University of Warsaw

Faculty of Mathematics, Informatics and Mechanics

Katarzyna Ewa Mazowiecka

Singularities of harmonic and biharmonic maps into compact manifolds

PhD dissertation

Supervisor

prof. dr hab. Paweł Strzelecki
Institute of Mathematics
University of Warsaw

Author's declaration:

I hereby declare that this dissertation is my own work.

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Katarzyna Mazowiecka

Supervisors' declaration:

The dissertation is ready to be reviewed

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prof. dr hab. Paweł Strzelecki

In loving memory of Julia Romanowska
dearest friend and a companion during
undergraduate and PhD studies

Abstract

This thesis is concerned with a study of singular points of *minimizing* harmonic and biharmonic maps.

In the first part we focus on harmonic maps. *Minimizing* harmonic maps with prescribed boundary conditions may have singularities. We focus on the model case of mappings from $\mathbb{B}^3$ to $\mathbb{S}^2$. For some boundary data it is known that all corresponding minimizers have singularities and the Dirichlet energy is strictly smaller than the infimum of the energy among the continuous extensions (the so called *Lavrentiev gap phenomenon* occurs). We prove that the Lavrentiev gap phenomenon for harmonic maps into spheres holds on a dense set in the set $\mathcal{S}$ of smooth boundary maps $\varphi\colon \mathbb{S}^2 \to \mathbb{S}^2$ endowed with the $W^{1,p}$-topology, where $1 \leq p < 2$. This result is sharp: it fails in the $W^{1,2}$-topology of $\mathcal{S}$.

In the second part we consider the case of *minimizing* biharmonic maps into compact manifolds. The first step in studying the singularities of such mappings is the question of regularity near the boundary. We obtain a conditional result — assuming a boundary monotonicity formula we prove that *minimizing* biharmonic maps are smooth in a full neighborhood of the boundary. We expect that the boundary monotonicity formula is satisfied by all minimizing biharmonic maps with sufficiently smooth boundary data.

Keywords: harmonic maps, biharmonic maps, singularities, boundary regularity

AMS Subject Classification: 58E20, 35J48, 35J58, 35J35

Streszczenie

Celem niniejszej pracy jest badanie osobliwości *minimalizujących* przekształceń harmonicznych oraz biharmonicznych.

W pierwszej części zajmujemy się problemem przekształceń harmonicznych. *Minimalizujące* przekształcenia harmoniczne z nałożonymi warunkami brzegowymi mogą być osobliwe. Koncentrujemy się na modelowym przypadku przekształceń z trójwymiarowej kuli w dwuwymiarową sferę. Dla pewnych warunków brzegowych wiadomo, że odpowiadające im minimalizujące przekształcenia harmoniczne muszą mieć osobliwości oraz że energia Dirichleta tych przekształceń jest ściśle mniejsza niż infimum energii wzięte po przekształceniach ciągłych (zachodzi tak zwane *zjawisko Ławrentiewa*). Dowodzimy, że zjawisko Ławrentiewa dla minimalizujących przekształceń harmonicznych w sfery zachodzi na gęstym zbiorze $\mathcal{S}$ gładkich przekształceń brzegowych $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ w topologii $W^{1,p}$, gdzie $1 \leq p < 2$. Wynik ten nie jest prawdziwy w topologii $W^{1,2}$, w tym sensie może być rozumiany jako optymalny.

W drugiej części analizujemy przypadek *minimalizujących* przekształceń biharmonicznych o wartościach w zwartych rozmaitościach. Pierwszym krokiem jaki należy podjąć studiując osobliwości takich przekształceń jest pytanie o brzegową regularność. Otrzymujemy warunkowy wynik — zakładając brzegową formułę monotoniczną pokazujemy regularność *minimalizujących* przekształceń biharmonicznych przy brzegu. Spodziewamy się, że wynik można wzmocnić, to znaczy, że dla dostatecznie gładkich warunków brzegowych formuła monotoniczna jest spełniona przez wszystkie minimalizujące przekształcenia biharmoniczne.

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Contents

Chapter 1

Introduction

The plan of the introduction is as follows. We start with a brief note about the Laplace equation in order to view harmonic maps as a natural generalization of the Dirichlet principle. As the thesis is divided into two parts, first devoted to harmonic maps and the second to biharmonic maps, so is the introduction. We introduce harmonic maps and discuss the related regularity, nonuniqueness and compactness results. Then we turn our attention to fourth order problems — biharmonic maps and state selected known results. Both of those classes of maps — biharmonic and harmonic maps — are special cases of k-polyharmonic maps, which we briefly discuss together with related open problems. Finally, in Section 1.4 we write about our main results: Lavrentiev gap phenomenon and instability of singularities for harmonic maps, and conditional boundary regularity for biharmonic maps.

The standard Dirichlet problem for the Laplace equation is to find a function $u : \Omega \to \mathbb{R}$ defined on a bounded, smooth domain $\Omega \subset \mathbb{R}^m$, which satisfies the equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.0.1}$$

It is well known that for a given $\varphi \in C^0(\partial\Omega)$, there exists a unique solution u such that $u \in C^\infty$. Moreover, any solution to (1.0.1) is a minimizer of the

Dirichlet integral

$$E(u) = \int_\Omega |\nabla u|^2 dx$$

in the class of functions $u : \Omega \to \mathbb{R}$ satisfying the boundary condition $u = \varphi$ on $\partial\Omega$. The latter fact is known as the Dirichlet principle (and was in fact proved in 1940 by H. Weyl, for more details on the historical perspective of the Laplace equation see [10]).

One might try to generalize this concept in several ways. Here we will deal with critical points with geometrical constraints. We will assume that our mappings have values in a manifold.

Let $\mathcal{N}$ be a smooth, compact Riemannian manifold without boundary of dimension n. According to J. Nash's embedding theorem [42], we may assume that $\mathcal{N}$ is isometrically embedded in some Euclidean space $\mathbb{R}^\ell$ for ℓ sufficiently large. For $k \in \mathbb{N}$ and $1 \leq p \leq \infty$ we define the Sobolev spaces

$$W^{k,p}(\Omega, \mathcal{N}) = \left\{ u \in W^{k,p}(\Omega, \mathbb{R}^\ell) \colon u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega \right\},$$

equipped with the topology inherited from the topology of the linear Sobolev space $W^{k,p}(\Omega, \mathbb{R}^\ell)$.

1.1 Harmonic maps

For $k = 1$ we define the Dirichlet energy for maps $u \in W^{1,2}(\Omega, \mathcal{N})$ as

$$E(u) = \int_\Omega |\nabla u|^2 \, dx. \tag{1.1.1}$$

Definition 1.1.1. We say that a map $u \in W^{1,2}(\Omega, \mathcal{N})$ is *(weakly) harmonic* if it is a critical point of the Dirichlet energy with respect to compactly supported variations in the target manifold, i.e.,

$$\left.\frac{d}{dt}\right|_{t=0} E(\Pi_{\mathcal{N}}(u + t\Phi)) = 0 \tag{1.1.2}$$

for all $\Phi \in C_0^\infty(\Omega, \mathbb{R}^\ell)$, where $\Pi_{\mathcal{N}} : O(\mathcal{N}) \to \mathcal{N}$ denotes the nearest point projection onto $\mathcal{N}$ from a neighborhood $O(\mathcal{N}) \subset \mathbb{R}^\ell$, for existence and properties of $\Pi_{\mathcal{N}}$ we refer the reader, e.g., to [52, Section 2.12.3].

Moreover, we say that a map $u \in W^{1,2}(\Omega, \mathcal{N})$ is *minimizing harmonic* if

$$E(u) \leq E(v)$$

for all $v \in W^{1,2}(\Omega, \mathcal{N})$ such that $u - v \in W_0^{1,2}(\Omega, \mathbb{R}^\ell)$.

Remark 1.1.2. There is an intermediate concept between harmonic and minimizing harmonic maps, i.e., the *stationary* harmonic maps. Those are maps $u \in W^{1,2}(\Omega, \mathcal{N})$, which in addition to (1.1.2) are critical points with respect to all variations of the domain, that is

$$\frac{d}{dt}\bigg|_{t=0} E(u(\cdot + t\xi(\cdot))) = 0 \tag{1.1.3}$$

whenever $\xi \in C_c^\infty(\Omega, \mathbb{R}^m)$.

Unlike the unconstrained case, the critical points of the Dirichlet energy may have singularities, are not necessarily minimizers of the energy, and there is no uniqueness in general in the class of prescribed boundary condition. Moreover, the corresponding system of Euler–Lagrange equations is a nonlinear system of second order equations, which reads as

$$-\Delta u = A(u)(\nabla u, \nabla u), \tag{1.1.4}$$

where A stands for the second fundamental form of an isometric imbedding of the target manifold $\mathcal{N} \subset \mathbb{R}^\ell$. Harmonic map equations are important as a simple and natural (but fairly nontrivial) model case of an elliptic system with a critically nonlinear right hand side, i.e., the right hand side of the equation is a priori only in L^1. The standard bootstrap methods do not yield any extra regularity of the solutions — regularity for such class of systems is a subtle issue.

Regularity

The topic of harmonic mappings has been extensively studied especially in the mid 80's of the last century. In case $m = 1$ harmonic maps are geodesics, therefore must be continuous. In the conformally invariant case $m = 2$ all harmonic maps are regular (see the celebrated result by F. Hélein [30]). Starting from $m = 3$ the singularities may appear even in the case of minimizing maps. There are examples of everywhere discontinuous harmonic maps (see the example of T. Rivière [45]).

If we restrict our attention the the class of *minimizing* harmonic maps we have partial regularity results. The seminal results by R. Schoen and K. Uhlenbeck [48, 49] states that Hausdorff dimension of the singular set is at most $m-3$ and in the case $m = 3$ the singular set consists of a finite number of points. Moreover, they proved that any minimizing harmonic maps is regular in a neighborhood of the boundary. The result is optimal, because the map $u(x) = \frac{x}{|x|} : \Omega = \mathbb{B}^m \to \mathbb{S}^{m-1}$ is minimizing harmonic for $m \geq 3$ (see [11, Section VII] for $m = 3$ and [35] for $m \geq 3$).

For *stationary* harmonic maps it is known that the singular set is of Hausdorff codimension 2 (see [14] for maps into spheres and a generalization to maps into any manifold [6]). It is an open problem if the estimate of the singular set for stationary harmonic maps is optimal. The boundary regularity in general is not known for *stationary* harmonic maps. For a conditional result see [55].

One of the basic tools in the study of regularity is the monotonicity formula which holds for all *stationary* harmonic maps. It states that the normalized energy is monotone with respect to the radius. More precisely, for all radii $0 < \sigma < \rho < \rho_0$

$$\rho^{2-m} \int_{B(y,\rho)} |\nabla u|^2 \, dx - \sigma^{2-m} \int_{B(y,\sigma)} |\nabla u|^2 \, dx = 2 \int_{B(y,\rho) \setminus B(y,\sigma)} r^{2-m} \left| \frac{\partial u}{\partial r} \right| \, dx,$$

where ρ_0 and y are such that $\overline{B(y, \rho_0)} \subset \Omega$, $r = |x-y|$ and $\frac{\partial}{\partial r}$ is the directional derivative in the direction of $\frac{x-y}{|x-y|}$.

A boundary analogue of this fact is known for *minimizing* harmonic maps and is one of the key-ingredients in the proof of boundary regularity. Up to our best knowledge no boundary monotonicity formula is known for *stationary* harmonic maps. See also [46] for a different class of harmonic maps and a boundary regularity result.

Compactness

One of the major problem in the study of regularity of harmonic maps is the question of compactness. In general, for an arbitrary target manifold $\mathcal{N}$ it is not known if a limit of weakly convergent in $W^{1,2}$ sequence of harmonic maps is harmonic. Partial results were obtained in this direction.

Here, the answer is positive in several cases. First it was proved by R. Schoen in K. Uhlenbeck, in a very simple situation, that limits of certain minimizing maps are minimizing and the convergence is in fact strong (see [48, Lemma 5.2]). Next, R. Hardt and F.H. Lin showed a similar result for minimizing harmonic maps into simply connected manifolds (see [28, Lemma 6.4]). Finally for energy minimizing maps into general target manifolds S. Luckhaus [37] (see also [38]) proved that if $\{u_j\}_{j\in\mathbb{N}}$ is a uniformly bounded in $W^{1,2}$ sequence of minimizing harmonic maps into general target manifold, then there exists a subsequence on which u_j converges (locally) strongly to a minimizing harmonic map. The answer is also positive for *all* maps into round spheres, or, more generally, into compact homogeneous Riemannian manifolds. The reason behind the latter result, observed initially by J. Shatah ([50]), is that the harmonic map equation in that case has its right hand side in the (local) Hardy space. This allows one to write the harmonic map equation in an equivalent form, as a system of first order conservation laws in divergence form. The general case is open.

Harmonic maps from $\mathbb{B}^3$ to $\mathbb{S}^2$

First of all let us mention that there is no uniqueness for minimizing harmonic maps with prescribed boundary condition. T. Rivière in [44] proved that for every nonconstant boundary condition $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ there exists infinitely many solutions to the harmonic map equation.

There are many examples of nonuniqueness of harmonic maps with prescribed boundary condition. One can find an example of a boundary condition $\varphi : \partial\mathbb{B}^3 \to \mathbb{S}^2$ for which there exists uncountably many minimizing harmonic mappings $u : \mathbb{B}^3 \to \mathbb{S}^2$ with $u = \varphi$ on $\partial\mathbb{B}^3$ (see [25, Corollary 2.2]).

Let us recall that in this case we know by [48, 49] that the energy minimizers are smooth except at a finite number of points. By [11] we know more, the topological degree of a singularity must be ± 1 and the behavior of u near a singularity is well understood (up to a rotation u behaves like $\frac{x}{|x|}$).

There are many examples of non-intuitive behavior of these maps. We mention below examples that concerns symmetry breaking. It turns out that there exists a boundary condition $\varphi : \partial\mathbb{B}^3 \to \mathbb{S}^2$ having values on the equator for which the energy minimizing map $u : \mathbb{B}^3 \to \mathbb{B}^3$ is continuous, has values on a single hemisphere but not only on the equator. It is clear that minimizers, to such boundary mappings, always come in “symmetric” pairs (see [26]). F. Almgren and E. Lieb construct in [3, Section 5] an example of a boundary mapping $\varphi : \partial\mathbb{B}^3 \to \mathbb{S}^2$, $\varphi(\partial\mathbb{B}^2) = \mathbb{S}^2$ such that φ is symmetric under the reflection through the equator of $\partial\mathbb{B}^3$ for which the energy minimizers have odd number of singularities, each of them lying close to either the north pole or the south pole. This example shows that the singularities also do not preserve the symmetry.

For more information, we would like to refer the interested reader to a very nice survey paper on singularities of harmonic maps (in any dimension) by R. Hardt [24].

1.2 Biharmonic maps

A generalization of harmonic maps we investigate here are biharmonic maps.

The Hessian energy (or extrinsic biharmonic energy) for maps $u \in W^{2,2}(\Omega, \mathcal{N})$ is defined as

$$H(u) = \int_\Omega |\Delta u|^2 \, dx, \tag{1.2.1}$$

where Δ is the standard Laplace operator on $\mathbb{R}^m$.

We point out that this energy depends on the embedding $\mathcal{N} \hookrightarrow \mathbb{R}^\ell$.

A map $u \in W^{2,2}(\Omega, \mathcal{N})$ is said to be *weakly biharmonic* if it is a critical point (with respect to the variations in the range) of the biharmonic energy, i.e., if it satisfies

$$\left.\frac{d}{dt}\right|_{t=0} H(\Pi_\mathcal{N}(u + t\zeta)) = 0, \tag{1.2.2}$$

where $\zeta \in C_0^\infty(\Omega, \mathbb{R}^\ell)$ and $\Pi_\mathcal{N}$ denotes the nearest point projection onto $\mathcal{N}$.

Remark 1.2.1. The so-defined maps are usually called *extrinsic biharmonic maps* in order to distinguish them from the *intrinsic biharmonic maps*. Here we will not consider the latter ones, therefore we will write simply *biharmonic maps* for *extrinsic biharmonic maps*.

Intrinsic biharmonic maps are defined as

$$H_T := \int_\Omega |(\Delta u)^T|^2 dx,$$

where the tension field $(\Delta u)^T$ is the component of Δu tangent to $T_u\mathcal{N}$. Intrinsic biharmonic mappings may be seen as a direct generalization of harmonic maps, because harmonic mappings have vanishing tension field. Therefore, they are also intrinsic biharmonic. Moreover, for $\mathcal{N}$ with nonpositive sectional curvature every intrinsic biharmonic map is harmonic. For targets with positive sectional curvature there exist intrinsic biharmonic maps which are not harmonic.

A survey on the intrinsic biharmonic maps can be found in S. Montaldo and C. Oniciuc's article [40].

Unlike the Hessian energy, the tension field does not depend on the embedding $\mathcal{N} \hookrightarrow \mathbb{R}^\ell$. Therefore, from a geometric point of view, the tension energy seems more natural. On the other hand, from the analytic point of view, the Hessian energy bounds the $W^{2,2}$ norm. This is no longer true for the tension energy.

Euler–Lagrange equations

In case $\mathcal{N} = \mathbb{R}^\ell$ the Euler–Lagrange system corresponding to biharmonic maps is simply

$$\Delta^2 u = 0$$

and the solution to this system is said to be a biharmonic *function.*

For $\mathcal{N} = \mathbb{S}^{\ell-1}$ the nearest point projection onto a sphere is given simply by $\Pi_{\mathbb{S}^{\ell-1}}(x) = \frac{x}{|x|}$, thus by (1.2.2) we get for any $\zeta \in C_c^\infty(\Omega, \mathbb{R}^\ell)$

$$\int_\Omega \langle \Delta u, \Delta\left(\zeta - \langle u, \zeta\rangle u\right)\rangle \, dx = 0. \tag{1.2.3}$$

The expression $\zeta - \langle u, \zeta\rangle u$ is the tangential part of ζ. Thus, the geometric interpretation of the equation is

$$\Delta^2 u \perp T_u \mathbb{S}^{\ell-1}. \tag{1.2.4}$$

Differentiating $|u|^2 = 1$ twice with respect to x_i and twice with respect to x_j it can be easily seen that

$$2u^k_{x_i x_j x_j} u^k_{x_i} + 2u^k_{x_i x_j} u^k_{x_i x_j} + u^k_{x_j x_j} u^k_{x_i x_i} + 2u^k_{x_j} u^k_{x_i x_i x_j} + u^k u^k_{x_i x_i x_j x_j} = 0, \tag{1.2.5}$$

and

$$\begin{aligned} 2u^k_{x_i x_j x_j} u^k_{x_i} &= \Delta(|\nabla u|^2), & u^k_{x_j x_j} u^k_{x_i x_i} &= |\Delta u|^2, \\ 2u^k_{x_i x_j} u^k_{x_i x_j} &= 2\nabla u \cdot \nabla \Delta u, & u^k u^k_{x_i x_i x_j x_j} &= \left\langle u, \Delta^2 u\right\rangle, \end{aligned}$$

where $\cdot$ denotes the inner product on $\mathbb{R}^\ell$. Multiplying (1.2.5) by u we obtain

$$-\Big(|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\nabla u \cdot \nabla \Delta u\Big)u = \left\langle u, \Delta^2 u\right\rangle u = \Delta^2 u,$$

where the last inequality is a consequence of (1.2.3). Thus, the Euler–Lagrange equations take in this situation the form

$$\Delta^2 u = -\Big(|\Delta u|^2 + \Delta(|\nabla u|^2) + 2\nabla u \cdot \nabla \Delta u\Big)u. \tag{1.2.6}$$

We note that, even though there appear third order derivatives on the right-hand side of the equation, they are in divergence form, the equation can be interpreted in the distributional sense for maps from $W^{2,2}$. The right hand side of the equation can be seen as $B(u, \nabla u, \nabla^2 u)$, where $|B| \lesssim |\nabla u|^2 + |\nabla^2 u|^2$.

For general target manifold, we similarly get, as in the situation of maps into spheres, the geometric interpretation of the Euler–Lagrange equation

$$\Delta^2 u \perp T_u\mathcal{N}. \tag{1.2.7}$$

Let us introduce the notation before stating an analytical equivalent form of the equation (1.2.7). Let $p \in \mathcal{N}$, $P(p) = \nabla \Pi_{\mathcal{N}}(p)$ be the orthonormal projection onto the tangent space $T_p\mathcal{N}$. The orthonormal projection onto the normal space will be denoted by $P^\perp$. We notice that $P + P^\perp = id$. Moreover, let $A(\cdot)(\cdot,\cdot)$ be the second fundamental form of $\mathcal{N}$ in $\mathbb{R}^\ell$, given by

$$A(p)(X,Y) = P^\perp(\nabla_X(Y)) \text{ for } X, Y \in T_p\mathcal{N},$$

where X, Y have been extended to tangent vector fields of $\mathcal{N}$ in a neighborhood of p. The corresponding Euler–Lagrange equation takes the form (for derivation of this system see for example [58])

$$\Delta^2 u = \langle \Delta(P(u)), \Delta u\rangle - \Delta(A(u)(\nabla u, \nabla u)) + 2\langle \nabla(P(u)), \nabla \Delta u\rangle, \tag{1.2.8}$$

in the sense of distributions.

One can see that the system (1.2.8) is a nonlinear fourth order system of equations of critical growth.

Partial regularity

We note that in case $m = 4$ the Hessian energy is conformally invariant, and hence conformal maps of the Euclidean four space are biharmonic[1]. The investigation of regularity of biharmonic maps was initiated by S.-Y. A. Chang et al. in [12]. They have investigated mappings with values in the sphere $\mathbb{S}^{\ell-1}$. In case $m = 4$ they proved the regularity of *all biharmonic* maps, while for $m \geq 5$ they have proved that stationary biharmonic maps are C^∞ except a closed set Σ of Hausdorff dimension at most $m - 4$. Their result was partially extended to general target manifold by C. Wang in [57, 56, 58]. Alternative proofs were given by P. Strzelecki [54] for $m = 4$, $\mathcal{N} = \mathbb{S}^{\ell-1}$, T. Lamm with T. Rivière [33] for $m = 4$ and arbitrary $\mathcal{N}$, M. Struwe [53] for $m \geq 5$ and arbitrary target manifold $\mathcal{N}$.

In [12] S.-Y. A. Chang, L. Wang and P. Yang derive from the stationary assumption a monotonicity formula, although only for sufficiently regular maps. That formula was crucial in the proof of partial regularity for $m \geq 5$. A rigorous proof of the monotonicity formula was given by G. Angelsberg in [5].

In the case of minimizing biharmonic maps the partial regularity results may be strengthened. First it was observed by M.-C. Hong and C. Wang in [31] that for $\mathcal{N} = \mathbb{S}^{\ell-1}$ the singular set Σ has Hausdorff dimension at most $m - 5$. One can prove the optimality of this result considering a map $\frac{x}{|x|} : \mathbb{B}^5 \to \mathbb{S}^4$ (see [31, Proposition A1.]). Finally, C. Scheven in [47] reduced the dimension of singular set of minimizing mappings for an arbitrary target manifold $\mathcal{N}$. His result states that, as in the case $\mathcal{N} = \mathbb{S}^{\ell-1}$, the singular set Σ of minimizing biharmonic maps has $\dim_{\mathcal{H}} \Sigma \leq m - 5$.

In a recent paper C. Breiner and T. Lamm [9] prove that each minimizing biharmonic map is *locally* in $W^{4,p}$ for $1 \leq p \leq 5/4$.

[1] This fact does not hold true if one tries to generalize biharmonic maps to arbitrary smooth manifolds in the domain in the most obvious way: as critical points of $\int_{\mathcal{M}} |\Delta_{\mathcal{M}} u|^2 dvol_{\mathcal{M}}$, where the standard Laplace operator was replaced by the Laplace–Beltrami operator $\Delta_{\mathcal{M}}$. In order to have conformal invariance in dimension 4 one should consider the so called *Paneitz* functional (see e.g [19, Section 1.5] and references therein).

The boundary regularity for minimizing biharmonic maps in general is not known.

Let us mention here two inconclusive results in this direction. Firstly, it was shown in [34] by T. Lamm and C. Wang that polyharmonic maps, in the conformal case $m = 2k$, enjoy the property of being continuous in a neighborhood of the boundary. Although, the proof is strongly dependent on the relation $m = 2k$ and one might not extend this method to the case $m > 2k$. The other results concerns partial boundary regularity for stationary maps. It was shown in [22] by H. Gong et al. that if we impose an additional condition on the boundary mapping then there exists a closed subset $\Sigma \subseteq \overline{\Omega}$, with $\mathcal{H}^{m-4}(\Sigma) = 0$ such that the stationary biharmonic map is smooth up to the boundary, except possibly the set Σ. The additional condition is the boundary monotonicity formula. Unlike the monotonicity formula, the boundary monotonicity formula is an artificial assumption — it is unknown whether it can be deduced for all stationary maps. The result [22] is a biharmonic counterpart of a result by Wang [55] for stationary harmonic maps.

Compactness

Unlike the case of minimizing harmonic maps it is not known if a sequence of weakly convergent minimizing biharmonic maps converges to a minimizing biharmonic map, unless $\mathcal{N} = \mathbb{S}^{\ell-1}$.

In [31] M.-C. Hong and C. Wang proved that the mapping $\frac{x}{|x|} : \mathbb{B}^5 \to \mathbb{S}^4$ is a unique *minimizing* map for its boundary condition. Moreover, they proved that there exist infinitely many solutions to system of biharmonic map equations for the boundary condition $\frac{x}{|x|}$. It seems that boundary regularity would allow us to establish a similar result for general boundary data.

The boundary regularity of biharmonic maps is a first step to understand which modifications of boundary maps $\varphi : \mathbb{S}^4 \to \mathbb{S}^4$ forces the singularities to appear.

1.3 k-polyharmonic maps

We mentioned above the polyharmonic maps. A k-polyharmonic map is defined for $u \in W^{k,2}(\Omega, \mathcal{N})$ as a critical point of

$$E^{k,2}(u) = \int_\Omega |\Delta^k u|^2 dx.$$

Complete interior and boundary regularity of k-polyharmonic maps in the critical dimension, i.e., in $W^{k,2}(\mathbb{R}^{2k}, \mathcal{N})$ was obtained in [18] and [34] respectively. A partial regularity result in the supercritical dimension $m > 2k$ was obtain for maps into simply (2k-1)-connected manifolds, see [17].

