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Analytic properties of operators on the non-reflexive spaces of smooth functions

PhD dissertation

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Author's declaration:

I hereby declare that this dissertation is my own work.

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The dissertation is ready to be reviewed

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In loving memory of Stanisław Kazaniecki, greatest father any son could ever ask for.

Abstract

This doctoral thesis consist of four parts, in which the properties of operators on non-reflexive spaces of smooth functions are investigated.

In the second chapter we investigate the existence of a priori estimates for differential operators in L^1 norm: for anisotropic homogeneous differential operators $T_1, \dots, T_\ell$, we study the conditions under which the inequality

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}$$

holds true. Properties of homogeneous rank one convex functions play a major role in the subject. We generalize the notions of quasi and rank one convexity to fit the anisotropic situation.

In the third chapter we prove that every Fourier multiplier on the homogeneous Sobolev space $\dot{W}_1^1(\mathbb{R}^d)$ is a continuous function. This theorem is a generalization of the result of A. Bonami and S. Poornima for Fourier multipliers, which are homogeneous functions of degree zero.

In the fourth chapter we construct a linear injection from the linear space of trigonometric polynomials on $\mathbb{T}^{dk}$ with bounded degrees with respect to each variable onto a suitable subspace $L^p_E \subset L^p(\mathbb{T}^d)$ spanned by characters from E. We establish a quantitative description of the set E, providing both necessary and sufficient conditions for the aforementioned injection to be an isomorphism in L^p norm for $1 \leqslant p < \infty$. One can choose the set E in such a way that the norm of the isomorphism is arbitrary close to one.

In the fifth chapter we study the properties of the trace operator $Tr : W_1^1(\Omega) \to X(\Omega)$. In the case of a domain Ω with a smooth boundary we give a new proof of Peetre theorem, i.e. we prove that there is no continuous, linear operator $S : L^1(\partial\Omega) \to W_1^1(\Omega)$ s.t. $Tr \circ S = Id$. The proof is amazingly simple and uses only the geometry of Whitney decomposition of Ω and basic properties of classical Banach spaces. In the case when Ω is von Koch's snowflake, we use a suitable Whitney decomposition to construct a continuous, linear right inverse of the trace operator.

Keywords: Sobolev spaces, Fourier multipliers, trace operator, Bellman function

AMS Subject Classification: 42B15, 42B05, 46B20, 46B25

Streszczenie

Poniższa rozprawa doktorska składa się z czterech rozdziałów, w których rozpatrujemy rózne własności operatorów na nierefleksywnych przestrzeniach funkcji gładkich.

W drugim rozdziale rozpatrujemy problem istnienia oszacowań a priori dla operatów różniczkowych w normie L^1: dla anizotropowo jednorodnych operatorów róźńiczkowcyh $T_1, \ldots, T_\ell$, badamy warunki dla których zachodzi oszacowanie

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}.$$

Własności funkcji jednorodnych pierwszego stopnia są kluczowe dla tego zagadnienia. W tym rozdziale podajemy uogólnienie quasiwypukłości i rank one wypukłości dostosowane do problemu anizotropowego.

W trzecim rozdziale dowodzimy, że mnożniki Fourierowskie na jednorodnej przestrzeni Sobolewa $\dot{W}_1^1(\mathbb{R}^d)$ są funkcją ciągłymi. Ten wynik jest rozszerzeniem wyniku A. Bonami i S. Poornimy dla mnożników Fouriera, które są funkcjami jednorodnymi stopnia zero.

W czwartym rozdziale konstruujemy liniowe przekształecnie różnowartościowe z przestrzeni liniowej wielomianów trygonometrycznych z ograniczonymi stopniami względem każdej współrzędnej na $\mathbb{T}^{dk}$ do odpowiedniej podprzestrzeni $L^p_E \subset L^p(\mathbb{T}^d)$ rozpiętej przez charaktery z pewnego zbioru E. Podajemy warunki konieczne i dostateczne na to, żeby powyższe przekształcenie było izomorfizmem przestrzeni Banacha w normie L^p dla $1 \leqslant p < \infty$. Zależnie od wyboru zbioru E norma tego izomorfizmu może być dowolnie bliska jedynki.

W rozdziale piątym badamy operator śladu $Tr : W_1^1(\Omega) \to X(\Omega)$. W przypadku obszaru Ω z wystarczająco gładkim brzegiem podajemy nowy dowód twierdzenia Peetre, tzn. dowodzimy, że nie istnieje ciągły operator liniowy $S : L^1(\partial\Omega) \to W_1^1(\Omega)$ takich, że $Tr \circ S = Id$. Ten dowód jest zaskakująco prosty. Używa jedynie własności geometrycznych rozkładu Whitney'a obszaru Ω i własności klasycznych przestrzeni Banacha. W przypadku gdy Ω będzie śnieżynką von Kocha, wykorzystujemy pokrycie Whitney'a do skonstruowania ciągłego, liniowego prawego odwrotnego operatora do operatora śladu.

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Contents

Chapter 1

Introduction

The dissertation consists of results on the properties of operators on function spaces of smooth functions equipped with a non-reflexive norm. In functional analysis spaces of analytic functions (e.g. Hardy spaces) and spaces of smooth functions (e.g. Sobolev spaces and Besov spaces) are especially interesting. While the operators on Hardy spaces are well studied, our knowledge about Sobolev spaces is unsatisfactory (except the case of reflexive spaces). The thesis consists of four parts. Each focuses on different properties of aforementioned operators. Let us briefly describe the content of chapters.

Anisotropic Ornstein noninequalities

The first part is a study of the existence of a priori estimates between differential operators in L^1 norm. Let T_j be differential operators with constant coefficients of order at most m, i.e.

$$T_j = \sum_{|\alpha| \leqslant m} a_{j,\alpha} \partial^\alpha.$$

For $d \geqslant 2$ we consider the existence of the following a priori estimate

$$\|T_1 f\|_{L_p(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_p(\mathbb{R}^d)}, \tag{1.1}$$

with constant independent on $f \in C_0^\infty(\mathbb{R}^d)$. Here and in what follows "$a \lesssim b$" means "there exists a constant c such that $a \leqslant cb$ uniformly", the meaning of the word "uniformly" will be clear from the context. Moreover, $a \simeq b$ will denote "$a \lesssim b$ and $b \lesssim a$".
In the reflexive case $1 < p < \infty$ there is a lot of a priori estimates of the type (1.1). For

example Calderon-Zygmund operators theory (eg. [46]) yields

$$\|\frac{\partial^2}{\partial x_1 \partial x_2} f\|_p \lesssim \|\Delta f\|_p$$

for $1 < p < \infty$. However in the non-reflexive case the above inequality is not satisfied. K. deLeeuw and H. Mirkil [10] have found a necessary and sufficient condition in the case $p = \infty$. Inequality (1.1) is satisfied for $p = \infty$ iff

$$\mathscr{F}(T_1) = \sum_{j=2}^{\ell} \mathscr{F}(T_j)\, \mathscr{F}(\mu_j),$$

where $\mathscr{F}(\cdot)$ denotes the Fourier transform and μ_j are bounded measures. The existence of a priori estimates for $p = 1$ is much more difficult than for $p = \infty$. In his seminal paper D. Ornstein [36] considered the case $p = 1$ and homogeneous differential operators of order m, i.e.

$$T_j = \sum_{|\alpha| = m} a_{j,\alpha} \partial^\alpha.$$

He proved that the estimate

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}$$

is satisfied only in the trivial case $T_1 \in \operatorname{span}\{T_j\}$. His proof was very technical and involved. Recently new, more comprehensible proofs of this fact appeared [8], [24], [23].

Let Λ be an affine hyperplane in $\mathbb{R}^d$ that intersects all the positive semi-axes. We call such a plane a *pattern of homogeneity*. We call a differential operator Λ-homogeneous if

$$T_j = \sum_{\alpha \in \Lambda} a_{j,\alpha} \partial^\alpha.$$

The aim of Chapter 2 is to give a proof of anisotropic version of Ornstein's theorem.

Theorem. *Let Λ be a pattern of homogeneity in $\mathbb{R}^d$, let $\{T_j\}_{j=1}^{\ell}$ be Λ-homogeneous differential operators. Suppose that all the monomials present in T_j have one and the same parity of degree. If the inequality*

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)} \tag{1.2}$$

holds true for any $f \in C_0^\infty(\mathbb{R}^d)$, then T_1 can be expressed as a linear combination of the other T_j.

The starting point of our argument mimics the approach from [23]. We introduce a notion of generalized rank one convexity and generalized gradient ∇. We define a Bellman function on a suitable space E by the formula

$$\boldsymbol{B}(e) = \inf_{\varphi \in C_0^\infty([0,1]^d)} \int\limits_{[0,1]^d} V(e + \nabla[\varphi](x))\, dx$$

for every $e \in E$. We study the properties of B and ultimately we prove that if $T_1 \notin \operatorname{span}\{T_j\}$, such function B does not exist. More precisely, in that case the above function has to be separately convex (i.e. convex with respect to each variable), homogeneous of degree one and sign changing. The whole problem reduces to the following theorem.

Theorem. *A function $F : \mathbb{R}^d \to \mathbb{R}$ that is separately convex and homogeneous of order one is non-negative.*

In contrast to Ornstein's original proof, we rather study the properties of Bellman function than construct a specific function built by a martingale approach. Contents of this chapter are taken from the article [19].

Continuity of Fourier multipliers on homogeneous Sobolev spaces

In the third chapter we study the properties of translation invariant operators. We call a function space $X(\mathbb{R}^n)$ translation invariant if every shift operators acts on this space as a isometry. An operator $T : X(\mathbb{R}^n) \to X(\mathbb{R}^n)$ is translation invariant if for every $v \in \mathbb{R}^n$

$$T \circ \tau_v = \tau_v \circ T,$$

where $\tau_v f(x) = f(x + v)$. The classical characterization of translation invariant operators on $L^1(\mathbb{R}^n)$ says that $T : L^1(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$ is translation invariant iff there exists a bounded measure μ such that $Tf = \mu * f$ for every $f \in L^1(\mathbb{R}^n)$ ([47]). The Fourier transform of a measure is a continuous function. Hence, every $f \in L^1(\mathbb{R}^n)$ satisfies the identity

$$\mathscr{F}(Tf) = m\mathscr{F}(f),$$

where m is a suitable continuous function. Let $W_1^1(\mathbb{R}^n)$ be a Sobolev space, i.e. completion of smooth functions with compact support on $\mathbb{R}^n$ with respect to the norm

$$\|f\|_{W_1^1(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)} + \|\nabla f\|_{L^1(\mathbb{R}^n)}.$$

From Ornstein's noninequality [36] it follows that the class of translation invariant operators on $W_1^1(\mathbb{R}^n)$ is wider than the class of convolutions with bounded measures [42]. Let $T : W_1^1(\mathbb{R}^n) \to W_1^1(\mathbb{R}^n)$ be translation invariant. From general theory there exists $m \in L^\infty$ s.t.

$$\mathscr{F}(Tf) = m\mathscr{F}(f).$$

However $W_1^1(\mathbb{R}^n)$ is a subset of $L^1(\mathbb{R}^n)$, hence the Fourier transform of a function from the Sobolev space $W_1^1(\mathbb{R}^n)$ is continuous. This yields that the above function m is continuous. The situation is much more delicate in case of homogeneous Sobolev spaces. We denote by $\dot{W}_1^1(\mathbb{R}^n)$ a space of weakly differentiable functions on $\mathbb{R}^n$ with integrable gradient. We a define seminorm on $\dot{W}_1^1(\mathbb{R}^n)$ by the formula

$$\|f\|_{\dot{W}_1^1(\mathbb{R}^n)} = \|\nabla f\|_{L^1(\mathbb{R}^n)}.$$

The quotient by constant functions $\dot{W}_1^1(\mathbb{R}^n)/P_0$ with the above norm is a Banach space. Abusing the notation, we denote this Banach space by $\dot{W}_1^1(\mathbb{R}^n)$.
As usual $\mathscr{S}$ denotes the space of Schwartz functions. Let T be a translation invariant operator on $\dot{W}_1^1(\mathbb{R}^n)$. For every such T there exists $m \in L^\infty$ s.t.

$$\mathscr{F}(Tf) = m\mathscr{F}(f) \qquad \forall f \in \mathscr{S}.$$

We denote by $\mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$ the space of all such functions m and we call them Fourier multipliers. We investigate the continuity of functions in $\mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$. The aim of this Chapter is to prove the continuity of functions from $\mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$. The special case when m is a homogeneous function of degree zero, i.e. $m(\lambda x) = m(x)$, was studied by A. Bonami and S. Poornima [3]. In their beautiful proof they show that m has to be a constant function. The main result of this Chapter is the following.

Theorem. *If $d \geqslant 2$ and $m \in \mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$ then $m \in C_b(\mathbb{R}^d)$.*

It is worth to mention that our proof uses the result by A. Bonami and B. Poornima. Contents of this chapter are taken from the article [22].

Isomorphism between sets of trigonometric polynomials

One of the essential tools used in the proof of the continuity of Fourier multipliers $m \in \mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$ from the third chapter is the estimation of the norm of a linear combination of finite Riesz products. Let $\{a_k\}$ be a sequence of natural numbers s.t. $a_{k+1} > 3a_k$. We define the finite Riesz product by the formula

$$R_n(x) = \Pi_{k=1}^n (1 + \cos 2\pi a_k x)$$

The key estimate used in Chapter 3 is an estimate (1.3) by R. Latała [29] valid for suitable (very) fast growing sequence $\{a_k\}$.

$$\|\sum_j b_j R_j\|_{L^1(\mathbb{T})} \simeq \sum_j |b_j|. \tag{1.3}$$

This inequality is a consequence of an inequality for random variables. The transference to trigonometric case is based on the observation that for sufficiently fast growing a_k's, functions $\cos(2\pi a_k x)$ mimic independent random variables. The problem is to find the specific conditions on $\{a_k\}$ such that R_j behave like independent random variables with respect to L^1 norm. This problem can be considered for much more general polynomials. In Chapter 4 we investigate what kind of conditions are sufficient for the behavior of trigonometric polynomials to be similar to that of independent random variables. To be precise, we give the following definitions. For $k \in \mathbb{N}$, $B \subset \mathbb{Z}^k$ let $L^p_B(\mathbb{T}^k) = \{f \in L^p(\mathbb{T}^k) : \operatorname{supp} \widehat{f} \subset B\}$.

Definition. *For a given sequence of integers $\tau = \{\tau_n\}_{n\in\mathbb{N}}$ and a set $A \subset \mathbb{Z}$ we define sets $E \subset \mathbb{Z}$ and $F \subset \mathbb{Z}^{\mathbb{N}}$ (here $\mathbb{Z}^{\mathbb{N}}$ is a dual group to $\mathbb{T}^{\omega}$), in the following way:*

$$
\begin{aligned}
F &:= \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots) \in \mathbb{Z}^{\mathbb{N}} \ : \ \lambda_n \in A\},\\
E &:= \{\beta \in \mathbb{Z} \ : \ \ \beta = \sum_{k=1} \tau_k \lambda_k \textit{ for } \boldsymbol{\lambda} \in F\}.
\end{aligned}
$$

For L^∞ norm the theorem below was proved by Y. Meyer [33]. The main result of Chapter 4 is a proof of sufficiency of the Meyer condition for L^1 norm.

Theorem. *For a given sequence of integers $\tau = \{\tau_n\}_{n\in\mathbb{N}}$ and finite set $A \subset [-r, r] \cap \mathbb{Z}$ satisfying*

$$
|\tau_{k+1}| > 2r \sum_{j=1}^{k} |\tau_j| \qquad \forall k \in \mathbb{N},
$$
$$
\sum_{j=1}^{\infty} \frac{|\tau_j|}{|\tau_{j+1}|} < \infty,
$$

the operator $T := L^p_F(\mathbb{T}^{\mathbb{N}}) \to L^p_E(\mathbb{T})$ given by the formula

$$
Tf(x) = \sum_{\boldsymbol{\lambda}\in F} \widehat{f}(\boldsymbol{\lambda}) e^{2\pi i \langle \sum_{j=1}^{\infty} \lambda_j \tau_j, x\rangle}
$$

is an isomorphism of Banach spaces.

In fact we prove a generalization of the above to a higher dimension. In [12] M. Déchamps gave a weaker condition

$$
\sum_{j=1}^{\infty} \frac{|\tau_j|^2}{|\tau_{j+1}|^2} < \infty.
$$

for the case L^∞. She claimed that this condition also works for L^1 norm, however her proof contained a flaw. Nevertheless we show that M. Déchamps condition is necessary for T to be an isomorphism in L^1 norm. Moreover in the last subsection we give an example of a sequence $\{\tau_k\}$ such that T defined as in the above theorem is an isomorphism for $p = 2$ and $p = 4$. However it is not an isomorphism for $p = 3$ and $p = \frac{4}{3}$. Therefore, the conditions on $\{\tau_k\}$ for which T is an isomorphism in L^p norm do not interpolate for $2 < p < 4$ and in general without additional conditions do not work for the dual space. Results of this chapter are based on the unpublished preprint [20]. It is worth to mention that in the special case of Riesz products the condition could be considerably weakened. In [30] R Latała, P. Nayar and T. Tkocz proved that $a_{k+1} > 1.2 \times 10^9 a_k$ is enough for the case L^p, $1 \leqslant p < \infty$. Finally, A. Bonami indicated a simple argument which gives $a_{k+1} > 3a_k$ for L^1 norm [4].

Trace operator and its right inverse on planar domains

In the last chapter of the thesis we study the properties of the trace operator. It was proven by E. Gagliardo [16] that for domains Ω with a smooth boundary $Tr : W_1^1(\Omega) \to L^1(\partial\Omega)$ is onto. It was proved by J. Petree [39] that the trace operator on $W_1^1(\mathbb{R}_+ \times \mathbb{R}^n)$ does not have continuous, linear right inverse. In the first part of the chapter we use the Whitney decomposition of a domain Ω to give a new proof of Peetre theorem.

Theorem. *Let Ω be a an open domain with Lipschitz boundary and $\partial\Omega$ be a Jordan curve. Let $Tr : W_1^1(\Omega) \to L^1(\partial\Omega)$ be a trace operator. Then there is no continuous, linear operator $S : L^1(\partial\Omega) \to W^{1,1}(\Omega)$ s.t. $TS = Id_{L^1(\partial\Omega)}$.*

The proof is amazingly simple. It uses just the geometry of Whitney covering and basic properties of classical Banach spaces.

In the second part we investigate the trace operator on von Koch's snowflake Ω_K. In [17] P. Hajłasz and O. Martio studied the existence of a right inverse operator to trace in the case of Sobolev spaces $W_1^p(\Omega)$ for $p > 1$. They characterized trace space as a generalized Sobolev space. In this part of thesis we will characterize the trace space of $W_1^1(\Omega_K)$. We use the density of restrictions of Lipschitz functions $Lip(\mathbb{R}^2)$ in $W_1^1(\Omega_K)$ to define the trace space. For Lipschitz functions the operator Tr is just a restriction to the boundary. We denote by $X(\Omega)$ the trace space - the completion of $Tr(Lip(\mathbb{R}^2))$ with respect to the norm

$$\|g\|_{X(\Omega_K)} := \inf\{\|f\|_{W_1^1(\Omega)} : Trf = g \text{ and } f \in Lip(\mathbb{R}^2)\}.$$

We prove that $X(\Omega)$ is isomorphic to Arens-Eels space with respect to the metric

$$d(x, y) = \inf\{|\nabla f\|_{L^1} : f \in W_1^1(\Omega),\ Trf = \mathbb{1}_{[x,y]}\}$$

on the boundary, where $\mathbb{1}_{[x,y]}$ is the characteristic function of an arc $[x, y]$.

Definition. *Let (Y, d_Y) be a metric space. We call a function $f : Y \to \mathbb{R}$ a molecule if it has finite support and $\sum_{y\in Y} f(y) = 0$. Let $x, y \in Y$. We define special type of a molecule - an atom : $m_{xy} = \mathbb{1}_{\{x\}} - \mathbb{1}_{\{y\}}$. Let m be a molecule, i.e. $m = \sum_{j=1}^{M} a_j m_{x_j y_j}$. Then the Arens-Eels norm of m is*

$$\|m\|_{AE(d_Y)} = \inf\left\{\sum_j |a_j| d_Y(x_j, y_j) : m := \sum_j a_j m_{x_j y_j}\right\},$$

where the infimum is taken over all possible representations of m as a linear combination of m_{pq}. The Arens-Eels space is the completion of molecules with respect to the norm $\|\cdot\|_{AE(d_Y)}$.

Using the structure of Whitney decomposition of the von Koch's snowflake we prove that there a exists metric d such that $\tilde{d} = d^{\alpha}$, where $0 < \alpha < 1$. The existence of the right inverse to trace operator is a consequence of this fact.

Theorem. *Let $Tr : W_1^1(\Omega_K) \to X(\Omega_K)$ be a trace operator, where $X(\Omega_K)$ is a trace space* (5.2)*. There exists a continuous, linear operator $S : X(\Omega_K) \to W_1^1(\Omega_K)$ s.t. $Tr \circ S = Id_{X(\Omega_K)}$.*

The results of this chapter are based on an unpublished joint work with my advisor M. Wojciechowski.

Chapter 2

Anisotropic Ornstein noninequalities

In his seminal paper [36], Ornstein proved the following: let $\{T_j\}_{j=1}^{\ell}$ be homogeneous differential operators of the same order in d variables (with constant coefficients); if the inequality

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}$$

holds true for any $f \in C_0^\infty(\mathbb{R}^d)$, then T_1 can be expressed as a linear combination of the other T_j. For example, in the statement above, the constant should be uniform with respect to all functions f. The aim of the present chapter is to extend this theorem to the case where the differential operators are anisotropic homogeneous; see also [21], where a partial progress in this direction was obtained by a simple Riesz product technique.

To formulate the results of this chapter, we have to introduce a few notions. Each differential polynomial $P(\partial)$ in d variables has a Newton diagram which matches a set of integral points in $\mathbb{R}^d$ to each such polynomial. The monomial $a\partial_1^{m_1}\partial_2^{m_2}\dots\partial_d^{m_d}$ corresponds to the point $m = (m_1, m_2, \dots, m_d)$; for an arbitrary polynomial, its Newton diagram is the union of the Newton diagrams of its monomials.

Let Λ be an affine hyperplane in $\mathbb{R}^d$ that intersects all the positive semi-axes. We call such a plane a *pattern of homogeneity*. We say that a differential polynomial is homogeneous with respect to Λ (or simply Λ-homogeneous) if its Newton diagram lies on Λ.

Conjecture 2.1. *Let Λ be a pattern of homogeneity in $\mathbb{R}^d$, let $\{T_j\}_{j=1}^{\ell}$ be a collection of Λ-homogeneous differential operators. If the inequality*

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)}$$

holds true for any $f \in C_0^\infty(\mathbb{R}^d)$, then T_1 can be expressed as a linear combination of the other T_j.

This conjecture may seem to be a simple generalization of Ornstein's theorem. We warn the reader that sometimes the anisotropic character of homogeneity brings new difficulties to inequalities for differential operators (the main is that one lacks geometric tools such as the isoperimetric inequality, or the coarea formula, etc.). For example, the classical embedding $W_1^1(\mathbb{R}^d) \hookrightarrow L_{\frac{d}{d-1}}$ due to Gagliardo and Nirenberg had been generalized to the anisotropic case only in [45] and finally in [27]; if one deals with similar embeddings for vector fields, the isotropic case was successfully considered in [49] (see also the survey [50]), and there is almost no progress for anisotropic case (however, see [25, 26]).

The method we use to attack the conjecture, differs from that of Ornstein (though there are some similarities). However, it is not new. It was noticed in [8] that Ornstein's theorem is related to the behavior of certain rank one convex functions (for some special operators this link had already been known, see [18]). The case $d = 2$ was considered there. As for the general case of Ornstein's (isotropic) theorem, its proof via rank one convexity was announced in [23] and the proofs are available in [24]. In a sense, we follow the plan suggested in [23]. However, the notions of quasi convexity, rank one convexity and others should be properly adjusted to the anisotropic world, we have not seen such an adjustment anywhere. For all these notions in the classical setting of the first gradient, their relationship with each other, properties, etc., we refer the reader to the book [9]. There are certain problems in the general anisotropic case that are not present in the classical setting. For example, the existence of the elementary laminate is not quite clear, at least, the classical reasoning does not work. Quasi convexity still implies the rank one convexity, but this requires a new proof. The approach of rank one convexity reduces Conjecture 2.1 to a certain geometric problem about separately convex functions (Theorem 2.14) that is covered by Theorem 1 announced in [23] (Theorem 1.1 in [24]). We give a simple proof of this fact, which may seem the second advantage of approach in this chapter (though our proof does not give more advanced Theorem 1 of [23]). We did not know the preprint [24] almost until the publication of the present text, and did our work independently. The discussion with the authors of [24] has shown that though the spirit of our approach in the geometric part is similar to that of [24], the presentation and details appear to be different.

We will prove a particular case of Conjecture 2.1, which still seems to be rather general (in particular, it covers the classical isotropic case).

Theorem 2.2. *Let Λ be a pattern of homogeneity in $\mathbb{R}^d$, let $\{T_j\}_{j=1}^{\ell}$ be Λ-homogeneous differential operators. Suppose that all the monomials present in the T_j have one and the same parity of degree. If the inequality*

$$\|T_1 f\|_{L_1(\mathbb{R}^d)} \lesssim \sum_{j=2}^{\ell} \|T_j f\|_{L_1(\mathbb{R}^d)} \tag{2.1}$$

holds true for any $f \in C_0^\infty(\mathbb{R}^d)$, then T_1 can be expressed as a linear combination of the other T_j.

