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Structural and algorithmic aspects of partial orderings of graphs

PhD dissertation

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Author’s declaration:
I hereby declare that this dissertation is my own work.

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This thesis falls within the field of Graph Theory. A central theme is the study of exclusion theorems and their uses in related topics. One of them is well-quasi-ordering: we identify well-quasi-ordered subclasses for several orderings of graphs using structural decompositions. A second one the the study of the relations between combinatorial invariants related to problems of packing and covering of combinatorial structures. In this direction, we establish new connections between these invariants for some classes of graphs. We also present algorithmic applications of the results.

Keywords: Graph Theory, exclusion theorems, forbidden substructures, packing and covering, well-quasi-ordering, combinatorial dichotomies.

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Chapter 1

Introduction

Consider a (possibly infinite) collection of objects. Among these objects, some may be more desirable that others. It might also happen that two objects are not comparable, that is, neither of them is more desirable than the other. For a particular object x of the collection, let us consider all the objects that are not more desirable than x. This includes objects that are less desirable than x, but also those that are not comparable with it. A central theme in this thesis, for various collections of objects and notions of *desirability*, is the study of these objects that are not more desirable than x. How do they look like? What is their *structure*? What are their properties?

The objects that we consider in this thesis are graphs and the desirability concept is expressed by several orderings of graphs: subgraphs, induced subgraphs, minors, topological minors, induced minors, immersions, etc., that most often express that a graph *contained* as a substructure of an other. One of the first theorems that established the properties of a class of graphs defined as above is the characterization of planar graphs by Kuratowski (see Figure 1.1 for a picture of the mentioned graphs).

Theorem 1.1 (Kuratowski's Theorem, 1930 [Kur30][1]). *A graph is planar*[2] *iff it does not contain a subgraph that is a subdivision of* K_5 *or* $K_{3,3}$*.*

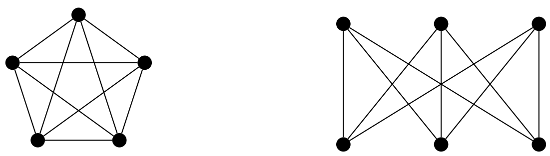


Figure 1.1: Kuratowski's graphs K_5 and $K_{3,3}$ (from left to right).

A feature that made this result famous is that it describes a topological property, being planar, in purely combinatorial terms: the absence of a substructure in the graph. Often cited as a precursor of modern Graph Theory, Kuratowski's Theorem has been followed by a long line of results on classes defined by *excluding* a given substructure. One of them is Turán's Theorem, which gives a an upper-bound on the number of edges of a graph not containing a large complete subgraph.

[1] According to [Bur78], this result has be independently obtained by Pontryagin, but never published.
[2] We say that a graph is *planar* if it can be drawn on the plane without crossing edges.

Theorem 1.2 (Turán's Theorem, 1941 [Tur41])**.** *Let $p \geq 2$ be an integer, and let G be a graph that does not contain a complete subgraph on p vertices. If G has n vertices, then its number of edges is at most:*

$$\frac{p-2}{2(p-1)} \cdot n^2.$$

In the long and fruitful series of papers Graph Minors, Robertson and Seymour gave a general depiction of the graphs not containing a large complete graph as a minor [RS03]. This result, known as the Graph Minor Structure Theorem, provides a structural description of these graphs: what they look like and how they can be decomposed. We do not give the statement of this theorem here as it would require to introduce many more definitions. Let us mention that in addition to its combinatorial value, this result is a cornerstone of the proof of the Graph Minor Theorem [RS04] that we will meet again in a few paragraphs.

The three theorems stated above yielded different conclusions about the considered classes: the first one provides us with topological information, the second one upper-bounds an invariant for all graphs in this class, and with the third one we learn about the structure of the graphs. Our purposes in introducing these theorems are to present possible outcomes of results on classes defined by excluded substructures, and to highlight the ones we consider in this thesis, that are bounds on invariants and structural decomposition of graphs.

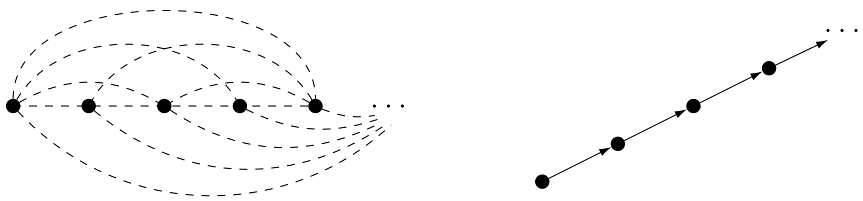
This thesis is centered around exclusion theorems. We now present two connected topics that will receive attention in this thesis: well-quasi-orderings and the Erdős-Pósa property.

Well-quasi-ordering

Let us go back to our initial example of a collection of objects, some of which are more desirable than others. Imagine now that we are asked to choose one object in this collection. What properties ensure that such a choice is always easy? If the collection contains several desirable objects that are pairwise not comparable, we can choose among them by rolling a dice. However this is not possible when these non comparable objects are infinitely many, assuming that we want a fair choice. An other situation that may occur is when the collection contains an infinite sequence of objects that are ever more desirable: in this case we cannot find a most desirable object among them (see Figure 1.2).

This leads us to consider collections where these two behaviors are forbidden. This is more or less the definition of well-quasi-orders. Formally, a *well-quasi-order* is a quasi-order (that is, a reflexive and transitive relation) that contains neither infinite collections of incomparable elements, not infinite decreasing sequences[3]. This concept is an extension of that of a well-order, usually defined for total orders (see [Mos06, Chapter 7] for

[3]While this definition slightly differs from the example we give above where infinite *increasing* sequences are forbidden, it is quite the same if we consider the dual order where x is at least y iff y is at least x in the original order.



Infinitely many incomparable objects. An infinite increasing sequence.

Figure 1.2: Two situation where the choice is hard. Dashed edges connect non-comparable elements and arrows point towards larger elements.

an introduction to well-ordering), to the setting of partial orders, where two elements are not always comparable. In a *well-order*, only infinite decreasing sequences are forbidden, as there is no collection of incomparable elements. The popularity of well-orders in mathematics is partly due to the fact that they are used (often implicitly) in several widely-spread proof techniques such as induction, proofs by minimal counterexample, and the related infinite descent technique. Well-orders can also be used to ensure program termination (or similarly, the termination of a rewriting system): if every state of the program execution is lower than its anterior step for some carefully chosen well-order, then the program will eventually terminate. This is facilitated by the use of recursively defined data types (as lists and trees), which are often well-ordered [MP67, Bur69].

In Graph Theory, one often deals with classes that are *closed* with respect to some ordering, i.e. every object that is lower than an object of the class also belongs to it. For instance, acyclic graphs are closed with respect to the subgraph relation, because no subgraph of an acyclic graph contains a cycle. The importance of well-quasi-orders in Graph Theory and Algorithms is due to the following fact: in a well-quasi-order, the complementary of a closed class has finitely many minimal elements. This follows from the definition, as these elements, being minimal, are incomparable. Therefore, in order to decide if an object x belongs to a closed subclass of a well-quasi-order, one simply needs to check that x is not greater than y, for finitely many objects y (the minimal elements of the complementary).

One of the most considerable results on well-quasi-ordering in Graph Theory is arguably the already mentioned Graph Minor Theorem of Robertson and Seymour.

Theorem 1.3 (Graph Minor Theorem, [RS04])**.** *Finite graphs are well-quasi-ordered by the minor relation.*

This result had a strong impact, in Combinatorics and Graph Theory, but also in related fields as Algorithms and their connections to Logic and Computational Complexity. For the aforementioned reasons, a consequence of the Graph Minor Theorem is that every class of graphs that is closed with respect to the minor relation has a characterization in terms of finitely many excluded minors. In some sense, this theorem provides a Kuratowski type exclusion result for every class that is closed with respect to minors. It should be noted that the finite list of excluded minors is not given by the theorem, which only provides its existence. Together with an algorithm that decides if a fixed graph is a minor of the input graph in polynomial time (also originating from the Graph Minors series [RS95]), the Graph Minor Theorem also implies (again, purely existentially) that decision problems associated to classes closed with respect to minors can be solved in polynomial time. The reader is refered to [Lov06, Joh87] for an account on Minor Theory and its algorithmic consequences.