The reason why the partial regularity results known for harmonic and biharmonic maps do not have their counterpart for k-polyharmonic maps is the lack of a monotonicity formula for $E^{k,2}$ with exception of partial results in this direction, see Blatt's [8] for $k = 3$ and a certain range of dimensions .

1.4 Main results

Harmonic maps. We prove in the model example $\Omega = \mathbb{B}^3$, $\mathcal{N} = \mathbb{S}^2$ that the singularities are unstable, when one takes into account small perturbations of the boundary data in the topology of each of the spaces $W^{1,p}$, $1 \leq p < 2$. More precisely, we prove that for each positive integer M the set of smooth maps $\psi \colon \mathbb{S}^2 \to \mathbb{S}^2$ of prescribed degree $d \in \mathbb{Z}$ which have the following three properties:

(i) there is a unique minimizing harmonic map $u \colon \mathbb{B}^3 \to \mathbb{S}^2$ which agrees with ψ on the boundary of the unit ball;

(ii) this map u has at least M singular points in $\mathbb{B}^3$;

(iii) the Lavrentiev gap phenomenon holds for ψ, i. e., the infimum of the Dirichlet energies $E(w)$ of all smooth extensions $w\colon \mathbb{B}^3 \to \mathbb{S}^2$ of ψ is strictly larger than the Dirichlet energy $\int_{\mathbb{B}^3} |\nabla u|^2$ of the (irregular) minimizer u,

is dense in the set $\mathcal{S}$ of all smooth maps $\phi\colon \mathbb{S}^2 \to \mathbb{S}^2$ of degree d endowed with the $W^{1,p}$–topology, where $1 \le p < 2$. This result is sharp: it fails in the $W^{1,2}$–topology of $\mathcal{S}$.

To do this we apply F. Almgren and E. Lieb's lemma [3] and show how to install singular points in the modified boundary map. One of the very important tools in this construction is the uniform boundary regularity of minimizing harmonic maps, which helps us control the degree and modify the boundary data in such a way that we can ensure that for the new map there exists precisely one minimizer.

Biharmonic maps. We prove that under the assumptions of boundary monotonicity formula biharmonic maps with Dirichlet boundary conditions are continuous in a full neighborhood of the boundary. Similarly as in the harmonic case [49] the complete boundary regularity is based on the nonexistence of nonconstant boundary tangent maps. The proof of nonexistence of such maps does not require any additional assumptions — a boundary monotonicity formula is not needed for this fact.

By Dirichlet boundary conditions we understand that for a given $\phi \in C^\infty(\Omega_\delta, \mathcal{N})$ and some $\delta > 0$, where $\Omega_\delta = \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) < \delta\}$ we have for a biharmonic map $u \in W^{2,2}(\Omega, \mathcal{N})$

$$u = \phi \text{ on } \partial\Omega, \quad \nabla u = \nabla\phi \text{ on } \partial\Omega.$$

Added in proof. When this thesis was completed the author has learned that S. Altuntas [4] proved that minimizing biharmonic maps satisfy boundary monotonicity formula for sufficiently smooth boundary conditions.

The dissertation is organized as follows. It is divided into two topics — harmonic and biharmonic maps. In the second chapter we present the results concerning the Lavrentiev gap phenomenon for harmonic maps from $\mathbb{B}^3$ into $\mathbb{S}^2$. The first two subsections 2.1 and 2.2 are the content of paper [39] coauthored by P. Strzelecki. They deal with the case of zero degree boundary conditions. The subsection 2.3 shows how to extend the result into boundary mappings of arbitrary degree. The third chapter is concerned with conditional boundary regularity of biharmonic maps. The general overview of the strategy of the proofs is specific to its chapter and is introduced at the beginning of each chapter. At the end we included appendices to both chapters. Appendix A is a construction of a map used in a proof in Chapter 2. Appendix B contains proofs of two additional facts about biharmonic maps near the boundary, which slightly differ from the interior case.

Chapter 2

The Lavrentiev gap phenomenon for harmonic maps into spheres

In this chapter, we revisit a well-known topic, the study of singularities of maps $u\colon \mathbb{B}^3 \to \mathbb{S}^2$ which minimize the Dirichlet integral

$$E(u) = \int_{\mathbb{B}^3} |\nabla u|^2 dx\,, \qquad u \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2) \tag{2.0.1}$$

under a prescribed boundary condition $u\,|_{\partial\mathbb{B}^3} = \varphi\colon \mathbb{S}^2 \to \mathbb{S}^2$. Here, $\mathbb{B}^3$ stands for the open unit ball in $\mathbb{R}^3$, $\mathbb{S}^2$ is the unit sphere, and

$$W^{1,2}(\mathbb{B}^3, \mathbb{S}^2) = \{v = (v_1, v_2, v_3) \in W^{1,2}(\mathbb{B}^3, \mathbb{R}^3)\colon |v(x)| = 1 \text{ for a.e. } x \in \mathbb{B}^3\}\,.$$

Moreover, for a map φ in the fractional Sobolev space $H^{1/2}(\mathbb{S}^2, \mathbb{S}^2)$ we write

$$W^{1,2}_\varphi(\mathbb{B}^3, \mathbb{S}^2) = \{v \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)\colon v\,|_{\partial\mathbb{B}^3} = \varphi \text{ in the trace sense}\}\,.$$

Minimizers of the Dirichlet integral (2.0.1) in $W^{1,2}_\varphi(\mathbb{B}^3, \mathbb{S}^2)$ satisfy the Euler–Lagrange system

$$\begin{cases} -\Delta u &= |\nabla u|^2 u \qquad \text{in } \mathbb{B}^3, \\ u\,|_{\partial\mathbb{B}^3} &= \varphi\,. \end{cases} \tag{2.0.2}$$

The main motivation behind the present work was to reach a deeper understanding of the mechanisms governing the onset of singularities of solutions, and the cardinality and structure of the set of minimizing solutions for a fixed boundary condition. We also wanted to know whether the *Lavrentiev gap phenomenon*, cf. (2.0.3) below, occurs typically (in a precise topological meaning). Despite the work of numerous experts over the last three decades, this topic is still not fully understood. Our main result states, roughly speaking, that even in the case when there is no purely topological reason for the solution of (2.0.2) to be discontinuous, singularities of u *do* occur under arbitrarily small (in the $W^{1,p}$ sense, for $1 \le p < 2$) perturbations of an *arbitrary* smooth boundary data φ.

Before giving formal statements of the results, let us sketch a broader perspective.

When $\deg \varphi \neq 0$, all solutions of (2.0.2) in $W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ obviously have singularities, as φ has no continuous extension $u\colon \mathbb{B}^3 \to \mathbb{S}^2$. By a celebrated classic theorem of Schoen and Uhlenbeck [48] the singular set of a *minimizing* solution of (2.0.2) consists of isolated points. By another theorem of Almgren and Lieb [3], if the boundary condition φ has square integrable derivatives on $\mathbb{S}^2$, then the number of these points does not exceed a constant multiple of the *boundary energy* $\int_{\mathbb{S}^2} |\nabla_T \varphi|^2 d\sigma$. (Non-minimizing solutions can behave in a wild way: Rivière [45] proves that for any non-constant boundary data φ there exists an everywhere discontinuous solution of the harmonic map system (2.0.2).)

However, even when $\varphi\colon \mathbb{S}^2 \to \mathbb{S}^2$ satisfies $\deg \varphi = 0$ — so that a priori there is no topological obstruction for a map $u \in W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ to be continuous — minimizers of E in $W^{1,2}_{\varphi}(\mathbb{B}^3, \mathbb{S}^2)$ might be singular because this is energetically preferable. Hardt and Lin [27] give an example of a smooth zero degree boundary data $\widetilde{\varphi}\colon \mathbb{S}^2 \to \mathbb{S}^2$ which is $H^{1/2}$-close to a constant map and has the following properties:

(a) Each minimizer v of E in $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$ has at least M singular points (the number M can be prescribed a priori);

(b) The Lavrentiev gap phenomenon holds for E in $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$. By this, we mean the following inequality:

$$\mu(\widetilde{\varphi}) := \min_{W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3,\mathbb{S}^2)} E(u) < \overline{\mu}(\widetilde{\varphi}) := \inf_{W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3,\mathbb{S}^2)\cap C^0(\overline{\mathbb{B}}^3)} E(u). \tag{2.0.3}$$

An immediate consequence of (2.0.3) is that $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3,\mathbb{S}^2)\cap C^0(\overline{\mathbb{B}}^3)$ is not dense in $W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3,\mathbb{S}^2)$.

As Bethuel, Brezis and Coron have shown, cf. [7, Theorem 5], for boundary conditions φ of zero degree, the Lavrentiev gap phenomenon is equivalent to the fact that all minimizing harmonic maps in $W^{1,2}_{\varphi}(\mathbb{B}^3,\mathbb{S}^2)$ have singularities. Other examples of unexpected and counterintuitive behavior of singularities of minimizing harmonic maps have been given by Almgren and Lieb in [3]. In particular, a minimizer u of E in $W^{1,2}_{\varphi}(\mathbb{B}^3,\mathbb{S}^2)$ can have a large number of singular points even if $\det \nabla_T\varphi \equiv 0$ on $\mathbb{S}^2$ and φ maps the whole sphere $\mathbb{S}^2$ to a smooth curve γ. The abstract of [3] ends with the phrase: "in particular, singularities in u can be unstable under small perturbations of φ."[1]

Our main result ascertains that the message of the last sentence, *singularities can be unstable*, may be strengthened, i. e., replaced with a firm *singularities* are *unstable*, at least when one takes into account small perturbations of the boundary data in the topology of each of the space $W^{1,p}$, $1 \le p < 2$. Here is the precise statement.

Theorem 2.0.1. *Assume that* $\varphi \in C^\infty(\mathbb{S}^2,\mathbb{S}^2)$ *is an arbitrary smooth map with* $\deg\varphi = 0$ *and* $1 \le p < 2$. *Then, for each* $\varepsilon > 0$ *and each* $M \in \mathbb{N}$ *there exists a map* $\widetilde{\varphi} \in C^\infty(\mathbb{S}^2,\mathbb{S}^2)$ *such that*

(i) $\deg\widetilde{\varphi} = 0$;

(ii) $\|\varphi - \widetilde{\varphi}\|_{W^{1,p}} < \varepsilon$ *and* $\mathcal{H}^2\big(\{x \in \mathbb{S}^2 \colon \varphi(x) \neq \widetilde{\varphi}(x)\}\big) < \varepsilon$;

(iii) *the Dirichlet integral* E *has precisely one minimizer* $\widetilde{u} \in W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3,\mathbb{S}^2)$; *moreover,* $\widetilde{u}$ *has at least* M *point singularities in* $\mathbb{B}^3$.

[1] We have just changed Almgren and Lieb's notation from φ, ψ to our u, φ respectively.

Combining the above result with Bethuel, Brezis and Coron, [7, Theorems 5–6], one immediately obtains the following.

Corollary 2.0.2. *Assume that* $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ *and* $\deg\varphi = 0$. *Let* $\widetilde{\varphi} \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ *be given by Theorem 2.0.1. Then the Lavrentiev gap phenomenon* (2.0.3) *holds for* $\widetilde{\varphi}$.

It is a natural question whether the occurrence of such boundary data is a *typical property* in the class of all maps of degree zero, i. e., whether the set of mappings $\widetilde{\varphi}\colon \mathbb{S}^2 \to \mathbb{S}^2$ such that conditions (i) and (iii) of Theorem 2.0.1 hold *contains a countable intersection of open and dense sets of maps of zero degree* in $H^{1/2}$ (or in some other topology). In spite of some efforts, we have not been able to settle that question.

The main novelty of Theorem 2.0.1 and its proof is that (1) we show that the singularities are unstable in a generic sense, (2) in order to achieve that, we show how to combine an appropriately modified idea of Hardt and Lin, applied by them only to *constant* boundary conditions $\phi\colon \mathbb{S}^2 \to \{*\}$, with a revisited version of Almgren and Lieb's method of installing new singular points, see [3, Theorem 4.3]. A bridge between these two ingredients is provided by a brief topological argument which guarantees that for each boundary condition φ with $\deg\varphi = 0$ there exist two antipodal points $\pm q \in \mathbb{S}^2$ such that φ maps them to the same point of $\mathbb{S}^2$, $\varphi(q) = \varphi(-q)$. We select any pair of such points and, roughly speaking, show how to insert numerous tiny bubbles into φ close to those two antipodal points to obtain the new boundary condition $\tilde{\varphi}$. This way, φ is changed only in two little spherical caps centered at $\pm q \in \mathbb{S}^2$, so that the second statement in (ii) in Theorem 2.0.1 does hold.

To control the degree of $\tilde{\varphi}$ and to guarantee the uniqueness of minimizers of the Dirichlet integral in $W^{1,2}_{\tilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$, we employ the uniform boundary regularity of minimizing harmonic maps combined with the fact that harmonic maps are real analytic in the interior of the regular set.

Finally, the distance from φ to $\widetilde{\varphi}$ in $W^{1,p}$ is estimated by a technical, explicit analysis of the small bubbles. It is crucial here that $p < 2$: the computations in Lemma 2.2.7 break down for $p = 2$, and an application of Almgren and Lieb's [3, Theorem 2.12] shows that Theorem 2.0.1 indeed fails for $p = 2$, see Remark 2.2.9. On the other hand, Hardt and Lin's Stability Theorem [29] asserts that for a Lipschitz boundary mapping ψ with unique energy minimizer v, each minimizer u for a boundary mapping $\widetilde{\psi}$ sufficiently close to ψ in the Lipschitz norm has the same number of singularities as v. In that sense, the $W^{1,p}$ topology for $1 \leq p < 2$ in Theorem 2.0.1 is optimal.

In Section 2.3 we extend the result into maps of arbitrary degree. The method presented there avoids the topological argument in the situation of mappings of zero degree. This time we will not modify the boundary mapping on antipodal points, instead we will change it only on a small disk around a point we choose. In order to preserve the topological degree of the modified map we use a composition with an orientation reversing map — a rotation whose determinant is negative.

The notation throughout the chapter is standard. For the standard Euclidean open ball we write $B(x_0, r) = \{x \in \mathbb{R}^3 : |x - x_0| < r\}$ and for fixed $q \in \mathbb{S}^2$ we write $D(q, r) = B(q, r) \cap \mathbb{S}^2$ for the spherical cap formed by the intersection of the ball $B(q, r)$ and the unit sphere. We denote by $\overline{A}$ the closure of the set A. We write

$$\partial E(\varphi) = \int_{\mathbb{S}^2} |\nabla_T \varphi|^2 d\sigma$$

to denote the *boundary energy* of a map $\varphi\colon \mathbb{S}^2 \to \mathbb{S}^2$. For a map $u\colon \mathbb{B}^3 \to \mathbb{S}^2$ we set

$$J(x) \equiv J(u)(x) = \sqrt{\det\left(Du(x)Du(x)^T\right)}.$$

If the rank of $Du(x)$ is maximal, i. e., equal to 2, then $J(u)(x)$ measures how $Du(x)\,|_V$, where V is the orthogonal complement of $\ker Du(x)$, distorts the surface measure: for an arbitrary ball B centered at x, the Jacobian $J(u)(x)$ is equal to the ratio of $\mathcal{H}^2(Du(x)B)$ to $\mathcal{H}^2(B \cap V)$.

We recall the standard fact, which will be used in several places, that if $\phi\colon U \subseteq \mathbb{S}^2 \to \mathbb{S}^2$ is conformal, then $\int_U |\nabla_T \phi|^2 d\sigma = 2\mathcal{H}^2(\phi(\mathbb{S}^2))$.

Throughout this chapter the term "minimizer" will always refer to an $\mathbb{S}^2$ valued mapping minimizing the Dirichlet energy to given boundary data.

This chapter, with the exception of Section 2.3, is based upon joint work with Paweł Strzelecki (see [39]).

2.1 Installing new singularities

We start with a theorem of Almgren and Lieb, see [3, Theorem 4.3], which describes how to modify the boundary mapping so that its energy minimizer would have a singularity and the energy of the new minimizer would be almost the same as the energy of the initial one. This result will serve as a main tool in constructing $\widetilde{\varphi}$ in the proof of Theorem 2.0.1.

Before giving the statement, we introduce the notation which will be useful in several places below.

Definition 2.1.1. For a fixed map $\psi : \mathbb{S}^2 \to \mathbb{S}^2$, which is smooth near a point $q \in \mathbb{S}^2$, and for a fixed number $\varrho > 0$, we let $[\psi]_{q,\varrho} \colon \mathbb{S}^2 \to \mathbb{S}^2$ denote any smooth boundary map which arises from ψ by a small deformation in a neighborhood of q so that the following four conditions are satisfied:

(a) $[\psi]_{q,\varrho}(x) = \psi(x)$ whenever $|x - q| \geq \varrho$;

(b) $[\psi]_{q,\varrho}(x) \equiv \psi(q)$ if $|x - q| = \varrho/2$;

(c) The restriction of $[\psi]_{q,\varrho}$ to the annular region $\frac{\varrho}{2} < |x - q| < \varrho$ satisfies the Lipschitz condition with a Lipschitz constant L which depends only on ψ and <u>not on</u> ϱ;

(d) $[\psi]_{q,\varrho}$ is a diffeomorphism of the spherical cap $\{|x - q| < \varrho/2\} \cap \mathbb{S}^2$ onto the punctured sphere $\mathbb{S}^2 \setminus \{\psi(q)\}$ such that the boundary Dirichlet integral energy of $[\psi]_{q,\varrho}$ on this cap equals $8\pi + \mathrm{o}(1)$ as $\varrho \to 0$.

It is well known that such maps exist, e.g. a modification of the mapping obtained in [2, Appendix A.2]. If we identify the spherical cap from (d) with a disk and assume that $\psi(q) = (0,0,1) = x_N$ we can map a concentric annulus, properly contained in the disk, to the whole sphere without two spherical caps centered at x_N and $-x_N$ consisting of points whose angular distance (in radians) from the point x_N are smaller or equal $\frac{\varrho}{2}$ and from the point $-x_N$ are greater or equal $\pi - \frac{\varrho}{2}$, respectively. To do this we use a rescaled and rotated inverse stereographic projection. It is a smooth conformal mapping and therefore its Dirichlet energy is equal twice the Hausdorff measure of the image (and hence approaches $2 \cdot 4\pi$ as $\varrho \to 0$). The remaining disk and annuli from the domain can be mapped into both punctured spherical caps left in the image without changing the Dirichlet integral too much.

We shall sometimes say that $[\psi]_{q,\varrho}$ arises from ψ by inserting a smooth bubble at q.

Theorem 2.1.2 ([3, Theorem 4.3]). *Suppose $u : \mathbb{B}^3 \to \mathbb{S}^2$ is a minimizer which is unique for its boundary mapping $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ and which has an interior singularity at $p \in \mathbb{B}^3$. Assume ψ has finite boundary Dirichlet integral energy and is smooth near $q \in \mathbb{S}^2$ and let $\psi_j : \mathbb{S}^2 \to \mathbb{S}^2$ be any sequence of continuous boundary mappings such that $\psi_j = [\psi]_{q,2/j}$ for all j sufficiently large.*

Finally, let u_j be any minimizer in $\mathbb{B}^3$ with boundary mapping ψ_j. Then, for all sufficiently large j, the mapping u_j will have at least two interior singular points q_j and p_j such that $q_j \to q$ and $p_j \to p$ as $j \to \infty$.

Since we had some trouble to follow the argument in [3] — in particular the lines 11–14 on page 521 — in full detail, we include here a more detailed variant of Almgren and Lieb's proof, explaining the parts which were unclear for us.

Proof. The proof consists of five steps.

Step 1. We first show that $u_j \to u$ strongly in H^1. By [3, Theorem 1.1],

$$E(u_j) < C\sqrt{\partial E(\psi_j)} < C\sqrt{\partial E(\psi) + 8\pi + L},$$

so $\sup_j E(u_j) < \infty$ and $\sup_j \partial E(\psi_j) < \infty$. Passing to a subsequence, without changing the notation, ψ_j converges weakly in H^1, strongly in L^2, and pointwise almost everywhere to a map ψ_0. Moreover, by [3, Theorem 1.2 part (4)] (after passing to a subsequence) u_j converge strongly in H^1 to some u_0 and u_0 is a minimizer for its boundary mapping ψ_0. However, by its very definition $\psi_j(x) \to \psi(x)$ for all $x \in \mathbb{S}^2 \setminus \{q\}$, so that $\psi_0 = \psi$ a.e. and by the uniqueness of u we obtain that $u_0 = u$.

Step 2. Now the existence of interior singular points p_j of u_j for sufficiently large j, as well as the convergence $p_j \to p$, follows from [3, Theorem 1.8 part (2)]. (In short, if all u_j were regular in a small neighborhood of p, the scaled energy of u over a small ball $B(p, 2/j)$ would be small enough to guarantee the regularity of u at p.)

Step 3. By the Boundary Regularity Theorem [49] and monotonicity formula (see e.g. [2, Corollary 1.7]), we may choose an $R > 0$ such that for each $r < R/2$ we have $\int_{B(q,2r)} |\nabla u(x)|^2\, dx < 2\pi r$.

Step 4. As $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ is continuous near q, for any $\varepsilon > 0$ we may find a $\delta > 0$ such that if $|x - q| < \delta$, $x \in \mathbb{S}^2$, then $|\psi(x) - \psi(q)| < \varepsilon$. Let us fix $\varepsilon > 0$ and assume that for a fixed small $r = \min(\frac{\delta}{2}, \frac{1}{2}R)$ independent of j there is no singularity for each u_j in the region $|x - q| < 2r$.

Combining the elementary inequality $2|J(x)| \le |\nabla u(x)|^2$ and the co-area formula

$$\int_{\mathbb{B}^3} |J(u)(x)|\, dx = \int_{w \in \mathbb{S}^2} \mathcal{H}^1(u^{-1}\{w\})\, d\mathcal{H}^2(w)\,,$$

see [15, Chapter 3], one obtains

$$\int\limits_{\{r<|x-q|<2r\}} |\nabla u(x)|^2\,dx \geq 2\int_{w\in\mathbb{S}^2} \mathcal{H}^1\left(u^{-1}\{w\}\cap\{r<|x-q|<2r\}\right)d\mathcal{H}^2(w). \tag{2.1.1}$$

For $q\in\mathbb{S}^2$, to shorten the notation, we write $A(q;a,b)=\big(B(q,b)\backslash B(q,a)\big)\cap\mathbb{S}^2$ for the intersection of the annulus $B(q,b)\setminus B(q,a)$ with the unit sphere. We also write $\mathcal{U}_t=\partial\left(B(q,t)\cap\mathbb{B}^3\right)$ for the boundary of the intersection of the unit ball and the ball centered at q of radius t, and $\mathcal{U}_t^-=\partial B(q,t)\cap\mathbb{B}^3$ for the boundary of the ball centered at q of radius t intersected with the unit ball. Finally, $\mathcal{V}_\varepsilon=B(\psi(q),\varepsilon)\cap\mathbb{S}^2$ stands for the spherical cap established by the intersection of a ball centered at $\psi(q)$ of radius ε and a unit sphere.

We will use (2.1.1) to estimate the energy of u_j for sufficiently large j's in the region $r<|x-q|<2r$. We consider $j>2/r$, so that the strict inclusion $D(q,2/j)\subsetneq D_r:=D(q,r)$ holds. By assumption (d), $\psi_j\big(D(q,1/j)\big)=\mathbb{S}^2\setminus\{\psi(q)\}$ and ψ_j is injective in this small spherical cap, i. e., for any $y\in\mathbb{S}^2\backslash\{\psi(q)\}$ the set $\psi|^{-1}_{D(q,1/j)}(y)$ consists of only one point. By (a) and (c), we also have $\psi_j\big(A(q;1/j,2/j)\big)\subsetneq\mathcal{V}_\varepsilon$ and $\psi_j\big(A(q;2/j,2r\big)\subseteq\mathcal{V}_\varepsilon$.

Since, by the assumption above, u_j is continuous in the region $\{|x-q|<2r\}$, we have $\deg\left(u_j|_{\mathcal{U}_t}\right)=0$ for every $t<2r$ because the set $\mathcal{U}_t$ is topologically a sphere. Now, choose a number $t\in(r,2r)$, fix a point $y\in\mathbb{S}^2\setminus\{\psi\big(A(q;1/j,t)\big)\}$ and consider the set $(u_j|_{\mathcal{U}_t})^{-1}(y)$ of all its preimages. We know that there exists precisely one point $a\in D(q,1/j)$ such that $\psi_j(a)=u_j(a)=y$; since the degree is 0 we deduce that there must be another point $b\in\mathcal{U}_t^-$ such that $u_j(b)=y$ (with a reverse orientation than at a). This degree consideration shows that for each $t\in(r,2r)$ there exists a point $x_t\in\mathcal{U}_t^-$ such that $u_j(x_t)=y$. Since $\mathbb{S}^2\setminus\{\psi\big(A(q;1/j,r)\big)\}\supset\mathbb{S}^2\setminus\mathcal{V}_\varepsilon$, we have $\mathcal{H}^1\left(u_j^{-1}\{w\}\cap\{r<|x-q|<2r\}\right)\geq r$ for all $w\in\mathbb{S}^2\setminus\mathcal{V}_\varepsilon$.