We note that the differential operators here are not necessarily scalar, i.e., one can prove the same theorem for the case where operators act on vector fields. It is one of the advantages of the general rank one convexity approach. However, to facilitate the notation, we work on the scalar case.

We outline the structure of the chapter. We begin with restating inequality (2.1) as an extremal problem described by a certain Bellman function (if inequality (2.1) holds, then the corresponding Bellman function is non-negative). We also study the properties of our Bellman function (they are gathered in Theorem 2.6), the most important of which is the quasi convexity. All this material constitutes Section 2.1. It turns out, that quasi convexity leads to a softer, but easier to work with, property of rank one convexity. The proof of this fact is given in Section 2.2, see Theorem 2.9. So, the Bellman function in question is rank one convex. In Section 2.3, we prove that rank one convex functions homogeneous of order one are non-negative, which gives us Theorem 2.2. In fact, it suffices to show a similar principle for separately convex functions on $\mathbb{R}^d$, which is formalized in Theorem 2.14. This theorem is purely convex geometric. Finally, we discuss related questions in Section 2.4.

2.1 Bellman function and its properties

Inequality (2.1) can be rewritten as

$$\inf_{\varphi \in C_0^\infty([0,1]^d)} \Big(\sum_{j=2}^{\ell} \|T_j\varphi\|_{L_1(\mathbb{R}^d)} - c\|T_1\varphi\|_{L_1(\mathbb{R}^d)} \Big) = 0, \tag{2.2}$$

where c is a sufficiently small positive constant.

Definition 2.3. *Suppose that ∂^α, $\alpha \in A$ are all the partial derivatives that are present in the T_j (thus A is a subset of $\Lambda \cap \mathbb{Z}^d$). Consider the Hilbert space E with an orthonormal basis e_α indexed with the set A. For each function φ and each point x, we have a mapping*

$$[0,1]^d \ni x \mapsto \nabla[\varphi](x) = \sum_{\alpha \in A} \partial^\alpha[\varphi](x) e_\alpha \in E.$$

We call the function $\nabla[\varphi]$ the generalized gradient of φ.

The operator $\nabla[\cdot]$ is an analogue of the usual gradient suitable for our problem.

Example 2.4. *Let $T_j = \partial_{x_j}$ for $j = 1, \ldots, d$. In this case the generalized gradient turns out to be the usual gradient on the Euclidean space $\mathbb{R}^d$.*

Example 2.5. *Let us take the differential operators*

$$\begin{aligned} T_1[\varphi] = \partial^{(2,0,1)}[\varphi] - \partial^{(0,3,1)}[\varphi], \quad T_2[\varphi] = \partial^{(4,0,0)}\varphi, \\ T_3[\varphi] = \partial^{(0,6,0)}[\varphi], \quad T_4[\varphi] = \partial^{(0,0,2)}[\varphi]. \end{aligned} \tag{2.3}$$

We can list all the partial derivatives present in the operators:

$$A = \{\partial^{(0,0,2)}, \partial^{(0,6,0)}, \partial^{(4,0,0)}, \partial^{(0,3,1)}, \partial^{(2,0,1)}\}.$$

All the operators T_j are Λ-homogeneous, where $\Lambda = \{x \in R^3 \colon \langle x,\, (3,2,6)\rangle = 12\}$. In this case the generalized gradient is of the following form:

$$\nabla[\varphi] = (\partial^{(0,0,2)}[\varphi], \partial^{(0,6,0)}[\varphi], \partial^{(4,0,0)}[\varphi], \partial^{(0,3,1)}[\varphi], \partial^{(2,0,1)}[\varphi]) \in \mathbb{R}^5.$$

We also consider the function $V : E \to \mathbb{R}$ given by the rule

$$V(e) = \Big(\sum_{j=2}^{\ell} |\tilde{T}_j e| - c|\tilde{T}_1 e|\Big), \tag{2.4}$$

here $\tilde{T}_j$ are the linear functionals on E such that $\tilde{T}_j(e) = \sum_A c_{\alpha,j} e_\alpha$ if $T_j = \sum_A c_{\alpha,j}\partial^\alpha$. With this portion of abstract linear algebra, we rewrite formula (2.2) as

$$\inf_{\varphi \in C_0^\infty([0,1]^d)} \int\limits_{[0,1]^d} V(\nabla[\varphi](x))\,dx = 0.$$

The main idea is to consider a perturbation of this extremal problem, i.e., the function $\boldsymbol{B} : E \to \mathbb{R}$ given by the formula

$$\boldsymbol{B}(e) = \inf_{\varphi \in C_0^\infty([0,1]^d)} \int\limits_{[0,1]^d} V(e + \nabla[\varphi](x))\,dx. \tag{2.5}$$

Theorem 2.6. *Suppose that inequality* (2.2) *holds true. Then, the function $\boldsymbol{B}$ possesses the properties listed below.*

1. *It satisfies the inequalities $-\|e\| \lesssim \boldsymbol{B}(e) \lesssim \|e\|$ and $\boldsymbol{B} \leqslant V$.*
2. *It is one homogeneous, i.e. $\boldsymbol{B}(\lambda e) = |\lambda|\boldsymbol{B}(e)$.*
3. *It is a Lipschitz function.*

4. *It is a generalized quasi convex function, i.e. for any $\varphi \in C_0^\infty([0,1]^d)$ and any $e \in E$ the inequality*

$$\boldsymbol{B}(e) \leqslant \int\limits_{[0,1]^d} \boldsymbol{B}(e + \nabla[\varphi](x))\, dx \tag{2.6}$$

holds true.

Proof. 1) We get the upper estimates on the function $\boldsymbol{B}$ by plugging $\varphi \equiv 0$ in the formula for it:

$$\boldsymbol{B}(e) \leqslant \int\limits_{[0,1]^d} V(e + \nabla[\varphi]) = V(e) \lesssim \|e\|.$$

We obtain the lower bounds on the function $\boldsymbol{B}$ from inequality (2.2) and the triangle inequality:

$$\begin{aligned}
&\int\limits_{[0,1]^d} \left(\sum_{j=2}^{\ell} \left|\tilde{T}_j\left(e + \nabla[\varphi]\right)\right| - c\left|\tilde{T}_1\left(e + \nabla[\varphi]\right)\right| \right) \\
&\qquad\geqslant \int\limits_{[0,1]^d} \left(\sum_{j=2}^{\ell} \left|\tilde{T}_j\left(e + \nabla[\varphi]\right)\right| - c\left|\tilde{T}_1\left(\nabla[\varphi]\right)\right| - c|\tilde{T}_1 e| \right) \\
&\qquad\geqslant \int\limits_{[0,1]^d} \left(\sum_{j=2}^{\ell} |\tilde{T}_j(e + \nabla[\varphi])| - \sum_{j=2}^{\ell} |\tilde{T}_j(\nabla[\varphi])| - c|\tilde{T}_1 e| \right) \\
&\qquad= \int\limits_{[0,1]^d} \left(\sum_{j=2}^{\ell} \left(|\tilde{T}_j\left(e + \nabla[\varphi]\right)| - |\tilde{T}_j(\nabla[\varphi])| \right) - c|\tilde{T}_1 e| \right) \\
&\qquad\geqslant -\sum_{j=2}^{\ell} |\tilde{T}_j e| - c|\tilde{T}_1 e|,
\end{aligned}$$

where $\varphi \in C_0^\infty([0,1]^d)$ is an arbitrary function. We take infimum of the above inequality over all admissible φ:

$$-\|e\| \lesssim -\sum_{j=2}^{\ell} |\tilde{T}_j e| - c|\tilde{T}_1 e| \leqslant \boldsymbol{B}(e).$$

2) Since V is a one homogeneous function, the following equality holds for every $\lambda \neq 0$:

$$\boldsymbol{B}(\lambda e) = \inf_{\varphi \in C_0^\infty([0,1]^d)} \int\limits_{[0,1]^d} V\left(\lambda e + \nabla[\varphi]\right) = \inf_{\varphi \in C_0^\infty([0,1]^d)} \int\limits_{[0,1]^d} |\lambda| V\left(e + \nabla[\lambda^{-1}\varphi]\right).$$

We know that $\lambda^{-1}C_0^\infty([0,1]^d) = C_0^\infty([0,1]^d)$ for every $\lambda \neq 0$, therefore

$$\begin{aligned}\boldsymbol{B}(\lambda e) &= \inf_{\varphi\in C_0^\infty([0,1]^d)} \int\limits_{[0,1]^d} |\lambda| V\left(e + \nabla[\lambda^{-1}\varphi]\right) \\ &= |\lambda| \inf_{\varphi\in C_0^\infty([0,1]^d)} \int\limits_{[0,1]^d} V\left(e + \nabla[\varphi]\right) = |\lambda|\boldsymbol{B}(e).\end{aligned}$$

3) In order to get the Lipschitz continuity of $\boldsymbol{B}$, we rewrite the formula for it:

$$\forall e \in E \qquad \boldsymbol{B}(e) = \inf_{\varphi\in C_0^\infty([0,1]^d)} V_\varphi(e),$$

where

$$V_\varphi(e) = \int\limits_{[0,1]^d} V(e + \nabla[\varphi](x))dx.$$

It follows from the Lipschitz continuity of V that every function V_φ is a Lipschitz function with the Lipschitz constant bounded by L, where L is the Lipschitz constant of the function V. For every two points $v_1, v_2 \in E$, we can find a sequence of functions V_{φ_n} such that $\boldsymbol{B}(v_j) = \inf_{n\in\mathbb{N}} V_{\varphi_n}(v_j)$ for $j \in \{1,2\}$. We define

$$f_k(e) = \min_{n=1,2,\ldots,k} V_{\varphi_n}(e).$$

For every $k \in \mathbb{N}$ the function f_k is the Lipschitz function with the Lipschitz constant bounded by L. Hence

$$|\boldsymbol{B}(v_1) - \boldsymbol{B}(v_2)| = \lim_{k\to\infty} |f_k(v_1) - f_k(v_2)| \leqslant L\|v_1 - v_2\|.$$

4) Before we prove the generalized quasi convexity of this function, we need to introduce some notation. We know that all $\alpha \in A$ have common pattern of homogeneity Λ, thus we can find a vector $\gamma \in \mathbb{N}^d$ and a number $k \in \mathbb{N}$ such that $\langle \alpha, \gamma\rangle = k$ for every $\alpha \in A$.

For every $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^d$ we denote

$$x_\lambda = (\lambda^{\gamma_1}x_1, \lambda^{\gamma_2}x_2, \ldots, \lambda^{\gamma_d}x_d).$$

For every $\lambda \in \mathbb{N}$ we define the partition of the unit cube $[0,1]^d$ into small parallelepipeds:

$$Q_y = y + \Pi_{j=1}^d[0, \lambda^{-\gamma_j}] \quad \text{for every } y \in Y, \quad \text{where}$$

$$Y = \left\{ y \in [0,1]^d : y = \left(\frac{\kappa_1}{\lambda^{\gamma_1}}, \frac{\kappa_2}{\lambda^{\gamma_2}}, \ldots, \frac{\kappa_d}{\lambda^{\gamma_d}}\right) \quad \text{for } \kappa_j \in \mathbb{N}\cup\{0\} \text{ and } \kappa_j < \lambda^{\gamma_j} \right\}.$$

Here Y is the set of "leftmost lowest" vertices of the parallelepipeds Q_y. The parallelepipeds Q_y are disjoint up to sets of measure zero and $\bigcup_{y\in Y} Q_y = [0,1]^d$. Let us fix $\varphi \in C_0^\infty([0,1]^d)$.

Since $\nabla[\varphi]$ is a uniformly continuous function on $[0,1]^d$ and the diameter of the parallelepipeds Q_y tends to zero uniformly with the growth of λ, we can choose λ sufficiently large to obtain

$$\forall\, y \in Y \;\; \forall z, v \in Q_y \qquad |\nabla[\varphi](z) - \nabla[\varphi](v)| \leqslant \frac{\varepsilon}{L}, \tag{2.7}$$

where L is the Lipschitz constant of the function V. Let $\{\psi_y\}_{y\in Y}$ be a family of functions in $C_0^\infty([0,1]^d)$. For these functions, we use the following rescaling:

$$\psi_{y,\lambda}(x) = \lambda^{-k}\psi_y((x-y)_\lambda).$$

Let us observe that the rescaling $(x-y)_\lambda$ transforms the cube $[0,1]^d$ into Q_y, thus $\operatorname{supp}\psi_{y,\lambda} \subset Q_y$. Moreover, we know that

$$\partial^\alpha[\psi_{y,\lambda}](x) = \lambda^{-k}\lambda^{(\sum_{j=1}^d \alpha_j\gamma_j)}\partial^\alpha[\psi_y]\left((x-y)_\lambda\right) = \partial^\alpha[\psi_y]\left((x-y)_\lambda\right)$$

for every $\alpha \in A$. By (2.5), we have

$$\begin{aligned}\boldsymbol{B}(e) &\leqslant \int\limits_{[0,1]^d} V\Big(e + \sum_{y\in Y}\nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](x)\Big)dx \\ &= \sum_{y\in Y}\int\limits_{Q_y} V\Big(e + \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](x)\Big)dx.\end{aligned}$$

We assumed that (2.7) holds, therefore, for arbitrary $v_y \in Q_y$ we have the following estimate:

$$\begin{aligned}&\int\limits_{Q_y} V\big(e + \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](x)\big)\,dx \\ &\qquad\qquad \leqslant \int\limits_{Q_y} V\left(e + \nabla[\psi_{y,\lambda}](x) + \nabla[\varphi](v_y)\right)dx + \varepsilon|Q_y| \\ &\qquad\qquad = \int\limits_{Q_y} V\left(e + \nabla[\psi_y]((x-y)_\lambda) + \nabla[\varphi](v_y)\right)dx + \varepsilon|Q_y|.\end{aligned}$$

Since $\lambda^{-\left(\sum_{j=1}^d \gamma_j\right)} = |Q_y|$, we have

$$\begin{aligned}&\int\limits_{Q_y} V\big(e + \nabla[\psi_y]((x-y)_\lambda) + \nabla[\varphi](v_y)\big)\,dx \\ &\qquad\qquad = |Q_y|\int\limits_{[0,1]^d} V\left(e + \nabla[\psi_y](z) + \nabla[\varphi](v_y)\right)dz\end{aligned}$$

for $z = (x-y)_\lambda$. Now for every $y \in Y, v_y \in Q_y$ we can choose ψ_y such that

$$\int\limits_{[0,1]^d} V\big(e + \nabla[\psi_y](z) + \nabla[\varphi](v_y)\big)\,dz \leqslant \boldsymbol{B}(e + \nabla[\varphi](v_y)) + \varepsilon$$

(this choice depends on v_y, however, we treat v_y as of a fixed parameter). We obtain

$$\boldsymbol{B}(e) \leqslant \sum_{y\in Y} |Q_y| \boldsymbol{B}(e + \nabla[\varphi](v_y)) + 2\varepsilon$$

from the above inequalities. We take mean integrals of this inequality over each cube Q_y with respect to v_y, which gives us

$$\boldsymbol{B}(e) \leqslant \sum_{y\in Y} \int_{Q_y} \boldsymbol{B}(e + \nabla[\varphi](v_y)) dv_y + 2\varepsilon = \int_{[0,1]^d} \boldsymbol{B}(e + \nabla[\varphi](x)) dx + 2\varepsilon.$$

Since ε was an arbitrary positive number, we have proved the generalized quasi convexity of $\boldsymbol{B}$. $\square$

The proof of the fourth point seems very similar to the standard *Bellman induction step* (see [34, 37, 48, 51] or any other paper on Bellman function method in probability or harmonic analysis); moreover, the function $\boldsymbol{B}$ itself is, in a sense, a Bellman function and inequality (2.6) is a Bellman inequality. We suspect that this "similarity" should be more well studied.

2.2 Rank one convexity

Inequality (2.6) looks like a convexity inequality. Sometimes it is really the case.

Definition 2.7. *We call a vector $e_x \in E$ a generalized rank one vector if it is of the form*

$$\sum_{\alpha\in A} i^{|\alpha|+|\alpha_0|} x^\alpha e_\alpha, \quad x \in \mathbb{R}^d,\ \alpha_0 \in A.$$

Remark 2.8. *In Theorem 2.2, we only consider the case where every $\alpha \in A$ has the same parity as the other elements of A. Therefore, $i^{|\alpha|+|\alpha_0|} \in \mathbb{R}$ for every $\alpha_0, \alpha \in A$. Hence the coefficients of the generalized rank one vector are real.*

Theorem 2.9. *The function $\boldsymbol{B}$ is a generalized rank one convex function, i.e. it is convex in the directions of generalized rank one vectors.*

To prove the theorem, we need two auxiliary lemmas.

Lemma 2.10. *For every $x \in \mathbb{R}^d$ and every $\varepsilon, \delta > 0$, there exists a function $l_{x,\varepsilon,\delta} \in C_0^\infty([0,1]^d)$ and a set $B \subset [0,1]^d$ such that the following holds.*

1. $\|\nabla[l_{x,\varepsilon,\delta}]\| \leqslant \|e_x\| + \varepsilon.$

2. $|B| \geqslant 1-\delta$.

3. *The function* $\nabla[l_{x,\varepsilon,\delta}]\big|_B$ *with respect to the measure* $\mu = |B|^{-1}dx\big|_B$ *is equimeasurable with the function* $\cos(2\pi t)e_x$, $t \in [0,1]$, *i.e.*

$$\mu\left(\{\nabla[l_{x,\varepsilon,\delta}] \in W\}\right) = \left|\left\{t \in [0,1] : \cos(2\pi t)e_x \in W\right\}\right|$$

for every Borel set W *in* E.

Proof. For a given $x \in \mathbb{R}^d$ we take the same γ and k as in the proof of the fourth point of Theorem 2.6. We consider the function

$$l_{x,\varepsilon,\delta}(\xi) = t^{-k}\cos(\sum_{j=1}^{d} t^{\gamma_j}x_j\xi_j)\Phi(\xi),$$

where Φ is the smooth hat function:

$$\Phi(\xi) = \begin{cases} 1 & \xi \in [2\delta', 1-2\delta']^d, \\ 0 & \xi \in [0,1]^d \backslash [\delta', 1-\delta']^d, \\ \Theta(\xi) \in [0,1] & \text{otherwise.} \end{cases}$$

for δ' sufficiently small (in particular, we need $2(2\delta')^d < \delta$). Similarly to the fourth point of Theorem 2.6, we define the set of proper parallelepipeds

$$Y_t = \left\{Q\colon Q = (k_jv_j)_{j=1,\ldots,d} + \Pi_{j=1}^d[0,w_j];\ k_j \in \{1\} \cup \left\{k_j \in \mathbb{N}\colon\ k_j < \frac{t^{\gamma_j}x_j}{2\pi} - 1\right\}\right\}.$$

where $v_j = w_j = 2\pi t^{-\gamma_j}x_j^{-1}$ if $x_j \neq 0$ and $v_j = \delta'$, $w_j = (1-2\delta')$ otherwise. For any δ', we can choose t to be so large that

$$\Big|\bigcup_{\substack{Q \in Y_t \\ Q \subset [2\delta', 1-2\delta']^d}} Q\Big| \geqslant 1-\delta.$$

We put B to be this union, i.e. the union of the parallelepipeds Q from the family Y_t that belong to $[2\delta', 1-2\delta']^d$ entirely.

If t is sufficiently large, then for every $\beta \in \mathbb{N}^d$ satisfying $0 \leqslant \langle\beta,\gamma\rangle < k$, we have

$$\sup_{\xi \in [0,1]^d} |t^{-1}\partial^\beta[\Phi](\xi)| \leqslant \varepsilon'. \tag{2.8}$$

For any $\beta \in \mathbb{N}^d$, the following holds:

$$\partial^\beta\left[\cos\left(\sum_{j=1}^{d} t^{\gamma_j}x_j\xi_j\right)\right] = t^{\langle\beta,\gamma\rangle}x^\beta\partial^\beta[\cos]\left(\sum_{j=1}^{d} t^{\gamma_j}x_j\xi_j\right).$$

Since all $\alpha \in A$ have the same parity, we either have $\partial^\alpha[\cos](\xi) = (-1)^{\frac{|\alpha|}{2}}\cos(\xi)$ for every $\alpha \in A$ or $\partial^\alpha[\cos](x) = (-1)^{\frac{|\alpha|+1}{2}}\sin(\xi)$ for every $\alpha \in A$. Without lost of generality we may assume $2\big||\alpha|$, because the functions sine and cosine are equimeasurable on their periodic domains. Therefore, for every $\xi \in [0,1]^d$ and $\alpha \in A$ we have

$$
\begin{aligned}
\partial^\alpha[l_{\xi,\varepsilon,\delta}](\xi) &= \Phi(\xi)\partial^\alpha[t^{-k}\cos\big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\big)] \\
&\quad + \sum_{\substack{\alpha'+\beta=\alpha \\ \beta\neq(0,0,\dots,0)}} c_{\alpha',\beta}\, t^{-k}\partial^{\alpha'}\left[\cos\big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j(x)\big)\right]\partial^\beta[\Phi] \\
&= \Phi(\xi)x^\alpha\partial^\alpha[\cos]\big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\big) \\
&\qquad + \sum_{\substack{\alpha'+\beta=\alpha \\ \beta\neq(0,0,\dots,0)}} c_{\alpha',\beta}\, t^{\langle\alpha',\gamma\rangle-k}\partial^{\alpha'}[\cos]\big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j(x)\big)\partial^\beta[\Phi] \\
&= (-1)^{\frac{|\alpha|}{2}}x^\alpha\cos\big(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j\big) + \text{error},
\end{aligned}
\tag{2.9}
$$

where the coefficients $c_{\alpha',\beta}$ come from the Leibniz formula. The error is $O(\varepsilon')$ in absolute value by (2.8) and equals to zero on the set $[2\delta', 1-2\delta']^d$ (because the function Φ is constant there). For every $\xi \in [0,1]^d$ we have

$$
\begin{aligned}
\nabla[l_{\xi,\varepsilon,\delta}](\xi) = \sum_{\alpha\in A}\partial^\alpha[l_{\xi,\varepsilon,\delta}](\xi)e_\alpha &= \sum_{\alpha\in A}\left((-1)^{\frac{|\alpha|}{2}}x^\alpha\cos(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j) + \text{error}\right)e_\alpha \\
&= e_x\cos(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j) + \text{error}.
\end{aligned}
$$

Thus, for every $\xi \in [0,1]^d$ and ε' sufficiently small, we obtain

$$\|\nabla[l_{\xi,\varepsilon,\delta}](\xi)\| \leqslant \|e_x\| + \|\text{error}\| \leqslant \|e_x\| + \varepsilon.$$

Since the error equals to zero on the set $[2\delta', 1-2\delta']^d$, it follows from (2.9) that for every $\xi \in B$ we have

$$\nabla[l_{\xi,\varepsilon,\delta}](\xi) = \cos(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j)e_x.$$

We note that the function $\cos(\sum_{j=1}^d t^{\gamma_j}x_j\xi_j)e_x$ restricted to any $Q \in Y_t$ is equimeasurable (with respect to the measure $\frac{dx}{|Q|}$ on Q) with the function $\cos(2\pi t)e_x$, $t \in [0,1]$, (one can verify this fact using an appropriate dilation). Since B is a union of several parallelepipeds Q, the same holds with Q replaced by B. $\square$

Lemma 2.11. *Suppose that $v : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that*

$$v(x) \leqslant \int_0^1 v(x + \lambda\cos(2\pi t))dt \tag{2.10}$$

for any $x, \lambda \in \mathbb{R}$. Then, v is convex.