Other natural orderings of graphs are not so generous; they usually do not well-quasi-order all graphs. An illustration of this fact is given in Figure 1.3: no cycle is a subgraph of a larger cycle. This raises the question of identifying the subclasses that

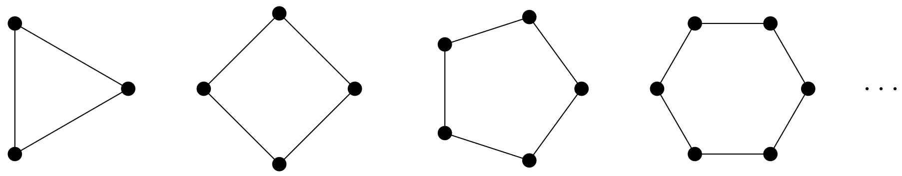


Figure 1.3: Cycles form an infinite set of graphs that are pairwise not comparable for the subgraph ordering.

are well-quasi-ordered.

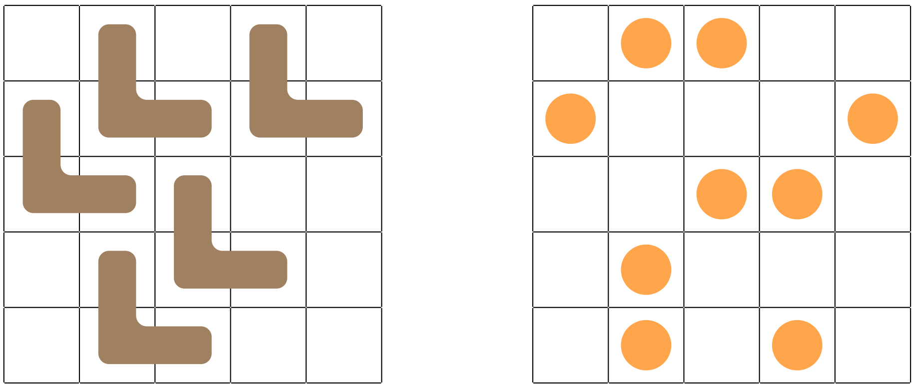
In this manuscript we provide a partial answer to this question for several orderings on graphs, complementing existing results. Essential parts of our proofs are decomposition theorems, which provide structural information about graphs excluding a certain substructure. Indeed, graphs in these classes often have a specific shape or structure, that can be used to dissect them into simpler objects, and eventually order them.

The Erdős–Pósa property

We now move to a seemingly unrelated topic and show how it is connected to exclusion theorems. Let us start with a concrete example, which is a mathematical puzzle for children (popularized by [Mod], see also [GOB06]). We consider a grid that represents a garden. Imagine that this garden is invaded by rats, which are represented by identical shapes of adjacent cells with a fixed orientation. The goal of the game is to defend the garden by disposing traps (that cover one cell each) so that any rat in the garden would meet a trap. A rat and a trap are depicted in Figure 1.4.

Figure 1.4: Rats and traps.

Naturally, we could place one on every cell, but we aim at using as few traps as possible. Figure 1.5 depicts five rats in the garden (can we have more?) and an arrangement of traps that defend the garden (can we use less?). Observe that this puzzle has



Rats invading the garden — Traps defending the garden

Figure 1.5: Hunting rats.

an infinity of variants: each choice of a garden (which could be a larger grid, or a more convoluted shape) and of a rat gives rise to a new problem.

This example allows us to define two numbers: the first one is the maximum number p of rats that fit in the garden without sharing a cell, and the second one is the minimum number c of traps required to defend the garden. Observe that c is always at least p. Indeed, if p rats are present in the garden, then for every rat, we need to place a trap on some of the cells it occupies. However, in some situations we may need more than p traps. This raises the following questions.

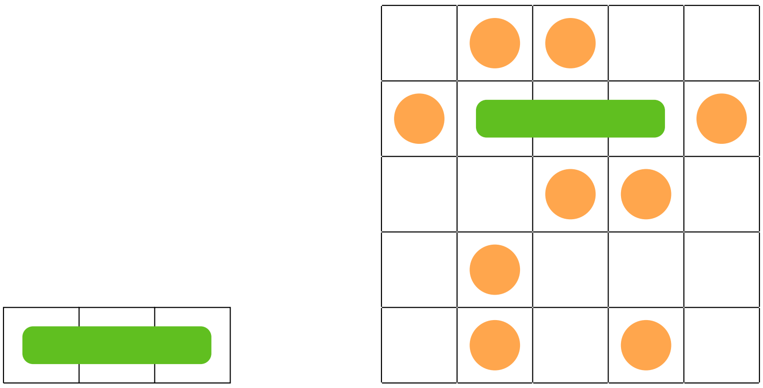
(1) What is the relation between c and p? In particular, can we bound c from above by a function of p?

(2) Can we easily compute the values of p and c?

It is easy to see that, if we consider p rats in the garden, placing a trap on each cell they occupy is enough to defend the garden. As a rat fills three cells, we get $c \leq 3p$. Can we do better? An variant of this puzzle is to protect the garden against several animals species with different shapes, for instance rats and snakes, defined as 3 consecutive cells. The problem then becomes different: as Figure 1.6 suggests, a trap arrangement protecting against rats may be inefficient against snakes. Also, the maximum number of rats or snakes that fit in the garden can potentially be larger than p.

Coming back to the setting of graphs, we can define similar invariants. Given a graph G and a class $\mathcal{H}$, the *packing number* of $\mathcal{H}$ in G is the maximum number of graphs in $\mathcal{H}$ (with repetitions allowed) that can be found in G without overlapping. Here, G plays the role of the garden, and $\mathcal{H}$ that of the list of potential pests. Similarly, the *covering number* of $\mathcal{H}$ in G is the minimum size of set of vertices (corresponding to traps) that meet every occurrence of a graph of $\mathcal{H}$ in G. Packing and covering numbers mirror the numbers p and c, respectively. The same questions as above can be asked in this setting. An example of a classic theorem answering question (1) is Kőnig's Theorem.

Theorem 1.4 (Kőnig's Theorem, 1931 [Kőn31])**.** *In a bipartite graph, the maximum*



A snake A snake avoiding traps

Figure 1.6: A solution for rats is not always a solution for snakes.

number of pairwise vertex-disjoint edges is equal to the minimum number of vertices that meet all edges.

This theorem can be restated as follows: the packing and covering numbers of $\{K_2\}$ are equal in bipartite graphs. For some class $\mathcal{H}$ of graphs, it may happen that these numbers are not equal, but that the covering number of $\mathcal{H}$ is bounded from above by a function of its packing number. In this case, $\mathcal{H}$ is said to have the *Erdős–Pósa property*. This name originates from the next result.

Theorem 1.5 (Erdős–Pósa Theorem, 1965 [EP65])**.** *There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that for every graph G and every positive integer k, either G has k pairwise vertex-disjoint cycles, or there is a set of $f(k)$ vertices, the removal of which yields an acyclic graph.*

In other words, this theorem states that the class of cycles have the Erdős–Pósa property. However, there are classes of graphs that do not have the Erdős–Pósa property. One of them is the class of cycles of odd length, as stated below.

Theorem 1.6 ([DL88, Ree99])**.** *There is a family $\{G_i\}_{i \in \mathbb{N}}$ such that for every $i \in \mathbb{N}$, the packing number of odd cycles in G_i is one and the covering number is at least i.*

An illustration of a graph in this family is provided by Figure 1.7. It has been proved that this graph does not contain two vertex-disjoint odd cycles. However one can easily check that removing any three vertices keeps some odd cycle intact.