A simple computation yields $\mathcal{H}^2(\mathbb{S}^2\setminus\mathcal{V}_\varepsilon)=\pi\big(3+\big(1-\frac{\varepsilon}{2}\big)^2\big)$Thus, for ε small,

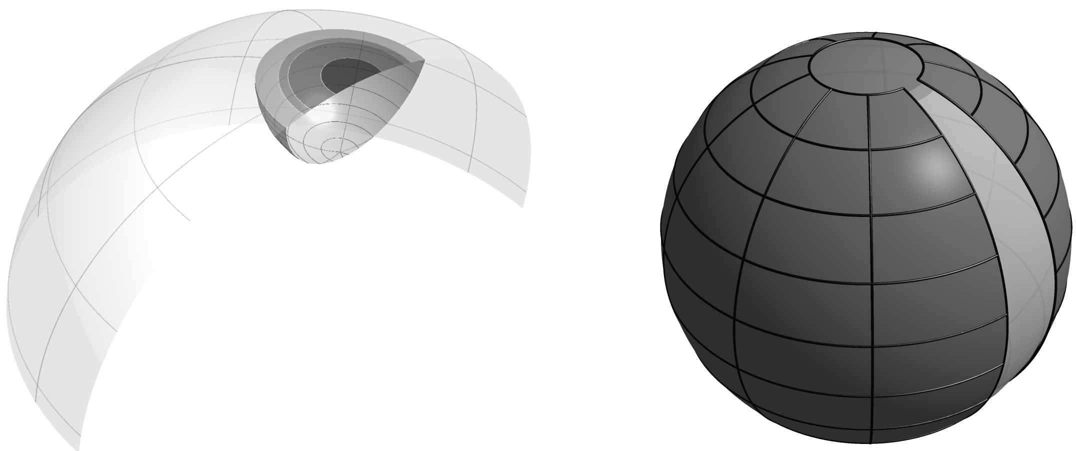

Figure 2.1: Left: the domain of u_j. On the top part of $\mathbb{S}^2$, four shaded areas are visible: the dark gray cap $D(q,1/j)$, a lighter annulus $A(q;1/j,2/j)$, still lighter narrow annulus $A(q;2/j,r)$, and the lightest $\mathcal{U}_t \setminus D_r$, with the rest of the boundary of $\mathbb{B}^3 \cap B(q,t)$ 'hanging below'. Right: the image of $u_j(\mathcal{U}_t)$, with corresponding shades of gray. The innermost dark cap $D(q,1/j)$ is mapped to almost the whole sphere, like a blown-up piece of the bubble gum.

by formula (2.1.1) we obtain

$$\begin{aligned}\int_{\{r<|x-q|<2r\}} |\nabla u_j(x)|^2 dx &\geq 2\int_{\mathbb{S}^2} \mathcal{H}^1\left(u_j^{-1}\{w\}\cap\{r<|x-q|<2r\}\right)\, d\mathcal{H}^2(w)\\ &\geq 2\cdot r\cdot \pi\left(3+\left(1-\frac{\varepsilon}{2}\right)^2\right)\\ &> 7\pi r.\end{aligned}$$

Having in mind the inequality $\int_{B(q,2r)} |\nabla u|^2 dx < 2\pi r$ from Step 3, this is a contradiction to the strong convergence obtained in Step 1. Thus in the region $|x-q|<2r$ for sufficiently large j's each u_j has a singularity q_j.

Step 5. Now it suffices to show that $q_j \to q$ as $j\to\infty$. Since $\varepsilon>0$ was arbitrary, we may choose a sequence of $\varepsilon_j \searrow 0$ such that the corresponding $r_j \searrow 0$ and the regions $B(q,2r_j)$ in which the singularity q_j appears will shrink to $\{q\}$. $\square$

Remark 2.1.3. The assertion of Theorem 2.1.2 holds true if we replace each ψ_j

by a smooth approximation $\widetilde{\psi}_j$ such that the modification in the region $|x-q| < \frac{1}{j}$ from Definition 2.1.1 (d) remains a diffeomorphism of the smaller disk to the whole sphere without a small cap centered at $\psi(q)$, such that for sufficiently large j's this cap is contained in $\mathcal{V}_\varepsilon$ from Step 4. One may easily check that it does not affect the proof.

2.2 Mappings of zero degree

Construction of $\widetilde{\varphi}$

The main idea is as follows: we will modify φ on two antipodal sets (in fact, on two little antipodal spherical caps in $\mathbb{S}^2$) of small measures. The modified $\widetilde{\varphi}$ will be arbitrarily close to φ in the space $W^{1,p}$, $1 \leq p < 2$ although its oscillations on these disks will be large in C^0. In the first step of the construction, we shall perturb the original mapping slightly, to make it constant on those two disks. Next, roughly speaking, we repeat the construction of Hardt and Lin in [27] in those regions to obtain our $\widetilde{\varphi}$.

At the beginning of this section we recall without proofs a few known results which will be used in the proof of Theorem 2.0.1. In the second part we construct our boundary condition and we close the section with the proof of Theorem 2.0.1.

Auxiliary propositions

The following theorem is a restatement of boundary regularity criterion of Schoen and Uhlenbeck [49]. This form, convenient for our purposes, is taken from [3, Theorem 1.10 (2)].

Theorem 2.2.1. *There exists $\varepsilon > 0$ with the following properties. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is three times continuously differentiable with $f(0) = 0$, $|\nabla f(0)| = 0$, and each partial derivative of f up to order 3 does not exceed ε^2 in absolute value. Suppose also that $\varphi_0 : \mathbb{R}^2 \to \mathbb{S}^2$ is three times continuously differentiable and that each partial derivative of φ_0 up to order 3 does not exceed ε^2 in absolute value. Finally suppose that u^* is a minimizer in the region*

$$\left\{(x, y, z) : x^2 + y^2 \leq 1 \text{ and } f(x, y) \leq z \leq 1\right\}$$

and the boundary mapping φ^* for u^* satisfies the condition that

$$\varphi^*(x, y, f(x, y)) = \varphi_0(x, y) \text{ whenever } x^2 + y^2 < 1.$$

Then there is a two times continuously differentiable mapping $u_0 : \mathbb{R}^3 \to \mathbb{S}^2$ such that each partial derivative of u_0 up to order 2 does not exceed ε in absolute value and u^* coincides with u_0 in the region

$$\left\{(x, y, z) : x^2 + y^2 \leq \frac{1}{2} \text{ and } f(x, y) \leq z \leq \varepsilon\right\}.$$

The next theorem was discovered by Almgren and Lieb; a precise statement can be found in [3, Theorem 4.1 (1)]. It asserts that the boundary mappings having unique minimizers are dense in $H^1(\partial\mathbb{B}^3)$. Theorem 2.2.2, and the trick used in its proof, will play an important role in our construction.

Theorem 2.2.2. *Suppose that q is a point in $\partial\mathbb{B}^3$, $\varepsilon > 0$, and that $\varphi : \partial\mathbb{B}^3 \to \mathbb{S}^2$ is a boundary mapping with $\partial E(\varphi) < \infty$. Then there is another mapping $\varphi^* : \partial\mathbb{B}^3 \to \mathbb{S}^2$ which coincides with φ except possibly on that part of $\partial\mathbb{B}^3$ within the ball $B(q, \varepsilon)$, which differs from φ in $H^1(\partial\mathbb{B}^3)$ norm by no more than ε, and for which there is exactly one minimizer $u^* : \mathbb{B}^3 \to \mathbb{S}^2$ having boundary mapping φ^*.*

The key observation in the proof of the above theorem is the following lemma which follows easily from the fact that harmonic maps into $\mathbb{S}^2$ are real analytic away from their singular points (see the proof of Theorem 4.1 in [3]).

Lemma 2.2.3. *Suppose Ω is a proper subdomain of a larger domain Ω^* and u is any minimizer (not necessary unique) in Ω^*. Then the restriction $u|_\Omega$ of u to Ω is the unique minimizer for its boundary mapping.*

To install singularities as in Theorem 2.1.2 we need to ensure that there exists precisely one minimizer for our boundary mapping. To this end, we have to modify the boundary mapping taking two issues into account. First, the $W^{1,p}(\partial\mathbb{B}^3)$ norm should not change too much; it turns out that we can control even the $W^{1,2}(\partial\mathbb{B}^3)$ norm. This ingredient is provided, basically, by Theorem 2.2.2. Its variant, adapted to our purposes, is proven later on in Lemma 2.2.8. Secondly, we need to make sure that our new mapping has degree zero. This follows from an argument based on Theorem 2.2.1.

We note here that we have already used the boundary regularity in Step 3 of the proof of Theorem 2.1.2.

Construction of $\widetilde{\varphi}$

We start with the observation that if $\deg\varphi = 0$ then there exist two antipodal points $q,\ -q \in \mathbb{S}^2$ such that $\varphi(q) = \varphi(-q)$. For the existence of such $\pm q \in \mathbb{S}^2$, see for instance Granas and Dugundji [23, Part II, p. 94, Thm. (6.1)]. For the convenience of the reader, we give here the gist of a quick argument: assume on the contrary that $\varphi(q) \neq \varphi(-q)$ for all $q \in \mathbb{S}^2$; one then easily constructs a homotopy from φ to another map φ_0 which preserves the antipodes, i. e., $\varphi_0(q) = -\varphi_0(-q)$ for each $q \in \mathbb{S}^2$. This is done as follows: for a given $q \in \mathbb{S}^2$, if we already have $\varphi(q) = -\varphi(-q)$ for some $q \in \mathbb{S}^2$, then the homotopy changes nothing; if $\varphi(q) \neq -\varphi(-q)$, then the two *distinct* points $\varphi(\pm q) \in \mathbb{S}^2$ determine a unique arc γ of the great circle such that the length of γ is smaller than π, and we let $\varphi(\pm q)$ travel at equal, constant speeds towards two antipodal points $\pm\widetilde{q}$ on that great circle (note that γ is located symmetrically on one of the half-circles joining $\pm\widetilde{q}$). However, it is well known that each map which preserves the antipodes must be of odd degree, a contradiction.

In the remaining part of this section, we simply say that $\pm q \in \mathbb{S}^2$ are the antipodal points of φ. First, we perturb φ slightly by making it constant close to $\pm q$.

Recall that $D(a, 2/j) \equiv B(a, 2/j) \cap \mathbb{S}^2$ denotes a spherical cap centered at a.

Definition 2.2.4. For each $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ with $\deg(\varphi) = 0$, having two antipodal points $\pm q \in \mathbb{S}^2$, and for a fixed number $\delta > 0$ such that

$$\mathcal{H}^2\left(\varphi(D(q, 2\delta)) \cup \varphi(D(-q, 2\delta))\right) < 4\pi,$$

we let $\varphi_1 : \mathbb{S}^2 \to \mathbb{S}^2$ denote any intermediate smooth mapping such that

(1) $\varphi_1(x) \equiv \varphi(q)$ for $x \in D(q, \delta) \cup D(-q, \delta)$;

(2) $\varphi_1(x) = \varphi(x)$ on $\mathbb{S}^2 \setminus \big(D(q, 2\delta) \cup D(-q, 2\delta)\big)$;

(3) On each of the two annuli $D(\pm q, 2\delta) \backslash \overline{D(\pm q, \delta)}$ the map φ_1 is given by a composition of φ with a smooth diffeomorphism from the annulus to the punctured disk $D(\pm q, 2\delta) \setminus \pm q$ with Lipschitz constant of both diffeomorphisms K. It is easy to see that one can have $K \leq C\delta^{-1}$ with an absolute constant C.

The parameter δ will be important in our further estimates. Therefore, we explain the choice of δ in the following lemma.

Lemma 2.2.5. *For each $\varepsilon > 0$ and each $1 \leq p < 2$ there is a $\delta > 0$ such that the map φ_1 specified in Definition 2.2.4 above has $\deg(\varphi_1) = 0$ and*

$$\|\nabla(\varphi - \varphi_1)\|_{L^p(\partial \mathbb{B}^3)} < \frac{\varepsilon}{4}.$$

Proof. By Sard's theorem (and the assumption that $\varphi(D(q, 2\delta)) \cup \varphi(D(-q, 2\delta))$ is not of full measure) we may choose a regular value y of φ_1 such that $y \notin \varphi(D(\pm q, 2\delta))$; by definition, the preimages of y under φ_1 are the same as its preimages under φ and both maps coincide in a small neighborhood of each of those preimages, so that $\deg(\varphi_1) = \deg(\varphi) = 0$. Since $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, we have $\max_{x\in\mathbb{S}^2} |\nabla_T \varphi(x)| < \infty$ and, as $\varphi \not\equiv \varphi_1$ only on the disks $D(\pm q, 2\delta)$ and $\nabla\varphi_1 \equiv 0$ on $D(\pm q, \delta)$,

$$\begin{aligned}
\|\nabla(\varphi - \varphi_1)\|^p_{L^p(\partial\mathbb{B}^3)} &\le \int\limits_{D(q,2\delta)} |\nabla_T\varphi(x)|^p d\sigma + \int\limits_{D(q,2\delta)\setminus\overline{D(q,\delta)}} |\nabla_T\varphi_1(x)|^p \, d\sigma \\
&\quad + \int\limits_{D(-q,2\delta)} |\nabla_T\varphi(x)|^p d\sigma + \int\limits_{D(-q,2\delta)\setminus\overline{D(-q,\delta)}} |\nabla_T\varphi_1(x)|^p \, d\sigma \\
&\le (8\pi\delta^2 + 6\pi\delta^2 K^p) \max_{x\in\mathbb{S}^2} |\nabla_T\varphi(x)|^p \qquad (2.2.1) \\
&\le 8C^p\pi\delta^{2-p} \max_{x\in\mathbb{S}^2} (|\nabla_T\varphi(x)|^p + 1),
\end{aligned}$$

where the last inequality holds provided that $\delta^p < C^p/4$, with C being the constant from Definition 2.2.4 (3). Thus choosing δ such that

$$\delta < \left(\frac{\varepsilon^p}{4^p \cdot 8C^p\pi \max_{x\in\mathbb{S}^2}(|\nabla_T\varphi(x)|^p + 1)} \right)^{1/(2-p)}$$

we obtain $\|\nabla(\varphi - \varphi_1)\|_{L^p(\partial\mathbb{B}^3)} < \frac{\varepsilon}{4}$.

□

We now fix φ_1 as above and, perturbing it, define a new intermediate map $\varphi_2 \colon \mathbb{S}^2 \to \mathbb{S}^2$. Let $\alpha = \arccos\left(1 - 2\delta^2 + \frac{\delta^4}{2}\right)$ denote the length of the arc $\gamma \cap B(q, \delta)$, where γ is any great circle through q. Without loss of generality suppose from now on that $q = x_N = (0, 0, 1) \in \mathbb{S}^2$. Roughly speaking, we are going to insert $2M$ appropriately small bubbles into φ_1, at points $\pm\xi_i$ close to $\pm q$, preserving the degree but forcing the minimizers to be singular at many points.

Definition 2.2.6. Let $\xi_i = \left(0, \sin\left(\frac{i\alpha}{M+1}\right), \cos\left(\frac{i\alpha}{M+1}\right)\right) \in B(q, \delta)$ for $i = 1, \dots, M$. For sufficiently large j's, with $2/j \ll \delta/2M$, we define $\varphi_2 : \mathbb{S}^2 \to \mathbb{S}^2$ as follows:

(1) $\varphi_2(x) = [\varphi_1]_{\xi_i, 2/j}(x)$ for $x \in D(\xi_i, 2/j)$;

(2) $\varphi_2(x) = \varphi_2(-x)$ for $x \in D(-\xi_i, 2/j)$;

(3) $\varphi_2 \equiv \varphi_1$ on $\mathbb{S}^2 \setminus \left(\bigcup_{i=1}^{M} D(\xi_i, 2/j) \cup D(-\xi_i, 2/j)\right)$,

where $[\psi]_{a,b}$ is the modification of ψ in the spherical cap $D(a, b)$, see Definition 2.1.1.

Note that φ_2 on each cap $D(\xi_i, 2/j)$ is either an orientation-preserving (degree 1) or an orientation-reversing (degree -1) map onto $\mathbb{S}^2$, while on $D(-\xi_i, 2/j)$ it is of opposite orientation (respectively degree -1 or degree 1) map onto $\mathbb{S}^2$. Since $\deg(\varphi_1) = 0$ we also have $\deg(\varphi_2) = 0$.

In the following lemma we will show that this procedure of inserting a single bubble to a map does not change the $W^{1,p}$ norm too much for $p < 2$.

Lemma 2.2.7. *Let $p < 2$, then for each $M \in \mathbb{N}$, $\varepsilon > 0$ there is a (sufficiently large) j such that $\left\|\nabla_T(\varphi_1 - [\varphi_1]_{\xi_i,2/j})\right\|_{L^p} < \frac{\varepsilon}{8M}$ for each i.*

Proof. Note that the mappings φ_1 and $[\varphi_1]_{\xi_i,2/j}$ differ only on $D(\xi_i, 2/j)$ and φ_1 is constant on that area. By Hölder's inequality and point (c) and (d) in the

Definition 2.1.1 we have

$$
\begin{aligned}
\int_{\mathbb{S}^2} |\nabla_T(\varphi_1 - [\varphi_1]_{\xi_i,2/j})|^p dx = \\
&\int_{D(\xi_i,1/j)} |\nabla_T(\varphi_1 - [\varphi_1]_{\xi_i,2/j})|^p dx \\
&\quad + \int_{D(\xi_i,2/j)\setminus D(\xi_i,1/j)} |\nabla_T(\varphi_1 - [\varphi_1]_{\xi_i,2/j})|^p dx \\
&\le \int_{D(\xi_i,1/j)} |\nabla_T[\varphi_1]_{\xi_i,2/j}|^p dx + L^p \cdot \mathcal{H}^2\big(D(\xi_i,2/j) \setminus D(\xi_i,1/j)\big) \\
&\le \left(\int_{D(\xi_i,1/j)} |\nabla_T[\varphi_1]_{\xi_i,2/j}|^2 dx\right)^{\frac{p}{2}} \cdot \big(\mathcal{H}^2\big(D(\xi_i,1/j)\big)\big)^{\frac{2-p}{2}} + L^p\pi\frac{3}{j^2} \\
&\le (8\pi + C)^{\frac{p}{2}}\, \pi^{\frac{2-p}{2}} \left(\frac{1}{j}\right)^{2-p} + L^p\pi\frac{3}{j^2}.
\end{aligned}
$$

Since $p < 2$ the last term of the inequality converges to 0 as $j \to \infty$. Therefore, by choosing j sufficiently large we get the assertion of the lemma. □

Lemma 2.2.8. *Fix $\delta_1 > 0$ sufficiently small. One may modify $\varphi_2 : \mathbb{S}^2 \to \mathbb{S}^2$ in a spherical cap of radius δ_1, located away from all $D(\xi_i, 2/j)$, obtaining a new map $\varphi_3 \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ such that $\|\varphi_2 - \varphi_3\|_{H^1(\partial\mathbb{B}^3)} < 10\delta_1$ and $\deg(\varphi_3) = 0$, for which there is exactly one minimizer $\widetilde{u} : \mathbb{B}^3 \to \mathbb{S}^2$ with $\widetilde{u}\,|_{\partial\mathbb{B}^3} = \varphi_3$.*

We essentially follow Almgren and Lieb's proof of Theorem 2.2.2; the only important difference is that we have to make sure that our φ_3 is of degree 0. For the sake of completeness we state the argument in full.

Proof. We extend the ball $\mathbb{B}^3$ slightly in a small neighborhood of $q^* \in \partial\mathbb{B}^3$ to obtain a new smooth domain $\Omega \supsetneq \mathbb{B}^3$. The domain is constructed in the following way: we choose

$$
q^* = \left(0, \sin\left(\frac{-\alpha}{2M}\right), \cos\left(\frac{-\alpha}{2M}\right)\right)
$$

away from all the ξ_i and from the caps where the bubbles are inserted into φ_1. Roughly speaking, the new Ω is the union of $\mathbb{B}^3$ and of a tiny and very flat bump

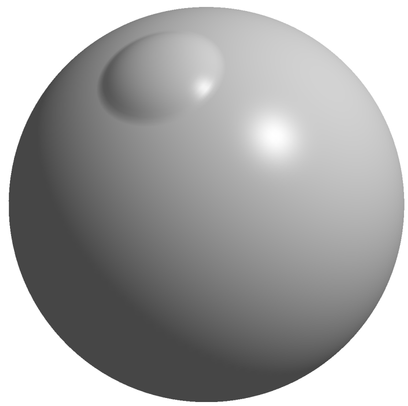

Figure 2.2: Ω is the union of a ball and a small flat bump.

of width $2\delta_1$ and height δ_1^5, which is centered at q^*, where $\delta_1 < \frac{1}{4}\text{dist}(q, q^*)$. It is convenient to imagine $\partial\Omega$ as the graph of a smooth nonnegative function $\theta\colon \mathbb{S}^2 \to [0, \infty)$ such that close to q^*, after we flatten the sphere locally,

$$\theta(\cdot) = \delta_1^5 \eta\left(\frac{\cdot}{\delta_1}\right) : \mathbb{R}^2 \to [0, \infty), \qquad \theta \text{ vanishes on } \mathbb{S}^2 \setminus B(q^*, \delta_1),$$

where η is a smooth nonnegative cutoff function supported in the unit disk with $\eta(0) > 0$. Formally, we let $T : \mathbb{S}^2 \setminus \{-q^*\} \to \mathbb{R}^2$ be a stereographic projection such that $T(q^*) = 0$ and set

$$\Omega = \mathbb{B}^3 \cup \left\{ y : T(\Pi(y)) \in B(0, \delta_1) \subseteq \mathbb{R}^2 \text{ and } \text{dist}(y, \mathbb{S}^2) < \delta_1^5 \eta\left(\frac{T(\Pi(y))}{\delta_1}\right) \right\},$$

where Π stands for the nearest point projection from $\partial\Omega$ to $\partial\mathbb{B}^3$. Multiplying η by a positive constant, we may obviously assume that each partial derivative up to order 3 of $\delta_1^5 \eta\left(\frac{\cdot}{\delta_1}\right)$ does not exceed δ_1^2 in absolute value.

Next we define a new mapping on the boundary of Ω, $\varphi^* : \partial\Omega \to \mathbb{S}^2$, by setting $\varphi^*(x) = \varphi_2(\Pi(x))$. By this definition we have $\varphi^* \equiv \varphi(q)$ on $B(q^*, 4\delta_1) \cap \partial\Omega$. In particular, each partial derivative of φ^* is equal to 0 on that set and therefore does not exceed δ_1^2 in absolute value.

Let $u^* : \Omega \to \mathbb{S}^2$ be any minimizer for φ^*. Then, $u^*|_{\mathbb{B}^3} : \mathbb{B}^3 \to \mathbb{S}^2$ is the unique minimizer for its boundary mapping $\varphi_3 := u^*|_{\partial\mathbb{B}^3}$ by Lemma 2.2.3. Note that by Theorem 2.2.1 u^* is of class C^2 on $B(q^*, 2\delta_1) \cap \Omega$ up to the boundary. This regularity assertion can easily be improved. To this end, we fix any smooth bounded domain $V \subset B(q^*, 2\delta_1) \cap \Omega$ with, say, $V \supset \Omega \cap B(q^*, \frac{3}{2}\delta_1)$, and with a C^∞ boundary $\partial V \supset B(q^*, \frac{3}{2}\delta_1) \cap \partial\Omega$. An easy inductive argument using linear Schauder theory, see [21, Thm. 6.19], applied to $u^* |_V$ and the elliptic system $-\Delta u = |\nabla u|^2 u \equiv f$ on V, shows that in fact u^* is of class $C^\infty(\overline{V})$. Therefore, φ_3 is of class $C^\infty(\mathbb{S}^2, \mathbb{S}^2)$.

Next we show that $\deg(\varphi_3) = 0$. By the Uniform Boundary Regularity Theorem 2.2.1, the energy minimizer u^* is two times continuously differentiable at least on $B(q^*, \delta_1)$ and each of its partial derivatives does not exceed δ_1, so that $|u^*(x) - u^*(y)| \leq \sqrt{3}\,\delta_1 |x - y|$ by the mean value theorem. Thus, if $x \in \partial\Omega \cap B(q^*, \delta_1)$ and $y \in \mathbb{S}^2 \cap B(q^*, \delta_1)$, then

$$|\varphi(q) - \varphi_3(y)| = |u^*(x) - u^*(y)| \leq \sqrt{3}\,\delta_1 |x - y| < 4\delta_1^2\,.$$

To compute the degree of φ_3, choose any regular value of φ_3 away from $\mathbb{S}^2 \cap B\big(\varphi(q), 4\delta_1^2\big)$. Its preimages under φ_3 will be the same as those under φ_2, and (as in the proof of Lemma 2.2.5) both φ_2 and φ_3 are equal in a small neighborhood of each of those preimages. Thus, the degree of φ_3 must be the same as that of φ_2, i. e., equal to zero.

Finally, since by Theorem 2.2.1 each partial derivative of $u^* |_{\partial\mathbb{B}^3} = \varphi_3$ does not exceed δ_1 on $B(q^*, \delta_1)$, and on the set $\{\varphi_2 \neq \varphi_3\}$ the mapping φ_2 is constant, we have the estimate

$$\begin{aligned}
\|\varphi_2 - \varphi_3\|^2_{H^1(\partial\mathbb{B}^3)} &= \int_{\{\varphi_2 \neq \varphi_3\}} \Big(|\nabla_T \varphi_2 - \nabla_T \varphi_3|^2 + |\varphi_2 - \varphi_3|^2 \Big)\, d\sigma \\
&\leq 2 \int_{\{\varphi_2 \neq \varphi_3\}} |\nabla_T \varphi_3|^2 \, d\sigma + 2^2 \mathcal{H}^2(\{\varphi_2 \neq \varphi_3\}) \\
&< 10\pi\delta_1^2 \qquad \text{for } \delta_1 < 1.
\end{aligned}$$

Therefore, for sufficiently small δ_1 we conclude that $\|\varphi_2 - \varphi_3\|_{H^1(\partial\mathbb{B}^3)} < 10\delta_1$. □

Proof of Theorem 2.0.1. It remains to check that the mapping φ_3 given in Lemma

2.2.8 has the properties (i)–(iii).

(i) and (iii): By Lemma 2.2.8, a minimizer u_3 for the boundary condition φ_3 is unique and of degree 0. The proof that u_3 has at least $2M$ singularities is essentially the same as in Theorem 2.1.2, therefore we skip it.

(ii): Fix $\varepsilon > 0$. We now attune δ, δ_1 and j to obtain $\|\varphi - \varphi_3\|_{W^{1,p}} < \varepsilon$. We first choose $\delta > 0$ as in the proof of Lemma 2.2.5, then j as in the proof of Lemma 2.2.7. Next, we fix $\delta_1 < \frac{1}{10}$. Finally, we recall that φ differs from φ_3 only on the two spherical caps $\mathbb{S}^2 \cap B(\pm q, 2\delta)$ whose $\mathcal{H}^2$ measure is $8\pi\delta^2$, shrinking δ if necessary we obtain $\mathcal{H}^2(\{x \in \mathbb{S}^2 : \varphi(x) \neq \varphi_3(x)\}) < \frac{\varepsilon}{4}$ and hence

$$\begin{aligned}\|\varphi - \varphi_3\|_{W^{1,p}} < \|\varphi - \varphi_3\|_{L^p} + \|\nabla(\varphi - \varphi_1)\|_{L^p} + \|\nabla(\varphi_1 - \varphi_2)\|_{L^p} \\ + \|\nabla(\varphi_2 - \varphi_3)\|_{L^2} \cdot \left(\mathcal{H}^2(\{\varphi_2 \neq \varphi_3\})\right)^{\frac{2-p}{2p}} \\ < \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2N \cdot \left\|\nabla(\varphi_1 - [\varphi_1]_{\xi_i, 2/j})\right\|_{L^p} + 10\delta_1 \cdot \frac{\varepsilon}{4} < \varepsilon.\end{aligned}$$

□

Remark 2.2.9. Theorem 2.0.1 does not hold if we replace the norm $W^{1,p}(\mathbb{S}^2, \mathbb{S}^2)$ for $1 \leq p < 2$ by $W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$.