Proof. We are going to verify that v is convex as a distribution, or what is the same, that the distribution v'' is non-negative. For that, we multiply inequality (2.10) by a positive function $\varphi \in C_0^\infty(\mathbb{R})$. Since v is a Lipschitz function, we can integrate it over $\mathbb{R}$:

$$\begin{aligned}
\int_{\mathbb{R}} v(x)\phi(x)dx &\leqslant \int_{\mathbb{R}} \int_0^1 v(x + \lambda\cos(2\pi t))\varphi(x)dtdx \\
&= \int_{\mathbb{R}} \int_0^1 v(x)\varphi(x - \lambda\cos(2\pi t))dtdx \\
&= \int_{\mathbb{R}} v(x) \int_0^1 \left(\varphi(x) - \lambda\cos(2\pi t)\varphi^{'}(x) + \frac{\lambda^2}{2}\cos^2(2\pi t)\varphi^{''}(x) + o(\lambda^2)\right) dtdx \\
&= \int_{\mathbb{R}} \left(v(x)\varphi(x) + v(x)\varphi^{''}(x)\frac{\lambda^2}{2}\left(\int_0^1 \cos^2(2\pi t)\right) + o(\lambda^2)\right) dx.
\end{aligned}$$

Therefore,

$$0 \leqslant \frac{1}{2}\left(\int_0^1 \cos^2(2\pi t)dt\right)\int_{\mathbb{R}} v(x)\phi^{''}(x)dx + \frac{o(\lambda^2)}{\lambda^2}.$$

Letting $\lambda \to 0$, we show that $v^{''}$ as a distribution satisfies $v^{''}(\phi) \geqslant 0$ for all $\phi \in C_0^\infty(\mathbb{R})$ and $\phi \geqslant 0$. From the Schwartz theorem it follows that $v^{''}$ is a non negative measure of locally finite variation. Thus v' is an increasing function and therefore v is convex. $\square$

Proof of Theorem 2.9. The function $\boldsymbol{B}$ is a generalized quasi convex function, hence it satisfies (2.6) for every $\varphi \in C_0^\infty([0,1]^d)$. Let us fix $x \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$. We plug $\lambda l_{x,\varepsilon,\delta}$ into (2.6). We get (for every $e \in E$)

$$\begin{aligned}
\boldsymbol{B}(e) \leqslant \int_{[0,1]^d} \boldsymbol{B}(e + \nabla[\lambda l_{x,\varepsilon,\delta}]) &= \int_B \boldsymbol{B}(e + \nabla[\lambda l_{x,\varepsilon,\delta}]) + \int_{[0,1]^d\setminus B} \boldsymbol{B}(e + \nabla[\lambda l_{x,\varepsilon,\delta}]) \\
&\leqslant \int_B \boldsymbol{B}(e + \nabla[\lambda l_{x,\varepsilon,\delta}]) + O\big(\lambda(\|e\| + \|e_x\| + \varepsilon)\delta\big)
\end{aligned}$$

from Lemma 2.10. Since $\nabla[l_{x,\varepsilon,\delta}]\big|_B$ is equimeasurable (B equipped with the measure $\frac{dx}{|B|}$) with $\cos(2\pi t)e_x$,

$$\int\limits_B \boldsymbol{B}(e + \nabla[\lambda l_{x,\varepsilon,\delta}])\frac{dx}{|B|} = \int\limits_{[0,1]} \boldsymbol{B}(e + \lambda\cos(2\pi t)e_x)dt.$$

Therefore,

$$\boldsymbol{B}(e) \leqslant |B| \int\limits_{[0,1]} \boldsymbol{B}(e + \lambda\cos(2\pi t)e_x)dt + O\big(\lambda(\|e\| + \|e_x\| + \varepsilon)\delta\big).$$

Since for $\delta \to 0$, we have $|B| \to 1$, and then

$$\boldsymbol{B}(e) \leqslant \int\limits_{[0,1]} \boldsymbol{B}(e + \lambda\cos(2\pi t)e_x)dt. \tag{2.11}$$

For a fixed $e \in E$, consider the function $\mathbb{R} \ni s \mapsto \boldsymbol{B}(e + se_x)$. By (2.11),

$$\boldsymbol{B}(e + se_x) \leqslant \int\limits_{[0,1]} \boldsymbol{B}(e + se_x + \lambda\cos(2\pi t)e_x)dt.$$

Thus, by Lemma 2.11, the function $\mathbb{R} \ni s \mapsto \boldsymbol{B}(e + se_x)$ is convex (one simply applies lemma to this function). Since $e \in E$ and $x \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$ were arbitrary, it proves the generalized rank one convexity of the function $\boldsymbol{B}$. $\square$

2.3 Separately convex homogeneous functions and proof of Theorem 2.2

Lemma 2.12. *Generalized rank one vectors span E.*

Proof. Since E is a finite dimensional Hilbert space, every functional on E is of the form $\phi^*(\cdot) = \langle \sum_{\alpha\in A} a_\alpha e_\alpha,\ \cdot\ \rangle$. We get

$$\phi^*(e_x) = \sum_{\alpha\in A} a_\alpha x^\alpha i^{|\alpha|+|\alpha_0|}$$

for every $x \in \mathbb{R}^d$. If E is not a span of generalized rank one vectors, then there exists a non trivial ϕ^* such that

$$0 = \phi^*(e_x) = \sum_{\alpha\in A} a_\alpha x^\alpha i^{|\alpha|+|\alpha_0|}$$

for every $x \in \mathbb{R}^d$. However, x^α are linearly independent monomials. Therefore, $a_\alpha = 0$ for every $\alpha \in A$. Hence $\phi^* \equiv 0$ and the generalized rank one vectors span E. $\square$

We recall that our aim was to show that T_1 is a linear combination of the other T_j. By comparing the kernels of the $\tilde{T}_j$, its is equivalent to the fact that $V \geqslant 0$ everywhere. By the evident inequality $B \leqslant V$, it suffices to prove that B is non-negative. By Lemma 2.12 and Theorem 2.9, this will follow from the theorem below. Hence it suffices to prove Theorem 2.14 to get Theorem 2.2.

Definition 2.13. *A function $F : \mathbb{R}^d \to \mathbb{R}$ is separately convex if it is convex with respect to each variable.*

Theorem 2.14. *A function $F : \mathbb{R}^d \to \mathbb{R}$ that is separately convex and homogeneous of order one is non-negative*

Before passing to the proof, we cite Theorem 2.31 of the book [9], which says that a separately convex function is continuous. This fact will be implicitly used several times in the reasoning below.

Proof. We proceed by induction. Suppose that the statement of the theorem holds true for the dimension $d-1$, we then prove it for the dimension d. Construct the function $G : \mathbb{R}^{d-1} \to \mathbb{R}$ by the formula

$$G(x) = F(x, 1), \quad x \in \mathbb{R}^{d-1}.$$

This function is separately convex and convex with respect to radius, i.e. for every $x \in \mathbb{R}^{d-1}$ the function $\mathbb{R}_+ \ni t \mapsto G(tx)$ is a convex function. Indeed, the function F is one homogeneous and separately convex, thus for $t, r > 0$ and $\tau \in (0,1)$ we have:

$$\begin{aligned}
\tau G(tx) + (1-\tau)G(rx) &= \tau F(tx, 1) + (1-\tau)F(rx, 1) \\
&= (\tau t + (1-\tau)r)\left(\frac{\tau t F(x, \frac{1}{t}) + (1-\tau) r F(x, \frac{1}{r})}{\tau t + (1-\tau)r}\right) \\
&\geqslant (\tau t + (1-\tau)r) F\left(x, \frac{1}{\tau t + (1-\tau)r}\right) \\
&= F\left((\tau t + (1-\tau)r)x, 1\right) = G\left((\tau t + (1-\tau)r)x\right).
\end{aligned}$$

We claim that for each $x \in \mathbb{R}^{d-1}$ the function $\mathbb{R} \ni t \mapsto G(tx)$ is convex. Since the function G is continuous, it suffices to prove that $G(tx) + G(-tx) \geqslant G(0)$ for all $t \in \mathbb{R}$. Consider another function V:

$$V(x) = \lim_{t \to 0+} \frac{G(tx) + G(-tx) - 2G(0)}{t}, \quad x \in \mathbb{R}^{d-1}.$$

The limit exists due to the convexity with respect to radius. This function V is one homogeneous and separately convex. However, it may have attained the value $-\infty$. Fortunately, this is not the case. If there exists $x \in \mathbb{R}^d$ such that $V(x) = -\infty$ then the following holds:

$$2V(0, x_2, \ldots, x_d) \leqslant V(x_1, \ldots, x_d) + V(-x_1, \ldots, x_d) = -\infty.$$

Therefore $V(0, x_2, \dots, x_d) = -\infty$. We repeat the above reasoning with $x_2, \dots, x_d$ instead of x_1 and we get that $V(0) = -\infty$, but from the definition of V we know that

$$V(0) = \lim_{t \to 0+} \frac{G(0) + G(0) - 2G(0)}{t} = 0.$$

Hence $V(x)$ is finite for every $x \in \mathbb{R}^{d-1}$. Thus, by the induction hypothesis, V is non-negative. So, $\mathbb{R} \ni t \mapsto G(tx)$ is a convex function.

By symmetry, $G(x) + G(-x) \geqslant 2F(x, 0)$. On the other hand, $\lim_{t\to\pm\infty} \frac{G(tx)}{t} = F(x, 0)$. So, the convexity of $t \mapsto G(tx)$ gives the inequality $|G(x) - G(-x)| \leqslant 2F(x, 0)$. Adding these two inequalities, we get that $F(x, 1) \geqslant 0$. $\square$

Proof of Theorem 2.2. Assume that inequality (2.1) holds. Then, by Theorem 2.6, the function $\boldsymbol{B}$ given by (2.5) is Lipschitz, one homogeneous, generalized quasi convex, and satisfies the inequality $B \leqslant V$, where the function V is given by formula (2.4). Then, by Theorem 2.9, $\boldsymbol{B}$ is a generalized rank one convex function.

Let $e \in E$ be an arbitrary point. By Lemma 2.12, e is a linear combination of generalized rank one vectors $e_{x_1}, e_{x_2}, \dots, e_{x_k}$. We may assume that they are linearly independent. Consider the function $F : \mathbb{R}^k \to \mathbb{R}$ given by the rule

$$F(z_1, z_2, \dots, z_k) = \boldsymbol{B}(z_1 e_{x_1} + z_2 e_{x_2} + \ldots + z_k e_{x_k}).$$

By the generalized rank one convexity of $\boldsymbol{B}$, F is separately convex. It is also one homogeneous, thus $F \geqslant 0$ by Theorem 2.14. Therefore, $\boldsymbol{B}(e)$ is also non-negative for arbitrary $e \in E$.

Since $\boldsymbol{B} \geqslant 0$, we have $V \geqslant 0$. In such a case, it follows from formula (2.4) that $\operatorname{Ker} \tilde{T}_1 \supset \cap_{j=2}^{\ell} \operatorname{Ker} \tilde{T}_j$. Therefore, T_1 is a linear combination of the other T_j. $\square$

2.4 Related questions

Towards Conjecture 2.1. The following statement plays the same role in view of Conjecture 2.1, as Theorem 2.14 plays in the proof of Theorem 2.2.

Conjecture 2.15. *Let $F : \mathbb{R}^{2d} \to \mathbb{R}$ be a Lipschitz homogeneous function of order one. Suppose that for any $j = 1, 2, \dots, d$ the function F is subharmonic with respect to the variables (x_j, x_{j+d}). Then, F is non-negative.*

Indeed, plugging the cosine function into (2.6) as we did in the proof of Theorem 2.9 leads to "subharmonicity"[1] of the function $\boldsymbol{B}$ in the directions of projections of a generalized rank one vector onto subspaces generated by odd and even monomials in A correspondingly. Therefore, Conjecture 2.1 follows from Conjecture 2.15.

We are not able to prove Conjecture 2.15. However, we know the following: in the case $d = 1$, the function F is not only non-negative, but, in fact, convex (i.e. a one homogeneous subharmonic function is convex). On the other hand, there is no much hope for simplifications: a subharmonic one homogeneous function in $\mathbb{R}^3$ (and thus in $\mathbb{R}^d$, $d \geqslant 3$) can attain negative values, e.g. in $\mathbb{R}^4$ one may take the function $\frac{x_1^2+x_2^2+x_3^2-x_4^2}{\sqrt{x_1^2+x_2^2+x_3^2}}$.

There are also reasons that differ from the ones discussed in the present chapter that may "break" inequality (2.1). One of them is a certain geometric property of the spaces generated by the operators T_j. Not stating any general theorem or conjecture, we treat an instructive example. Consider the non-inequality

$$\|\partial_1^2\partial_2 f\|_{L_1} \lesssim \|\partial_1^4 f\|_{L_1} + \|\partial_2^2 f\|_{L_1}. \tag{2.12}$$

Conjecture 2.1 hints us that it cannot be true. We will disprove it on the torus $\mathbb{T}^2$ and leave to the reader the rigorous formulation and proof of the corresponding transference principle, whose heuristic form is "inequalities of the sort (2.1) are true or untrue simultaneously on the torus and the Euclidean space". Consider two anisotropic homogeneous Sobolev spaces W_1 and W_2, which are obtained from the set of trigonometric polynomials by completion and factorization over the null-space with respect to the seminorms

$$\|f\|_{W_1} = \|\partial_1^4 f\|_{L_1} + \|\partial_2^2 f\|_{L_1}, \quad \|f\|_{W_2} = \|\partial_1^2\partial_2 f\|_{L_1} + \|\partial_1^4 f\|_{L_1} + \|\partial_2^2 f\|_{L_1}.$$

If inequality (2.12) holds true, then these two spaces are, in fact, equal (the identity operator is a Banach space isomorphism between these spaces). However, it follows from the result of [40] (see [53, 54] as well) that W_2 has a complemented translation-invariant Hilbert subspace[2], whereas W_1 does not, a contradiction.

Martingale transforms. Let $S = \{S_n\}_n$, $n \in \{0\} \cup \mathbb{N}$, be an increasing filtration of finite algebras on the standard probability space. We suppose that it differentiates L_1 (i.e. for any $f \in L_1(\Omega)$ the sequence $\mathbb{E}(f \mid S_n)$ tends to f almost surely). We will be working with martingales adapted to this filtration.

Definition 2.16. *Let $\alpha = \{\alpha_n\}_n$ be a bounded sequence. The linear operator*

$$T_\alpha[f] = \sum_{j=1}^{\infty} \alpha_{j-1}(f_j - f_{j-1}), \quad f = \{f_n\}_n \text{ is an } L_1 \text{ martingale},$$

[1] The "subharmonicity" means that $D\boldsymbol{B} \geqslant 0$ as a distribution, where D is an elliptic symmetric differential operator of second order (with constant real coefficients); one can then pass to usual subharmonicity by an appropriate change of variable.

[2] That means that there exists a subspace $X \subset W_2$ such that $g \in X$ whenever $g(\cdot + t) \in X$, $t \in \mathbb{T}^2$, X is isomorphic to an infinite dimensional Hilbert space, and there exists a continuous projector $P : W_2 \to X$.

is called a martingale transform.

Our definition is not as general as the usual one, and we refer the reader to the book [37] for the information about such type operators. We only mention that martingale transforms serve as a probabilistic analogue for the Calderón–Zygmund operators. The probabilistic version of Conjecture 2.1 looks like this.

Conjecture 2.17. *Suppose $\alpha^1, \alpha^2, \dots, \alpha^\ell$ are bounded sequences. Suppose that the algebras S_n uniformly grow, i.e. there exists $\gamma < 1$ such that each atom a of S_n is split in S_{n+1} into atoms of probability not greater than $\gamma|a|$ each. The inequality*

$$\|T_{\alpha^1} f\|_{L_1} \lesssim \sum_{j=2}^{\ell} \|T_{\alpha^j} f\|_{L_1} \tag{2.13}$$

holds for any martingale f adapted to $\{S_n\}_n$ if and only if α^1 is a sum of a linear combination of the α^j and an ℓ_1 sequence.

We do not know whether the condition of uniform growth fits this conjecture. Anyway, it is clear that one should require some condition of this sort (otherwise one may take $S_n = S_{n+1} = \ldots = S_{n+k}$ very often and loose all the control of the sequences α^j on this time intervals). Again, we are not able to prove the conjecture in the full generality, but will deal with an important particular case.

Theorem 2.18. *Suppose $\alpha^1, \alpha^2, \dots, \alpha^\ell$ to be bounded periodic sequences. The inequality*

$$\|T_{\alpha^1} f\|_{L_1} \lesssim \sum_{j=2}^{\ell} \|T_{\alpha^j} f\|_{L_1}$$

holds if and only if α^1 is a linear combination of the other α^j.

Proof. To avoid technicalities, we will be working with finite martingales (denote the class of such martingales by $\mathcal{M}$). The general case can be derived by stopping time. Assume that inequality (2.13) holds true. Consider the Bellman function $\boldsymbol{B} : \mathbb{R}^\ell \to \mathbb{R}$ given by the formula

$$\boldsymbol{B}(x) = \inf_{f \in \mathcal{M}} \Big(\sum_{j=2}^{\ell} \big\| x_j + T_{\alpha^j}[f] \big\|_{L_1} - c \big\| x_1 + T_{\alpha^1}[f] \big\|_{L_1} \Big).$$

It is easy to verify that this function is one homogeneous and Lipschitz. Moreover, $\boldsymbol{B}$ is convex in the direction of $(\alpha_n^1, \alpha_n^2, \dots, \alpha_n^\ell)$ for each n (by the assumption of periodicity, there is only a finite number of these vectors); the proof of this assertion is a simplification of Theorem 2.9 (here we do not have to make additional approximations; however, see [48], Lemma 2.17 for a very similar reasoning). Thus, by Theorem 2.14, $\boldsymbol{B}$ is non-negative on the span of $\{(\alpha_n^1, \alpha_n^2, \dots, \alpha_n^\ell)\}_n$. Since $\boldsymbol{B}(x) \leqslant \sum_{j \geqslant 2} |x_j| - c|x_1|$, the aforementioned span does not contain the x_1-axis. Therefore, α^1 is a linear combination of the other α^j. $\square$

Case $p > 1$. Inequality (2.1) may become valid provided one replaces the L_1-norm with the L_p one, $1 < p < \infty$. Let c_p be the best possible constant in the inequality

$$\|T_1 f\|_{L_p(\mathbb{R}^d)}^p \leqslant c_p \sum_{j=2}^{\ell} \|T_j f\|_{L_p(\mathbb{R}^d)}^p. \tag{2.14}$$

It is interesting to compute the asymptotics of c_p as $p \to 1$. Some particular cases have been considered in [2], we also refer the reader there for a discussion of similar questions.

Conjecture 2.19. *Let Λ be a pattern of homogeneity in $\mathbb{R}^d$, let $\{T_j\}_{j=1}^{\ell}$ be a collection of Λ-homogeneous differential operators. If T_1 cannot be expressed as a linear combination of the other T_j, then $c_p \gtrsim \frac{1}{p-1}$.*

The conjecture claims that if there is no continuity at the endpoint, then the inequality behaves at least as if it had a weak type $(1,1)$ there (it is also interesting to study when there is a weak type $(1,1)$ indeed). First, we note that this question is interesting even when there are only two polynomials. Second, this is only a bound from below for c_p. Even in the case of two polynomials, c_p can be as big as $(p-1)^{1-d}$ (and thus the endpoint inequality may not be of weak type $(1,1)$, at least when $d \geqslant 3$), see [2] for the example.

As in the previous point, Conjecture 2.19 will follow from the corresponding geometric statement in the spirit of Theorem 2.14.

Conjecture 2.20. *Let $F : \mathbb{R}^d \to \mathbb{R}$ be separately convex p-homogeneous function (i.e., $F(\lambda x) = |\lambda|^p F(x)$). Suppose that $F(x) \leqslant |x|^p$. Then, $F(x) \gtrsim (1-p)|x|^p$.*

Conjecture 2.19 is derived from Conjecture 2.20 in the same way as Theorem 2.2 derived from Theorem 2.14: one considers the Bellman function (2.5) with the function V given by the formula

$$V(e) = \Big(c_p \sum_{j=2}^{\ell} |\tilde{T}_j e|^p - |\tilde{T}_1 e|^p\Big),$$

proves its generalized quasi convexity, which leads to the generalized rank one convexity, and then uses Conjecture 2.20 to estimate c_p from below.

It is not difficult to verify the case $d = 2$ of Conjecture 2.20. Therefore, there exists a C_0^∞-function f_p such that

$$(p-1)\|\partial_1\partial_2 f_p\|_{L_p(\mathbb{R}^2)} \gtrsim \Big(\|\partial_1^2 f_p\|_{L_p(\mathbb{R}^2)} + \|\partial_2^2 f_p\|_{L_p(\mathbb{R}^2)}\Big).$$

Chapter 3

Continuouity of Fourier multipliers on homogenous Sobolev Spaces

We consider the invariant operators on the homogeneous Sobolev spaces on $\mathbb{R}^d$ given by Fourier multipliers. The homogeneous Sobolev space $\dot{W}_1^1(\mathbb{R}^d)$ consists of functions on $\mathbb{R}^d$ whose distributional gradient is integrable. A measurable function $m : \mathbb{R}^d \to \mathbb{R}$ is called a (Fourier) multiplier if the operator given by the formula $T_m f = \mathscr{F}^{-1}(m \cdot \mathscr{F}(f))$ is bounded. Fourier transforms of bounded measures are examples of multipliers. Indeed, the convolution with a bounded measure is a bounded operator on every translation invariant space with continuous shifts operators, in particular on the homogeneous Sobolev space. However, the class of Fourier multipliers on $\dot{W}_1^1(\mathbb{R}^d)$ is wider than the class of Fourier transforms of measures (Proposition 2.2 in [42]). One of the important questions about the invariant subspaces of L^1 is a description of bounded singular operators acting on it e.g. the Calderon-Zygmund operators are given by multiplier with singularity at zero. Therefore, the question of the continuity of a multiplier arises quite naturally in the theory.

The simplest case of noncontinuous multipliers was settled by A. Bonami and S. Poornima who proved that the only homogeneous multipliers of degree zero are the constants. In their beautiful proof they use very delicate result by Ornstein (cf. [36]) on the non-majorization of a partial derivative by other derivatives of the same order. While the class of homogeneous multipliers, containing e.g. Riesz transforms, is the most important one, the question of the continuity of general multipliers remained open. The aim of this chapter is to fill the gap. We prove that any multiplier acting on the homogeneous Sobolev space with integral norm is a continuous function.

Our proof uses three main ingredients. The first one is the Bonami - Poornima result.

The second is the Riesz product technique which allows us to make the crucial estimates on the torus group. This would be sufficient for our purpose, provided we are able to transfer the problem from $\mathbb{R}^d$ to $\mathbb{T}^d$. Such transference in the case of multipliers on L^p spaces is the subject of the theorem of deLeeuw (cf. [11]). However, in the case of multipliers on the homogeneous Sobolev space no version of the deLeeuw transference theorem is known. We are able to overcome this difficulty due to the special form of functions on which the multiplier reaches its norm. The question of general deLeeuw type theorem for the homogeneous Sobolev spaces remains open.
One can ask whether a similar approach could be used to prove the Ornstein's non-inequality. Indeed in some special cases this technique works, for more details one can check [21].

For a formal statement of the main theorem we use standard definitions and notations for classical spaces in particular

- $\mathscr{D}(\mathbb{R}^d)$ - space of $C^\infty(\mathbb{R}^d)$ functions with compact support on $\mathbb{R}^d$.
- $\mathscr{D}'(\mathbb{R}^d)$ - space of distributions on $\mathbb{R}^d$.
- $\mathscr{S}(\mathbb{R}^d)$ - Schwartz function space on $\mathbb{R}^d$.
- $\mathscr{F}(\cdot)$ - Fourier transform on the space of tempered distributions.
- $\mathscr{F}^{-1}(\cdot)$ - inverse Fourier transform on the space of tempered distributions.

One can find more details on the function spaces mentioned above in [44]. For the definition of the Fourier transform we follow [47]. As usual, C will denote a generic constant, whose value can change from line to line.
We write $W^p_k(\mathbb{R}^d)$ for the Sobolev space, given by

$$W^p_k(\mathbb{R}^d) := \left\{ f \in L^p(\mathbb{R}^d) \,:\, D^{\boldsymbol{\alpha}} f \in L^p(\mathbb{R}^d) \text{ for } |\boldsymbol{\alpha}| \leqslant k \right\}$$

with the norm

$$\|f\|_{W^p_k(\mathbb{R}^d)} := \sum_{0 \leqslant |\boldsymbol{\alpha}| \leqslant k} \|D^{\boldsymbol{\alpha}} f\|_{L^p(\mathbb{R}^d)}$$

where $\boldsymbol{\alpha}$ is a multi-index and $D^{\boldsymbol{\alpha}}$ is the corresponding distributional derivative and $k \in \mathbb{N}^+$. Analogously we write $\dot{W}^k_p(\mathbb{R}^d)$ for the homogeneous Sobolev space, given by

$$\dot{W}^p_k(\mathbb{R}^d) := \left\{ f \in \mathscr{D}'(\mathbb{R}^d) \,:\, D^{\boldsymbol{\alpha}} f \in L^p(\mathbb{R}^d) \text{ for } |\boldsymbol{\alpha}| = k \right\}$$

with the seminorm

$$\|f\|_{\dot{W}^p_k(\mathbb{R}^d)} := \sum_{|\boldsymbol{\alpha}| = k} \|D^{\boldsymbol{\alpha}} f\|_{L^p(\mathbb{R}^d)}$$

where $\boldsymbol{\alpha}$, $D^{\boldsymbol{\alpha}}$ and k are the same as above. The homogeneous Sobolev spaces are special cases of Beppo-Levi spaces which are discussed in [13]. Later we will use the symbol $\dot{W}^p_k(\mathbb{R}^d)$ to denote the quotient space $\dot{W}^p_k(\mathbb{R}^d)/\mathscr{P}^k$, where $\mathscr{P}^k$ stands for the space of polynomials of degree strictly less than k. The space $\dot{W}^p_k(\mathbb{R}^d)/\mathscr{P}^k$ with the quotient norm is a Banach space.
We say that the function $m \in L^\infty(\mathbb{R}^d)$ is a Fourier multiplier on X, where X is either the Lebesgue space, the Sobolev space or the homogeneous Sobolev space $\dot{W}^1_1(\mathbb{R}^d)$, if there exists a bounded operator $T_m : X \to X$ such that

$$\mathscr{F}\left(T_m f\right) = m\mathscr{F}\left(f\right) \qquad \forall\, f \in \mathscr{S}(\mathbb{R}^d).$$

We use the symbol $\mathscr{M}(X, X)$ to denote the space of the Fourier multipliers on X with the norm

$$\|m\|_{\mathscr{M}(X,X)} := \|T_m\| \qquad \forall m \in \mathscr{M}(X, X).$$

Now we can state the main result of this chapter

Theorem 3.1. *If $d \geqslant 2$ and $m \in \mathscr{M}(\dot{W}^1_1(\mathbb{R}^d), \dot{W}^1_1(\mathbb{R}^d))$ then $m \in C_b(\mathbb{R}^d)$.*

In the proof we will use the theorem of A. Bonami and S. Poornima on the homogeneous Fourier multipliers on $\dot{W}^1_1(\mathbb{R}^d)$.

