on these generators.

LEMMA 5.3.25. *$\mathcal{R}(\mathbb{C}^3/G) = \mathbb{C}[x,y,z]^{[G,G]}$ is generated by $x^k+y^k$, $x^k-y^k$, $xy$, $z$, where $\varepsilon$ acts trivially on $x^k+y^k$ and $xy$ and multiplies by $-1$ the remaining two generators, and $\varepsilon\rho$ acts trivially on $x^k-y^k$ and $z$ and multiplies by $-1$ the remaining two generators.*

LEMMA 5.3.26. *The values of monomial valuations on given generators of $\mathcal{R}(\mathbb{C}^3/G)$ are as follows:*

| *val**gen* | $x^k+y^k$ | $x^k-y^k$ | $xy$ | $z$ |
|---|---|---|---|---|
| $\nu_{\rho^i}$ | $\frac{ik}{\gcd(n,i)}$ | $\frac{ik}{\gcd(n,i)}$ | $\frac{n}{\gcd(n,i)}$ | *0* |
| $\nu_\varepsilon$ | *0* | *1* | *0* | *1* |
| $\nu_{\varepsilon\rho}$ | *1* | *0* | *0* | *1* |

Let $\varphi : X \to \mathbb{C}^3/G$ be a crepant resolution.

THEOREM 5.3.27. *The Cox ring $\mathcal{R}(X)$ as a $\mathbb{C}$-subalgebra of $\mathbb{C}[x,y,z]^{[G,G]}[t_\varepsilon^{\pm 1}, t_{\varepsilon\rho}^{\pm 1}, t_i^{\pm 1} : i \in \{1,\ldots,k\}]$ is generated by:*

$$(x^k+y^k)t_{\varepsilon\rho}\prod_{i=1}^{k} t_i^{\frac{ik}{\gcd(n,i)}},\ (x^k-y^k)t_\varepsilon\prod_{i=1}^{k} t_i^{\frac{ik}{\gcd(n,i)}},\ xy\prod_{i=1}^{k} t_i^{\frac{n}{\gcd(n,i)}},\ zt_\varepsilon t_{\varepsilon\rho},$$
$$t_\varepsilon^{-2},\ t_{\varepsilon\rho}^{-2},\ \{t_i^{-\frac{n}{\gcd(n,i)}} : i \in \{1,\ldots,k\}\}.$$

We define the surjective map

$$\kappa : \mathbb{C}[Z_1,Z_2,Z_3,Z_4,T_\varepsilon,T_{\varepsilon\rho},T_i : i \in \{1,\ldots,k\}] \longrightarrow \mathcal{R}(X) \tag{5.3.28}$$

which sends each variable to the respective generator of $\mathcal{R}(X)$.

COROLLARY 5.3.29. *The ideal $I_k = \ker\kappa$ of $\operatorname{Spec}\mathcal{R}(X) \subset \mathbb{C}^{k+6}$ is generated by*

$$Z_1^2T_{\varepsilon\rho} - Z_2^2T_\varepsilon - 4Z_3^k\prod_{i=1}^{k-1} T_i^{k-i}.$$

*The Mov cone and the set of all crepant resolutions.* The matrix of weights of the Picard torus action on $\operatorname{Spec}\mathcal{R}(X) \subset \mathbb{C}^{k+6}$, given by values of monomial valuations listed above, subdivided into groups of columns corresponding to monomials in the trinomial defining $\operatorname{Spec}\mathcal{R}(X)$, is

$$U_k = \left(\begin{array}{cc|cc|ccccc|cc}
1 & 0 & 1 & 0 & 2 & -2 & 0 & & 0 & 0 & 0\\
2 & 0 & 2 & 0 & 2 & 0 & -2 & & 0 & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
k-1 & 0 & k-1 & 0 & 2 & 0 & 0 & & -2 & 0 & 0\\
k & 0 & k & 0 & 2 & 0 & 0 & & 0 & -2 & 0\\
0 & 0 & 1 & -2 & 0 & 0 & 0 & & 0 & 0 & 1\\
1 & -2 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & 1
\end{array}\right)$$

This follows from the description of $\mathrm{Cl}(X)$ as the subgroup of $\Lambda_X$ generated by $E_1,\ldots,E_k$ and $\frac{1}{2}E_\varepsilon, \frac{1}{2}E_{\varepsilon\rho}$.

Now we describe $\mathrm{Mov}(X) \subset N^1(X) = \mathrm{Cl}(X) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^{k+2}$. We also use the following vectors:

$$\begin{aligned} &q_1 = (0,\dots,0,1,0), \qquad q_2 = (0,\dots,0,0,1), \qquad q_3 = (0,\dots,0,1,1)\\ &u_i = (1,2,\dots,i-1,i,\dots,i,0,0) \text{ for } i \in \{1,\dots,k\},\\ &v_i = u_i + q_1 = (1,2,\dots,i-1,i,\dots,i,1,0) \text{ for } i \in \{1,\dots,k\},\\ &v_i' = u_i + q_2 = (1,2,\dots,i-1,i,\dots,i,0,1) \text{ for } i \in \{1,\dots,k\},\\ &w_i = 2u_i + q_3 = (2,4,\dots,2i-2,2i,\dots,2i,1,1) \text{ for } i \in \{1,\dots,k\}. \end{aligned} \tag{5.3.30}$$

The next two statements can be proved based on the same ideas as for $n$ odd, see 5.3.10 and 5.3.11, but they require more cases to check, hence we skip the details.

PROPOSITION 5.3.31. *The cone* $\mathrm{Mov}(\mathcal{R}) = \mathrm{Mov}(X)$ *is spanned by rays* $q_1, q_2, u_1, \dots, u_k$ *and defined by inequalities*

$$\{(a_1,\dots,a_{k+2})\colon a_{k+1} \geq 0,\ a_{k+2} \geq 0,\ 2a_1 \geq a_2,\ a_k \geq a_{k-1},\ 2a_i \geq a_{i-1} + a_{i+1},\ 1 < i < k\}.$$

PROPOSITION 5.3.32. *There are* $(k+1)^2$ *GIT chambers of* $\mathrm{Mov}(\mathcal{R}) = \mathrm{Mov}(X)$ *which are relative interiors of the cones:*

$$\begin{aligned} &\sigma_{0,0} = \mathit{cone}(q_1, q_3, v_1, \dots, v_k),\ \sigma_{0,0}' = \mathit{cone}(q_2, q_3, v_1', \dots, v_k'),\\ &\sigma_{0,i} = \mathit{cone}(q_3, w_1, \dots, w_i, v_i, \dots, v_k) \text{ for } i \in \{1,\dots,k-1\},\\ &\sigma_{0,i}' = \mathit{cone}(q_3, w_1, \dots, w_i, v_i', \dots, v_k') \text{ for } i \in \{1,\dots,k-1\},\\ &\sigma_{i,j} = \mathit{cone}(u_1, \dots, u_i, w_i, \dots, w_j, v_j, \dots, v_k) \text{ for } i,j \in \{1,\dots,k-1\},\ i \leq j,\\ &\sigma_{i,j}' = \mathit{cone}(u_1, \dots, u_i, w_i, \dots, w_j, v_j', \dots, v_k') \text{ for } i,j \in \{1,\dots,k-1\},\ i \leq j,\\ &\sigma_0'' = \mathit{cone}(q_3, v_k, v_k', w_1, \dots, w_{k-1}),\ \sigma_k'' = \mathit{cone}(u_1, \dots, u_k, v_k, v_k'),\\ &\sigma_i'' = \mathit{cone}(u_1, \dots, u_i, w_i, \dots, w_{k-1}, v_k, v_k') \text{ for } i \in \{1,\dots,k-1\}. \end{aligned}$$

*where all the rays are as defined in* (5.3.30).

REMARK 5.3.33. Similarly as in section 5.3.1 for $n$ odd, one can analyze the structure of the central fibre of crepant resolutions and the flops between different resolutions for $n = 2k$. The following diagram shows which chambers of the subdivision from proposition 5.3.32 have common facet, i.e. which resolutions corresponding to chambers differ by a flop. Here $(k+1)^2$ crepant resolutions are pictured in the form of an isosceles triangle with $2k+1$ resolutions at the base and the number of resolution in consecutive rows parallel to the base decreasing by 2. Flops can be performed between adjacent resolutions in rows and in columns.

$$\begin{tikzcd}[column sep=small]
\sigma_{0,0} \arrow[r, dash, "q_1/w_1"] & \sigma_{0,1} \arrow[r, dash, "v_1/w_2"] \arrow[d, dash, "q_3/u_1"] & \cdots \arrow[r, dash, "v_{k-2}/w_{k-1}"] & \sigma_{0,k-1} \arrow[r, dash, "v_{k-1}/v_k'"] \arrow[d, dash, "q_3/u_1"] & \sigma_0'' \arrow[r, dash, "v_k/v_{k-1}'"] \arrow[d, dash, "q_3/u_1"] & \sigma_{0,k-1}' \arrow[r, dash, "w_{k-1}/v_{k-2}'"] \arrow[d, dash, "q_3/u_1"] & \cdots \arrow[r, dash, "w_2/v_1'"] & \sigma_{0,1}' \arrow[r, dash, "w_1/q_2"] \arrow[d, dash, "q_3/u_1"] & \sigma_{0,0}' \\
& \sigma_{1,1} \arrow[r, dash, "v_1/w_2"] & \cdots \arrow[r, dash, "v_{k-2}/w_{k-1}"] & \sigma_{1,k-1} \arrow[r, dash, "v_{k-1}/v_k'"] \arrow[d, dash, "w_1/u_2"] & \sigma_1'' \arrow[r, dash, "v_k/v_{k-1}'"] \arrow[d, dash, "w_1/u_2"] & \sigma_{1,k-1}' \arrow[r, dash, "w_{k-1}/v_{k-2}'"] \arrow[d, dash, "w_1/u_2"] & \cdots \arrow[r, dash, "w_2/v_1'"] & \sigma_{1,1}' & \\
& & & \vdots \arrow[d, dash, "w_{k-2}/u_{k-1}"] & \vdots \arrow[d, dash, "w_{k-2}/u_{k-1}"] & \vdots \arrow[d, dash, "w_{k-2}/u_{k-1}"] & & & \\
& & & \sigma_{k-1,k-1} \arrow[r, dash, "v_{k-1}/v_k'"] & \sigma_{k-1}'' \arrow[r, dash, "v_k/v_{k-1}'"] \arrow[d, dash, "w_{k-1}/u_k"] & \sigma_{k-1,k-1}' & & & \\
& & & & \sigma_k'' & & & &
\end{tikzcd}$$

For the resolutions corresponding to $\sigma_{i,j}$ and $\sigma'_{i,j}$ with $i, j = 0, \dots, k$, $i \leq j$ the exceptional fibre is a chain of $k+2$ smooth rational curves. The passage from $\sigma_{i,j}$ to $\sigma_{i,j+1}$ flops the $(j+2)$-th curve of the chain, the passage from $\sigma_{i,k-1}$ to $\sigma''_i$ flops $(k+2)$-th curve and the passage from $\sigma_{i,j}$ to $\sigma_{i+1,j}$ flops the $(i+1)$-th curve of the chain. The change of models for $\sigma'_{i,j}$'s is analogous.

For the resolutions corresponding to $\sigma''_i$ with $i = 0, \dots, k-1$ the central fibre consists of a chain of $k$ smooth rational curves with two additional curves intersecting the $k$-th curve at the same point. The passage from $\sigma''_i$ to $\sigma''_{i+1}$ flops the $(i+1)$-th curve of the chain.

Finally, the central fibre of resolution corresponding to $\sigma''_k$ consists of smooth rational curves with dual graph of intersection equal to the Dynkin diagram $D_{k+2}$.

The $G$-Hilb resolution can be identified based on the proof of [**79**, Thm 5.2] as the central resolution on the base of the triangle, corresponding to the chamber inside the cone $\sigma''_0$.

## 5.4. Reducible examples with a divisor contracted to a point

In this section we apply the results of previous parts of the work to two examples in which the resolution contracts a divisor to a point. Recall that by McKay correspondence of Ito and Reid such divisors correspond to age two conjugacy classes of the group (2.2.27).

Example 5.4.1. Let $\overline{G} = (\mu_4 \mid \mu_2,\ \mathrm{BD}_{16} \mid \mathrm{BD}_8) = \left\langle \left(\begin{smallmatrix} 0 & i \\ i & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} \zeta_8^3 & 0 \\ 0 & \zeta_8 \end{smallmatrix}\right) \right\rangle$ and let $G$ be the corresponding subgroup of $\mathrm{SL}_3(\mathbb{C})$. Using GAP [**41**] we compute that $G$ has order 16 and has six nontrivial conjugacy classes, five of which have age 1. One may choose the following representatives of junior classes:

$$g_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad g_2 = \begin{pmatrix} 0 & \zeta_8 & 0 \\ \zeta_8^7 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and $g_3 = \mathrm{diag}(-1,-1,1)$, $g_4 = \mathrm{diag}(-i,i,1)$, $g_5 = \mathrm{diag}(\zeta_8^3, \zeta_8, -1)$.

Here $H = \mathrm{BD}_{16}$, so $[G,G] \simeq [\overline{G}, \overline{G}] = [\mathrm{BD}_{16}, \mathrm{BD}_{16}] = \left\langle \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right) \right\rangle$. By theorem 2.2.19, the generators of algebra of invariants $\mathbb{C}[x,y]^{[\overline{G},\overline{G}]}$ homogeneous with respect to $\mathrm{Ab}(H)$-action are $p_1 = x^4+y^4$, $p_2 = x^4-y^4$, $p_3 = xy$ with relation $Z_1^2 - Z_2^2 - 4Z_3^4$. Diagonalizing representatives of conjugacy classes of age 1 we compute values of corresponding monomial valuations $\nu_1, \dots, \nu_5$ on generators $p_1, p_2, p_3$ and $z$ of $\mathbb{C}[x,y,z]^{[G,G]}$:

| val\gen | $x^4+y^4$ | $x^4-y^4$ | $xy$ | $z$ |
|---|---|---|---|---|
| $\nu_1$ | 4 | 6 | 2 | 0 |
| $\nu_2$ | 1 | 0 | 0 | 1 |
| $\nu_3$ | 4 | 4 | 2 | 0 |
| $\nu_4$ | 4 | 4 | 4 | 0 |
| $\nu_5$ | 4 | 4 | 4 | 4 |

By theorem 5.2.3, the Cox ring $\mathcal{R}$ of a crepant resolution $X \to \mathbb{C}^3/G$ is isomorphic to $\mathbb{C}[Z_1, Z_2, Z_3, Z_4, T_1, \dots, T_5]/(Z_1^2T_2 - Z_2^2T_1 - 4Z_3^4T_4^2T_5)$. As in lemma 5.3.8 we see that $\mathrm{Cl}(X) \subset \Lambda_X$ is generated by $\frac{1}{2}(E_1+E_2), \frac{1}{2}(E_1+E_5), \frac{1}{2}(E_2+E_5), E_3, E_4$. Identifying $E_i = -2e_i, i = 1,2,5$ and $E_3 = -e_3, E_4 = -e_4$ for the canonical basis $e_1, \dots, e_5$ of $\mathbb{Q}^5$ we have the degree matrix for indicated generators of $\mathcal{R}$:

$$\left(\begin{array}{cc|cc|ccc|cc}
2 & 0 & 3 & -2 & 1 & 0 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
2 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & -1 \\
1 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & -2 & 1 & 0
\end{array}\right),$$

where the first three groups of columns correspond to variables in monomials of the relation, and the last two columns to variables $Z_4$ and $T_3$, not involved in the relation.

Now we compute the movable cone of $X$ and its chamber decomposition. The rays of $\mathrm{Mov}(X) \subset \mathbb{R}^5$ are

$$\begin{array}{lll}
v_1 = (0,0,0,0,1), & v_2 = (0,1,0,0,1), & v_3 = (2,0,2,1,1), \\
v_4 = (1,0,1,1,1), & v_5 = (1,1,1,1,1), & v_6 = (3,0,2,1,1), \\
v_7 = (4,2,3,2,2), & v_8 = (2,1,2,1,1), & v_9 = (6,3,4,2,3).
\end{array}$$

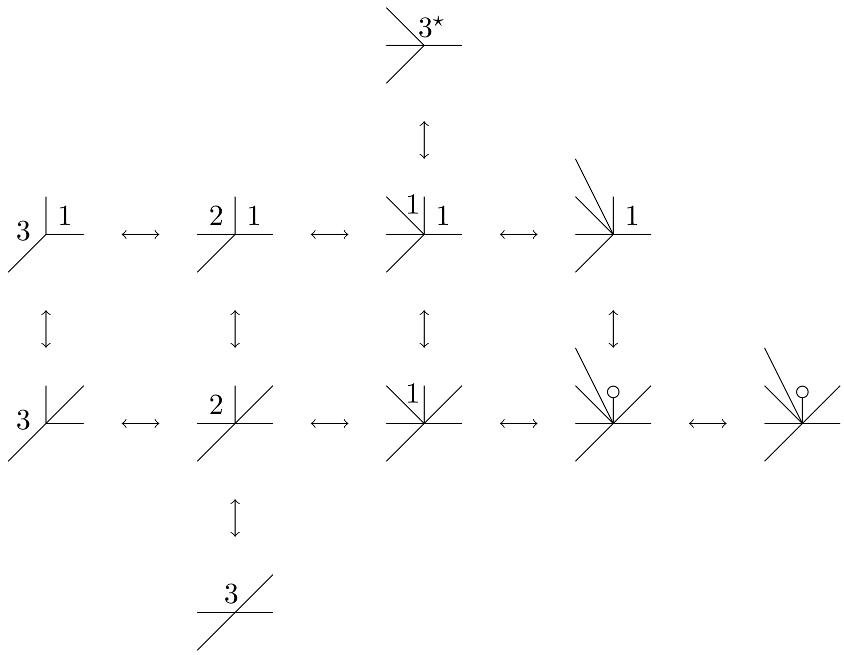
Let $w_1 = (2,0,2,1,2)$, $w_2 = (3,0,2,1,3)$, $w_3 = (3,1,2,1,2)$. Then the 11 (simplicial) chambers in $\mathrm{Mov}(X)$, corresponding to all crepant resolutions of $\mathbb{C}^3/G$, are the relative interiors of the cones:

$$\begin{array}{ll}
\sigma_1 = cone(v_6, v_7, v_8, v_9, w_3), & \sigma_2 = cone(v_2, v_7, v_8, v_9, w_3), \\
\sigma_3 = cone(v_4, v_6, v_7, v_8, w_3), & \sigma_4 = cone(v_2, v_4, v_7, v_8, w_3), \\
\sigma_5 = cone(v_4, v_6, v_8, w_1, w_3), & \sigma_6 = cone(v_4, v_6, w_1, w_2, w_3), \\
\sigma_7 = cone(v_2, v_4, v_8, w_1, w_3), & \sigma_8 = cone(v_2, v_4, v_5, v_7, v_8), \\
\sigma_9 = cone(v_2, v_4, w_1, w_2, w_3), & \sigma_{10} = cone(v_3, v_4, v_6, v_8, w_1), \\
\sigma_{11} = cone(v_1, v_2, v_4, w_1, w_2). &
\end{array}$$

Flops between resolutions, i.e. pairs of adjacent chambers, are shown in the diagram below; a label $v/w$ at an edge means that in the set of rays $v$ is replaced by $w$.

$$\begin{tikzcd}
 & & \sigma_{10} \arrow[d, dash, "v_3/w_3"] & & \\
\sigma_1 \arrow[r, dash, "v_9/v_4"] \arrow[d, dash, "v_6/v_2"] & \sigma_3 \arrow[r, dash, "v_7/w_1"] \arrow[d, dash, "v_6/v_2"] & \sigma_5 \arrow[r, dash, "v_8/w_2"] \arrow[d, dash, "v_6/v_2"] & \sigma_6 \arrow[d, dash, "v_6/v_2"] & \\
\sigma_2 \arrow[r, dash, "v_9/v_4"] & \sigma_4 \arrow[r, dash, "v_7/w_1"] \arrow[d, dash, "w_3/v_5"] & \sigma_7 \arrow[r, dash, "v_8/w_2"] & \sigma_9 \arrow[r, dash, "w_3/v_1"] & \sigma_{11} \\
 & \sigma_8 & & &
\end{tikzcd}$$

The following picture presents the changes of the central fibre between resolutions. Denote by $X_i \to \mathbb{C}^3/G$ the resolution corresponding to the chamber inside the cone $\sigma_i$. It turns out that the exceptional divisor contained in the central fibre of $X_i$ is a smooth projective toric surface $S_i$. The remaining part of the central fibre consists of chains of smooth rational curves attached to this toric surface at some fixed points of the torus action. For each resolution we draw the fan of $S_i$ and mark the maximal cones corresponding to appropriate fixed points with the number equal to the length of chain attached at this point. For the chain marked by $3^\star$ the point of intersection with the surface $S_{10}$ lies at the second curve of the chain. The remaining chains intersect the surface in the point lying at the curve which is the end of the chain. All flops except $X_9 \dashrightarrow X_{11}$ contract the curve from a chain and blow-up the toric surface in the intersection point. Passage from $X_9$ to $X_{11}$ flops the curve corresponding to the marked ray of the fan. Note that $S_1 \cong \mathbb{P}^2$, $S_8 \cong \mathbb{P}^1 \times \mathbb{P}^1$, $S_{10}$ is the second Hirzebruch surface and the remaining $S_i$'s are toric blow-ups of $\mathbb{P}^2$.



It would be interesting to relate this description to the previous work on the crepant resolutions of quotient singularities in dimension three by answering the following:

Open question 5.4.2. Which cone $\sigma_i$ corresponds to the resolution given by the $G$-Hilbert scheme?

Example 5.4.3. Consider $\overline{G} = (\mu_8 \mid \mu_4,\ \mathrm{BD}_{12} \mid C_6) = \left\langle \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^5 \end{pmatrix}, \begin{pmatrix} 0 & \zeta_8^3 \\ \zeta_8^3 & 0 \end{pmatrix} \right\rangle$ of order 24. Let $G$ be the corresponding subgroup of $\mathrm{SL}_3(\mathbb{C})$. It has seven conjugacy classes $[g_1], \dots, [g_7]$ of age 1 and four of age 2.

The representatives of junior classes are $g_1 = \mathrm{diag}(-1,-1,1)$, $g_2 = \mathrm{diag}(\zeta_6^5, \zeta_6, 1)$, $g_3 = \mathrm{diag}(\zeta_3^2, \zeta_3, 1)$, $g_4 = \mathrm{diag}(i, i, -1)$, $g_5 = \mathrm{diag}(\zeta_{12}, \zeta_{12}^5, -1)$ and

$$g_6 = \begin{pmatrix} 0 & \zeta_8^5 & 0 \\ \zeta_8^5 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}, \qquad g_7 = \begin{pmatrix} 0 & \zeta_8 & 0 \\ \zeta_8 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

In this case $H = \mathrm{BD}_{12}$, so $[G,G] \simeq [\overline{G},\overline{G}] = [\mathrm{BD}_{12}, \mathrm{BD}_{12}] = \left\langle \begin{pmatrix} \zeta_3 & 0 \\ 0 & \zeta_3^2 \end{pmatrix} \right\rangle$. Theorem 2.2.19 implies that generators of the ring of invariants $\mathbb{C}[x,y]^{[\overline{G},\overline{G}]}$ homogeneous with respect to $\mathrm{Ab}(H)$-action are $p_1 = x^3 + y^3$, $p_2 = x^3 - y^3$, $p_3 = xy$ with relation $Z_1^2 - Z_2^2 - 4Z_3^3$. As before, diagonalizing representatives of conjugacy classes of elements of age 1 we compute values of corresponding monomial valuations $\nu_1, \dots, \nu_7$ on generators of $\mathbb{C}[x,y,z]^{[G,G]}$.

| val\gen | $x^3+y^3$ | $x^3-y^3$ | $xy$ | $z$ |
|---|---|---|---|---|
| $\nu_1$ | 3 | 3 | 2 | 0 |
| $\nu_2$ | 3 | 3 | 6 | 0 |
| $\nu_3$ | 3 | 3 | 3 | 0 |
| $\nu_4$ | 3 | 3 | 2 | 2 |
| $\nu_5$ | 3 | 3 | 6 | 6 |
| $\nu_6$ | 7 | 3 | 2 | 2 |
| $\nu_7$ | 3 | 7 | 2 | 2 |

By Theorem 5.2.3, the Cox ring of a crepant resolution $X \to \mathbb{C}^3/G$ is isomorphic to $\mathbb{C}[Z_1, Z_2, Z_3, Z_4, T_1, \dots, T_7]/(Z_1^2 T_6 - Z_2^2 T_7 - 4Z_3^3 T_2^2 T_3 T_5)$. As in lemma 5.3.8 we see that

$\mathrm{Cl}(X) \subset \Lambda_X$ is generated by $E_1, E_2, E_3, E_4, E_5, \frac{1}{2}(E_4+E_5)+\frac{1}{4}(E_6+E_7), \frac{1}{2}(E_1+E_2)+\frac{1}{4}(E_4-E_5)+\frac{1}{8}(E_6-E_7)$. Identifying $E_i=-2e_i,\ i=1,2,\ E_3=-e_3,\ E_i=-4e_i,\ i=4,5,\ E_i=-8e_i,\ i=6,7$ we obtain the degree matrix of indicated generators of $\mathcal{R}$:

$$\left(\begin{array}{cc|cc|cccc|ccc}
3 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & -2 & 0\\
1 & 0 & 1 & 0 & 2 & -2 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0\\
3 & 0 & 3 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & -4\\
1 & 0 & 1 & 0 & 2 & 0 & 0 & -4 & 2 & 0 & 0\\
7 & -8 & 3 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0\\
3 & 0 & 7 & -8 & 2 & 0 & 0 & 0 & 2 & 0 & 0
\end{array}\right),$$

where the first three groups of columns correspond to variables in monomials of the relation and the last three columns to variables $Z_4, T_1, T_4$, not involved in the relation.

The movable cone $\mathrm{Mov}(X) \subset \mathbb{R}^7$ has 17 rays:

$$\begin{array}{llll}
(0,0,0,3,1,3,3), & (0,0,0,1,1,1,1), & (0,0,0,3,1,3,7), & (0,0,0,3,1,7,3),\\
(3,1,1,3,1,3,3), & (3,1,1,3,3,3,3), & (2,1,1,2,1,2,2), & (2,1,1,2,2,2,2),\\
(2,1,1,3,1,3,3), & (2,1,1,3,1,3,7), & (2,1,1,3,1,7,3), & (2,2,1,2,2,2,2),\\
(2,2,1,6,2,6,6), & (2,2,1,6,2,6,14), & (2,2,1,6,2,14,6), & (3,1,1,3,1,3,7),\\
(3,1,1,3,1,7,3). & & &
\end{array}$$

It is subdivided into 34 chambers, corresponding to all crepant resolutions of $\mathbb{C}^3/G$.

As in the previous example it would be interesting to find the answer to the following:

Open question 5.4.4. Which chamber corresponds to the $G$-Hilbert scheme?

### 5.5. Irreducible example with a divisor contracted to a point

While for reducible 3-dimensional representations we have proved that the Cox ring of a crepant resolution is defined by a single trinomial equation, the irreducible case is much more interesting from the point of view of the structure of the Cox ring.

Thus we intended to investigate the irreducible case not only to understand the geometry of the resolutions, but also to get new insight into the structure of the Cox ring in this much more intriguing setting. To be able to finish our computations, we have chosen the smallest possible group with elements of age 2 in order to work with singularities where not much is known about the set of crepant resolutions. There are two more examples of the similar nature presented in the joint paper with Maria Donten-Bury [**33**].

Consider a trihedral group $G$ generated by

$$\begin{pmatrix} \zeta_7 & 0 & 0\\ 0 & \zeta_7^2 & 0\\ 0 & 0 & \zeta_7^4 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & 0\\ 0 & 1 & 0 \end{pmatrix},$$

where $\zeta_7$ is the seventh root of unity. The commutator subgroup is $[G,G] \simeq \mathbb{Z}_7$, generated by the first group generator.

We compute the ring of invariants of the commutator subgroup $[G,G]$ and find its generating set consisting of eigenvectors of the $\mathrm{Ab}(G)$-action. The main difficulty is in the next step: extending this generating set to a (minimal) one satisfying the valuative criterion from theorem 4.1.15. We use the following simple algorithm:

(1) for a fixed (standard) degree, starting from the smallest, compute the space of linear relations between leading forms $L_{1,i}, \ldots, L_{k,i}$ of $[G,G]$-invariants $P_1, \ldots, P_k$ in this degree for different monomial valuations $\nu_i$ corresponding to junior classes,

(2) intersect spaces of relations computed for different valuations in order to check if there is a relation $R$ which increases values of more than one valuation simultaneously (i.e. $R(L_{1,i}, \dots, L_{k,i}) = 0 = R(L_{1,j}, \dots, L_{k,j})$ for some valuations $\nu_i, \nu_j$),
(3) check whether such a relation $R$ produces an element $R(P_1, \dots, P_k)$ not satisfying valuation compatibility property with respect to current generating set – if yes then add it to the set of generators,
(4) check whether the current generating set is minimal with the valuation compatibility property (adding a new element may cause some redundancies),
(5) check whether the spectrum of the ring determined by current generating set has smooth GIT quotients – if not then go back to step 1, increasing the degree.