Erdős–Pósa type results are interesting as they link together invariants related to apparently orthogonal problems. Let us also briefly mention that they have been used in algorithm design (see e.g. [FLMS12a, Corollary 2] for a simple proof using the Erdős–Pósa Theorem) and in bioinformatics [ARS16, ADG04, Ara08]. Therefore, considerable effort has been put in identifying classes that have the Erdős–Pósa property, i.e. answering question (1). It appears that in several situations, a proof of (1) can be obtained by considering two cases:

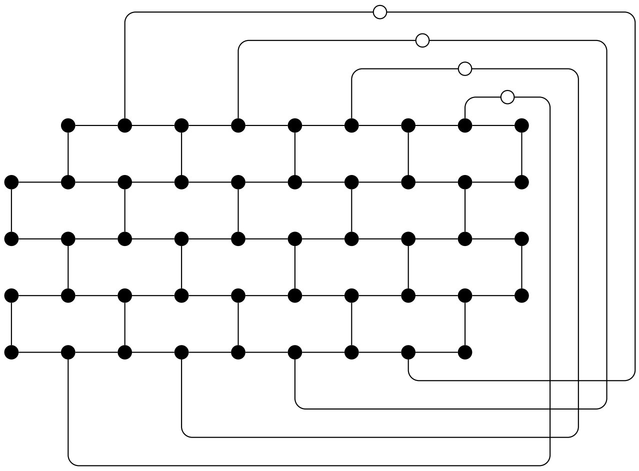


Figure 1.7: An Escher wall of height 4.

1. the considered graph has a large packing number; or

2. some structural invariant of the graph (most often the treewidth) is bounded.

From the first case we can directly conclude, whereas the bound provided by the second case usually yields structural information on the graph that indicates how to construct a small cover. However, to apply this proof scheme, we first need to ensure that at least one of these two cases is true for the considered graphs. Such a *dichotomy* is provided by exclusion theorems of the type of Turán's Theorem: in the case where a substructure is absent from the graph, they provide a bound on some parameter.

Question (2) can be asked in the setting of graphs as well. However, it appears that for several graph classes, computing packing an covering numbers is an NP-hard problem [GJ79]. The two natural options to attack them are thus parameterized algorithms and approximation (see e.g. [FLMS12a, FLM+16]).

In this thesis we continue the long line of research on the Erdős–Pósa property by answering (1) for several classes of graphs. We also consider the problems of (2), for which we design an approximation algorithm on certain classes of graphs. These algorithms rely on exclusion theorems, but also on the fact that the considered classes have the Erdős–Pósa property.

Overview of the results and organization of the thesis

Most of the definitions and notions used in this thesis are defined on Chapter 2.

Well-quasi-orderings. Chapter 3 is devoted to questions related to well-quasi-orderings. In this chapter, we present general tools to prove well-quasi-ordering results (Section 3.2). We then use these techniques together with antichains introduced in Section 3.4 to show three results on well-quasi-ordering:

1. a characterization of the classes of simple graphs defined by one forbidden induced minor that are well-quasi-ordered with respect to this relation, in Section 3.5;
2. a similar characterization for the relation of contraction, in Section 3.6; and
3. a characterization of the closed classes of multigraphs that are well-quasi-ordered by the multigraph contraction relation.

These results complement the similar characterizations known for most of the usual orderings of graphs that are not well-quasi-orders in general. They are obtained using decomposition theorems for graphs excluding a substructure.

Exclusion theorems. Chapter 4 is concerned with exclusion theorems that upper-bounds a parameter of the graphs excluding a substructure. We obtain:

1. bounds on a girth-like parameter of graphs excluding large complete graphs as a minor, in Section 4.1;
2. bounds on the degree of graphs that exclude a large collection of multiedges as a minor, in Section 4.2;
3. (low) bounds on the treewidth of graphs excluding various planar graphs, in Section 4.3 as a minor; and
4. bounds on the tree-cut width of graphs excluding a planar subcubic graph as an immersion, in Section 4.4.

Most of these results are used in the following chapters.

The Erdős–Pósa property. Chapter 5 deals with connections between invariants of packing and covering. It is split into three parts:

1. general techniques for proving Erdős–Pósa type results, either from tree-like decompositions of graphs, or using invariants as the girth, in Section 5.2;
2. an application of these techniques to several classes of graphs in Section 5.3, most often using exclusion results proved in Chapter 4;
3. a summary of positive and negative results on the Erdős–Pósa property, in Section 5.4.

Algorithmic applications. We conclude this thesis by presenting in Chapter 6 an approximation algorithm for packing and covering numbers of certain classes of graphs, relying on exclusion theorems proved Section 4.1 and on the properties of the considered classes. The proof is using a notion of equivalence of graphs with respect to the considered problem which allows us to reduce the graph when some structure appears. Using a result of Section 4.1, we can then either reduce further the graph, or use the arguments presented in Section 5.2 to conclude.

Chapter 2

Definitions

2.1 Preliminaries

In this thesis, logarithms are binary. If $\mathcal{S}$ is a set of sets, then $\pmb{\cup}\mathcal{S} = \bigcup_{S \in \mathcal{S}} S$. For every integers i and j, the notation $[\![i, j]\!]$ stands for the interval of integers $\{i, \ldots, j\}$. For every positive integer k, we denote by $\mathbb{N}_{\geq k}$ the set $\mathbb{N} \setminus [\![0, k-1]\!]$. We denote by $\mathcal{P}(S)$ the power set of a set S and by $\mathcal{P}^{<\omega}(S)$ the set of all its finite subsets. Given a function $\phi : A \to B$ and a subset $C \subseteq A$, we define $\phi(C) = \{\phi(x) \mid x \in C\}$. Let $\mathbf{t} = (x_1, \ldots, x_l) \in \mathbb{N}^l$ and $\chi, \psi : \mathbb{N} \to \mathbb{N}$. We say that $\chi(n) = O_{\mathbf{t}}(\psi(n))$ if there exists a computable function $\phi : \mathbb{N}^l \to \mathbb{N}$ such that $\chi(n) = O(\phi(\mathbf{t}) \cdot \psi(n))$. We set $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$.

By $\mathrm{polylog}(t)$ we denote some function that is a polynomial in the logarithm of t. More formally, we write $f(t) = O(\mathrm{polylog}\, t)$ if there are constants t_0, A, α such that $\forall t > t_0,\ f(t) \leq A \log^{\alpha}(t)$.

If $\mathcal{H}$ is a set of set, we define $\pmb{\cup}\mathcal{H} = \bigcup_{H \in \mathcal{H}} H$.

2.2 Orders

In this section we introduce basic definitions that are related to order theory.

Sequences. A *sequence* of elements of a set A is an ordered countable collection of elements of A. Unless otherwise stated, sequences are finite. The sequence of elements $s_1, \ldots, s_k \in A$ in this order is denoted by $\langle s_1, \ldots, s_k \rangle$. We use the notation $A^\star$ for the class of all finite sequences over A (including the empty sequence). The length of a finite sequence $s \in A^\star$ is denoted by $|s|$.

Ordered sets. A *quasi-order* (qoset for short) is a pair $(A, \preceq)$ where A is a set and $\preceq$ is a binary relation on A that is reflexive and transitive.

Two elements x and y of a qoset $(A, \preceq)$ are said to be *non-comparable* if none of $x \preceq y$ and $y \preceq x$ holds. In a qoset, an *antichain* is a sequence of pairwise non-comparable elements. In a sequence $\langle x_i \rangle_{i \in I \subseteq \mathbb{N}}$ of a qoset $(A, \preceq)$, a pair (x_i, x_j), $i, j \in I$ is a *good pair* if $x_i \preceq x_j$ and $i < j$. A qoset $(A, \preceq)$ is a *well-quasi-order* (*wqo* for short), and A is said to be *well-quasi-ordered* by $\preceq$, if every infinite sequence has a good pair, or equivalently, if

$(A, \preceq)$ has neither an infinite decreasing sequence, nor an infinite antichain. An infinite sequence containing no good pair is called an *bad sequence*.