Theorem 3.2 (A. Bonami, S. Poornima)**.** *Let Ω be a continuous function on $\mathbb{R}^d\backslash\{0\}$, homogeneous of degree zero i.e.*

$$\Omega(\varepsilon \boldsymbol{x}) = \Omega(\boldsymbol{x}) \qquad \forall\, \boldsymbol{x} \in \mathbb{R}^d.$$

Then

$$\Omega \in \mathscr{M}(\dot{W}^1_1(\mathbb{R}^d), \dot{W}^1_1(\mathbb{R}^d)) \Leftrightarrow \Omega \equiv K \in \mathbb{C}.$$

For the proof see [3].

In the next section we prove Theorem 3.1. To focus the attention on the main line of the proof, some technical lemmas are formulated there without proofs. For the reader's convenience proofs of the technical lemmas are given in the last section.

Proof of the Theorem 3.1

Let the function $m \in \mathscr{M}(\dot{W}^1_1(\mathbb{R}^d), \dot{W}^1_1(\mathbb{R}^d))$. Hence $\xi_i m(\boldsymbol{\xi})\mathscr{F}\left(f\right)(\boldsymbol{\xi})$ is a Fourier transform of an integrable function for every $f \in \mathscr{S}(\mathbb{R}^d)$. Therefore m is a continuous function on $\mathbb{R}^d\backslash\{0\}$. Thus it is enough to show the existence of the limit $\lim_{\boldsymbol{x}\to 0} m(\boldsymbol{x})$.
Prior to the proof we need one more definition. Let $f : \mathbb{R}^d \to \mathbb{R}$.

(*) We say that the function f has *almost radial limits* at 0 iff for every vector $\boldsymbol{w} \in \mathbb{S}^{d-1}$ there exists a scalar $g(\boldsymbol{w}) \in \mathbb{R}$ such that for every sequences $t_k \to 0$ and $\boldsymbol{w}^k \to \boldsymbol{w}$ ($t_k \in \mathbb{R}$; $\boldsymbol{w}^k \in \mathbb{S}^{d-1}$) we have

$$\lim_{k\to\infty} f(t_k \boldsymbol{w}^k) = g(\boldsymbol{w}).$$

Proof of Theorem 3.1. Since m is bounded, there are three possibilities:

Case I The multiplier m has almost radial limits at 0 (*).

Case II The multiplier m does not satisfy condition (*). Then there exists a sequence $\{\boldsymbol{a}^n\}_{n\in\mathbb{N}} \subset \mathbb{R}^d$, $\boldsymbol{a}^n \to 0$, a vector $v \in \mathbb{S}^1$ and two different scalars a and b such that

$$\lim_{n\to\infty} \frac{\boldsymbol{a}^n}{|\boldsymbol{a}^n|} = \boldsymbol{v}$$

and one of the following is satisfied

(a) Symmetric case.

$$\begin{aligned} \lim_{n\to\infty} m(\boldsymbol{a}^{2n}) &= \lim_{n\to\infty} m(-\boldsymbol{a}^{2n}) &= a, \\ \lim_{n\to\infty} m(\boldsymbol{a}^{2n+1}) &= \lim_{n\to\infty} m(-\boldsymbol{a}^{2n+1}) &= b. \end{aligned} \tag{3.1}$$

(b) Asymmetric case.

$$\begin{aligned} \lim_{n\to\infty} m(\boldsymbol{a}^n) &= a, \\ \lim_{n\to\infty} m(-\boldsymbol{a}^n) &= b. \end{aligned}$$

Proof in the Case I

We will use the following lemma, stated in [3], on the pointwise convergence of multipliers.

Lemma 3.3. *Let $\{m_k\}$ be a sequence of Fourier multipliers on $\dot{W}_1^1(\mathbb{R}^d)$ and assume that the corresponding operators have commonly bounded norms. If m_k converge pointwise to a function $m(\cdot)$ then $m(\cdot)$ is a Fourier multiplier on $\dot{W}_1^1(\mathbb{R}^d)$.*

In the next lemma we use Theorem 3.2 to show that the multipliers satisfying condition (*) are continuous.

Lemma 3.4. *If $d \geqslant 2$ and $m \in \mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$ satisfies condition (*), then $\lim_{\boldsymbol{\xi}\to 0} m(\boldsymbol{\xi})$ exists and is finite.*

Proof. Note first that m has the radial limit at 0 (we apply (*) to fixed $\boldsymbol{v} = \boldsymbol{v}^k = \boldsymbol{w}^k$). Hence the formula

$$\Omega(\boldsymbol{\xi}) := \lim_{n\to\infty} m(\frac{1}{n}\boldsymbol{\xi}).$$

defines a homogeneous function on $\mathbb{R}^d\backslash\{0\}$. The condition (*) implies continuity of Ω on $\mathbb{R}^d\backslash\{0\}$. Indeed

$$\lim_{\boldsymbol{\xi}_k\to\boldsymbol{\xi}} \Omega(\boldsymbol{\xi}_k) = \lim_{\boldsymbol{\xi}_k\to\boldsymbol{\xi}} \lim_{n\to\infty} m(\frac{1}{n}\boldsymbol{\xi}_k) \overset{(*)}{=} \lim_{n\to\infty} m(\frac{1}{n}\boldsymbol{\xi}) = \Omega(\boldsymbol{\xi}) \tag{3.2}$$

Since the norm of multipliers from $\mathscr{M}(\dot{W}_1^1(\mathbb{R}^d), \dot{W}_1^1(\mathbb{R}^d))$ is invariant under rescaling, the functions $m(\frac{1}{n}\cdot)$ are Fourier multipliers with equal norms. By Lemma 3.3 their pointwise limit, being bounded and continuous on $\mathbb{R}^d\backslash\{0\}$, is a Fourier multiplier on $\dot{W}_1^1(\mathbb{R}^d)$. Then Theorem 3.2 implies that Ω is a constant function which in turn means that all radial limits of m are equal. In similar way as in (3.2) we check that a function which has all radial limits equal and satisfies condition (*) is continuous at zero. Hence multiplier m is a continuous function. $\square$

Proof in the Case IIa

From now on we assume that $d = 2$. This allows us to simplify the notation yet not loosing the generality. We can also assume, transforming linearly if necessary, that $a = 1$, $b = -1$ and $\boldsymbol{v} = (1, 0)$. We will estimate the norm of the multiplier m from the following lemma:

Lemma 3.5. *(cf. [55]) There exists constant $C > 0$ such that for every $s \in \mathbb{N}^+$, there exists M_s such that*

$$\left\| \sum_{j=1}^{s} (-1)^j \cos\left(2\pi\langle \boldsymbol{c}^j, \boldsymbol{\xi}\rangle\right) \prod_{1\leqslant k<j} \left(1 + \cos\left(2\pi\langle \boldsymbol{c}^k, \boldsymbol{\xi}\rangle\right)\right) \right\|_{L^1(\mathbb{T}^d)} \geqslant Cs \tag{3.3}$$

whenever $\{\boldsymbol{c}^k\}_{k=1}^s \subset \mathbb{Z}^d$ satisfies

$$|\boldsymbol{c}^{k+1}| > M_s|\boldsymbol{c}^k|.$$

Remark 3.6. *The value of M_s could be derived from [33], where it is proved that whenever $\sum_{k=1}^s \left(\frac{|\boldsymbol{c}^k|}{|\boldsymbol{c}^{k+1}|}\right) < \infty$ then the expression appearing in the inequality* (3.3) *is equivalent to the similar one with functions $\boldsymbol{\xi} \mapsto \cos(2\pi\langle \boldsymbol{c}^j, \boldsymbol{\xi}\rangle)$ replaced by cosines of certain independent random variables, for which it follows by the theorem by R. Latała (Theorem 1 in [29]). In [12] the weaker condition $\sum_{k=1}^s \left(\frac{|\boldsymbol{c}^k|}{|\boldsymbol{c}^{k+1}|}\right)^2 < \infty$ is claimed to be sufficient (see Chapter 4). Similar inequality was obtained and used by M. Wojciechowski in [55].*

In the rest of the chapter we put $N := \left(|\frac{\log(M_s)}{\log(2)}| + 2\right)$.

Let us assume that the operator T_m corresponding to the multiplier m is bounded. For every $s \in \mathbb{N}$ we will construct a function h_s with norm bounded by a constant independent of s, such that

$$\|T_m h_s\|_{\dot{W}_1^1(\mathbb{R}^d)} \geqslant Cs.$$

Let $\varepsilon > 0$, be fixed later. We construct the sequence of balls $\mathbf{B}(\boldsymbol{c}^k, r_k)$ and $\mathbf{B}(-\boldsymbol{c}^k, r_k)$ for $k \in \{1, 2, \dots, s\}$, such that the following conditions hold:

II-A. $|m(\boldsymbol{\xi}) - (-1)^k| < \varepsilon$ for $\boldsymbol{\xi} \in \boldsymbol{B}(\boldsymbol{c}^k, r_k) \cup \boldsymbol{B}(-\boldsymbol{c}^k, r_k)$ for $k = 1, 2, \dots, s$,

II-B. $r_n \leqslant 2^{-N} r_{n+1}$ for $n = 1, 2, \dots, s-1$,

II-C. $\boldsymbol{c}^n \in \mathbb{Q} \times \mathbb{Q}$ for $n = 1, 2, \dots, s$,

II-D. $|\boldsymbol{c}^{n+1}| < 2^{-N} r_n$ for $n = 1, 2, \dots, s-1$,

II-E. $|c_2^n|/|c_1^n| \leqslant \frac{1}{3^{s+2}s}$ for $n = 1, 2, \dots, s$,

II-F. $|\boldsymbol{c}^n| < 2^{-N} |\boldsymbol{c}^{n+1}|$ for $n = 1, 2, \dots, s-1$,

II-G. $|c_i^n| > r_n$ for $n = 1, 2, \dots, s$ and $i = 1, 2$,

II-H. $|c_i^n| < 2^{-N} |c_i^{n+1}|$ for $n = 1, 2, \dots, s$ and $i \in \{1, 2\}$.

II-I. $\boldsymbol{B}(\sum_{j=1}^n \zeta_j \boldsymbol{c}^j, r_1) \subset \boldsymbol{B}(\zeta_n \boldsymbol{c}^k, r_k)$ for $\zeta_k \in \{-1, 1\}$, $\zeta_j \in \{-1, 0, 1\}$. and $n = 1, 2, \dots, s$.

We define sequences $\{\boldsymbol{c}^k\}$ and $\{r_k\}$ by backward induction. There is no problem with r_n because it is chosen always after $\boldsymbol{c}^n$ and for II-B and II-G we take it sufficiently small. For $\boldsymbol{c}^n$ note that the conditions II-D and II-F require only that $\boldsymbol{c}^n$ is small enough. Conditions II-A, II-E, II-H will be satisfied if we take as $\boldsymbol{c}^n$ a vector $\boldsymbol{a}^k$ with sufficiently large index k s.t. $k \equiv n \mod 2$. At the end we adjust our choice to the condition II-C: since the rationals are dense in $\mathbb{R}$ and all other inequalities are strict, we can do this in such a way that inequalities remain valid.
The condition II-I follows from II-B, II-D and II-F. Indeed for $k \in \{1, \dots, s-1\}$, $\zeta_j \in \{-1, 0, 1\}$, $j \in \{1, ..., k-1\}$ and $\zeta_k \in \{-1, 1\}$ we have

$$\sum_{j=1}^{k-1} r_j < 2^{-N} \sum_{j=2}^{k-1} r_j + 2^{-N} r_k < \dots < \left(\sum_{j=1}^{k} 2^{-Nj} \right) r_k < \frac{1}{2} r_k.$$

Hence

$$\left| \zeta_k \boldsymbol{c}^k - \sum_{j=1}^{k} \zeta_j \boldsymbol{c}^j \right| = \left| \sum_{j=1}^{k-1} \zeta_j \boldsymbol{c}^j \right| < \sum_{j=1}^{k-1} \left| \zeta_j \boldsymbol{c}^j \right| < \sum_{j=1}^{k-1} 2^{-N} r_j < \frac{1}{2} r_k. \tag{3.4}$$

By condition II-B we have $r_l < \frac{r_k}{4}$ for $k > l$. Therefore by (3.4)

$$\boldsymbol{B}(\sum_{j=1}^{k} \zeta_j \boldsymbol{c}^j, r_1) \subset \boldsymbol{B}(\zeta_k \boldsymbol{c}^k, r_k) \qquad \forall k \in \{1, 2, \dots, s\}.$$

The norm of T_m is invariant under rescaling. Then by condition II-C for fixed s multiplying $\boldsymbol{c}^j$'s by suitable scalar and rescaling multiplier m by the same scalar, we may assume that $\boldsymbol{c}^1, \dots, \boldsymbol{c}^s \in \mathbb{Z}^2$ and the conditions II-A – II-I are still satisfied. Note that if $\boldsymbol{q} \in \mathbb{Z}^2$ has the representation

$$\boldsymbol{q} = \sum_{j=1}^{s} \zeta_j(\boldsymbol{q}) \boldsymbol{c}^j \quad \text{ where } \zeta_j(\boldsymbol{q}) \in \{-1, 0, 1\}, \tag{3.5}$$

it is unique. For $\boldsymbol{q} \in \mathbb{Q}^2$ we denote by $\chi(\boldsymbol{q})$ the number of non zero terms in the representation (3.5). We define the set

$$\Lambda_s := \{\boldsymbol{q} : \boldsymbol{q} = \sum_{j=1}^{s} \zeta_j(\boldsymbol{q}) \boldsymbol{c}^j ; \boldsymbol{q} \neq 0 \ \text{ where } \zeta_j(\boldsymbol{q}) \in \{-1, 0, 1\}\}. \tag{3.6}$$

If $\boldsymbol{q}$, $\tilde{\boldsymbol{q}} \in \Lambda_s$ are two different vectors then

$$|\boldsymbol{q} - \tilde{\boldsymbol{q}}| \geqslant \inf |\boldsymbol{c}^j| \geqslant 1. \tag{3.7}$$

We will construct a function h_s in such a way that one of its derivatives behaves like a Riesz product. Let

$$g(t) := \max\{1 - |t|, 0\}^2$$

and

$$G(\boldsymbol{\xi}) := g(\xi_1) g(\xi_2).$$

We denote by R_s the modified Riesz product:

$$R_s(\boldsymbol{t}) := -1 + \Pi_{k=1}^{s}(1 + \cos(2\pi \langle \boldsymbol{t}\,, \boldsymbol{c}^k \rangle))$$

For fixed $\theta \in \mathbb{N}^+$ we define a function $H^\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by the formula

$$H^\theta(\boldsymbol{\xi}) := \sum_{\boldsymbol{q} \in \Lambda_s} \frac{1}{2^{\chi(\boldsymbol{q})}} G\left(2^\theta(\boldsymbol{\xi} - \boldsymbol{q})\right) = \sum_{\boldsymbol{q} \in \mathbb{Z}^2} \widehat{R}_s(\boldsymbol{q}) G\left(2^\theta(\boldsymbol{\xi} - \boldsymbol{q})\right). \tag{3.8}$$

Since R_s are densities of periodic measures with uniformly bounded norms and the inverse Fourier transform of the function G decays sufficiently fast at infinity we get:

Corollary 3.7. *For every $\theta \in \mathbb{N}^+$ the following inequality is satisfied*

$$\|\mathscr{F}^{-1}(H^\theta)\|_{L^1(\mathbb{R}^2)} \leqslant C \|R_s\|_{L^1(\mathbb{T}^2)} \leqslant C,$$

where the constant C is independent of s.

In the next lemma we state another property of H^θ.

Lemma 3.8. *There exists $\theta = \theta(s) \in \mathbb{N}^+$ such that*

$$\left\| \mathscr{F}^{-1}\left(\frac{\xi_2}{\xi_1} H^\theta\right)\right\|_{L^1(\mathbb{R}^2)} \leqslant C$$

where the constant C is independent of s.

The proof of this fact one can find in the Appendix. From now on we put $H := H^{\theta(s)}$.

Remark 3.9. *Note that homogeneous, non-constant functions are never multipliers on $L^1(\mathbb{R}^d)$. The above lemma holds true only due to the special form of H^θ, mainly the strong concentration of its support near x_1-axis and because of small size of its support.*

Since H is bounded, continuous and has compact support separated from the axis $\{\xi_1 = 0\}$, the function $\frac{H}{\xi_1}$ is a tempered distribution. We define a tempered distribution h by the formula

$$h(\psi) := \frac{H}{x_1}(\mathscr{F}^{-1}\psi) \qquad \forall \psi \in \mathscr{S}.$$

By standard properties of the Fourier transform on the space of tempered distributions, we get

$$\begin{aligned} \mathscr{F}\left(\frac{\partial}{\partial x_1} h\right) &= H \\ \mathscr{F}\left(\frac{\partial}{\partial x_2} h\right) &= \frac{\xi_2}{\xi_1} H. \end{aligned} \tag{3.9}$$

We proved that both H and $\frac{\xi_2}{\xi_1}H$ are the Fourier transforms of L^1 functions. Hence equalities (3.9) mean that $h \in \dot{W}_1^1(\mathbb{R}^d)$ with the norm bounded by a constant independent of s.
Now we estimate the norm of $T_m h$ from below. Since $T_m : \dot{W}_1^1(\mathbb{R}^2) \to \dot{W}_1^1(\mathbb{R}^2)$, obviously $\frac{\partial}{\partial x_1} T_m h \in L^1(\mathbb{R}^2)$. We denote by P the periodization of the function $\frac{\partial}{\partial x_1} T_m h$. It is only the fact that, when the function is in $L^1(\mathbb{R}^d)$, then its periodization is in $L^1(\mathbb{T}^d)$. We have

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant \left\| \frac{\partial}{\partial x_1} T_m h\right\|_{L^1(\mathbb{R}^2)} \geqslant \|P\|_{L_1(\mathbb{T}^2)}. \tag{3.10}$$

One can check that the function P is a polynomial given by the formula

$$P(\boldsymbol{\xi}) = \sum_{\boldsymbol{p} \in \Lambda_s} m(\boldsymbol{p}) H(\boldsymbol{p}) e^{2\pi i \langle \boldsymbol{p}, \boldsymbol{\xi}\rangle}. \tag{3.11}$$

We put

$$a(\boldsymbol{p}) := \begin{cases} (-1)^k H(\boldsymbol{p}) & \text{when } \boldsymbol{p} \in \Lambda_s \text{ and } \boldsymbol{p} \in \boldsymbol{B}(\boldsymbol{c}^k, r_k) \cup \boldsymbol{B}(-\boldsymbol{c}^k, r_k), \\ 0 & \text{otherwise.} \end{cases}$$

Since Λ_s is a finite set, the function

$$Z(\boldsymbol{\xi}) := \sum_{\boldsymbol{p}\in\mathbb{Z}^2} a(\boldsymbol{p}) e^{2\pi i\langle \boldsymbol{p},\boldsymbol{\xi}\rangle}$$

is a polynomial. By the triangle inequality,

$$\|P\|_{L_1(\mathbb{T}^2)} \geqslant \|Z\|_{L^1(\mathbb{T}^2)} - \|P-Z\|_{L^1(\mathbb{T}^2)}. \tag{3.12}$$

By the conditions II-I and II-A, all coefficients of Z differ by at most ε from the corresponding coefficients of P. Since both polynomials have no more then 3^s non-zero coefficients, we get

$$\|Z-P\|_{L^1(\mathbb{T}^2)} \leqslant \varepsilon 3^s. \tag{3.13}$$

It is easy to verify that

$$Z(\boldsymbol{\xi}) = \sum_{j=1}^{s} (-1)^j \cos\left(2\pi\langle \boldsymbol{c}^j,\, \boldsymbol{\xi}\rangle\right) \prod_{1\leqslant k<j} \left(1+\cos\left(2\pi\langle \boldsymbol{c}^k,\, \boldsymbol{\xi}\rangle\right)\right).$$

By the condition II-F and Lemma 3.5,

$$\|Z\|_{L^1(\mathbb{T}^2)} \geqslant Cs.$$

Combining now successively (3.10), (3.12) and (3.13), we get

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant Cs - \varepsilon 3^s.$$

Setting $\varepsilon = C3^{-s-1}s$

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant Cs$$

which by the uniform boundedness of $\|h\|_{\dot{W}_1^1(\mathbb{R}^2)}$ proves that T is unbounded.

Proof in Case IIb

The proof in this case is very similar to Case IIa. The only difference is that, due to lack of symmetry, we have to replace Lemma 3.5 by its asymmetric counterpart. We will use the following result from [55].

Lemma 3.10. *There exist $C>0$ such that for every $n\in\mathbb{N}^+$ there exists $M=M(n)$ such that for any sequence $\{\boldsymbol{c}^k\}_{k=1}^n \subset \mathbb{Z}^d$, which satisfies*

$$|\boldsymbol{c}^{k+1}| > M|\boldsymbol{c}^k|,$$

following inequality holds

$$\|\sum_{j=1}^{n} e^{2\pi i\langle \boldsymbol{c}^j,\boldsymbol{\xi}\rangle} \prod_{1\leqslant k<j} \left(1+\cos\left(\langle 2\pi \boldsymbol{c}^k, \boldsymbol{\xi}\rangle\right)\right) \|_{L_1(\mathbb{T}^r)} \geqslant Cn.$$

For fixed $\varepsilon > 0$ we construct the sequence of balls $\boldsymbol{B}(\boldsymbol{c}^n, r_n)$ and $\boldsymbol{B}(-\boldsymbol{c}^n, r_n)$ satisfying conditions II-B – II-I and

II-A'. $|m(\boldsymbol{\xi}) - 1| < \varepsilon$ for $\boldsymbol{B}(\boldsymbol{c}^n, r_n)$ and $|m(\boldsymbol{\xi})| < \varepsilon$ for $\boldsymbol{\xi} \in \boldsymbol{B}(-\boldsymbol{c}^n, r_n)$ and $n = 1, 2, \dots, s$.

The inductive construction is similar as in the Case IIa. Then, similarly as in the Case IIa, we define $\theta(s)$ and h, and we get

$$\|h\|_{\dot{W}_1^1(\mathbb{R}^2)} \leqslant C,$$

where the constant $C > 0$ independent of s. Analogously as in the Case IIa we define polynomial P by (3.11) and by similar reasons

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant \|P\|_{L^1(\mathbb{T}^2)}.$$

Then we put

$$a(\boldsymbol{p}) := \begin{cases} H(\boldsymbol{p}) & \text{when } \boldsymbol{p} \in \Lambda_s \text{ and } \boldsymbol{p} \in \boldsymbol{B}(\boldsymbol{c}^k, r_k), \\ 0 & \text{otherwise }, \end{cases}$$

where $k \in \{1, 2, \dots, s\}$. The function $a(\cdot)$ differs from its analogue from Case IIa. We define a polynomial Z by

$$Z(\boldsymbol{\xi}) := \sum_{\boldsymbol{p} \in \mathbb{Z}^2} a(\boldsymbol{p}) e^{2\pi i \langle \boldsymbol{p}, \boldsymbol{\xi} \rangle}.$$

It is easy to check that

$$Z(\xi) = \sum_{j=1}^{2n} e^{2\pi i \langle \boldsymbol{c}^j, \boldsymbol{\xi} \rangle} \prod_{1 \leqslant k < j} \left(1 + \cos\left(2\pi \langle \boldsymbol{c}^k,\ \boldsymbol{\xi} \rangle\right)\right),$$

and similar reasoning as in the Case IIa (3.13) gives

$$\|P\|_{L^1(\mathbb{T}^2)} \geqslant \|Z\|_{L^1(\mathbb{T}^2)} - \varepsilon 3^s.$$

By Lemma 3.10,

$$\|Z\|_{L_1(\mathbb{T}^2)} \geqslant Cs.$$

Hence

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant Cs - \varepsilon 3^s,$$

and setting $\varepsilon = C 3^{-s-1} s$ we get

$$\|T_m h\|_{\dot{W}_1^1(\mathbb{R}^2)} \geqslant Cs$$

which by uniform boundedness of $\|h\|_{\dot{W}_1^1(\mathbb{R}^2)}$ proves that T is unbounded. $\square$

Proof of Lemma 3.8

We begin with two lemmas. We study the operator given by sufficiently smooth multiplier acting on a subspace of L^1 functions with compactly supported Fourier transform. Let k be the smallest even number greater then $\lceil \frac{d}{2} \rceil$, $d \geqslant 2$. We fix function $\eta \in C_0^\infty$ supported in ball of radius 1.

Lemma 3.11. *Let $0 < \varepsilon \leqslant r < 1$ and $f \in C^{k+1}(\boldsymbol{B}(\mathbf{0}, r))$ with all derivatives of order less than or equal to k vanishing at $\mathbf{0}$. Then the following inequality holds*

$$\|\mathscr{F}^{-1}(\eta_\varepsilon f)\|_{L^1(\mathbb{R}^d)} \leqslant C(\eta, d)\, \varepsilon \left(\sum_{|\boldsymbol{\alpha}|=k+1} |D^{\boldsymbol{\alpha}} f(\mathbf{0})| + o(\varepsilon) \right), \tag{3.14}$$

where $\eta_\varepsilon(\boldsymbol{x}) := \eta(\varepsilon \boldsymbol{x})$.