Finally, one has to verify that the ring generated by the obtained set is really a Cox ring. This can be done either with the Singular [**27**] library `quotsingcox.lib` [**34**] accompanying [**35**] or with the algorithm for finding the Cox ring of minimal models of quotient singularities from [**94**]. The source code for our computations is available at www.mimuw.edu.pl/∼marysia/research/3dimcox. Note that, however, the algorithm from [**94**] does not behave very well in the case when the candidate for the generating set of the Cox ring is not correct, hence we use it only for verification, not for determining elements of the generating set.

The group $G$ has just 4 nontrivial conjugacy classes: 3 of age 1 and 1 of age 2, the last one containing the cube of the first group generator. Hence we have to find generators corresponding to $[G, G]$-invariants and add just 3 other ones, corresponding to exceptional divisors. However, $\mathbb{C}[x, y, z]^{[G,G]}$ needs already at least 13 generators, in degrees from 3 to 7. The requirement of being eigenvectors with respect to the action of $\mathrm{Ab}(G) \simeq \mathbb{Z}_3$ causes that they cannot be taken monomials.

Our computations show that the initial set of $[G, G]$-invariants which are $\mathrm{Ab}(G)$-eigenvectors, returned by the library [**34**] based on Singular's computation of finite group invariants, is almost suitable for constructing the generating set of $\mathcal{R}(X)$. It suffices to modify one invariant in degree 6 and three in degree 7 by a correction term, which is a product of lower degree generators, to increase their valuations (associated with conjugacy classes of elements of order 3). Thus, we have 16 generators of $\mathcal{R}(X)$ in total: 13 coming from $[G, G]$-invariants and 3 corresponding to exceptional divisors.

Proposition 5.5.1. *The following set of generators of $\mathbb{C}[x, y, z]^{[G,G]}$ satisfies valuation compatibility property, i.e. it produces a generating set of the Cox ring $\mathcal{R}(X)$ via theorem 4.1.15.*

$$
\begin{aligned}
&F_1 = xyz, \qquad G_1 = xy^3 + x^3z + yz^3,\\
&G_2 = (-\zeta_3 - 2)xy^3 + (2\zeta_3 + 1)x^3z + (-\zeta_3 + 1)yz^3,\\
&G_3 = (\zeta_3 - 1)xy^3 + (-2\zeta_3 - 1)x^3z + (\zeta_3 + 2)yz^3,\\
&H_1 = x^3y^2 + y^3z^2 + x^2z^3, \qquad H_2 = \zeta_3x^3y^2 + (-\zeta_3 - 1)y^3z^2 + x^2z^3,\\
&H_3 = (-\zeta_3 - 1)x^3y^2 + \zeta_3y^3z^2 + x^2z^3, \qquad L_1 = x^5y + y^5z + xz^5 - 3x^2y^2z^2,\\
&L_2 = \zeta_3x^5y + (-\zeta_3 - 1)y^5z + xz^5, \qquad L_3 = (-\zeta_3 - 1)x^5y + \zeta_3y^5z + xz^5,\\
&M_1 = x^7 + y^7 + z^7 - x^2y^4z - x^4yz^2 - xy^2z^4,\\
&M_2 = (-3\zeta_3 - 2)x^7 + (\zeta_3 + 3)y^7 - 7x^2y^4z + (7\zeta_3 + 7)x^4yz^2 - 7\zeta_3xy^2z^4 + (2\zeta_3 - 1)z^7,\\
&M_3 = (-3\zeta_3 - 1)x^7 + (\zeta_3 - 2)y^7 + 7x^2y^4z + 7\zeta_3x^4yz^2 + (-7\zeta_3 - 7)xy^2z^4 + (2\zeta_3 + 3)z^7.
\end{aligned}
$$

*The matrix of values of the monomial valuations is (columns corresponds to generators, ordered as above)*

$$\begin{pmatrix} 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 & 7 \\ 0 & 0 & 2 & 1 & 0 & 2 & 1 & 3 & 2 & 1 & 3 & 2 & 4 \\ 0 & 0 & 1 & 2 & 0 & 1 & 2 & 3 & 1 & 2 & 3 & 4 & 2 \end{pmatrix}$$

PROOF. To obtain given set of $[G,G]$-invariants, one preforms necessary computations along the following scheme explained above. The linear algebra part, that is investigating relations between leading forms, is done using simple scripts in Macaulay2 [**45**].

However, in this case checking that the constructed ring is a Cox ring, is hard. The algorithm in the library [**34**] is not efficient enough to finish the computation on a standard computer (the problematic step is computing preimages of generators of the ideal of relations). The approach via the algorithm in [**94**] works, though some parts of it have to be implemented in Macaulay2 and some in Singular due to certain restrictions of these systems. □

We also determine the subdivision of $\mathrm{Mov}(X)$ into chambers, using proposition 5.1.1.

PROPOSITION 5.5.2. *The number of chambers in the GIT chamber subdivision of the cone* $\mathrm{Mov}(X)$*, i.e. the number of projective crepant resolutions of* $\mathbb{C}^3/G$*, is 4. There is a central chamber, from which one can pass to each of the remaining (corner) ones, which are not connected to each other.*

We collected some data on the structure of central fibres of crepant resolutions of $\mathbb{C}^3/G$. By proposition 5.5.2 the Mov cone decomposes into the central chamber inside the GIT cone $\sigma_0$ and three *corner* chambers inside GIT cones $\sigma_1, \sigma_2, \sigma_3$. By $I$ we denote the ideal of $\mathrm{Spec}\,\mathcal{R} \subset \mathbb{C}^{16}$ given by generators produced from the invariants in proposition 5.5.1.

We start with a description of the components of the subset $S_0 \subset \mathrm{Spec}\,\mathcal{R}(X)$ mapped to the central fibre (its ideal can be computed as for dihedral groups, see argument preceding corollary 5.3.21). Then we present four tables of orbits of the action of $(\mathbb{C}^*)^{16}$ on $\mathbb{C}^{16}$, which cover $S_0$ and are stable with respect to a linearisation chosen from a chamber. In each table, *equations* are vanishings of coordinates describing the closure of the orbit, *dim* is the dimension of the orbit, and *dim*$(\cap)$ is the dimension of the intersection of the orbit with $\mathrm{Spec}\,\mathcal{R}(X)$.

PROPOSITION 5.5.3. *The subset* $S_0 \subset \mathrm{Spec}\,\mathcal{R}(X) \subset \mathbb{C}^{16} = \mathrm{Spec}\,\mathbb{C}[T_1, \dots, T_{16}]$ *mapped to the central fibre of a resolution* $X \to \mathbb{C}^3/G$ *has three components*

$$V(I + T_{14}),\ V(T_1, T_2, T_4, T_5, T_6, T_7, T_9, T_{10}, T_{11}, T_{12}, T_{15}),$$
$$V(T_1, T_2, T_3, T_5, T_6, T_7, T_9, T_{10}, T_{11}, T_{13}, T_{16}).$$

*and two more, which are unstable for any chamber. General points on the first one are always stable; it corresponds to an exceptional divisor. General points on the second one are stable only for* $\sigma_1$ *and general points on the third one are stable only for* $\sigma_2$*.*

PROOF. One decomposes, e.g. in Singular [**27**], the ideal generated by $I$ and the generators of the ring of invariants of the Picard torus action (which can be computed in 4ti2 [**1**]). The information on stability can be read out from the tables given below. □

Once again, it would be interesting to find the answer to the following:

OPEN QUESTION 5.5.4. Which cone among $\sigma_i,\ i = 0, 1, 2, 3$ corresponds to the resolution given by the $G$-Hilbert scheme?

**Stable orbits for the central fibre for $\sigma_0$.**

| equation | dim | dim ($\cap$) |
|---|---|---|
| $T_{14} = 0$ | 15 | 5 |
| $T_1 = T_{14} = 0$ | 14 | 4 |
| $T_2 = T_{14} = 0$ | 14 | 4 |
| $T_3 = T_{14} = 0$ | 14 | 4 |
| $T_4 = T_{14} = 0$ | 14 | 4 |
| $T_1 = T_5 = T_6 = T_7 = T_8 = T_9 = T_{10} = T_{14} = 0$ | 8 | 4 |
| $T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_7 = T_8 = T_9 = T_{10} = T_{14} = 0$ | 5 | 3 |
| $T_2 = T_5 = T_7 = T_9 = T_{10} = T_{11} = T_{12} = T_{14} = T_{15} = 0$ | 7 | 4 |
| $T_2 = T_3 = T_5 = T_7 = T_9 = T_{10} = T_{11} = T_{12} = T_{14} = T_{15} = 0$ | 6 | 3 |
| $T_2 = T_5 = T_6 = T_9 = T_{10} = T_{11} = T_{13} = T_{14} = T_{16} = 0$ | 7 | 4 |
| $T_2 = T_4 = T_5 = T_6 = T_9 = T_{10} = T_{11} = T_{13} = T_{14} = T_{16} = 0$ | 6 | 3 |
| $T_1 = T_2 = T_5 = T_6 = T_7 = T_9 = T_{10} = T_{14} = T_{15} = T_{16} = 0$ | 6 | 4 |
| $T_1 = T_2 = T_5 = T_6 = T_7 = T_8 = T_9 = T_{10} = T_{14} = T_{15} = T_{16} = 0$ | 5 | 3 |
| $T_1 = T_2 = T_5 = T_6 = T_7 = T_9 = T_{10} =$ $= T_{11} = T_{12} = T_{13} = T_{14} = T_{15} = T_{16} = 0$ | 3 | 3 |

**Stable orbits for the central fibre for $\sigma_1$.**

| equation | dim | dim ($\cap$) |
|---|---|---|
| $T_{14} = 0$ | 15 | 5 |
| $T_1 = T_{14} = 0$ | 14 | 4 |
| $T_2 = T_{14} = 0$ | 14 | 4 |
| $T_3 = T_{14} = 0$ | 14 | 4 |
| $T_4 = T_{14} = 0$ | 14 | 4 |
| $T_1 = T_5 = T_6 = T_7 = T_8 = T_9 = T_{10} = T_{14} = 0$ | 8 | 4 |
| $T_1 = T_2 = T_3 = T_4 = T_5 = T_6 = T_7 = T_8 = T_9 = T_{10} = T_{14} = 0$ | 5 | 3 |
| $T_1 = T_2 = T_4 = T_5 = T_6 = T_7 = T_9 = T_{10} = T_{11} = T_{12} = T_{15} = 0$ | 5 | 4 |
| $T_1 = T_2 = T_4 = T_5 = T_6 = T_7 = T_9 = T_{10} = T_{11} = T_{12} = T_{13} = T_{15} = 0$ | 4 | 3 |
| $T_2 = T_5 = T_7 = T_9 = T_{10} = T_{11} = T_{12} = T_{14} = T_{15} = 0$ | 7 | 4 |
| $T_2 = T_3 = T_5 = T_7 = T_9 = T_{10} = T_{11} = T_{12} = T_{14} = T_{15} = 0$ | 6 | 3 |
| $T_1 = T_2 = T_5 = T_6 = T_7 = T_9 = T_{10} = T_{14} = T_{15} = T_{16} = 0$ | 6 | 4 |
| $T_1 = T_2 = T_5 = T_6 = T_7 = T_8 = T_9 = T_{10} = T_{14} = T_{15} = T_{16} = 0$ | 5 | 3 |
| $T_1 = T_2 = T_4 = T_5 = T_6 = T_7 = T_9 =$ $= T_{10} = T_{11} = T_{12} = T_{14} = T_{15} = T_{16} = 0$ | 3 | 3 |

**Stable orbits for the central fibre for $\sigma_2$.**

| equation | dim | dim ($\cap$) |
|---|---|---|
| $T_{14}=0$ | 15 | 5 |
| $T_1=T_{14}=0$ | 14 | 4 |
| $T_2=T_{14}=0$ | 14 | 4 |
| $T_3=T_{14}=0$ | 14 | 4 |
| $T_4=T_{14}=0$ | 14 | 4 |
| $T_1=T_5=T_6=T_7=T_8=T_9=T_{10}=T_{14}=0$ | 8 | 4 |
| $T_1=T_2=T_3=T_4=T_5=T_6=T_7=T_8=T_9=T_{10}=T_{14}=0$ | 5 | 3 |
| $T_1=T_2=T_3=T_5=T_6=T_7=T_9=T_{10}=T_{11}=T_{13}=T_{16}=0$ | 5 | 4 |
| $T_1=T_2=T_3=T_5=T_6=T_7=T_9=T_{10}=T_{11}=T_{12}=T_{13}=T_{16}=0$ | 4 | 3 |
| $T_2=T_5=T_6=T_9=T_{10}=T_{11}=T_{13}=T_{14}=T_{16}=0$ | 7 | 4 |
| $T_2=T_4=T_5=T_6=T_9=T_{10}=T_{11}=T_{13}=T_{14}=T_{16}=0$ | 6 | 3 |
| $T_1=T_2=T_5=T_6=T_7=T_9=T_{10}=T_{14}=T_{15}=T_{16}=0$ | 6 | 4 |
| $T_1=T_2=T_5=T_6=T_7=T_8=T_9=T_{10}=T_{14}=T_{15}=T_{16}=0$ | 5 | 3 |
| $T_1=T_2=T_3=T_5=T_6=T_7=T_9=$<br>$=T_{10}=T_{11}=T_{13}=T_{14}=T_{15}=T_{16}=0$ | 3 | 3 |

**Stable orbits for the central fibre for $\sigma_3$.**

| equation | dim | dim ($\cap$) |
|---|---|---|
| $T_{14}=0$ | 15 | 5 |
| $T_1=T_{14}=0$ | 14 | 4 |
| $T_2=T_{14}=0$ | 14 | 4 |
| $T_3=T_{14}=0$ | 14 | 4 |
| $T_4=T_{14}=0$ | 14 | 4 |
| $T_1=T_5=T_6=T_7=T_8=T_9=T_{10}=T_{14}=0$ | 8 | 4 |
| $T_1=T_2=T_3=T_4=T_5=T_6=T_7=T_8=T_9=T_{10}=T_{14}=0$ | 5 | 3 |
| $T_5=T_6=T_7=T_8=T_9=T_{10}=T_{11}=T_{12}=T_{13}=T_{14}=0$ | 6 | 4 |
| $T_1=T_5=T_6=T_7=T_8=T_9=T_{10}=T_{11}=T_{12}=T_{13}=T_{14}=0$ | 5 | 3 |
| $T_2=T_5=T_7=T_9=T_{10}=T_{11}=T_{12}=T_{14}=T_{15}=0$ | 7 | 4 |
| $T_2=T_3=T_5=T_7=T_9=T_{10}=T_{11}=T_{12}=T_{14}=T_{15}=0$ | 6 | 3 |
| $T_2=T_5=T_6=T_9=T_{10}=T_{11}=T_{13}=T_{14}=T_{16}=0$ | 7 | 4 |
| $T_2=T_4=T_5=T_6=T_9=T_{10}=T_{11}=T_{13}=T_{14}=T_{16}=0$ | 6 | 3 |
| $T_2=T_5=T_6=T_7=T_8=T_9=T_{10}=$<br>$=T_{11}=T_{12}=T_{13}=T_{14}=T_{15}=T_{16}=0$ | 3 | 3 |

CHAPTER 6

# Background on torus actions

In this chapter we collect various results on actions of algebraic tori that we use in the next chapter. There, we study four-dimensional symplectic quotient sigularities, and we need additional tools to the ones presented in chapter 4 – our idea is to develop methods employing torus action on a resolution. In 6.1 we show that torus actions on quotient singularities could be lifted to crepant resolutions. In 6.2 we analyze weights of $\mathbb{C}^*$-action on a variety with a symplectic structure compatible with the action. In 6.3 we collect results on the local structure of the action. In section 6.3.1 we define the compass at a fixed point of a variety with a torus action. We also present the Białynicki-Birula theorem on the (partial) decomposition of a smooth algebraic variety with $\mathbb{C}^*$-action into $\mathbb{C}^*$-fibrations over components of the fixed point locus. The last part 6.4 of this chapter introduces the notion of the equivariant Euler characteristic in the context of varieties projective over an affine base.

## 6.1. Torus actions on crepant resolutions

Let $T \cong (\mathbb{C}^*)^r$ be an algebraic torus acting linearly on $\mathbb{C}^n$. Assume that $G \subset \mathrm{SL}_n(\mathbb{C})$ is a finite group and that the actions of $T$ and $G$ commute. Then the action of $T$ descends to the action on the quotient $\mathbb{C}^n/G$ since the grading on $\mathbb{C}[x_1, \ldots, x_n]$ corresponding to the $T$-action induce the grading on the invariant ring of $G$. Moreover we have the following theorem:

Proposition 6.1.1. *If $\varphi\colon X \to \mathbb{C}^n/G$ is a crepant resolution then there is a unique action of $T$ on $X$ such that $\varphi$ is a $T$-equivariant map.*

Proof. The argument given in the proof of [**7**, Proposition 8.2] shows the assertion for the action of every one-parameter subgroup of $T$. We conclude since the action of $T$ can be expressed as a product of actions of its one-parameter subgroups. □

## 6.2. $\mathbb{C}^*$-actions on symplectic varieties

Let $X$ be a smooth variety with symplectic structure $\omega$. Assume that one-dimensional algebraic torus $\mathbb{C}^*$ acts on $X$.

Definition 6.2.1 (Homogeneous symplectic form). We say that $\omega$ is homogeneous of degree $k$ with respect to the $\mathbb{C}^*$-action if $t^*\omega = t^k\omega$ for every $t \in \mathbb{C}^*$.

Proposition 6.2.2. *If $X, Y$ are two smooth varieties with symplectic structures $\omega_X, \omega_Y$ respectively and $\mathbb{C}^*$-action, and if $\varphi\colon X \to Y$ is a $\mathbb{C}^*$-equivariant morphism such that $\varphi^*\omega_Y = \omega_X$, then $\omega_Y$ is homogeneous of degree $k$ if and only if $\omega_X$ is homogeneous of degree $k$.*

PROOF. This can be proven by explicit calculation:

$$\begin{aligned}t^*\omega_X(u,v) &= \omega_X(Dt(u),Dt(v)) = \varphi^*\omega_Y(Dt(u),Dt(v)) = \omega_Y(D\varphi Dt(u),D\varphi Dt(v)) = \\ &= \omega_Y(D(\varphi\circ t)(u),D(\varphi\circ t)(v)) = \omega_Y(D(t\circ\varphi)(u),D(t\circ\varphi)(v)) = \\ &= \omega_Y(DtD\varphi(u),DtD\varphi)(v)) = t^*\omega_Y(D\varphi(u),D\varphi(v)).\end{aligned}$$

Here $u,v\in T_xX$ and $x\in X$, $t\in\mathbb{C}^*$. By abuse of notation we denoted the bilinear forms on $T_xX$ and $T_{\varphi(x)}Y$ induced by $\omega_X$ and $\omega_Y$ by the same symbols. Similarly $D\varphi\colon TX\to TY$, $Dt\colon TX\to TX$, $Dt\colon TY\to TY$ are the maps induced by $\varphi$ and $t$ respectively. □

We will be using the results of this section in the following setting:

EXAMPLE 6.2.3. The standard symplectic form $\omega=\sum_{i=1}^n dx_i\wedge dy_i$ on $\mathbb{C}^{2n}$ is homogeneous of weight two with respect to standard $\mathbb{C}^*$-action by homothety: $t\cdot(x_1,y_1,\dots,x_n,y_n)=(tx_1,ty_1,\dots,tx_n,ty_n)$. Moreover, if $G\subset\mathrm{Sp}_{2n}(\mathbb{C})$ and $\varphi\colon X\to\mathbb{C}^{2n}/G$ is a symplectic resolution, then the symplectic form on $X$ is homogeneous of weight two.

PROPOSITION 6.2.4. *If $X$ is a smooth $2n$-dimensional variety with a $\mathbb{C}^*$-action and a symplectic form $\omega$ of weight $k$ then at each fixed point $x$ of $X$ there is a basis $u_1,v_1,\dots,u_n,v_n$ of $T_xX$ consisting of eigenvectors of $\mathbb{C}^*$-action, such that the sum of weights of the $\mathbb{C}^*$-action on $u_i$ and $v_i$ is equal to $-k$ for each $i=1,\dots,n$.*

PROOF. Since a linear $\mathbb{C}^*$-action is diagonalizable there exists a basis $e_1,\dots,e_{2n}$ of eigenvectors of the action on $T_xX$. As $\omega_x\colon T_xX\times T_xX\to\mathbb{C}$ is nondegenerate there exists $j$ such that $\omega_x(e_1,e_j)\neq 0$. Moreover, the vector subspace $\mathrm{span}\{e_1,e_j\}^\perp=\{v\in T_xX\colon \omega_x(e_1,v)=\omega_x(e_j,v)=0\}$ is $\mathbb{C}^*$-invariant, because $\omega$ is homogeneous. Now note that $T_xX=\mathrm{span}\{e_1,e_j\}^\perp\oplus\mathrm{span}\{e_1,e_j\}$. Taking $u_1=e_1$, $v_1=e_j$ and arguing by induction on $n$ we obtain the basis $u_1,v_1,\dots u_n,v_n$ of $T_xX$ consisting of eigenvectors of $\mathbb{C}^*$-action and such that $\omega_x=u_1^*\wedge v_1^*+\ldots+u_n^*\wedge v_n^*$, where $u_1^*,v_1^*,\dots,u_n^*,v_n^*$ is the dual basis of $T_x^*X$. It follows that the sum of weights of $u_i$ and $v_i$ is equal to $-k$ for each $i=1,\dots,n$. □

## 6.3. Local properties of algebraic torus action

Assume that an algebraic torus $T\cong(\mathbb{C}^*)^r$ acts on algebraic variety $X$. In the next chapter we will need the following local properties of the $T$-action on $X$. The first is a simple lemma from work of Luna and the second is a celebrated result of Sumihiro.

PROPOSITION 6.3.1 ([**69**, III.1, Lemme]). *Assume that $X$ is affine. Let $x$ be a smooth point of $X$ and denote by $T'\subset T$ the isotropy subgroup of $x$. Then there exists a $T'$-equivariant étale map $\phi\colon X\to T_xX$ such that $\phi(x)=0$.*

SKETCH OF PROOF. One uses the fact that the map $d\colon\mathfrak{m}\to\mathfrak{m}/\mathfrak{m}^2\cong T_x^*X$ is $T'$-equivariant to find a subspace of $W\subset\mathfrak{m}$ such that $d|_W$ is an isomorphism. Then applying the functor of symmetric algebra to the inverse isomorphism $(d|_W)^{-1}$ one obtains a morphism with desired properties. □

THEOREM 6.3.2 ([**87**, Corollary 2]). *If $X$ is normal then there exists a cover of $X$ by $T$-invariant affine open subsets.*

We will use the following corollaries of the results above.

COROLLARY 6.3.3 (cf. [**58**, Proposition 1.3]). *If $X$ is smooth then $X^T$ is smooth.*

COROLLARY 6.3.4. *Assume that $X$ is smooth. Let $x$ be a fixed point of $X$. Then there exists an open $T$-invariant affine neighbourhood $U\subset X$ of $x$ and a $T$-invariant étale map $\phi\colon U\to T_xX$ such that $\phi(x)=0$.*

**6.3.1. Compasses and $\mathbb{C}^*$-fibrations.** Let $T \cong (\mathbb{C}^*)^r$ be an algebraic torus acting on a smooth algebraic variety $X$ of dimension $d$. Denote the connected components of $X^T$ by $Y_1, \dots, Y_s$. If $x \in X^T$ the torus $T$ acts on the cotangent space $T^*_x X = \mathfrak{m}_x/\mathfrak{m}_x^2$ with weights which are the same for $x$ in fixed connected component $Y_j$. By corollary 6.3.4 there are $d_i := \dim Y_i$ weights equal to zero. Denote the remaining nonzero $d - d_i$ weights by $\nu_{i,1}, \dots, \nu_{i,d-d_i}$.

DEFINITION 6.3.5. The multiset of weights $\nu_{i,1}, \dots, \nu_{i,d-d_i}$ is called *the compass of $X$ at $Y_i$*.

Assume now additionally that $X$ is also quasiprojective variety and $T = \mathbb{C}^*$ is a one-dimensional algebraic torus acting on $X$. For a fixed point $x \in X^T$ we denote by $(T_xX)^+, (T_xX)^-, (T_xX)^0$ eigenspaces of the tangent space $T_xX$ on which $T$ acts with positive, negative and zero weights respectively. We will describe the local structure of the (part of the) variety $X$ along fixed locus of $T$. We need the following concept.

DEFINITION 6.3.6 ($\mathbb{C}^*$-fibration). A *trivial $\mathbb{C}^*$-fibration* of dimension $n$ over a variety $Z$ is a variety with a $\mathbb{C}^*$-action equivariantly isomorphic to $V \times Z$, where $V$ is a $\mathbb{C}^*$-representation of dimension $n$ and $\mathbb{C}^*$ acts trivially on $Z$. A variety $E$ with a $\mathbb{C}^*$-action and a morphism $E \to Z$ is a *$\mathbb{C}^*$-fibration* $E$ of dimension $n$ over $Z$ if there exists an open cover $\{Z_i\}_i$ such that $E \times_Z Z_i$ is equivariantly isomorphic over $Z_i$ to a trivial $\mathbb{C}^*$-fibration of dimension $n$ over $Z_i$.

THEOREM 6.3.7 ([**14**, Theorem 4.1]). *For any $i = 1, \dots, r$ there exist locally closed subvariety $X_i^+ \subset X$, (resp. $X_i^- \subset X$) and a morphism $\gamma_i^+ : X_i^+ \to Y_i$ (resp. $\gamma_i^- : X_i^- \to Y_i$) such that*

*(a) $Y_i$ is a closed subvariety of $X_i^+$ (resp. $X_i^-$), $\gamma_i^+|_{Y_i}$ (resp. $\gamma_i^-|_{Y_i}$) is the identity.*
*(b) $X_i^+$ (resp. $X_i^-$) with the action of $\mathbb{C}^*$ (induced from the action on $X$) and with $\gamma_i^+$ (resp. $\gamma_i^-$) is a $\mathbb{C}^*$-fibration over $Y_i$.*
*(c) For any $x \in Y_i$, $T_x(X_i^+) = (T_xX)^0 \oplus (T_xX)^+$ (resp. $T_x(X_i^-) = (T_xX)^0 \oplus (T_xX)^-$).*

*The dimension of the $\mathbb{C}^*$-fibration defined in (b) is equal to* $\dim(T_xX_i)^+$ *(resp.* $\dim(T_xX_i)^-$*).*

If $X$ is complete, then each of finite families $\{X_i^+\}$ and $\{X_i^-\}$ cover $X$ and one gets two decompositions of $X$ into $\mathbb{C}^*$-fibrations that are locally closed in $X$ (this is the celebrated ABB decomposition, see [**14**, Theorem 4.3]).

## 6.4. Equivariant Euler characteristic and the Lefschetz-Riemann-Roch theorem

In this section we give the definition and prove necessary properties of the notion of equivariant Euler characteristic for algebraic torus actions on varieties that are projective over an affine variety. Unfortunately, we did not find a reference that covers this notion in such setting. In the later chapters we need only corollary 6.4.22. For an introduction to equivariant K-theory and a discussion of projective case see [**20**, Chapter 5]. The basic difference between the absolutely projective case and the relatively projective case is that in the former the equivariant Euler characteristic is a polynomial over integers, and in the latter one has to make an additional assumption on the action (see assumption 6.4.15) to get the Euler characteristic as a Laurent power series with integer coefficients. Nevertheless, after making the necessary assumption, the expression of this series in terms of weights of the action still can be obtained by the localization formula as in the absolutely projective case (see corollary 6.4.21).

Let $X$ be a smooth quasiprojective algebraic variety with an action of an algebraic torus $T \cong (\mathbb{C}^*)^r$. Denote the structure morphism of the action by $\sigma \colon T \times X \to X$ and the projection onto $X$ by $p_X$.

Definition 6.4.1 ($T$-sheaf). A $T$-sheaf on $X$ is a coherent sheaf $\mathcal{F}$ together with an isomorphism $\phi \colon \sigma^*\mathcal{F} \cong p_X^*\mathcal{F}$, satisfying cocycle condition: $p_{23}^*\phi \circ (\mathrm{id}_T \times \sigma)^*\phi = (m \times \mathrm{id}_X)^*\phi$, where $m : T \times T \to T$ is the multiplication morphism and $p_{23} : T \times T \times X \to T \times X$ is the projection onto product of second and third factor $T \times X$.