Proof. Let $\psi : \mathbb{S}^2 \to \mathbb{S}^2$ be a constant map. For this boundary condition of degree 0 there exists exactly one minimizer $u : \mathbb{B}^3 \to \mathbb{S}^2$, $u \equiv const$ for which the Lavrentiev gap phenomenon does not hold. If we modify the boundary map into $\widetilde{\psi}$, without changing its degree, so that the gap phenomenon would hold we would have to install at least 2 singular points into each minimizer. By [3, Theorem 2.12] the number of singularities is bounded by a universal constant C_{AL} times the boundary energy, which in our case would give $\int_{\mathbb{S}^2} |\nabla_T \widetilde{\psi}|^2 d\sigma \geq \frac{2}{C_{AL}}$. Therefore, the modified boundary mapping $\widetilde{\psi}$ with singularities for each corresponding minimizer cannot be arbitrary close to the constant map ψ in the $W^{1,2}$ norm. □

2.3 Mappings of nonzero degree

We will show how to modify a boundary mapping $\varphi : \mathbb{S}^2 \to \mathbb{S}^2$ of any degree, so that new singularities will appear for the corresponding (unique) minimizer. This modification will not change the topological degree of the boundary map. This will allow us to generalize Theorem 2.0.1 into maps of any degree.

Theorem 2.3.1. *Assume that* $\varphi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ *is an arbitrary smooth map and* $1 \le p < 2$. *Then, for each* $\varepsilon > 0$ *and each* $M \in \mathbb{N}$ *there exists a map* $\widetilde{\varphi} \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ *such that*

(i) $\deg \widetilde{\varphi} = \deg \varphi$;

(ii) $\|\varphi - \widetilde{\varphi}\|_{W^{1,p}} < \varepsilon$ *and* $\mathcal{H}^2\big(\{x \in \mathbb{S}^2 \colon \varphi(x) \neq \widetilde{\varphi}(x)\}\big) < \varepsilon$;

(iii) *the Dirichlet integral* E *has precisely one minimizer* $\widetilde{u} \in W^{1,2}_{\widetilde{\varphi}}(\mathbb{B}^3, \mathbb{S}^2)$; *moreover,* $\widetilde{u}$ *has at least* $\deg(\varphi) + M$ *point singularities in* $\mathbb{B}^3$.

Remark 2.3.2. As showed by Bethuel, Brezis and Coron in [7, Theorem 5] the Lavrentiev gap phenomenon always holds for maps of nonzero degree.

In the proof of Theorem 2.0.1 we found antipodal points $\pm q$ such that $\varphi(q) = \varphi(-q)$. This allowed us to make a modification on small disks in a neighborhood of one of those points and then an *opposite* modification on the antipode by using an orientation reversing map $-Id$. This time we will not have such antipodal points. The difference is that we will modify the boundary map φ only on one small disk centered at a point $q \in \mathbb{S}^2$. In order to preserve the degree of the map after inserting a bubble we will use another orientation reversing map — a rotation whose determinant is negative — to describe the modification on another disk.

Proof. We proceed similarly as in the proof of Theorem 2.0.1. Again the singularities will be inserted as described in Section 2.1. We recall that by [11] the number of singularities is estimated from below by the topological degree of the boundary map. We divide the proof into 3 steps corresponding to consecutive modifications of the boundary map. One can easily check that all estimates of $\|\varphi - \widetilde{\varphi}\|_{W^{1,p}(\partial \mathbb{B}^3)}$ remain unchanged, therefore, we skip them.

Step 1. We choose an arbitrary point on the sphere, say the north pole $x_N = (0,0,1)$ and we modify φ in such a way that the new map has a fixed value on a small disk cantered at x_N.

Definition 2.3.3 (cf. Definition 2.2.4)**.** Let $\delta > 0$ be any number such that $\mathcal{H}^2\left(\varphi(D(x_N, 2\delta))\right) < 4\pi$ and let $\varphi_1 : \mathbb{S}^2 \to \mathbb{S}^2$ denote any intermediate smooth mapping such that

(1) $\varphi_1(x) \equiv \varphi(x_N)$ for $x \in D(x_N, \delta)$;

(2) $\varphi_1(x) = \varphi(x)$ on $\mathbb{S}^2 \setminus \big(D(x_N, 2\delta)\big)$;

(3) On the remaining annulus $D(x_N, 2\delta)\setminus\overline{D(x_N, \delta)}$ the map φ_1 is given by a composition of φ with a smooth diffeomorphism from the annulus to the punctured disk $D(x_N, 2\delta) \setminus \{x_N\}$ with Lipschitz constant K. It is easy to see that one can have $K \leq C\delta^{-1}$ with an absolute constant C.

Similarly as in Lemma 2.2.5 the map φ_1 specified in Definition 2.3.3 above has the same degree as φ.

Step 2. Next we modify the map φ_1 on small disks inside $D(x_N, \delta)$. We will insert small bubbles. For clarity we will place all of the centers of the bubbles on the great arc $\gamma = \{(0, y, z) \in \mathbb{R}^3 : y^2 + z^2 = 1\}$. Each point from γ can be transformed into another point on γ in several ways, for example by a rotation around the x-axis by a certain angle. The most natural one would be a rotation which preserves the orientation:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

In order to control the degree of the boundary mapping we will use a rotation which reverses the orientation:

$$\widetilde{R}_x(\theta) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

Set $\alpha = \arccos\left(1 - 2\delta^2 + \frac{\delta^4}{2}\right)$ the length of the arc $\gamma \cap B(q, \delta)$.

Definition 2.3.4. Let $\xi_i = \left(0, \sin\left(\frac{i\alpha}{M+1}\right), \cos\left(\frac{i\alpha}{M+1}\right)\right) \in B(q, \delta)$ for $i = 1, \ldots, M$. For sufficiently large j's, with $2/j \ll \delta/2M$, we define $\varphi_2 : \mathbb{S}^2 \to \mathbb{S}^2$ as follows:

(1) $\varphi_2(x) = [\varphi_1]_{\xi_i, 2/j}(x)$ for $x \in D(\xi_i, 2/j)$;

(2) $\varphi_2(x) = \varphi_2(R_i x)$ for $x \in D(\widetilde{\xi}_i, 2/j)$;

(3) $\varphi_2 \equiv \varphi_1$ on $\mathbb{S}^2 \setminus \left(\bigcup_{i=1}^{\ell} D(\xi_i, 2/j) \cup D(\widetilde{\xi}_i, 2/j)\right)$,

where $R_i = \widetilde{R}_x(\theta_i)$, θ_i is the angle by which the point ξ_i is rotated by $\widetilde{R}_x$ into $\widetilde{\xi}_i$, and for $\xi_i = (0, y_i, z_i)$ we denote by $\widetilde{\xi}_i = (0, -y_i, z_i)$. We recall that $[\psi]_{a,b}$ is the modification of ψ in the spherical cap $D(a, b)$, see Definition 2.1.1.

Note that this modification does not change the degree of the boundary map. As φ_2 on each cap $D(\xi_i, 2/j)$ is either an orientation-preserving (degree 1) or an orientation-reversing (degree -1) map onto $\mathbb{S}^2$, while, because R_i reverses the orientation, on $D(\widetilde{\xi}_i, 2/j)$ it is of opposite orientation (respectively degree -1 or degree 1) map onto $\mathbb{S}^2$.

Step 3. The remaining part is to modify the map φ_2 to obtain uniqueness of the minimizer for the new map $\widetilde{\varphi}$. We do it as in Lemma 2.2.8, the only difference is that we must choose the point q^* away from the inserted bubbles. One can obtain it by choosing, e.g.,

$$q^* = \left(0, \sin\left(\frac{-\alpha}{4M}\right), \cos\left(\frac{-\alpha}{4M}\right)\right).$$

The proof is complete. $\square$

2.4 A remark on the nonuniqueness in the class of minimizing mappings

In the following we explain how the boundary mapping constructed in Theorem 2.0.1 leads to a nonuniqueness example, similar (in the construction) to that of [27, Section 5].

Remark 2.4.1. Fix any $M \in \mathbb{N}$. For each number $k \in \mathbb{Z}$ there exists a mapping $\varphi_\tau \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, $\deg(\varphi_\tau) = k$, which serves as a boundary data for at least two energy minimizing maps from $\mathbb{B}^3$ to $\mathbb{S}^2$ having different number of singularities (one of them at most M; the other one at least $M+2$).

Indeed, let $\psi \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ be any mapping having exactly $M \in \mathbb{N}$ singular points such that $\deg(\psi) = k$ and for which there exists unique energy minimizer $w \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$. We construct $\widetilde{\psi} \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$ as in Theorem 2.3.1 for which $\deg(\widetilde{\psi}) = k$ and there exists precisely one energy minimizing mapping $\widetilde{w} \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ with at least $M+2$ singularities.

Since the mappings ψ and $\widetilde{\psi}$ are homotopic, there exists a smooth family of smooth mappings $\{\varphi_t\}_{t\in[0,1]}$ such that $\varphi_0 = \psi$ and $\varphi_1 = \widetilde{\psi}$.

From the Stability Theorem obtained in [29] we deduce that for t sufficiently close to 0 each energy minimizer with boundary data φ_t has exactly M singular points. Let

$$\tau = \sup\big\{t \in [0,1] : \text{ each energy minimizer with boundary data } \varphi_t \\ \text{has at most } M \text{ singular points in } \mathbb{B}^3\big\}.$$

Then, $0 < \tau < 1$. We may choose a sequence $s_i \nearrow \tau$ and a sequence of energy minimizing maps $u_i \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ having at most M singular points such that $u_i\,|_{\mathbb{S}^2}=\varphi_{s_i}$. Similarly we choose $t_i \searrow \tau$ with a sequence of minimizing mappings $v_i \in W^{1,2}(\mathbb{B}^3, \mathbb{S}^2)$ having at least $M+2$ singularities, $v_i\,|_{\mathbb{S}^2}=\varphi_{t_i}$. (Since we consider boundary maps of the same degree and it is known that the degree of a minimizing harmonic map on a small sphere around a singular point is ± 1, the number of singular points must jump at least by 2.) Passing to subsequences without changing notation, we obtain $u_i \to u$ and $v_i \to v$, the convergence is strong in $W^{1,2}$ and $u\,|_{\mathbb{S}^2}=\varphi_\tau = v\,|_{\mathbb{S}^2}$.

The mapping u has at most M singularities. (It is plausible that one might prove that the number of singularities equals M, by choosing the homotopy appropriately.) Indeed, assume u has at least $M+2$ singular points. Then, by [3, Theorem 1.8 (2)], in an arbitrarily small ball around each singularity of u there would be a singularity of u_i for i sufficiently large, a contradiction.

On the other hand, v has at least $M+2$ point singularities. Recall that each v_i has at least $M+2$ singularities and again by [3, Theorem 1.8] we know that singular points converge to singular points. To see that v has at least $M+2$ singularities we must exclude the possibility that some singularities of the v_i's come together and cancel. By [3, Theorem 2.1] there exists a universal constant C such that if d denotes the distance from a singularity a to the boundary of the ball then there is no other singularity within distance Cd from a. Thus, the singularities of v_i cannot merge in the interior of $\mathbb{B}^3$. Moreover, by Theorem 2.2.1, there is a neighborhood of the boundary which contains no singularities of v and of the v_i's sufficiently close to v (as the φ_{t_i}'s and φ_τ are close to each other in C^∞). This precludes the case of singularities merging in the limit at the boundary.

At the end, we wish to state a problem related to the aforementioned nonuniqueness example.

Problem 2.4.2. Fix $1 \le p < 2$. Does there exists a constant $C = C(p)$ with the following property:

For each pair of smooth maps $\psi_i \colon \mathbb{S}^2 \to \mathbb{S}^2$, $i = 1, 2$ of the same degree, such that $\{\psi_1 \neq \psi_2\} \subset B(x,r) \cup B(-x,r)$ for some $x \in \mathbb{S}^2$ and $r > 0$ small, there is a homotopy $\psi_t \colon \mathbb{S}^2 \to \mathbb{S}^2$, $\psi_t \in C^\infty$ for $t \in [0,1]$, such that

$$\sup_{t\in[0,1]} \|\psi_0 - \psi_t\|_{W^{1,p}} \leq C(p)\|\psi_0 - \psi_1\|_{W^{1,p}}\,?$$

(The case of $p \geq 2$ is also interesting but not related in a direct way to our problem.)

A positive answer would allow one to strengthen Remark 2.4.1 in the following way: for each smooth $\psi_0 \colon \mathbb{S}^2 \to \mathbb{S}^2$ of any degree, each $M \in \mathbb{N}$ and each $\varepsilon > 0$ there exists $\psi_\tau \in C^\infty(\mathbb{S}^2, \mathbb{S}^2)$, $\deg(\psi_\tau) = \deg(\psi_0)$ and $\|\psi_0 - \psi_\tau\|_{W^{1,p}} < \varepsilon$, which serves as a boundary data for at least two energy minimizing maps from $\mathbb{B}^3$ to $\mathbb{S}^2$ having different number of singularities (one of them at most M; the other one at least $M+2$).

Chapter 3

Conditional boundary regularity for minimizing biharmonic maps

In this chapter we focus on the boundary regularity for *minimizing* biharmonic maps. Our original motivation to study this topic was the desire to understand how, in the model case $u : B^5 \to \mathbb{S}^4$, to modify a boundary map in order to force singularities to appear in the corresponding minimizer of the biharmonic energy. A possible applications of the boundary regularity are wide: We expect that such a result can be used to obtain general nonuniqueness of biharmonic maps as well as examples of nonuniqueness in the class of minimizing biharmonic maps. Furthermore, we suspect a result stating that the boundary data having unique minimizing map are dense in some boundary norms stronger than the natural trace norm.

In the case of second order problems a boundary regularity result for *minimizing* harmonic maps was proved by Schoen and Uhlenbeck [49], for *minimizing* p-harmonic maps[1] by Hardt and Lin [28] and independently by Fuchs [16]. There is also a conditional result for *stationary* harmonic maps [55], which under the assumptions of a boundary monotonicity formula for stationary maps yields a partial regularity at the boundary. See also [46] for a boundary regu-

[1] p-harmonic maps are defined similarly as harmonic maps, they are critical points of the p-energy, i.e., $E_p(u) = \int_\Omega |\nabla u|^p \, dx$ among maps in $W^{1,p}(\Omega, \mathcal{N})$.

larity result for another class of harmonic maps.

The main reason for which no partial boundary regularity result is known for *stationary* harmonic maps is the lack of a boundary monotonicity formula. The boundary regularity results for *minimizing* harmonic and p-harmonic maps crucially depend on the existence of a monotonicity formula at the boundary. Such a formula is obtained by reflecting a comparison map used in the proof of a monotonicity formula for minimizing maps, see [49, Lemma 1.3]. A boundary monotonicity formula may be obtained for sufficiently smooth *stationary* harmonic maps. According to [36] such a formula was obtained first by W.Y. Ding, see also [13] and references therein.

Now let us pass to biharmonic setting. Let $\Omega \subset \mathbb{R}^m$ be a smooth, bounded domain and assume that $\mathcal{N}$ is a compact C^3-manifold with $\partial\mathcal{N} = \emptyset$. We recall that

$$W^{2,2}(\Omega, \mathcal{N}) = \{u \in W^{2,2}(\Omega, \mathbb{R}^\ell) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega\}$$

and the Hessian energy is given by

$$H(u) = \int_\Omega |\Delta u|^2 \, dx.$$

A map $u \in W^{2,2}(\Omega, \mathcal{N})$ is called *minimizing biharmonic* if, for all maps $v \in W^{2,2}(\Omega, \mathcal{N})$ satisfying $u - v \in W_0^{2,2}$, it holds

$$H(u) \leq H(v).$$

We will be interested in the boundary regularity of minimizing biharmonic maps, so we assume that u satisfies the Dirichlet boundary condition. More precisely, let $\varphi \in C^\infty(\Omega_\delta, \mathcal{N})$ be given for a $\delta > 0$, where

$$\Omega_\delta = \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

We assume that u satisfies

$$\left(u, \frac{\partial u}{\partial \nu}\right)\Bigg|_{\partial\Omega} = \left(\varphi, \frac{\partial \varphi}{\partial \nu}\right)\Bigg|_{\partial\Omega}, \tag{3.0.1}$$

where ν denotes the outer normal vector.

Similarly as in the case of harmonic maps a boundary monotonicity formula may be proved for sufficiently smooth biharmonic maps. Gong, Lamm, and Wang gave a biharmonic counterpart of the conditional boundary regularity result for stationary harmonic maps [55] and proved that under the assumption of a boundary monotonicity formula stationary harmonic maps are smooth up to the boundary with off a singular set of codimension 4.

We show that the conditional partial regularity result of Gong et al. can be strengthen to full boundary regularity in the case of *minimizing* biharmonic maps.

Theorem 3.0.3. *Let $m \geq 5$, $\varphi \in C^\infty(\Omega_\delta, \mathcal{N})$ for some $\delta > 0$, assume that $u \in W^{2,2}(\Omega, \mathcal{N})$ is a minimizing biharmonic map, which satisfies the boundary monotonicity inequality* (3.1.6). *Then, u is smooth on a full neighborhood of the boundary $\partial\Omega$.*

We conjecture that the assumption (3.1.6) is satisfied by all minimizing biharmonic maps with sufficiently smooth boundary data.

Similarly as in the case of harmonic [49] and p-harmonic [28] maps the complete boundary regularity is based on the nonexistence of nonconstant boundary tangent maps. We will consider tangent maps at the boundary and prove that they arise as strong limits of rescaled maps on some smaller domain, containing a portion of the boundary. In order to obtain a strong convergence from a sequence we initially only know is uniformly bounded in $W^{2,2}$ we will prove an analogue of Scheven's compactness result.

Scheven, following the result for harmonic maps [36], has based his argument on an analysis of defect measures. We follow his general strategy, modifying numerous technical details so that the proof works for a map obtained via a higher order reflection across a flat portion of the boundary.

We will not prove that a limit u of a weakly convergent sequence of minimizing maps $(u_j)_{j\in\mathbb{Z}}$ is again minimizing. Such a result, is known only in the case when

$\mathcal{N} = \mathbb{S}^{\ell-1}$ (see [31, Lemma 3.3.]). In the case of harmonic maps, such a result is known for minimizing maps into arbitrary target manifolds. Since the maps u_j and u slightly differ on the boundary one may not use directly the definition of minimizing map to compare their energies. A tool for comparing those energies was provided by Luckhaus and his lemma in [37]. Unfortunately we may not use directly Luckhaus's lemma to maps from $W^{2,2}$. An analogue of this lemma is not known in the biharmonic setting.

Instead, similarly as in [48, 49] and [28], for us it will be sufficient that in very simple situations a limit of *minimizing* maps is again *minimizing*. By a repeated formation of tangent boundary maps we arrive at a boundary tangent map which has a special form — it is independent of the first $(m-5)$-variables, homogeneous of degree 0, whose only discontinuity may occur at the origin. It was proved by Scheven that such maps are in fact minimizing (cf. Lemma 3.3.4).

The chapter is organized as follows. In Section 3.1 we state various facts about biharmonic maps which will be needed in the following proofs. In Section 3.2 we give a boundary analogue of Scheven's compactness result for minimizing biharmonic maps. In Section 3.3 we focus on the tangent maps at the boundary. Without any additional assumptions we prove that there exist no nonconstant boundary tangent maps and finally give the proof of the main result.

Notation. We use the following notation

$$\int_{\partial B_r \setminus \partial B_\rho} f \, d\mathcal{H}^{m-1} := \int_{\partial B_r} f \, d\mathcal{H}^{m-1} - \int_{\partial B_\rho} f \, d\mathcal{H}^{m-1}.$$

For balls centered at the origin we often write $B_r(0) = B_r$, for B_1 we simply write B. Sometimes to emphasize the dimension of a ball we will write B^k, for a k-dimensional ball. We also write $\mathbb{R}^m_+ = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_m > 0\}$, $\mathbb{R}^m_- = \{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_m < 0\}$, $B_r^+(a) = B_r(a) \cap \mathbb{R}^m_+$, and $B_r^-(a) = B_r(a) \cap \mathbb{R}^m_-$. For the the flat part of the boundary of ∂B_σ^+ we use

$$T_\sigma = \{x \in B_\sigma \colon x_m = 0\}.$$

We denote the average of u over $B_r(a)$ by

$$(u)_{B_r(a)} = \fint_{B_r(a)} u(x)\, dx := \frac{1}{|B_r(a)|} \int_{B_r(a)} u(x)\, dx.$$

In what follows we will use sequences and partial derivatives, for partial derivatives we write

$$\frac{\partial}{\partial x_i} u = \partial_i u = u_{x_i},$$

while u_i will denote the i-th element of a sequence of maps $(u_j)_{j\in\mathbb{N}}$. For simplicity we will try to use the following convention: Letters $u,\ v,\ w$ will be used to denote maps from B^+ into $\mathcal{N}$, whereas $\widetilde{u},\ \widetilde{v}, \widetilde{w}$ will denote maps from B into $\mathbb{R}^\ell$. The constant C traditionally stands for a general constant and may vary from line to line.

Added in proof. When this thesis was completed the author has learned that S. Altuntas [4] proved that minimizing biharmonic maps satisfy boundary monotonicity formula for sufficiently smooth boundary conditions. Therefore, the result is no longer conditional and might be strengthened to boundary regularity for minimizing maps.

3.1 Facts about regularity of biharmonic maps

In this section we gather facts from the regularity theory of biharmonic maps, which will be needed later on. We begin by recalling the definition of Morrey spaces, for more details see, e.g., [20, Chapter 3].

Let $p \geq 1$, $\lambda > 0$, and Ω be a bounded domain in $\mathbb{R}^m$. We say that a function $f \in L^p(\Omega)$ belongs to the Morrey space $L^{p,\lambda}(\Omega)$ if

$$\|f\|^p_{L^{p,\lambda}(\Omega)} := \sup_{a\in\Omega,\, r>0} r^{-\lambda} \int_{B_r(a)\cap\Omega} |f(x)|^p\, dx < \infty. \tag{3.1.1}$$

The following boundary decay estimate for biharmonic maps that satisfy a smallness condition in Morrey norm is due to Gong, Lamm, and Wang, see [22, Lemma 3.1, p. 179].

Lemma 3.1.1. *There exists* $\varepsilon > 0$ *and* $\theta \in \left(0, \frac{1}{2}\right)$ *such that if* $u \in W^{2,2}(B^+, \mathcal{N})$ *is a biharmonic map satisfying*

$$u\Big|_{T_1} = \varphi\Big|_{T_1} \quad \textit{and} \quad \frac{\partial u}{\partial x_m}\Big|_{T_1} = \frac{\partial \varphi}{\partial x_m}\Big|_{T_1} \quad \textit{for some} \quad \varphi \in C^\infty(\overline{B^+}, \mathcal{N}) \tag{3.1.2}$$

and

$$\left\|\nabla^2 u\right\|^2_{L^{2,m-4}(B^+)} + \|\nabla u\|^4_{L^{4,m-4}(B^+)} \le \varepsilon, \tag{3.1.3}$$

then

$$\|\nabla u\|_{L^{2,m-2}(B^+_\theta)} \le \frac{1}{2}\|\nabla u\|_{L^{2,m-2}(B^+_\theta)} + C\theta\|\nabla\varphi\|_{C^1(B^+)}. \tag{3.1.4}$$

In particular, $u \in C^\infty\left(\overline{B^+_{\frac{1}{2}}}, \mathcal{N}\right)$.

The following theorem is a key-ingredient in the regularity theory. It was first proved for sufficiently regular maps by Chang, Wang, and Yang in [12, Proposition 3.2.] and for the general case by Angelsberg in [5]. We employ the notation for Φ_u from [47, Theorem 2.3].

Theorem 3.1.2 (Monotonicity formula)**.** *Let* $u \in W^{2,2}(B^+_R, \mathcal{N})$ *be stationary biharmonic and* $a \in B_{R/4}$*. Then the expression*

$$\begin{aligned}\Phi_u(a,r) :=& \\ & r^{4-m}\int_{B_r(a)} |\Delta u|^2\,dx \\ & + 2\int_{\partial B_r(a)} \left(\frac{(x^i-a^i)u_{x_j}u_{x_ix_j}}{|x-a|^{m-3}} - 2\frac{\left((x^i-a^i)u_{x_i}\right)^2}{|x-a|^{m-1}} + 2\frac{|\nabla u|^2}{|x-a|^{m-3}}\right)\,d\mathcal{H}^{m-1}\end{aligned}$$

is well defined for a.e. $0 < r \leq R/4$ *and monotonously nondecreasing for all* r *outside a set of measure zero. more precisely, there holds for a.e.* $0 < \rho < r \leq R/4$

$$\Phi_u(a,r) - \Phi_u(a,\rho) = \\ 4\int_{B_r(a)\setminus B_\rho(a)} \left(\frac{\left(u_{x_j} + (x^i - a^i)u_{x_ix_j}\right)^2}{|x-a|^{m-2}} + (m-2)\frac{\left((x^i-a^i)u_{x_i}\right)^2}{|x-a|^m} \right) dx. \tag{3.1.5}$$

Remark 3.1.3. The Angelsberg's proof of monotonicity formula, roughly speaking, is based on inserting a correct test function in the so-called first variational formula (an equation which follows from the definitions of stationary harmonic maps). This idea follows the proof of monotonicity formula for stationary *harmonic* maps (see, e.g., [52]), which in turn is based on the proof of the monotonicity formula for Yang–Mills fields, see[43]. The first publication of a monotonicity formula for *minimizing* harmonic maps seems to be [48, Proposition 2.4.], which by reflection arguments can be extended to boundary monotonicity formula for minimizing harmonic maps, see [49, Lemma 1.3.]. The proof in [48] proof relies on constructing a comparison map. It would be interesting to obtain an analogous proof for *minimizing* biharmonic maps.