Proof. We recall that for such k the left hand side is bounded up to a constant by $\|\eta f\|_{W_1^k}$ (cf. [47]). By the Leibniz Formula, it is sufficient to prove that all derivatives $D^{\boldsymbol{\beta}} f$ are dominated by $\sum_{|\boldsymbol{\alpha}|=k+1} |D^{\boldsymbol{\alpha}} f(0)| + o(\varepsilon)$ for $|\boldsymbol{\beta}| \leqslant k$ on $\boldsymbol{B}(\mathbf{0}, \varepsilon)$. This is a consequence of Taylor's Formula.

$$\begin{aligned} D^{\boldsymbol{\beta}} f(\boldsymbol{x}) &= \sum_{|\boldsymbol{\alpha}| \leqslant k+1-|\boldsymbol{\beta}|} D^{\boldsymbol{\alpha}+\boldsymbol{\beta}} f(\mathbf{0}) \boldsymbol{x}^{\boldsymbol{\alpha}} + o(|\boldsymbol{x}|^{k+1-|\boldsymbol{\beta}|}) \\ &= \sum_{|\boldsymbol{\alpha}|=k+1} D^{\boldsymbol{\alpha}} f(\mathbf{0}) + o(\varepsilon). \end{aligned} \tag{3.15}$$

□

Lemma 3.12. *Let $0 < \varepsilon \leqslant r \leqslant 1$ and $f \in C^{k+1}(\boldsymbol{B}(0, r))$ then the following inequality holds*

$$\|\mathscr{F}^{-1}(\eta_\varepsilon f)\|_{L^1(\mathbb{R}^d)} \leqslant C(\eta, d) \left(|f(\mathbf{0})| + \varepsilon \left(\sum_{|\boldsymbol{\alpha}| \leqslant k+1} |D^{\boldsymbol{\alpha}} f(\mathbf{0})| \right) + o(\varepsilon) \right). \tag{3.16}$$

Proof. Writing f as the sum of a polynomial of degree k and a function satisfying the assumptions of the previous lemma, we see that it is sufficient to consider only polynomials and by linearity monomials. For $f(\boldsymbol{\xi}) = (2i\pi\boldsymbol{\xi})^{\boldsymbol{\alpha}}$, we have

$$\|\mathscr{F}^{-1}(\eta_\varepsilon f)(\boldsymbol{x})\|_{L_1} = \|\varepsilon^{d+|\boldsymbol{\alpha}|} D^{\boldsymbol{\alpha}} \eta(\frac{\boldsymbol{x}}{\varepsilon})\|_{L_1} \leqslant C(\eta) \varepsilon^{\boldsymbol{\alpha}}. \tag{3.17}$$

Hence inequality (3.16) follows. □

Now we can prove the Lemma 3.8.

Proof. of Lemma 3.8.
By the definition of H^θ we see that its support is contained in the union of disjoint balls of radius r with centered in points of Λ_s. Radius r depends only on the parameter θ, so we can choose it as small as we wish. Let $\eta_{\boldsymbol{q}} \in C^\infty$ be rescaled and translated copies of the same function η with $\operatorname{supp} \eta_{\boldsymbol{q}} \subset B(\boldsymbol{q}, 2r)$ and $\eta_{\boldsymbol{q}}(\boldsymbol{\xi}) = 1$, $\boldsymbol{\xi} \in B(\boldsymbol{q}, r)$ for every $\boldsymbol{q} \in \Lambda_s$. The following identity holds

$$\frac{\xi_2}{\xi_1} H^\theta(\boldsymbol{\xi}) = \sum_{\boldsymbol{\phi} \in \Lambda_s} \eta_{\boldsymbol{q}}(\boldsymbol{\xi}) \frac{\xi_2}{\xi_1} H^\theta(\boldsymbol{\xi}). \tag{3.18}$$

By the condition II-G (page 30) the function $f = \frac{\xi_2}{\xi_1}$ satisfies conditions of Lemma 3.12 on these balls. Hence for r small enough by the triangle inequality, (3.16), and (3.18)

$$\begin{aligned}\|\mathscr{F}^{-1}(\eta_{\boldsymbol{q}} f H^\theta)\|_{L^1(\mathbb{R}^2)} \leqslant C(\eta) \sum_{\boldsymbol{q} \in \Lambda_s} \left(|f(\boldsymbol{q})| + \varepsilon \left(\sum_{|\alpha| \leqslant k+1} |D^\alpha f(\boldsymbol{q})| \right) + o(\varepsilon) \right) \\ \cdot \|\mathscr{F}^{-1}(H^\theta)\|_{L^1(\mathbb{R}^2)}.\end{aligned}$$

By conditions II-E and II-H,

$$\left| \frac{q_2}{q_1} \right| = \left| \frac{c_2^k + \sum_{j=1}^{k-1} \zeta_j c_2^j}{c_1^k + \sum_{j=1}^{k-1} \zeta_j c_1^j} \right| \leqslant \frac{k|c^k|}{|c_1^k| - \sum_{j=1}^{k-1} |c_1^j|} \leqslant \frac{s}{3} \left| \frac{c_1^k}{c_2^k} \right| \leqslant \frac{1}{2 \cdot 3^s}.$$

Since $|\Lambda_s| \leqslant 3^s$ we can choose sufficiently small $\varepsilon > 0$ such that

$$\|\mathscr{F}^{-1}(\frac{\xi_2}{\xi_1} H^\theta)\|_{L_1(\mathbb{R}^2)} \leqslant C \|\mathscr{F}^{-1}(H^\theta)\|_{L_1(\mathbb{R}^2)},$$

where the constant C does not depend on s. $\square$

Chapter 4

Isomorphism between sets of trigonometric polynomials

It is an easy observation that for any two trigonometric polynomials of one variable $p(t)$ and $q(t)$ the L^p norm of the polynomial $p(t) + e^{2\pi i N t}q(t)$ tends to the L^p norm of the two variables polynomial $p(t)+e^{2\pi i s}q(t)$. It is a consequence of the fact that character with high oscillation mimics an independent Steinhaus variable. Based on this observation Y. Meyer proposed in [33] that the formula

$$f(x) = \sum_{\boldsymbol{n}\in J} a_{\boldsymbol{n}} e^{2\pi i \sum_{j=1}^{k} n_j x_j} \to Tf(z) = \sum_{n\in J} a_n e^{2\pi i \sum_{i=1}^{k} \tau_j n_j z},$$

where $x = (x_1, ..., x_j)$, provides an L^p-isomorphism between the space of k-variables trigonometric polynomials with spectrum contained in k-cube $J = [-1,1]^k \cap \mathbb{Z}^k$ and the invariant subspace of one variable polynomials if the sequence τ_j increase rapidly enough. This statement is established in [33] for $p = \infty$ and sequence of positive integers τ_j s.t.

$$\sum_{j=1}^{\infty} \frac{\tau_j}{\tau_{j+1}} < \infty \qquad \text{and} \qquad \tau_{j+1} \geqslant 3\tau_j$$

Later M. Déchamps [12] improved this result extending the spectrum to k-hyperrectangle $\prod_j [-r_j, r_j] \cap \mathbb{Z}^k$ and the sequence τ_j s.t.

$$\sum_{j=1}^{\infty} \left(\frac{r_j\tau_j}{\tau_{j+1}}\right)^2 < \infty \qquad \text{and} \qquad \tau_{j+1} \geqslant \frac{\pi}{2}(r_j+1)\tau_j$$

In both articles the extensions to $1 \leqslant p < \infty$ seams to be incomplete. In the last section of this chapter we indicate why the use of duality and interpolation in this problem is a delicate matter.
The purpose of this chapter is to prove that under the Meyer's condition, the formula from

the beginning of the chapter for $1 \leqslant p < \infty$ defines an isomorphism. We also give an extensions of this fact to the isomorphism of multidimensional tori and we clarify the role of size and cardinality in quantitative condition.

We study the equivalence between finite dimensional spaces of trigonometric polynomials defined on $\mathbb{T}^d$ with spaces of trigonometric polynomials defined on $\mathbb{T}^{dk}$. Let $\tau = \{\tau_i\}_{i=1}^k$ be a given family of integer vectors, where $\tau_i = (\tau_{i,1}, \dots, \tau_{i,d})$. We investigate the operator

$$f(x) = \sum_\lambda a_\lambda e^{2\pi i\langle(\lambda_1,\dots,\lambda_k),(x_1,\dots,x_k)\rangle} \to Tf(z) = \sum_\lambda a_\lambda e^{2\pi i\langle\sum_{i=1}^k \tau_j\lambda_j, z\rangle} \quad \forall\ x \in \mathbb{T}^{dk}, z \in \mathbb{T}^d, \tag{4.1}$$

where $x = (x_1, \dots, x_k)$, with suitably chosen $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^{dk}$. To be more precise we introduce following notation.

Definition 4.1. *Let $k \in \mathbb{N}$ and $A \subset \mathbb{Z}^k$*

$$L_A^p(\mathbb{T}^k) = \{f \in L^p(\mathbb{T}^k) : \operatorname{supp} \widehat{f} \subset A\}.$$

Definition 4.2. *For a given sequence of d-tuples of integers $\tau = \{\tau_n\}_{n\in\mathbb{N}}$ and family of sets $\mathscr{A} = \{A_n\}_{n\in\mathbb{N}}$ ($A_n \subset \mathbb{Z}^d$) we define sets $E \subset \mathbb{Z}^d$ and $F \subset \mathbb{Z}^\mathbb{N}$ (here $\mathbb{Z}^\mathbb{N}$ is a dual group to $\mathbb{T}^\mathbb{N}$), in the following way:*

$$\begin{aligned} F &:= F(\mathscr{A}) = \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots) \in (\mathbb{Z}^d)^\mathbb{N} \ : \ \lambda_n \in A_n\}, \\ E &:= E(\mathscr{A}, \tau) = \{\beta \in \mathbb{Z}^d \ : \ \beta = \sum_{k=1} \tau_k\lambda_k \text{ for } \boldsymbol{\lambda} \in F\}, \end{aligned} \tag{4.2}$$

where by $\tau_k\lambda_k$ we denote pointwise product i.e. $\tau_k\lambda_k = (\tau_{k,1}\lambda_{k,1}, \dots, \tau_{k,d}\lambda_{k,d})$, $\tau_k = (\tau_{k,1}, \dots, \tau_{k,d})$ and $\lambda_k = (\lambda_{k,1}, \dots, \lambda_{k,d})$.

The main result of this chapter is following:

Theorem 4.3. *For a given sequence of d-tuples of integers $\tau = \{\tau_n\}_{n\in\mathbb{N}}$ and family of sets $\mathscr{A} = \{A_n\}_{n\in\mathbb{N}}$ ($A_n \subset \mathbb{Z}^d$) satisfying*

$$\begin{gathered} A_n \subset [-r_n, r_n]^d, \\ \|\tau_{k+1}\|_\infty > 2\sum_{j=1}^k r_j\|\tau_j\|_\infty \qquad \forall k \in \mathbb{N}, \\ \sum_{j=1}^\infty \frac{\#A_{j+1}\|\tau_j\|_\infty r_j}{\min\limits_{k\in\{1,\dots,d\}} |\tau_{j+1,k}|} < \infty, \end{gathered} \tag{4.3}$$

the operator $T := T_{\mathscr{A},\tau} : L_F^p(\mathbb{T}^\mathbb{N}) \to L_E^p(\mathbb{T}^d)$ given by the formula

$$Tf(x) = \sum_{\boldsymbol{\lambda}\in F} \widehat{f}(\boldsymbol{\lambda}) e^{2\pi i\langle\sum_{j=1}^\infty \lambda_j\tau_j, x\rangle} \tag{4.4}$$

is an isomorphism of Banach spaces.

Remark 4.4. If we assume that $\cup A_n$ is bounded set it is enough to assume geometric growth of τ and the l^1 summability of a sequence $j \mapsto \frac{\|\tau_j\|_\infty}{\min\limits_{k\in\{1,\dots,d\}} |\tau_{j+1,k}|}$. If

$$2d\frac{\#A_{j+1}\|\tau_j\|_\infty(r_j+1)}{\min\limits_{k\in\{1,\dots,d\}} |\tau_{j+1,k}|} < 1$$

for every $j \in \mathbb{N}$ then

$$K^{-1}\|f\|_{L^1_F(\mathbb{T}^{\mathbb{N}})} \leqslant \|Tf\|_{L^1_E(\mathbb{T})} \leqslant K\|f\|_{L^1_F(\mathbb{T}^{\mathbb{N}})}$$

where the constant K depends only on the value of $\sum_{j=1}^\infty \frac{\#A_{j+1}\|\tau_j\|_\infty r_j}{\min\limits_{k\in\{1,\dots,d\}} |\tau_{j+1,k}|}$.

Remark 4.5. Theorem 1 holds for polynomials with values in Banach spaces as well. All the steps of the proof could be repeated verbatim for trigonometric polynomials with Banach space coefficients.

For fixed finite set of characters one can choose a sequence τ growing so fast that the following holds.

Corollary 4.6. *Let $A \subset \mathbb{Z}^d$ be a finite set. Then there exists $\phi : A \to \mathbb{Z}$ s.t. the operator*

$$\sum_{\lambda\in A} a_\lambda e^{2\pi i\langle\lambda,z\rangle} \to \sum_{\lambda\in A} a_\lambda e^{2\pi i\phi(\lambda)x}$$

is a $(1+\varepsilon)$-isometry.

The chapter is organized as follows. In the first section we will prove three elementary lemmas about approximation of trigonometric polynomials by simple functions. Section 2 contains the proof of the Theorem 4.3. In Section 3 we establish a necessary condition for p=1. In the last section we show an example of a sequence τ and family of sets $\mathscr{A}$ such that T is an isomorphism for $p=2$ and $p=4$ but fails to be an isomorphism for $p=3$ and $p=4/3$. This shows that one has to be very careful using interpolation and duality arguments to find this type of criteria. For examples of applications of such criterion we refer the reader to [29],[55].

Auxiliary lemmas

We begin with the estimate on the approximation of trigonometric polynomial by simple functions.

Lemma 4.7. *Let $s, N \in \mathbb{Z}^d$ and f be a trigonometric polynomial on $\mathbb{T}^{dk}$. Assume that the degrees with respect to the last d variables of the polynomial f are controlled by the coordinates of s i.e. $\deg_{z_i}(f) \leqslant |s_i|$, where $z = (z_1, \ldots, z_i)$. Let*

$$\tilde{f}(y', z) = \sum_{\substack{n \in \mathbb{N}^d \\ n_j \leqslant |N_j|-1}} \chi_{I_n}(z) \fint_{I_n} f(y', z)\, dz \qquad \forall\, y' \in \mathbb{T}^{d(k-1)}, z \in \mathbb{T}^d,$$

where $I_n = \left[\frac{n_1}{|N_1|}; \frac{n_1+1}{|N_1|}\right] \times \cdots \times \left[\frac{n_d}{|N_d|}; \frac{n_d+1}{|N_d|}\right]$. Then

$$\left| \|f\|_{L^p(\mathbb{T}^{dk})} - \|\tilde{f}\|_{L^p(\mathbb{T}^{dk})} \right| \leqslant \frac{d\, \|s\|_\infty}{\min\limits_{j \in \{1,\ldots,d\}} |N_j|} \|f\|_{L^p(\mathbb{T}^{dk})}.$$

Proof. We estimate the difference of L^p norms of the functions by the norm of its partial derivatives. We use the fact that the optimal constant in Poincare's inequality on a convex set is dominated by its diameter (see eg. [38], [1]).

$$\begin{aligned}
\left| \|f\|_{L^p(\mathbb{T}^{dk})} - \|\tilde{f}\|_{L^p(\mathbb{T}^{dk})} \right|^p &\leqslant \|f - \tilde{f}\|^p_{L^p(\mathbb{T}^{dk})} \\
&= \sum_{\substack{n \in \mathbb{N}^d \\ n_j \leqslant |N_j|-1}} \int_{\mathbb{T}^{d(k-1)}} \int_{I_n} \left| f(y', x) - \fint_{I_n} f(y', z)\, dz \right|^p dx dy' \\
&\leqslant \sum_{\substack{n \in \mathbb{N}^d \\ n_j \leqslant |N_j|-1}} \int_{\mathbb{T}^{d(k-1)}} \operatorname{diam}(I_n)^p \int_{I_n} |\nabla_z f(y', z)|^p\, dz dy'.
\end{aligned}$$

Since $\operatorname{diam}(I_n) \geqslant \min\limits_{j \in \{1,\ldots,d\}} |N_j|$ we get

$$\left| \|f\|_{L^p(\mathbb{T}^{dk})} - \|\tilde{f}\|_{L^p(\mathbb{T}^{dk})} \right|^p \leqslant \frac{1}{\min\limits_{j \in \{1,\ldots,d\}} |N_j|^p} \|\nabla_z f\|^p_{L^p(\mathbb{T}^{dk})},$$

where $\nabla_z f$ is the gradient of the function f with respect to last d variables. Using Bernstein's inequality (see eg. [56])

$$\left\| \frac{\partial}{\partial z_i} f \right\|_{L^p(\mathbb{T}^{dk})} \leqslant \deg_{z_i}(f)\, \|f\|_{L^p(\mathbb{T}^{dk})}.$$

We get from the triangle inequality

$$\begin{aligned}
\|\nabla_z f\|_{L^p(\mathbb{T}^{dk})} &\leqslant d \max_{j \in \{1,\ldots,d\}} \left\| \frac{\partial}{\partial z_j} f \right\|_{L^p(\mathbb{T}^{dk})} \\
&\leqslant d \max_{j \in \{1,\ldots,d\}} \deg_{z_j}(f)\, \|f\|_{L^p(\mathbb{T}^{dk})} \leqslant d\, \|s\|_\infty \|f\|_{L^p(\mathbb{T}^{dk})}.
\end{aligned} \tag{4.5}$$

Hence

$$\left|\|f\|_{L^p(\mathbb{T}^{dk})} - \|\tilde{f}\|_{L^p(\mathbb{T}^{dk})}\right| \leqslant \frac{1}{\min\limits_{j\in\{1,\ldots,d\}} |N_j|} \|\nabla_z f\|_{L^p(\mathbb{T}^{dk})} \leqslant \frac{d\,\|s\|_\infty}{\min\limits_{j\in\{1,\ldots,d\}} |N_j|} \|f\|_{L^p(\mathbb{T}^{dk})}$$

□

Lemma 4.8. *Let $f_n \in L^p(\mathbb{T}^d)$ be a trigonometric polynomial for every $n \in A$. We define*

$$w(y', y, z) = \sum_{n\in A} e^{2\pi i\langle n,y\rangle} f_n(y', z) \qquad \forall\ y' \in \mathbb{T}^{d(k-1)}\ \forall\ y, z \in \mathbb{T}^d.$$

Then

$$\|w\|_{L^p(\mathbb{T}^{d(k+1)})} \leqslant \sum_{n\in A} \|f_n\|_{L^p(\mathbb{T}^{dk})} \leqslant\ \#A\|w\|_{L^p(\mathbb{T}^{d(k+1)})}.$$

Proof. The left hand side is just a triangle inequality. We get the right hand side by summing for $n \in A$ the following inequalities.

$$\begin{aligned}\int_{\mathbb{T}^{dk}} |f_n|^p dy' dz &= \int_{\mathbb{T}^{dk}} \left|\int_{\mathbb{T}^d} e^{-2\pi i\langle n,y\rangle} w dy\right|^p dy' dz \\ &\leqslant \int_{\mathbb{T}^{dk}} \int_{\mathbb{T}^d} |w|^p \, dy dy' dz = \|w\|^p_{L^p(\mathbb{T}^{d(k+1)})}.\end{aligned}$$

□

Lemma 4.9. *Let trigonometric polynomials $f_n \in L^p(\mathbb{T}^{dk})$ satisfy $\deg_{z_j}(f_n) \leqslant s_j$ for $n \in A$ and $j \in \{1, \ldots, d\}$. Let*

$$w(y', z) := \sum_{n\in A} e^{2\pi i\langle Nn,z\rangle} f_n(y', z) \qquad \forall\ y' \in \mathbb{T}^{d(k-1)}\ \forall\ vz \in \mathbb{T}^d,$$

$$w(y', y, z) := \sum_{n\in A} e^{2\pi i\langle n,y\rangle} f_n(y', z) \qquad \forall\ y' \in \mathbb{T}^{d(k-1)}\ \forall\ y, z \in \mathbb{T}^d.$$

This pair of functions satisfies the estimates

$$\left(1 - 2d\frac{\#A\|s\|_\infty}{\min\limits_{j\in\{1,\ldots,d\}} |N_j|}\right) \|w\|_{L^1(\mathbb{T}^{d(k+1)})} \leqslant \|w\|_{L^1(\mathbb{T}^{dk})} \tag{4.6}$$

and

$$\|w\|_{L^1(\mathbb{T}^{dk})} \leqslant \left(1 + 2d\frac{\#A\|s\|_\infty}{\min\limits_{j\in\{1,\ldots,d\}} |N_j|}\right) \|w\|_{L^1(\mathbb{T}^{d(k+1)})}.$$

It could happen that the constant in (4.6) is negative. We consider such situation in Remark 4.10.

Proof. Let us define functions

$$\tilde{w}(y', y, z) := \sum_{n \in A} e^{2\pi i \langle n, y \rangle} \tilde{f}_n(y', z) \qquad \forall\, y' \in \mathbb{T}^{d(k-1)}, \forall\, y, z \in \mathbb{T}^d,$$
$$\bar{w}(y', z) := \sum_{n \in A} e^{2\pi i \langle Nn, z \rangle} \tilde{f}_n(y', z) \qquad \forall\, y' \in \mathbb{T}^{d(k-1)}, \forall\, z \in \mathbb{T}^d,$$

where functions $\tilde{f}_n$ are defined as in Lemma 4.7. By the triangle inequality,

$$\begin{aligned} \big| \|w\|_{L^p(\mathbb{T}^{dk})} - \|w\|_{L^p(\mathbb{T}^{d(k+1)})} \big| \leqslant\ & \big| \|w\|_{L^p(\mathbb{T}^{dk})} - \|\bar{w}\|_{L^p(\mathbb{T}^{dk})} \big| \\ & + \big| \|\tilde{w}\|_{L^p(\mathbb{T}^{d(k+1)})} - \|\bar{w}\|_{L^p(\mathbb{T}^{dk})} \big| \\ & + \big| \|w\|_{L^p(\mathbb{T}^{d(k+1)})} - \|\tilde{w}\|_{L^p(\mathbb{T}^{d(k+1)})} \big| . \end{aligned} \tag{4.7}$$

Again by the triangle inequality,

$$\big| \|w\|_{L^p(\mathbb{T}^{dk})} - \|\bar{w}\|_{L^p(\mathbb{T}^{dk})} \big| + \big| \|w\|_{L^p(\mathbb{T}^{d(k+1)})} - \|\tilde{w}\|_{L^p(\mathbb{T}^{d(k+1)})} \big| \leqslant 2 \sum_{n \in A} \|f_n - \tilde{f}_n\|_{L^p(\mathbb{T}^{dk})}.$$

Hence by the definition of $\tilde{f}_{k,n}$, Lemma 4.7 and Lemma 4.8,

$$\begin{aligned} & \big| \|w\|_{L^p(\mathbb{T}^k d)} - \|\bar{w}\|_{L^p(\mathbb{T}^{kd})} \big| + \big| \|w\|_{L^p(\mathbb{T}^{d(k+1)})} - \|\tilde{w}\|_{L^p(\mathbb{T}^{d(k+1)})} \big| \\ & \qquad \leqslant 2d \frac{\#A \|s\|_\infty}{\min\limits_{j \in \{1,\dots,d\}} |N_j|} \|w\|_{L^1(\mathbb{T}^{d(k+1)})}. \end{aligned} \tag{4.8}$$

For the second term of the right hand side of (4.7) we have

$$\begin{aligned} \|\bar{w}\|^p_{L^p(\mathbb{T}^{dk})} &= \int_{\mathbb{T}^{kd}} |\bar{w}(y', z)|^p dy'\, dz \\ &= \int_{\mathbb{T}^{d(k-1)}} \sum_{\substack{m \in \mathbb{N}^d \\ m_u \leqslant |N_u| - 1}} \int_{I_m} \left| \sum_{n \in A} e^{2\pi i \langle Nn, z \rangle} \tilde{f}_n(y', z) \right|^p dz\, dy' \end{aligned}$$

The function $\tilde{f}_n(y', \cdot)$ is a constant on every d-parallelotope I_m for $m \in \mathbb{Z}^d \cap \{m_j \leqslant |N_j| - 1\}$ and every $y' \in \mathbb{T}^{d-1}$. We denote this value by $h_n(m, y')$. We have following identity.

$$\begin{aligned} \|\bar{w}\|^p_{L^p(\mathbb{T}^{dk})} &= \int_{\mathbb{T}^{d(k-1)}} \sum_{\substack{m \in \mathbb{N}^d \\ m_u \leqslant |N_u| - 1}} \int_{I_m} \left| \sum_{n \in A} e^{2\pi i \langle Nn, z \rangle} h_n(m, y') \right|^p dz\, dy' \\ &= \prod_{u=1}^d |N_u|^{-1} \int_{\mathbb{T}^{d(k-1)}} \sum_{\substack{m \in \mathbb{N}^d \\ m_u \leqslant |N_u| - 1}} \int_{\mathbb{T}^d} \left| \sum_{n \in A} e^{2\pi i \langle n, y \rangle} h_n(m, y') \right|^p dy\, dy' \\ &= \int_{\mathbb{T}^{d(k-1)}} \int_{\mathbb{T}^d} \sum_{\substack{m \in \mathbb{N}^d \\ m_u \leqslant |N_u| - 1}} |I_m| \left| \sum_{n \in A} e^{2\pi i \langle n, y \rangle} h_n(m, y') \right|^p dy\, dy'. \end{aligned}$$