Union and product. If $(A, \preceq_A)$ and $(B, \preceq_B)$ are two qosets, then

- their union $(A \cup B, \preceq_A \cup \preceq_B)$ is the qoset defined as follows: for every $x, y \in A \cup B$, we have $x \preceq_A \cup \preceq_B y$ if

$$(x, y \in A \text{ and } x \preceq_A y), \text{ or } (x, y \in B \text{ and } x \preceq_B y);$$

- their Cartesian product $(A \times B, \preceq_A \times \preceq_B)$ is the qoset defined by:

$$\forall (a, b), (a', b') \in A \times B,\ (a, b) \preceq_A \times \preceq_B (a', b') \text{ if } a \preceq_A a' \text{ and } b \preceq_B b'.$$

Ordering sequences. For any qoset $(A, \preceq)$, we define the relation $\preceq^\star$ on $A^\star$ as follows: for every $r = \langle r_1, \ldots, r_p\rangle$ and $s = \langle s_1, \ldots, s_q\rangle$ of $A^\star$, we have $r \preceq^\star s$ if there is a increasing function $\varphi\colon [\![1, p]\!] \to [\![1, q]\!]$ such that for every $i \in [\![1, p]\!]$ we have $r_i \preceq s_{\varphi(i)}$. Observe that $=^\star$ is then the subsequence relation. This order relation is extended to the class $\mathcal{P}^{<\omega}(A)$ of finite subsets of A as follows, generalizing the subset relation: for every $B, C \in \mathcal{P}^{<\omega}(A)$, we write $B \preceq^{\mathcal{P}} C$ if there is an injection $\varphi\colon B \to C$ such that $\forall x \in B,\ x \preceq \varphi(x)$. Observe that $=^{\mathcal{P}}$ is the subset relation.

In order to stress that domain and codomain of a function are qosets, we sometimes use, in order to denote a function φ from a qoset $(A, \preceq_A)$ to a qoset $(B, \preceq_B)$, the following notation: $\varphi\colon (A, \preceq_A) \to (B, \preceq_B)$.

Monotonicity. A function $\varphi\colon (A, \preceq_A) \to (B, \preceq_B)$ is said to be *monotone* if it satisfies the following property:

$$\forall x, y \in A,\ x \preceq_A y \Rightarrow f(x) \preceq_B f(y).$$

Informally, a monotone function preserves the order. A function $\varphi\colon (A, \preceq_A) \to (B, \preceq_B)$ is a *qoset epimorphism* (*epi* for short) if it is surjective and monotone.

Closed sets. Let $(A, \preceq)$ be a qoset. A subset $B \subseteq A$ is said to be *downward $\preceq$-closed* (sometimes shortened as *$\preceq$-closed*) if for every $x, y \in A$ such that $x \preceq y$ and $y \in B$ we have $x \in B$. Symmetrically, we say that B is *upward $\preceq$-closed* if for every $x, y \in A$ such that $x \preceq y$ and $x \in B$ we have $y \in B$.

Incl and Excl. Let $(A, \preceq)$ be a qoset. For every $x \in A$, we define

$$\begin{aligned} \mathrm{Excl}_{\preceq}(x) &= \{y \in A,\ x \not\preceq y\} \quad \text{and} \\ \mathrm{Incl}_{\preceq}(x) &= \{y \in A,\ y \preceq x\}. \end{aligned}$$

Informally, $\mathrm{Excl}_{\preceq}(x)$ is the class of elements *excluding* x (with respect to $\preceq$), and $\mathrm{Incl}_{\preceq}(x)$ the class of elements *included* in x. We extend these definitions to any subset $B \subseteq A$ as follows:

$$\mathrm{Excl}_{\preceq}(B) = \{y \in A,\ \forall x \in B,\ x \not\preceq y\} \quad \text{and}$$
$$\mathrm{Incl}_{\preceq}(B) = \{y \in A,\ \exists x \in B,\ y \preceq x\}.$$

The set $\mathrm{Incl}_{\preceq}(B)$ contains all the elements of B and all the elements that are smaller than some element of B. It is sometimes referred to as the *closure* of B with respect to the relation $\preceq$. Observe both $\mathrm{Incl}_{\preceq}(B)$ and $\mathrm{Excl}_{\preceq}(B)$ are downward $\preceq$-closed sets. A subset $B \subseteq A$ is said to be *uniquely defined* if $B = \mathrm{Excl}_{\preceq}(x)$ for some $x \in A$. A part of Chapter 3 is devoted to the study of well-quasi-ordered subsets of qosets on graphs that are uniquely defined.

Canonical antichains. An antichain A of a qoset $(S, \preceq)$ is said to be *canonical* if it is such that every contraction-closed subclass J of S has an infinite antichain iff $J \cap A$ is infinite. If $\mathrm{Incl}(A)$ has no infinite antichains, then A is a *fundamental antichain*. Note that canonical antichains can be used to characterize the $\preceq$-closed subsets of $(S, \preceq)$ and also to describe the variety of its antichains. Canonical antichains have been introduced by Ding in [Din09] and then studied for several graph orderings. We will meet them again in Chapter 3.

2.3 Graphs

2.3.1 Basics

A graph G is a pair (V, E), where V is a set referred to as the *set of vertices* of G and E is a multiset, the underlying set of which is a subset of $\binom{V}{2}$, that we call the *set of edges* of G (even if it is a multiset). In this thesis, we do not consider graphs with loops, i.e. edges connecting a vertex to itself. This definition corresponds to what is sometimes called a “loopless multigraph”. The *order* of a G is its number of vertices, that we denote by $n(G)$, whereas we use $m(G)$ for its number of edges (counting multiplicities). We denote by $\mathrm{mult}_G(\{u, v\})$ the function that gives the multiplicity in G of a given edge (which might be zero if $\{u, v\}$ does not belong to E). We drop the subscript when it is clear from the context. To refer to a particular edge of a multiedge $\{u, v\}$, we use the subscript $\{u, v\}_i$, where $i \in [\![1, \mathrm{mult}_G(\{u, v\})]\!]$.

We also deal with *simple graphs*, which are graphs where every edge has multiplicity one. We use the notations $V(G)$ and $E(G)$ for the vertex and edge sets of a graph, respectively. The *underlying simple graph* of a graph G is the graph G' such that $V(G') = V(G)$ and $E(G')$ is the underlying set of $E(G)$. We sometimes use the notation $|G| = |V(G)|$ and $\|G\| = |E(G)|$ (counting multiplicities). All the graphs we consider in this thesis are finite, i.e. both $V(G)$ and the values taken by mult_G are finite. For H and G graphs, we write $H + G$ for the disjoint union of H and G. Also, for every $k \in \mathbb{N}$, $k \cdot G$ is the disjoint union of k copies of G. The Cartesian product $H \times G$ of H and G

is the graph on vertex set $V(H) \times V(G)$ and where, given two vertices $(u,v),(u',v') \in V(H) \times V(G)$, either $u = u'$ and $\mathrm{mult}_{H\times G}(\{(u,v),(u',v')\}) = \mathrm{mult}_G(\{v,v'\})$, or $v = v'$ and $\mathrm{mult}_{H\times G}(\{(u,v),(u',v')\}) = \mathrm{mult}_H(\{u,u'\})$.

For every $X \subseteq V(G)$, the graph *induced* by X, that we write $G[X]$, is the subgraph with vertex set X and edge set $\{e \in E(G),\ e \subseteq X\}$. We denote by $G \setminus X$ the subgraph of G induced by $V(G) \setminus X$. If $X \subseteq E(G)$, then $G \setminus X$ is the subgraph $(V(G), E(G) \setminus X)$. The complement of a simple graph G, denoted $\overline{G}$ is the graph obtained by replacing every edge by a non-edge and vice-versa in G. Given a graph class $\mathcal{C}$ and a graph G, we call *$\mathcal{C}$-subgraph* of G any subgraph of G that is isomorphic to some graph in $\mathcal{C}$. If the class of $\mathcal{C}$-subgraphs of G is empty, then we say that G is *$\mathcal{C}$-free*.

Neighbors and degree. Two vertices $u,v \in V(G)$ are said to be *adjacent* if $\mathrm{mult}(\{u,v\}) \geq 1$. An edge $e \in E(G)$ is *incident* to a vertex $v \in V(G)$ if $v \in e$. Two edges are *incident* if they share some endpoint. The *neighborhood* of a vertex $v \in V(G)$, denoted $N_G(v)$, is the set of all vertices of G that are adjacent to v. For every subset $S \subseteq V(G)$, we set $N_G(S) = \bigcup_{v\in S} N_G(v) \setminus S$ (all vertices of $V(G) \setminus S$ that have a neighbor in S). We extent the definition of mult_G as follows, for every subsets $X, Y \subseteq V(G)$:

$$\mathrm{mult}_G(X,Y) = \sum_{(x,y)\in X\times Y} \mathrm{mult}_G(\{x,y\}).$$

If $\mathrm{mult}_G(X,Y) \geq 1$, we say that X and Y are *adjacent* in G.