The notion of a $T$-sheaf generalizes the notion of a $T$-equivariant vector bundle ([**20**, Sect. 5.1]). The cocycle condition can be expressed as the commutativity of the diagram:

$$\begin{tikzcd}
(m \times \mathrm{id}_X)^*\sigma^*\mathcal{F} \arrow[rr, "(m\times \mathrm{id}_X)^*\phi"] \arrow[d, equal] & & (m \times \mathrm{id}_X)^*p_X^*\mathcal{F} \arrow[d, equal] \\
(\mathrm{id}_T \times \sigma)^*\sigma^*\mathcal{F} \arrow[r, "(\mathrm{id}_T \times\sigma)^*\phi"] & (\mathrm{id}_T \times \sigma)^*p_X^*\mathcal{F} = p_{23}^*\sigma^*\mathcal{F} \arrow[r, "p_{23}^*\phi"] & p_{23}^*p_X^*\mathcal{F}
\end{tikzcd}$$

where identifications follow from commutativity of the diagrams:

$$\begin{tikzcd}
T \times T \times X \arrow[r, "m\times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_T \times\sigma"] & T \times X \arrow[d, "\sigma"] \\
T \times X \arrow[r, "\sigma"] & X,
\end{tikzcd}
\quad
\begin{tikzcd}
T \times T \times X \arrow[r, "p_{23}"] \arrow[d, "\mathrm{id}_T \times\sigma"] & T \times X \arrow[d, "\sigma"] \\
T \times X \arrow[r, "p_X"] & X,
\end{tikzcd}
\quad
\begin{tikzcd}
T \times T \times X \arrow[r, "m\times \mathrm{id}_X"] \arrow[d, "p_{23}"] & T \times X \arrow[d, "p_X"] \\
T \times X \arrow[r, "p_X"] & X.
\end{tikzcd}$$

Proposition 6.4.2 ([**20**, Lemma 5.1.23]). *If $(\mathcal{F}, \phi)$ is a $T$-sheaf then there is an induced $T$-action on $H^0(X, \mathcal{F})$ defined by the composition:*

$$H^0(X, \mathcal{F}) \xrightarrow{\sigma^*} H^0(T \times X, \sigma^*\mathcal{F}) \xrightarrow{\phi} H^0(T \times X, p_X^*\mathcal{F}) = H^0(T, \mathcal{O}_T) \otimes_{\mathbb{C}} H^0(X, \mathcal{F}),$$

*so that $t \cdot s = \sum_i f_i(t)s_i$, where $\phi(\sigma^*(s)) = \sum_i f_i \otimes s_i$.*

Proposition 6.4.3 (Twisting $T$-sheaf structure by a character). *Given $T$-sheaf $(\mathcal{F}, \phi)$ and a character $t^m$ of $T$ we define $t^m \cdot \phi$ by:*

$$H^0(U_1 \times U_2, \sigma^*\mathcal{F}) \ni s \mapsto t^m|_{U_1} \cdot \phi(s) \in H^0(U_1 \times U_2, p_X^*\mathcal{F}) = H^0(U_1, \mathcal{O}_T) \otimes_{\mathbb{C}} H^0(U_2, \mathcal{F}),$$

*for open affine $U_1 \subset T$, $U_2 \subset X$. Then $(\mathcal{F}, t^m \cdot \phi)$ is a $T$-sheaf.*

For our convenience we will abuse notation by writing $t^m \cdot \mathcal{F}$ instead of $(\mathcal{F}_X, t^m \cdot \phi)$, whenever it will be clear from the context what the map $\phi$ is.

Example 6.4.4 ($T$-sheaf structures on a trivial line bundle). There is a natural $T$-sheaf structure on the structure sheaf $\mathcal{O}_X$ given by the composition of natural isomorphisms $\phi : \sigma^*\mathcal{O}_X \cong \mathcal{O}_X \cong p_X^*\mathcal{O}_X$. By [**63**, Lemma 2.2, Proposition 2.3] any other structure of $T$-sheaf on $\mathcal{O}_X$ is of the form $t^m \cdot \phi$, where $t^m$ is a character of $T$.

Definition 6.4.5 (Morphism of $T$-sheaves). A *morphism of $T$-sheaves* $(\mathcal{F}, \phi) \to (\mathcal{F}', \phi')$ on $X$ is a morphism: $\alpha : \mathcal{F} \to \mathcal{F}'$ of underlying coherent modules such that the following diagram is commutative:

$$\begin{tikzcd}
\sigma^*\mathcal{F} \arrow[r, "\phi"] \arrow[d, "\sigma^*\alpha"] & p_X^*\mathcal{F} \arrow[d, "p_X^*\alpha"] \\
\sigma^*\mathcal{F}' \arrow[r, "\phi'"] & p_X^*\mathcal{F}'.
\end{tikzcd}$$

With such a definition of morphism $T$-sheaves on $X$ form a category. Moreover direct sum of $T$-sheaves, kernel, image and cokernel of a morphism of $T$-sheaves have natural structures of a $T$-sheaves. With these notions the category of $T$-sheaves on $X$ is an abelian category [**89**, 1.4].

DEFINITION 6.4.6 (Equivariant $K_0$). An equivariant $K_0$-group of $X$ is the quotient $K_0^T(X)$ of the free abelian group generated by $T$-sheaves on $X$ divided by relations $\mathcal{F} = \mathcal{F}' + \mathcal{F}''$ for each exact sequence: $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of $T$-sheaves.

Let $R(T)$ be the representation ring of $T$. Note that $R(T) \cong \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$. It turns out that $K_0^T(X)$ admits a natural $R(T)$-algebra structure. Let us first introduce the structure of an $R(T)$-module.

PROPOSITION 6.4.7 ($R(T)$-module structure on $K_0^T$). *$K_0^T(X)$ is an $R(T)$-module with multiplication given by $t^m \cdot [\mathcal{F}] = [t^m \cdot \mathcal{F}]$ for characters $t^m$ of $T$.*

The following important result allows one to restrict attention to locally free $T$-sheaves while studying $K_0^T(X)$. It uses the assumption of smoothness of $X$.

THEOREM 6.4.8 ([**20**, Proposition 5.1.28]). *For every $T$-module $\mathcal{F}$ there exists a finite equivariant resolution of $\mathcal{F}$ by locally free $T$-sheaves, i.e. an exact sequence of the form:*

$$0 \to \mathcal{E}_p \to \dots \to \mathcal{E}_1 \to \mathcal{F} \to 0,$$

*where maps are morphisms of $T$-sheaves.*

COROLLARY 6.4.9. *$K_0^T(X)$ is the quotient of the free abelian group generated by locally free $T$-sheaves on $X$ divided by relations $\mathcal{E} = \mathcal{E}' + \mathcal{E}''$ for each exact sequence: $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ of free $T$-sheaves.*

Now we are ready to describe multiplication in $K_0^T(X)$.

PROPOSITION 6.4.10 (Multiplication in $K_0^T$, [**20**, 5.2.11(iii)]). *We define multiplication in $K_0^T(X)$ by setting $[\mathcal{E}] \cdot [\mathcal{E}'] := [\mathcal{E} \otimes \mathcal{E}']$ for locally free $T$-sheaves $\mathcal{E}, \mathcal{E}'$ which generate $K_0^T(X)$ by theorem 6.4.8 and extending by linearity. With this multiplicative structure $K_0^T(X)$ is a commutative $R(T)$-algebra.*

Let us look at the simplest possible example, when $X$ is a point.

EXAMPLE 6.4.11 (Equivariant $K_0$ for a point). If $X = pt$ is a point then $K_0^T(X)$ is isomorphic to the representation ring $R(T)$ of $T$, i.e. to the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$. Indeed, by taking global sections $T$-sheaves on a point may be identified with the finite-dimensional $T$-representations. This identification preserves sum and multiplication.

Projective maps induce maps in $K$-theory.

PROPOSITION 6.4.12 (Projective pushforward for $K_0^T$,[**20**, 5.2.13]). *Let $p : X \to Y$ be a projective $T$-equivariant map of smooth quasiprojective $T$-varieties. Then we define the proper pushforward $f_*([\mathcal{F}]) = \sum_{i=0}^{\dim X} (-1)^i [R^i p_* \mathcal{F}]$. This gives a well-defined $R(T)$-module homomorphism: $f_* : K_0^T(X) \to K_0^T(Y)$.*

The following fundamental theorem allows one to express equivariant $K_0$ in terms of $K_0$ of fixed point loci. Let $T$ be an algebraic torus and let $S \subset R(T)$ be a multiplicative system generated by the set $\{1 - t^m \colon t^m \text{ character of } T\}$. Let $X, Y$ be smooth quasiprojective varieties with action of $T$. Assume that $X^T, Y^T$ are projective over $\mathbb{C}$. Note that they are smooth by corollary 6.3.3. Denote by $N^*_{X^T/X}$ the conormal bundle of $X^T$ in $X$ and let $\lambda_{-1}(N^*_{X^T/X}) = \sum_{i=0}^{\dim X}(-1)^i\Lambda^i(N^*_{X^T/X})$. Let $p : X \to Y$ be an equivariant projective map.

THEOREM 6.4.13 (Localization theorem for $K_0^T$,[**88**, Theorem 6.4, Corollary 6.7]). *With above assumptions and notation:*

*(i) The element $\lambda_{-1}(N^*_{X^T/X})$ is invertible in the localized ring $S^{-1}K_0^T(X^T)$.*
*(ii) The map $\mathrm{Loc}_X\colon S^{-1}K_0^T(X) \to S^{-1}K_0^T(X^T)$ given by $[\mathcal{E}] \mapsto [\mathcal{E}|_T/\lambda_{-1}(N^*_{X^T/X})]$ for locally free sheaves $\mathcal{E}$ is an isomorphism of $R(T)$-algebras.*
*(iii) The following diagram commutes:*

$$\begin{tikzcd} S^{-1}K_0^T(X) \arrow[r, "p_*"] \arrow[d, "\mathrm{Loc}_X"] & S^{-1}K_0^T(Y) \arrow[d, "\mathrm{Loc}_Y"] \\ S^{-1}K_0^T(X^T) \arrow[r, "(p|_{X^T})_*"] & S^{-1}K_0^T(Y^T). \end{tikzcd}$$

For our purposes the most important will be the basic case of a linear action on an affine space. Let $T = (\mathbb{C}^*)^r$ act linearly on $\mathbb{C}^n$.

THEOREM 6.4.14.

*(i) Every locally free $T$-sheaf on $\mathbb{C}^n$ is equivariantly isomorphic to a direct sum of rank one free $T$-sheaves.*
*(ii) Every $T$-module on $\mathbb{C}^n$ is a quotient of a direct sum of rank one free $T$-sheaves.*
*(iii) Every $T$-module on $\mathbb{C}^n$ has an equivariant finite resolution by direct sums of rank one free sheaves.*

PROOF. (i) is an equivariant version of the Quillen-Suslin theorem, proven for abelian groups in [**72**].
For (ii) let $s_1, \dots, s_k \in H^0(X, \mathcal{F})$ be homogeneous generators over $H^0(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = \mathbb{C}[x_1, \dots, x_r]$ of degrees $w_1, \dots, w_r$. Then we have a surjective map $\bigoplus_{i=1}^k (t^{w_i} \cdot \mathcal{O}_{\mathbb{C}^n})e_i \to \mathcal{F}$ defined by $e_i \mapsto s_i$.
For (iii) we use the Hilbert syzygy theorem [**20**, Theorem 5.1.30] together with (i) and (ii). Using (ii) inductively we construct a free equivariant resolution of $\mathcal{F}$:

$$\dots \to \mathcal{E}_p \to \dots \to \mathcal{E}_1 \to \mathcal{F} \to 0,$$

where maps are morphisms of $T$-sheaves. By the Hilbert syzygy theorem $\mathcal{E}_p$ is a locally free $T$-sheaf and we may apply (i). ☐

From now on we will work under the following assumption.

ASSUMPTION 6.4.15. There exists an embedding $\mathbb{C}^* \subset T$ such that the weights of the coordinates $x_1, \dots, x_n$ on $\mathbb{C}^n$ are positive.

Note that when assumption 6.4.15 is satisfied we have $(\mathbb{C}^n)^T = \{0\}$. The second important consequence of this assumption is the next proposition.

PROPOSITION 6.4.16. *Assume that assumption 6.4.15 is satisfied. Let $\mathcal{F}$ be a $T$-sheaf on $\mathbb{C}^n$. Then:*

*(i) The graded pieces with respect to the grading on $H^0(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n}) = \mathbb{C}[x_1, \dots, x_n]$ induced by the $T$-action are finite dimensional vector spaces over $\mathbb{C}$.*
*(ii) In the weight-space decomposition:*

$$H^0(\mathbb{C}^n, \mathcal{F}) = \bigoplus_{w \in \mathbb{Z}^r} H^0(\mathbb{C}^n, \mathcal{F})_w,$$

*each component $H^0(\mathbb{C}^n, \mathcal{F})$ is a finite dimensional vector space over $\mathbb{C}$.*

Proof. (i) is immediate as there are only finitely many monomials in $x_1, \dots, x_n$ of fixed degree. (ii) follows from (i) as $\mathcal{F}$ is a quotient of a free $T$-sheaf on $\mathbb{C}^n$ by theorem 6.4.8 and by theorem 6.4.14. $\square$

The previous proposition allows us to formulate the following definition.

Definition 6.4.17 (Equivariant Euler characteristic for a $T$-sheaf on an affine space). Under the assumption 6.4.15 *the equivariant Euler characteristic* of a $T$-sheaf $\mathcal{F}$ on $\mathbb{C}^n$ is a Laurent power series

$$\chi^T(\mathbb{C}^n, \mathcal{F}) = \sum_{w \in \mathbb{Z}^r} (\dim H^0(\mathbb{C}^n, \mathcal{F})_w) t^w \in \mathbb{Z}((t_1, \dots, t_r)).$$

Proposition 6.4.18. *Equivariant Euler characteristic is a well-defined map of rings:*

$$\chi^T(\mathbb{C}^n, \cdot) : K_0^T(\mathbb{C}^n) \to \mathbb{Z}((t_1, \dots, t_r)),$$

*that factorizes through the localization map from theorem 6.4.13, i.e. we have a commutative diagram:*

$$\begin{tikzcd}
S^{-1}K_0^T(\mathbb{C}^n) \arrow[rr, "\mathrm{Loc}"] \arrow[dr, "S^{-1}\chi^T(\mathbb{C}^n,\cdot)"'] & & S^{-1}K_0^T(\{0\}) = S^{-1}\mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}] \arrow[dl] \\
& \mathbb{Z}((t_1, \dots, t_r)), &
\end{tikzcd}$$

*where $S \subset R(T) = \mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ is the same multiplicative system as in the localization theorem 6.4.13 and $S^{-1}\mathbb{Z}[t_1^{\pm 1}, \dots, t_r^{\pm 1}] \to \mathbb{Z}((t_1, \dots, t_r))$ is the inclusion of rational functions into Laurent power series ring.*

Proof. By the vanishing of the higher cohomology on an affine space [**46**, III.3] $\chi^T(\mathbb{C}^n, \cdot)$ is a well-defined map from $K_0^T(\mathbb{C}^n)$. Since the composition of ring homomorphisms is a ring homomorphisms it suffices to check the commutativity of the diagram on additive generators of $K_0^T(\mathbb{C}^n)$. By corollary 6.4.9 and theorem 6.4.14 as a group $K_0^T(\mathbb{C}^n)$ is generated by the classes of rank-one free $T$-sheaves. These are of the form $t^m \cdot \mathcal{O}_{\mathbb{C}^n}$ by example 6.4.4. The proposition follows now by the explicit computation of the equivariant Euler characteristic of $t^m \cdot \mathcal{O}_{\mathbb{C}^n}$ and by the fact that the restriction to the fibre over zero gives an isomorphism $K_0^T(\mathbb{C}^n) \cong K_0^T(\{0\})$ (the latter is a consequence of theorem 6.4.14 or localization formula 6.4.13 but in fact it is much easier, see e.g. [**20**, Theorem 5.4.17]). Indeed, $\mathrm{Loc}([t^m \cdot \mathcal{O}_{\mathbb{C}}]) = [(t^m \cdot \mathcal{O}_{\mathbb{C}^n})/\lambda_{-1}(T_0^*\mathbb{C}^n)] = \frac{t^m}{\prod_{i=1}^n (1-t^{\nu_i})} \in \mathbb{Z}((t_1, \dots, t_r))$, where $\nu_1, \dots, \nu_n$ are weights of the $T$-action on $T_0^*\mathbb{C}^n$. As generators of $H^0(\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ have weights $\nu_1, \dots, \nu_r$ and are algebraically independent we get precisely the same Laurent power series by the explicit calculation of $\chi(\mathbb{C}^n, t^m \cdot \mathcal{O}_{\mathbb{C}^n})$. $\square$

Now assume that $Y \subset \mathbb{C}^n$ is a $T$-invariant affine variety.

Definition 6.4.19 (Equivariant Euler characteristic for a $T$-sheaf on an affine variety). Under the assumption 6.4.15 *the equivariant Euler characteristic* of a $T$-sheaf $\mathcal{F}$ on $Y \subset \mathbb{C}^n$ is a Laurent power series

$$\chi^T(Y, \mathcal{F}) = \chi^T(\mathbb{C}^n, \iota_*\mathcal{F}) \in \mathbb{Z}((t_1, \dots, t_r)),$$

where $\iota : Y \to \mathbb{C}^n$ is the closed embedding.

Assume that $X \xrightarrow{p} Y \subset \mathbb{C}^n$ is an equivariant projective map from a smooth $T$-variety to an affine $T$-variety.

Definition 6.4.20 (Equivariant Euler characteristic for a $T$-sheaf). Under the assumption 6.4.15 *the equivariant Euler characteristic* of a $T$-sheaf $\mathcal{F}$ on $X$ is a Laurent power series

$$\chi^T(X,\mathcal{F}) = \sum_{i=0}^{\dim X}(-1)^i\chi^T(Y, R^i p_*\mathcal{F}) = \sum_{i=0}^{\dim X}(-1)^i\chi^T(\mathbb{C}^n, R^i(\iota\circ p)_*\mathcal{F}) \in \mathbb{Z}((t_1,\dots,t_r)).$$

Denote the connected components of $X^T$ by $X_1,\dots,X_s$. By corollary 6.3.3 $X_i$ are smooth varieties. Let $N^*_{X_i/X}$ be a conormal bundle to $X_i$ in $X$. Denote $p_i : X_i \to \{0\} = \operatorname{Spec}\mathbb{C}$ the constant map.

Corollary 6.4.21 (Lefschetz-Riemann-Roch theorem). *The equivariant Euler characteristic of $T$-vector bundle $E$ on $X$ is equal to:*

$$\chi^T(X,E) = \sum_{i=1}^{s} p_{i*}\left(\frac{E|_{X_i}}{\lambda_{-1}(N^*_{X_i/X})}\right) \in S^{-1}R(T),$$

*where $S \subset R(T)$ is the same multiplicative system as in the localization theorem 6.4.13 .*

Proof. This follows immediately by the definition of $\chi^T(X,E)$ when we apply theorem 6.4.13(iii) and proposition 6.4.18. □

Suppose further that in the above situation each component $X_i$ is a point and $\{\nu_{i,j}\}_{j=1}^{\dim X}$ is the compass (definition 6.3.5) of $X_i$ in $X$. Let $L$ be a $T$-line bundle on $X$ and let $\mu_i(L)$ be the weight of $T$-action on the fibre over the point $X_i$.

Corollary 6.4.22. *The equivariant Euler characteristic of $T$-line bundle $L$ on $X$ is equal to:*

$$\chi^T(X,L) = \sum_{i=1}^{s}\frac{t^{\mu_i(L)}}{\prod_{j=1}^{\dim X}(1-t^{\nu_{i,j}})}.$$

Proof. This follows from corollary 6.4.21 since the cotangent space to the $i$-th $T$-fixed point is in this case precisely $N^*_{X_i/X}$. □

CHAPTER 7

# Resolutions of symplectic quotient singularities in dimension four

In this final chapter we apply general results from chapter 4 and the results on torus actions from the previous chapter to crepant resolutions of symplectic quotient singularities in dimension four admitting an action of two-dimensional torus. In section 7.1 we introduce briefly the setup in which we work and we classify all the crepant resolutions of a given symplectic quotient singularity as GIT quotients of the Cox ring of a single resolution. Then, in section 7.2 we outline the strategy which we use to analyze examples in the next three sections. The order of examples correspond to their level of complexity. In section 7.3 we study the symplectic action of the group $S_3$ on $\mathbb{C}^4$ – it is the simplest of the examples considered since there is only one irreducible exceptional divisor on the crepant resolution. Then, in section 7.4 we consider the symplectic action of the wreath product $\mathbb{Z} \wr S_2$ on $\mathbb{C}^4$. It is the simplest case of the wreath product of the group defining Du Val singularity with $S_2$ – the one corresponding to the $A_1$ singularity. Finally, in section 7.5 we study the most challenging of the three examples, the binary tetrahedral group acting on $\mathbb{C}^4$ (see 7.5.1 for a description of the action), which according to the work of Bellamy and Schedler [**11**] and Lehn and Sorger [**68**], gives the quotient admitting a symplectic resolution. In the last section 7.6 we present thoughts on the possibility of generalization of the method applied to the three examples worked out in this chapter.

Parts of this chapter are based on preprints [**32**] (joint work with Maria Donten-Bury) and [**44**].

## 7.1. General setting

Let $G \subset \mathrm{Sp}_4(\mathbb{C})$ be a finite group. Assume that there exists a symplectic resolution $\varphi\colon X \to \mathbb{C}^4/G$. Let $E_1, \ldots, E_m$ be the components of the exceptional divisor of the resolution $\varphi\colon X \to \mathbb{C}^4/G$. Let $C_i$ denote the generic fibre of $\varphi|_{E_i}\colon E_i \to \varphi(E_i)$. By theorem 2.3.13, the classes of $C_1, \ldots, C_m$ form a basis of the vector space $N_1(X/(\mathbb{C}^4/G))$ which in our case is equal to $N_1(X)$. The dual basis of $N^1(X)$ (via the intersection pairing) will be denoted by $L_1, \ldots, L_m$. Then, the coefficients of a divisor $D$ on $X$ in this basis are the intersection numbers $(C_i.D)_i$.

Proposition 7.1.1 ([**37**, 2.16]). *We have a commutative diagram, whose rows are exact sequences:*

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i=1}^{m} \mathbb{Z}E_i \arrow[r] \arrow[d, equal] & \mathrm{Cl}(X) \arrow[r, "\varphi_*"] \arrow[d, hook] & \mathrm{Cl}(\mathbb{C}^4/G) \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \bigoplus_{i=1}^{m} \mathbb{Z}E_i \arrow[r] & \bigoplus_{i=1}^{m} \mathbb{Z}L_i \arrow[r] & Q \arrow[r] & 0,
\end{tikzcd}$$

*Here the homomorphism $\bigoplus_i \mathbb{Z}E_i \to \bigoplus_i \mathbb{Z}L_i$ takes $E_i$ to $\sum_i (E_i.C_j)L_j$, and the group $Q$ is defined as its cokernel. The image of $D \in \mathrm{Cl}(\mathbb{C}^4/G)$ in $Q$ is given by $D \mapsto \sum_i (\overline{D}.C_i)[L_i]$.*

By proposition 2.3.12 we know that $\mathbb{C}^4/G$ has Du Val singularities in codimension 2 and the intersection matrix $(E_i.C_j)_{i,j}$ is a direct sum of corresponding Cartan matrices, see also [**92**, Theorem. 1.4] and [**3**, Thm. 4.1]. In particular it is invertible and so the lattice $\mathrm{Cl}(X) = \mathrm{Pic}(X) \subset N^1(X)$ is a finite index sublattice of the lattice $\bigoplus_{i=1}^m \mathbb{Z}L_i \subset N^1(X)$.

Combining the proposition 7.1.1 with results of section 4.2 we have an embedding

$\Theta\colon \mathcal{R}(X) \to \mathcal{R}(\mathbb{C}^4/G)\otimes\mathbb{C}[\mathrm{Cl}(X)] \subset \mathcal{R}(\mathbb{C}^4/G)\otimes\mathbb{C}\left[\oplus_i \mathbb{Z}L_i\right] = \mathbb{C}[x_1,\dots,x_4]^{[G,G]}[t_1^{\pm 1},\dots,t_m^{\pm 1}]$.

In cases of our interest it will be true that $\mathrm{Cl}(X) = \bigoplus_i \mathbb{Z}L_i$.

Given the system of $\mathrm{Ab}(G)^\vee$-homogeneous generators $\phi_1,\dots,\phi_s$ of $\mathbb{C}[x_1,y_1,x_2,y_2]^{[G,G]}$ we consider the subalgebra $\mathcal{R} \subset \Theta(\mathcal{R}(X))$ generated by elements of the form $\phi_i t^{\overline{D}_{\phi_i}}$ and $t^{E_j}$, where $\overline{D}_{\phi_i}$ is the strict transform of the divisor $D_{\phi_i} \in \mathrm{WDiv}(\mathbb{C}^4/G)$ corresponding to $\phi_i$. By corollary 4.2.5 $\nu_{E_i} = \frac{1}{r_i}\nu_{g_i}$, where $g_i \in G$ is an element in the conjugacy class corresponding to $E_i$ via McKay correspondence and $r_i$ is the order of $g_i$. On the other hand, using proposition 7.1.1 we can express $\Theta$ directly by $\Theta(\phi t^D) = \varphi_*(\phi)\prod_{i=1}^m t^{D.C_i}$.

Remark 7.1.2. In section 4.2 as well as in chapter 5 we considered the embedding $\Theta$ in coordinates given by the basis $-\frac{1}{r_1}E_1,\dots,-\frac{1}{r_m}E_m$ of the lattice $\Lambda_X \subset N^1(X)$. To pass from those coordinates to the coordinates given by a basis $L_1,\dots,L_m$ of the lattice $\bigoplus_{i=1}^m \mathbb{Z}L_i$ one uses the matrix $(E_i.C_j)_{i,j}$. More precisely, if $\phi_D \in \mathbb{C}[x_1,y_1,x_2,y_2]^{[G,G]}$ is an element corresponding to a divisor $D \in \mathrm{WDiv}(\mathbb{C}^4/G)$ then we have:

$$\left(\frac{1}{r_1}\nu_1(\phi_D),\dots,\frac{1}{r_m}\nu_m(\phi_D)\right)\cdot(E_i.C_j)_{i,j} = -(\overline{D}.C_1,\dots,\overline{D}.C_m). \tag{7.1.3}$$

Recall the cone $\mathrm{Mov}(\mathcal{R})$ introduced in (4.1.19). The next proposition is the analogue of proposition 5.1.1 and again it gives a geometric motivation for seeking the generators of $\mathcal{R}(X)$. It is true for symplectic resolutions in arbitrary dimension.

Proposition 7.1.4. *Assume that $G \subset \mathrm{Sp}_{2n}(G)$ is a finite group and there exists a symplectic resolution $\varphi\colon X \to \mathbb{C}^{2n}/G$. If $\mathcal{R} = \Theta(\mathcal{R}(X))$ then* $\mathrm{Mov}(X) = \mathrm{Mov}(\mathcal{R})$. *Moreover there is a one-to-one correspondence between the crepant resolutions of $\mathbb{C}^{2n}/G$ and GIT chambers of the action of the Picard torus $\mathbb{T}$ on* $\mathrm{Spec}\,\mathcal{R}$ *contained in* $\mathrm{Mov}(\mathcal{R})$. *Namely, taking GIT quotients corresponding to GIT chambers of this action we obtain all pairwise nonisomorphic symplectic resolutions of $\mathbb{C}^{2n}/G$.*

Proof. Like in the proof of the proposition 5.1.1 it is a corollary from proposition 4.1.20 if we show that every minimal model of $X$ is smooth. In the symplectic case, assuming that the smooth minimal model exists, all the other minimal models are smooth by [**76**, Corollary 31] or by a more general result [**76**, Corollary 25]. The smoothness of at least one minimal model follows from our assumption on the existence of a symplectic resolution and by propositions 2.1.11 and 2.3.5. □

## 7.2. Outline of the strategy

In what follows we consider three examples of symplectic quotient singularities admitting an action of a two-dimensional algebraic torus and we study geometry of their crepant resolutions. Each of the examples is treated according to the same strategy that we outline briefly in this section.

To avoid confusion we emphasize the fact that throughout the remaining part of this chapter we consider simultaneously two different tori. One is the Picard torus $\mathbb{T} = \mathrm{Hom}(\mathrm{Cl}(X),\mathbb{C}^*)$ of a resolution $X \to \mathbb{C}^4/G$ and the other one is the torus $T = (\mathbb{C}^*)^2$ acting on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ by multiplication of scalars on each component $\mathbb{C}^2$. Torus $T$ acts also on the quotient and (equivariantly) on resolutions.

First, we compute the data related to the general method from section 4.2. We find $\mathrm{Ab}(G)^\vee$-homogeneous generators of $\mathbb{C}[x_1, y_1, x_2, y_2]^{[G,G]}$, which, by 4.2.1, is the Cox ring of the quotient $\mathbb{C}^4/G$. We also find monomial valuations corresponding to symplectic reflections, which in the symplectic case are the elements of age one and hence, by McKay correspondence (theorem 2.2.18), they give the valuations of crepant divisors over the quotient. Using this data, we find a 'candidate set' for a set of generators of the image of the embedding $\Theta$ from section 4.1. In all examples except the last one we stick to the set arising from $\mathrm{Ab}(G)^\vee$-homogeneous invariants of the commutator subgroup computed in Singular [**27**]. In the last example it is easy to see that such set of invariants does not satisfy the valuation compatibility criterion, but we modify it by a simple linear transformation involving two generators. For such a candidate set we form a subalgebra $\mathcal{R}$ of the Cox ring that they generate and compute the cone $\mathrm{Mov}(\mathcal{R})$ together with its GIT subdivision with respect to the Picard torus action on $\operatorname{Spec}\mathcal{R}$ induced by the $\mathrm{Cl}(X)$-grading. This step, together with the formulation of results in each example, is done in sections 7.3.1, 7.4.1 and section 7.5.1 respectively.

Then, we study geometry of the GIT quotient with respect to a linearization $\lambda$ from a chamber of GIT subdivision of $\mathrm{Mov}(\mathcal{R})$. We start with the study of the central fibre, i.e. the fibre over the point $[0] \in \mathbb{C}^4/G$ given by a natural morphism $(\operatorname{Spec}\mathcal{R})^{ss}(\lambda)/\!/\mathbb{T} \to \mathbb{C}^4/G$. This is done in sections 7.3.2, 7.4.2 and 7.5.2.