The summands of the inner sum equal to the integrals of a constant functions. Since $h_n(m, y') = f_n(y', z)$ for $z \in I_m$,

$$\|\bar{w}\|^p_{L^p(\mathbb{T}^{dk})} = \int_{\mathbb{T}^{d(k-1)}} \int_{\mathbb{T}^d} \sum_{\substack{m \in \mathbb{N}^d \\ m_u \leqslant |N_u|-1}} \int_{I_m} \left| \sum_{n \in A} e^{2\pi i \langle n, y \rangle} f_n(y', z) \right|^p dz\, dy\, dy'$$

$$= \int_{\mathbb{T}^{d(k+1)}} \left| \sum_{n \in A} e^{2\pi i \langle n, y \rangle} \bar{f}^d_n(y', z) \right|^p dz\, dy\, dy'$$

$$= \int_{\mathbb{T}^{d(k+1)}} |\tilde{w}|^p dz\, dy\, dy' = \|\tilde{w}\|^p_{L^p(\mathbb{T}^{d(k+1)})}.$$

The above equality together with (4.7) and (4.8) gives

$$\left| \|w\|_{L^p(\mathbb{T}^{dk})} - \|w\|_{L^p(\mathbb{T}^{d(k+1)})} \right| \leqslant 2d \frac{\#A \|s\|_\infty}{\min\limits_{j \in \{1,\dots,d\}} |N_j|} \|w\|_{L^p(\mathbb{T}^{d(k+1)})}$$

which implies the lemma. □

4.1 Proof of Theorem 4.3

Let $f \in L^p_F(\mathbb{T}^{\mathbb{N}})$ be a trigonometric polynomial, which depends only on first $s \times d$ variables. Then, by the definition of T we have,

$$Tf(z) = \sum_{(\lambda_1, \dots, \lambda_s, 0, 0, \dots) \in F} \widehat{f}(\lambda_1, \dots, \lambda_s) e^{2\pi i \langle \sum_{j=1}^s \tau_j \lambda_j, z \rangle} \qquad \forall\, z \in \mathbb{T}^d,$$

which can be rewritten in the form

$$w_1(z) := Tf(z) = \sum_{n \in A_s} e^{2\pi i \langle \tau_s n, z \rangle} g_{1,n}(z),$$

where $g_{1,n}$ are suitable polynomials with $\deg_{z_i}(g_{1,n}) \leqslant \sum_{t=1}^{s-1} r_t |\tau_{t,i}| \leqslant \sum_{t=1}^{s-1} r_t ||\tau_t||_\infty$. By Lemma 4.9,

$$\left(1 - \frac{2d\, \#A_s \sum_{t=1}^{s-1} r_t ||\tau_t||_\infty}{\min\limits_{i \in \{1,\dots,d\}} |\tau_{s,i}|} \right) \|w_2\|_{L^p(\mathbb{T}^{2d})} \leqslant \|w_1\|_{L^p(\mathbb{T}^d)} \tag{4.9}$$

and

$$\|w_1\|_{L^p(\mathbb{T}^d)} \leqslant \left(1 + \frac{2d\, \#A_s \sum_{i=1}^{s-1} r_t \|\tau_t\|_\infty}{\min\limits_{i \in \{1,\dots,d\}} |\tau_{s,i}|} \right) \|w_2\|_{L^p(\mathbb{T}^{2d})},$$

where
$$w_2(y_1,z)=\sum_{(\lambda_1,\dots,\lambda_s,0,0,\dots)\in F}\widehat{f}(\lambda_1,\dots,\lambda_s)e^{2\pi i\langle\lambda_s,y_1\rangle}e^{2\pi i\langle\sum_{j=1}^{s-1}\tau_j\lambda_j,z\rangle},$$
for $y_1,z\in\mathbb{T}^d$. Similarly as in the case $k=1$ we proceed for $k>1$. We obtain trigonometric polynomials of the form
$$w_k(y',z):=\sum_{(\lambda_1,\dots,\lambda_s,0,0,\dots)\in F}\widehat{f}(\lambda_1,\dots,\lambda_s)e^{2\pi i\langle(\lambda_s,\dots,\lambda_{s-k+1}),y'\rangle}e^{2\pi i\langle\sum_{j=1}^{s-k}\tau_j\lambda_j,z\rangle},\tag{4.10}$$
for all $y'\in\mathbb{T}^{d(k-1)}$ and $z\in\mathbb{T}^d$. This could be rewritten as
$$w_k(y',z)=\sum_{j\in A_k}e^{2\pi i\langle\tau_{s-k+1}j,z\rangle}g_{k,j}(y',z)$$
where polynomials $g_{k,j}$ satisfy $\deg_{z_u}(g_{k,j})\leqslant\sum_{j=1}^{s-k}r_j|\tau_{j,u}|$. By Lemma 4.9,
$$(1-K(k))\,\|w_{k+1}\|_{L^p(\mathbb{T}^{d(k+1)})}\leqslant\|w_k\|_{L^p(\mathbb{T}^{dk})}\leqslant(1+K(k))\,\|w_{k+1}\|_{L^p(\mathbb{T}^{d(k+1)})},$$
where constant $K(k)$ is given by
$$K(k)=2d\,\#A_{s-k+1}\frac{\sum_{j=1}^{s-k}r_j\|\tau_j\|_\infty}{\min\limits_{i\in\{1,\dots,d\}}|\tau_{s-k+1,i}|}.$$
Combining the above inequalities for $k=1,\dots,s-1$ we get
$$\prod_{j=1}^{s-1}(1-K(j))\|w_s\|_{L^p(\mathbb{T}^{ds})}\leqslant\|Tf\|_{L^p(\mathbb{T}^d)}\leqslant\prod_{j=1}^{s-1}(1+K(j))\|w_s\|_{L^p(\mathbb{T}^{ds})},$$
The constant $1-K(j)$ could be negative for some j. We consider such case in Remark 4.10. For now we assume that $1-K(j)>0$. Note that w_s equals f up to a permutation of variables. Hence $\|w_s\|_{L^p(\mathbb{T}^{ds})}=\|f\|_{L^p(\mathbb{T}^{ds})}$. Since A_j,r_j,τ_j satisfy (4.3) we have
$$\sum_{k=1}^\infty K(k)=\sum_{k=1}^\infty\frac{2\sqrt{d}\,\#A_{n-k+1}\sum_{j=1}^{n-k}r_j\|\tau_j\|_\infty}{\min\limits_{i\in\{1,\dots,d\}}\|\tau_{n-k+1,i}\|_\infty}\leqslant C\sum_{k=1}^\infty\frac{\#A_k\;r_{k-1}\|\tau_{k-1}\|_\infty}{\min\limits_{i\in\{1,\dots,d\}}|\tau_{k,i}|}<\infty.$$
Hence there exists a constant K independent on n such that
$$K^{-1}\|f\|_{L^p(\mathbb{T}^{dn})}\leqslant\|Tf\|_{L^p(\mathbb{T}^d)}\leqslant K\|f\|_{L^p(\mathbb{T}^{dn})},\tag{4.11}$$
Remark 4.10. It could happen that the first few constants $K(j)$ are larger than 1. Since $\sum K(j)$ is convergent, there is only a finite number of them. In this case we replace the respective induction steps by trivial estimates of the norm depending on isomorphism of finitely dimensional spaces

4.2 Necessary condition for L^1 isomorphism

In this section we investigate a necessary condition on the operator T of the form (4.1) to be an isomorphism in L^1 norm. We are looking for a condition expressed in terms of a summability of a sequence $\frac{|\tau_j|}{|\tau_{j+1}|}$. We show that ℓ^2 summability is necessary. More precisely for every family $\mathscr{A}$ of uniformly bounded sets A_i s.t. $\#A_i \geqslant 2$ there exists a sequence τ_j such that $\sum \frac{|\tau_j|^2}{|\tau_{j+1}|^2} = +\infty$ but $\sum \frac{|\tau_j|^p}{|\tau_{j+1}|^p} < +\infty$ for every $p > 2$ and the operator T corresponding to τ is not an isomorphism. Our argument is a slight modification of an idea of F. Nazarov, which proves the necessity of a geometric growth of $|\tau_k|$ ([35]).
For simplicity we assume that $d = 1$ and $A_i = \{-1, 1\}$ but the similar construction works in general case. We choose sequence

$$\tau_k = \begin{cases} (\,l!\,)^3 & k = 2l, \\ (\,l!\,)^3\, 2\lfloor\sqrt{k}\rfloor & k = 2l+1. \end{cases}$$

For $n \in \mathbb{N}$ we define a trigonometric polynomial f_n by

$$f_n(x) = \prod_{j=1}^{2n} \sin(2\pi\tau_j x) \qquad \forall x \in \mathbb{T}.$$

If T given by (4.1) is an isomorphism then there is a constant independent on n such that

$$\|f_n\|_{L^1_E(\mathbb{T})} \geqslant C_1 \|Tf\|_{L^1_F(\mathbb{T}^{2n})} = \int_{\mathbb{T}^{2n}} \left| \prod_{j=1}^{2n} \sin(2\pi x_j) \right| dx_1 \cdots dx_{2n} = \|\sin(2\pi x)\|^{2n}_{L^1(\mathbb{T})}.$$

We are going to apply now Theorem 4.3 for the sequence $\widetilde{\tau} = \{\tau_{2n}\}_{n\in\mathbb{N}}$ and the sequence of sets $\mathscr{B} = \{B_k\}$, where $B_k = \left\{-2\lfloor\sqrt{k}\rfloor, -1, 1,\ 2\lfloor\sqrt{k}\rfloor\right\}$. We get the following bounds on a norm of f_n with constants independent on n.

$$\begin{aligned} \|f_n\|_{L^1(\mathbb{T})} &\leqslant C_2 \int_{\mathbb{T}^n} \left| \prod_{j=1}^{n} \sin(2\pi x_j) \sin(2\pi\, 2\lfloor\sqrt{j}\rfloor x_j) \right| dx_1 \cdots dx_n \\ &= \prod_{j=1}^{n} \|\sin(2\pi x) \sin(4\pi\lfloor\sqrt{j}\rfloor x_j)\|_{L^1(\mathbb{T})}. \end{aligned}$$

It is now enough to show that for sufficiently large k,

$$\|\sin(2\pi x)\sin(4\pi k x)\|_{L^1(\mathbb{T})} \leqslant (1 - \frac{c}{k^2}) \|\sin(2\pi x)\|^2_{L^1(\mathbb{T})} = (1 - \frac{c}{k^2}) \left(\frac{2}{\pi}\right)^2, \tag{4.12}$$

because this implies a contradiction:

$$C_3 \leqslant \prod_{j=1}^{n} \frac{\|\sin(2\pi x)\sin(4\pi\lfloor\sqrt{j}\rfloor x_j)\|_{L^1(\mathbb{T})}}{\|\sin(2\pi x)\|^2_{L^1(\mathbb{T})}} \leqslant C_4 \prod_{j=n_0}^{n} (1 - \frac{c}{j}) \xrightarrow{n\to\infty} 0.$$

To show (4.12) we split the integral over the interval [0,1] into a sum of integrals over intervals of length equal to the period of the second function.

$$\begin{aligned}\int_0^1 |\sin(2\pi x)\sin(4\pi kx)|dx &= \frac{1}{2\pi}\int_0^{2\pi} |\sin(t)\sin(2kt)|dt\\ &= \frac{1}{\pi}\int_0^{\pi} \sin(t)|\sin(2kt)|dt\\ &= \frac{1}{\pi}\sum_{j=0}^{k-1}\int_{\frac{j}{k}\pi}^{\frac{j+1}{k}\pi} \sin(t)|\sin(2kt)|dt\\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{k}}\sum_{j=0}^{k-1}\sin(t+\frac{j\pi}{k})|\sin(2kt)|dt.\end{aligned}$$

By the formula for the sum of sines.

$$\begin{aligned}\int_0^1 |\sin(2\pi x)\sin(4\pi kx)|dx &= \frac{1}{\pi}\int_0^{\frac{\pi}{k}} \frac{\cos(t-\frac{\pi}{2k})}{\sin(\frac{\pi}{2k})}|\sin(2kt)|dt\\ &= \frac{1}{\pi^2}\int_0^{2\pi} \frac{\pi\cos(\frac{x-\pi}{2k})}{2k\sin(\frac{\pi}{2k})}|\sin(x)|dx\\ &\leqslant \frac{1}{\pi^2}\int_0^{2\pi} \frac{\pi\cos(\frac{\pi}{k})}{2k\sin(\frac{\pi}{2k})}|\sin(x)|dx\\ &= \frac{4}{\pi^2}+\int_0^{2\pi}\left(\frac{\pi\cos(\frac{\pi}{k})}{2k\sin(\frac{\pi}{2k})}-1\right)|\sin(x)|dx\end{aligned}$$

Since

$$\lim_{k\to\infty} k^2\left(\frac{\pi\cos(\frac{\pi}{k})}{2k\sin(\frac{\pi}{2k})}-1\right)=\frac{-11\pi^2}{24}$$

the Lebesgue dominated convergence theorem gives for sufficiently large k

$$\int_0^1 |\sin(2\pi x)\sin(4\pi kx)|dx \leqslant \frac{4}{\pi^2}\left(1-\frac{11\pi^4}{12k^2}\right).$$

4.3 Interesting counterexample

In this section we will find a family of sets $\mathscr{A}=\{A_j\}_{\mathbb{N}}$, sequence $\tau=\{\tau_j\}_{j\in\mathbb{N}}$ and $2\leqslant p<r<q$ such that $T_{\mathscr{A},\tau}$ (4.1) is an isomorphism between $L^p_E(\mathbb{T})$ and $L^p_F(\mathbb{T}^{\mathbb{N}})$ and between $L^q_E(\mathbb{T})$ and $L^q_F(\mathbb{T}^{\mathbb{N}})$ but $T_{\mathscr{A},\tau}$ is not an isomorphism between $L^r_E(\mathbb{T})$ and $L^r_F(\mathbb{T}^{\mathbb{N}})$ for every $p<r<q$. Moreover it fails to be an isomorphism for a dual exponent $r=\frac{q}{q-1}$.
We start by proving stronger version of the Theorem 4.3 for exponents q, which are even natural numbers, and sets $A_j=\{0,1\}$. We will show that in this case it is enough to have

geometrical growth of the sequence τ for T being isometry between spaces $L^q_E(\mathbb{T})$ and $L^q_F(\mathbb{T}^{\mathbb{N}})$. We proof auxiliary lemma which is counterpart of Lemma 4.9 for the special case $q = 2j$ and a sets of analytic polynomials.

Lemma 4.11. *For $j \in \mathbb{N}$,$q = 2j$ and f_1, f_0 analytic trigonometric polynomials on $\mathbb{T}$ satisfying $j \deg_z (f_i) < N_k$ we have*

$$\|f_0(z) + e^{2\pi i N_k z} f_1(z)\|_{L^q(\mathbb{T})} = \|f_0(z) + e^{2\pi i y} f_1(z)\|_{L^q(\mathbb{T}^2)} \qquad \forall\, z, y \in \mathbb{T}.$$

Proof. Let us observe this simple equality

$$\begin{aligned} \|f_0(z) + e^{2\pi i N_k z} f_1(z)\|_{L^{2j}(\mathbb{T})} &= \| \left(f_0(z) + e^{2\pi i N_k z} f_1(z)\right)^j \|^2_{L^2(\mathbb{T})} \\ &= \left\| \sum_{s=0}^{j} e^{2\pi i s N_k z} f_1^s(z) f_0^{j-s}(z) \begin{pmatrix} j \\ s \end{pmatrix} \right\|^2_{L^2(\mathbb{T})} \end{aligned} \tag{4.13}$$

Since

$$\deg_z \left(f_1^s(z) f_0^{j-s}(z)\right) \leqslant j \max\{\deg_z (f_0), \deg_z (f_1)\} < N_k$$

we have

$$\deg_z \left(f_1^s(z) f_0^{j-s}(z)\right) + sN_k < (s+1)N_{k+1}.$$

Hence every exponent occurs in no more than one element of the sum on the right hand side of (4.13), because f_0, f_1 are analytic. Changing $e^{2\pi i s N_k z}$ to $e^{2\pi i s y}$ in this sum is an injective operation on exponents. Hence from Plancherel's formula we have

$$\begin{aligned} \left\| \sum_{s=0}^{j} e^{2\pi i s N_k z} f_1^s(z) f_0^{j-s}(z) \begin{pmatrix} j \\ s \end{pmatrix} \right\|^2_{L^2(\mathbb{T})} &= \left\| \sum_{s=0}^{j} e^{2\pi i s y} f_1^s(z) f_0^{j-s}(z) \begin{pmatrix} j \\ s \end{pmatrix} \right\|^2_{L^2(\mathbb{T}^2)} \\ &= \| \left(f_0(z) + e^{2\pi i y} f_1(z)\right)^j \|^2_{L^2(\mathbb{T}^2)} \\ &= \|f_0(z) + e^{2\pi i y} f_1(z)\|_{L^{2j}(\mathbb{T}^2)} \end{aligned}$$

□

We proof following theorem

Theorem 4.12. *Let $q = 2j$ be an even natural number. For sequence of natural numbers $\{\tau_n\}_{n\in\mathbb{N}}$ and family of sets $\{A_n\}_{n\in\mathbb{N}}$ satisfying*

$$\begin{aligned} A_n &= \{0, 1\}, \\ \tau_{k+1} &\geqslant (j+1)\tau_k \qquad \forall k \in \mathbb{N}, \end{aligned} \tag{4.14}$$

the operator $T : L^q_F(\mathbb{T}^{\mathbb{N}}) \to L^q_E(\mathbb{T})$ given by the formula (4.1) *is an isometry.*

Proof. The line of argument is analogous to the proof of the Theorem 4.3. We take a trigonometric polynomial $f \in L^q_F(\mathbb{T}^{\mathbb{N}})$ which depends only on first s variables. We define polynomials w_k the same way as in Theorem 4.3. So $w_1(z) = Tf(z)$ and they satisfy formula (4.10). Since $A_j = \{0,1\}$ we use following representation of the polynomial w_k

$$w_k(y', z) = g_{k,0}(y', z) + e^{2\pi i \tau_k z} g_{k,1}(y', z) \qquad \forall\, y \in \mathbb{T}^k\ \forall z \mathbb{T}$$

and for $j \in \{0,1\}$

$$\deg_z (g_{k,j}) \leqslant \sum_{u=1}^{s-k+1} \tau_u \leqslant \sum_{u=1}^{k-1} (j+1)^{-u} \tau_{s-k+1} < \tau_{s-k+1} \frac{j+1}{j} < \frac{1}{j} \tau_{s-k+2}$$

Hence from Lemma 4.11 we get

$$\begin{aligned} \|w_{k+1}(y', y_k, z)\|^p_{L^p(\mathbb{T}^{k+2})} &= \int_{\mathbb{T}^{k-1}} \|w_{k+1}(y', \cdot, \cdot)\|^p_{L^p(\mathbb{T}^2)} dy' \\ &= \int_{\mathbb{T}^{k-1}} \|g_{k,0}(y', z) + e^{2\pi i y_k} g_{k,1}(y', z)\|^p_{L^p(\mathbb{T}^2)} dy' \\ &= \int_{\mathbb{T}^{k-1}} \|g_{k,0}(y', z) + e^{2\pi i \tau_k z} g_{k,1}(y', z)\|^p_{L^p(\mathbb{T})} dy' \\ &= \|w_k(y', z)\|^p_{L^p(\mathbb{T}^k)} \end{aligned}$$

We remember that $\|w_s\|_{L^p(\mathbb{T}^s)} = \|f\|_{L^p_F(\mathbb{T}^{\mathbb{N}})}$ and we get

$$\|Tf\|_{L^p_E(\mathbb{T})} = \|w_1\|_{L^p(\mathbb{T})} = \|w_2\|_{L^p(\mathbb{T}^2)} = \cdots = \|w_s\|_{L^p(\mathbb{T}^s)} = \|f\|_{L^p_F(\mathbb{T}^{\mathbb{N}})}$$

□

If we take $p = 2$ and $q = 4$, $A_j = \{0,1\}$ and sequence

$$\tau_k = \begin{cases} (l!)^2 & k = 2l, \\ (l!)^2\, 5 & k = 2l+1. \end{cases} \tag{4.15}$$

Obviously sequence τ and family $\mathscr{A}$ satisfy assumptions of Theorem 4.12 for $p = 2$ and $q = 4$. Now we adapt the reasoning from third section to this case. For every $n \in \mathbb{N}$ we take trigonometric polynomial f

$$f_n(x) = \prod_{j=1}^{2n} (1 + e^{2\pi i\, \tau_j\, x}) \qquad \forall x \in \mathbb{T}.$$

Once again since if $T_{\mathscr{A},\tau}$ is an isomorphism for the sequence τ_k and L^r-norm, we get

$$C_1^{-1} \|1 + e^{2\pi i x}\|^{2n}_{L^r(\mathbb{T})} \leqslant \|f_n\|_{L^r(\mathbb{T})} \leqslant C_1 \|1 + e^{2\pi i x}\|^{2n}_{L^r(\mathbb{T})}$$

Let us observe that

$$f_n(x) = \prod_{j=1}^{n} (1 + e^{2\pi i\, \tau_{2j}\, x})(1 + e^{2\cdot 5\pi i\, \tau_{2j}\, x}) \qquad \forall x \in \mathbb{T}.$$

Let $\widetilde{\tau} = \{\tau_{2k}\}_{k\in\mathbb{N}}$ and $\mathscr{B} = \{B_k\}_{k\in\mathbb{N}}$, where $B_k = \{0, 1, 5, 6\}$. We have chosen τ_{2k} in such a way that the assumptions of the Theorem 4.3 are satisfied for $T_{\mathscr{B},\widetilde{\tau}}$. Similarly as in last section we get

$$C_2^{-1}\|(1+e^{2\pi ix})(1+e^{2\cdot 5\pi ix})\|^n_{L^r(\mathbb{T}^2)} \leqslant \|f_n\|_{L^r(\mathbb{T})} \leqslant C_2\|(1+e^{2\pi ix})(1+e^{2\cdot 5\pi ix})\|^n_{L^r(\mathbb{T}^2)}$$

Therefore for every natural number n we get with constant $C_3 > 0$ independent on n s.t.

$$C_3^{-1} \leqslant \left(\frac{\|(1+e^{2\pi ix})(1+e^{2\cdot 5\pi ix})\|_{L^r(\mathbb{T}^2)}}{\|1+e^{2\pi ix}\|^2_{L^r(\mathbb{T})}}\right)^n \leqslant C_3.$$

Above inequality could be satisfied for every $n \in \mathbb{N}$ if only if

$$\|(1+e^{2\pi ix})(1+e^{2\cdot 5\pi ix})\|_{L^r(\mathbb{T}^2)} = \|1+e^{2\pi ix}\|^2_{L^r(\mathbb{T})}.$$

From numerical approximation it follows that

$$\begin{aligned}\|1+e^{2\pi ix}\|^2_{L^3(\mathbb{T})} &\approx 2.25901,\\ \|(1+e^{2\pi ix})(1+e^{2\cdot 5\pi ix})\|_{L^3(\mathbb{T}^2)} &\approx 2.25812\end{aligned}$$

and

$$\begin{aligned}\|1+e^{2\pi ix}\|^2_{L^{\frac{4}{3}}(\mathbb{T})} &\approx 1.76593473,\\ \|(1+e^{2\pi ix})(1+e^{2\cdot 5\pi ix})\|_{L^{\frac{4}{3}}(\mathbb{T}^2)} &\approx 1.77176422.\end{aligned}$$

Hence for $r = 3$ and $r = 4/3$ from (4.15) we get that $T_{\mathscr{A},\tau}$ of the form (4.1) fails to be an isomorphism.

Chapter 5

Trace operator and its right inverse on planar domains

It was shown by Gagliardo ([16]) that the trace operator transforms space $W_1^1(\Omega)$ onto $L^1(\partial\Omega)$ for domains with regular boundary. From this theorem imediately arises a question whether there exists a right inverse operator to the trace, i.e. a continuous, linear operator $S : L^1(\partial\Omega) \to W_1^1(\Omega)$ s.t. $Tr \circ S = Id$. It turns out that in general such operator does not exist. This was proved by Peetre ([39]). In his paper he has shown the non-existence of right inverse to the trace for a half plane. From that by straightening the boundary one can deduce non-existence for Ω with a smooth boundary. More recent proofs can be found in [41], [5]. In this chapter we present an extraordinary simple proof based on geometry of a Whitney covering and basic properties of classical Banach spaces.

Theorem 5.1. *Let Ω be an open domain with Lipschitz boundary and $\partial\Omega$ be a Jordan curve. Let $Tr : W_1^1(\Omega) \to L^1(\partial\Omega)$ be a trace operator. Then there is no continuous, linear operator $S : L^1(\partial\Omega) \to W^{1,1}(\Omega)$ s.t. $TS = Id_{L^1(\partial\Omega)}$.*

In [17] Hajlasz and Martio studied the existence of a right inverse to trace operator in the case of Sobolev spaces $W_1^p(\Omega)$ for $p > 1$. They characterize trace space as a generalized Sobolev space. However this characterization does not work for p=1. The behavior of the trace space changes dramatically for the domains with fractal boundary. In the third section we use the structure of a specific Whitney covering of Ω_K - domain bounded by the von Koch's curve, we show that in this case the trace space of $W_1^1(\Omega_K)$ is isomorphic to Arens-Eels space with a suitable metric. Surprisingly, based on this observation we are able to construct a right inverse operator to the trace operator. The theorem below is the main result of this chapter.

Theorem 5.2. *Let $Tr : W_1^1(\Omega_K) \to X(\Omega_K)$ be a trace operator, where $X(\Omega_K)$ is a trace space* (5.2)*. There exists a continuous, linear operator $S : X(\Omega_K) \to W_1^1(\Omega_K)$ s.t. $Tr \circ S = Id_{X(\Omega_K)}$.*

In the following section we define the trace operator, trace space and auxiliary properties $BV(\Omega)$ needed in the proof.