The *degree* $\deg(v)$ of a vertex $v \in V(G)$ is the cardinality of $N_G(v)$, i.e. the number of vertices of G that are adjacent to v. On the other hand, the multidegree $\mathrm{mdeg}_G(v)$ of v is defined as the cardinality of $\mathrm{mult}_G(\{v\}, N_G(v))$, i.e. the number of edges incident with v. Observe that these values are equal in simple graphs. The minimum degree over all vertices of a graph G is denoted by $\delta(G)$, and the maximum degree by $\Delta(G)$.

Distances. For a given graph G and two vertices $u,v \in V(G)$, $\mathbf{dist}_G(u,v)$ denotes *the distance between u and v*, which is the number of edges on a shortest path between u and v, and $\mathbf{diam}(G)$ denotes $\max\{\mathbf{dist}_G(u,v) \mid u,v \in V(G)\}$. For a set $S \subseteq V(G)$ and a vertex $w \in V$, $\mathbf{dist}_G(S,w)$ denotes $\min\{\mathbf{dist}_G(v,w) \mid v \in S\}$. Also, for a given vertex $u \in V(G)$, $\mathbf{ecc}_G(u)$ denotes the *eccentricity* of vertex v, that is, $\max\{\mathbf{dist}_G(u,v) \mid v \in V(G)\}$. The *girth* of a graph G, denoted $\mathbf{girth}(G)$, is the length of a shortest cycle in G if G is not a forest, and ∞ otherwise.

Connectivity and separations. A graph G is *connected* if, for every $x,y \in V(G)$, there is a path starting in x and ending in y. A pair (A,B) of subsets of $V(G)$ is a called a *separation of order k in G* if $k = |A \cap B|$, none of A,B is a subset of the other, and there is no edge of G between $A \setminus B$ and $B \setminus A$.

For every $k \in \mathbb{N}$, $k \geq 2$, we say that G is *k-connected* if there is no separation of order less than k in G. A *connected component* of a graph is a maximal connected subgraph. We denote by $\mathbf{cc}(G)$ the number of connected component of a graph G. We say that a subgraph is a *2-connected component* is a maximal 2-connected subgraph. A 2-connected component is also called a *block*. A *cut* $C = (X,Y)$ of G is a partition of

$V(G)$ into two subsets X and Y. The *cut-set* of C is $E(X, Y)$ and the *width* of the cut is $|E(X, Y)|$.

Trees, paths, and cycles. A *tree* is a connected acyclic simple graph. Given a tree T we denote by $L(T)$ the set of its *leaves*, that are the vertices of degree one.

For every two vertices $u, v \in V(T)$, there is exactly one path in T between u and v, that we denote by uTv. Also, given that uTv has at least 2 vertices, we denote by $\mathring{u}Tv$ (resp. $uT\mathring{v}$) the path uTv with the vertex u (resp. v) deleted. Let C be a cycle on which we fixed some orientation. Then, there is exactly one path following this orientation between any two vertices $u, v \in V(C)$. Similarly, we denote this path by uCv and we define $\mathring{u}Cv$ and $uC\mathring{v}$ as we did for the tree. Two paths in a graph are *internally disjoint* if they do not share any internal vertex. In a rooted tree T with root r, the *least common ancestor* of two vertices u and v, written $\mathbf{lca}\,T(u, v)$, is the first common vertex of the paths uTr and vTr.

2.3.2 Special graphs and graph classes

In this section we define specific graphs that will appear all along this thesis. Let n be a positive integer. We denote by:

- K_n the complete graph on n vertices;
- P_n the path on n vertices;
- C_n the cycle on n vertices (when $n \geq 2$);
- θ_n the graph obtained by connecting two vertices with an edge of multiplicity n (see Subsection 2.3.2 for a picture of θ_5).

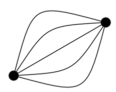


Figure 2.1: The graph θ_5.

We call *prism* the Cartesian product of K_3 and K_2. A *cograph* is a graph not containing the path on four vertices as induced subgraph. A *linear forest* is a disjoint union of paths. A graph is *subcubic* its maximum degree is upper-bounded by 3.

Grids and Walls. Let k and r be positive integers where $k, r \geq 2$. The $(k \times r)$-*grid* $\Gamma_{k,r}$ is the Cartesian product of two paths of lengths $k-1$ and $r-1$ respectively. We denote by Γ_k the $(k \times k)$-grid. The k-*wall* W_k is the graph obtained from a $((k+1) \times (2 \cdot k+2))$-grid with vertices (x, y), $x \in \{1, \ldots, k+1\}$, $y \in \{1, \ldots, 2k+2\}$, after the removal of the "vertical" edges $\{(x, y), (x, y+1)\}$ for odd $x + y$, and then the removal of all vertices of degree 1. The graphs Γ_4 and W_4 are depicted on Figure 2.2.

Let W_k be a wall. We denote by $P_j^{(v)}$ the shortest path connecting vertices $(1, 2j)$ and $(k+1, 2j)$, $j \in [k]$ and call these paths the *vertical paths of* W_k, with the assumption that $P_j^{(v)}$ contains only vertices (x, y) with $y = 2j, 2j-1$. Note that these paths are vertex-disjoint. Similarly, for every $i \in [k+1]$ we denote by $P_i^{(h)}$ the shortest path connecting vertices $(i, 1)$ and $(i, 2k+2)$ (or $(i, 2k+1)$ if $(i, 2k+2)$ has been removed) and call these paths the *horizontal paths of* W_k.

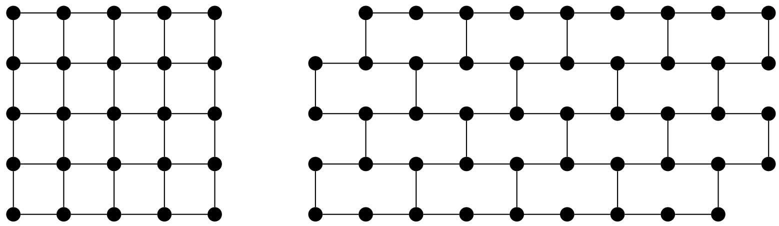


Figure 2.2: The (4×4)-grid (left) and the 4-wall (right).

Wheels. For every positive integer n, a *n-wheel*, also called *wheel of order n*, is a simple graph obtained by connecting a (new) vertex to n distinct vertices of an induced cycle C. This cycle is said to be the *cycle* of the n-wheel, whereas the new vertex is its *center*. A *double wheel of order n* is obtained from a cycle of order by adding two non-adjacent, each connected to n vertices of the cycle.

Yurts. For every integer $n > 0$, the *yurt graph* of order n the graph Y_n of the form

$$\begin{aligned} V(Y_n) &= \{x_1, \ldots, x_n, y_1, \ldots, y_n, o\} \\ E(Y_n) &= \big\{\{x_i, x_{i+1}\}_{i \in [\![1, n-1]\!]}\big\} \\ &\quad \cup \big\{\{y_i, y_{i+1}\}_{i \in [\![1, n-1]\!]}\big\} \\ &\quad \cup \{\{x_i, y_i\}\}_{i \in [\![1, n]\!]} \\ &\quad \cup \{\{y_i, o\}\}_{i \in [\![1, n]\!]} \end{aligned}$$

(see Figure 2.3 for an example).

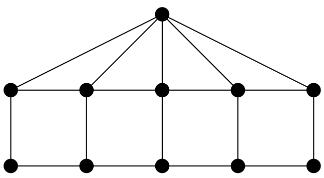


Figure 2.3: The yurt graph of order 5.