Definition 3.1.4. Consider $u \in W^{2,2}(\Omega, \mathcal{N})$ a minimizing biharmonic map under the boundary condition (3.0.1). We say that u satisfies the boundary monotonicity inequality if there exist $R_0 > 0$ and $C = C(m, \partial\Omega, \delta, \|\varphi\|_{C^4(\Omega_\delta)})$ such that for any $a \in \partial\Omega$ and $0 < \rho \leq r \leq R_0$, there holds

$$H_u^+(a,\rho) + e^{C\rho}R_u^+(a,\rho) + P_u^+(a,\rho,r) \\ \leq e^{Cr}H_u^+(a,r) + e^{Cr}R_u^+(a,r) + Cre^{Cr}, \tag{3.1.6}$$

where

$$H_u^+(a,\tau) := \tau^{4-m} \int\limits_{B_\tau(a)\cap\Omega} |\nabla^2(u-\varphi)|^2\, dx, \tag{3.1.7}$$

$$P_u^+(a,\rho,r) := \int_{(B_r(a)\setminus B_\rho(a))\cap\Omega} \frac{|(u-\varphi)_{x_j} + (x-a)^i(u-\varphi)_{x_ix_j}|^2}{|x-a|^{m-2}}\, dx \\ + (m-2)\int_{(B_r(a)\setminus B_\rho(a))\cap\Omega} \frac{|(x-a)^i(u-\varphi)_{x_i}|^2}{|x-a|^m}\, dx \tag{3.1.8}$$

and

$$R_u^+(a,\tau) = (F_u^+(a,\tau) + G_u^+(a,\tau)) \tag{3.1.9}$$

for

$$\begin{aligned}
F_u^+(a,\tau) := 2\tau^{3-m} & \int\limits_{\partial B_\tau(a)\cap\Omega} (x-a)^i (u-\varphi)_{x_j} (u-\varphi)_{x_i x_j}\, d\mathcal{H}^{m-1} \\
& - 4\tau^{3-m} \int\limits_{\partial B_\tau(a)\cap\Omega} \left(\frac{|(x-a)^i (u-\varphi)_{x_i}|^2}{|x-a|^2} - |\nabla(u-\varphi)|^2 \right) d\mathcal{H}^{m-1}, \\
G_u^+(a,\tau) := 2\tau^{4-m} & \int\limits_{\partial B_\tau(a)\cap\Omega} \left(\left\langle \Delta(u-\varphi), \frac{\partial}{\partial r}(u-\varphi) \right\rangle \right. \\
& \qquad \left. - \left\langle \nabla(u-\varphi), \frac{\partial}{\partial r}(\nabla(u-\varphi)) \right\rangle \right) d\mathcal{H}^{m-1}.
\end{aligned}$$

In the latter $\frac{\partial}{\partial r}$ is the directional derivative in the direction of the outward pointing unit normal for $\partial B_\tau(a)$.

Remark 3.1.5. We conjecture that the boundary monotonicity formula (3.1.6) is satisfied by all minimizing biharmonic maps with sufficiently smooth boundary data. Despite some efforts we were not able to prove (or disprove) this conjecture. The case of *stationary* biharmonic maps seems to be more complicated. Up to our best knowledge it is not known even in the corresponding second order problems whether in general *stationary harmonic* maps satisfy a boundary monotonicity formula.

Remark 3.1.6. Any $W^{4,2}$ biharmonic map satisfies inequality (3.1.6). For derivation see [22, Section 2].

The following facts hold also for *stationary* biharmonic maps under the assumptions of boundary monotonicity formula. But, as we are focused on the boundary regularity for *minimizing* maps and expect that for such maps the condition (3.1.6) is automatically satisfied we state them for minimizing harmonic maps.

The following result is a consequence of the boundary monotonicity formula and for the proof we refer to the appendix B.

Lemma 3.1.7. *Let $u \in W^{2,2}(B^+, \mathcal{N})$ be a minimizing biharmonic map with boundary value φ as in* (3.0.1)*, satisfying the boundary monotonicity formula* (3.1.6) *and let additionally $\|u-\varphi\|_{W^{2,2}(B^+)} < \infty$. Then, for some $\Lambda > 0$ we have $\left\|\nabla^2(u-\varphi)\right\|_{L^{2,m-4}(B^+)} < \Lambda$.*

The following is also a consequence of the boundary monotonicity formula, the proof can be found in [22, Lemma 4.1]. (Compare also in the interior case in [12, Lemma 4.8], [58, Lemma 5.3], [53, Appendix B]).

Lemma 3.1.8. *Assumptions that the assumptions of Theorem 3.0.3 are fulfilled. There exist $\varepsilon_1 > 0$, $\theta \in (0,1)$, $C_1 = C_1(m, \Omega, \mathcal{N})$, and $R_1 = R_1(R_0, \varepsilon_1)$ such that if for $a \in \partial\Omega$ and $0 < \tau \leq R_1$*

$$\tau^{4-m} \int_{B_\tau(a)\cap\Omega} \left(|\nabla^2 u|^2 + \tau^{-2}|\nabla u|^2\right) dx \leq \varepsilon_1^2, \tag{3.1.10}$$

then

$$\sup_{B_\rho(b)\subset(B_{\theta\tau}(a)\cap\Omega)} \rho^{4-m} \int_{B_\rho(b)\cap\Omega} (|\nabla^2 u|^2 + |\nabla u|^4)\, dx \leq C_1\varepsilon_1. \tag{3.1.11}$$

The following epsilon regularity result is the main result of [22].

Theorem 3.1.9. *Let $m \geq 5$, $\varphi \in C^\infty(\Omega_\delta, \mathcal{N})$ for some $\delta > 0$, assume that $u \in W^{2,2}(\Omega, \mathcal{N})$ is a minimizing biharmonic map, which satisfies the boundary monotonicity inequality* (3.1.6)*. Then, there exists an $\varepsilon_2 > 0$ such that $u \in C^\infty(\overline{\Omega} \setminus \Sigma)$, where the singular set is given by*

$$\Sigma := \left\{ a \in \overline{\Omega} : \liminf_{r\searrow 0} r^{4-m} \int_{B_r(a)\cap\Omega} \left(|\nabla^2 u|^2 + |\nabla u|^4\right) dx \geq \varepsilon_2 \right\}$$

and $\mathcal{H}^{m-4}(\Sigma) = 0$.

Lemma 3.1.10. *There are constants $\varepsilon_3 > 0$ and $\theta \in (0,1)$ such that under the assumptions of Theorem 3.0.3 a minimizing biharmonic map $u \in C^\infty(B_r^+, \mathcal{N})$ with boundary values φ as in 3.0.1 with*

$$r^{4-m} \int_{B_r^+(a)} \left(|\nabla^2 u|^2 + r^{-2}|\nabla u|^2\right) dx \leq \varepsilon_3$$

satisfies $\|\nabla u\|_{C^2(B_{\theta r}(a)\cap\Omega)} \leq 1$.

Proof. The proof follows [47, proof of Theorem 2.6]. We list the following boundary analogues needed to replace the interior facts used in [47]:

- Lemma 2.4 (i) in [47] by Lemma 3.1.7;
- Lemma 2.4 (ii) in [47] by Lemma 3.1.8;
- Theorem 2.5 in [47] by Theorem 3.1.9;
- Theorem 3.1 from [41] by Theorem 4.1 from [41].

□

Similarly as in [47, Corollary 2.7], we obtain the following consequence.

Corollary 3.1.11. *Let* $\varepsilon_0 := \min(\varepsilon_1^2, C_1\varepsilon_1, \varepsilon_2, \varepsilon_3)$ *with constants introduced in Lemma 3.1.8, Theorem 3.1.9 and Lemma 3.1.10. Then there exists* $\theta \in (0,1)$ *such that for any minimizing biharmonic map* $u \in W^{2,2}(B_r^+(a), \mathcal{N})$*, the estimate*

$$r^{4-m} \int_{B_r^+(a)} \left(|\nabla^2 u|^2 + r^{-2}|\nabla u|^2\right) \, dx \leq \varepsilon_0 \tag{3.1.12}$$

implies $u \in C^3(B_{\theta r}(a) \cap \Omega, \mathcal{N})$ *and* $\|u\|_{C^3(B_{\theta r}(a)\cap\Omega,\mathcal{N})}$ *is bounded by a constant depending only on* $\mathcal{N}$*.*

3.2 Compactness at the boundary

From now on we will assume that minimizing biharmonic maps satisfy the boundary monotonicity formula, cf. Definition 3.1.4. We would like to emphasize once again that this is an artificial assumption and we conjecture that this formula is satisfied by all minimizing biharmonic maps with sufficiently smooth boundary data.

For simplicity we will assume that $\Omega = B_4^+$. In this situation our boundary condition states that

$$\left(u, \frac{\partial u}{\partial x_m}\right)\Bigg|_{T_4} = \left(\varphi, \frac{\partial \varphi}{\partial x_m}\right)\Bigg|_{T_4}. \tag{3.2.1}$$

The following compactness theorem is due to Scheven, cf. [47, Theorem 1.5.]. Here we present a boundary analogue of this statement.

Theorem 3.2.1. *There is a constant $C_\varphi = C_\varphi(m)$ with the following property.*

Let $M(B_4^+) \subseteq W^{2,2}(B_4^+, \mathcal{N})$ be the closure with respect to the $W^{2,2}_{loc}$-topology of the set of minimizing biharmonic maps satisfying the boundary monotonicity formula (3.1.6). *Let $u_i \in M(B_4^+)$, be a sequence of maps with boundary values φ_i in the sense of* (3.0.1) *and $\varphi \in C^\infty(B_4^+)$. Moreover, let*

$$\begin{aligned}
&\varphi_i \to \varphi \quad \textit{strongly in } W^{2,2}_{loc} \textit{ and } L^6_{loc},\\
&\sup_{i\in\mathbb{N}} \|\varphi_i\|_{C^2} < \frac{\varepsilon_0}{2}, \quad \sup_{i\in\mathbb{N}} \|\varphi_i\|_{C^3(B_4^+)} < C(\mathcal{N}),\\
&\sup_{i\in\mathbb{N}} \|u_i\|_{W^{2,2}(B_4^+)} < \infty, \;\; \textit{and} \qquad \int_{B^+} |\Delta\varphi|^2\, dx < C_\varphi,
\end{aligned} \tag{3.2.2}$$

where ε_0 is the constant from Corollary 3.1.11. *Then, there is a map $u \in M(B_4^+)$ such that, up to a subsequence, $u_i \to u$ in $W^{2,2}(B^+_{1/2}, \mathcal{N})$ as $i \to \infty$.*

Remark 3.2.2. In fact the L^6 convergence of $\nabla\varphi_i$ can be relaxed to $L^{4+\epsilon}$ for any $\epsilon > 0$. For this purpose, in the proof of Theorem 3.2.1, one should replace Young's inequality with exponents 3 and $\frac{3}{2}$ in the estimate (3.2.15), by

$$|\nabla\varphi_i|^2 |\nabla u_i|^2 \le \frac{2|\nabla\varphi_i|^{4+\epsilon}}{4+\epsilon} + \frac{(2+\epsilon)|\nabla\widetilde{u}_i|^{\frac{8+2\epsilon}{2+\epsilon}}}{4+\epsilon}.$$

To proceed with the proof, the only important thing for us in this estimate is that the exponent at $|\nabla\widetilde{u}_i|$ stays below 4.

We will extend $(u - \varphi)$ onto the whole ball by a higher order reflection, for properties of the reflection see, e.g, [1, proof of Theorem 4.26]. We choose such a reflection, which preserves C^3 continuity of a map. Let $u \in W^{2,2}(B_4^+, \mathcal{N})$ with boundary values φ as in (3.0.1), then the reflection $\widetilde{u}$ is given by

$$\widetilde{u}(x) = \begin{cases} u(x) - \varphi(x) & \text{for } x_m \geq 0, \\ \sum_{i=1}^4 \lambda_i (u - \varphi)\left(x', -\frac{x_m}{i}\right) & \text{for } x_m < 0, \end{cases} \tag{3.2.3}$$

where $x' = (x_1, \ldots, x_{m-1})$ denotes the first $(m-1)$-coordinates and the constants λ_i are determined by the system

$$\sum_{i=1}^4 \lambda_i \left(-\frac{1}{i}\right)^j = 1 \quad \text{for } j = 0, 1, 2, 3.$$

Here $\lambda_1 = -10$, $\lambda_2 = 160$, $\lambda_3 = -405$, and $\lambda_4 = 256$. We note that $\widetilde{u}$ in general does not have values in $\mathcal{N}$. Next, observe that since $u - \varphi \in W^{2,2}(B_4^+, \mathbb{R}^\ell)$ we have $\widetilde{u} \in W^{2,2}(B_4, \mathbb{R}^\ell)$. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be such that $|\alpha| \leq 3$ and

$$E_\alpha \widetilde{u}(x) = \begin{cases} u(x) - \varphi(x) & \text{for } x_m \geq 0, \\ \sum_{i=1}^4 \lambda_i \left(-\frac{1}{i}\right)^{\alpha_m} (u - \varphi)\left(x', -\frac{x_m}{i}\right) & \text{for } x_m < 0. \end{cases}$$

It follows easily that, if $u - \varphi \in C^3(\overline{B}_4^+, \mathbb{R}^\ell)$, then $\widetilde{u} \in C^3(B_4, \mathbb{R}^\ell)$ and

$$D^\alpha \widetilde{u}(x) = E_\alpha D^\alpha \widetilde{u}(x).$$

Moreover, if $a \in T_4$ and $r > 0$ is such that $B_r(a) \subset B_4$, then for $x_m \in B_r^-(a)$ we have $-\frac{x_m}{i} \in B_r^+(a)$. Thus, for $|\alpha| = 2$, we have the following estimate

$$\begin{aligned} \int_{B_r(a)} |\nabla^2 \widetilde{u}|^2 \, dx &= \int_{B_r^+(a)} |\nabla^2 (u - \varphi)|^2 \, dx \\ &\quad + \int_{B_r^-(a)} \left| \sum_{i=1}^4 \lambda_i \left(-\frac{1}{i}\right)^{\alpha_m} D^\alpha (u - \varphi)\left(x', -\frac{x_m}{i}\right) \right|^2 dx \\ &\leq C_{ref} \int_{B_r^+(a)} |\nabla^2 (u - \varphi)|^2 \, dx \end{aligned} \tag{3.2.4}$$

with $C_{ref} \leq 832$.

For reflected maps as in (3.2.3), in order to prove Theorem 3.2.1 we closely follow Scheven's Section 3.1 of [47], adjusting numerous technical details whenever necessary. All of the tools used by Scheven in proofs of Lemmata, Theorem and Corollaries have their boundary analogues, therefore, we will be rather brief in most of the proofs below. The difference here is that instead of working with minimizing maps themselves defined on half balls, we will work with higher order reflections of the differences of the mappings and their boundary data. Moreover, the boundary monotonicity formula has a little bit different form from the interior one and yields an additional term (which can still be well controlled).

We shall work with the following definition of convergence of pairs of sequences of maps and measures (slightly different from the one used in [47]).

Definition 3.2.3. For $i \in \mathbb{N}$, let $u_i \in W^{2,2}(\Omega, \mathcal{N})$ and ν_i be Radon measures on $\overline{\Omega}$. We abbreviate $\mu_i := (u_i, \nu_i)$. For a map $u_0 \in W^{2,2}(\Omega, \mathcal{N})$, a Radon measure ν_0 on $\overline{\Omega}$ and $\mu_0 := (u_0, \nu_0)$ we write $\mu_i \rightrightarrows \mu_0$ as $i \to \infty$ if and only if

$$\begin{aligned} u_i &\rightharpoonup u_0 && \text{weakly in } W^{2,2}(\Omega) \\ u_i &\to u_0 && \text{strongly in } W^{1,2}(\Omega) \text{ and for a.e. } x \in \Omega \\ |\nabla^2 u_i|^2 dx + \nu_i &\rightharpoonup |\nabla^2 u_0|^2 dx + \nu_0 && \text{in the sense of measures.} \end{aligned}$$

We recall that, by Lemma 3.1.7, any sequence of minimizing biharmonic maps u_i satisfying the boundary monotonicity formula (3.1.6) with boundary conditions φ_i as in (3.0.1), such that $\sup_{i\in\mathbb{N}} \|\widetilde{u}_i\|_{W^{2,2}(B_4^+)} < \infty$, where $\widetilde{u}_i$ is the reflection given by (3.2.3), satisfies also $\sup_{i\in\mathbb{N}} \|\widetilde{u}_i\|_{L^{2,m-4}}(B_4^+) < \Lambda$ for some $\Lambda > 0$.

We modify Scheven's set $\mathcal{B}_\Lambda^M$ to our purposes and let

$$\widetilde{\mathcal{B}}_\Lambda := \left\{ (\widetilde{u}, \widetilde{\nu}) \;\middle|\; \begin{array}{l} (\widetilde{u}_i, 0) \rightrightarrows (\widetilde{u}, \widetilde{\nu}), \text{ where } \widetilde{u}_i \in W^{2,2}(B_4, \mathbb{R}^\ell), \\ \widetilde{u}_i \text{ are the reflections given in (3.2.3)} \\ \text{of minimizing biharmonic maps} \\ u_i \in W^{2,2}(B^+, \mathcal{N}) \text{ with boundary values } \varphi_i \\ \text{as in (3.0.1), satisfying boundary} \\ \text{monotonicity formula (3.1.6),} \\ \text{assumptions (3.2.2), and } \left\|\nabla^2 \widetilde{u}_i\right\|^2_{L^{2,m-4}(B)} \le \Lambda \end{array} \right\}. \tag{3.2.5}$$

Combining boundary monotonicity (3.1.6) with (3.2.4) we obtain

$$\begin{aligned}
&\rho^{4-m} \int_{B_\rho(a)} |\nabla^2 \widetilde{u}|^2 \, dx + Ce^{C\rho} R_u^+(a,\rho) \\
&\quad \leq C \left(\rho^{4-m} \int_{B_\rho^+(a)} |\nabla^2 (u-\varphi)|^2 \, dx + e^{C\rho} R_u^+(a,\rho) \right) \\
&\quad \leq C \left(e^{Cr} r^{4-m} \int_{B_r^+(a)} |\nabla^2 (u-\varphi)|^2 \, dx + e^{Cr} R_u^+(a,r) + Cre^{Cr} \right) \\
&\quad \leq C \left(e^{Cr} r^{4-m} \int_{B_r(a)} |\nabla^2 \widetilde{u}|^2 \, dx + e^{Cr} R_u^+(a,r) + Cre^{Cr} \right).
\end{aligned} \tag{3.2.6}$$

Lemma 3.2.4. *Assume $\widetilde{\mathcal{B}}_\Lambda \ni (\widetilde{u}_i, 0) \rightrightarrows (c, \widetilde{\nu})$ as $i \to \infty$ for a constant $c \in \mathbb{R}^\ell$ and a Radon measure $\widetilde{\nu}$ on $\overline{B}_4$. Then for each $a \in B$, there is a subsequence $\{i_k\}_{k \in \mathbb{N}}$ such that for a.e. $0 < r < 1$ we have*

$$r^{4-m} \int_{B_r(a)} |\nabla^2 \widetilde{u}|^2 \, dx + Ce^{Cr} R_u^+(a,r) \to r^{4-m} \widetilde{\nu}(B_r(a)) \text{ as } k \to \infty \tag{3.2.7}$$

and for every $a \in B$ and for a.e. $0 < \rho \leq r < 1$

$$\rho^{4-m} \widetilde{\nu}(B_\rho(a)) \leq Cr^{4-m} \widetilde{\nu}(B_r(a)) + Cre^{Cr}. \tag{3.2.8}$$

If for a minimizing sequence of biharmonic maps $\{u_i\}$ we have $\widetilde{u}_i \to \widetilde{u}_0$ strongly in $W^{2,2}(B_4, \mathcal{N})$ as $i \to \infty$, then for every $a \in B$ there is a subsequence $\{i_k\} \subset \mathbb{N}$ such that

$$\begin{aligned}
H_{u_{i_k}}^+(a,r) &\to H_{u_0}^+(a,r) \quad \text{for a.e. } 0 < r < 1 \text{ as } k \to \infty; \\
R_{u_{i_k}}^+(a,r) &\to R_{u_0}^+(a,r) \quad \text{for a.e. } 0 < r < 1 \text{ as } k \to \infty,
\end{aligned} \tag{3.2.9}$$

where H^+ and R^+ are the quantities from the boundary monotonicity formula and are defined in (3.1.7) *and* (3.1.9) *respectively.*

Proof. The proof is essentially the same as the proof of [47, Lemma 3.2]. We briefly note the following differences. In order to obtain the convergence in (3.2.7) we need to ensure that the term $Ce^{Cr} R_{u_i,\varphi_i}^+(a,r)$ converges on a subsequence to 0. With addition to the argument used by Scheven we let $a \in B$ be fixed and

$$\widetilde{g}_i(\tau) := \int\limits_{\partial B_\tau(a)\cap B_4^+} \left(\left\langle \Delta\widetilde{u}_i, \frac{\partial}{\partial r}\widetilde{u}_i \right\rangle - \left\langle \nabla\widetilde{u}_i, \frac{\partial}{\partial r}(\nabla\widetilde{u}_i)) \right\rangle \right) d\mathcal{H}^{m-1}$$

for all $i \in \mathbb{N}$ and a.e $\tau \in (0,1]$. Then

$$\|g_i\|_{L_1([0,1])} \leq C \|\widetilde{u}_i\|_{W^{2,2}} \|\nabla\widetilde{u}_i\|_{L^2} \to 0 \quad \text{as} \quad i \to \infty,$$

which, after the same arguments as Scheven's, yields (3.2.7).

The inequality (3.2.8) is a consequence of the monotonicity for the reflected map $\widetilde{u}$ (3.2.6).

In the second case, in addition to Scheven's argument, by the strong convergence we get strong convergence in $L^1([0,1])$ of f_i (defined as in [47, proof of Lemma 3.2.] with u_i replaced by the difference $(u_i - \varphi_i)$) and of g_i to some f_0 and g_0. We may choose a subsequence so that $f_{i_i} \to f_0$ a.e and $g_{i_k} \to g_0$ a.e. as $k \to \infty$. Together with the strong convergence of $u_i \to u_0$ we obtain (3.2.9).

$\square$

We employ Scheven's definitions of rescaled pairs to our case of reflected maps. First, we observe that for every $\widetilde{\mu} = (\widetilde{u}, \widetilde{\nu}) \in \widetilde{\mathcal{B}}_\Lambda$ we have by definition

$$\sup_{a\in B,\,\rho<1} \rho^{4-m} \left(\int_{B_\rho(a)} |\Delta\widetilde{u}|^2\,dx + \widetilde{\nu}(B_\rho(a)) \right) \leq \Lambda. \tag{3.2.10}$$

Definition 3.2.5. The tangent data of $\widetilde{\mu}$ are defined as follows. Let $a \in B$ and $0 < r < 1$. For a pair $\widetilde{\mu} = (\widetilde{u}, \widetilde{\nu}) \in \widetilde{\mathcal{B}}_\Lambda$ we define the rescaled pair $\widetilde{\mu}_{a,r} := (\widetilde{u}_{a,r}, \widetilde{\nu}_{a,r})$ by

$$\begin{aligned} \widetilde{u}_{a,r}(x) &:= \widetilde{u}(a + rx) && \text{for } x \in B_{1/r}(0) \\ \widetilde{\nu}_{a,r}(A) &:= r^{4-m}\widetilde{\nu}(a + rA) && \text{for every Borel set } A \subset B_{1/r}(0), \end{aligned}$$

in the first definition we have chosen some representative of $\widetilde{u}$. The pair $\widetilde{\mu}_*$ is said to be a tangent pair to $\widetilde{\mu}$ in the point a if there exists a sequence $r_i \searrow 0$ with $\widetilde{\mu}_{a,r_i} \rightrightarrows \widetilde{\mu}_*$. Observe that (3.2.10) is scaling invariant, therefore (3.2.10) holds as well for the rescaled pairs $\widetilde{\mu}_{a,r}$. Thus, up to a subsequence, the limit always exists.

The next lemma is an immediate consequence of [47, Lemma 3.3].

Lemma 3.2.6. *Let $\widetilde{\mu}_i \in \widetilde{\mathcal{B}}_\Lambda$, $\widetilde{u} \in W^{2,2}(B_4, \mathcal{N})$ and $\widetilde{\nu}$ be a Radon measure on $\overline{B}_4$. If $\widetilde{\mu}_i \rightrightarrows \widetilde{\mu} = (\widetilde{u}, \widetilde{\nu})$ as $i \to \infty$, then $\widetilde{\mu} \in \widetilde{\mathcal{B}}_\Lambda$. In particular, if $\widetilde{\mu}_* = (\widetilde{u}_*, \widetilde{\nu}_*)$ is a tangent pair of $\widetilde{\mu} = (\widetilde{u}, \widetilde{\nu}) \in \widetilde{\mathcal{B}}_\Lambda$ in some point $a \in B_1$, then $\widetilde{\mu}_* \in \widetilde{\mathcal{B}}_\Lambda$.*

For a pair $\widetilde{\mu} = (\widetilde{u}, \widetilde{\nu}) \in \widetilde{\mathcal{B}}_\Lambda$ we define the set $\Sigma_{\widetilde{\mu}}$ as the set of points $a \in \overline{B}_1$ with

$$\liminf_{\rho \searrow 0} \left(\rho^{4-m} \int_{B\rho(a)} \left(|\nabla^2 \widetilde{u}|^2 + \rho^{-2} |\nabla \widetilde{u}|^2 \right) dx + \rho^{4-m} \widetilde{\nu}(B_\rho(a)) \right) \geq \frac{\varepsilon_0}{2}, \tag{3.2.11}$$

where the constant ε_0 is the constant introduced in Corollary 3.1.11. We observe that theorem on the structure of defect measures [47, Theorem 3.4] carries over directly to our setting to yield

Theorem 3.2.7. *For any $\widetilde{\mu} = (\widetilde{u}, \widetilde{\nu}) \in \widetilde{\mathcal{B}}_\Lambda$, there holds $\Sigma_{\widetilde{\mu}} = \mathrm{sing}\, \widetilde{u} \cup \mathrm{spt}\, \widetilde{\nu}$, in particular $\Sigma_{\widetilde{\mu}}$ is a closed set. Moreover, there are constants c and C depending only on m such that for every $\widetilde{\mu} = (\widetilde{u}, \widetilde{\nu}) \in \widetilde{\mathcal{B}}_\Lambda$, we have*

$$c\, \varepsilon_0 \mathcal{H}^{m-4} \llcorner \Sigma_{\widetilde{\mu}} \leq \widetilde{\nu} \llcorner \overline{B}_1 \leq C \Lambda \mathcal{H}^{m-4} \llcorner \Sigma_{\widetilde{\mu}}. \tag{3.2.12}$$

For any sequence $\widetilde{\mathcal{B}}_\Lambda \ni (\widetilde{u}_i, 0) \rightrightarrows (\widetilde{u}, \widetilde{\nu})$ as $i \to \infty$, we have subconvergence $\widetilde{u}_i \to \widetilde{u}$ in $C^2_{loc}(B \setminus \Sigma_{\widetilde{\mu}})$.