5.1 Properties of $BV(\Omega)$ and trace operator

From now on we assume that $\Omega \subset \mathbb{R}^2$, $\partial\Omega$ is a Jordan curve. Our approach to Theorem 5.1 up to technical differences works in higher dimensions. However in the proof of the Theorem 5.2 the properties of two dimensional euclidean space are crucial. We define the trace operator and the trace space for $W^1_1(\Omega)$. Let us recall a notion of (slightly generalized) Whitney covering of Ω.

Definition 5.3. *We call the family of polygons $\mathcal{A}$ a Whitney decomposition of an open set $\Omega \subset \mathbb{R}^2$ if it satisfies:*

1. *For $A \in \mathcal{A}$ the boundaries ∂A are uniformly bi-lipschitz.*
2. *$\bigcup_{Q \in \mathcal{A}} A = \Omega$ and elements of $\mathcal{A}$ have pairwise disjoint interiors.*
3. *$C^{-1} \operatorname{vol}_2 A \leqslant dist(A, \partial\Omega)^n \leqslant C \operatorname{vol}_2(A)$*
4. *If $\partial A \cap \partial B$ has a positive one dimensional Hausdorff measure then*
   (a) *$C^{-1} \leqslant \frac{\operatorname{vol}_2(A)}{\operatorname{vol}_2(B)} \leqslant C$.*
   (b) *$C^{-1} \leqslant \frac{l(\partial A)}{l(\partial B)} \leqslant C$*
   (c) *$C^{-1} l(\partial A) \leqslant l(\partial A \cap \partial B) \leqslant C^{-1} l(\partial A)$,*

   where $l(\cdot)$ denotes length of a curve, and vol_2 denotes the area of the polygon.
5. *For a given polygon $A \in \mathcal{A}$ there exists at most N polygons $B \in \mathcal{A}$ s.t. $\partial A \cap \partial B \neq \emptyset$.*

For the purpose of this chapter we will also assume that polygons of $\mathcal{A}$ are uniformly star shaped in the following sense

6. *For every $A \in \mathcal{A}$ there exists a point $x \in A$ and positive numbers λ, τ s.t. $B(x, \lambda) \subset A \subset B(x, \lambda)$, $\frac{\lambda}{\tau}$ is fixed and the polygon is star shaped with respect to x. We call such point a center of A.*

Let $\mathcal{A}$ be such covering then we can define a graph describing it's geometry.

Definition 5.4. *Let $\mathcal{A}$ be a Whitney decomposition. We call a graph $G := G(\mathcal{A}) = (V(\mathcal{A}), E(\mathcal{A})) =: (V, E)$ a graph of $\mathcal{A}$ if $V := \mathcal{A}$ and $\{A, B\} \in E$ only if boundaries of A and B have intersection of positive one dimensional Hausdorff measure.*

Then we introduce a special subspace of $BV(\Omega)$.

Definition 5.5. *Let $\mathcal{A}$ be a Whitney decomposition of Ω. We define the following subspaces of $BV(\Omega)$*

$$BV_{\mathcal{A},0} = \{F \in BV(\Omega) : \forall\, A \in \mathcal{A} \quad \int_A F(x)dx = 0\}$$

and

$$BV_G = \{f \in BV(\Omega) : \forall\, A \in \mathcal{A} \quad f|_A = f_A \in \mathbb{R}\}$$

It is a known fact that for a given Whitney decomposition the space $BV_{\mathcal{A},0}$ is a complemented subspace of $BV(\Omega)$. A proof of this fact can be found in ([43],[14]).

Lemma 5.6. *For any domain Ω:*

$$BV(\Omega) = BV_{\mathcal{A},0} \oplus BV_G.$$

Let us observe that we can easily calculate the norm of function $f \in BV_G$.

$$\|f\|_{BV_G} := \|f\|_{BV(\Omega)} \simeq \sum_{A \in V} |f_A| \ \mathrm{vol}_2(A) + \sum_{\{A,B\} \in E} |f_A - f_B|\, l(\partial A \cap \partial B)$$

In their unpublished preprint Derezinski, Nazarov, Wojchiechowski [15] have proven that there is a spanning tree of the graph $G(\mathcal{A})$ with a desirable properties i.e.

Lemma 5.7. *If Ω is simply connected planar domain and A is its Whitney decomposition. Then there exists spanning tree $T = (V_T, E_T)$ of the graph $G(\mathcal{A})$ s.t.*

1. *for every $f \in \dot{BV}_G(\Omega)$*

$$\|f\|_{BV_G} \simeq \|f\|_{BV_T} := \sum_{A \in V_T} |f_A| \ \mathrm{vol}_2(A) + \sum_{\{A,B\} \in E_T} |f_A - f_B|\, l(\partial A \cap \partial B) \tag{5.1}$$

2. *for every point x on the boundary there is a infinite branch $\mathrm{br}(x)$ of T s.t. $\mathrm{br}(x) \cong \mathbb{Z}_+$ and $\mathrm{dist}(A_n, x) \to 0$ as $n \to \infty$, where $A_n \in \mathrm{br}(x)$. For a sequence of real numbers $\{a_{A_n}\}$ we call a limit $\lim_{n\to\infty} a_{A_n}$ a limit along the branch $\mathrm{br}(x)$.*

We will call such tree a Whitney tree of $\mathcal{A}$

It follows immediately that $BV_G \cong BV_T$, where BV_T is a set BV_G with the norm $\|\cdot\|_{BV_T}$. Using the above notation we define trace of $f \in W_1^1(\Omega)$. Since Ω is a domain with a Jordan curve as boundary it follows from Koskela, Zhang theorem ([28]) that restrictions of Lipschitz function $\mathrm{Lip}(\mathbb{R}^2)$ are dense in $W_1^1(\Omega)$. For $f \in C(\overline{\Omega}) \cap W_1^1(\Omega)$ we define the trace operator as a restriction of f to the boundary. We define a trace space $X(\Omega)$ as completion of a space $Tr(C(\overline{\Omega}) \cap W_1^1(\Omega))$ with respect to the norm $\|\cdot\|_X$, where

$$\|g\|_{X(\Omega)} := \inf\{\|f\|_{W_1^1(\Omega)} : Trf = g \text{ and } f \in C(\overline{\Omega}) \cap W_1^1(\Omega)\}. \tag{5.2}$$

Since Lipschitz functions on Ω are dense in $W_1^1(\Omega)$ we can define trace operator on a whole space $W_1^1(\Omega)$. It is obvious that $Tr : W_1^1(\Omega) \to X(\Omega)$ is a continuous linear operator and it is surjective. We want to extend the trace operator to the $BV(\Omega)$.

Lemma 5.8. *There exists a continuous, linear operator $\Phi : BV_G \to W_1^1(\Omega)$ s.t. for every $A \in \mathcal{A}$*

$$f_A = \fint_A f(y)dy = \fint_A \Phi(f)(y)dy + o(\mathrm{dist}(A, \partial\Omega)). \tag{5.3}$$

Proof. Let ϕ be a mollifier, i.e. $\phi \in C^\infty(\mathbb{R}^2, \mathbb{R}_+)$, $\operatorname{supp}\phi \subset B(0,1)$ and $\int_{B(0,1)} \phi = 1$. We define an operator Φ with the formula

$$\Phi(f)(x) = \int_\Omega f(x-t)\phi\left(\frac{t}{c\,\mathrm{dist}^2(x,\Omega)}\right)\frac{1}{c^2\,\mathrm{dist}^4(x,\Omega)}dt$$

This formula defines a continuous operator from BV_G to $W_1^1(\Omega)$. Let us observe that by the definition of Φ, $\Phi(f)(x) = f_A$ for every $x \in H(A, 1 - c\,\mathrm{dist}(\mathrm{x},\Omega)^2)$ which implies (5.3). Therefore the trace spaces of $W_1^1(\Omega)$ and $BV(\Omega)$ are identical and $Tr : BV(\Omega) \to X(\Omega)$. $\square$

Let $P : BV(\Omega) \to BV_G$ be a projection from $BV(\Omega)$ onto BV_G. We define $\widetilde{Tr} : BV(\Omega) \to X(\Omega)$ by the formula

$$\widetilde{Tr} f = Tr\,\Phi\,(Pf) \qquad \forall\, f \in BV(\Omega).$$

If $f \in C(\overline{\Omega}) \cap W_1^1(\Omega)$ then the function $\Phi\,(Pf)$ is continuous on $\overline{\Omega}$. Therefore its trace is a restriction of $\Phi\,(Pf)$ to the boundary. However the value of the restriction at point $x \in \partial\Omega$ for the function from $C(\overline{\Omega})$ is equal to the limit of $\fint_A \Phi(Pf(y))dy$ along the branch $\mathrm{br}(x)$. From (5.3) and the definition of the space BV_T

$$\Phi(P(f))(x) = f(x) \qquad \forall\, x \in \partial\Omega.$$

Since $f \in C(\overline{\Omega}) \cap W_1^1(\Omega)$ are dense in $W_1^1(\Omega)$ and $\widetilde{Tr} f = Trf$ the operator $\widetilde{Tr}$ is an extension of the trace operator to $BV(\Omega)$. We will abuse the notation and from now on we will denote $\widetilde{Tr}$ by Tr. From the definition of trace it follows that

$$Trf = 0 \qquad \forall\, f \in BV_{\mathcal{A},0} \tag{5.4}$$

5.2 Proof of Peetre's theorem

In this section we will give a proof of Theorem 5.1.

Proof. Since Ω has Lipschitz boundary by theorem of Gagliardo $X(\Omega) \cong L^1(\partial\Omega)$ - space of functions integrable with respect to the 1-dimensional Hausdorff measure. Let us denote by $P : BV(\Omega) \to BV_G$ the projection onto BV_G. Assume there exist $S : L^1(\partial\Omega) \to W_1^1(\Omega) \subset BV(\Omega)$ s.t. $Tr \circ S = Id_{L^1(\partial\Omega)}$. Then the following diagram is commutative

$$\begin{tikzcd}
L^1(\partial\Omega) \arrow[r, "S"] & BV(\Omega) \arrow[r, "Tr"] \arrow[d, "P"'] & L^1(\partial\Omega) \\
& BV_G \arrow[ur, "Tr"'] &
\end{tikzcd}$$

From (5.4) and Gagliardo theorem we conclude that $Tr|_{BV_{\mathcal{A}}(\Omega)}$ is onto $L^1(\partial\Omega)$. On the other hand, $Tr \circ P \circ S = Id_{L^1(\partial\Omega)}$. Hence $L^1(\partial\Omega)$ is isomorphic to a subspace of BV_G. The definition of BV_G implies that BV_G is isomorphic to a subspace of $\ell^1(V) \oplus \ell^1(E) \cong \ell^1$. Since the measure on the boundary is non atomic, $L^1(\partial\Omega) \cong L^1(\mathbb{T})$. However, it is well known that L^1 could not be embedded in ℓ^1. (To see this, note that by Khintchine inequality, Radamacher functions span ℓ^2 in L^1 space. The space ℓ^2 could not be embedded in ℓ^1 because, every subspace of ℓ^1 contains a copy ℓ^1 ([32], Proposition 1.a.11). $\square$

5.3 Trace operator on von Koch's snowflake

Let Ω_K be a domain bounded by von Koch's curve. Since Ω_K is simply connected and von Koch's curve is a Jordan curve, we can use all the properties from the first section. It is enough to show that there exists a right inverse $S : X(\Omega) \to BV_G$ to the trace on BV_G because then $\Phi \circ S : X(\Omega) \to W_1^1(\Omega)$ and $Tr \circ \Phi \circ S = Id_{X(\Omega)}$, where Φ is an operator from Lemma 5.8.
It is a well known fact that Ω_K satisfies Poincare inequality (eg. [6]). Therefore

$$\left\| f - \fint_{\Omega_K} f(y)dy \right\|_{L^1(\Omega)} \leqslant |\nabla f|_\Omega,$$

where $|\mu|_\Omega$ is a total variation of a measure μ on Ω. This inequality implies

$$BV_G \cong \dot{BV}_T \oplus \mathbb{R},$$

where

$$\dot{BV}_T = \{f \in BV_G : \fint_{\Omega_K} f(y)dy = 0\}$$

with the total variation of gradient as the norm. In this case the norm is equal to

$$\|f\|_{\dot{BV}_T} = \sum_{\{A,B\}\in E_T} |f_A - f_B| l(\partial A \cap \partial B).$$

Similarly $X(\Omega) = \mathbb{R} \oplus \dot{X}(\Omega)$ for a quotient space $\dot{X}(\Omega) = X/P_0$, where P_0 is the space of constant functions on Ω_K. We reduce the problem to finding the right inverse operator to the trace $Tr : \dot{BV}_T \to \dot{X}(\Omega)$. We know that for all $g \in \dot{X}(\Omega)$,

$$\|g\|_{\dot{X}(\Omega)} = \inf\{\|f\|_{\dot{BV}_T} : Trf := g\}$$

We will show that for a carefully chosen Whitney covering. We introduce the following notation

Definition 5.9. *For a given tree T by $R := R(T)$ we will denote the root of T. For a vertex $A \in V_T$ by $D_n(A)$ we denote descendants of A of order exactly n and we put $D_n = D_n(R)$. For a vertex $A \in V_T$ by $A \downarrow$ we denote its unique father. We will denote by $D\uparrow(A)$ the set of all descendants of A i.e. $D\uparrow(A) = \bigcup_n D_n(A)$.*

We take a covering $\mathcal{A}_{\mathcal{K}}$ as shown on the Figure 5.1. This covering of von Koch's snowflake is easy to describe if we look at its Whitney tree T_K. The root of T_K is a six pointed star with six "pants" shaped descendants. We denote it by R. In this tree there are three types of polygons/vertices. The aforementioned root, "pants" shaped polygons and "palace" shaped polygons. The type of a vertex describes direct descendants of this vertex (Figure 5.2). Polygons in D_{n+1} are similar to polygons from D_n with a scale $\frac{1}{3}$. The tree T_K is the tree from Lemma 5.7. Hence for such Whitney covering the norm of $\dot{BV}_{T_K}$ satisfies

$$\|f\|_{\dot{BV}_{T_K}} \simeq \sum_{n=1}^{\infty} \sum_{A\in D_n} |f_A - f_{A\downarrow}| 3^{-n}$$

Further we will use above formula as a norm on $\dot{BV}_{T_K}$. We want to study the norm on $\dot{X}(\Omega)$. To be precise, we want to define and calculate the norm of $\|\sum_j a_j \mathbb{1}_{[x_j,y_j]}\|_{\dot{X}(\Omega)}$.

Definition 5.10. *Let us denote by $D_\infty(A)$ a cylinder of A, i.e. $D_\infty(A) = \{x \in \partial\Omega : A \in \mathrm{br}(x)\}$. We call an arc rational if there exists a finite sequence $A_1, ..., A_k \in V_{T_K}$ s.t. $[x, y] := \cup_{n=1}^k D_\infty(A_n)$ and we say that points x,y are rational points.*

For a given arc $[x, y]$ there exist a sequence of vertices $A_k \in V_{T_K}$ s.t. $[x, y] = \bigcup_k D_\infty(A_k)$ and sets $D_\infty(A_k)$ are pairwise disjoint. Moreover this sequence can be taken maximal in the sense that if vertex A is in the sequence then there exists $z \in D_\infty(A\downarrow)$, which is

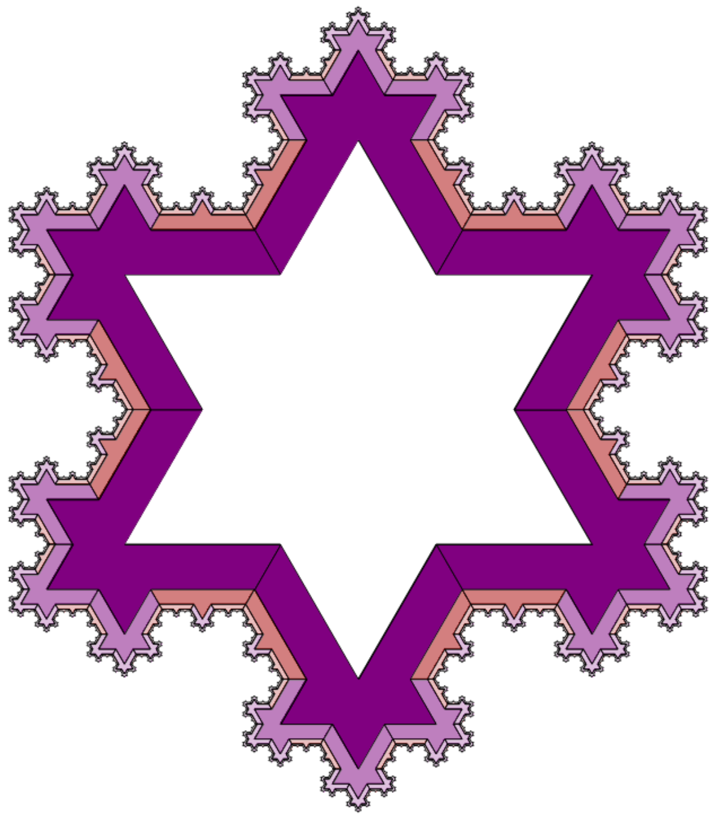


Figure 5.1: Self similar Whitney decomposition of von Koch's snowflake

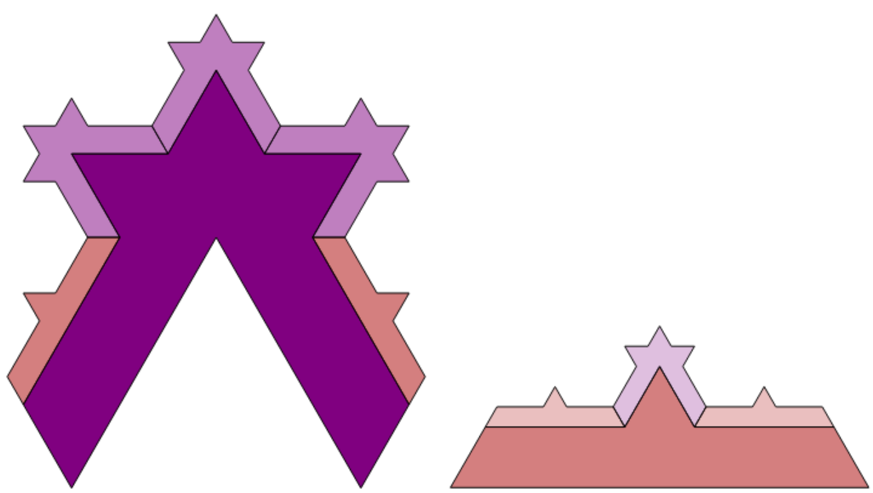


Figure 5.2: On the left "pants" shaped polygon and its descendants, on the right "palace" shaped polygon and its descendants

not in $[x,y]$. Such sequence is unique for $[x,y]$. Let $n(k)$ be a natural number such that $A_k \in D_{n(K)}$. We introduce a auxiliary metric on the boundary $\partial\Omega_K$

$$d_K(x,y) = \sum_k 3^{-n(k)}.$$

It is easy to check that $d_K(x,y)$ is a metric equivalent to the two dimensional euclidean metric. We prefer this metric over euclidean metric because it is a monotone function on an arc $[x,y]$ with respect to the natural order on the arc.

Definition 5.11. *We call a function F linear on the arc $[x,y]$ if there exists $a \in \mathbb{R}$ such that $F(t) = a\, d_K(x,t)$ for every $t \in [x,y]$.*

In the lemma below we show that for every rational arc and linear function on this arc there exists its "good" extension to $\dot{BV}_{T_K}$.

Lemma 5.12. *For every rational points $x, y \in \partial\Omega$, let function F be a linear function on the arc $[x,y]$. There exists $h \in \dot{BV}_{T_K}$ such that*

1. $\|h\|_{\dot{BV}(\Omega)} \lesssim \|F\|_\infty d_K(x,y)$,
2. $F(z) = \lim\limits_{\substack{A\in \mathrm{br}(z)\\ A\to z}} ⨍ h(y)dy \qquad \forall\, z \in \partial\Omega.$

Proof. Without loss of generality we assume that $F(t) = d_K(t,x)$. Every rational arc $[x,y]$ can be written as a finite sum $\bigcup_{k=1}^M D_\infty(A_k)$ in a unique way s.t. sets $D_\infty(A_k)$ are pairwise disjoint and the sequence A_k has the minimal cardinality of all sequence which cover $[x,y]$. From this assumption it is clear that $\#\{A_k : A_k \in D_n\} \leqslant 10$. Let us put

$$h_A = \begin{cases} \inf\{F(z) : \mathrm{br}(z) \in D_\infty(A)\} & A \in \cup_k D\!\uparrow\!(A_k), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly along every infinite branch $\mathrm{br}(z)$ the limit of $\lim\limits_{\substack{A\in \mathrm{br}(z)\\ A\to z}} h_A$ exists and it is equal to $F(z)$. We need to estimate the total variation of h. Observe that due to linearity of the function F there are positive numbers $\{b_i\}_{i=1}^5, \{c_i\}_{i=1}^3$ s.t. for every pants shaped polygon $A \in D_n \cap \bigcup_{k=1}^M D_\infty(A_k)$ we have

$$\frac{1}{3^n} \sum_{Q\in D_1(A)} |h(A) - h(Q)| = \frac{1}{3^n} \sum_{i=1}^5 \frac{b_i}{3^n} \leqslant \max_i b_i \frac{\#D_1(A)}{9^n}.$$

Similarly for palace shaped polygon B

$$\frac{1}{3^n} \sum_{Q\in D_1(B)} |h(B) - h(Q)| = \frac{1}{3^n} \sum_{i=1}^3 \frac{c_i}{3^n} \leqslant \max_i c_i \frac{\#D_1(B)}{9^n}.$$

Let $\rho := \max\{b_1, \ldots, b_5, c_1, c_2, c_3\}$. We can prove inductively that $\#D_j(A) \lesssim 4^j$. Therefore we have following estimate on the variation on the subtree $D\!\uparrow(A_k)$, starting with $A\!\downarrow$

$$\begin{aligned}\frac{1}{3^{n(k)-1}}|f_{A_k} - 0| + \sum_{i=1}^{\infty} \sum_{Q\in D_i(A_k)} |h_B - h_{B\downarrow}| \frac{1}{3^{n(k)-1+i}} &\leqslant \rho \sum_{j=n(k)}^{\infty} \frac{\#D_{j-n(k)}(A)}{9^j} \\ &\lesssim \sum_{j=n(k)}^{\infty} \frac{4^{j-n(k)}}{9^j} \lesssim \frac{1}{9^{n(k)}} \\ &\lesssim \|F\|_\infty \frac{1}{3^{n(k)}}\end{aligned}$$

We sum above inequalities over all A_k and we get

$$\|h\|_{\dot{BV}(\Omega)} \leqslant \|F\|_\infty d_K(x, y).$$

□

In the lemma below we prove the existence of a class of functions in $\dot{BV}$, which have desirable properties and every function from this class provides a good approximation of the norm of its trace on the boundary.