Next, we present an open cover of the GIT quotient consisting of affine spaces, thus proving the smoothness of the quotient. At this point we are also able to prove that the GIT quotient is a crepant resolution. This step is done in sections 7.3.3, 7.4.3 and 7.5.3.

Then, in sections 7.3.4, 7.4.4 and 7.5.4 we explain how we found the open cover. Working under the assumption of smoothness of the quotient, we find compasses at fixed points (see definition 6.3.5) of two-dimensional torus action on the GIT quotient and give the heuristic argument that was used to predict the covers in sections 7.3.3, 7.4.3 and 7.5.3.

To prove that $\mathcal{R}$ is actually the whole Cox ring, we employ results of sections 6.4 and 4.4. First, we prove that $\mathrm{Mov}(\mathcal{R}) = \mathrm{Mov}(X)$ and that the GIT subdivision of $\mathrm{Mov}(\mathcal{R})$ is the same as the subdivision of $\mathrm{Mov}(X)$ induced by the Picard torus action on $\mathcal{R}(X)$. We also calculate the Hilbert series for the subring $\mathcal{R}(X)_{\geq 0} = \bigoplus_{L \in \mathrm{Mov}(X)} \mathcal{R}(X)_L$, using corollary 6.4.22. This is done in sections 7.3.5, 7.4.5 and 7.5.5. Finally, in sections 7.3.6, 7.4.6 and 7.5.6 we apply the results of section 4.4 to deduce that $\mathcal{R} = \mathcal{R}(X)$.

## 7.3. Symmetric group $G = S_3$

**7.3.1. The setup and results.** In this section we apply the strategy presented in the previous section to investigate an action of $G = S_3$ on $\mathbb{C}^4$ with coordinates $(x_1, y_1, x_2, y_2)$. We take the direct sum of two copies of the standard two-dimensional representation of $S_3$, which preserves the symplectic form $\omega = dx_1 \wedge dy_2 + dy_1 \wedge dx_2$ on $\mathbb{C}^4$. That is, we identify $G$ with the matrix group generated by

$$\varepsilon = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & \epsilon^{-1} & 0 & 0 \\ 0 & 0 & \epsilon & 0 \\ 0 & 0 & 0 & \epsilon^{-1} \end{pmatrix} \qquad \alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $\epsilon = e^{2\pi i/3}$ is a 3rd root of unity. The following proposition summarizes basic properties of this representation; all of them are immediate.

Proposition 7.3.1. *With the notation above:*

(1) *There are three conjugacy classes of elements of $G$ of which one consists of symplectic reflections.*
(2) *The commutator subgroup $H = [G, G]$ is a cyclic group of order 3 generated by $\varepsilon$.*
(3) $\mathrm{Ab}(G) = G/[G, G]$ *is a group of order 2 generated by the class of $\alpha$.*
(4) *Symplectic reflections in $G$ are $\varepsilon^i\alpha$ for $i = 0, 1, 2$. These elements correspond to transpositions in $S_3$ and as such generate $G$.*
(5) *The representation of $G$ defined as above is reducible. It decomposes into a direct sum of two 2-dimensional representations $\mathbb{C}^4 = V_1 \oplus V_2$. In particular the $(\mathbb{C}^*)^2$-action on $\mathbb{C}^4$ induced by multiplication by scalars on $V_i$ commutes with $G$.*

Let $\Sigma \subset \mathbb{C}^4/G$ be the singular locus of the quotient. It can be described as follows.

Proposition 7.3.2. *The preimage of $\Sigma$ via the quotient map $\mathbb{C}^4 \to \mathbb{C}^4/G$ consists of six planes, each of which maps onto $\Sigma$. Outside the image of 0 the singular locus is a transversal $A_1$-singularity.*

Proof. This is a consequence of a direct computation of the subspaces fixed by symplectic reflections of $G$ – these are the only elements that stabilize a proper nontrivial subspace of $\mathbb{C}^4$. □

Let $\varphi \colon X \to \mathbb{C}^4/G$ be a symplectic resolution. It is known that such a resolution exists (see [**11**, Sect. 1]) but we will also prove it independently in section 7.3.3. Using the symplectic McKay correspondence (theorem 2.3.11) and propositions 7.3.1 and 7.3.2 we obtain the following facts about the geometry of $X$.

Proposition 7.3.3. *There is a single irreducible exceptional divisor $E$ of $X$ which is mapped onto $\Sigma$. The central fibre $\varphi^{-1}([0])$ consist of an irreducible surface. The fibre of $\varphi$ over any point in $\Sigma \setminus [0]$ consists of one curve isomorphic to $\mathbb{P}^1$.*

By section 6.1 and proposition 7.3.1 we have also:

Corollary 7.3.4. *There is a natural $T := (\mathbb{C}^*)^2$-action on $X$ making $\varphi$ an equivariant map.*

As we noted earlier, we consider two different tori – one is the Picard torus $\mathbb{T}$ which in this case is one-dimensional and the other one is the two-dimensional torus $T$ acting on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ by multiplication of scalars on each component $\mathbb{C}^2$.

Let $C$ be the numerical class of a complete curve which is a generic fibre of the morphism $\varphi|_E \colon E \to \Sigma$. We may describe the generator of the Picard group of $X$ in terms of its intersection with curve $C$. By proposition 7.1.1 we have:

Proposition 7.3.5. *The Picard group of $X$ is a free rank one abelian group generated by line bundle $L$ such that $L.C = 1$.*

One may see the Cox ring $\mathcal{R}(X)$ of $X$ as a subring of $\mathbb{C}[x_1, y_1, x_2, y_2]^{[G,G]}[t_1^{\pm 1}]$ (see 7.1). It follows that the action of the two-dimensional torus $T = (\mathbb{C}^*)^2$ on $X$ induces the action on $\operatorname{Spec} \mathcal{R}(X)$.

Proposition 7.3.6. *The elements $\phi_{ij}$ in the table below are the eigenvectors of the action of $\mathrm{Ab}(G)$ which generate the ring of invariants $\mathbb{C}[x_1, y_1, x_2, y_2]^{[G,G]} \subset \mathbb{C}[x_1, y_1, x_2, y_2]$.*

| *eigenvalue* | *generators* |
|---|---|
| 1 | $\phi_{01} = x_1y_1, \quad \phi_{02} = x_2y_2, \quad \phi_{03} = x_1y_2 + x_2y_1$ |
| | $\phi_{04} = x_1^3 + y_1^3,\ \phi_{05} = x_2^3 + y_2^3,\ \phi_{06} = x_1^2x_2 + y_1^2y_2,\ \phi_{07} = x_1x_2^2 + y_1y_2^2$ |
| $-1$ | $\phi_{11} = x_1y_2 - x_2y_1$ |
| | $\phi_{12} = x_1^3 - y_1^3,\ \phi_{13} = x_2^3 - y_2^3,\ \phi_{14} = x_1^2x_2 - y_1^2y_2,\ \phi_{15} = x_1x_2^2 - y_1y_2^2.$ |

Proof. We compute this generating set using Singular, [**27**]. First we find the invariants of the action of $[G, G]$. Then we split their linear span into smaller $\mathrm{Ab}(G)$-invariant subspaces on which the action of $\mathrm{Ab}(G)$ is easy to diagonalize. $\square$

Let $\nu$ be the monomial valuation $\mathbb{C}(x_1, y_1, x_2, y_2)^* \to \mathbb{Z}$ associated with symplectic reflection $\alpha$.

Remark 7.3.7. We have $\nu(\phi_{0j}) = 0$ and $\nu(\phi_{1j}) = 1$. Moreover $\nu(\phi_{ij}) = \overline{D}_{\phi_{ij}}.C$ by (7.1.3).

In the spirit of section 4.1 we may form a candidate set for a generating set of the Cox ring $\mathcal{R}(X)$ using generators $\phi_{ij}$. Our goal in this few sections is to prove that this is indeed the generating set. As we pointed out in section 7.2 we will not use the valuative criterion from 4.1, but multigraded Castelnuovo-Mumford regularity from 4.4 and the Lefschetz-Riemann-Roch theorem from 6.4. See [**32**, Sect. 4.2] for an alternative argument using valuative criterion.

Theorem 7.3.8. *The Cox ring $\mathcal{R}(X) \subset \mathcal{R}(\mathbb{C}^4/G)[t_1^{\pm 1}]$ of the symplectic resolution $X \to \mathbb{C}^4/G$ is generated by the elements $w_{01}, \ldots, w_{07}, w_{11}, \ldots, w_{15}, t$, where $w_{0i} = \phi_{0i}$, $w_{1j} = \phi_{1j}t_1$, $t = t_1^{-2}$. In particular the degree matrix with respect to the generator $L$ of $\mathrm{Pic}(X)$ (the first row) and to the $T$-action (remaining two rows) is:*

$$\begin{array}{c} \begin{array}{ccccccccccccc} w_{01} & w_{02} & w_{03} & w_{04} & w_{05} & w_{06} & w_{07} & w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & t \end{array} \\ \left(\begin{array}{ccccccccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & -2 \\ \hline 2 & 0 & 1 & 3 & 0 & 2 & 1 & 1 & 3 & 0 & 2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 3 & 1 & 2 & 1 & 0 & 3 & 1 & 2 & 0 \end{array}\right) \end{array}$$

We will prove this theorem in section 7.3.6. From now on we will denote the ring generated by the elements from the theorem 7.3.8 by $\mathcal{R}$.

It is a general principle that using the degrees of generators of the Cox ring of $X$ we can describe the movable cone $\mathrm{Mov}(X)$ and find the number of resolutions and the corresponding subdivision of $\mathrm{Mov}(X)$ into the nef cones of resolutions of $X$ (see proposition 7.1.4). However, since we use the description of movable cone and its subdivision to prove that $\mathcal{R}$ is a Cox ring of $X$ we give independent proofs.

Proposition 7.3.9.

(1) *The cone $\mathrm{Mov}(X)$ of movable divisors of $X$ is equal to $\mathrm{Mov}(\mathcal{R})$ and is generated by the line bundle $L$ and it is equal to the ample cone of $X$.*
(2) *There is a unique symplectic resolution of $\mathbb{C}^4/G$.*

Proof.

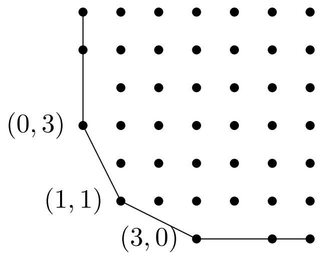
(1) This follows from theorem 2.3.13 which allows us to find $\mathrm{Mov}(X)$ and by definition of $\mathrm{Mov}(\mathcal{R})$ (see e.g. the paragraph preceeding proposition 7.1.4).
(2) This is an immediate consequence of the previous point and proposition 7.1.4.

$\square$

The next theorem also follows from theorem 7.3.8, but as we need it before we prove this theorem it is proven independently in the section 7.3.3.

Theorem 7.3.10. *Taking the GIT quotient of $\mathrm{Spec}\,\mathcal{R}$ by the Picard torus action with respect to the linearization given by the character corresponding to $L \in \mathrm{Cl}(X)$ one obtains the symplectic resolution of $\mathbb{C}^4/G$.*

Remark 7.3.11. The weights of the $T$-action on global sections of the fixed line bundle on $X$ are lattice points in $\mathbb{Z}^2$. Taking a convex hull one obtains a lattice polyhedron in $\mathbb{R}^2$. For example taking the line bundle $L$ one gets a polyhedron with the tail (or recession cone) equal to the positive quadrant of $\mathbb{R}^2$ and with a head spanned by the lattice points from the picture below (see [**2**, 4.2] for terminology):




By [**18**, Lemma 2.4(c)] if $L$ is globally generated, then marked vertices of this polyhedron correspond to $T$-fixed points of $X$. We will see in lemma 7.3.19 that indeed fixed points of this polytope correspond to points in $X^T$ and in lemma 7.3.16 that $L$ is globally generated.

**7.3.2. The central fibre of the resolution.** In this section we study the structure of the central fibre $\varphi^{-1}([0])$ of a resolution $\varphi\colon X \to \mathbb{C}^4/G$ using the ideal of relations between generators of the ring $\mathcal{R}$, under the assumption that $X$ is a GIT quotient of $\operatorname{Spec}\mathcal{R}$. The results of this section are used in the next one, where we investigate the action of the two-dimensional torus $T$ on $X$ with the fixed point locus $X^T$ contained in the central fibre. The additional assumption that $X$ is a GIT quotient of $\operatorname{Spec}\mathcal{R}$ is dealt with in section 7.3.3.

Lemma 7.3.12. *We have an isomorphism* $\operatorname{Spec}\mathcal{R}^{\mathbb{T}} \cong \mathbb{C}^4/G$. *In particular the inclusion of invariants* $\mathcal{R}^{\mathbb{T}} \subset \mathcal{R}$ *induce a map* $p\colon \operatorname{Spec}\mathcal{R} \to \mathbb{C}^4/G$.

Proof. This follows from lemma 4.3.1. □

Let $Z = p^{-1}([0])$. Decomposing the ideal of relations from the presentation of $\operatorname{Spec}\mathcal{R}$ one obtains the decomposition of $Z$ into irreducible components. We consider the closed embedding $\operatorname{Spec}\mathcal{R} \subset \mathbb{C}^{13}$ given by the generators of $\mathcal{R}$ from statement of theorem 7.3.8.

Proposition 7.3.13. *The components of $Z$ are the following subvarieties of $\mathbb{C}^{13}$:*

$$Z_u = V(w_{ij} \mid (i,j) \in (0,1), \dots, (0,7), (1,1), \dots, (1,5)),$$
$$Z_0 = V(t, w_{07}, w_{06}, w_{05}, w_{04}, w_{03}, w_{02}, w_{01}, w_{14}^2 - w_{12}w_{15},\ w_{13}w_{14} - w_{15}^2,\ w_{12}w_{13} - w_{14}w_{15}).$$

*The component $Z_u$ is contained in the locus of unstable points with respect to any linearization of the Picard torus via a character from the movable cone.*

Proof. The first part follows by decomposing ideal of $Z$ in Singular. The statements concerning stability are the consequences of lemma 7.3.14 below. □

Lemma 7.3.14. *The unstable locus of* $\operatorname{Spec}\mathcal{R}$ *with respect to a linearization of the Picard torus action on the trivial line bundle by the character corresponding to* $L \in \operatorname{Cl}(X) \setminus 0$ *is cut out by equations* $w_{11} = w_{12} = w_{13} = 0$. *Moreover, all the semistable points of $Z$ are stable and have trivial isotropy groups.*

Proof. This can be done by a computer calculation, using the Singular library gitcomp by Maria Donten-Bury (see `www.mimuw.edu.pl/~marysia/gitcomp.lib`). □

The following theorem gives a description of the central fibre. Let $W$ be the locus of stable points of $\operatorname{Spec}\mathcal{R}$ with respect to the $\mathbb{T}$-action linearized by the character corresponding to $L$ and consider the quotient map $W \to X$. Denote by $S$ the image of the set of stable points of the component $Z_0$. Note that this is the central fibre of $\varphi$.

THEOREM 7.3.15. *$S$ is a toric surface isomorphic to the projective cone over the image of the third Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$.*

PROOF. We may assume that $Z_0$ is the toric variety embedded into $\mathbb{C}^5$ with coordinates $w_{11}, w_{12}, w_{13}, w_{14}, w_{15}$ defined by the toric ideal generated by binomials:

$$w_{14}^2 - w_{12}w_{15},\ w_{13}w_{14} - w_{15}^2,\ w_{12}w_{13} - w_{14}w_{15}.$$

This coincides with homogeneous equations of the twisted cubic curve which is the image of the third Veronese embedding $\mathbb{P}^1 \to \mathbb{P}^3$. Since $\mathbb{T} = \mathbb{C}^*$ acts on the coordinates $w_{1i}$ with weights equal to 1 we are done. $\square$

LEMMA 7.3.16. *$L$ is a globally generated line bundle on $X$.*

PROOF. Since $L$ is invariant with respect to the $T$-action and the base point locus of a linear system is closed it either has to be empty or it has a nontrivial intersection with the central fibre $S$. But $L|_S$ is an ample line bundle on a toric variety and as such it has to be globally generated by [**23**, Theorem 6.1.7, Theorem 6.1.14]. $\square$

**7.3.3. Smoothness of the GIT quotient.** Let $\mathcal{R}$ be the subring of the Cox ring of the crepant resolution generated by the elements from the statement of theorem 7.3.8. In this section we show that the GIT quotient $\operatorname{Spec}\mathcal{R}//_L\mathbb{T}$ with respect to the linearization of the trivial line bundle by the character of the Picard torus $\mathbb{T}$ corresponding to $L$ is smooth. In consequence we see that $\operatorname{Spec}\mathcal{R}//_L\mathbb{T} \to \mathbb{C}^4$ is a crepant resolution. This makes the results on geometry of crepant resolutions of $\mathbb{C}^4/G$ in the previous section unconditional and helps to conclude that $\mathcal{R}$ is the whole Cox ring in section 7.3.6.

We consider $\operatorname{Spec}\mathcal{R}$ as a closed subvariety of $\mathbb{C}^{13}$ via the embedding given by generators from statement of Theorem 7.3.8.

THEOREM 7.3.17. *The stable locus of* $\operatorname{Spec}\mathcal{R}$ *with respect to the linearization of the trivial line bundle by a character corresponding to $L$ is covered by three $T \times \mathbb{T}$-invariant open subsets $U_1, U_2, U_3$ such that $U_i/\mathbb{T} \cong \mathbb{C}^4$. More precisely:*

*(1)* $U_1 = \{w_{13} \neq 0\}$ *and* $(\mathcal{R}_{w_{13}})^{\mathbb{T}} = \mathbb{C}\left[w_{02}, w_{05}, \frac{w_{11}}{w_{13}}, \frac{w_{15}}{w_{13}}\right]$,

*(2)* $U_2 = \{w_{11} \neq 0\}$ *and* $(\mathcal{R}_{w_{11}})^{\mathbb{T}} = \mathbb{C}\left[\frac{w_{12}}{w_{11}}, \frac{w_{13}}{w_{11}}, \frac{w_{14}}{w_{11}}, \frac{w_{15}}{w_{11}}\right]$,

*(3)* $U_3 = \{w_{12} \neq 0\}$ *and* $(\mathcal{R}_{w_{12}})^{\mathbb{T}} = \mathbb{C}\left[w_{01}, w_{04}, \frac{w_{11}}{w_{12}}, \frac{w_{14}}{w_{12}}\right]$.

*In particular the GIT quotient* $\operatorname{Spec}\mathcal{R}//_L\mathbb{T}$ *with respect to the linearization of the trivial line bundle by the character corresponding to $L$ is smooth.*

The choice of an open cover will be explained in the next section (see 7.3.25).

PROOF. Lemma 7.3.14 implies that $\{U_i\}_{i=1,2,3}$ form an open cover of the quotient. It remains to prove equalities from points (1)–(3). Note that then in each case the four generators of the ring on the right-hand side of the equality have to be algebraically independent as the GIT quotient $\operatorname{Spec}\mathcal{R}//\mathbb{T}$ is irreducible and of dimension four.

By symmetry it suffices to consider only $U_i$ for $i = 1, 2$. In each case we calculate the invariants of the localization of the coordinate ring of the ambient $\mathbb{C}^{13}$ obtaining:

(1) $\operatorname{Spec} \mathcal{R}^{\mathbb{T}}_{w_{13}} = \mathbb{C}\left[w_{0i}, \frac{w_{1j}}{w_{13}}, w_{13}^2 t\right]_{i=1,\dots,7,\ j=1,\dots,5}$

(2) $\operatorname{Spec} \mathcal{R}^{\mathbb{T}}_{w_{11}} = \mathbb{C}\left[w_{0i}, \frac{w_{1j}}{w_{11}}, w_{11}^2 t\right]_{i=1,\dots,7,\ j=1,\dots,5}$

Then, using the Gröbner basis of the ideal of relations between generators of $\mathcal{R}$ with respect to an appropriate lexicographic order, we verify with Singular [**27**] in each case that each of the generators of these two rings can be expressed as a polynomial of the four generators from the statement. □

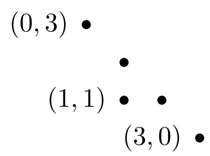
By the inclusion of invariants $\mathcal{R}^{\mathbb{T}} \subset \mathcal{R}$ (see lemma 7.3.12) we have the induced projective map $\operatorname{Spec} \mathcal{R}/\!/_L \mathbb{T} \to \operatorname{Spec} \mathcal{R}/\mathbb{T} \cong \mathbb{C}^4/G$.

Corollary 7.3.18. *The map* $\varphi\colon \operatorname{Spec} \mathcal{R}/\!/_L \mathbb{T} \to \operatorname{Spec} \mathcal{R}/\mathbb{T} \cong \mathbb{C}^4/G$ *is a crepant resolution.*

Proof. Denote $\overline{E} = \{t = 0\} \subset \operatorname{Spec} \mathcal{R}/\!/_L \mathbb{T}$. This is an irreducible divisor on $\operatorname{Spec} \mathcal{R}/\!/_L \mathbb{T}$. By the construction of $\mathcal{R}$ the map $\varphi$ is an isomorphism outside $\overline{E}$. Hence $\varphi$ is a resolution and it has to be crepant since there is only one crepant divisor over $\mathbb{C}^4/G$ by symplectic McKay correspondence and it has to be present on each resolution. □

**7.3.4. Compasses at fixed points.** In this section we obtain a local description of the action of the two-dimensional torus $T$ on a symplectic resolution $X = \operatorname{Spec} \mathcal{R}/\!/_L \mathbb{T}$ of $\mathbb{C}^4/G$ at fixed points of this action. We will not use the precise description of the smooth open cover of $X$ from theorem 7.3.17 as it was the compass calculation that originally led us to this open cover (see remark 7.3.25 at the end of this section for a detailed explanation). We find compasses of all fixed points of the action $T$ on $X$ (see definition 6.3.5).

Lemma 7.3.19. *The following diagram shows the weights of the action of $T$ on the space of sections of $H^0(X, L)$ which are nonzero after the restriction to the central fibre $S$.*




*The dots correspond to the weights of sections of $L$ restricted to $S$.*

Proof. This follows by a computer calculation (using Macaulay2 [**45**]) of dimensions of appropriate graded pieces of the coordinate ring of $Z_0$ from proposition 7.3.13. □

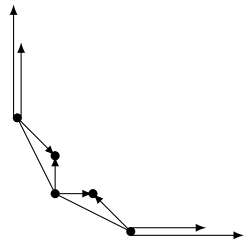
Remark 7.3.20. Lattice points from lemma 7.3.19 are contained in the polyhedron from remark 7.3.11. Moreover their convex hull form the minimal head of this polyhedron.

Remark 7.3.21. Considering the polytope which is the convex hull of weights marked by dots in lemma 7.3.19 we get the polytope of the line bundle $L$ pulled back to $S$ viewed as a toric variety (see [**23**, Chapters 2,6] for a reference on the polytopes in toric geometry). As $L$ is globally generated (see lemma 7.3.16) the vertices correspond to the fixed points of the action of $T$ on $S$, cf. [**18**, Lemma 2.4(3)]. In particular one obtains the weights of the action of $T$ on the tangent space to $S$ at fixed points.

Theorem 7.3.22. *The fixed points of the $T$-action correspond to the vertices of the polytope which is the convex hulls of weights marked by dots in lemma 7.3.19. The compasses of the points corresponding to the vertices are as in the table below:*

| *Point* | *Compass* | | | |
|---|---|---|---|---|
| $P_1 \leftrightarrow (0,3)$ | $\nu_{1,1} = (1,-2)$, | $\nu_{1,2} = (1,-1)$, | $\nu_{1,3} = (0,3)$, | $\nu_{1,4} = (0,2)$ |
| $P_2 \leftrightarrow (1,1)$ | $\nu_{2,1} = (-1,2)$, | $\nu_{2,2} = (2,-1)$, | $\nu_{2,3} = (0,1)$, | $\nu_{2,4} = (1,0)$ |
| $P_3 \leftrightarrow (3,0)$ | $\nu_{3,1} = (-2,1)$, | $\nu_{3,2} = (-1,1)$, | $\nu_{3,3} = (2,0)$, | $\nu_{3,4} = (3,0)$ |

The following picture illustrates the weights of the $T$-action calculated in the theorem. It is a directed graph. The points correspond to the sections of $H^0(X, L)$ for $L$ which are nonzero after the restriction to the central fibre together with vectors, as in lemma 7.3.19. The directed edges are the vectors from the compasses attached to the points which correspond to fixed points of $T$-action. In case when two vertices are connected by the two edges pointing in both ways we depict them by a single edge without any arrow.



In the proof of the theorem we will use two lemmas that are true in the following general setting. Let $G \subset \mathrm{Sp}_4(\mathbb{C})$ be a finite group, such that $\mathbb{C}^4 = V_1 \oplus V_2$ as $G$-representations, with $\dim V_i = 2$. Let $\varphi\colon X \to \mathbb{C}^4/G$ be a symplectic resolution. Then, by section 6.1 there is a two-dimensional torus $T$ acting on $X$ and $\mathbb{C}^4/G$ so that $\varphi$ is equivariant. Clearly $X^T \subset \varphi^{-1}([0])$. Let $x \in X^T$. Then we may decompose $T_xX = T_x(\varphi^{-1}([0])) \oplus V'$ where $V'$ is an eigenspace of the $T$-action. We will call the weights of $T$ on $V'$ the *remaining weights* on $T_xX$. Note that lemmas give an information on the weights of torus actions on $T_xX$ and in the proof of the theorem we will draw conclusions about weights on the dual space $T_x^*X$, by taking negatives of weights on $T_xX$.

LEMMA 7.3.23. *The remaining weights for the $T$-action on $T_xX$ are of the form $(a,0)$ or $(0,a)$.*

PROOF. First note that every orbit of the $T$-action on $\mathbb{C}^4 \setminus 0$ is either two-dimensional or has the isotropy group equal to $\mathbb{C}^* \times 1$ or $1 \times \mathbb{C}^*$.
Now take any $x \in X^T$, any remaining weight $\lambda$ of the $T$-action on $T_xX$ and a one-dimensional eigenspace $V_\lambda$ with this weight which is not contained in the tangent space to the central fibre. By corollary 6.3.4 such an eigenspace corresponds to the closure of an orbit $O$ of the $T$-action via an equivariant local étale map $U \to T_xX$, where $U$ is an invariant neighbourhood of $x$. In particular $\dim O = 1$, and $\dim \varphi(O) = 1$, as $O$ is not contained in the central fibre of the resolution $\varphi : X \to \mathbb{C}^4/G$. Therefore $\varphi(O)$ as well as $O$ and $V_\lambda$ are stabilized by either $\mathbb{C}^* \times 1$ or $1 \times \mathbb{C}^*$ and the claim follows. □

For the next claim, consider $\mathbb{C}^*$ as a subtorus of $T$ embedded with the weight $(1,-1)$.

LEMMA 7.3.24. *Let $x \in X^{\mathbb{C}^*}$. Among the weights of the induced $\mathbb{C}^*$-action on $T_xX$ two weights are positive and two are negative.*

PROOF. Let

$$X_x^+ = \{x \in X\colon \lim_{t\to 0} tx \in X_x\},$$
$$X_x^- = \{x \in X\colon \lim_{t\to 0} t^{-1}x \in X_x\},$$

where $X_x \subset X^{\mathbb{C}^*}$ is the connected component containing $x$. We will use the fact that $X_x^{\pm}$ are irreducible, locally closed subsets of $X$ (see theorem 6.3.7).

If at least three weights at $x$ were nonnegative then $\dim X_x^+ \geq 3$ by theorem 6.3.7. Similarly if at least three weights at $x$ are nonpositive then $\dim X_x^- \geq 3$. Suppose that $\dim X_x^+ \geq 3$. On the other hand $(\mathbb{C}^4)_0^+ = \mathbb{C}^2 \times 0$ is two-dimensional and hence also $(\mathbb{C}^4/G)_0^+ = (\mathbb{C}^2 \times 0)/G$ is two-dimensional. Since $\dim \varphi^{-1}([0]) = 2$ and the fibres of $\varphi\colon X \setminus \varphi^{-1}([0]) \to \mathbb{C}^4/G$ are of dimension at most one then $\dim \varphi(X_x^+) \geq \dim X_x^+ - 1 = 2$. As $\varphi(X_x^+) \subset (\mathbb{C}^4/G)_0^+$ we know that $\dim \varphi(X_x^+) = 2$ and hence $X_x^+$ has to be an exceptional divisor of the resolution $\varphi : X \to \mathbb{C}^4/G$. But the image of such an exceptional divisor is contained in the singular locus of $\mathbb{C}^4/G$ which consists of the image of four planes in $\mathbb{C}^4$ that have one-dimensional intersection with $\mathbb{C}^2 \times 0$ and so, $\dim \varphi(X_x^+) = 1$, a contradiction. The case $\dim X_x^- \geq 3$ is completely analogous. □

Proof of theorem 7.3.22. First, note that $X^T$ is contained in the central fibre. In particular $X^T = S^T$ and as we noted in the remark 7.3.21 the elements of $S^T$ correspond to the vertices of the polytope from the statement. Taking into account the natural inclusion of the tangent space to the central fibre into the tangent space of $X$ most of the weights can be deduced from the fact that the action of $T$ on $S$ is toric. The polytope spanned by the points marked by dots in the lemma 7.3.19 is the polytope of this toric variety. Thus using toric methods one can describe affine cover of $S$ and the weights of the $T$-action on the tangent space to its $T$-fixed points. Altogether the weights calculated in this way are the ones from the assertion except $\nu_{1,3}, \nu_{1,4}, \nu_{3,3}, \nu_{3,4}$.