Proof. We proceed as in [47, proof of Theorem 3.4]. The proof of the inclusion $\Sigma_{\widetilde{\mu}} \subset \mathrm{sing}\, \widetilde{u} \cup \mathrm{spt}\, \widetilde{\nu}$ and the estimates (3.2.12) remain unchanged, therefore we omit this part. To proof the inclusion $(\mathrm{sing}\, \widetilde{u} \cup \mathrm{spt}\, \widetilde{\nu}) \subset \Sigma_{\widetilde{\mu}}$ and the subconvergence we follow Scheven with the following modifications and adjustments.

We divide the proof into three cases: $a \in T_1 \setminus \Sigma_{\widetilde{\mu}}$, $a \in B^+ \setminus \Sigma_{\widetilde{\mu}}$, and $a \in B^- \setminus \Sigma_{\widetilde{\mu}}$.

First, if we choose $a \in T_1 \setminus \Sigma_{\widetilde{\mu}}$, then the difference in the proof is the following: We choose a radius $0 < \rho < 1$ with

$$(2\rho)^{4-m} \int_{B_{2\rho}(a)} \left(|\nabla^2 \widetilde{u}|^2 + (2\rho)^{-2} |\nabla \widetilde{u}|^2 \right) \, dx + (2\rho)^{4-m} \widetilde{\nu}(B_{2\rho}(a)) < \frac{\varepsilon_0}{2}.$$

Next, we choose a sequence of minimizing biharmonic maps $u_i \in W^{2,2}(B_4^+, \mathcal{N})$ with boundary data φ_i with $(\widetilde{u}_i, 0) \rightrightarrows (\widetilde{u}, \widetilde{\nu}) = \widetilde{\mu}$ with

$$\lim_{i\to\infty} (2\rho)^{4-m} \int_{B_{2\rho}(a)} \left(|\nabla^2 \widetilde{u}_i|^2 + (2\rho)^{-2} |\nabla \widetilde{u}_i|^2 \right) \, dx < \frac{\varepsilon_0}{2}.$$

Now, in order to apply Corollary 3.1.11 we estimate

$$\begin{aligned}
&\lim_{i\to\infty} (2\rho)^{4-m} \int_{B^+_{2\rho}(a)} \left(|\nabla^2 u_i|^2 + (2\rho)^{-2} |\nabla u_i|^2 \right) \, dx \\
&\qquad \leq \lim_{i\to\infty} (2\rho)^{4-m} \int_{B_{2\rho}(a)} \left(|\nabla^2 \widetilde{u}_i|^2 + (2\rho)^{-2} |\nabla \widetilde{u}_i|^2 \right) \, dx + C\rho^2 \left\| \varphi_i \right\|^2_{C^2} \\
&\qquad < \varepsilon_0.
\end{aligned}$$

Hence, we obtain uniform estimates $\sup_{i\in\mathbb{N}} \|u_i\|_{C^3(B^+_\sigma(a),\mathcal{N})} \leq C(\mathcal{N})$ on some smaller half-ball $B^+_\sigma(a) \subset B^+_\rho(a)$. Since the boundary conditions φ_i are smooth and uniformly bounded we obtain as well $\sup_{i\in\mathbb{N}} \|u_i - \varphi_i\|_{C^3(B^+_\sigma(a),\mathbb{R}^\ell)} < C(\mathcal{N})$. Now, due to the properties of the reflection (3.2.3), we have $\widetilde{u}_i \in C^3(B_\sigma(a))$ with the estimate

$$\sup_{i\in\mathbb{N}} \|\widetilde{u}_i\|_{C^3(B_\sigma(a),R^\ell)} < C(\mathcal{N}).$$

Now, similarly as in [47] by Arzelà-Ascoli theorem we find a subsequence, which converges $\widetilde{u}_{i_j} \to \widetilde{u}$ in $C^2(B_\sigma(a), \mathbb{R}^\ell)$, as $j \to \infty$, from which we deduce $\widetilde{\nu}(B_\sigma(a)) = 0$. Thus, $(\operatorname{sing} \widetilde{u} \cup \operatorname{spt} \widetilde{\nu}) \subset \Sigma_{\widetilde{\mu}}$.

If we choose $a \in B^+ \setminus \Sigma_{\widetilde{\mu}}$ then the proof is identical as in the case of interior points in [47].

Finally, if we choose $a \in B^- \setminus \Sigma_{\widetilde{\mu}}$ and sufficiently small ρ small enough to ensure $\frac{a_m}{4} + 2\sigma < 0$, where a_m is the m-th component of a, then $B_{2\rho}(a) \subset B^-$. By the definition the behavior of the reflected map on $B_{2\rho}(a) \subset B^-$ corresponds to the behavior of the map on four balls in the upper half: $B_{2\rho}(a_j)$, where $a_j = (a', -a_m/j)$ for $j = 1, 2, 3, 4$, and $a = (a', a_m)$. Thus from $a \notin \Sigma_{\widetilde{\mu}}$ we deduce that $a_j \notin \Sigma_{\widetilde{\mu}}$ and by repeating the proof in the interior case we obtain the desired inclusion. □

As a consequence of Theorem 3.2.7 we obtain, exactly as in [47, Corollary 3.6], the following.

Corollary 3.2.8. *If $\widetilde{\mathcal{B}}_\Lambda \ni (\widetilde{u}_i, 0) \rightrightarrows (\widetilde{u}, \widetilde{\nu}) = \widetilde{\mu}$ as $i \to \infty$, then $\mathcal{H}^{m-4}(\Sigma_{\widetilde{\nu}}) = 0$ implies $\widetilde{u}_i \to \widetilde{u}$ strongly in $W^{2,2}(B_{1/2}, \mathcal{N})$. Conversely, the strong convergence $\widetilde{u}_i \to \widetilde{u}$ in $W^{2,2}(B, \mathcal{N})$ implies $\mathcal{H}^{m-4}(\Sigma_{\widetilde{\mu}} \cap B) = 0$.*

We also have a counterpart of [47, Lemma 3.7], which makes it possible to restrict our attention to the case, when the limiting map is constantly equal 0 and the defect measure is flat.

Lemma 3.2.9. *Assume there is a pair $(\widetilde{u}, \widetilde{\nu}) \in \widetilde{\mathcal{B}}_\Lambda$ with $\mathcal{H}^{m-4}(\text{spt}\,\widetilde{\nu}) > 0$. Then there is a pair $(\widetilde{u}_*, \overline{\nu}) \in \widetilde{\mathcal{B}}_\Lambda$, such that $\widetilde{u}_* = 0 \in \mathbb{R}^\ell$ and*

$$\overline{\nu} = C\mathcal{H}^{m-4} \llcorner V,$$

where V is an $(m-4)$-dimensional subspace $V \subset \mathbb{R}^m$ and $C > 0$ is a constant.

Proof. The proof follows directly the proof of Scheven's Lemma 3.7 [47]. Identically as there, there is a point $a \in B$ and a sequence $r_i \searrow 0$ for which $\widetilde{\mu}_{a,r_i} \rightrightarrows \widetilde{\mu}_* = (\widetilde{u}_*, \widetilde{\nu}_*) \in \widetilde{\mathcal{B}}_\Lambda$, for which $\widetilde{u}_*$ is a constant. We know also that $\widetilde{u}_*$ is equal zero on T_4, thus $\widetilde{u}_* = 0$.

In the proof of the structure of the measure $\overline{\nu}$ the only difference from Scheven's proof we should observe is that, by inequality (3.2.8), the quantity

$$\widetilde{\Theta}^{m-4}(\widetilde{\nu}_*, a) := \lim_{\rho \searrow 0} \rho^{4-m} \widetilde{\nu}_*(B_\rho(a))$$

is well defined and a similar analysis to that in [47] shows that there exists a tangent measure $\overline{\nu}$ to $\widetilde{\nu}_*$, such that $\overline{\nu} = C\mathcal{H}^{m-4} \llcorner V$, where V is an $(m-4)$-dimensional subspace $V \subset \mathbb{R}^m$. $\square$

We are ready to prove the Compactness Theorem 3.2.1. The (rough) idea of the proof follows Scheven's proof of Theorem 1.5 in [47]. The results of this section yield that if the theorem was false we would obtain a sequence of reflections $\widetilde{u}_i$, converging to 0 off the support of a defect measure which up to a constant is an $(m-4)$-dimensional Hausdorff measure and which is flat. To show that it is impossible we construct a comparison map and use the minimizing property of u_i on a half-ball. We define a comparison map as an interpolation between u_i and its boundary data v_i with the exception of a tori of small radius in which the energy concentration set is included. To define the map on the remaining tori we use a kind of radially constant extension on a tori. The existence of such a map leads to a contradiction with the special form of the defect measure, if we choose sufficiently small outer annuli on which the comparison map is equal u_i and sufficiently small intermediate annuli on which the map is defined as an interpolation between u_i and φ_i.

Proof of Theorem 3.2.1. In the following, *we forego possibly more general and sophisticated estimates in favor of simple arithmetic*[2].

First lines of our proof are essentially the same as the first 19 lines of Scheven's proof: we argue by contradiction and collect the results of this section.

The theorem is equivalent to $\widetilde{\nu} \llcorner \overline{B} \equiv 0$ for all $(\widetilde{u}, \widetilde{\nu}) \in \widetilde{\mathcal{B}}_\Lambda$. Thus, we consider a sequence $w_i \in W^{2,2}(B_4^+, \mathcal{N})$ of minimizing biharmonic maps with some boundary values such that $\sup_i \|\widetilde{w}_i\|_{W^{2,2}(B_4^+)} < \infty$. By Lemma 3.1.7 we know that $\sup_i \|\widetilde{w}_i\|_{L^{2,m-4}} < \Lambda$ for some $\Lambda > 0$, so that after passing to a subsequence we have $(\widetilde{w}_i, 0) \rightrightarrows (\widetilde{w}, \widetilde{\nu}) = \widetilde{\mu} \in \widetilde{\mathcal{B}}_\Lambda$. Assume, on the contrary, that we do not have strong convergence $\widetilde{w}_i \to \widetilde{w}$ in $W^{2,2}(B_{1/2}, \mathcal{N})$. Then, by Corollary 3.2.8

[2] see Iwaniec [32, Note, p.607]

we know that $\mathcal{H}^{m-4}(\Sigma_{\widetilde{\mu}}) > 0$. By Lemma 3.2.9 we know that there are minimizing biharmonic maps $u_i \in W^{2,2}(B_4^+, \mathcal{N})$ with boundary values φ_i, such that $(\widetilde{u}_i, 0) \rightrightarrows (0, \overline{\nu})$ and $\overline{\nu} \llcorner \overline{B} = C\mathcal{H}^{m-4} \llcorner V$. Moreover, by Theorem 3.2.7 we get

$$\widetilde{u}_i \to 0 \quad \text{in } C^2_{loc}(B \setminus V) \text{ as } i \to \infty. \tag{3.2.13}$$

Now let us note that if the energy concentration set V would be a subset of T_1 we would be in the simplest situation as our map $\widetilde{u}_i$ vanish on T_1. Therefore, without loss of generality we may assume that

$$V = \{0\} \times \overline{B}^{m-4}.$$

Let κ, σ be suitable parameters, which will be specified later, satisfying

$$\frac{1}{2} < \kappa < 1, \quad 0 < \sigma < \frac{1}{16}, \quad \text{and } 0 < \kappa + 2\sigma < 1.$$

Let $\psi \in C^\infty(B^+, [0,1])$ be a cut-off function with $\psi \equiv 0$ on $B^+_{\kappa-\sigma}$, $\psi \equiv 1$ outside $B^+_{\kappa+\sigma}$ and $|\nabla\psi| \le \frac{C}{\sigma}$, $|\nabla^2\psi|^2 \le \frac{C}{\sigma^2}$ on B^+. We employ Scheven's notation of tori: for $(X_1, X_2) \in \mathbb{R}^4 \times \mathbb{R}^{m-4}$, we define

$$\mathbb{T}_s := \{x \in \mathbb{R}^m : [x] \le s\}, \quad \text{where } [x] := [|X_1|^2 + (|X_2| - \kappa)^2]^{1/2}.$$

Now let

$$\widetilde{v}_i(x) := \Pi_{\mathcal{N}}(\varphi_i + \psi(\widetilde{u}_i)) \quad \text{for } x \in B^+ \setminus \mathbb{T}_{2\sigma}. \tag{3.2.14}$$

We recall that $\Pi_{\mathcal{N}} : O(\mathcal{N}) \to \mathcal{N}$ is the nearest point projection of a neighborhood $O(\mathcal{N}) \subset \mathbb{R}^\ell$ of $\mathcal{N}$ onto $\mathcal{N}$. We observe that

$$\widetilde{v}_i \equiv \varphi_i \text{ on } B^+_{\kappa-\sigma} \quad \text{and} \quad \widetilde{v}_i \equiv u_i \text{ outside } B^+_{\kappa+\sigma}.$$

Moreover, the set $\{0 < \psi < 1\} \setminus \mathbb{T}_{2\sigma}$ has positive distance to the energy concentration set $\{0\} \times B^{m-4}$, so that we have convergence $\widetilde{u}_i \to 0$ in C^2 on the former set. Therefore, for sufficiently large $i \in \mathbb{N}$, the maps $\widetilde{v}_i(x)$ are well defined for $x \in B^+ \setminus \mathbb{T}_{2\sigma}$.

Simple computations yield

$$
\begin{aligned}
&\int_{B^+\setminus\mathbb{T}_{2\sigma}} |\Delta\widetilde{v}_i|^2\,dx \leq \int_{B^+\setminus B^+_\kappa} |\Delta u_i|^2\,dx + \int_{B^+_\kappa} |\Delta\varphi_i|^2\,dx \\
&\quad + C\int_{\{0<\psi<1\}\setminus\mathbb{T}_{2\sigma}} \left(|\Delta\varphi_i|^2 + |\nabla\varphi_i|^4\right)\,dx \\
&\quad + C\int_{\{0<\psi<1\}\setminus\mathbb{T}_{2\sigma}} \left(|\Delta\widetilde{u}_i|^2 + |\nabla\varphi_i|^2|\nabla\widetilde{u}_i|^2 + \frac{|\nabla\widetilde{u}_i|^2}{\sigma^2} + \frac{|\widetilde{u}_i|^4}{\sigma^4} + \frac{|\widetilde{u}_i|^2}{\sigma^4}\right)dx \\
&\leq m\int_{B\setminus B_\kappa} |\nabla^2\widetilde{u}_i|^2\,dx + \int_{B^+} |\Delta\varphi_i|^2\,dx + C\int_{\{0<\psi<1\}} \left(|\nabla\varphi_i|^4 + |\nabla\varphi_i|^6\right)\,dx \\
&\quad + C\int_{\{0<\psi<1\}\setminus\mathbb{T}_{2\sigma}} \left(|\Delta\widetilde{u}_i|^2 + |\nabla\widetilde{u}_i|^3 + \frac{|\nabla\widetilde{u}_i|^2}{\sigma^2} + \frac{|\widetilde{u}_i|^4}{\sigma^4} + \frac{|\widetilde{u}_i|^2}{\sigma^4}\right)\,dx,
\end{aligned}
\tag{3.2.15}
$$

where in the last estimate we used Young's inequality

$$
|\nabla\varphi_i|^2|\nabla\widetilde{u}_i|^2 \leq \frac{|\nabla\varphi_i|^6}{3} + \frac{2|\nabla\widetilde{u}_i|^3}{3}
$$

and the pointwise inequality $|\Delta u_i|^2 \leq m|\nabla^2 u_i|^2$.

By Gagliardo-Nirenberg's inequality we have for $p > 1$

$$
\|\widetilde{u}_i\|^2_{L^{2p}(B^+)} \leq C\,\|\widetilde{u}_i\|_{L^\infty(B^+)}\,\|\widetilde{u}_i\|_{W^{2,p}(B^+)}\,. \tag{3.2.16}
$$

For $1 < p < 2$ the uniform boundedness of $\|\widetilde{u}_i\|_{W^{2,p}(B^+)}$ combined with the above inequality implies $\sup_{i\in\mathbb{N}}\|\widetilde{u}_i\|_{L^{2p}(B^+)} < \infty$. Recall that $(\widetilde{u}_i, 0) \rightrightarrows (0, \overline{\nu})$, thus $\widetilde{u}_i \to 0$ strongly in $W^{1,p}(B^+)$. Now, by Hölder's inequality for exponent $\frac{7}{2}$

$$
\begin{aligned}
\int_{B^+} |\nabla\widetilde{u}_i|^3\,dx &= \int_{B^+} |\nabla\widetilde{u}_i|^{1/2}|\nabla\widetilde{u}_i|^{5/2}\,dx \\
&\leq \left(\int_{B^+} |\nabla\widetilde{u}_i|^{7/4}\,dx\right)^{2/7}\left(\int_{B^+} |\nabla\widetilde{u}_i|^{7/2}\,dx\right)^{5/7}.
\end{aligned}
\tag{3.2.17}
$$

Taking $p = \frac{7}{4}$ we see that the first term of the latter inequality converges to 0 and the second is bounded, hence $|\nabla\widetilde{u}_i|^3 \to 0$ strongly in $L^3(B^+)$. Now we are ready to pass to the limit in (3.2.15).

By the C^2-convergence $\widetilde{u}_i \to 0$ on the set $\{0 < \psi < 1\} \setminus \mathbb{T}_{2\sigma}$ and by the convergence $|\Delta\widetilde{u}_i|^2\,dx \rightharpoonup \overline{\nu}$ in the sense of measures, we get, since $\overline{\nu}(\partial B) = 0$,

$$\lim_{i\to\infty} \int_{B^+\setminus\mathbb{T}_{2\sigma}} |\Delta v_i|^2\,dx \le m\overline{\nu}(B\setminus B_\kappa) + C_\varphi + C(\sigma), \tag{3.2.18}$$

where $C(\sigma)$ is the limit of $C\int_{\{0<\psi<1\}} \left(|\nabla\varphi_i|^4 + |\nabla\varphi_i|^6\right)\,dx$. From the absolute continuity of the Lebesgue integral, by shrinking $\sigma > 0$, the constant $C(\sigma)$ can be taken arbitrary small.

Note that for $m = 4$ the above construction of $\widetilde{v}_i$ is possible for all $x \in B^+$. In this situation $\widetilde{v}_i \equiv u_i$ on T_1 and since the vector $\frac{\partial u_i}{\partial x_m}$ is perpendicular to the manifold $\mathcal{N}$ we have

$$\frac{\partial}{\partial x_m}\widetilde{v}_i\bigg|_{T_1} = \frac{\partial}{\partial x_m}u_i\bigg|_{T_1},$$

see, e.g., [52, Section 2.12.3]. Thus, $u_i - \widetilde{v}_i \in W_0^{2,2}(B^+)$ and we can use the minimality of u_i to compare it with $\widetilde{v}_i$.

Applying Lemma B.0.5 for $R = 1$ and $r = 1 - \sigma$, using the inequality (3.2.4), and the strong convergence of $\nabla\widetilde{u}_i$ in L^2, we get

$$\begin{aligned}
\overline{\nu}(B_{1-\sigma}) &= \lim_{i\to\infty} \int_{B_{1-\sigma}} |\nabla^2\widetilde{u}_i|^2\,dx \le \lim_{i\to\infty} 2\int_B \left(|\Delta\widetilde{u}_i|^2 + \frac{C}{\sigma^2}|\nabla\widetilde{u}_i|^2\right)\,dx \\
&\le \lim_{i\to\infty} 2C_{ref}\left(\int_{B^+} |\Delta\widetilde{u}_i|^2\,dx + \int_B \frac{C}{\sigma^2}|\nabla\widetilde{u}_i|^2\,dx\right) \\
&\le \lim_{i\to\infty} 4C_{ref}\left(\int_{B^+} |\Delta u_i|^2\,dx + \int_{B^+} |\Delta\varphi_i|^2\,dx + \int_B \frac{C}{\sigma^2}|\nabla\widetilde{u}_i|^2\,dx\right) \\
&\le \lim_{i\to\infty} 4C_{ref}\left(\int_{B^+} |\Delta\widetilde{v}_i|^2\,dx + \int_{B^+} |\Delta\varphi_i|^2\,dx + \int_B \frac{C}{\sigma^2}|\nabla\widetilde{u}_i|^2\,dx\right) \\
&\le 4m\,C_{ref}\overline{\nu}(B\setminus B_\kappa) + 4C_{ref}C_\varphi + C(\sigma) \\
&< \overline{\nu}(B_{1-\sigma}),
\end{aligned} \tag{3.2.19}$$

a contradiction with the form of $\widetilde{\nu}$. For the last inequality we needed $C_\varphi < \frac{1}{4C_{ref}}$ and a sufficiently small σ.

For $m \geq 5$ we apply a retraction $\Psi \in C^\infty(\mathbb{T}_{4\sigma} \setminus \mathbb{T}_0, \mathbb{T}_{4\sigma} \setminus \mathbb{T}_{2\sigma})$ from [47, Lemma 3.8] with the following properties: $\Psi = id$ and $\nabla\Psi \equiv Id$ on $\partial\mathbb{T}_{4\sigma}$, where id denotes the identity on $\partial\mathbb{T}_{4\sigma}$ and Id is the identity map on $\mathbb{R}^m$. Furthermore,

$$|\nabla\Psi(x)| \leq \frac{C\sigma}{[x]}, \quad |\nabla^2\Psi(x)| \leq \frac{C\sigma}{[x]^2}, \text{ and } \det(\nabla\Psi(x)) \geq \frac{C\sigma^4}{[x]^4} \tag{3.2.20}$$

for constants dependent only on m.

We are ready to define a comparison map. Let

$$v_i(x) := \begin{cases} \Pi_{\mathcal{N}}(\varphi_i(x) + \psi(x)\widetilde{u}_i(x)) & \text{for } x \in B^+ \setminus \mathbb{T}_{4\sigma}, \\ \Pi_{\mathcal{N}}(\varphi_i(x) + \psi(x)\widetilde{u}_i(\Psi(x))) & \text{for } x \in \mathbb{T}_{4\sigma}, \end{cases} \tag{3.2.21}$$

i.e., $v_i(x) = \widetilde{v}_i(x)$ on $B^+ \setminus \mathbb{T}_{4\sigma}$. Due to the properties of the retraction Ψ we have $v \in W^{2,2}(B^+, \mathcal{N})$. We immediately have

$$\left(v_i, \frac{\partial}{\partial x_m} v_i\right)\Bigg|_{T_1 \setminus \mathbb{T}_{4\sigma}} = \left(u_i, \frac{\partial}{\partial x_m} u_i\right)\Bigg|_{T_1 \setminus \mathbb{T}_{4\sigma}}.$$

To see that the trace of v_i is the same as u_i's on $T_1 \cap \mathbb{T}_{4\sigma}$ we note that for Ψ from Scheven's Lemma 3.8 we have

$$x \in T_1 \cap \mathbb{T}_{4\sigma} \Rightarrow \Psi(x) \in T_1.$$

Thus,

$$u_i(\Psi(x)) - \varphi_i(\Psi(x)) = 0, \quad \nabla u_i(\Psi(x)) = \nabla(\Psi(x)) = 0 \quad \text{for } x \in T_1 \cap \mathbb{T}_{4\sigma}.$$

Hence, after simple computations, for $x \in T_1 \cap \mathbb{T}_{4\sigma}$,

$$v_i(x) = u_i(x), \quad \frac{\partial}{\partial x_m} v_i(x) = (\Pi_{\mathcal{N}})_{x_k} \frac{\partial u_i^k(x)}{\partial x_m} = \frac{\partial u_i(x)}{\partial x_m}.$$

The last equality is, again, a consequence of the fact that $\frac{\partial u_i}{\partial x_m} \perp \mathcal{N}$.

Similarly as in (3.2.15) we compute

$$
\begin{aligned}
&\int_{\mathbb{T}^+_{4\sigma}} |\Delta v_i|^2\,dx \\
&\le C\int_{\mathbb{T}^+_{4\sigma}} \left(|\Delta\varphi_i|^2 + |\nabla\varphi_i|^4 + |\nabla\varphi_i|^6\right)\,dx \\
&\quad + C\int_{\mathbb{T}^+_{4\sigma}} \Bigg(|\nabla^2\widetilde{u}_i\circ\Psi|^2|\nabla\Psi|^4 + |\nabla\widetilde{u}_i\circ\Psi|^3|\nabla\Psi|^3 \\
&\qquad\qquad + |\nabla\widetilde{u}_i\circ\Psi|^2\left(\frac{|\nabla\Psi|^2}{\sigma^2} + |\nabla^2\Psi|^4\right) + \frac{|\widetilde{u}_i\circ\Psi|^4}{\sigma^4} + \frac{|\widetilde{u}_i\circ\Psi|^2}{\sigma^4}\Bigg)\,dx.
\end{aligned}
\tag{3.2.22}
$$

Using the properties (3.2.20) of Ψ and the fact that $[x] \le \frac{1}{4}$ we get

$$
\begin{aligned}
&\int_{\mathbb{T}^+_{4\sigma}} |\Delta v_i|^2\,dx - C\int_{\mathbb{T}^+_{4\sigma}} \left(|\Delta\varphi_i|^2 + |\nabla\varphi_i|^4 + |\nabla\varphi_i|^6\right)\,dx \\
&\le C\int_{\mathbb{T}^+_{4\sigma}} \Bigg(\frac{\sigma^4}{[x]^4}|\nabla^2\widetilde{u}_i\circ\Psi|^2 + \frac{\sigma^3}{[x]^4}|\nabla\widetilde{u}_i\circ\Psi|^3 + \frac{1}{[x]^4}|\nabla\widetilde{u}_i\circ\Psi|^2 \\
&\qquad\qquad + \frac{\sigma^2}{[x]^4}|\nabla\widetilde{u}_i\circ\Psi|^2 + \frac{\sigma^{-4}}{[x]^4}|\widetilde{u}_i\circ\Psi|^4 + \frac{\sigma^{-4}}{[x]^4}|\widetilde{u}_i\circ\Psi|^2\Bigg)\,dx \\
&\le C\int_{\mathbb{T}^+_{4\sigma}} \Bigg(|\nabla^2\widetilde{u}_i\circ\Psi|^2 + \sigma^{-1}|\nabla\widetilde{u}_i\circ\Psi|^3 + \sigma^{-4}|\nabla\widetilde{u}_i\circ\Psi|^2 \\
&\qquad\qquad + \sigma^{-2}|\nabla\widetilde{u}_i\circ\Psi|^2 + \sigma^{-8}|\widetilde{u}_i\circ\Psi|^4 + \sigma^{-8}|\widetilde{u}_i\circ\Psi|^2\Bigg)\det(\nabla\Psi)\,dx \\
&\le C\int_{\mathbb{T}^+_{4\sigma}\setminus\mathbb{T}^+_{2\sigma}} \left(|\nabla^2\widetilde{u}_i|^2 + \sigma^{-1}|\nabla\widetilde{u}_i|^3 + \sigma^{-4}|\nabla\widetilde{u}_i|^2 + \sigma^{-8}|\widetilde{u}_i|^4 + \sigma^{-8}|\widetilde{u}_i|^2\right)\,dx.
\end{aligned}
\tag{3.2.23}
$$

In order to pass with i to the limit in the above inequality we note that similarly as in (3.2.18), we have $\int_{\mathbb{T}^+_{4\sigma}\setminus\mathbb{T}^+_{2\sigma}} |\nabla^2\widetilde{u}_i|^2\,dx \le \int_{\mathbb{T}_{4\sigma}\setminus\mathbb{T}_{2\sigma}} |\nabla^2\widetilde{u}_i|^2\,dx$. Thus,

$$
\begin{aligned}
\lim_{i\to\infty}\int_{\mathbb{T}^+_{4\sigma}} |\Delta v_i|^2\,dx &\le C\int_{\mathbb{T}^+_{4\sigma}} (|\Delta\varphi|^2 + |\nabla\varphi|^4 + |\nabla\varphi|^6)\,dx + C\,\overline{\nu}(\mathbb{T}_{4\sigma}) \\
&= C(\sigma) + C\,\overline{\nu}(\mathbb{T}_{4\sigma}).
\end{aligned}
\tag{3.2.24}
$$

Once again, from the absolute continuity of the Lebesgue integral, by shrinking $\sigma > 0$, the constant $C(\sigma)$ can be taken arbitrary small.