2.3.3 Annotating graphs

Labeled graphs. As detailed in Chapter 3, labeling the vertices of graphs may simplify well-quasi-ordering proofs. Let us introduce some definitions related to graph labeling. For this, we consider a qoset $(S, \preceq)$. A $(S, \preceq)$-*labeled graph* is a pair (G, λ) such that G is a graph, and $\lambda\colon V(G) \to \mathcal{P}^{<\omega}(S)$ is a function referred as the *labeling of the graph*. For the sake of simplicity, we will refer to the labeled graph of a pair (G, λ) by G and to λ by λ_G. If $\mathcal{G}$ is a class of (unlabeled) graphs, $\mathrm{lab}_{(S,\preceq)}(\mathcal{G})$ denotes the class of $(S, \preceq)$-labeled graphs of $\mathcal{G}$. The notion of labeled graph generalizes the one of graph as any (unlabeled) graph can be seen as a $\emptyset$-labeled graph.

This definition of a labeled graph differs from the commonly used one where vertices are assigned single elements of S instead of finite subsets. Whereas the usual definition can be easily used to prove well-quasi-ordering results with the induced subgraph ordering, the definition we introduced is suited for proofs on graph orderings involving contractions. The extensions of these orderings to labelled graphs will be given in Section 2.4.

Rooted graphs. In several proofs, the labels that will be assigned to vertices of graphs will be sets of rooted graphs. This will be used in order to *encode* a class of connected graphs as labeled 2-connected graphs (cf. Lemma 3.2 and Lemma 3.3). Roots will also be used on trees to defined a partial order on the vertices.

A *rooted graph* is a couple (G, r) where G is a graph and r is a vertex of G, called *root* of G. For the sake of simplicity, we sometimes denote by G the rooted graph (G, r) and refer to its root by $\mathrm{root}(G)$. If $\mathcal{H}$ is a class of graphs, we define its rooted closure, denoted $\mathcal{H}_r$ as the class of rooted graphs $\mathcal{H}_r = \{(G, v) : G \in \mathcal{H}, v \in G\}$.

Rooted trees. Let (T, s) be a rooted tree. Given a vertex $x \in V(T)$, the *descendants* of x in (T, s), denoted by $\mathrm{desc}_{(T,s)}(x)$, is the set containing each vertex w such that the unique path from w to s in T contains x. Given a rooted tree (T, s) and a vertex $x \in V(G)$, the *height* of x in (T, s) is the maximum distance between x and a vertex in $\mathrm{desc}_{(T,s)}(x)$. The *height* of (T, s) is the height of s in (T, s). The *children* of a vertex $x \in V(T)$ are the vertices in $\mathrm{desc}_{(T,s)}(x)$ that are adjacent to x. A *leaf* of (T, s) is a vertex of T without children. The *parent* of a vertex $x \in V(T) \setminus \{s\}$, denoted by $\mathbf{p}(x)$, is the unique vertex of T that has x as a child.

2.4 Graph operations and orderings

Most of the common order relations on graphs can be defined in two equivalent ways: either in terms of *graph operations*, or by using *models*. Let us have a closer look at them.

2.4.1 Local operations

Let G be a graph. We here describe the effects of the following local operations when applied to G: the vertex deletion, the edge deletion, the vertex dissolution, the vertex identification and the lift.

Deleting a vertex v (resp. an edge e) yields the graph $G \setminus \{v\}$ (resp. $G \setminus \{e\}$). For every $\{u, v\} \in E(G)$, the *contraction* of the edge $\{u, v\}$ adds a new vertex w, sets mult($\{w, w'\}$) = mult($\{u, w'\}$) + mult($\{v, w'\}$) for every $w \in N(u) \cup N(v)$ and then deletes u and v (see Figure 2.4). If G is a simple graph, we do not keep the multiple edges that might be created during this process (i.e. we set their multiplicity to one). In the case where G is labeled, we label the new vertex w in the obtained graph with $\lambda_G(u) \cup \lambda_G(v)$.

Figure 2.4: The contraction of the edge e creates a double edge.

If u is a vertex of degree two, the *dissolution* of u is the contraction of one edge incident with u. On the other hand, a *subdivision* of the edge $\{u, v\}$ adds a new vertex adjacent to u and v and decreases the multiplicity of the edge $\{u, v\}$ by one (i.e., removes this edge in the case were G is a simple graph). The vertex added during this process is called a *subdivision vertex*. These two operations are depicted on Figure 2.5.

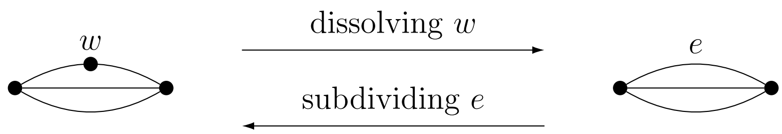


Figure 2.5: Dissolution and subdivision as complementary operations.

The *identification* of two vertices u and v adds the edge $\{u, v\}$ if it was not already existing, and contracts it. If G is $(\Sigma, \preceq)$-labeled (for some qoset $(\Sigma, \preceq)$), a *label contraction* is the operation of relabeling a vertex $v \in V(G)$ with a label l such that $l \preceq^{\mathcal{P}} \lambda_G(v)$. This operation will be used when dealing with well-quasi-orders in Chapter 3. The *lift* of two incident edges $\{u, v\}$ and $\{v, w\}$ decreases by one the multiplicities of these edges and increases by one the multiplicity of $\{u, w\}$ (or create the edge if it was not existing).

The *closure* of a class $\mathcal{G}$ by a given operation is the class obtained from graphs of $\mathcal{G}$ by a finite application of this operation.

2.4.2 Containment models

In this section we define models, which are functions witnessing the presence of a substructure in a graph. They come in different flavours, depending on the type of substructure considered.

Containment models. Let G and H be two graphs and let us consider a function $\mu\colon V(H) \to \mathcal{P}^{<\omega}(V(G))$ and the following properties:

(M1): for every two distinct $u, v \in V(H)$, the sets $\mu(u)$ and $\mu(v)$ are vertex-disjoint;

(M2): for every $u \in V(H)$, the subgraph of G induced by $\mu(u)$ is connected;

(M3): $\lambda_H(u) \preceq^\star \bigcup_{v\in\mu(u)} \lambda_G(v)$;

(M4): $\text{root}(G) \in \mu(\text{root}(G))$;

(M5): $\forall u, v \in V(H),\ \text{mult}_G(\mu(u), \mu(v)) \geq \text{mult}_H(\{u, v\})$;

(M6): $\forall u, v \in V(H),\ \text{mult}_G(\mu(u), \mu(v)) = \text{mult}_H(\{u, v\})$;

(M7): $\bigcup_{v\in V(H)} \mu(v) = V(G)$;

(M8): $\forall v \in V(H),\ |\mu(v)| = 1$.

...-model	(M1)	(M2)	(M3)	(M4)	(M5)	(M6)	(M7)	(M8)
minor	✓	✓	(✓)	(✓)	✓			
induced minor	✓	✓	(✓)	(✓)	✓	✓		
contraction	✓	✓	(✓)	(✓)	✓	✓	✓	
subgraph	✓	✓	(✓)	(✓)	✓		✓	✓
induced subgraph	✓	✓	(✓)	(✓)	✓	✓	✓	✓

Table 2.1: Requirements for containment models

If μ satisfies (M1) up to (M5), then we call it an *H-minor model* in G, or a *minor model of H*. A minor model that also satisfies (M6) is an *induced minor model*, and an induced minor model where (M7) holds is a *contraction* model. A minor model that additionally satisfies (M7) and (M8) is a *subgraph model*; it is an *induced subgraph model* if it satisfies (M1) up to (M8). These definitions are summarized by Table 2.1. Items (M3) and (M4) are required only when H and G are labeled or rooted, respectively. An example of a minor model is given in Figure 2.6. When (M3) holds, μ is said to be *label-preserving*.

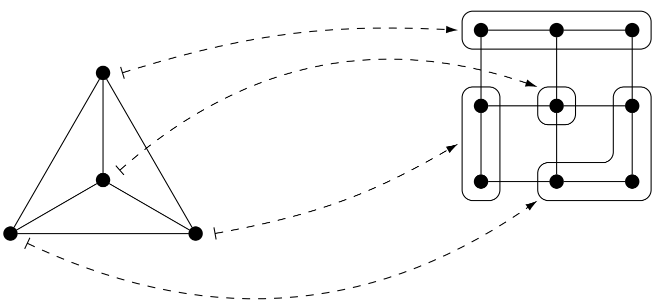


Figure 2.6: A K_4-minor model (dashed arrows) in the 3×3 grid, that is also a topological minor model and an induced minor model.