Now the calculation of all the weights for the homothety action is easy, since the symplectic form is of weight two with respect to this action, and we can compute at least two weights at each $T$-fixed point, by summing components of each known weight $\nu_{i,j}$. For the remaining weights of the $T$-action we combine lemmas 7.3.23 and 7.3.24.

For example we know that $\nu_{1,1} = (1, -2)$ and $\nu_{1,2} = (1, -1)$, which gives the weights of the homothety action $-1$ and $0$ at point corresponding to $(0, 3)$. By proposition 6.2.4 the remaining weights for homothety are equal to 3 and 2. By the lemma 7.3.23 the remaining weight $\nu_{1,3}$ is of the form $(3, 0)$ or $(0, 3)$ and $\nu_{1,4}$ is of the form $(2, 0)$ or $(0, 2)$. Since $\nu_{1,1}$ and $\nu_{1,2}$ yield two positive weights for the $\mathbb{C}^*$-action considered in the lemma 7.3.24 and so do $(3, 0)$ and $(2, 0)$ we have $\nu_{1,3} = (0, 3)$ and $\nu_{1,4} = (0, 2)$. Other weights are computed analogously. □

Remark 7.3.25. Note that theorem 7.3.22 follows immediately by theorem 7.3.17 when we note that $U_i \cong \mathbb{C}^4$ is a $T$-invariant neighbourhood of the $i$-th $T$-invariant point (we order points as in rows of the table from the assertion). Nevertheless, since we used the statement of theorem 7.3.22 to guess the open cover $U_i$, we decided to give an independent proof to preserve the logical consequence of our considerations.

More precisely, to find isomorphisms $U_i \cong \mathbb{C}^4$ in theorem 7.3.17 we used the compass calculation from this section with a priori assumption on smoothness of the quotient $\operatorname{Spec} \mathcal{R}/\!/_L \mathbb{T}$ as a heuristic. To guess the coordinates on each invariant open subset $U_i$ we picked an element $f \in \mathcal{R}$ of a degree corresponding to the $i$-th fixed point and four elements of $(\mathcal{R}_f)^{\mathbb{T}}$ of degrees equal to the predicted weights of the action on the cotangent space.

### 7.3.5. Dimensions of movable linear systems.

In this section we use the torus $T$ action on $X$ to give a formula for dimensions of these graded pieces of $\mathcal{R}(X)$ which correspond to the movable linear systems on the resolution.

Let $X \to \mathbb{C}^4/G$ be the resolution corresponding to the linearization of the Picard torus action by a character $L \in \mathrm{Cl}(X)$.

Denote by $P_i$ the fixed points of the $T$-action on $X$ as in the table from theorem 7.3.22. Let $\{\nu_{i,j}\}_{j=1}^{4}$ denote the compass of $P_i$ in $X$. Let us also denote by $\mu_i(L)$ the weight of the $T$-action on the fibre of $L$ over $P_i$. Note that $\mu_i$ is linear i.e. $\mu_i(A+B) = \mu_i(A) + \mu_i(B)$.

REMARK 7.3.26. In section 7.3.4 we computed the weights $\mu_i$ for the line bundle $L$:

| $i$ | $\mu_i(L)$ |
|---|---|
| 1 | $(0,3)$ |
| 2 | $(1,1)$ |
| 3 | $(3,0)$ |

THEOREM 7.3.27. *If $h^0(X,pL)_{(a,b)}$ is the dimension of the subspace of sections $H^0(X,pL)$ on which $T$ acts with the weight $(a,b)$, then we have the following generating function for such dimensions for line bundles inside the movable cone:*

$$\sum_{a,b,p\geq 0} h^0(X,pL)_{(a,b)} y^p t_1^a t_2^b = \sum_{i=1}^{3} \frac{1}{(1-t^{\mu_i(L)}y)\prod_{j=1}^{4}(1-t^{\nu_{i,j}})}.$$

REMARK 7.3.28. The computed generating function may be interpreted as the multivariate Hilbert series of a $\mathbb{Z}_{\geq 0}\times\mathbb{Z}^2_{\geq 0}$-graded subalgebra of $\mathcal{R}(X)$ consisting of the graded pieces of $\mathcal{R}(X)$ corresponding to movable line bundles on $X$. We will use this interpretation in the next section.

PROOF. By a corollary of the Lefschetz-Riemman-Roch theorem (corollary 6.4.22) we have:

$$\chi^T(X,L) = \sum_{i=1}^{3} \frac{t^{\mu_i(L)}}{\prod_{j=1}^{4}(1-t^{\nu_{i,j}})}.$$

Using the linearity of $\mu_i$:

$$\sum_{p\geq 0}\chi^T(X,pL)y^p = \sum_{p\geq 0}\sum_{i=1}^{3} \frac{t^{p\mu_i(L)}}{\prod_{j=1}^{4}(1-t^{\nu_{i,j}})}y^p = \sum_{i=1}^{3} \frac{1}{(1-t^{\mu_i(L)}y)\prod_{j=1}^{4}(1-t^{\nu_{i,j}})}.$$

The assertion follows now by Kawamata-Viehweg vanishing, which for $p \geq 0$ implies

$$\chi^T(X,pL) = \sum_{a,b\geq 0} h^0(X,pL)_{(a,b)} t_1^a t_2^b.$$

$\square$

EXAMPLE 7.3.29. The dimensions of the weight spaces corresponding to the lattice points in a head of the polyhedron spanned by weights for the line bundle $L$ considered in remark 7.3.11 and in section 7.3.4 can be depicted on the following diagram:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1 | 3 | 4 | 5 | 7 | 8 | 9 | 11 |
| 1 | 2 | 3 | 5 | 5 | 7 | 8 | 9 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| | 2 | 2 | 3 | 4 | 5 | 5 | 7 |
| 1 | 1 | 2 | 3 | 3 | 4 | 5 | 5 |
| | 1 | 1 | 2 | 2 | 3 | 3 | 4 |
| | 1 | 1 | 1 | 2 | 2 | 2 | 3 |
| | | | 1 | | 1 | 1 | 1 |

**7.3.6. The Cox ring.** Using the generating function from theorem 7.3.27 we may prove theorem 7.3.8. In this section we outline an argument using the methods of section 4.4. Denote by $\mathcal{R}(X)$ the Cox ring of $X$ and by $\mathcal{R}$ the subring of $\mathcal{R}(X)$ generated by the elements from the statement of theorem 7.3.8.

LEMMA 7.3.30. *The Cox ring of $X$ is generated by $t$ and by*

$$\mathcal{R}(X)_{\geq 0} := \bigoplus_{p \geq 0} H^0(X, pL).$$

PROOF. This is a particular case of proposition 4.4.5. □

Thus to prove that $\mathcal{R} = \mathcal{R}(X)$ it suffices to show that $\mathcal{R}$ contains $\mathcal{R}(X)_{\geq 0}$. The following lemma reduces the problem further, to finitely many graded pieces with respect to $\mathbb{Z}^2$-grading by characters of the Picard torus of $X$.

LEMMA 7.3.31. *$\mathcal{R}(X)_{\geq 0}$ is generated by the sections of all linear spaces $H^0(X, M)$ for $M \in \mathcal{S}$ where $\mathcal{S} := \{\mathcal{O}_X, L, 2L\}$.*

PROOF. By virtue of lemma 7.3.16 this follows from proposition 4.4.4 with $m_1 = 1$ and $r = 2$ for cone $\sigma = \mathrm{Mov}(X)$. □

Hence we reduced the problem to showing that $\mathcal{R}$ contains spaces of global sections only for these finitely many line bundles which are elements of $\mathcal{S}$ in lemma 7.3.31. This, with the help of computer algebra, can be done with the use of the previous section, namely by theorem 7.3.27 in which we computed the Hilbert series of $\mathcal{R}(X)_{\geq 0}$.

LEMMA 7.3.32. *$\mathcal{R}$ contains $H^0(X, M)$ for each $M \in \mathcal{S}$, where $\mathcal{S}$ is as in the lemma 7.3.31.*

PROOF. We calculate the Hilbert series of $\mathcal{R}$ in Macaulay2 [**45**]. It is of the form:

$$\frac{1}{1 - y^{-2}} \cdot F(y, t_1, t_2),$$

where:

$$F(y, t_1, t_2) = \\ = \frac{1 + yt_1^2t_2 + yt_1t_2^2 + t_1t_2 + t_1^3 + t_1^2t_2 + t_1t_2^2 + t_2^3 - yt_1^4t_2 - yt_1^3t_2^2 - yt_1^2t_2^3 - yt_1t_2^4 - yt_1^3t_2^3 - t_1^3t_2^2 - t_1^2t_2^3 - yt_1^4t_2^4}{(1 - t_2^2)(1 - t_1^2)(1 - yt_2^3)(1 - yt_1^3)(1 - yt_1t_2)}.$$

Then, using Singular [**27**] we extract from it the Hilbert series for each of the vector spaces $\mathcal{R}_M$, $M \in \mathcal{S}$, graded by characters of $T$. The Hilbert series $F_0, F_1, F_2$ for $\mathcal{O}_X, L, 2L$ respectively are:

$$\begin{aligned} F_0(t_1, t_2) &= \tfrac{F(1,t_1,t_2) + F(-1,t_1,t_2)}{2}, \\ F_1(t_1, t_2) &= \tfrac{F(1,t_1,t_2) - F(1,t_1,t_2)}{2}, \\ F_2(t_1, t_2) &= F_0(t_1, t_2) - F(0, t_1, t_2). \end{aligned}$$

Using Singular we check that for each $M \in \mathcal{S}$ the Hilbert series of $\mathcal{R}_M$ agrees with the Hilbert series for $\mathcal{R}(X)_M$ which we calculated in theorem 7.3.27. Since $\mathcal{R} \subset \mathcal{R}(X)$ it means that $\mathcal{R}_M = \mathcal{R}(X)_M$. □

## 7.4. Wreath product $G = \mathbb{Z}_2 \wr S_2$

**7.4.1. The setup and results.** In this part we consider the symplectic action of the wreath product $G = \mathbb{Z}_2 \wr S_2$ on $\mathbb{C}^4$ with coordinates $(x_1, y_1, x_2, y_2)$ and symplectic form $dx_1 \wedge dx_2 + dy_1 \wedge dy_2$. The action is given by the embedding $\mathbb{Z}_2 \cong \langle -\mathrm{id}\rangle \subset \mathrm{SL}_2(\mathbb{C})$ and by permutation of factors $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$. More precisely, $G$ is generated by matrices

$$\alpha = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad \beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Remark 7.4.1. This is a subgroup of the 32-element group investigated in [**37**]. In the notation used therein $\alpha = T_0,\ \beta = T_2$.

The treatment follows the outline given in section 7.2 and is analogous to the one applied in case $G = S_3$ in section 7.3. Hence, whenever possible, we will omit full proofs and point out that arguments are analogous.

Below we list basic properties of $G$. The arguments are straightforward.

Proposition 7.4.2. *With notation as above:*

- *$G$ is isomorphic to the dihedral group $D_8$ of order 8, the isomorphism is given by identifying elements $\pm\alpha, \pm\beta$ with reflections and $\alpha\beta$ with rotation.*
- *The commutator subgroup $[G, G] \cong \mathbb{Z}_2$ is generated by matrix $-\mathrm{id}$.*
- *$\mathrm{Ab}(G) = G/[G, G] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ is generated by classes of $\alpha$ and $\beta$.*
- *Matrices $\pm\alpha$ and $\pm\beta$ are the only symplectic reflections in $G$. Their conjugacy classes are $\{\pm\alpha\},\ \{\pm\beta\}$.*
- *The representation of $G$ defined as above is reducible. It decomposes into a direct sum of two 2-dimensional representations $\mathbb{C}^4 = V_1 \oplus V_2$. In particular the $(\mathbb{C}^*)^2$-action on $\mathbb{C}^4$ induced by multiplication by scalars on $V_i$ commutes with $G$.*

Let $\Sigma \subset \mathbb{C}^4/G$ be the singular locus of the quotient. By analysis of the fixed-point loci of symplectic reflections in $G$ it can be described as follows.

Proposition 7.4.3. *$\Sigma$ consists of two irreducible components, and outside of the image of $0$ each of them is transversal $A_1$-singularity. The preimage of each component of $\Sigma$ via the quotient map $\mathbb{C}^4 \to \mathbb{C}^4/G$ consists of two planes.*

Let $\varphi : X \to \mathbb{C}^4/G$ be a symplectic resolution. It is known that such a resolution exists (see [**11**, Sect. 1]) but we will also prove it independently in section 7.4.3. As in the case $G = S_3$ considered in section 7.3.1 we have:

Proposition 7.4.4. *There are two exceptional divisors $E_1, E_2$ of $X$ each of which is mapped onto an irreducible component of $\Sigma$. The central fibre $\varphi^{-1}([0])$ consist of four surfaces. The fibre of $\varphi$ over any point in $\Sigma \setminus [0]$ consists of a curve isomorphic to $\mathbb{P}^1$ which is contained in exactly one of two exceptional divisors.*

Corollary 7.4.5. *There is a natural $T := (\mathbb{C}^*)^2$-action on $X$ making $\varphi$ an equivariant map.*

Again we have two important two-dimensional tori – one is the Picard torus $\mathbb{T} := \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ and the other one is the torus $T = (\mathbb{C}^*)^2$ acting on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ by multiplication of scalars on each component $\mathbb{C}^2$.

Let $C_i$ be the numerical class of a complete curve which is a generic fibre of the morphism $\varphi|_{E_i} : E_i \to \varphi(E_i)$. Again we describe the generators of the Picard group of $X$ in terms of their intersections with curves $C_i$ using proposition 7.1.1:

PROPOSITION 7.4.6. *The Picard group of $X$ is a free rank two abelian group generated by line bundles $L_1, L_2$ such that the intersection matrix $(L_i.C_j)_{i,j}$ is equal to identity matrix.*

Again using the computer algebra system Singular [**27**] to compute the invariants of $[G, G]$ we obtain the following proposition.

PROPOSITION 7.4.7. *The elements $\phi_{ij}$ in table below are eigenvectors of the action of $\mathrm{Ab}(G)$, which generate the ring of invariants $\mathbb{C}[x_1, y_1, x_2, y_2]^{[G,G]} \subset \mathbb{C}[x_1, y_1, x_2, y_2]$.*

| $\alpha$ | $\beta$ | *generators* |
|---|---|---|
| $1$ | $1$ | $\phi_{01} = x_1^2 + y_1^2,\ \phi_{02} = x_2^2 + y_2^2,\ \phi_{03} = x_1x_2 + y_1y_2$ |
| $-1$ | $1$ | $\phi_{11} = x_1y_2 + x_2y_1,\ \phi_{12} = x_1y_1,\ \phi_{13} = x_2y_2$ |
| $1$ | $-1$ | $\phi_{21} = x_1x_2 - y_1y_2,\ \phi_{22} = x_1^2 - y_1^2,\ \phi_{23} = x_2^2 - y_2^2$ |
| $-1$ | $-1$ | $\phi_3 = x_1y_2 - x_2y_1$ |

Let $\nu_1, \nu_2 \colon \mathbb{C}(x_1, y_1, x_2, y_2)^* \to \mathbb{Z}$ denote monomial valuations corresponding to symplectic reflections $\alpha, \beta$ respectively.

REMARK 7.4.8. We have

$$\nu_i(\phi_{jk}) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j \end{cases}$$

and $\nu_i(\phi_3) = 1$. Moreover $\nu_i(\phi_{jk}) = \overline{D}_{\phi_{jk}}.C_i$ by (7.1.3).

As for $G = S_3$ we have (cf. theorem 7.3.8).

THEOREM 7.4.9. *The Cox ring $\mathcal{R}(X) \subset \mathcal{R}(\mathbb{C}^4/G)[t_1^{\pm 1}, t_2^{\pm 1}]$ of the symplectic resolution $X \to \mathbb{C}^4/G$ is generated by the elements $w_{01}, w_{02}, w_{03}, w_{11}, w_{12}, w_{13}, w_{21}, w_{22}, w_{23}, w_3, s, t$, where $w_{0i} = \phi_{0i},\ w_{jk} = \phi_{jk}t_j,\ s = t_1^{-2},\ t = t_2^{-2}$. In particular the degree matrix with respect to the generators $L_1, L_2$ of $\mathrm{Pic}(X)$ (the first two rows) and to the $T$-action (remaining two rows) is:*

$$\left(\begin{array}{cccccccccccc} w_{01} & w_{02} & w_{03} & w_{11} & w_{12} & w_{13} & w_{21} & w_{22} & w_{23} & w_3 & s & t \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & -2 \\ \hline 2 & 0 & 1 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 & 0 \end{array}\right)$$

We prove this theorem in section 7.4.6. From now on we will denote the ring generated by the elements from the theorem 7.4.9 by $\mathcal{R}$.

Similarly as in the case $G = S_3$ we need to prove independently of the previous theorem the description of $\mathrm{Mov}(X)$ and its subdivision. The proposition below is originally due to the work of [**3**].

PROPOSITION 7.4.10.

(1) *The cone $\mathrm{Mov}(X)$ of movable divisors of $X$ is equal to $\mathrm{Mov}(\mathcal{R})$ and is generated by line bundles $L_1$ and $L_2$.*
(2) *There are two symplectic resolutions of $\mathbb{C}^4/G$. The chambers in $\mathrm{Mov}(X)$ corresponding to the nef cones of these resolutions are relative interiors of cones cone$(L_1, L_1 + L_2)$ and cone$(L_2, L_1 + L_2)$. The Mori cones of corresponding resolutions are cone$(C_2, C_1 - C_2)$ and cone$(C_1, C_2 - C_1)$.*

PROOF.

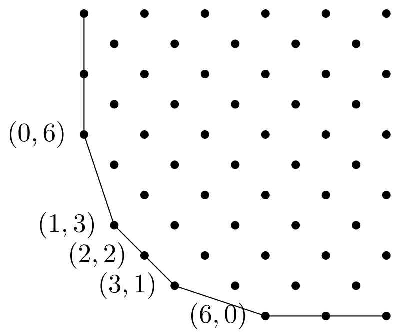
(1) This follows from theorem 2.3.13 which allows us to find $\mathrm{Mov}(X)$ and by definition of $\mathrm{Mov}(\mathcal{R})$ (see e.g. the paragraph preceeding proposition 7.1.4).
(2) We will prove the first part of the claim in section 7.4.5 as proposition 7.4.26 (we will not use this result until then). The part on Mori cones then follows by taking dual cones.

□

The theorem below will be proven independently of theorem 7.4.9 in section 7.4.3.

Theorem 7.4.11. *Taking a GIT quotient of* $\operatorname{Spec}\mathcal{R}$ *by the Picard torus action with respect to the linearization given by a character* $(a, b)$ *with* $a > b > 0$ *and with* $b > a > 0$ *one obtains the two symplectic resolutions of* $\mathbb{C}^4/G$.

Remark 7.4.12. The weights of the $T$-action on global sections of the fixed line bundle $L$ on $X$ are lattice points in $\mathbb{Z}^2$. Taking a convex hull one obtains a lattice polyhedron in $\mathbb{R}^2$. For example fixing a line bundle $L = 2L_1 + L_2$ one gets a polyhedron with the tail equal to the positive quadrant of $\mathbb{R}^2$ and with a head spanned by the lattice points from the picture below:




By [**18**, Lemma 2.4(c)] if $L$ is globally generated, then marked vertices of this polyhedron correspond to $T$-fixed points of $X$. We will see in lemma 7.4.21 that indeed fixed points of this polytope correspond to points in $X^T$ and in lemma 7.4.18 that $L$ is globally generated.

**7.4.2. The structure of the central fibre.** In this section we study the structure of the central fibre $\varphi^{-1}([0])$ of a resolution $\varphi\colon X \to \mathbb{C}^4/G$ using the ideal of relations between generators of the ring $\mathcal{R}$, under the assumption that $X = \operatorname{Spec}\mathcal{R}//_L\mathbb{T}$ for a certain linearization $L$. The results of this section are used in the next one, where we investigate the action of the two-dimensional torus $T$ on $X$ with the fixed point locus $X^T$ contained in the central fibre. The additional assumption that $X = \operatorname{Spec}\mathcal{R}//_L\mathbb{T}$ is dealt with in section 7.4.3. Note that the approach adopted here is different than the one adopted in the paper [**32**].

The reasoning is analogous to the case of $G = S_3$ except that we have to consider stability with respect to more than one linearization of $\mathbb{T}$ and there are more components of the central fibre:

Lemma 7.4.13. *We have an isomorphism* $\operatorname{Spec}\mathcal{R}^{\mathbb{T}} \cong \mathbb{C}^4/G$. *In particular the inclusion of invariants* $\mathcal{R}^{\mathbb{T}} \subset \mathcal{R}$ *induce map* $p\colon \operatorname{Spec}\mathcal{R} \to \mathbb{C}^4/G$.

Let $Z = p^{-1}([0])$. By decomposing its ideal in computer algebra system [**27**] and using lemma 7.4.15 below we obtain:

PROPOSITION 7.4.14. *The components of $Z$ are the following subvarieties of $\mathbb{C}^{12}$:*

$$\begin{aligned}
Z_u &= V(w_3, w_{23}, w_{22}, w_{21}, w_{13}, w_{12}, w_{11}, w_{03}, w_{02}, w_{01}),\\
Z_0 &= V(w_{21}^2 - w_{22}w_{23}, -w_{13}w_{22} + w_{12}w_{23}, -2w_{13}w_{21} + w_{11}w_{23},\\
&\quad - 2w_{12}w_{21} + w_{11}w_{22}, w_{11}w_{21} - 2w_{12}w_{23}, w_{11}^2 - 4w_{12}w_{13}, t, s, w_{03}, w_{02}, w_{01}),\\
Z_P &= V(-w_3^2 t + w_{11}^2 - 4w_{12}w_{13}, s, w_{23}, w_{22}, w_{21}, w_{03}, w_{02}, w_{01}),\\
Z'_P &= V(-w_3^2 s + w_{21}^2 - w_{22}w_{23}, t, w_{13}, w_{12}, w_{11}, w_{03}, w_{02}, w_{01}).
\end{aligned}$$

*The component $Z_u$ is contained in the locus of unstable points with respect to any linearization of the Picard torus via character from the movable cone. Points in the component $Z'_P$ are unstable with respect to any linearization by the character $(2,1) \in \mathrm{Cl}(X)$ and points in the component $Z_P$ are unstable with respect to any linearization by the character $(1,2) \in \mathrm{Cl}(X)$.*

Note that in the above proposition we may replace characters $(2,1)$ and $(1,2)$ by any other pair of characters from the interiors of the two top-dimensional GIT cones. We chose characters $(2,1)$, and $(1,2)$ to keep any computations involving the chosen characters as simple as possible.

LEMMA 7.4.15. *The unstable locus of $\operatorname{Spec}\mathcal{R}$ with respect to a linearization of the trivial line bundle by a character $(2,1) \in \mathrm{Cl}(X)$ is cut out by equations:*

$$w_{13}w_{23} = w_{13}w_3 = w_{11}w_3 = w_{12}w_3 = w_{12}w_{22} = 0.$$

*Moreover, all the semistable points of $Z$ are stable and have trivial isotropy groups.*

The following theorem gives a description of components of the central fibre. Let $W$ be the locus of stable points of $\operatorname{Spec}\mathcal{R}$ with respect to the $\mathbb{T}$-action linearized by a character $(2,1)$ (the case $(1,2)$ is analogous) and consider the quotient map $W \to X$. Denote by $S_0, P$ the images of sets of stable points of the components $Z_0, Z_P$. Note that these are precisely the components of the central fibre of $X$.

THEOREM 7.4.16.

(a) *$S_0$ is a toric surface isomorphic to the Hirzebruch surface $\mathcal{H}_4$. The action of $T$ on $S_0$ is given by characters in the columns of the matrix $\begin{pmatrix} 1 & 0 \\ -1 & -2 \end{pmatrix}$.*
(b) *$P$ is isomorphic to $\mathbb{P}^2$. The action of $T$ on $P$ in homogeneous coordinates is given by the matrix $\begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.*

PROOF OF $(a)$.

CLAIM 1. *By rescaling variables we may assume that $Z_0$ is the toric variety embedded into $\mathbb{C}^7$ with coordinates $w_{11}, w_{12}, w_{13}, w_{21}, w_{22}, w_{23}, w_3$ defined by the toric ideal generated by binomials:*

$$\begin{gathered}
w_{21}^2 - w_{22}w_{23}, w_{12}w_{23} - w_{13}w_{22}, w_{11}w_{23} - w_{13}w_{21},\\
w_{11}w_{22} - w_{12}w_{21}, w_{11}w_{21} - w_{12}w_{23}, w_{11}^2 - w_{12}w_{13}.
\end{gathered}$$

CLAIM 2. *$Z_0$ is the affine variety of the cone $\sigma^\vee = cone(e_1 - 2e_2 + 2e_3, e_1, e_3, 2e_2 - e_3, e_4)$ in space $\mathbb{R}^4$ spanned by the character lattice $M = \mathbb{Z}^4$ of four-dimensional torus.*

PROOF OF THE CLAIM. One may do the calculation as in the proof of [**23**, Thm 1.1.17] or use computer algebra system Polymake [**42**] to check that the toric variety of the cone from the statement has the binomial ideal from claim 1. □

Claim 3. *The Picard torus acting on $Z_0$ may be viewed as $(\mathbb{C}^*)^2$ embedded into $T_M = (\mathbb{C}^*)^4$ by characters in the columns of the matrix:*

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}.$$

Proof of the claim. The $i$-th column corresponds to $e_i \in M$, $i = 1, 2, 3, 4$ and these correspond to $w_{13}, w_{21}, w_{22}, w_3$ respectively. $\square$

To obtain $S_0$ from $Z_0$ we have to remove unstable orbits of $Z_0$ and divide the remaining open subset by the action of the Picard torus. By lemma 7.4.15 we have:

Claim 4. *The unstable locus of $Z_0$ is the closure of two orbits:*

$$O_1 = \{w_{11} = w_{12} = w_{13} = 0\}, \qquad O_2 = \{w_{11} = w_{13} = w_{21} = w_{22} = w_{23} = w_3 = 0\}.$$

One checks by [**23**, 3.2.7] that:

Claim 5. *The orbits $O_1$, $O_2$ correspond respectively to the following faces of the cone $\sigma = cone(2e_1 + e_2, e_1, e_2 + 2e_3, e_2 + e_3, e_4)$ dual to $\sigma^\vee$:*

$$\tau_1 = cone(e_1), \qquad \tau_2 = cone(2e_1 + e_2, e_2 + e_3, e_4).$$

To obtain the fan of the toric variety $S_0$ we consider the fan of $Z_0$, remove cones $\tau_1$ and $\tau_2$ together with all the cones containing them and take the family of the images of the remaining cones via the dual of the kernel of the matrix from claim 3, i.e. by:

$$Q := \begin{pmatrix} 0 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{pmatrix}.$$

Now $Q(2e_1 + e_2) = (-1, -3)$, $Q(e_1) = (0, -1)$, $Q(e_2 + 2e_3) = (1, -1)$, $Q(e_2 + e_3) = (0, -1)$, $Q(e_4) = (0, 1)$ and changing coordinates in $\mathbb{Z}^2$ by $(1, 0) \mapsto (1, -1)$, $(0, 1) \mapsto (0, -1)$ we obtain the standard fan for Hirzebruch surface $\mathcal{H}_4$.

The matrix of the $T$-action on $S_0$ is given by the product of the matrix $J$ of embedding of $T$ into the big torus of $Z_0$ with the transpose of the matrix $Q$. Here, inspecting the construction of the generators of $\mathcal{R}$, we obtain:

$$J = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 0 & 1 \end{pmatrix}.$$

$\square$

Proof of $(b)$. Denote $x := w_{11}$, $y := w_{12}$, $z := w_{13}$, $w := w_3$. Rescaling coordinates we may assume that $Z_P$ is embedded in $\mathbb{C}^5$ as the hypersurface $x^2 - w^2t + 4yz = 0$. Then the unstable locus is described by equations $xw = yw = zw = 0$, see lemma 7.4.15. This gives an open cover of the set of semistable points, given by the union of three open sets $\{xw \neq 0\} \cup \{yw \neq 0\} \cup \{zw \neq 0\}$. Gluing the quotients of these open sets coincides with the standard construction of $\mathbb{P}^2$ (with coordinates $x, y, z$) by gluing three affine planes. Finally, note that $T$ acts on $w_{11}, w_{12}, w_{13}$ with weights $(1, 1), (2, 0), (0, 2)$ respectively. $\square$

Since, contrary to the case $G = S_3$, we have more than one component of the central fibre we would like to describe the incidence relation between various components. By the analysis of intersections of stable loci of $Z_0$ and $Z_P$ we can systematically describe the intersections of the components of the central fibre in terms of the identifications from theorem 7.4.16.

Theorem 7.4.17. *$S_0 \cap P$ is the curve corresponding to $(0, -1)$ on $S_0$ and the quadric curve $x^2 - yz = 0$ on $P$ with homogeneous coordinates $x, y, z$.*

PROOF. We will be using the notation from the proof of theorem 7.4.16. On $Z_0$ the intersection $Z_0 \cap Z_P$ is cut out by equations $w_{21} = w_{22} = w_{23} = 0$. Hence it contains as a dense subset the orbit of the toric variety $Z_0$ which corresponds to the one-dimensional cone $\tau = cone(e_2 + e_3)$, since $\sigma^\vee \setminus \tau^\perp \ni v_{21}, v_{22}, v_{23}$. Then we have $Q(e_2 + e_3) = (0, -1)$.
On $Z_P$ the intersection $Z_0 \cap Z_P$ is cut out by the equation $t = 0$. By the construction of the isomorphism $P \cong \mathbb{P}^2$ this equation yields the curve $x^2 - yz = 0$ on $\mathbb{P}^2$. $\square$

The next lemma shows that all nef line bundles on $X$ are globally generated as for $G = S_3$, which again will be important in the next sections. Note that it is already known that all nef line bundles on $X$ are globally generated by work of [**3**].