Now let $0 < \gamma < 1$ be a small number. We have by Lemma B.0.5

$$\int_{B^+_{1-\gamma}} |\nabla \widetilde{u}_i|^2 \, dx \leq 2 \int_{B^+} \left(|\Delta \widetilde{u}_i|^2 + \frac{C}{\gamma^2} |\nabla \widetilde{u}_i|^2 \right) \, dx. \tag{3.2.25}$$

Combining (3.2.4) and (3.2.25) we obtain

$$\begin{aligned}
\overline{\nu}(B_{1-\gamma}) &= \lim_{i\to\infty} \int_{B_{1-\gamma}} |\nabla^2 \widetilde{u}_i|^2 \, dx \leq \lim_{i\to\infty} C_{ref} \int_{B^+_{1-\gamma}} |\nabla^2 \widetilde{u}_i|^2 \, dx \\
&\leq \lim_{i\to\infty} 2\, C_{ref} \int_{B^+} \left(|\Delta \widetilde{u}_i|^2 + \frac{C}{\gamma^2} |\nabla \widetilde{u}_i|^2 \right) \, dx \\
&\leq \lim_{i\to\infty} 4\, C_{ref} \left(\int_{B^+} |\Delta u_i|^2 \, dx + \int_{B^+} |\Delta \varphi_i|^2 \, dx + \int_{B^+} \frac{C}{\gamma^2} |\nabla \widetilde{u}_i|^2 \, dx \right) \\
&\leq \lim_{i\to\infty} 4\, C_{ref} \left(\int_{B^+} |\Delta v_i|^2 \, dx + \int_{B^+} |\Delta \varphi_i|^2 \, dx + \int_{B^+} \frac{C}{\gamma^2} |\nabla \widetilde{u}_i|^2 \, dx \right) \\
&\leq 4m\, C_{ref} \overline{\nu}(B \setminus B_\kappa) + 4m\, C_{ref}(C_\varphi + C(\sigma) + C\, \overline{\nu}(\mathbb{T}_{4\sigma})).
\end{aligned} \tag{3.2.26}$$

To get a contradiction we use the special form of the measure

$$\overline{\nu} \llcorner \overline{B} = C \mathcal{H}^{m-4} \llcorner \left(\{0\} \times \overline{B}^{m-4} \right).$$

We choose the number κ so that $4m\, C_{ref} \overline{\nu}(B \setminus B_\kappa) < \overline{\nu}(B_{1-\gamma})$, for example

$$\kappa = \sqrt[m-4]{1 - \frac{(1-\gamma)^{m-4}}{3500m}}.$$

Next we observe that if C_φ is sufficiently small, e.g, is such that

$$4m\, C_{ref} C_\varphi < \frac{1}{2} \left(\overline{\nu}(B_{1-\gamma}) - 4m\, C_{ref} \overline{\nu}(B \setminus B_\kappa) \right)$$

then by shrinking $\sigma > 0$ the number $4m\, C_{ref}(C(\sigma) + C\, \overline{\nu}(\mathbb{T}_{4\sigma}))$ can be arbitrary small and thus

$$4m\, C_{ref} \overline{\nu}(B \setminus B_\kappa) + 4m\, C_{ref}(C_\varphi + C(\sigma) + C\, \overline{\nu}(\mathbb{T}_{4\sigma})) < \overline{\nu}(B),$$

contradicting (3.2.26). This finishes the proof of Theorem 3.2.1.

□

3.3 Tangent maps at the boundary

In this section we prove, using the compactness result from the previous section, that limits of rescaled maps converge strongly to boundary tangent maps, which are homogeneous of degree 0 and have constant values on the flat part of the boundary ∂B^+. Next, we show how to rule out the possibility of existence of nonconstant minimal minimizing biharmonic maps from a half ball, that are constant on the flat part of the boundary T_1. Finally, combining results of this section, Scheven's lemma, which states that the tangent maps that occur in the dimension reduction argument are minimal, and Gong, Lamm, and Wang's epsilon regularity result — Lemma 3.1.1 — we give the proof of the main result.

Definition 3.3.1. Let $a \in T_1$ and $x \in \frac{1}{\lambda}(B_1^+ - a)$. We define the rescaled map by

$$u_{a,\lambda}(x) := u(a + \lambda x).$$

A map $v \in W^{2,2}_{loc}(\mathbb{R}^m, \mathcal{N})$ is called a *tangent map at the boundary* of u at the point a if there exists a sequence $\lambda_i \searrow 0$ with $u_{a,\lambda_i} \to v$ in $W^{2,2}_{loc}(\mathbb{R}^m, \mathcal{N})$ as $i \to \infty$.

Lemma 3.3.2. *Let u be as before with boundary values $\varphi \in C^\infty$ and let $a \in T_1$. Then for each sequence $\{\lambda_i\}_{i\in\mathbb{N}}$ for which $0 < \lambda_i < 1$, there exists a subsequence $\lambda_{i_j} \to 0$ such that the maps $u_{\lambda_{i_j}}$ converge strongly in $W^{2,2}(B^+_{1/2}, \mathcal{N})$ to a map $u_0 \in W^{2,2}(B^+, \mathcal{N})$ that is biharmonic, homogeneous of degree 0, and has constant boundary values on T_1.*

Proof. Step 1: Strong convergence. Observe that $\sup_i \|u_{a,\lambda_i}\|_{W^{2,2}(B^+)} < \infty$.

Indeed, by a change of variables

$$\left\|\nabla^2 u_{a,\lambda_i}\right\|^2_{L^2(B_1^+)} = \int_{B_1^+} |\lambda_i^2 \nabla^2 u(a+\lambda_i x)|^2\, dx = \lambda_i^{4-m} \int_{B_{\lambda_i}^+(a)} |\nabla^2 u|^2\, dy.$$

The supremum of the latter one is bounded by Lemma 3.1.7. Moreover, $u_0\big|_{T_1} = \varphi(a)$ by the continuity of φ. Thus the assumptions of Theorem 3.2.1 are satisfied and we obtain the strong subconvergence to u_0.

Step 2: Homogeneity of degree 0. By strong convergence and Lemma 3.2.4 we have

$$\begin{aligned} H^+_{u_0}(0,r) &= \lim_{i\to\infty} H^+_{u_{a,r_i}}(0,r) = \lim_{i\to\infty} H^+_u(a, rr_i) = \lim_{\rho\searrow 0} H^+_u(a,\rho) \\ R^+_{u_0}(0,r) &= \lim_{i\to\infty} R^+_{u_{a,r_i}}(0,r) = \lim_{i\to\infty} R^+_u(a, rr_i) = \lim_{\rho\searrow 0} H^+_u(a,\rho). \end{aligned}$$

Thus, $H^+_{u_0}(0,r),\ R^+_{u_0}(0,r)$ do not depend on r and we denote

$$H^+_{u_0}(0) = \lim_{\rho\searrow 0} H^+_u(a,\rho), \quad R^+_{u_0}(0) = \lim_{\rho\searrow 0} H^+_u(a,\rho). \tag{3.3.1}$$

Now by monotonicity formula (3.1.6)

$$\begin{aligned} P^+_{u,\varphi}(a,\rho,r) \le e^{Cr} H^+_{u,\varphi}(a,r) + e^{Cr} R^+_{u,\varphi}(a,r) + Cre^{Cr} \\ - H^+_{u,\varphi}(a,\rho) - e^{C\rho} R^+_{u,\varphi}(a,\rho). \end{aligned}$$

In particular

$$\begin{aligned} \int\limits_{B_r^+(a)\setminus B_\rho^+(a)} \frac{\left|(x-a)^i (u-\varphi)_{x_i}\right|^2}{|x-a|^m}\, dx &\le e^{Cr} H^+_{u,\varphi}(a,r) + e^{Cr} R^+_{u,\varphi}(a,r) + Cre^{Cr} \\ &\quad - H^+_{u,\varphi}(a,\rho) - e^{C\rho} R^+_{u,\varphi}(a,\rho). \end{aligned}$$

By (3.3.1) passing to the limit in the last inequality with $\rho \searrow 0$ we obtain

$$\begin{aligned} \int_{B_r^+(a)} \frac{\left|(x-a)^i (u-\varphi)_{x_i}\right|^2}{|x-a|^m}\, dx &\le e^{Cr} H^+_u(a,r) + e^{Cr} R^+_u(a,r) \\ &\quad - H^+_{u_0}(0) - R^+_{u_0}(0) + Cre^{Cr}. \end{aligned} \tag{3.3.2}$$

By a change of variables

$$\int_{B_r^+(a)} \frac{\left|(x-a)^i(u-\varphi)_{x_i}\right|^2}{|x-a|^m}\,dx = \int_{B_1^+} \frac{|x^i\,(u_{a,r})_{x_i}\,|^2}{|x|^m}\,dx.$$

Thus, passing with r to zero in (3.3.2) we get

$$\lim_{r\to 0}\int_{B_1^+} \frac{|x^i\,(u_{a,r})_{x_i}\,|^2}{|x|^m}\,dx = 0$$

and as a consequence $x^i(u_0)_{x_i} \equiv 0$ a.e., which implies the desired homogeneity.

□

Lemma 3.3.3. *Any minimizing biharmonic map $u_0 \in W^{2,2}(B_1^+, \mathcal{N})$ that is homogeneous of degree 0 and that is constant on $B_1 \cap \{x_m = 0\}$ must be a constant.*

Proof. For $m < 4$ by Sobolev embedding theorem a mapping in $W^{2,2}$ must be continuous. Being homogeneous of degree 0, it must be a constant.

For $m = 4$ assume, contrary to our claim, that u_0 is a nonconstant minimizing map from $\mathbb{B}^+$ to $\mathcal{N}$. Let $y = \beta(x) = 2x$. Simple calculation gives

$$\begin{aligned}
0 &< \int_{B_1^+} |\Delta u_0|^2\,dx = \int_{B_{\frac{1}{2}}^+} |\Delta u_0(2x)|^2 \cdot 2^4\,dx \\
&= \int_{B_{\frac{1}{2}}^+} |\Delta(u_0 \circ \beta)(x)|^2 |\nabla^2\beta(x)|^{-2} \cdot 2^4\,dx = \int_{B_{\frac{1}{2}}^+} |\Delta u_0(x)|^2\,dx \\
&< \int_{B_1^+} |\Delta u_0(x)|^2\,dx < \infty,
\end{aligned}$$

which is impossible.

For $m > 4$, we shall consider the energy of a comparison function v_α, the same as in [28, proof of Theorem 5.7]. We use spherical coordinates to represent a point X on the hemisphere $\partial B_1 \cap \{x_m \geq 0\}$ by a point $\omega \in \mathbb{S}^{m-2}$ and the angle

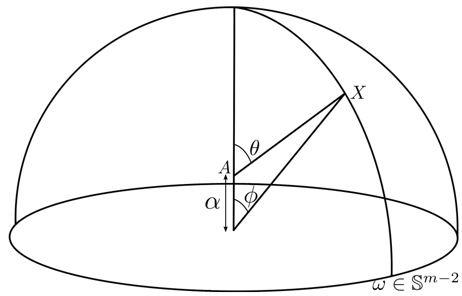

Figure 3.1: Proof of Lemma 3.3, the relation between α, θ and ϕ

$\phi \in [0, \frac{1}{2}\pi]$. Let $0 < \alpha < 1$, $A = (0, \dots, 0, \alpha)$ and θ denote the angle between vectors AX and AN (where $N = (0, \dots, 0, 1)$ is the north pole). The angle θ satisfies the relation

$$\theta = \phi + \arcsin(\alpha \sin\theta). \tag{3.3.3}$$

As the angle ϕ ranges between 0 and $\frac{1}{2}\pi$, the angle θ ranges between 0 and $\Theta(\alpha) = \operatorname{arcctg}(-\alpha) = \pi - \arcsin((1+\alpha^2)^{-\frac{1}{2}})$. The distance between x and $(0, \dots, \alpha)$ is $R(\phi, \alpha) = [(\alpha - \cos\phi)^2 + \sin^2\phi]^{\frac{1}{2}}$. The desired comparison mapping is given by

$$v_\alpha(\theta, \omega) = u_0(\phi, \omega) \quad \text{for } \theta \in [0, \Theta] \text{ and } \omega \in \mathbb{S}^{m-2}. \tag{3.3.4}$$

Let $J(\alpha) = \int_{B^+} |\Delta v_\alpha|^2 \, dx$ denote the Hessian energy of v_α. One can compute

$$\begin{aligned}
J(\alpha) &= \\
&= \int_0^{\Theta(\alpha)} \int_0^{R(\phi,\alpha)} \int_{\mathbb{S}^{m-2}} \frac{1}{r^4 \sin^2\theta} \sum_{i=1}^k \left| \cos\theta \frac{\partial v_\alpha^i}{\partial\theta} + \sin\theta \frac{\partial^2 v_\alpha^i}{\partial\theta^2} + \sin^{-1}\theta \frac{\partial^2 v_\alpha^i}{\partial\omega^2} \right|^2 \\
&\qquad\qquad \cdot \sin^{m-2}\theta r^{m-1} \, d\omega \, dr \, d\theta \\
&= \int_0^{\Theta(\alpha)} \int_{\mathbb{S}^{m-2}} \sum_{i=1}^k \left| \cos\theta \frac{\partial v_\alpha^i}{\partial\theta} + \sin\theta \frac{\partial^2 v_\alpha^i}{\partial\theta^2} + \sin^{-1}\theta \frac{\partial^2 v_\alpha^i}{\partial\omega^2} \right|^2 \\
&\qquad\qquad \cdot \sin^{m-4}\theta \frac{1}{m-4} R^{m-4}(\phi, \alpha) \, d\omega \, d\theta.
\end{aligned}$$

Changing variables according to $\theta = \theta(\phi, \alpha) : [0, \frac{1}{2}\pi] \times [0, 1) \to [0, \Theta(\alpha))$, we find that $J(\alpha)$ equals

$$
\begin{aligned}
&J^i(\alpha) = \\
&\frac{1}{m-4}\int_0^{\frac{\pi}{2}}\int_{\mathbb{S}^{m-2}} \Bigg| \cos\theta(\phi,\alpha)\frac{\partial u_0^i}{\partial \phi}\frac{\partial \phi}{\partial \theta} \\
&\qquad\qquad + \sin\theta(\phi,\alpha)\left[\frac{\partial^2 u_0^i}{\partial \phi^2}\left(\frac{\partial \phi}{\partial \theta}\right)^2 + \frac{\partial u_0^i}{\partial \phi}\frac{\partial^2 \phi}{\partial \theta^2}\right] + \sin^{-1}\theta\frac{\partial^2 u_0^i}{\partial \omega^2}\Bigg|^2 \\
&\qquad\qquad \cdot \sin^{m-4}\theta(\phi,\alpha)R^{m-4}(\phi,\alpha)\left|\frac{\partial \theta}{\partial \phi}\right| \, d\omega \, d\phi.
\end{aligned}
$$

We denote

$$
K(\alpha,\omega,\phi) = \cos\theta\frac{\partial u_0^i}{\partial \phi}\frac{\partial \phi}{\partial \theta} + \sin\theta\left[\frac{\partial^2 u_0^i}{\partial \phi^2}\left(\frac{\partial \phi}{\partial \theta}\right)^2 + \frac{\partial u_0^i}{\partial \phi}\frac{\partial^2 \phi}{\partial \theta^2}\right] + \sin^{-1}\theta\frac{\partial^2 u_0^i}{\partial \omega^2}.
$$

Since $J(\alpha)$ has a minimum at $\alpha = 0$, the one-sided derivative $J'(0^+)$ is non-negative (we cannot strengthen this into $J'(0) = 0$ as v_α is not necessary differentiable on an open interval containing $\alpha = 0$). We compute this derivative

$$
\begin{aligned}
&(m-4)\frac{d}{d\alpha}J^i(\alpha) = \\
&\int_0^{\frac{\pi}{2}}\int_{\mathbb{S}^{m-2}} 2K(\alpha,\omega,\phi)\frac{\partial}{\partial \alpha}K(\alpha,\omega,\phi)\sin^{m-4}\theta(\phi,\alpha)R^{m-4}(\phi,\alpha)\left|\frac{\partial \theta}{\partial \phi}\right| \\
&\qquad + |K(\alpha,\omega,\phi)|^2 \cdot \Bigg((m-4)\sin^{m-5}\theta\cos\theta\frac{\partial \theta}{\partial \alpha}R^{m-4}\left|\frac{\partial \theta}{\partial \phi}\right| \\
&\qquad\quad + (m-4)\sin^{m-4}\theta R^{m-5}\frac{\partial R}{\partial \alpha}\left|\frac{\partial \theta}{\partial \phi}\right| + \sin^{m-4}\theta R^{m-4}\frac{\partial^2 \theta}{\partial \phi^2}\Bigg) \, d\omega \, d\phi,
\end{aligned}
$$

where

$$\begin{aligned}\frac{\partial}{\partial\alpha}K(\alpha,\omega,\phi) &= -\sin\theta(\phi,\alpha)\frac{\partial\theta}{\partial\alpha}\frac{\partial u_0^i}{\partial\phi}\frac{\partial\phi}{\partial\theta} + \cos\theta(\phi,\alpha)\frac{\partial u_0^i}{\partial\phi}\frac{\partial^2\phi}{\partial\theta\partial\alpha}\\ &+ \cos\theta(\phi,\alpha)\frac{\partial\theta}{\partial\alpha}\left[\frac{\partial^2 u_0^i}{\partial\phi^2}\left(\frac{\partial\phi}{\partial\theta}\right)^2 + \frac{\partial u_0^i}{\partial\phi}\frac{\partial^2\phi}{\partial\theta^2}\right]\\ &+ \sin\theta(\phi,\alpha)\left[2\frac{\partial^2 u_0^i}{\partial\phi^2}\frac{\partial\phi}{\partial\theta}\frac{\partial^2\phi}{\partial\theta\partial\alpha} + \frac{\partial u_0^i}{\partial\phi}\frac{\partial^3\phi}{\partial\theta^2\partial\alpha}\right]\\ &- \sin^{-2}\theta(\phi,\alpha)\cos\theta(\phi,\alpha)\frac{\partial\theta}{\partial\alpha}\frac{\partial^2 u_0^i}{\partial\omega^2}.\end{aligned}$$

Using the following observations:

(i) $R(\phi,\alpha)\,|_{\alpha=0} = 1$,

(ii) $[\partial\theta/\partial\alpha]_{\alpha=0} = \sin\phi$,

(iii) $[\partial\theta/\partial\phi]_{\alpha=0} = 1 = [\partial\phi/\partial\theta]_{\alpha=0}$,

(iv) $[\partial R/\partial\alpha]_{\alpha=0} = -\cos\phi$,

(v) $[\partial^2\theta/\partial\phi\partial\alpha]_{\alpha=0} = \cos\phi$,

(vi) $[\partial^2\phi/\partial\theta\partial\alpha]_{\alpha=0} = -\cos\phi$,

(vii) $[\partial^3\phi/\partial\theta^2\partial\alpha]_{\alpha=0} = \sin\phi$,

(viii) $[\partial^2\phi/\partial\theta^2]_{\alpha=0} = 0$,

(ix) $\sin\theta(\phi,\alpha)\,|_{\alpha=0} = \sin\phi$,

and letting $e(u_0) = \sum_{i=1}^k \left|\cos\phi\frac{\partial u_0^i}{\partial\phi} + \sin\phi\frac{\partial^2 u_0^i}{\partial\phi^2} + \sin^{-1}\phi\frac{\partial^2 u_0^i}{\partial\omega^2}\right|^2 \sin^{m-4}\phi$, we conclude that

$$
\begin{aligned}
0 \leq (m-4)J'(0^+) = -2 &\int_0^{\pi/2} \cos\phi \int_{\mathbb{S}^{m-2}} e(u_0)\, d\omega\, d\phi \\
&+ (m-4) \int_0^{\pi/2} \cos\phi \int_{\mathbb{S}^{m-2}} e(u_0)\, d\omega\, d\phi \\
&- (m-4) \int_0^{\pi/2} \cos\phi \int_{\mathbb{S}^{m-2}} e(u_0)\, d\omega\, d\phi \\
&+ \int_0^{\pi/2} \cos\phi \int_{\mathbb{S}^{m-2}} e(u_0)\, d\omega\, d\phi \\
= - &\int_0^{\pi/2} \cos\phi \int_{\mathbb{S}^{m-2}} e(u_0)\, d\omega\, d\phi \leq 0.
\end{aligned}
$$

Hence, $e(u_0) = 0$ for almost all (φ, ω) and u_0 must be continuous, therefore constant. □

We will need the following lemma due to Scheven [47, Lemma 4.2].

Lemma 3.3.4. *Assume that* $\widehat{v} \in W^{2,2}_{loc}(\mathbb{R}^m, \mathcal{N})$ *is a tangent map of a minimizing biharmonic map and for some* $5 \leq k \leq m$ *it satisfies* $sing(\widehat{v}) = \mathbb{R}^{m-k} \times \{0\}$ *and* $\partial_i \widehat{v} \equiv 0$ *for all* $1 \leq i \leq m-k$*. Then the restriction* $v := \widehat{v}\,|_{\{0\}\times\mathbb{R}^k} \in C^\infty(\mathbb{R}^k \setminus \{0\}, \mathcal{N})$ *is a minimizing biharmonic map and homogeneous of degree zero.*

In the following proof of the boundary regularity by the above lemma we will get that the maps that appear in Federer dimension reduction argument are minimal. We will not repeat the whole argument, as it is known for experts. Instead we refer the interested reader to [51, Theorem A.4.] and in the case of harmonic maps [48, pp. 332–334]

Proof of Theorem 3.0.3. We note that the boundary regularity of biharmonic maps follows for $m \leq 3$ by Sobolev embedding and in the critical dimension $m = 4$ is already known (see [34]). We follow the proof of [28, Corollary 5.8., p. 579].

For $m = 5$ every map which is homogeneous of degree 0 map must be smooth away from the origin.

For $m \geq 5$ we make an $(m-4)$ repeated formulation of boundary tangent maps (see [48, Proof of Theorem II and IV, pp.333–334]), until we obtain a boundary tangent map at a point $b \in T_1$ in the form $u_0(x, y) = v_0(y)$, where $(x, y) \in \mathbb{R}^{m-5} \times \mathbb{R}^5$ and v_0 is a map whose only discontinuity occurs at the origin. In this case, it follows from Lemma 3.3.4 that v_0 and hence u_0 is minimizing. By Lemma 3.3.4 u_0 is homogeneous of degree 0 and constant at T_1. Thus, by 3.3.3 u_0 is constant.

In order to obtain u_0 we constructed a formulation of boundary tangent map, each time getting a sequence of maps converging strongly to a boundary tangent map. Now applying a diagonal sequence argument we extract a subsequence λ_i and rescaled maps u_{b,λ_i} which converge strongly to u_0 as $\lambda_i \searrow 0$. Therefore, because u_0 is constant, for each $\epsilon > 0$ there exists a number $M > 0$ such that for each $i > M$

$$\left(\frac{\lambda_i}{2}\right)^{4-m} \int_{B^+_{\lambda_i}(b)} |\nabla^2 u|^2 \, dx < \epsilon, \qquad \left(\frac{\lambda_i}{2}\right)^{2-m} \int_{B^+_{\lambda_i}(b)} |\nabla u|^2 \, dx < \epsilon. \tag{3.3.5}$$

We claim now that for every $\epsilon > 0$ there exists $\widetilde{R} > 0$ such that for each $\lambda < \widetilde{R}$

$$\lambda^{4-m} \int_{B^+_{\lambda}(b)} |\nabla^2 u|^2 \, dx + \lambda^{2-m} \int_{B^+_{\lambda}(b)} |\nabla u|^2 \, dx < \epsilon. \tag{3.3.6}$$

Indeed, assume on the contrary that there exists an $\epsilon > 0$ such that for each $j \in \mathbb{N}$ there exists a $\lambda_j < \frac{1}{j}$ such that

$$\begin{aligned} \epsilon &\leq \lambda_j^{4-m} \int_{B^+_{\lambda_j}(b),} |\nabla^2 u|^2 \, dx + \lambda_j^{2-m} \int_{B^+_{\lambda_j}(b)} |\nabla u|^2 \, dx \\ &\approx \int_{B^+_{\frac{1}{2}}(b)} |\nabla^2 u_{\lambda_j}(y)|^2 \, dy + \int_{B^+_{\frac{1}{2}}(b)} |\nabla u_{\lambda_j}(y)|^2 \, dy. \end{aligned} \tag{3.3.7}$$

But this contradicts the strong convergence of u_{λ_n} in $W^{2,2}\big(B^+_{1/2}, \mathcal{N}\big)$ to a constant map.