Lemma 5.13. *Let $x, y \in \partial\Omega$. There are a sequences of functions $f_n \in \dot{BV}(\Omega)$, $g_n \in C(\overline{\Omega_K}) \cap \dot{BV}(\Omega_K)$, and $h_n \in \dot{BV}(\Omega)$ s.t.*

1. *$f_n = h_n + g_n$,*
2. *For every $z \in \partial\Omega$, $\mathbb{1}_{[x,y]}(z) = \lim_{\substack{A\in \mathrm{br}(z)\\ A\to z}} ⨍_A f(y)dy$,*
3. *$\|g_n\|_{\dot{BV}(\Omega)} \leqslant (1 + \frac{1}{n^2})\|Trg_n\|_{\dot{X}(\Omega_K)}$.*
4. *Trg_n is a Cauchy sequence in $\dot{X}(\Omega)$*
5. *$\|h_n\|_{\dot{BV}(\Omega)} \leqslant \frac{1}{n^2}$.*
6. *$\|f_n\|_{\dot{BV}(\Omega)} \leqslant (1 + \frac{1}{n^2})\|Trg_n\|_{\dot{X}(\Omega_K)} + \frac{1}{n^2}$.*

Proof. We use Lemma 5.12. For every ε and every rational arc $[x, y]$, the characteristic function of $[x, y]$ can be written as sum of a Lipschitz function g and a four linear functions $p_1, \ldots, p_4$, with supports in arcs $[t_1, x]$, $[x, s_1]$, $[t_2, y]$, $[y, t_2]$ respectively. Moreover t_i, s_i are rational, $|t_i - s_i| \leqslant \varepsilon$ and the linear functions p_i are bounded uniformly with respect to

ε. Hence from the above lemma for every linear function p_i there exists a function f^i s.t $\|f^i\|_{\dot{BV}(\Omega)} \leqslant C\varepsilon$ and for every $z \in \partial\Omega$

$$\lim_{\substack{A\in \mathrm{br}(z)\\ A\to z}} f^i_A = p_i(z).$$

Any Lipschitz extension of g to Ω_K is in $W^1_1(\Omega_K)$. Hence g is in the trace space. From the definition of the trace space there exists a $g_\varepsilon \in C(\overline{\Omega_K}) \cap \dot{BV}(\Omega_K)$ such that

$$\|g_\varepsilon\|_{\dot{BV}(\Omega_K)} \leqslant (1+\varepsilon)\|g\|_{\dot{X}(\Omega_K)},$$
$$Trg_\varepsilon = g.$$

Since g_ε is in $C(\overline{\Omega})$ we have $\lim_{\substack{A\in \mathrm{br}(x)\\ A\to x}} ⨍_A g_\varepsilon(y)dy = g(x)$. Therefore the function $f = g_\varepsilon + \sum_{i=1}^4 f^i = g_\varepsilon + h_\varepsilon$ has desired properties. The limits along $\mathrm{br}(z)$ of $⨍_A g_\varepsilon(y)dy$ exists and are equal to $\mathbb{1}_{[x,y]}(z)$ for every $z \in \partial\Omega$ and

$$\|f\|_{\dot{BV}} \leqslant (1+\varepsilon)\|g\|_{\dot{X}(\Omega_K)} + C\varepsilon,$$

where the term $C\varepsilon$ is the estimate on the norms of the functions f^i. For every n we choose suitable ε and we get desired properties. The sequence Trg_n is Cauchy sequence. Indeed for a given function g_n and $m > n$ there exists a continuous piecewise linear function q with support on a small set on the boundary s.t.

$$q + Trg_n = Trg_m,$$

From Lemma 5.12 there exist a function $\tilde{q} \in \dot{BV}(\Omega)$ with a small norm such that

$$Tr(g_n + \tilde{q}) = Trg_m$$

The size of the support of q depends only on g_n. Therefore

$$\|Trg_n - Trg_m\|_{\dot{X}(\Omega)} \leqslant \epsilon$$

for sufficiently large n, m. □

The Cauchy sequence $\{g_n\}$ defines an element in $g \in \dot{X}(\Omega)$. From the analogous argument as in the above Lemma if $f \in \dot{BV}(\Omega)$ satisfies $\mathbb{1}_{[x,y]}(z) = \lim_{\substack{A\in \mathrm{br}(z)\\ A\to z}} ⨍_A f(y)dy$ for every z on the boundary then $Trf = g$. To simplify notation we denote $g = \mathbb{1}_{[x,y]}$. From the point 6. of the Lemma 5.13 it follows

$$\|g\|_{\dot{X}(\Omega)} = \lim_{n\to\infty} \|Trg_n\|_{\dot{X}(\Omega)} = \lim_{n\to\infty} \|f_n\|_{\dot{BV}(\Omega)}$$

Since the projection from $\dot{BV}$ onto $\dot{BV}_{T_K}$ preserves the trace, we may assume that functions f_n are from $\dot{BV}_{T_K}$. Therefore the function $g = \sum_j a_j \mathbb{1}[x_j, y_k]$, whose arcs $[x_j, y_j]$ are rational, satisfies

$$\|g\|_{\dot{X}(\Omega_K)} \simeq \inf\{\|f\|_{\dot{BV}_{T_K}} : f \in L \text{ and } Trf = g\},$$

where $L \subset \dot{BV}_{T_K}$ consists of such f that the limit $\lim\limits_{\substack{A \in \mathrm{br}(x) \\ A \to x}} f_A$ exist for every $x \in \partial\Omega$ and it is equal to $Trf(x)$.

Remark 5.14. *In the above lemmas we abuse the notation a bit. For rational points x there are two branches $\mathrm{br}(x)$. If we look at a finite linear combination of characteristic functions of arcs, the are finitely many points (endpoints of segments) on which the limits over this two the branches are different. However they are equal to the value of the trace either on left or right side of that endpoint. Further in the chapter we are only interested in branches which contain some specific vertex A. Hence we are interested only in one of the problematic branches and it is clear what we mean by the limit.*

We want to characterize the space $\dot{X}(\Omega_K)$. We introduce, a metric on von Koch's curve by formula

$$\tilde{d}(x,y) := \|\mathbb{1}_{[x,y]}\|_{\dot{X}(\Omega_K)},$$

where $\mathbb{1}_{[x,y]}$ is a characteristic function of an arc on the von Koch's curve which connects x and y. It does not matter which one of the two arcs we take because the difference between their characteristic functions is a constant. Further in the proof it will be clear which one of arc is considered. Since $\|\cdot\|_{\dot{X}(\Omega_K)}$ is a norm, $\tilde{d}$ is a metric on the boundary. For a given metric space (Y, d_Y) we define the Arens-Eels space ([52]).

Definition 5.15. *Let (Y, d_Y) be a metric space. We call a function $f : Y \to \mathbb{R}$ a molecule if it has finite support and $\sum_{y \in Y} f(y) = 0$. Let $x, y \in Y$. We define special type of a molecule - an atom : $m_{xy} = \mathbb{1}_x - \mathbb{1}_y$, where $\mathbb{1}_a$ is a characteristic of a set a. Let m be a molecule, i.e. $m = \sum_{j=1}^{M} a_j m_{x_j y_j}$, then the Arens-Eels norm of m is*

$$\|m\|_{AE(d_Y)} = \inf\left\{\sum_j |a_j| d_Y(x_j, y_j) : m := \sum_j a_j m_{x_j y_j}\right\},$$

where the infimum is taken over all possible representations of m as a sum of m_{pq}. The Aerens-Eels space is the completion of molecules with respect to the norm $\|\cdot\|_{AE}$.

We want to show that $\dot{X}(\Omega_K)$ is isomorphic to the Arens-Eels space with the metric $\tilde{d}$. We will denote by $M(\tilde{d})$ linear space of molecules. It is a non complete norm space. By the definition its dense it is dense in $AE(\tilde{d})$. We define the candidate for the isomorphism on the a linearly dense subsets of both spaces. We set $\Psi : AE(\tilde{d}) \to \dot{X}(\Omega_K)$ by the formula

$$\Psi(m_{xy}) = \mathbb{1}_{[x,y]} \qquad \forall\, x, y \in \partial\Omega_K. \tag{5.5}$$

Lemma 5.16. $\Psi : AE(\tilde{d}) \to \dot{X}(\Omega_K)$ *is an isomorphism between Banach spaces.*

Proof. By triangle inequality and the definitions of $\tilde{d}(x,y)$ and Arens-Eels space, it follows that Ψ is continuous

$$\|\Psi(f)\|_{\dot{X}(\Omega_K)} \leqslant \|f\|_{AE(\tilde{d})}. \tag{5.6}$$

In the trace space we have following density result.

Lemma 5.17. $\Phi(M(\tilde{d}))$ *is dense in* $\dot{X}(\Omega_K)$.

Proof. From [28] we know that restrictions of Lipschitz functions on $\mathbb{R}^2$ are dense in $W_1^1(\Omega_K)$. Therefore Lipschitz functions are dense in $\dot{X}(\Omega_K)$. Hence for any $f \in \dot{X}(\Omega_K)$ there exist a sequence of Lipschitz functions f_n s.t.

$$\lim_{n\to\infty} \|f - f_n\|_{\dot{X}(\Omega_K)} = 0.$$

So it is enough to approximate Lipschitz functions with piecewise constant functions. Let f be a Lipschitz function. We define function $g_k = \sum f(x_j)\mathbb{1}_{[x_j,x_{j+1}]}$, where x_j are rational points of order k i.e. $\exists A \in D_k$ s.t. $[x_j, x_{j+1}] = D_\infty(A)$. Similarly as in Lemma 5.12 we define function

$$h_A = \inf\{f(z) - g_k(z) : z \in D_\infty(A)\}$$

Let K be Lipschitz constant of a function of f. It is easy to check inductively that $\#D_k \lesssim 4^k$. We repeat the approach from Lemma 5.12. Following estimate is satisfied

$$\|h\|_{BV_{T_K}} \lesssim K\frac{4^k}{3^{2k}}$$

Left hand side tens to zero with $k \to \infty$. Hence $\Psi(M(\tilde{d}))$ is dense in $\dot{X}(\Omega_K)$. □

To show that Ψ is an isomorphism we need to prove the estimate from below on the norm of $\Psi(m)$. The next auxiliary lemma reduces our problem to a finite tree.

Lemma 5.18. *Let* $f \in L$ *and* $Trf(z) = c$ *for every* $z \in [x, y]$. *Function* $\tilde{f} \in L$ *given by the formula*

$$\tilde{f}_A = \begin{cases} c & D_\infty(A) \subset [x,y], \\ f_A & \textit{in a opposite case}, \end{cases}$$

satisfies

$$\|\tilde{f}\|_{\dot{BV}_{T_K}} \leqslant \|f\|_{\dot{BV}_{T_K}}.$$

Proof. Fix $A_0 \in V_T$ such that $D_\infty(A_0) \subset [x, y]$. Without loss of generality we assume that $f_{A_0} = 0$ and $c = 1$. If B is a descendant of A_0 it follows from the definition that $D_\infty(B) \subset [x, y]$.

We can assume that for $B \in D\!\uparrow (A_0)$ the value f_B does not exceed one. Indeed if B is such that $f_{B\downarrow} \leqslant 1$ and $f_B > 1$ then we define an auxiliary function h

$$h_Q = \begin{cases} 1 & Q = B \text{ or } Q \in D\!\uparrow (B) \subset [x,y], \\ f_Q & \text{in a opposite case}, \end{cases}$$

Function h has the same trace as f and differs from f only on $D\uparrow(B)$. Since

$$|f_{B\downarrow} - f_B| > |f_{B\downarrow} - 1|$$

and h is constant on $D\uparrow(B)$ it follows that

$$\|h\|_{\dot{BV}_{T_K}} < \|f\|_{\dot{BV}_{T_K}}.$$

We can assume that f is monotone (non-decreasing) on $D\uparrow(A_0)$ with respect to descendancy relation i.e. if $B \in D\uparrow(A_0)$ and C is a descendant of B then $f_B \leqslant f_C$. Indeed suppose that $f_C < f_B < 1$ for some $C \in D_1(B)$. Since for functions in L the value of trace $Tr f(x)$ is defined as limit along $\mathrm{br}(x)$, but for $x \in D_\infty(A)$ the limit is one. Therefore on every branch $\mathrm{br}(x)$ s.t. $x \in D_\infty(C)$ there exists a vertex Q such that $f_Q \geqslant f_B$ and $f_{Q\downarrow} < f_B$. We denote by $\omega(C)$ the set of all such vertices. Let $T(C)$ be a tree with a root C and set of leafs equal to $\{Q\downarrow : Q \in \omega(C)\}$. We define auxiliary function p by the formula

$$p_Q = \begin{cases} f_B & Q \in V_{T(C)}, \\ f_Q & \text{in a opposite case}, \end{cases}$$

On the tree $T(C)$ the variation of p is equal to the weighted sum of differences on leafs. However for every $Q \in \omega(C)$

$$|p_Q - p_{Q\downarrow}| = |f_Q - f_B| \geqslant |f_Q - f_{Q\downarrow}|.$$

Therefore

$$\|p\|_{\dot{BV}_{T_K}} < \|f\|_{\dot{BV}_{T_K}}.$$

We have reduced our problem to the set of functions $Y(f) \subset L$ s.t. $h \in Y(f)$ iff it is a non-decreasing function on $D\uparrow(A_0)$ with respect to descendancy relation, $h_B = f_B$ for every $B \in V_{T_K} \setminus D\uparrow(A_0)$ and $Tr\, h(x) = 1$ for $x \in D_\infty(A_0)$. We introduce a partial order on $Y(f)$. For $h,\ z \in Y(f)$

$$h \preceq z \quad \Leftrightarrow \quad \forall A \in V_{T_K} \quad h_A \leqslant z_A \quad \text{and} \quad \|z\|_{\dot{BV}_{T_K}} \leqslant \|h\|_{\dot{BV}_{T_K}}.$$

If $C \subset Y(f)$ is a chain with respect to the relation $\preceq$ then it has an upper bound in $Y(f)$. Indeed the function $z \in Y(f)$ defined by the formula

$$z_A = \sup_{u \in C} u_A$$

is an upper bound. Function z is a supremum of non-decreasing functions hence it is non-decreasing. If every non-decreasing sequence b_α^k is convergent to one as $k \to \infty$ then $\sup_\alpha b_\alpha^k$ converges to one. Therefore z has the same trace as functions in $Y(f)$. In particular $Tr\, h := 1$ for $x \in D_\infty(A_0)$. By the definition if $u \preceq v$ then $u_Q \leqslant v_Q$ for every $Q \in V_{T_K}$ and the total variation $\|v\|_{\dot{BV}_{T_K}} \leqslant \|u\|_{\dot{BV}_{T_K}}$. Therefore for every n we can choose a sequence $f^k \in Y(f)$ s.t.

$$\lim_{k \to \infty} \|f^k\|_{\dot{BV}_{T_K}} = \inf_{u \in C} \|u\|_{\dot{BV}_{T_K}}.$$

and $\lim f_Q^k = z_Q$ for every $Q \in \bigcup_{j=1}^n D_j$. Therefore following estimate is satisfied

$$\sum_{j=1}^{n} \sum_{Q \in D_j} \frac{1}{3^j} |z_Q - z_{Q\downarrow}| \leqslant \inf_{u \in C} \|u\|_{\dot{BV}_{T_K}}$$

Taking limit with $n \to \infty$ we get

$$\|z\|_{\dot{BV}_{T_K}} < \inf_{u \in C} \|u\|_{\dot{BV}_{T_K}}.$$

Since every chain in $Y(f)$ has an upper bound in $Y(f)$ by the Kuratowski-Zorn Lemma, there exists element of $Y(f)$ maximal with respect to $\preceq$. Let $w \in Y(f)$ be a maximal element. By the monotonicity of w, it follows that $w_{Q\downarrow} \leqslant w_Q$ for every $Q \in D\uparrow(A_0)$. Since for every $Q \in V_{T_K}$ the set of direct descendants $D_1(Q)$ has at least three elements,

$$\begin{aligned}
|w_{Q\downarrow} - w_Q| + \sum_{B \in D_1(Q)} \frac{1}{3} |w_B - w_Q| &= w_Q - w_{Q\downarrow} + \sum_{B \in D_1(Q)} \frac{1}{3} w_B - w_Q \\
&= (1 - \frac{\# D_1(Q)}{3}) w_Q - w_{Q\downarrow} + \sum_{B \in D_1(Q)} \frac{1}{3} w_B \\
&\geqslant (1 - \frac{\# D_1(Q)}{3}) \min_{B \in D_1(Q)} (w_B) \\
&\quad - w_{Q\downarrow} + \sum_{B \in D_1(Q)} \frac{1}{3} w_B.
\end{aligned}$$

Function w is maximal with respect to $\preceq$, hence $w_Q = \min_{B \in D_1(Q)} w_B$ for every $Q \in D\uparrow(A_0)$. Therefore there is an infinite branch $\mathrm{br}(x)$ s.t $x \in D_\infty(A_0)$ and w is constant on $\mathrm{br}(x) \cap D\uparrow(A_0)$. However for $x \in D_\infty(A_0)$ the limit over any branch $\mathrm{br}(x)$ is equal to one. Hence $h_B = 1$ for every $B \in D\uparrow(A_0)$. We have proven that changing the values of f to one on the descendants of A_0 does not increase the total variation. It remains to consider the value at the point A_0. By the triangle inequality and the fact that for every vertex Q, $\# D_1(Q) \geqslant 3$ we have

$$|f_{A_0\downarrow} - f_{A_0}| + \sum_{B \in D_1(A)} \frac{1}{3} |1 - f_{A_0\downarrow}| \geqslant |f_{A_0\downarrow} - 1|.$$

Therefore changing the value of f on A_0 and its descendants to one, will not increase the total variation. Since only assumption on A_0 was that $D_\infty(A_0) \subset [x, y]$ we have desired estimate

$$\|\tilde{f}\|_{\dot{BV}_{T_K}} \leqslant \|f\|_{\dot{BV}_{T_K}}.$$

$\square$

Lemma 5.19. *Let $A_0 \in D_n$ and $[x, y] = D_\infty(A_0)$ then*

$$\tilde{d}(x, y) = 3^{-n}. \tag{5.7}$$

Proof. For any $f \in \dot{BV}_{T_K}$ s.t. $Trf = \mathbb{1}_{[x,y]}$ we define

$$\tilde{f} = \begin{cases} 1 & D_\infty(A) \subset [x,y], \\ f_A & A \in D_k, k \leqslant n_0, \\ 0 & \text{in a opposite case.} \end{cases}$$

From the Lemma 5.18 it follows that

$$\|\tilde{f}\|_{\dot{BV}_{T_K}} \leqslant \|f\|_{\dot{BV}_{T_K}}$$

However

$$\|\tilde{f}\|_{\dot{BV}_{T_K}} \geqslant \frac{1}{3^n} \sum_{B \in D_1(A_0\downarrow)} |f_{A_0\downarrow} - f_B| \geqslant \frac{1}{3^n} \left(|f_{A_0\downarrow} - 1| + |f_{A_0}|\right) \geqslant \frac{1}{3^n}.$$

The right hand side of the inequality is a total variation of a function p

$$p_A = \begin{cases} 1 & D_\infty(A) \subset [x,y], \\ 0 & \text{in a opposite case.} \end{cases}$$

□

Let us observe that the set of functions $\sum_j a_j \mathbb{1}_{[x_j,y_j]}$, where x_j, y_j are rational, is dense in $\dot{X}(\Omega_K)$. Indeed for every irrational arc $[x,y]$ there exist a sequence of points t_n, z_n s.t.

$$\|\mathbb{1}_{[x,y]} - \mathbb{1}_{[t_n,z_n]}\|_{\dot{X}(\Omega_K)} \lesssim \frac{1}{3^n}.$$

Similarly we observe that molecules $\sum_j a_j m_{x_j y_j}$, where x_j, y_j are rational, are dense in Arens-Eels space.
We fix $g = \sum_j a_j \mathbb{1}_{[x_j,y_j]}$, where arcs $[x_j, y_j]$ are rational and pairwise disjoint. Let $f \in L$ be any function such that $Trf = g$. There exists $n_0 = n_0(g)$ such that for $A \in D_{n_0}$ either there exist an arc $[x_j, y_j]$ s.t. $D_\infty(A) \subset [x_j, y_j]$ or $D_\infty(A)$ and $\bigcup [x_k, y_k]$ are disjoint. We define function $Wf \in L$ by

$$Wf_A = \begin{cases} a_j & D_\infty(A) \subset [x_j, y_j] \\ 0 & D_\infty(A) \cap \bigcup_j [x_j, y_j] = \emptyset, \\ f_A & \text{in other cases.} \end{cases}$$

It is easy to observe that $Trf = TrWf$. Moreover from Lemma 5.18 it follows that

$$\|Wf\|_{\dot{BV}_{T_K}} \leqslant \|f\|_{\dot{BV}_{T_K}}.$$

Therefore

$$\inf\{\|f\|_{\dot{BV}_{T_K}} : Trf = g\} = \inf\{\|f\|_{\dot{BV}_{T_K}} : Trf = g \text{ and } f = Wf\}.$$

Since we minimize the total variation over the set $\{Trf = g \text{ and } f = Wf\}$, the values f_A are fixed for $A \in D_k$, $k > n_0$. Therefore the total variation on this set is a function of finitely many variables. Moreover it is a piecewise linear function with finitely many pieces. Therefore the minimum is attained. We denote the total variation minimizer by ψ. We define by $\gamma^A \in \dot{BV}_{T_K}$

$$\gamma^A_B = \begin{cases} 1 & B \in D \uparrow (A), \\ 0 & \text{in other cases.} \end{cases}$$

Therefore from Abel summation formula

$$\psi = \psi_R + \sum_{j=1}^{n_0} \sum_{A \in D_n} \left(\psi_A - \psi_{A\downarrow}\right) \gamma^A.$$

A simple calculation gives us

$$\|\psi\|_{\dot{BV}_{T_K}} = \sum_{j=1}^{n_0} \sum_{A \in D_n} |\psi_A - \psi_{A\downarrow}| \, \|\gamma^A\|_{\dot{BV}_{T_K}}. \tag{5.8}$$

The function $\|\psi\|_{\dot{BV}_{T_K}}$ minimize the variation for a given trace, hence

$$\|Trf\|_{\dot{X}(\Omega_K)} = \|\psi\|_{\dot{BV}_{T_K}}.$$

Therefore from (5.8), (5.7)

$$\begin{aligned} \|Tr\,\psi\|_{\dot{X}(\Omega_K)} &\simeq \sum_{j=1}^{n_0} \sum_{A \in D_n} |\psi_A - \psi_{A\downarrow}| \, d(x(A), y(A)) \\ &\geqslant \|\sum_j a_j m_{x_j y_j}\|_{AE(\tilde{d})} \end{aligned}$$

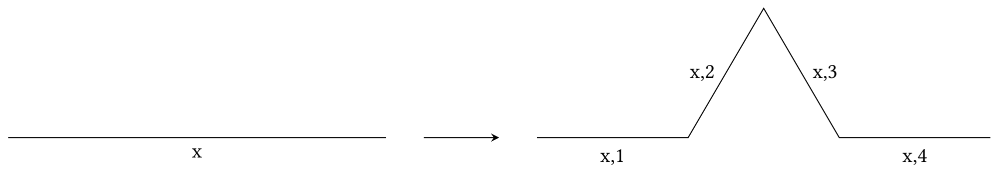
Therefore Ψ is an isomorphism of Banach spaces. $\square$

We have proven that the trace space is isomorphic to the Arens-Eels space.

We will characterize $AE(\tilde{d})$ further.

Lemma 5.20. *$AE(\tilde{d})$ is isomorphic to ℓ^1*

Proof. In order to characterize $AE(\tilde{d})$ we introduce another metric on the von Koch's curve. The von Koch's curve is constructed inductively. The induction starts with a triangle and every segment of the triangle is replaced with a piecewise linear curve w. This curve is made of from 4 segments. In the next step every old segment is replaced with a rescaled copy of w. Every segment is indexed in the following way. The segment S_x is replaced with segments $S_{x,0}, S_{x,1}, S_{x,2}, S_{x,3}$.



$I = \{x = (x_1, x_2 \ldots) : x_1 \in \{0,1,2\}, x_i \in \{0,1,2,3\}$ for $i > 1\}$ is a set of all infinite indices of segments in the von Koch's curve construction. For every point $x \in \partial\Omega_K$ there is a corresponding index $i(x) \in I$ such that segments $S_{i(x)_1,\ldots,i(x)_k} \to x$ as $k \to \infty$. We define a bijection between set of indices and a one dimensional Torus with the euclidean metric

$$\mathbb{T} = \{y : y = \frac{i(x)_1}{3} + \sum_{j=2}^{\infty} \frac{i(x)_j}{4^j} \quad i(x) \in I\}.$$

Every $x \in \partial\Omega_K$ has a unique index in $\mathbb{T}$. Abusing notation we denote it by $i(x)$. We can define a metric on $\partial\Omega_K$ by

$$d(x,y) := d_{\mathbb{T}}(i(x), i(y)).$$

As is easily on a Figure 5.1, if $A \in D_n$ is a "pants" shaped polygon then $D_\infty(A) = [x,y]$, where $d(i(y), i(x)) = \frac{2}{4^n}$. It is so because its descendants cover two segments of n-th generation. Similarly if A is a "palace" shaped polygon, $d(i(y), i(x)) = \frac{1}{4^n}$. In any of the above cases we have

$$\tilde{d}(x,y) \simeq \frac{1}{3^n} = \frac{1}{4^{n \log_4(3)}} \simeq d(x,y)^{log_4(3)}.$$

For rational points x, y we define

$$f_A^{[x,y]} := \begin{cases} 1 & D_\infty(A) \subset [x,y], \\ 0 & \text{otherwise.} \end{cases}$$

Obviously $Trf^{[x,y]} := \mathbb{1}_{[x,y]}$. Since x, y are rational, there exists unique finite sequence of $\{A_k\}_{k \in I} \subset V_T$, such that $f^{[x,y]} = \sum_k \gamma^{A_k}$. Let $m = \min\{n : \exists\ k\ A_k \in D_n\}$. From the definition of $f^{[x,y]}$ we deduce that γ^{A_k} have disjoint support, and for every n there are at most 10 polygons in $\{A_k\}_{k \in I} \cap D_n$. Therefore

$$d(x,y) = \sum_k d(x(A_k), y(A_k)) \leqslant 10 \sum_{i=m} \frac{1}{4^i} \simeq \frac{1}{4^m}.$$

and we have analogous estimate for $\tilde{d}$. Hence

$$\tilde{d}(x,y) \simeq \frac{1}{3^m} = \frac{1}{4^{m \log_4(3)}} \simeq d(x,y)^{\log_4(3)}.$$

Therefore $AE(\tilde{d}) \cong AE(d^{\log_4(3)})$. Since $0 < \log_4(3) < 1$ the claim of the lemma follows from the theorem below,

Theorem 5.21. *Let $N \in \mathbb{N}$ and X is isometric to infinite compact subset of $\mathbb{R}^N$. If d, $\tilde{d}$ are metrics on X s.t $\tilde{d} \simeq d^\alpha$ for $0 < \alpha < 1$ then the space $AE(\tilde{d})$ is isomorphic to ℓ^1.*

The case N=1 was proven by Z. Ciesielski [7] and for $N > 1$ the above Theorem follows from Theorem 3.5.5 and Theorem 3.3.3 in [52]. $\square$

Therefore $\dot{X}(\Omega_K)$ is isomorphic to ℓ^1. Let $\dot{X}(\Omega_K) = \operatorname{span}\{e_i\}$. From the definition of the trace space for every e_i there exists $f_i \in \dot{BV}_{T_K}$ such that $\|f_i\|_{\dot{BV}_{T_K}} \leqslant 2\|e_i\|_{\dot{X}(\Omega_K)}$ and $Trf_i = e_i$. Hence the S given by the formula

$$S\left(\sum_i a_i e_i\right) = \sum_i a_i f_i$$

is the desired right inverse operator with $\|S\| = 2$. Indeed

$$Tr\left(S\left(\sum_i a_i e_i\right)\right) = Tr\left(\sum_i a_i f_i\right) = \sum_i a_i e_i.$$

This concludes the proof of Theorem 5.2.

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