LEMMA 7.4.18. *$L_1 + L_2$ and $L_1$ are globally generated line bundles on $X$.*

PROOF. Since $L_1 + L_2$ and $L_1$ are invariant with respect to the $T$-action and the base point locus of a linear system is closed for both linear systems $|L_1 + L_2|$ and $|L_1|$ it either has to be empty or it has a nontrivial intersection with the central fibre. The assertion follows by inspecting the weights of the generators of the Cox ring with respect to the Picard torus action and the equations of components of the fibre $p^{-1}([0])$ where $p\colon \operatorname{Spec} \mathcal{R} \to \mathbb{C}^4/G$ is as in proposition 7.4.14. It turns out that the intersections of the zero sets of elements of each of these two linear systems with $p^{-1}([0])$ are contained in the unstable locus. $\square$

**7.4.3. Smoothness of the GIT quotient.** Let $\mathcal{R}$ be the subring of the Cox ring of the crepant resolution generated by the elements from the statement of theorem 7.4.9. In this section we show that the GIT quotient $\operatorname{Spec} \mathcal{R}//_L \mathbb{T}$ with respect to the linearization of the trivial line bundle by the character $L = 2L_1 + L_2$ of the Picard torus $\mathbb{T}$ is smooth. In consequence we see that $\operatorname{Spec} \mathcal{R}//_L \mathbb{T} \to \mathbb{C}^4$ is a crepant resolution. This makes the results on the geometry of crepant resolutions of $\mathbb{C}^4/G$ in the previous section unconditional and helps to conclude that $\mathcal{R}$ is the whole Cox ring in the final section 7.4.6.
We consider $\operatorname{Spec} \mathcal{R}$ as a closed subvariety of $\mathbb{C}^{12}$ via the embedding given by generators from statement of Theorem 7.4.9. The next two results are analogous to the one in the case of $G = S_3$ (theorem 7.3.17 and corollary 7.3.18).

THEOREM 7.4.19. *The stable locus of* $\operatorname{Spec} \mathcal{R}$ *with respect to the linearization of the trivial line bundle by a character $(a, b)$, $a > b > 0$ is covered by seven $T \times \mathbb{T}$-invariant open subsets $U_1, \ldots, U_5$ such that $U_i/\mathbb{T} \cong \mathbb{C}^4$. More precisely if $(a, b) = (2, 1)$ then:*

*(1)* $U_1 = \{w_{13}w_{23} \neq 0\}$ *and* $(\mathcal{R}_{w_{13}w_{23}})^{\mathbb{T}} = \mathbb{C}\left[w_{13}^2 s, w_{02}, \frac{w_{11}}{w_{13}}, \frac{w_3}{w_{13}w_{23}}\right]$,
*(2)* $U_2 = \{w_{13}w_3 \neq 0\}$ *and* $(\mathcal{R}_{w_{13}w_3})^{\mathbb{T}} = \mathbb{C}\left[\frac{w_{13}w_{23}}{w_3}, w_{02}, \frac{w_{11}}{w_{13}}, \frac{w_{12}}{w_{13}}\right]$,
*(3)* $U_3 = \{w_{11}w_3 \neq 0\}$ *and* $(\mathcal{R}_{w_{11}w_3})^{\mathbb{T}} = \mathbb{C}\left[\frac{w_{13}}{w_{11}}, \frac{w_{11}w_{23}}{w_3}, \frac{w_{11}w_{22}}{w_3}, \frac{w_{12}}{w_{11}}\right]$,
*(4)* $U_4 = \{w_{12}w_3 \neq 0\}$ *and* $(\mathcal{R}_{w_{12}w_3})^{\mathbb{T}} = \mathbb{C}\left[\frac{w_{12}w_{22}}{w_3}, w_{01}, \frac{w_{11}}{w_{12}}, \frac{w_{13}}{w_{12}}\right]$,
*(5)* $U_5 = \{w_{12}w_{22} \neq 0\}$ *and* $(\mathcal{R}_{w_{13}w_3})^{\mathbb{T}} = \mathbb{C}\left[w_{12}^2 s, w_{01}, \frac{w_{11}}{w_{12}}, \frac{w_3}{w_{12}w_{22}}\right]$.

*In particular the GIT quotient* $\operatorname{Spec} \mathcal{R}//_{(a,b)} \mathbb{T}$ *with respect to the linearization of the trivial line bundle by a character $(a, b)$, $a > b > 0$ is smooth.*

The choice of an open cover is based on the reasoning as in the case $G = S_3$, see remark 7.3.25.

PROOF. Lemma 7.4.15 implies that $\{U_i\}_{i=1,\ldots 5}$ form an open cover of the quotient. It remains to prove equalities from points (1)–(5). Note that then in each case the four generators of the ring on the right-hand side of the equality have to be algebraically independent as the GIT quotient $\operatorname{Spec} \mathcal{R}/\!/_{(a,b)}\mathbb{T}$ is irreducible and of dimension four.

By symmetry it suffices to consider only $U_i$ for $i = 1, 2, 3$. In each case we calculate the invariants of the localization of the coordinate ring of the ambient $\mathbb{C}^{12}$, with the help of 4ti2 [**1**] obtaining in consequence:

(1) $\operatorname{Spec} \mathcal{R}^{\mathbb{T}}_{w_{13}w_{23}} = \mathbb{C}\left[w_{0i}, \frac{w_{1i}}{w_{13}}, \frac{w_{2i}}{w_{23}}, w_{13}^2 s, w_{23}^2 t, \frac{w_3}{w_{13}w_{23}}\right]_{i=1,2,3}$

(2) $\operatorname{Spec} \mathcal{R}^{\mathbb{T}}_{w_{13}w_3} = \mathbb{C}\left[w_{0i}, \frac{w_{1i}}{w_{13}}, \frac{w_{13}w_{2i}}{w_3}, w_{13}^2 s, \frac{w_3^2 t}{w_{13}^2}\right]_{i=1,2,3}$

(3) $\operatorname{Spec} \mathcal{R}^{\mathbb{T}}_{w_{11}w_3} = \mathbb{C}\left[w_{0i}, \frac{w_{1i}}{w_{11}}, \frac{w_{11}w_{2i}}{w_3}, w_{11}^2 s, \frac{w_3^2 t}{w_{11}^2}\right]_{i=1,2,3}$

Then, using the Gröbner basis of the ideal of relations between generators of $\mathcal{R}$ with respect to an appropriate lexicographic order, we verify with Singular [**27**] in each case that each of the generators of this three rings can be expressed as a polynomial of the four generators from the assertion. $\square$

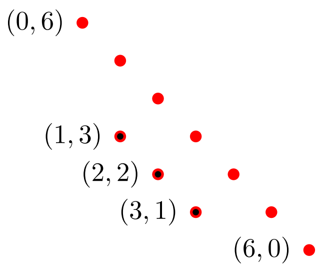
By the inclusion of invariants $\mathcal{R}^{\mathbb{T}} \subset \mathcal{R}$ (see lemma 7.4.13) we have the induced projective map $\operatorname{Spec} \mathcal{R}/\!/_L\mathbb{T} \to \operatorname{Spec} \mathcal{R}/\mathbb{T} \cong \mathbb{C}^4/G$.

COROLLARY 7.4.20. *The map* $\varphi\colon \operatorname{Spec} \mathcal{R}/\!/_L\mathbb{T} \to \operatorname{Spec} \mathcal{R}/\mathbb{T} \cong \mathbb{C}^4/G$ *is a crepant resolution.*

**7.4.4. Compasses of fixed points.** In this section we obtain a local description of the action of the two-dimensional torus $T$ on a symplectic resolution $X = \operatorname{Spec} \mathcal{R}/\!/_L\mathbb{T}$ of $\mathbb{C}^4/G$ at fixed points of this action, where $L = 2L_1 + L_2$. As for $G = S_3$ the results of this section served as a guide in section 7.4.3, where we proved that indeed $X$ is of the form $\operatorname{Spec} \mathcal{R}/\!/_L\mathbb{T}$. This is why we are not using here the open cover from theorem 7.4.19, but only the smoothness of the quotient (see also remark 7.3.25).

We find compasses of all fixed points of the action $T$ on $X$ (see definition 6.3.5). The arguments are analogous as these of section 7.3.4.

LEMMA 7.4.21. *The following diagram shows the weights of the action of $T$ on the space of sections of $H^0(X, L)$ for $L = 2L_1 + L_2$ which are nonzero after the restriction to some irreducible component of the central fibre.*




*The black dots correspond to the weights of sections of $L$ restricted to $P$ and red ones to $S_0$ (note that lattice points marked by multiple colours correspond to weights occurring in restriction to more than one component).*

PROOF. This follows by a computer calculation (using Macaulay2 [**45**]) of dimensions of appropriate graded pieces of the coordinate rings of $Z_0$ and $Z_P$ from proposition 7.4.14. $\square$

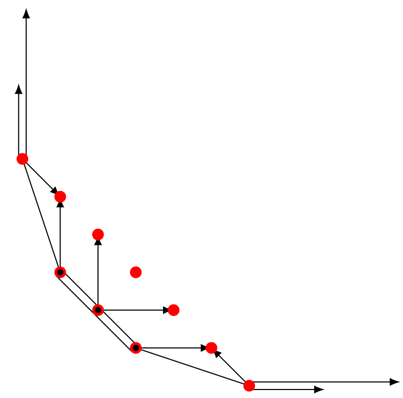
REMARK 7.4.22. Lattice points from lemma 7.4.21 are contained in the polyhedron from remark 7.4.12. Moreover, their convex hull form the minimal head of this polyhedron.

REMARK 7.4.23. Considering the polytope which is a convex hull of weights marked by the red colour in lemma 7.4.21 we get the polytope of the line bundle $L$ pulled back to the corresponding component $S_0$ of the central fibre viewed as a toric variety. As $L$ is globally generated (see lemma 7.4.18) the vertices correspond to the fixed points of the action of $T$ on $S$, cf. [**18**, Lemma 2.4(3)].

THEOREM 7.4.24. *The fixed points of the $T$-action correspond to the vertices of the polytopes which are convex hulls of weights marked by the colour fixed in lemma 7.4.21. The compasses of the points corresponding to the vertices of these polytopes are as in the table below:*

| *Point* | *Compass* | | | |
|---|---|---|---|---|
| $P_1 \leftrightarrow (0,6)$ | $\nu_{1,1} = (1,-3)$, | $\nu_{1,2} = (1,-1)$, | $\nu_{1,3} = (0,4)$, | $\nu_{1,4} = (0,2)$ |
| $P_2 \leftrightarrow (1,3)$ | $\nu_{2,1} = (-1,3)$, | $\nu_{2,2} = (1,-1)$, | $\nu_{2,3} = (2,-2)$, | $\nu_{2,4} = (0,2)$ |
| $P_3 \leftrightarrow (2,2)$ | $\nu_{3,1} = (-1,1)$, | $\nu_{3,2} = (1,-1)$, | $\nu_{3,3} = (2,0)$, | $\nu_{3,4} = (0,2)$ |
| $P_4 \leftrightarrow (3,1)$ | $\nu_{4,1} = (3,-1)$, | $\nu_{4,2} = (-1,1)$, | $\nu_{4,3} = (-2,2)$, | $\nu_{4,4} = (2,0)$ |
| $P_5 \leftrightarrow (6,0)$ | $\nu_{5,1} = (-3,1)$, | $\nu_{5,2} = (-1,1)$, | $\nu_{5,3} = (4,0)$, | $\nu_{5,4} = (2,0)$ |

The following picture illustrates the weights of the $T$-action calculated in the theorem. It is a directed graph. The points correspond to the sections of $H^0(X, L)$ for $L = 2L_1 + L_2$ which are nonzero after the restriction to the central fibre together with vectors, as in lemma 7.4.21. The directed edges are the vectors from the compasses attached to the points which correspond to fixed points of $T$-action. In case when two vertices are connected by the two edges pointing in both ways we depict them by a single edge without any arrow.



PROOF OF THEOREM 7.4.24. First, note that $X^T$ is contained in the central fibre. In particular $X^T = S_0^T \cup P^T$. Now all the weights corresponding to fixed points, except $(2,2)$, together with most elements of compasses can be deduced from the fact that the action of $T$ on $S_0$ is toric. In the calculation of compasses we take into account the natural inclusions of the tangent spaces to the components of the central fibre into the tangent space of $X$. The polytope spanned by the points marked by red colour in the lemma 7.4.21 is the polytope of $S_0$. Thus using toric methods one can describe affine cover of $S_0$ and the weights of the $T$-action on the tangent space to its $T$-fixed points. In the case of weights at $(1,3), (2,2)$ and $(3,1)$ which come from the action of $T$ on $P$ we use the explicit description of $P$ from the proof of theorem 7.4.16(b). Altogether the weights calculated up to this point are the ones from the assertion except $\nu_{1,3}, \nu_{1,4}, \nu_{2,3}, \nu_{3,3}, \nu_{3,4}, \nu_{4,4}, \nu_{5,3}, \nu_{5,4}$. Now the calculation of all the remaining weights is analogous as in the proof of theorem 7.3.22, in particular we use lemmas 7.3.23 and 7.3.24. □

**7.4.5. Dimensions of movable linear systems.** In this section we use the torus $T$ action on $X$ to give a formula for dimensions of these graded pieces of $\mathcal{R}(X)$ which correspond to movable linear systems on some of the resolutions.

Let $X \to \mathbb{C}^4/G$ be the resolution corresponding to the linearization of the Picard torus action by a character $(2,1)$. Let $X' \to \mathbb{C}^4/G$ be the resolution corresponding to the linearization $(1,2)$.

Denote by $P_i$ the fixed points of the $T$-action on $X$ as in the table from theorem 7.4.24. Let $\{\nu_{i,j}\}_{j=1}^4$ denote the compass of $P_i$ in $X$. Let us also denote by $\mu_i(L)$ the weight of the $T$-action on the fibre of $L$ over $P_i$. Note that $\mu_i$ is linear i.e. $\mu_i(A+B) = \mu_i(A) + \mu_i(B)$.

REMARK 7.4.25. By lemma 7.4.18 we may compute the weights $\mu_i$ for line bundles $L_1+L_2$ and $L_1$ similarly as for $2L_1+L_2$ in section 7.4.4 to obtain (the last column is calculated from the first two ones by the linearity of $\mu_i$):

| $i$ | $\mu_i(L_1)$ | $\mu_i(L_1+L_2)$ | $\mu_i(L_2)$ |
|---|---|---|---|
| 1 | $(0,2)$ | $(0,4)$ | $(0,2)$ |
| 2 | $(0,2)$ | $(1,1)$ | $(0,-1)$ |
| 3 | $(1,1)$ | $(1,1)$ | $(0,0)$ |
| 4 | $(2,0)$ | $(1,1)$ | $(-1,1)$ |
| 5 | $(2,0)$ | $(4,0)$ | $(2,0)$ |

We may now prove the observation on the subdivision of the cone of movable divisors on $X$, see proposition 7.4.10 (which is originally due to the work of [**3**]).

PROPOSITION 7.4.26. *There are two symplectic resolutions of $\mathbb{C}^4/G$. The chambers in* $\mathrm{Mov}(X)$ *corresponding to the nef cones of these resolutions are relative interiors of cones* $cone(L_1, L_1+L_2)$ *and* $cone(L_2, L_1+L_2)$.

PROOF. Consider the homomorphisms $\mu_i : N^1(X) \to \mathbb{R}^2$. The two walls of the chamber $\mathcal{C}$ containing $2L_1+L_2$ are corresponding to the contractions of $X$, in particular they identify some $T$-fixed points of $X$. Hence each wall has to be spanned by an element $v \in \mathrm{Mov}(X)$ satisfying $\mu_i(v) = \mu_j(v)$ for some $i \neq j$. Now the only such elements in $cone(L_1, L_1+L_2)$ are lying on the rays spanned by $L_1+L_2$ and $L_1$. Therefore $\mathcal{C} = cone(L_1, L_1+L_2)$. The analogous argument, using the homomorphisms $\mu_i'\colon N^1(X') \to \mathbb{R}^2$ corresponding to the $T$-fixed points of $X'$, shows that the chamber containing $L_1+2L_2$ is equal to the relative interior of $cone(L_1+L_2, L_2)$. $\square$

THEOREM 7.4.27. *If $h^0(X, pL_1+qL_2)_{(a,b)}$ is the dimension of the subspace of sections $H^0(X, pL_1+qL_2)$ on which $T$ acts with the weight $(a,b)$, then we have the following generating function for such dimensions for line bundles inside the movable cone:*

$$\sum_{a,b,p,q \geq 0} h^0(X, pL_1+qL_2)_{(a,b)} y_1^p y_2^q t_1^a t_2^b =$$

$$= \sum_{i=1}^{7} \frac{1}{(1-t^{\mu_i(L_1)}y_1)(1-t^{\mu_i(L_2)}y_2)\prod_{j=1}^4(1-t^{\nu_{i,j}})}.$$

Again, the computed generating function may be interpreted as the multivariate Hilbert series of a $\mathbb{Z}^2_{\geq 0} \times \mathbb{Z}^2_{\geq 0}$-graded subalgebra of $\mathcal{R}(X)$ consisting of the graded pieces of $\mathcal{R}(X)$ corresponding to movable line bundles on $X$. This is the interpretation of the theorem that we will use in the next section.

PROOF. By a corollary of the Lefschetz-Riemman-Roch theorem (corollary 6.4.22) we have:

$$\chi^T(X, pL_1+qL_2) = \sum_{i=1}^{7} \frac{t^{\mu_i(pL_1+qL_2)}}{\prod_{j=1}^4(1-t^{\nu_{i,j}})}.$$

Using the linearity of $\mu_i$:

$$\sum_{p,q\geq 0} \chi^T(X, pL_1 + qL_2) y_1^p y_2^q = \sum_{p,q\geq 0} \sum_{i=1}^{7} \frac{t^{p\mu_i(L_1)} \cdot t^{q\mu_i(L_2)}}{\prod_{j=1}^{4}(1 - t^{\nu_{i,j}})} y_1^p y_2^q =$$
$$\sum_{i=1}^{7} \frac{1}{(1 - t^{\mu_i(L_1)} y_1)(1 - t^{\mu_i(L_2)} y_2) \prod_{j=1}^{4}(1 - t^{\nu_{i,j}})}.$$

The assertion follows now by Kawamata-Viehweg vanishing, which implies

$$\chi^T(X, pL_1 + qL_2) = \sum_{a,b\geq 0} h^0(X, pL_1 + qL_2)_{(a,b)} t_1^a t_2^b = \sum_{a,b\geq 0} h^0(X', pL_1 + qL_2)_{(a,b)} t_1^a t_2^b.$$

if $p \geq q \geq 0$ and likewise for $q \geq p \geq 0$ on $X'$. $\square$

Example 7.4.28. The dimensions of the weight spaces corresponding to the lattice points in a head of the polyhedron spanned by weights for the line bundle $2L_1 + L_2$ considered in remark 7.4.12 and in section 7.4.4 can be depicted on the following diagram:

$$\begin{array}{ccccccccccc}
2 & & 7 & & 13 & & 18 & & 24 & & 29 \\
 & 4 & & 9 & & 14 & & 19 & & 24 & \\
1 & & 6 & & 10 & & 15 & & 19 & & 24 \\
 & 3 & & 7 & & 11 & & 15 & & 19 & \\
1 & & 4 & & 8 & & 11 & & 15 & & 18 \\
 & 2 & & 5 & & 8 & & 11 & & 14 & \\
 & & 3 & & 5 & & 8 & & 10 & & 13 \\
 & 1 & & 3 & & 5 & & 7 & & 9 & \\
 & & 1 & & 3 & & 4 & & 6 & & 7 \\
 & & & 1 & & 2 & & 3 & & 4 & \\
 & & & & & & 1 & & 1 & & 2
\end{array}$$

### 7.4.6. The Cox ring.

Using the generating function from theorem 7.4.27 we may prove theorem 7.4.9 i.e. we reconfirm that the elements from the statement are indeed sufficient to generate the Cox ring of $X$. In this section we outline an argument using the methods of section 4.4.

Denote by $\mathcal{R}(X)$ the Cox ring of $X$ and by $\mathcal{R}$ the subring of $\mathcal{R}(X)$ generated by the elements from the statement of theorem 7.4.9.

Lemma 7.4.29. *The Cox ring of $X$ is generated by $s, t$ and by*

$$\mathcal{R}(X)_{\geq 0} := \bigoplus_{p,q\geq 0} H^0(X, pL_1 + qL_2).$$

Proof. This is a particular case of proposition 4.4.5. $\square$

Thus to prove that $\mathcal{R} = \mathcal{R}(X)$ it suffices to show that $\mathcal{R}$ contains $\mathcal{R}(X)_{\geq 0}$. The following lemma reduces the problem further, to finitely many graded pieces with respect to $\mathbb{Z}^2$-grading by characters of the Picard torus of $X$.

Lemma 7.4.30. *$\mathcal{R}(X)_{\geq 0}$ is generated by the sections of all linear spaces $H^0(X, L)$ for $L \in \mathcal{S} \cup \mathcal{S}'$ where $\mathcal{S} := \{\mathcal{O}_X, L_1, L_1+L_2, 2L_1, 2L_1+L_2, 2L_1+2L_2, 3L_1+L_2, 3L_1+2L_2, 4L_1+2L_2\}$ and $\mathcal{S}' := \{L_2, 2L_2, L_1 + 2L_2, L_1 + 3L_2, 2L_1 + 3L_2, 2L_1 + 4L_2\}$.*

Proof. By virtue of lemma 7.4.18 this follows from proposition 4.4.4 with $r = 2$ and all $m_1 = m_2 = 1$ for cones $cone(L_1, L_1 + L_2)$ and $cone(L_2, L_1 + L_2)$. $\square$

Hence we reduced the problem to showing that $\mathcal{R}$ contains spaces of global sections only for these finitely many line bundles which are elements of $\mathcal{S} \cup \mathcal{S}'$ in lemma 7.4.30. This, with the help of computer algebra, can be done with the use of the previous section, namely by theorem 7.4.27 in which we computed the Hilbert series of $\mathcal{R}(X)_{\geq 0}$.

LEMMA 7.4.31. *$\mathcal{R}$ contains $H^0(X, L)$ for each $L \in \mathcal{S} \cup \mathcal{S}'$, where $\mathcal{S}$ is as in the lemma 7.4.30.*

PROOF. We calculate the Hilbert series of $\mathcal{R}$ in Macaulay2 [**45**]. It is of the form:

$$\frac{1}{(1-y_1^{-2})(1-y_2^{-2})} \cdot G(y_1, y_2, t_1, t_2),$$

where:

$$F(y_1,y_2,t_1,t_2) = \frac{1 + y_1t_1t_2 + y_2t_1t_2 + t_1^2 + t_1t_2 + t_2^2 - y_1y_2t_1^3t_2 - y_1y_2t_1^2t_2^2 - y_1y_2t_1t_2^3 - y_1t_1^2t_2^2 - y_2t_1^2t_2^2 - y_1y_2t_1^3t_2^3}{(1-y_2t_2^2)(1-y_2t_1^2)(1-y_1t_2^2)(1-y_1t_1^2)(1-y_1y_2t_1t_2)}.$$

Then, using Singular [**27**] we extract from it Hilbert series for each of the vector spaces $\mathcal{R}_L$, $L \in \mathcal{S}$, graded by characters of $T$. Denote the Hilbert series corresponding to $pL_1 + qL_2$ by $F_{p,q}$. Then we have:

$$\begin{aligned}
F_{0,0}(t_1,t_2) &= \tfrac{F(1,1,t_1,t_2)+F(-1,1,t_1,t_2)+F(1,-1,t_1,t_2)+F(-1,-1,t_1,t_2)}{4},\\
F_{1,0}(t_1,t_2) &= \tfrac{F(1,1,t_1,t_2)-F(-1,1,t_1,t_2)+F(1,-1,t_1,t_2)-F(-1,-1,t_1,t_2)}{4},\\
F_{0,1}(t_1,t_2) &= \tfrac{F(1,1,t_1,t_2)+F(-1,1,t_1,t_2)-F(1,-1,t_1,t_2)-F(-1,-1,t_1,t_2)}{4},\\
F_{1,1}(t_1,t_2) &= \tfrac{F(1,1,t_1,t_2)-F(-1,1,t_1,t_2)-F(1,-1,t_1,t_2)+F(-1,-1,t_1,t_2)}{4},\\
F_{2,0}(t_1,t_2) &= F_{0,0}(t_1,t_2) - \tfrac{F(0,1,t_1,t_2)+F(0,-1,t_1,t_2)}{2},\\
F_{2,1}(t_1,t_2) &= F_{0,1}(t_1,t_2) - \tfrac{F(0,1,t_1,t_2)-F(0,-1,t_1,t_2)}{2},\\
F_{1,2}(t_1,t_2) &= F_{1,0}(t_1,t_2) - \tfrac{F(1,0,t_1,t_2)-F(-1,0,t_1,t_2)}{2},\\
F_{2,2}(t_1,t_2) &= F_{2,0}(t_1,t_2) - \tfrac{F(1,0,t_1,t_2)+F(-1,0,t_1,t_2)}{2} + F(0,0,t_1,t_2),\\
F_{3,1}(t_1,t_2) &= F_{1,1}(t_1,t_2) - \tfrac{\frac{\partial F}{\partial y_1}(0,1,t_1,t_2)-\frac{\partial F}{\partial y_1}(0,-1,t_1,t_2)}{2},\\
F_{3,2}(t_1,t_2) &= F_{1,2}(t_1,t_2) - \tfrac{\frac{\partial F}{\partial y_1}(0,1,t_1,t_2)+\frac{\partial F}{\partial y_1}(0,-1,t_1,t_2)}{2} + \frac{\partial F}{\partial y_1}(0,0,t_1,t_2),\\
F_{4,2}(t_1,t_2) &= F_{2,2}(t_1,t_2) - \tfrac{\frac{\partial^2 F}{\partial y_1^2}(0,1,t_1,t_2)+\frac{\partial^2 F}{\partial y_1^2}(0,1,t_1,t_2)}{4} + \tfrac{1}{2}\frac{\partial^2 F}{\partial y_1^2}(0,0,t_1,t_2).
\end{aligned}$$

Using Singular we check that for each $L \in \mathcal{S}$ the Hilbert series of $\mathcal{R}_L$ agrees with the Hilbert series for $\mathcal{R}(X)_L$ which we calculated in theorem 7.4.27.

Since $\mathcal{R} \subset \mathcal{R}(X)$ it means that $\mathcal{R}_L = \mathcal{R}(X)_L$ for $L \in \mathcal{S}$.

Now note that by symmetry of the Hilbert series of $\mathcal{R}$ and of Hilbert series of $\mathcal{R}(X)_{\geq 0}$ (see theorem 7.4.27) if $H^0(X, aL_1 + bL_2) \subset \mathcal{R}$ then $H^0(X, bL_1 + aL_2) \subset \mathcal{R}$. In particular from what we proven it follows also that $\mathcal{R}_L = \mathcal{R}(X)_L$ for $L \in \mathcal{S}'$. □

## 7.5. Binary tetrahedral group

**7.5.1. The setup and results.** In this section we collect known facts on the group $G \subset \mathrm{Sp}_4(\mathbb{C})$ investigated by Bellamy and Schedler in [**11**] and by Lehn and Sorger in [**68**], the corresponding symplectic quotient singularity $\mathbb{C}^4/G$ and its symplectic resolution $X$. In particular we introduce two $\mathbb{Z}^2$-gradings on the Cox ring of $X$, one induced by the Picard torus action and one induced by the action of a two-dimensional torus on $X$.

The treatment is according to the program from section 7.1, and the execution follows analogously to the examples in sections 7.3 and 7.4.

Let $G \subset \mathrm{Sp}_4(\mathbb{C})$ be the symplectic representation of binary tetrahedral group generated by the matrices:

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}, \qquad -\frac{1}{2}\begin{pmatrix} (1+i)\epsilon & (-1+i)\epsilon & 0 & 0 \\ (1+i)\epsilon & (1-i)\epsilon & 0 & 0 \\ 0 & 0 & (1+i)\epsilon^2 & (-1+i)\epsilon^2 \\ 0 & 0 & (1+i)\epsilon^2 & (1-i)\epsilon^2 \end{pmatrix},$$

where $\epsilon = e^{2\pi i/3}$ is a third root of unity.

PROPOSITION 7.5.1.