Now by Lemma 3.1.8, (3.3.6) implies for an $r < \widetilde{R}$

$$\left\|\nabla^2 u\right\|^2_{L^{2,m-4}(B_r^+(b))} + \|\nabla u\|^4_{L^{4,m-4}(B_r^+(b))} < C_1\sqrt{\epsilon}. \tag{3.3.8}$$

Thus, by Theorem 3.1.1 we finally conclude that $u \in C^\infty(\overline{B_{\frac{r}{2}}}(b), \mathcal{N})$. $\square$

Appendices

Appendix A

Harmonic maps

We shall extract the formula for $[\psi]_{p,q}$ given in Definition 2.1.1. As mentioned in Section 2.1 this follows the construction of Almgren, Browder and Lieb in [2, Appendix A.2].

Rotate $\mathbb{S}^2$ so that p is the south pole $x_S = (0, 0, -1)$, let $q = 2/j$, and work in the spherical coordinates (ϕ, θ), where $\phi \in [0, \pi]$ stands for the polar angle ($\phi = \pi$ corresponds to x_S) and $\theta \in [0, 2\pi]$ – for the azimuthal angle.

Let $\Phi := [\varphi_1]_{x_S, 2/j}$ on the spherical cap $D(x_S, 2/j)$. Without loss of generality we assume that $\Phi(x_S) = \varphi_1(x_S) = (0, 0, 1)$. On the annulus $A(x_S; 1/j, 2/j)$ the map φ_1 is constant, and Φ is Lipschitz with a constant not depending on j. The main task is to estimate the p-energy of Φ on a smaller disk $D(x_S, 1/j)$, blown by Φ onto a punctured sphere. For the sake of explicit estimates, we shall extract the formula for Φ on $D(x_S, 1/j)$.

In the spherical coordinates on $\mathbb{S}^2$ and the polar coordinates on the equatorial $\mathbb{R}^2$, the stereographic projection Φ_1 is given by $\Phi_1(\phi, \theta) = \left(\cot\left(\frac{\phi}{2}\right), \theta\right)$. We have $D(x_S, 1/j) = \{(\phi, \theta) \colon \gamma_j < \phi \leq \pi\}$ with the latitude angle

$$\gamma_j = 2 \arccos \frac{1}{2j} \quad \text{ on } \partial D(x_S, 1/j).$$

Thus, in the polar coordinates (ρ, ϑ) in $\mathbb{R}^2$,

$$\Delta_j := \Phi_1(D(x_S, 1/j)) = \{(\rho, \vartheta) \colon 0 \le \rho < d_j := (4j^2 - 1)^{-1/2}\} .$$

We now set $\Phi|_{D(x_S,1/j)} = \Phi_2 \circ \Phi_1|_{D(x_S,1/j)}$, where Φ_2 sends an annulus

$$\tilde{A}_j \Subset \Delta_j \setminus \{0\}$$

onto the whole $\mathbb{S}^2$ without two small caps (by rescaling $\tilde{A}_j$ and then applying Φ_1^{-1}), and Φ_2 is Lipschitz with an absolute constant on $\Delta_j \setminus \tilde{A}_j$ (so that the resulting map Φ satisfies all the requirements of Definition 2.1.1).

Specifically, fix $0 < \beta = \beta_j = \frac{d_j}{2}$, and let $R = R_j = \cot \frac{\beta_j}{2}$ be the radius of the circle in $\mathbb{R}^2$ which is mapped to $\{\phi = \beta_j\}$ on $\mathbb{S}^2$ by the inverse stereographic projection Φ_1^{-1}. Set

$$\lambda_j = \frac{R_j}{\beta_j}\,, \qquad r_j = \beta_j \tan^2 \frac{\beta_j}{2}\,, \qquad \text{so that} \quad \lambda_j r_j = \tan \frac{\beta_j}{2} = \cot \frac{\pi - \beta_j}{2}.$$

The circle $\partial B^2(0, \lambda_j r_j) \subset \mathbb{R}^2$ is mapped by the inverse stereographic projection to the latitude circle $\pi - \beta_j$ near the south pole in $\mathbb{S}^2$. Hence,

$$\tilde{A}_j := \{(\rho, \vartheta) \colon r_j < \rho < \beta_j\} \subset \Delta_j$$

satisfies $(\Phi_1^{-1} \circ \lambda_j \,\mathrm{Id})\big(\tilde{A}_j\big) = \mathbb{S}^2 \cap \{\beta_j < \phi < \pi - \beta_j\}$. We define the whole map $\Phi_2 \colon \Delta_j \to \mathbb{S}^2$ by setting

$$\Phi_2(\rho, \vartheta) = \begin{cases} \left(\pi - \rho\frac{\beta_j}{r_j}, \vartheta\right) & \text{for } 0 \le \rho \le r_j, \\ \left(2 \operatorname{arc\,cot}\left(\frac{R_j}{\beta_j}\, \rho\right), \vartheta\right) & \text{for } r_j < \rho < \beta_j \quad \text{(i.e., on } \tilde{A}_j\text{)}, \\ (d_j - \rho, \vartheta) & \text{for } \beta_j \le \rho < d_j. \end{cases}$$

Finally, $\Phi = \Phi_2 \circ \Phi_1$ on $D(x_S, 1/j)$.

We make here a few comments:

(a) Φ is a composition of a conformal map (Φ_1) and a Lipschitz map (Φ_2) with constant β_j/r_j on the spherical cap $D_j := D(x_S, 2r_j/\sqrt{r_j^2 + 1})$;

(b) The p-energy of Φ on

$$A_j^{(1)} = \Phi^{-1}\big(\mathbb{S}^2 \cap \{\beta_j < \phi < \pi - \beta_j\}\big) = A\left(x_S; \frac{2r_j}{\sqrt{r_j^2+1}}, \frac{2\beta_j}{\sqrt{\beta_j^2+1}}\right) \subset \mathbb{S}^2$$

goes to zero as $j \to \infty$,

(c) Φ is a composition of a conformal map (Φ_1) and a Lipschitz map (Φ_2) with constant 1 on the annular region $A_j^{(2)} := A(x_S; \frac{2\beta}{\sqrt{\beta^2+1}}, \frac{1}{j})$.

Thus, by Hölder's inequality and conformality of Φ_1

$$\int_{D_j} |\nabla_T \Phi|^p d\sigma \leq \left(\frac{\beta_j}{r_j}\right)^p \left(2\mathcal{H}^2\left(\Delta_j\right)\right)^{p/2} \cdot \left(\mathcal{H}^2\left(D_j\right)\right)^{(2-p)/2}$$

and

$$\int_{A_j^{(2)}} |\nabla_T \Phi|^p d\sigma \leq \left(2\mathcal{H}^2\big(\Phi_1\big(A_j^{(2)}\big)\big)\right)^{p/2} \cdot \mathcal{H}^2\big(A_j^{(2)}\big)^{(2-p)/2}.$$

It is easy now to observe that both those terms converge to 0 as $j \to \infty$.

Appendix B

Biharmonic maps

Lemma B.0.5. *There is a constant C depending only on m such that for any $0 < r < R$ and any map $u \in W^{2,2}(B_R^+, \mathbb{R}^\ell)$ with vanishing $W^{2,2}$ trace on $T_R = \{x \in B_R \colon x_m = 0\}$ we have*

$$\int_{B_r^+} |\nabla^2 u|^2 \, dx \leq 2 \int_{B_R^+} \left(|\Delta u|^2 + \frac{C}{(R-r)^2} |\nabla u|^2 \right) \, dx. \tag{B.0.1}$$

Proof. The proof is exactly as in [47, Lemma A.1]. We choose the same cut-off function $\eta \in C_c^\infty(B_R, [0,1])$ such that $\eta \equiv 1$ on B_r and $|\nabla \eta| < \frac{C}{(R-r)}$. If we assume that $u \in C^\infty(B_R^+, \mathbb{R}^l)$ and integrate twice by parts the following integral

$$\int_{B_R^+} \eta^4 |\Delta|^2 \, dx,$$

the boundary term will vanish on the flat part of ∂B_R^+, because the $W^{2,2}$ trace of u vanishes there. It will also vanish on the curved part of the boundary, for

η vanishes there. Thus,

$$\begin{aligned}\int_{B_R^+} \eta^4 u_{x_i x_i} u_{x_j x_j}\, dx &= -4 \int_{B_R^+} \eta^3 \eta_{x_j} u_{x_i x_i} u_{x_j}\, dx - \int_{B_R^+} \eta^4 u_{x_i x_i x_j} u_{x_j}\, dx \\ &= -4 \int_{B_R^+} \eta^3 \eta_{x_j} u_{x_i x_i} u_{x_j}\, dx + 4 \int_{B_R^+} \eta^3 \eta_{x_i} u_{x_i x_j} u_{x_j}\, dx \\ &\quad + \int_{B_R^+} \eta^4 u_{x_i x_j} u_{x_i x_j}\, dx.\end{aligned}$$

Hence,

$$\begin{aligned}\int_{B_R^+} \eta^4 |\nabla^2 u|^2\, dx &\leq \int_{B_R^+} \eta^4 |\Delta u|^2\, dx + C \int_{B_R^+} \eta^3 |\nabla \eta| |\nabla u| |\nabla^2 u|\, dx \\ &\leq \int_{B_R^+} \eta^4 |\Delta u|^2\, dx + \frac{1}{2} \int_{B_R^+} \eta^4 |\nabla^2 u|^2\, dx \\ &\quad + \frac{C}{(R-r)^2} \int_{B_R^+} \eta^2 |\nabla u|^2\, dx.\end{aligned}$$

The desired inequality for smooth u follows by subtracting $\frac{1}{2} \int_{B_R^+} \eta^4 |\nabla^2 u|^2\, dx$ from both sides. Now an approximation argument yields the the same argument for $W^{2,2}$ maps. $\square$

The next Lemma shows that by boundary monotonicity formula a bound in $W^{2,2}$ implies a bound in the Morrey space $L^{2,m-4}$. The proof is almost identical to the proof in the interior case, but as the boundary monotonicity formula yields an additional term we sketch the proof below.

Lemma B.0.6. *Let $u \in W^{2,2}(B_4^+, \mathcal{N})$ be a minimizing biharmonic map with boundary value φ as in* (3.0.1), *satisfying the boundary monotonicity formula* (3.1.6) *and $\|u - \varphi\|_{W^{2,2}(B^+)} < \infty$. Let $\widetilde{u}$ be the reflection of $u - \varphi$ given in 3.2.3, then*

$$\sup_{y \in B,\ \rho < 1} \rho^{4-m} \int_{B_\rho(y)} |\nabla^2 \widetilde{u}|^2\, dx \leq C \int_{B_2} |\nabla^2 \widetilde{u}|^2\, dx + \widetilde{C}, \tag{B.0.2}$$

for constants $C = C(m)$ and $\widetilde{C} = \widetilde{C}(m, \mathcal{N})$.

Proof of Lemma B.0.6. We give the necessary modification of [47, Lemma A.2].

We note that since u satisfies the boundary monotonicity formula (3.1.6) we have for $\widetilde{u}$ and $a \in T_1$ the following

$$\begin{aligned} &\rho^{4-m} \int_{B_\rho(a)} |\nabla^2 \widetilde{u}|^2 \, dx + C e^{C\rho} R_u^+(a, \rho) \\ &\quad \le C \left(e^{Cr} r^{4-m} \int_{B_r(a)} |\nabla^2 \widetilde{u}|^2 \, dx + e^{Cr} R_u^+(a, r) + C r e^{Cr} \right). \end{aligned} \tag{B.0.3}$$

Let $0 < s < 1/8$ be given. By Fubini theorem we may choose good radii $\rho < r$ with $s \le \rho \le 2s < \frac{1}{2} \le r \le 1$ such that

$$\begin{aligned} \rho^{3-m} \int_{B_\rho(a)} |\nabla \widetilde{u}|^2 \, d\mathcal{H}^{m-1} &\le C s^{2-m} \int_{B_{2s(a)}} |\nabla \widetilde{u}|^2 \, dx \\ \rho^{5-m} \int_{B_\rho(a)} |\nabla^2 \widetilde{u}|^2 \, d\mathcal{H}^{m-1} &\le C s^{4-m} \int_{B_{2s(a)}} |\nabla^2 \widetilde{u}|^2 \, dx \\ \int_{B_r(a)} \left(|\nabla^2 \widetilde{u}|^2 + |\nabla \widetilde{u}|^2 \right) \, d\mathcal{H}^{m-1} &\le C \int_{B_1(a)} \left(|\nabla^2 \widetilde{u}|^2 + |\nabla \widetilde{u}|^2 \right) \, dx, \end{aligned}$$

where the constant C depends only on the dimension m.

One can easily observe that

$$\begin{aligned} \left| R_{u,\varphi}^+(a, \tau) \right| &\le C \tau^{4-m} \int_{\partial B_\tau^+(a)} \left(|\nabla^2 u| |\nabla u| + \frac{1}{\tau} |\nabla u|^2 \right) \, d\mathcal{H}^{m-1} \\ &\le C \tau^{4-m} \int_{\partial B_\tau(a)} \left(|\nabla^2 \widetilde{u}| |\nabla \widetilde{u}| + \frac{1}{\tau} |\nabla \widetilde{u}|^2 \right) \, d\mathcal{H}^{m-1}. \end{aligned}$$

Combining this observation with (B.0.3) we get

$$\begin{aligned}\rho^{4-m}\int_{B_\rho(a)}|\nabla^2\widetilde{u}|^2\,dx \le Ce^{Cr}r^{4-m}&\int_{B_r(a)}|\nabla^2\widetilde{u}|^2\,dx + Cre^{Cr}\\ &+ C\rho^{4-m}\int_{\partial B_\rho(a)}\left(|\nabla^2\widetilde{u}||\nabla\widetilde{u}| + \frac{1}{\rho}|\nabla\widetilde{u}|^2\right)\,d\mathcal{H}^{m-1}\\ &+ Ce^{Cr}r^{4-m}\int_{\partial B_r(a)}\left(|\nabla^2\widetilde{u}||\nabla\widetilde{u}| + \frac{1}{r}|\nabla\widetilde{u}|^2\right)\,d\mathcal{H}^{m-1}\end{aligned}$$

Thus, since $s < \rho < 2s$ and by Young's inequality with ϵ

$$\begin{aligned}s^{4-m}\int_{B_s(a)}|\nabla^2\widetilde{u}|^2\,dx &\le C\rho^{4-m}\int_{B_\rho(a)}|\nabla^2\widetilde{u}|^2\,dx\\ &\le \frac{1}{4}(2s)^{4-m}\int_{B_{2s}(a)}|\nabla^2\widetilde{u}|^2\,dx + Cs^{2-m}\int_{B_{2s}(a)}|\nabla\widetilde{u}|^2\,dx\\ &\quad + C\int_{B_1(a)}|\nabla^2\widetilde{u}|^2\,dx + C\int_{B_1(a)}|\nabla\widetilde{u}|^2\,dx + C.\end{aligned}\tag{B.0.4}$$

Next, we proceed exactly as in [47]. Observe that by Nirenberg's interpolation inequality

$$\|\nabla f\|^2_{L^4(\Omega)} \le C(\Omega)\|f\|_{L^\infty(\Omega)}\|f\|_{W^{2,2}(\Omega)}$$

we have after a few transformations

$$C\tau^{2-m}\int_{B_\tau(y)}|\nabla\widetilde{u}|^2\,dx \le \frac{1}{4}\tau^{4-m}\int_{B_\tau(y)}|\nabla\widetilde{u}|^2\,dx + \widetilde{C},\tag{B.0.5}$$

where $\widetilde{C}$ is a constant dependent on the target manifold $\mathcal{N}$. Applying (B.0.5) into (B.0.4) for $\tau = 1$ and $\tau = 2s$, denoting $\widehat{H}(\tau) := \tau^{4-m}\int_{B_\tau(y)}|\nabla^2\widetilde{u}|^2\,dx$, we arrive at

$$\widehat{H}(s) \le \frac{1}{2}\widehat{H}(2s) + C\widehat{H}(1) + \widetilde{C}.$$

for all $0 < s < \frac{1}{4}$.

Thus, for all small $\sigma > 0$

$$\sup_{\sigma<s<1}\widehat{H}(s) \le \sup_{\sigma<s<1/4}\widehat{H}(s) + C\widehat{H}(1) \le \frac{1}{2}\sup_{\sigma<s<1/}\widehat{H}(2s) + C\widehat{H}(1) + \widetilde{C}.$$

Since $\sigma > 0$ the term $\frac{1}{2}\sup_{\sigma<s<1/4}\widehat{H}(4s)$ is finite and can be absorbed by the left hand side of the inequality giving

$$\sup_{\sigma<s<1}\widehat{H}(\rho) \leq C\widehat{H}(1) + \widetilde{C}.$$

The estimate is independent of $\sigma > 0$ and thus the claimed inequality follows.

$\square$

Remark B.0.7. In the last proof we did not need a higher order reflection. An odd reflection is enough to ensure that if $u - \varphi \in W_0^{2,2}(B^+, \mathbb{R}^\ell)$, then the reflected map is in $W^{2,2}(B, \mathbb{R}^\ell)$.

Bibliography

[1] Robert A. Adams. *Sobolev spaces.* Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] Frederick J. Almgren, Jr., William Browder, and Elliott H. Lieb. Co-area, liquid crystals, and minimal surfaces. In *Partial differential equations (Tianjin, 1986)*, volume 1306 of *Lecture Notes in Math.*, pages 1–22. Springer, Berlin, 1988.

[3] Frederick J. Almgren, Jr. and Elliott H. Lieb. Singularities of energy minimizing maps from the ball to the sphere: examples, counterexamples, and bounds. *Ann. of Math. (2)*, 128(3):483–530, 1988.

[4] Serdar Altuntas. Private communication. *PhD thesis in preparation at the University of Duisburg-Essen under supervision of Christoph Scheven.*

[5] Gilles Angelsberg. A monotonicity formula for stationary biharmonic maps. *Math. Z.*, 252(2):287–293, 2006.

[6] Fabrice Bethuel. On the singular set of stationary harmonic maps. *Manuscripta Math.*, 78(4):417–443, 1993.

[7] Fabrice Bethuel, Haïm Brezis, and Jean-Michel Coron. Relaxed energies for harmonic maps. In *Variational methods (Paris, 1988)*, volume 4 of *Progr. Nonlinear Differential Equations Appl.*, pages 37–52. Birkhäuser Boston, Boston, MA, 1990.

[8] Simon Blatt. Monotonicity formulas for extrinsic triharmonic maps and the triharmonic Lane-Emden equation. *J. Differential Equations*, 262(12):5691–5734, 2017.

[9] Christine Breiner and Tobias Lamm. Quantitative stratification and higher regularity for biharmonic maps. *Manuscripta Math.*, 148(3-4):379–398, 2015.

[10] Haim Brezis. The interplay between analysis and topology in some nonlinear PDE problems. *Bull. Amer. Math. Soc. (N.S.)*, 40(2):179–201, 2003.

[11] Haïm Brezis, Jean-Michel Coron, and Elliott H. Lieb. Harmonic maps with defects. *Comm. Math. Phys.*, 107(4):649–705, 1986.

[12] Sun-Yung A. Chang, Lihe Wang, and Paul C. Yang. A regularity theory of biharmonic maps. *Comm. Pure Appl. Math.*, 52(9):1113–1137, 1999.

[13] Yun Mei Chen and Fang-Hua Lin. Evolution of harmonic maps with Dirichlet boundary conditions. *Comm. Anal. Geom.*, 1(3-4):327–346, 1993.

[14] Lawrence C. Evans. Partial regularity for stationary harmonic maps into spheres. *Arch. Rational Mech. Anal.*, 116(2):101–113, 1991.

[15] Herbert Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[16] Martin Fuchs. p-harmonic obstacle problems. III. Boundary regularity. *Ann. Mat. Pura Appl. (4)*, 156:159–180, 1990.

[17] Andreas Gastel. Partial regularity of polyharmonic maps to targets of sufficiently simple topology. *Z. Anal. Anwend.*, 35(4):397–410, 2016.

[18] Andreas Gastel and Christoph Scheven. Regularity of polyharmonic maps in the critical dimension. *Comm. Anal. Geom.*, 17(2):185–226, 2009.

[19] Filippo Gazzola, Hans-Christoph Grunau, and Guido Sweers. *Polyharmonic boundary value problems*, volume 1991 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 2010. Positivity preserving and nonlinear higher order elliptic equations in bounded domains.

[20] Mariano Giaquinta. *Introduction to regularity theory for nonlinear elliptic systems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

[21] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order.* Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[22] Huajun Gong, Tobias Lamm, and Changyou Wang. Boundary partial regularity for a class of biharmonic maps. *Calc. Var. Partial Differential Equations*, 45(1-2):165–191, 2012.

[23] Andrzej Granas and James Dugundji. *Fixed point theory.* Springer Monographs in Mathematics. Springer-Verlag, New York, 2003.

[24] Robert Hardt. Singularities of harmonic maps. *Bull. Amer. Math. Soc. (N.S.)*, 34(1):15–34, 1997.

[25] Robert Hardt, David Kinderlehrer, and Fang Hau Lin. The variety of configurations of static liquid crystals. In *Variational methods (Paris, 1988)*, volume 4 of *Progr. Nonlinear Differential Equations Appl.*, pages 115–131. Birkhäuser Boston, Boston, MA, 1990.

[26] Robert Hardt, David Kinderlehrer, and Mitchell Luskin. Remarks about the mathematical theory of liquid crystals. In *Calculus of variations and partial differential equations (Trento, 1986)*, volume 1340 of *Lecture Notes in Math.*, pages 123–138. Springer, Berlin, 1988.

[27] Robert Hardt and Fang-Hua Lin. A remark on H^1 mappings. *Manuscripta Math.*, 56(1):1–10, 1986.

[28] Robert Hardt and Fang-Hua Lin. Mappings minimizing the L^p norm of the gradient. *Comm. Pure Appl. Math.*, 40(5):555–588, 1987.

[29] Robert Hardt and Fang-Hua Lin. Stability of singularities of minimizing harmonic maps. *J. Differential Geom.*, 29(1):113–123, 1989.

[30] Frédéric Hélein. Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(8):591–596, 1991.

[31] Min-Chun Hong and Changyou Wang. Regularity and relaxed problems of minimizing biharmonic maps into spheres. *Calc. Var. Partial Differential Equations*, 23(4):425–450, 2005.

[32] Tadeusz Iwaniec. p-harmonic tensors and quasiregular mappings. *Ann. of Math. (2)*, 136(3):589–624, 1992.

[33] Tobias Lamm and Tristan Rivière. Conservation laws for fourth order systems in four dimensions. *Comm. Partial Differential Equations*, 33(1-3):245–262, 2008.

[34] Tobias Lamm and Changyou Wang. Boundary regularity for polyharmonic maps in the critical dimension. *Adv. Calc. Var.*, 2(1):1–16, 2009.

[35] Fang-Hua Lin. A remark on the map $x/|x|$. *C. R. Acad. Sci. Paris Sér. I Math.*, 305(12):529–531, 1987.

[36] Fang-Hua Lin. Gradient estimates and blow-up analysis for stationary harmonic maps. *Ann. of Math. (2)*, 149(3):785–829, 1999.

[37] Stephan Luckhaus. Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold. *Indiana Univ. Math. J.*, 37(2):349–367, 1988.

[38] Stephan Luckhaus. Convergence of minimizers for the p-Dirichlet integral. *Math. Z.*, 213(3):449–456, 1993.

[39] Katarzyna Mazowiecka and Paweł Strzelecki. The Lavrentiev gap phenomenon for harmonic maps into spheres holds on a dense set of zero degree boundary data. *Adv. Calc. Var.*, 10(3):303–314, 2017.

[40] S. Montaldo and C. Oniciuc. A short survey on biharmonic maps between Riemannian manifolds. *Rev. Un. Mat. Argentina*, 47(2):1–22 (2007), 2006.

[41] Roger Moser. Pointwise estimates by scaling for quasilinear equations. *Unpublished manuscript*, 2005.

[42] John Nash. The imbedding problem for Riemannian manifolds. *Ann. of Math. (2)*, 63:20–63, 1956.

[43] Peter Price. A monotonicity formula for Yang-Mills fields. *Manuscripta Math.*, 43(2-3):131–166, 1983.

[44] Tristan Rivière. Existence of infinitely many weakly harmonic maps into spheres for non- constant boundary datas. *Thèse, Université Paris 6*, 1993.

[45] Tristan Rivière. Everywhere discontinuous harmonic maps into spheres. *Acta Math.*, 175(2):197–226, 1995.

[46] Christoph Scheven. Variationally harmonic maps with general boundary conditions: boundary regularity. *Calc. Var. Partial Differential Equations*, 25(4):409–429, 2006.

[47] Christoph Scheven. Dimension reduction for the singular set of biharmonic maps. *Adv. Calc. Var.*, 1(1):53–91, 2008.

[48] Richard Schoen and Karen Uhlenbeck. A regularity theory for harmonic maps. *J. Differential Geom.*, 17(2):307–335, 1982.

[49] Richard Schoen and Karen Uhlenbeck. Boundary regularity and the Dirichlet problem for harmonic maps. *J. Differential Geom.*, 18(2):253–268, 1983.

[50] Jalal Shatah. Weak solutions and development of singularities of the $\mathrm{SU}(2)$ σ-model. *Comm. Pure Appl. Math.*, 41(4):459–469, 1988.

[51] Leon Simon. *Lectures on geometric measure theory*, volume 3 of *Proceedings of the Centre for Mathematical Analysis, Australian National University.* Australian National University, Centre for Mathematical Analysis, Canberra, 1983.

[52] Leon Simon. *Theorems on regularity and singularity of energy minimizing maps.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996. Based on lecture notes by Norbert Hungerbühler.

[53] Michael Struwe. Partial regularity for biharmonic maps, revisited. *Calc. Var. Partial Differential Equations*, 33(2):249–262, 2008.

[54] Paweł Strzelecki. On biharmonic maps and their generalizations. *Calc. Var. Partial Differential Equations*, 18(4):401–432, 2003.

[55] Changyou Wang. Boundary partial regularity for a class of harmonic maps. *Comm. Partial Differential Equations*, 24(1-2):355–368, 1999.

[56] Changyou Wang. Biharmonic maps from $\mathbb{R}^4$ into a Riemannian manifold. *Math. Z.*, 247(1):65–87, 2004.

[57] Changyou Wang. Remarks on biharmonic maps into spheres. *Calc. Var. Partial Differential Equations*, 21(3):221–242, 2004.

[58] Changyou Wang. Stationary biharmonic maps from $\mathbb{R}^m$ into a Riemannian manifold. *Comm. Pure Appl. Math.*, 57(4):419–444, 2004.