Models for immersions and topological minors. In order to deal with the orderings of immersion and topological minor, to be defined in Subsection 2.4.3, we need a different kind of model. An *H-immersion model* is a pair of functions (ϕ, ψ) satisfying the following properties:

1. $\phi\colon V(H) \to V(G)$ is an injection;

2. ψ sends $\{u, v\}_i$ to a path of G between $\phi(u)$ and $\phi(v)$, for every $\{u, v\} \in E(H)$ and every $i \in [\![1, \mathrm{mult}_H(\{u, v\})]\!]$, in a way such that distinct edges are sent to edge-disjoint paths.

Every vertex in the image of ϕ is called a *branch vertex* and every path in $\psi(E(H))$ a *certifying path*. If it also holds that no branch vertex is an internal vertex of any certifying path, then the function (ϕ, ψ) is an *H-strong-immersion model*. If moreover, the paths in the image of ψ are internally disjoint, then (ϕ, ψ) is an H-topological minor model.

With these definitions the following observation is straightforward.

Observation 2.1. Let H and G be graphs. If (ϕ, ψ) is an H-topological-minor model in G then (ϕ, ψ) is also an H-strong-immersion model in G.

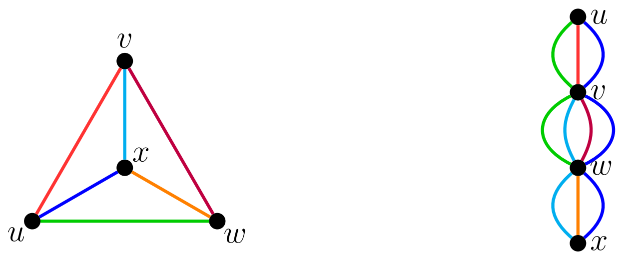


Figure 2.7: A K_4-immersion model in a graph that has no K_4-minor model: vertices are sent on vertices with the same name, and edges are sent to paths of the same color.

The next section links together models and local operations in the definition of graph orderings.

2.4.3 Graph orderings

Local operations and models can be used to express that a graph is *contained* as a substructure of an other one. The graph ordering where this notion of *containment* is the most obvious is perhaps the subgraph relation which can be defined as follows: H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

The orderings we consider here can be defined in two equivalent ways: either using models, of using local operations. Let us start with the definition involving models.

- H is a *subgraph* of G, what we note $H \leq_{\mathrm{sg}} G$, if there is an H-subgraph model in G;

- H is an *induced subgraph* of G, what we note $H \leq_{\mathrm{isg}} G$, if there is an H-induced subgraph model in G;

- H is a *contraction* of G, what we note $H \leq_{\mathrm{c}} G$, if there is an H-contraction model in G;

- H is a *minor* of G, what we note $H \leq_{\mathrm{m}} G$, if there is an H-minor model in G;

- H is an *induced minor* of G, what we note $H \leq_{\mathrm{im}} G$, if there is an H-induced minor model in G;

- H is a *topological minor* of G, what we note $H \leq_{\mathrm{tm}} G$, if there is an H-topological minor model in G;

- H is an *immersion* of G, what we note $H \leq_{\mathrm{imm}} G$, if there is an H-immersion model in G;

- H is a strong *immersion* of G, what we note $H \leq_{\mathrm{sim}} G$, if there is an H-strong immersion model in G;

We add μ as superscript of the aforementioned order symbols when we want to specify that μ is a model witnessing the relation (like in $H \leq_{\mathrm{c}}{}^{\mu} G$). Observe that each of the aforementioned relation defines a qoset on the class of graphs. For every $\preceq \in \{\leq_{\mathrm{sg}}, \leq_{\mathrm{isg}}, \leq_{\mathrm{c}}, \leq_{\mathrm{m}}, \leq_{\mathrm{im}}\}$, the graph G is said to be *H-$\preceq$-free*, or to *exclude* H with respect to $\preceq$, if $H \preceq G$ does not hold.

We now give the equivalent definition of these containment relations in terms of local operations. The list below also contains the definitions of dissolution, induced immersions, and induced topological minors, that we did not defined in terms of models as we will not consider them (see e.g. [KO04b] for a definition).

- $H \leq_{\mathrm{isg}} G$ iff there is a sequence of vertex deletions transforming G into H;

- $H \leq_{\mathrm{sg}} G$ iff there is a sequence of vertex deletions and edge deletions transforming G into H;

- $H \leq_{\mathrm{c}} G$ iff there is a sequence of edge contractions transforming G into H;

- $H \leq_{\mathrm{im}} G$ iff there is a sequence of vertex deletions and edge contractions transforming G into H;

- $H \leq_{\mathrm{m}} G$ iff there is a sequence of vertex deletions, edge deletions and edge contractions transforming G into H;

- $H \leq_{\mathrm{imm}} G$ iff there is a sequence of vertex deletions, edge deletions and lifts transforming G into H;

- $H \leq_{\mathrm{tm}} G$ iff there is a sequence of vertex deletions, edge deletions, and vertex dissolutions transforming G into H;

- H is an induced topological minor of G iff there is a sequence of vertex deletions and vertex dissolutions transforming G into H;
- H is an induced immersion of G iff there is a sequence of vertex deletions and lifts transforming G into H.

Note that the aforementioned sequences of operations are allowed to be empty. Table 2.2 summarizes these definitions. In this table, VD stands for "vertex deletion", ED for "edge deletion", C^- for vertex dissolution, C for edge contraction and L for lift. The relations between the different containment relations are depicted on Figure 2.8 where the same abbreviations are used.

Relation	VD	ED	C^-	C	L
isomorphism					
spanning subgraph		✓			
induced subgraph	✓				
subgraph	✓	✓			
dissolution			✓		
contraction			✓	✓	
induced minor	✓		✓	✓	
induced topological minor	✓		✓		
topological minor	✓	✓	✓		
minor	✓	✓	✓	✓	
induced immersion	✓				✓
immersion	✓	✓			✓

Table 2.2: Containment relations defined in terms of local operations.

The aforementioned orderings allow us to define classes of graphs.

Classes defined by graph orderings. Let H be a graph. An *major of H* (*H-major* for short) is a subgraph-minimal graph that contains H as a minor. We denote the class of H-majors by $\mathcal{M}(H)$.

A *subdivision of H* (*H-subdivision* for short) is a graph obtained from H by subdividing edges. Observe that a graph G has a subgraph isomorphic to subdivision of H iff it contains H as a topological minor. We denote by $\mathcal{T}(H)$ the class of all subdivisions of H.

Also, an *immersion expansion of H* (*H-immersion expansion* for short) is a subgraph-minimal graph that contain H as an immersion. Again, a graph G has a subgraph isomorphic to an immersion expansion of H iff it contains H as an immersion. We write $\mathcal{I}(H)$ for the class of immersions expansions of H. These definitions are extended to classes: $\mathcal{M}(\mathcal{H})$ is the class of all subgraph-minimal graphs that contain some member of $\mathcal{H}$ as a minor (and similarly for $\mathcal{T}(\mathcal{H})$ and $\mathcal{I}(\mathcal{H})$).