*(1) There are 7 conjugacy classes of elements of $G$ among which two consist of symplectic reflections.*
*(2) The commutator subgroup $[G,G]$ has order 8 and it is isomorphic to the quaternion group. In particular the abelianization $G/[G,G]$ is cyclic of order 3.*
*(3) The representation $G$ defined above is reducible. It decomposes into two two-dimensional representations $V_1 \oplus V_2$. In particular the $(\mathbb{C}^*)^2$-action on $\mathbb{C}^4$ induced by multiplication by scalars on $V_i$ commutes with $G$.*

PROOF. Points (1) and (2) can be quickly verified with GAP Computer Algebra System [**41**]. Point (3) follows directly by the definition of $G$. $\square$

Let $\Sigma \subset \mathbb{C}^4/G$ be the singular locus of the quotient. Analogously as in sections 7.3.1 and 7.4.1 we have:

PROPOSITION 7.5.2. *The preimage of $\Sigma$ via the quotient map $\mathbb{C}^4 \to \mathbb{C}^4/G$ consists of four planes, each of which maps onto $\Sigma$. Outside the image of 0 the singular locus is a transversal $A_2$-singularity.*

Let $\varphi : X \to \mathbb{C}^4/G$ be a projective symplectic resolution which exists by [**11**] or by [**68**] (we will prove it independently in 7.5.3).

PROPOSITION 7.5.3. *There are two exceptional divisors $E_1, E_2$ of $X$ each of which is mapped onto $\Sigma$. The central fibre $\varphi^{-1}([0])$ consist of four surfaces. The fibre of $\varphi$ over any point in $\Sigma \setminus [0]$ consists of two curves isomorphic to $\mathbb{P}^1$ intersecting in one point and each of which is contained in exactly one of two exceptional divisors.*

COROLLARY 7.5.4. *There is a natural $T := (\mathbb{C}^*)^2$-action on $X$ making $\varphi$ an equivariant map.*

Once again we have two different two-dimensional tori – one is the Picard torus $\mathbb{T} := \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$ and the other one is the torus $T$ acting on $\mathbb{C}^4 = \mathbb{C}^2 \times \mathbb{C}^2$ by multiplication of scalars on each component $\mathbb{C}^2$.

Let $C_i$ be the numerical class of a complete curve which is a generic fibre of the morphism $\varphi|_{E_i} : E_i \to \Sigma$.

PROPOSITION 7.5.5. *The Picard group of $X$ is a free rank two abelian group generated by line bundles $L_1, L_2$ such that the intersection matrix $(L_i.C_j)_{i,j}$ is equal to identity matrix.*

Using Singular computer algebra system [**27**] we may obtain the candidate set for the generating set of $\mathcal{R}(X)$. Using similar methods as in section 5.5 it is easy to verify that the original outcome of the program does not satisfy the valuative compatibility criterion, so it will not suffice to generate $\mathcal{R}(X)$. However after a linear transformation involving two generators of $[G,G]$-invariants we obtain the following candidate set, which turns out to be sufficient, as the next theorem asserts. Denote the following elements of the (Laurent) polynomial ring $\mathbb{C}[x_1,y_1,x_2,y_2][t_1^{\pm 1},t_2^{\pm 1}]$:

$$\begin{aligned}
w_{01} &= y_1x_2 - x_1y_2,\\
w_{02} &= x_2^5y_2 - x_2y_2^5,\\
w_{03} &= x_1^5y_1 - x_1y_1^5,\\
w_{04} &= x_1^4 + (-4b+2)x_1^2y_1^2 + y_1^4,\\
w_{05} &= x_2^4 + (4b-2)x_2^2y_2^2 + y_2^4,\\
w_{06} &= x_1x_2^3 + (-2b+1)y_1x_2^2y_2 + (-2b+1)x_1x_2y_2^2 + y_1y_2^3,\\
w_{07} &= x_1^3x_2 + (2b-1)x_1y_1^2x_2 + (2b-1)x_1^2y_1y_2 + y_1^3y_2,
\end{aligned}$$

$$\begin{aligned}
w_{11} &= (-3bx_1^2x_2^2 + (-b+2)y_1^2x_2^2 + (-4b+8)x_1y_1x_2y_2 + (-b+2)x_1^2y_2^2 - 3by_1^2y_2^2)t_1,\\
w_{12} &= (x_2^4 + (-4b+2)x_2^2y_2^2 + y_2^4)t_1,\\
w_{13} &= (x_1^3x_2 + (-2b+1)x_1y_1^2x_2 + (-2b+1)x_1^2y_1y_2 + y_1^3y_2)t_1,\\
w_{14} &= (-5x_1^4y_1x_2 + y_1^5x_2 - x_1^5y_2 + 5x_1y_1^4y_2)t_1,\\
w_{15} &= (x_1y_1x_2^4 + 2x_1^2x_2^3y_2 - 2y_1^2x_2y_2^3 - x_1y_1y_2^4)t_1,
\end{aligned}$$

$$\begin{aligned}
w_{21} &= ((3b-3)x_1^2x_2^2 + (b+1)y_1^2x_2^2 + (4b+4)x_1y_1x_2y_2 + (b+1)x_1^2y_2^2 + (3b-3)y_1^2y_2^2)t_2,\\
w_{22} &= (x_1^4 + (4b-2)x_1^2y_1^2 + y_1^4)t_2,\\
w_{23} &= (x_1x_2^3 + (2b-1)y_1x_2^2y_2 + (2b-1)x_1x_2y_2^2 + y_1y_2^3)t_2,\\
w_{24} &= (y_1x_2^5 + 5x_1x_2^4y_2 - 5y_1x_2y_2^4 - x_1y_2^5)t_2,\\
w_{25} &= (-2x_1^3y_1x_2^2 - x_1^4x_2y_2 + y_1^4x_2y_2 + 2x_1y_1^3y_2^2)t_2,
\end{aligned}$$

$$\begin{aligned}
w_3 &= (9x_1^2y_1x_2^3 + (-2b+1)y_1^3x_2^3 + 9x_1^3x_2^2y_2 + (6b-3)x_1y_1^2x_2^2y_2 + (-6b+3)x_1^2y_1x_2y_2^2 - 9y_1^3x_2y_2^2+\\
&+(2b-1)x_1^3y_2^3 - 9x_1y_1^2y_2^3)t_1t_2
\end{aligned}$$

$$\begin{aligned}
s &= t_1^{-2}t_2,\\
t &= t_1t_2^{-2},
\end{aligned}$$

where $b$ is a primitive root of unity of order 6.

THEOREM 7.5.6. *The Cox ring $\mathcal{R}(X)$ of $X$ is generated by 20 generators:*

$$w_{01},\dots,w_{07},w_{11},\dots,w_{15},w_{21},\dots,w_{25},w_3,s,t.$$

*The degree matrix of these generators with respect to the generators $L_1, L_2$ of $\operatorname{Pic}(X)$ (first two rows) and with respect to the $T$-action (remaining two rows) is:*

$$\left(\begin{array}{cccccccccccccccccccc}
w_{01} & w_{02} & w_{03} & w_{04} & w_{05} & w_{06} & w_{07} & w_{11} & w_{12} & w_{13} & w_{14} & w_{15} & w_{21} & w_{22} & w_{23} & w_{24} & w_{25} & w_3 & s & t\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -2\\
\hline
1 & 0 & 6 & 4 & 0 & 1 & 3 & 2 & 0 & 3 & 5 & 2 & 2 & 4 & 1 & 1 & 4 & 3 & 0 & 0\\
1 & 6 & 0 & 0 & 4 & 3 & 1 & 2 & 4 & 1 & 1 & 4 & 2 & 0 & 3 & 5 & 2 & 3 & 0 & 0
\end{array}\right)$$

We prove this theorem in section 7.5.6. From now on we will denote the ring generated by elements from the statement of theorem 7.5.6 by $\mathcal{R}$.

As in two previous examples we need to describe the movable cone $\operatorname{Mov}(X)$ and find its GIT subdivision independently of the theorem above.

PROPOSITION 7.5.7.

*(1) The cone $\operatorname{Mov}(X)$ of movable divisors of $X$ is equal to $\operatorname{Mov}(\mathcal{R})$ and is the cone generated by the line bundles $L_1$ and $L_2$.*

*(2) There are two symplectic resolutions of $\mathbb{C}^4/G$. The chambers in $\operatorname{Mov}(X)$ corresponding to the nef cones of these resolutions are relative interiors of cones cone$(L_1, L_1+L_2)$ and cone$(L_2, L_1+L_2)$. The Mori cones of corresponding resolutions are cone$(C_2, C_1-C_2)$ and cone$(C_1, C_2-C_1)$.*

PROOF.

(1) This follows from theorem 2.3.13 which allows us to find $\operatorname{Mov}(X)$ and by definition of $\operatorname{Mov}(\mathcal{R})$ (see e.g. the paragraph preceeding proposition 7.1.4).

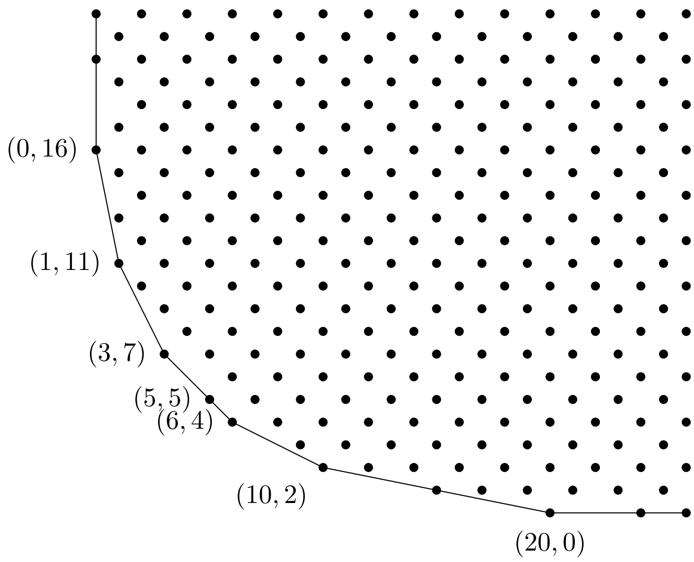
(2) We will prove the first part of the claim in section 7.5.5 as proposition 7.5.24 (we will not use this result until then). The part on Mori cones then follows by taking dual cones.

□

We also will prove the following result independently in section 7.5.3.

THEOREM 7.5.8. *Taking a GIT quotient of* $\operatorname{Spec}\mathcal{R}$ *by the Picard torus action with respect to the linearization given by a character* $(a, b)$ *with* $a > b > 0$ *and with* $b > a > 0$ *one obtains the two symplectic resolutions of* $\mathbb{C}^4/G$.

REMARK 7.5.9. The weights of the $T$-action on global sections of the fixed line bundle $L$ on $X$ are lattice points in $\mathbb{Z}^2$. Taking a convex hull one obtains a lattice polyhedron in $\mathbb{R}^2$. For example fixing a line bundle $L = 2L_1 + L_2$ one gets a polyhedron with the tail equal to the positive quadrant of $\mathbb{R}^2$ and with a head spanned by the lattice points from the picture below:




By [**18**, Lemma 2.4(c)] if $L$ is globally generated, then marked vertices of this polyhedron correspond to $T$-fixed points of $X$ where $X$ is the resolution on which $L$ is relatively ample. We will see in 7.5.20 that indeed fixed points of this polytope correspond to points in $X^T$ and in lemma 7.5.17 that $L$ is globally generated.

**7.5.2. The structure of the central fibre.** In this section we study the structure of the central fibre $\varphi^{-1}([0])$ of such a resolution $\varphi\colon X \to \mathbb{C}^4/G$ using the ideal of relations between generators of the ring $\mathcal{R}$, under the assumption that $X = \operatorname{Spec}\mathcal{R}/\!/_L\mathbb{T}$ for some linearization $L$. Analogously to arguments given in two previous examples, the results of this section are used in the next one, where we investigate the action of the two-dimensional torus $T$ on $X$ with the fixed point locus $X^T$ contained in the central fibre. The additional assumption that $X = \operatorname{Spec}\mathcal{R}/\!/_L\mathbb{T}$ is dealt with in section 7.5.3. The omitted proofs are similar to the ones in section 7.4.2.

LEMMA 7.5.10. *We have an isomorphism* $\operatorname{Spec}\mathcal{R}^{\mathbb{T}} \cong \mathbb{C}^4/G$. *In particular the inclusion of invariants* $\mathcal{R}^{\mathbb{T}} \subset \mathcal{R}$ *induce map* $p\colon \operatorname{Spec}\mathcal{R} \to \mathbb{C}^4/G$.

Let $Z = p^{-1}([0])$. Again, decomposition of its ideal with use of computer algebra system Singular [**27**] and application of lemma 7.5.12 yields:

Proposition 7.5.11. *The components of $Z$ are the following subvarieties of $\mathbb{C}^{20}$:*

$$\begin{aligned}
Z_u &= V(w_3, w_{ij} \mid (i,j) \in (0,1), \ldots, (0,7), (1,1), \ldots, (1,5), (2,1), \ldots, (2,5)),\\
Z_0 &= V(s, t, w_{25}, w_{24}, w_{15}, w_{14}, w_{07}, w_{06}, w_{05}, w_{04}, w_{03}, w_{02}, w_{01},\\
&\quad w_{12}w_{22} - w_{13}w_{23},\ w_{11}w_{21} - 9w_{13}w_{23},\ w_{21}^3 - 27w_{22}w_{23}^2,\ w_{13}w_{21}^2 - 3w_{11}w_{22}w_{23},\\
&\quad w_{12}w_{21}^2 - 3w_{11}w_{23}^2,\ 3w_{13}^2w_{21} - w_{11}^2w_{22},\ 3w_{12}w_{13}w_{21} - w_{11}^2w_{23},\ w_{11}^3 - 27w_{12}w_{13}^2)\\
Z_P &= V(s, w_{25}, w_{24}, w_{23}, w_{22}, w_{21}, w_{15}, w_{14}, w_{07}, w_{06}, w_{05}, w_{04}, w_{03}, w_{02}, w_{01},\\
&\quad w_{11}^3 - 27w_{12}w_{13}^2 - i\sqrt{3}w_3^2t),\\
Z'_P &= V(t, w_{25}, w_{24}, w_{15}, w_{14}, w_{13}, w_{12}, w_{11}, w_{07}, w_{06}, w_{05}, w_{04}, w_{03}, w_{02}, w_{01},\\
&\quad w_{21}^3 - 27w_{22}w_{23}^2 + i\sqrt{3}w_3^2s),\\
Z_1 &= V(s, w_{25}, w_{24}, w_{23}, w_{21}, w_{15}, w_{12}, w_{07}, w_{06}, w_{05}, w_{04}, w_{03}, w_{02}, w_{01},\\
&\quad 2w_3w_{22}t + \zeta_3w_{11}w_{14},\ 2w_{11}^2w_{22} + \zeta_{12}^7\sqrt{3}w_3w_{14},\ w_{11}^3 - i\sqrt{3}w_3^2t,\ 4w_{11}w_{22}^2t + \zeta_{12}^5\sqrt{3}w_{14}^2),\\
Z_2 &= V(t, w_{25}, w_{22}, w_{15}, w_{14}, w_{13}, w_{11}, w_{07}, w_{06}, w_{05}, w_{04}, w_{03}, w_{02}, w_{01},\\
&\quad 2w_3w_{12}s + \zeta_6w_{21}w_{24},\ w_{21}^3 + i\sqrt{3}w_3^2s,\ 2w_{12}w_{21}^2 + \zeta_{12}^{11}\sqrt{3}w_3w_{24},\ 4w_{12}^2w_{21}s + \zeta_{12}^7\sqrt{3}w_{24}^2),
\end{aligned}$$

*where $\zeta_3, \zeta_6, \zeta_{12}$ are primitive 3rd, 6th and 12th roots of unity. The component $Z_u$ is contained in the locus of unstable points with respect to any linearization of the Picard torus via character from the movable cone. Points in the component $Z'_P$ are unstable with respect to any linearization by a character $(2,1)$ and points in the component $Z_P$ are unstable with respect to any linearization by a character $(1,2)$.*

Lemma 7.5.12. *The unstable locus of $\operatorname{Spec}\mathcal{R}$ with respect to a linearization of the trivial line bundle by a character $(2,1)$ is cut out by equations:*

$$w_{12}s = w_{12}w_{23} = w_{11}w_3 = w_{12}w_3 = w_{13}w_3 = w_{13}w_{22} = w_{22}t = 0.$$

*Moreover, all the semistable points of $Z$ are stable and have trivial isotropy groups.*

The following theorem gives a description of components of the central fibre. Let $W$ be the locus of stable points of $\operatorname{Spec}\mathcal{R}$ with respect to the $\mathbb{T}$-action linearized by a character $(2,1)$ (the case $(1,2)$ is analogous) and consider the quotient map $W \to X$. Denote by $S_0, S_1, S_2, P$ the images of sets of stable points of the components $Z_0, Z_1, Z_2, Z_P$. Note that these are precisely the components of the central fibre of $X$.

Theorem 7.5.13.

*(a) $S_0$ is a non-normal toric surface whose normalization is isomorphic to the Hirzebruch surface $\mathcal{H}_6$. The action of $T$ on the normalization of $S_0$ is given by characters in the columns of the matrix $\left(\begin{smallmatrix} 1 & -1 \\ -1 & -1 \end{smallmatrix}\right)$.*

*(b) $S_1$ is a non-normal toric surface whose normalization is the toric surface of a fan spanned by rays: $(0,1), (1,0), (1,-1), (-1,-2)$. The action of $T$ on the normalization of $S_1$ is given by characters in the columns of the matrix $\left(\begin{smallmatrix} 3 & -1 \\ 1 & -1 \end{smallmatrix}\right)$.*

*(c) $S_2$ is a non-normal toric surface whose normalization is the toric surface of a fan spanned by rays: $(0,1), (1,-1), (-1,-2)$. The action of $T$ on the normalization of $S_2$ is given by characters in the columns of the matrix $\left(\begin{smallmatrix} -1 & 3 \\ -1 & 1 \end{smallmatrix}\right)$.*

*(d) $P$ is isomorphic to $\mathbb{P}^2$. The action of $T$ on $P$ in homogeneous coordinates is given by the matrix $\left(\begin{smallmatrix} 2 & 0 & 3 \\ 2 & 4 & 1 \end{smallmatrix}\right)$.*

Proofs of (a)–(c) are analogous to the proof of theorem 7.4.16(a), and (d) is similar to 7.4.16(b).

We can also describe the non-normal locus of components of the central fibre.

Theorem 7.5.14. *If $\nu_i : \widetilde{S}_i \to S_i$ is the normalization of the component $S_i$ of the central fibre $i = 0, 1, 2$ and $\mathcal{N}_i \subset S_i$ is the locus of the non-normal points of $S_i$, then*

(a) $\nu_0^{-1}(\mathcal{N}_0)$ *is the sum of the closures of the orbits corresponding to* $(-1,-3)$ *and* $(1,-3)$ *i.e. it is the sum of invariant fibres of* $\mathcal{H}_6$.
(b) $\nu_1^{-1}(\mathcal{N}_1)$ *is the sum of the closures of the orbits corresponding to* $(-1,-2)$ *and* $(1,-1)$.
(c) $\nu_2^{-1}(\mathcal{N}_2)$ *is the sum of the closures of the orbits corresponding to* $(-1,-2)$ *and* $(1,-1)$.

PROOF. We use lemma 7.5.15 and the description of $\mathbb{T}$-stable orbits of $Z_i$ analogous to the one in the proof of theorem 7.5.13 for $Z_0$. Altogether, $\mathbb{T}$-stable orbits of $Z_i$ which consist of normal points correspond to the cones which do not contain the cones $\tau_j$ and are not contained in $\omega_k$, where $\tau_j$'s are defined analogously as in the proof of 7.4.16(a) and $\omega_k$'s are:

- For $i=0$: $\omega_1 = cone(e_4, e_2+e_3)$ and $\omega_2 = cone(e_1, e_4)$.
- For $i=1,2$: $\omega := \omega_1 = cone(e_2+e_3, e_4)$.

In each case one easily finds all such cones and their images via the map $Q$ (again, notation after proof of 7.5.13) turn out to be the cones parametrizing orbits in the statement. We conclude since the non-normal points of $S_i = Z_i//\mathbb{T}$ are precisely the images of the non-normal points of $Z_i$ which are $\mathbb{T}$-stable. Here we use the fact that all semistable points of $Z_i$ are stable and the isotropy groups of the $\mathbb{T}$-action are trivial by lemma 7.5.12, so that the quotient $Z_i^s \to Z_i//\mathbb{T}$ is a torsor. □

LEMMA 7.5.15. *Let $S$ be a subsemigroup of the lattice $M$ and let $U = \operatorname{Spec}\mathbb{C}[S]$ be the corresponding (not necessarily normal) affine toric variety. Let $\sigma = cone(S)$. Let $\widetilde{U} = \operatorname{Spec}\mathbb{C}[M\cap\sigma]$ be the toric normalization of $U$. Then for each subcone $\tau \prec \sigma$ the orbit $O(\tau) \subset U$ consists of the normal points of $U$ if and only if we have the equality of semigroups $M\cap\langle\tau\rangle + S = M\cap\langle\tau\rangle + M\cap\sigma$, where $\langle\tau\rangle$ is the linear span of $\tau$.*

PROOF. The toric variety $U(\tau) = \operatorname{Spec}\mathbb{C}[M\cap\langle\tau\rangle + S]$ is the open subvariety of $\operatorname{Spec}\mathbb{C}[S]$ obtained by removing these torus orbits which does not contain $O(\tau)$ in their closure. Similarly $\widetilde{U}(\tau) = \operatorname{Spec}\mathbb{C}[M\cap\langle\tau\rangle + M\cap\sigma]$ is the open subvariety of $\operatorname{Spec}\mathbb{C}[M\cap\sigma]$ obtained by removing all the torus orbits whose closure does not contain the orbit corresponding to $\tau$. Moreover $\widetilde{U}(\tau) \to U(\tau)$ is the normalization. Hence the equality $M\cap\langle\tau\rangle + S = M\cap\langle\tau\rangle + M\cap\sigma$ holds if and only if $U(\tau)$ is normal, which is precisely the case when $O(\tau)$ consists of the normal points, since it is contained in the closure of each orbit of $U(\tau)$ and the non-normal locus of a variety is closed. □

The analysis of intersections among the components of the central fibre is analogous to the one in case $G = \mathbb{Z}_2 \wr S_2$ (cf. theorem 7.4.17).

THEOREM 7.5.16.

(a) $S_0 \cap S_1$ *is the curve corresponding to* $(-1,-3)$ *on the normalization of* $S_0$ *and to* $(1,-1)$ *on a normalization of* $S_1$.
(b) $S_0 \cap S_2$ *is the curve corresponding to* $(1,-3)$ *on the normalization of* $S_0$ *and to* $(1,-1)$ *on a normalization of* $S_2$.
(c) $S_0 \cap P$ *is the curve corresponding to* $(0,-1)$ *on the normalization of* $S_0$ *and the cuspidal cubic curve* $x^3 - yz^2 = 0$ *on* $P$ *with homogeneous coordinates* $x, y, z$.
(d) $S_1 \cap P$ *is the curve corresponding to* $(1,0)$ *on the normalization of* $S_1$ *and to the line* $y = 0$ *on* $P$ *with homogeneous coordinates* $x, y, z$ *(note that this is the flex tangent of the cuspidal cubic curve* $S_0 \cap P$*).*

(e) $S_2 \cap P$ *is the point corresponding to the cone spanned by rays* $(0,1),(1,-1)$ *on the normalization of* $S_2$ *and to the point* $x = z = 0$ *on* $P$ *with homogeneous coordinates* $x, y, z$ *(note that this is the cusp of the cubic curve* $S_0 \cap P$*).*

The next lemma shows that all nef line bundles on $X$ are globally generated, which will be important in the next sections. The proof is analogous to the one in the previous example (lemma 7.4.18).

LEMMA 7.5.17. $L_1 + L_2$ *and* $L_1$ *are globally generated line bundles on* $X$.

**7.5.3. Smoothness of the GIT quotient.** Let $\mathcal{R}$ be the subring of the Cox ring of the crepant resolution generated by the elements from the statement of theorem 7.5.6. In this section we show that the GIT quotient $\operatorname{Spec}\mathcal{R}//_L\mathbb{T}$ with respect to the linearization of the trivial line bundle by the character $L = 2L_1+L_2$ of the Picard torus $\mathbb{T}$ is smooth. In consequence we see that $\operatorname{Spec}\mathcal{R}//_L\mathbb{T} \to \mathbb{C}^4$ is a crepant resolution. This makes the results on the geometry of crepant resolutions of $\mathbb{C}^4/G$ in the previous section unconditional and helps to conclude that $\mathcal{R}$ is the whole Cox ring in the final section 7.5.6.
We consider $\operatorname{Spec}\mathcal{R}$ as a closed subvariety of $\mathbb{C}^{20}$ via the embedding given by generators from statement of Theorem 7.5.6.
The proofs of results in this section are as in the case $G = S_3$ presented in section 7.3.3.

THEOREM 7.5.18. *The stable locus of* $\operatorname{Spec}\mathcal{R}$ *with respect to the linearization of the trivial line bundle by a character* $(a,b)$, $a > b > 0$ *is covered by seven* $T \times \mathbb{T}$*-invariant open subsets* $U_1,\ldots,U_7$ *such that* $U_i/\mathbb{T} \cong \mathbb{C}^4$. *More precisely if* $(a,b) = (2,1)$ *then:*

(1) $U_1 = \{w_{12}s \neq 0\}$ *and* $(\mathcal{R}_{w_{12}s})^{\mathbb{T}} = \mathbb{C}\left[w_{02}, w_{05}, \frac{w_{23}}{w_{12}s}, \frac{w_{24}}{w_{12}s}\right]$,
(2) $U_2 = \{w_{12}w_{23} \neq 0\}$ *and* $(\mathcal{R}_{w_{12}w_{23}})^{\mathbb{T}} = \mathbb{C}\left[\frac{w_{12}^2 s}{w_{23}}, \frac{w_{21}}{w_{23}}, \frac{w_{24}}{w_{23}}, \frac{w_3}{w_{12}w_{23}}\right]$,
(3) $U_3 = \{w_{12}w_3 \neq 0\}$ *and* $(\mathcal{R}_{w_{12}w_3})^{\mathbb{T}} = \mathbb{C}\left[\frac{w_{11}}{w_{12}}, \frac{w_{13}}{w_{12}}, \frac{w_{12}w_{23}}{w_3}, \frac{w_{12}w_{24}}{w_3}\right]$,
(4) $U_4 = \{w_{11}w_3 \neq 0\}$ *and* $(\mathcal{R}_{w_{11}w_3})^{\mathbb{T}} = \mathbb{C}\left[\frac{w_{12}}{w_{11}}, \frac{w_{13}}{w_{11}}, \frac{w_{11}w_{22}}{w_3}, \frac{w_{11}w_{23}}{w_3}\right]$,
(5) $U_5 = \{w_{13}w_3 \neq 0\}$ *and* $(\mathcal{R}_{w_{13}w_3})^{\mathbb{T}} = \mathbb{C}\left[\frac{w_{11}}{w_{13}}, \frac{w_{12}}{w_{13}}, \frac{w_{12}w_{21}}{w_3}, \frac{w_{12}w_{22}}{w_3}\right]$,
(6) $U_6 = \{w_{13}w_{22} \neq 0\}$ *and* $(\mathcal{R}_{w_{13}w_{22}})^{\mathbb{T}} = \mathbb{C}\left[\frac{w_{22}^2 t}{w_{13}}, \frac{w_{11}}{w_{13}}, \frac{w_{14}}{w_{13}}, \frac{w_3}{w_{13}w_{22}}\right]$,
(7) $U_7 = \{w_{22}t \neq 0\}$ *and* $(\mathcal{R}_{w_{22}t})^{\mathbb{T}} = \mathbb{C}\left[w_{03}, w_{04}, \frac{w_{13}}{w_{22}^2 t}, \frac{w_{14}}{w_{12}s}\right]$.

*In particular the GIT quotient* $\operatorname{Spec}\mathcal{R}//_{(a,b)}\mathbb{T}$ *with respect to the linearization of the trivial line bundle by a character* $(a,b)$, $a > b > 0$ *is smooth.*

PROOF. Lemma 7.5.12 implies that $\{U_i\}_{i=1,\ldots 7}$ form an open cover of the quotient. It remains to prove equalities from points (1)–(7). Note that then in each case the four generators of the ring on the right-hand side of the equality have to be algebraically independent as the GIT quotient $\operatorname{Spec}\mathcal{R}//_{(a,b)}\mathbb{T}$ is irreducible and of dimension four.
By symmetry it suffices to consider only $U_i$ for $i = 1,2,3,4$. In each case we calculate the invariants of the localization of the coordinate ring of the ambient $\mathbb{C}^{20}$, with the help of 4ti2 [**1**] obtaining in consequence:

(1) $\operatorname{Spec}\mathcal{R}^{\mathbb{T}}_{w_{12}s} = \mathbb{C}\left[w_{0i}, \frac{w_{1j}}{w_{12}}, \frac{w_{2j}}{w_{12}^2 s}, \frac{w_3}{w_{12}^3 s}, w_3 st\right]_{i=1,\ldots,7,\ j=1,\ldots,5}$
(2) $\operatorname{Spec}\mathcal{R}^{\mathbb{T}}_{w_{12}w_{23}} = \mathbb{C}\left[w_{0i}, \frac{w_{1j}}{w_{12}}, \frac{w_{2j}}{w_{23}}, \frac{w_{12}^2 s}{w_{23}}, \frac{w_{23}^2 t}{w_{12}}, \frac{w_3}{w_{12}w_{23}}\right]_{i=1,\ldots,7,\ j=1,\ldots,5}$
(3) $\operatorname{Spec}\mathcal{R}^{\mathbb{T}}_{w_3 w_{12}} = \mathbb{C}\left[w_{0i}, \frac{w_{1j}}{w_{12}}, \frac{w_{12}w_{2j}}{w_3}, \frac{w_{12}^3 s}{w_3}, \frac{w_3^2 t}{w_{12}^3}\right]_{i=1,\ldots,7,\ j=1,\ldots,5}$

(4) $\operatorname{Spec}\mathcal{R}^{\mathbb{T}}_{w_3 w_{11}} = \mathbb{C}\left[w_{0i}, \frac{w_{1j}}{w_{11}}, \frac{w_{11}w_{2j}}{w_3}, \frac{w_{11}^3 s}{w_3}, \frac{w_3^2 t}{w_{11}^3}\right]_{i=1,\dots,7,\ j=1,\dots,5}$

Then, using the Gröbner basis of the ideal of relations between generators of $\mathcal{R}$ with respect to an appropriate lexicographic order, we verify with Singular [**27**] in each case that each of the generators of these four rings can be expressed as a polynomial of the four generators from the assertion. □

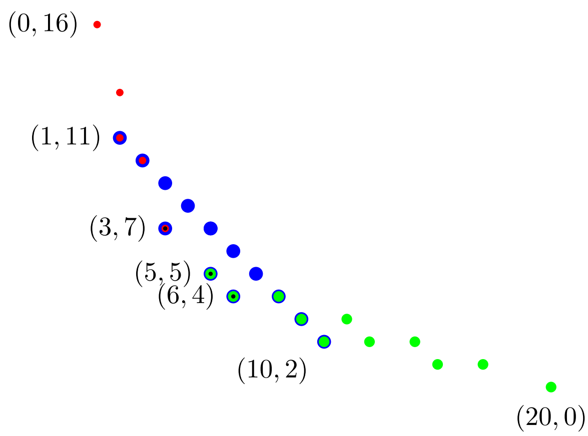
By the inclusion of invariants $\mathcal{R}^{\mathbb{T}} \subset \mathcal{R}$ (see lemma 7.5.10) we have the induced projective map $\operatorname{Spec}\mathcal{R}//_L\mathbb{T} \to \operatorname{Spec}\mathcal{R}/\mathbb{T} \cong \mathbb{C}^4/G$.

Corollary 7.5.19. *The map* $\varphi\colon \operatorname{Spec}\mathcal{R}//_L\mathbb{T} \to \operatorname{Spec}\mathcal{R}/\mathbb{T} \cong \mathbb{C}^4/G$ *is a crepant resolution.*

**7.5.4. Compasses of fixed points.** In this section we obtain a local description of the action of the two-dimensional torus $T$ on a symplectic resolution $X = \operatorname{Spec}\mathcal{R}//_L\mathbb{T}$ of $\mathbb{C}^4/G$ at fixed points of this action, where $L = 2L_1 + L_2$. The arguments and remarks are similar to the ones given for $G = \mathbb{Z}_2 \wr S_2$ in section 7.4.4.

We describe compasses of all fixed points of the action $T$ on $X$ (see definition 6.3.5).

Lemma 7.5.20. *The following diagram shows the weights of the action of $T$ on the space of sections of $H^0(X, L)$ for $L = 2L_1 + L_2$ which are nonzero after the restriction to some irreducible component of the central fibre.*



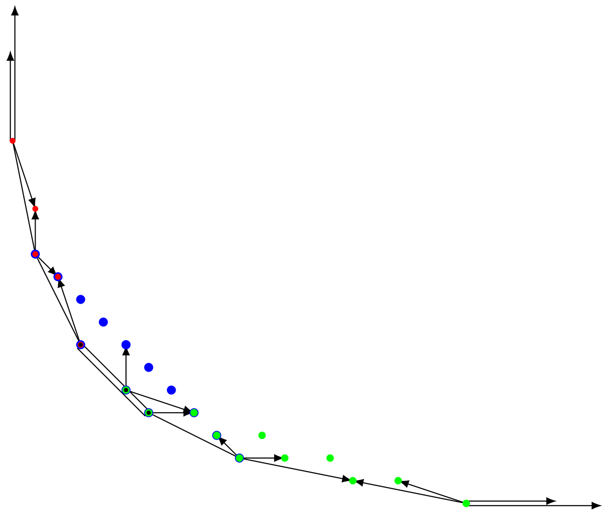

*The black dots correspond to the weights of sections of $L$ restricted to $P$, blue ones to $S_0$, green ones to $S_1$ and red ones to $S_2$ (note that lattice points marked by multiple colours correspond to weights occurring in restriction to more than one component).*

Remark 7.5.21. Lattice points from lemma 7.5.20 are contained in the polyhedron from remark 7.5.9. Moreover, their convex hull form the minimal head of this polyhedron.

Theorem 7.5.22. *The fixed points of the $T$-action correspond to the vertices of the polytopes which are convex hulls of weights marked by the colour fixed in lemma 7.5.20. The compasses of the points corresponding to the vertices of these polytopes are as in the table below:*

| *Point* | *Compass* | | | |
|---|---|---|---|---|
| $P_1 \leftrightarrow (0,16)$ | $\nu_{1,1} = (1,-3)$, | $\nu_{1,2} = (1,-5)$, | $\nu_{1,3} = (0,4)$, | $\nu_{1,4} = (0,6)$ |
| $P_2 \leftrightarrow (1,11)$ | $\nu_{2,1} = (-1,5)$, | $\nu_{2,2} = (2,-4)$, | $\nu_{2,3} = (1,-1)$, | $\nu_{2,4} = (0,2)$ |
| $P_3 \leftrightarrow (3,7)$ | $\nu_{3,1} = (-2,4)$, | $\nu_{3,2} = (-1,3)$, | $\nu_{3,3} = (2,-2)$, | $\nu_{3,4} = (3,-3)$ |
| $P_4 \leftrightarrow (5,5)$ | $\nu_{4,1} = (-2,2)$, | $\nu_{4,2} = (1,-1)$, | $\nu_{4,3} = (3,-1)$, | $\nu_{4,4} = (0,2)$ |
| $P_5 \leftrightarrow (6,4)$ | $\nu_{5,1} = (-1,1)$, | $\nu_{5,2} = (4,-2)$, | $\nu_{5,3} = (-3,3)$, | $\nu_{5,4} = (2,0)$ |
| $P_6 \leftrightarrow (10,2)$ | $\nu_{6,1} = (-4,2)$, | $\nu_{6,2} = (5,-1)$, | $\nu_{6,3} = (-1,1)$, | $\nu_{6,4} = (2,0)$ |
| $P_7 \leftrightarrow (20,0)$ | $\nu_{7,1} = (-3,1)$, | $\nu_{7,2} = (-5,1)$, | $\nu_{7,3} = (4,0)$, | $\nu_{7,4} = (6,0)$ |

The following picture illustrates the weights of the $T$-action calculated in the theorem. It is a directed graph. The points correspond to the sections of $H^0(X,L)$ for $L=2L_1+L_2$ which are nonzero after the restriction to the central fibre together with vectors, as in lemma 7.5.20. The directed edges are the vectors from the compasses attached to the points which correspond to fixed points of $T$-action. In case when two vertices are connected by the two edges pointing in both ways we depict them by a single edge without any arrow.



**7.5.5. Dimensions of movable linear systems.** In this section we use the torus $T$ action on $X$ to give a formula for dimensions of these graded pieces of $\mathcal{R}(X)$ which correspond to the movable linear systems on some of the resolutions. The arguments are analogous to the ones in the case $G=\mathbb{Z}_2 \wr S_2$ in section 7.4.5.

Let $X \to \mathbb{C}^4/G$ be the resolution corresponding to the linearization of the Picard torus action by a character $(2,1)$. Let $X' \to \mathbb{C}^4/G$ be the resolution corresponding to the linearization $(1,2)$.

Denote by $P_i$ the fixed points of the $T$-action on $X$ as in the table from theorem 7.5.22. Let $\{\nu_{i,j}\}_{j=1}^4$ denote the compass of $P_i$ in $X$. Let us also denote by $\mu_i(L)$ the weight of the $T$-action on the fibre of $L$ over $P_i$. Note that $\mu_i$ is linear i.e. $\mu_i(A+B)=\mu_i(A)+\mu_i(B)$.

Remark 7.5.23. By lemma 7.5.17 we may compute the weights $\mu_i$ for line bundles $L_1+L_2$ and $L_1$ similarly as for $2L_1+L_2$ in section 7.5.4 to obtain (the last column is calculated from the first two ones by the linearity of $\mu_i$):

| $i$ | $\mu_i(L_1)$ | $\mu_i(L_1+L_2)$ | $\mu_i(L_2)$ |
|---|---|---|---|
| 1 | $(0,4)$ | $(0,12)$ | $(0,8)$ |
| 2 | $(0,4)$ | $(1,7)$ | $(1,3)$ |
| 3 | $(0,4)$ | $(3,3)$ | $(3,-1)$ |
| 4 | $(2,2)$ | $(3,3)$ | $(1,1)$ |
| 5 | $(3,1)$ | $(3,3)$ | $(0,2)$ |
| 6 | $(3,1)$ | $(7,1)$ | $(4,0)$ |
| 7 | $(8,0)$ | $(12,0)$ | $(4,0)$ |

Proposition 7.5.24. *There are two symplectic resolutions of $\mathbb{C}^4/G$. The chambers in* $\mathrm{Mov}(X)$ *corresponding to the nef cones of these resolutions are relative interiors of cones* $cone(L_1, L_1+L_2)$ *and* $cone(L_2, L_1+L_2)$.

Theorem 7.5.25. *If $h^0(X,pL_1+qL_2)_{(a,b)}$ is the dimension of the subspace of sections $H^0(X,pL_1+qL_2)$ on which $T$ acts with the weight $(a,b)$, then we have the following* *generating function for such dimensions for line bundles inside the movable cone:*

$$\sum_{a,b,p,q\geq 0} h^0(X, pL_1+qL_2)_{(a,b)} y_1^p y_2^q t_1^a t_2^b =$$

$$= \sum_{i=1}^{7} \frac{1}{(1-t^{\mu_i(L_1)}y_1)(1-t^{\mu_i(L_2)}y_2)\prod_{j=1}^{4}(1-t^{\nu_{i,j}})}.$$

As in the two previous examples, the computed generating function may be interpreted as the multivariate Hilbert series of a $\mathbb{Z}^2_{\geq 0}\times\mathbb{Z}^2_{\geq 0}$-graded subalgebra of $\mathcal{R}(X)$ consisting of the graded pieces of $\mathcal{R}(X)$ corresponding to movable line bundles on $X$. This is the interpretation of the theorem that we will use in the next section.

EXAMPLE 7.5.26. The dimensions of the weight spaces corresponding to the lattice points in a head of the polyhedron spanned by weights for the line bundle $2L_1+L_2$ considered in remark 7.5.9 and in section 7.5.4 can be depicted on the following diagram:

```
  1     4     7    12    15    19    23    27    30    35    38    42    46    50
     2     5     9    12    16    20    23    27    31    34    38    42    45
  1     3     7    10    14    17    21    24    28    31    35    38    42    45
     2     5     8    11    15    18    21    25    28    31    35    38    41
        3     5     9    12    15    18    22    24    28    31    34    37    41
     2     4     7    10    13    16    19    22    25    28    31    34    37
  1     2     5     8    11    13    17    19    22    25    28    30    34    36
     1     3     6     8    11    14    16    19    22    24    27    30    32
        2     4     7     9    12    14    17    19    22    24    27    29    32
     1     3     5     7    10    12    14    17    19    21    24    26    28
        1     3     5     8     9    12    14    16    18    21    22    25    27
     1     2     4     6     8    10    12    14    16    18    20    22    24
        1     2     5     6     8    10    12    13    16    17    19    21    23
           1     3     4     6     8     9    11    13    14    16    18    19
              2     3     5     6     8     9    11    12    14    15    17    18
           1     2     3     5     6     7     9    10    11    13    14    15
                    2     3     4     5     7     7     9    10    11    12    14
                 1     2     3     4     5     6     7     8     9    10    11
                    1     2     2     4     4     5     6     7     7     9     9
                             1     2     2     3     4     4     5     6     6
                                1     1     2     2     3     3     4     4     5
                                               1     1     1     2     2     2
                                                              1           1     1
```

**7.5.6. The Cox ring.** Using the generating function from theorem 7.5.25 we may prove theorem 7.5.6 i.e. we prove that $\mathcal{R}=\mathcal{R}(X)$. In this section we outline an argument using the methods of section 4.4.

Denote by $\mathcal{R}(X)$ the Cox ring of $X$ and by $\mathcal{R}$ the subring of $\mathcal{R}(X)$ generated by the elements from the statement of theorem 7.5.6.

LEMMA 7.5.27. *The Cox ring of $X$ is generated by $s,t$ and by*

$$\mathcal{R}(X)_{\geq 0} := \bigoplus_{p,q\geq 0} H^0(X, pL_1+qL_2).$$

PROOF. This is a particular case of proposition 4.4.5. □

Thus to prove that $\mathcal{R}=\mathcal{R}(X)$ it suffices to show that $\mathcal{R}$ contains $\mathcal{R}(X)_{\geq 0}$. The following lemma reduces the problem further, to finitely many graded pieces with respect to $\mathbb{Z}^2$-grading by characters of the Picard torus of $X$.

LEMMA 7.5.28. *$\mathcal{R}(X)_{\geq 0}$ is generated by the sections of all linear spaces $H^0(X,L)$ for $L\in \mathcal{S}\cup\mathcal{S}'$ where $\mathcal{S} := \{\mathcal{O}_X, L_1, L_1+L_2, 2L_1, 2L_1+L_2, 2L_1+2L_2, 3L_1+L_2, 3L_1+2L_2, 4L_1+2L_2\}$ and $\mathcal{S}' := \{L_2, 2L_2, L_1+2L_2, L_1+3L_2, 2L_1+3L_2, 2L_1+4L_2\}$.*

PROOF. By virtue of lemma 7.5.17 this follows from proposition 4.4.4 with $r = 2$ and all $m_1 = m_2 = 1$ for cones $cone(L_1, L_1 + L_2)$ and $cone(L_2, L_1 + L_2)$.. □

Hence we reduced the problem to showing that $\mathcal{R}$ contains spaces of global sections only for these finitely many line bundles which are elements of $\mathcal{S} \cup \mathcal{S}'$ in lemma 7.5.28. This, with the help of computer algebra, can be done with the use of the previous section, namely by theorem 7.5.25 in which we computed the Hilbert series of $\mathcal{R}(X)_{\geq 0}$.

LEMMA 7.5.29. *$\mathcal{R}$ contains $H^0(X, L)$ for each $L \in \mathcal{S} \cup \mathcal{S}'$, where $\mathcal{S}$ is as in the lemma 7.5.28.*

PROOF. We calculate the Hilbert series of $\mathcal{R}$ in Macaulay2 [**45**]. It is of the form:

$$\frac{1}{(1 - y_1^{-2} y_2)(1 - y_1 y_2^{-2})} \cdot F(y_1, y_2, t_1, t_2),$$

where:

$$F(y_1, y_2, t_1, t_2) = \frac{f(y_1, y_2, t_1, t_2)}{(1 - t_2^4)(1 - t_1^4)(1 - y_2 t_1 t_2^3)(1 - y_2 t_1^4)(1 - y_1 t_2^4)(1 - y_1 t_1^3 t_2)(1 - y_1 y_2 t_1^3 t_2^3)}$$

and

$$\begin{aligned} f = {} & 1 + y_1 t_1^2 t_2^2 + y_2 t_1^2 t_2^2 + t_1 t_2 + y_1^2 t_1^4 t_2^4 + y_1 y_2 t_1^4 t_2^4 + y_2^2 t_1^4 t_2^4 + y_1 t_1^5 t_2 + y_1 t_1^3 t_2^3 + y_1 t_1^2 t_2^4 + y_2 t_1^4 t_2^2 + y_2 t_1^3 t_2^3 + y_2 t_1 t_2^5 + t_1^3 t_2 + \\ & t_1^2 t_2^2 + t_1 t_2^3 - y_1^2 y_2 t_1^8 t_2^4 - y_1^2 y_2 t_1^6 t_2^6 - y_1^2 y_2 t_1^5 t_2^7 - y_1 y_2^2 t_1^7 t_2^5 - y_1 y_2^2 t_1^6 t_2^6 - y_1 y_2^2 t_1^4 t_2^8 - y_1^2 t_1^5 t_2^5 - y_1 y_2 t_1^7 t_2^3 - 2 y_1 y_2 t_1^6 t_2^4 - \\ & y_1 y_2 t_1^5 t_2^5 - 2 y_1 y_2 t_1^4 t_2^6 - y_1 y_2 t_1^3 t_2^7 - y_2^2 t_1^5 t_2^5 - y_1 t_1^6 t_2^2 - y_1 t_1^4 t_2^4 - y_1 t_1^3 t_2^5 - y_1 t_1^2 t_2^6 - y_2 t_1^6 t_2^2 - y_2 t_1^5 t_2^3 - y_2 t_1^4 t_2^4 - y_2 t_1^2 t_2^6 + \\ & t_1^6 + t_1^4 t_2^2 + t_1^3 t_2^3 + t_1^2 t_2^4 + t_2^6 + y_1^2 y_2^2 t_1^9 t_2^7 + y_1^2 y_2^2 t_1^7 t_2^9 + y_1^2 y_2 t_1^6 t_2^8 + y_1 y_2^2 t_1^8 t_2^6 - y_1^2 t_1^8 t_2^4 - y_1^2 t_1^4 t_2^8 - y_1 y_2 t_1^9 t_2^3 - y_1 y_2 t_1^7 t_2^5 - \\ & 2 y_1 y_2 t_1^6 t_2^6 - y_1 y_2 t_1^5 t_2^7 - y_1 y_2 t_1^3 t_2^9 - y_2^2 t_1^8 t_2^4 - y_2^2 t_1^4 t_2^8 - y_1 t_1^9 t_2 - y_1 t_1^7 t_2^3 - 2 y_1 t_1^6 t_2^4 - 2 y_1 t_1^5 t_2^5 - y_1 t_1^4 t_2^6 - 2 y_1 t_1^3 t_2^7 - y_1 t_1^2 t_2^8 - \\ & y_2 t_1^8 t_2^2 - 2 y_2 t_1^7 t_2^3 - y_2 t_1^6 t_2^4 - 2 y_2 t_1^5 t_2^5 - 2 y_2 t_1^4 t_2^6 - y_2 t_1^3 t_2^7 - y_2 t_1 t_2^9 + y_1^2 y_2 t_1^{12} t_2^4 + y_1^2 y_2 t_1^{10} t_2^6 + 2 y_1^2 y_2 t_1^9 t_2^7 + 2 y_1^2 y_2 t_1^8 t_2^8 + \\ & y_1^2 y_2 t_1^7 t_2^9 + 2 y_1^2 y_2 t_1^6 t_2^{10} + y_1^2 y_2 t_1^5 t_2^{11} + y_1 y_2^2 t_1^{11} t_2^5 + 2 y_1 y_2^2 t_1^{10} t_2^6 + y_1 y_2^2 t_1^9 t_2^7 + 2 y_1 y_2^2 t_1^8 t_2^8 + 2 y_1 y_2^2 t_1^7 t_2^9 + y_1 y_2^2 t_1^6 t_2^{10} + \\ & y_1 y_2^2 t_1^4 t_2^{12} + y_1^2 t_1^9 t_2^5 + y_1^2 t_1^5 t_2^9 + y_1 y_2 t_1^{10} t_2^4 + y_1 y_2 t_1^8 t_2^6 + 2 y_1 y_2 t_1^7 t_2^7 + y_1 y_2 t_1^6 t_2^8 + y_1 y_2 t_1^4 t_2^{10} + y_2^2 t_1^9 t_2^5 + y_2^2 t_1^5 t_2^9 - y_1 t_1^5 t_2^7 - \\ & y_2 t_1^7 t_2^5 - t_1^6 t_2^4 - t_1^4 t_2^6 - y_1^2 y_2^2 t_1^{13} t_2^7 - y_1^2 y_2^2 t_1^{11} t_2^9 - y_1^2 y_2^2 t_1^{10} t_2^{10} - y_1^2 y_2^2 t_1^9 t_2^{11} - y_1^2 y_2^2 t_1^7 t_2^{13} + y_1^2 y_2 t_1^{11} t_2^7 + y_1^2 y_2 t_1^9 t_2^9 + \\ & y_1^2 y_2 t_1^8 t_2^{10} + y_1^2 y_2 t_1^7 t_2^{11} + y_1 y_2^2 t_1^{11} t_2^7 + y_1 y_2^2 t_1^{10} t_2^8 + y_1 y_2^2 t_1^9 t_2^9 + y_1 y_2^2 t_1^7 t_2^{11} + y_1^2 t_1^8 t_2^8 + y_1 y_2 t_1^{10} t_2^6 + 2 y_1 y_2 t_1^9 t_2^7 + y_1 y_2 t_1^8 t_2^8 + \\ & 2 y_1 y_2 t_1^7 t_2^9 + y_1 y_2 t_1^6 t_2^{10} + y_2^2 t_1^8 t_2^8 + y_1 t_1^9 t_2^5 + y_1 t_1^7 t_2^7 + y_1 t_1^6 t_2^8 + y_2 t_1^8 t_2^6 + y_2 t_1^7 t_2^7 + y_2 t_1^5 t_2^9 - y_1^2 y_2^2 t_1^{12} t_2^{10} - y_1^2 y_2^2 t_1^{11} t_2^{11} - \\ & y_1^2 y_2^2 t_1^{10} t_2^{12} - y_1^2 y_2 t_1^{12} t_2^8 - y_1^2 y_2 t_1^{10} t_2^{10} - y_1^2 y_2 t_1^9 t_2^{11} - y_1 y_2^2 t_1^{11} t_2^9 - y_1 y_2^2 t_1^{10} t_2^{10} - y_1 y_2^2 t_1^8 t_2^{12} - y_1^2 t_1^9 t_2^9 + y_1 y_2 t_1^9 t_2^9 - y_2^2 t_1^9 t_2^9 - \\ & y_1^2 y_2^2 t_1^{12} t_2^{12} - y_1^2 y_2 t_1^{11} t_2^{11} - y_1 y_2^2 t_1^{11} t_2^{11} - y_1^2 y_2^2 t_1^{13} t_2^{13}. \end{aligned}$$

Then, using Singular [**27**] we extract from it Hilbert series for each of the vector spaces $\mathcal{R}_L$, $L \in \mathcal{S}$, graded by characters of $T$. Denote the Hilbert series corresponding to $pL_1 + qL_2$ by $F_{p,q}$. Then we have:

$$\begin{aligned} F_{0,0}(t_1, t_2) &= \tfrac{F(1,1,t_1,t_2) + F(\epsilon,\epsilon^2,t_1,t_2) + F(\epsilon^2,\epsilon,t_1,t_2)}{3}, \\ F_{1,0}(t_1, t_2) &= \tfrac{F(1,1,t_1,t_2) + \epsilon^2 F(\epsilon,\epsilon^2,t_1,t_2) + \epsilon F(\epsilon^2,\epsilon,t_1,t_2)}{3}, \\ F_{0,1}(t_1, t_2) &= \tfrac{F(1,1,t_1,t_2) + \epsilon F(\epsilon,\epsilon^2,t_1,t_2) + \epsilon^2 F(\epsilon^2,\epsilon,t_1,t_2)}{3}, \\ F_{1,1}(t_1, t_2) &= F_{0,0}(t_1, t_2) - F(0, 0, t_1, t_2), \\ F_{2,0}(t_1, t_2) &= F_{0,1}(t_1, t_2) - \tfrac{\partial F}{\partial y_2}(0, 0, t_1, t_2), \\ F_{2,1}(t_1, t_2) &= F_{1,0}(t_1, t_2) - \tfrac{\partial F}{\partial y_1}(0, 0, t_1, t_2) - \tfrac{1}{2} \tfrac{\partial^2 F}{\partial y_2^2}(0, 0, t_1, t_2), \end{aligned}$$

$$F_{2,2}(t_1,t_2) = F_{0,0}(t_1,t_2) - \frac{\partial^2 F}{\partial y_1 \partial y_2}(0,0,t_1,t_2) - \frac{1}{3!}\frac{\partial^3 F}{\partial y_1^3}(0,0,t_1,t_2) - \frac{1}{3!}\frac{\partial^3 F}{\partial y_2^3}(0,0,t_1,t_2),$$

$$F_{3,1}(t_1,t_2) = F_{2,0}(t_1,t_2) - \frac{1}{2}\frac{\partial^3 F}{\partial y_1 \partial y_2^2}(0,0,t_1,t_2) - \frac{1}{2}\frac{\partial^2 F}{\partial y_1^2}(0,0,t_1,t_2) - \frac{1}{4!}\frac{\partial^4 F}{\partial y_2^4}(0,0,t_1,t_2),$$

$$F_{3,2}(t_1,t_2) = F_{2,1}(t_1,t_2) - \frac{1}{2}\frac{\partial^3 F}{\partial y_1^2 \partial y_2}(0,0,t_1,t_2) - \frac{1}{3!}\frac{\partial^4 F}{\partial y_1 \partial y_2^3}(0,0,t_1,t_2) - \frac{1}{4!}\frac{\partial^4 F}{\partial y_1^4}(0,0,t_1,t_2) - \\ - \frac{1}{5!}\frac{\partial^5 F}{\partial y_2^5}(0,0,t_1,t_2),$$

$$F_{4,2}(t_1,t_2) = F_{3,1}(t_1,t_2) - \frac{1}{3!}\frac{\partial^4 F}{\partial y_1^3 \partial y_2}(0,0,t_1,t_2) - \frac{1}{2!3!}\frac{\partial^5 F}{\partial y_1^2 \partial y_2^3}(0,0,t_1,t_2) - \frac{1}{5!}\frac{\partial^6 F}{\partial y_1 \partial y_2^5}(0,0,t_1,t_2) - \\ - \frac{1}{5!}\frac{\partial^5 F}{\partial y_1^5}(0,0,t_1,t_2) - \frac{1}{7!}\frac{\partial^7 F}{\partial y_2^7}(0,0,t_1,t_2).$$

Using Singular we check that for each $L \in \mathcal{S}$ the Hilbert series of $\mathcal{R}_L$ agrees with the Hilbert series for $\mathcal{R}(X)_L$ which we calculated in theorem 7.5.25. Since $\mathcal{R} \subset \mathcal{R}(X)$ it means that $\mathcal{R}_L = \mathcal{R}(X)_L$.

As in the case $G = \mathbb{Z}_2 \wr S_2$ by symmetry of the Hilbert series of $\mathcal{R}$ and of Hilbert series of $\mathcal{R}(X)_{\geq 0}$ (see theorem 7.5.25) if $H^0(X, aL_1 + bL_2) \subset \mathcal{R}$ then $H^0(X, bL_1 + aL_2) \subset \mathcal{R}$. In particular from what we proven it follows also that $\mathcal{R}_L = \mathcal{R}(X)_L$ for $L \in \mathcal{S}'$. □

## 7.6. General scheme

The considerations from this chapter suggest the following geometric approach to the verification whether a 'candidate' for generating set of $\mathcal{R}(X)$ of the form as in situation 4.1.10 actually generates the Cox ring of a crepant resolution $X$ for a quotient singularity. The idea is to use a torus $T$ acting on a resolution.

- *Step 1* Calculate the fixed point locus $X^T$ and the invariants needed to compute the equivariant Euler characteristic of line bundles on the resolution with Lefschetz-Riemman-Roch formula. In case when $X^T$ is discrete it suffices to know the compass at each point of $X^T$ and the weights of the $T$-action on fibres over fixed points of line bundles generating $\mathrm{Pic}(X)$.
- *Step 2* Use the Lefschetz-Riemman-Roch formula (corollary 6.4.22) combined with the Kawamata-Viehweg vanishing theorem to compute the Hilbert series of the non-negatively graded part of the Cox ring $\mathcal{R}(X)_{\geq 0}$. For the case when $\dim X^T \leq 1$ one may use [**18**, Corollary A.3].
- *Step 3* Use multigraded Castelnuovo-Mumford regularity and the Kawamata-Viehweg vanishing as outlined in section 4.4 to obtain a finite set $\mathcal{S}$ of line bundles whose global sections generate the Cox ring.
- *Step 4* Compute the Hilbert series for $\mathcal{R}$ and extract from it Hilbert series of the vector spaces $\mathcal{R}_L$, $L \in \mathcal{S}$, graded by characters of $T$. Check whether they are equal to Hilbert series $\mathcal{R}(X)_L$ calculated in Step 2.

Note that if we know the Hilbert series for $\mathcal{R}(X)_{\geq 0}$ calculated in step 1 then the above procedure suggests also where to look for additional generators of $\mathcal{R}(X)$ if we do not have equality in step 4.

The implementation of the procedure proposed above is challenging, since one has to obtain enough information on the geometry of resolutions without knowing the Cox ring. The required data are of two kinds – in step 1, one has to find data associated with the torus action. And in step 3, to successfully use regularity, one has to find sufficiently many line bundles which are globally generated on one of the resolutions. Nevertheless, we plan on developing these ideas and and applying them in the future study of other examples.

One family of potential candidates to study with use of torus action methods arising from the family of groups $\mathbb{Z}_n \wr S_2 \subset \mathrm{Sp}_4(\mathbb{C})$. The corresponding family of quotient singularities is a subset of the family studied recently by Bellamy and Craw [**8**]. It might be worthwhile to study it with techniques involving torus actions and compare the two methods. It would be also interesting to adapt methods related to torus actions to the case of three-dimensional quotient singularities. Finally, every crepant resolution of a quotient singularity admits a $\mathbb{C}^*$-action induced by homothety action on $\mathbb{C}^n$. One may hope to develop our methods to use this action in the study of geometry of crepant resolutions.

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