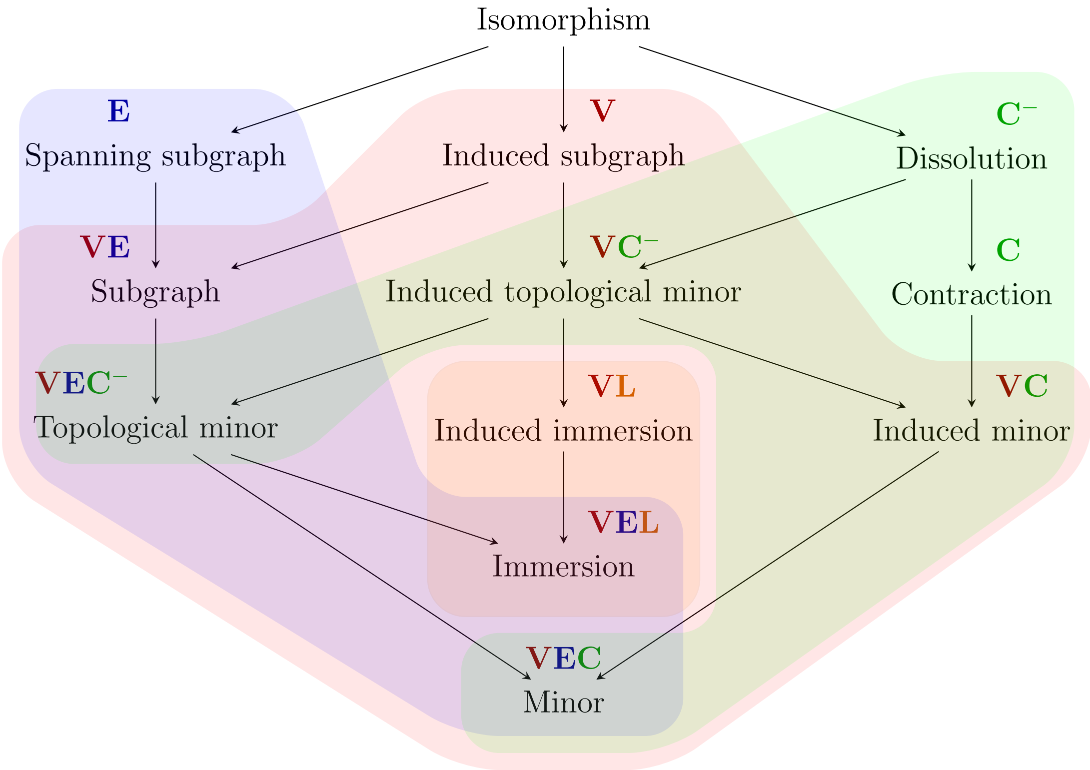


Figure 2.8: Connections between common orderings of graphs.

2.5 Tree-like decompositions and width parameters

A considerable amount of the recent algorithmic advances relies on tree-like decompositions, that are a way to decompose the graph into subsets organized in a tree-like fashion. These decompositions give rise to graph parameters usually called *width parameters*. Among the existing decompositions, tree-decompositions are certainly those that received the most considerable attention.

Definition 2.1. A *tree decomposition* of a graph G is a pair $(T, \mathcal{X})$ where T is a tree and $\mathcal{X}$ a family $\{X_t\}_{t \in V(T)}$ of subsets of $V(G)$ (called *bags*) indexed by elements of $V(T)$ and such that the following holds

(i) $\bigcup_{t \in V(T)} X_t = V(G)$;

(ii) for every edge e of G there is an element of $\mathcal{X}$ containing both ends of e;

(iii) for every $v \in V(G)$, the subgraph of T induced by $\{t \in V(T),\ v \in X_t\}$ is connected.

The *width* of a tree decomposition T is defined as equal to $\max_{t \in V(T)} |X_t| - 1$. The *treewidth* of G, written $\mathbf{tw}(G)$, is the minimum width of any of its tree decompositions.

The *torso* of a bag X_t of a tree decomposition $(T, \{X_t\}_{t \in V(T)})$ is the underlying simple graph of the graph obtained from $G[X_t]$ by adding all the edges $\{x, y\}$ such that $x, y \in X_t \cap X_{t'}$ for some neighbor t' of t in T.

Treewidth has been extensively used in Algorithmics and Combinatorics. The Graph Minor series of Robertson and Seymour provides several results and tools related to this parameter. In particular, Robertson and Seymour proved that every graph of *big enough* treewidth contains as a minor a *big* grid. This result will be discussed in Chapter 4. Also, it appears that several algorithmic problems, that are hard in general, become tractable on graphs of bounded treewidth. In this direction, Courcelle proved [Cou90] that a large family of problems (those that can be expressed in monadic second-order logic[1]) can be solved in linear time on graphs of bounded treewidth.

In order to deal with tree decompositions, we sometimes consider *nice tree decomposition*, which are defined as follows.

Definition 2.2 ([Klo94a])**.** A triple $(T, r, \{X_t\}_{t\in V(T)})$ is said to be a *nice* tree decomposition of a graph G if $(T, \{X_t\}_{t\in V(T)})$ is a tree-decomposition where the following holds:

1. every vertex of T has degree at most 3;

2. (T, r) is a rooted tree and the bag of the root r is empty ($X_r = \emptyset$);

3. every vertex t of T is

   - either a *base node*, i.e. a leaf of T whose bag is empty ($X_t = \emptyset$) and different from the root;
   - or an *introduce node*, i.e. a vertex with only one child t' such that $X_{t'} = X_t \cup \{u\}$ for some $u \in V(G)$;
   - or a *forget node*, i.e. a vertex with only one child t' such that $X_t = X_{t'} \cup \{u\}$ for some $u \in V(G)$;
   - or a *join node*, i.e. a vertex with two child t_1 and t_2 such that $X_t = X_{t_1} = X_{t_2}$.

It is known that every graph G has an nice tree decomposition with width $\mathbf{tw}(G)$ and at most $4n$ nodes [Klo94a]. A more restrictive notion is the one of path decomposition, defined as follows.

Definition 2.3 ([Klo94b])**.** A tree decomposition $(T, \{X_t\}_{t\in V(T)})$ is a *path decomposition* if T is a path, and the *pathwidth* of G, that we write $\mathbf{pw}(G)$, is the minimum width of a path decomposition of G.

A path decomposition $\big(p_1p_2\dots p_k, \{X_{p_i}\}_{i\in[\![1,k]\!]}\big)$ of a graph G is said to be *nice* if $|X_{p_1}| = 1$ and

$$\forall i \in [\![2,k]\!]\,,\ \big|(X_{p_i} \setminus X_{p_{i-1}}) \cup (X_{p_{i-1}} \setminus X_{p_i})\big| = 1$$

It is known [BT04] that every graph have an optimal path decomposition which is nice and that in such decomposition, every node X_i is either an *introduce node* (i.e. either $i = 1$ or $\big|X_{p_i} \setminus X_{p_{i-1}}\big| = 1$) or a *forget node* (i.e. $\big|X_{p_{i-1}} \setminus X_{p_i}\big| = 1$).

[1]We do not define this logic here as we will not use it.

Remark 2.1. For every graph G on n vertices, there is an optimal path decomposition with n introduce nodes and n forget nodes (one of each for each vertex of G), thus of length $2n$.

Tree partitions have proven useful when dealing with problems related to vertices. For instance, deciding if the deletion of a certain number of vertices yields a given property is such a problem. However, these decompositions are not well-suited for the study of problems related to edges. Therefore, several attempts have been done to design an *edge-analogues* of treewidth. Let us present two of them.

Definition 2.4. A *tree partition* of a graph G is a pair $\mathcal{D} = (T, \mathcal{X})$ where T is a tree and $\mathcal{X} = \{X_t\}_{t \in V(T)}$ is a partition of $V(G)$ (the elements of which are called *bags*) such that either $n(T) = 1$ or for every $\{x, y\} \in E(G)$, there exists an edge $\{t, t'\} \in E(T)$ where $\{x, y\} \subseteq X_t \cup X_{t'}$. In other words, the endpoints of every edge of G either belong to the same bag or they belong to bags of adjacent vertices of T. Given an edge $f = \{t, t'\} \in E(T)$, we define E_f as the set of edges with one endpoint in X_t and the other in $X_{t'}$. The *width* of $\mathcal{D}$ is defined as $\max\{\max_{t \in V(T)} |X_t|, \max_{f \in E(T)} |E_f|\}$. The *tree partition width* of G is the minimum width over all tree partitions of G and is denoted by $\mathbf{tpw}(G)$.

A *rooted tree partition* of a graph G is a triple $\mathcal{D} = (T, s, \mathcal{X})$ where (T, s) is a rooted tree and $(T, \mathcal{X})$ is a tree partition of G.

Tree partitions have been introduced in [See85] (see also [Hal91] and tree partition width has been defined for simple graphs in [DO96]. The above definition is an extension of the original definition to the setting of (multi)graphs. Tree partitions will in particular be used in Section 4.1 and Subsection 5.2.1. However, a drawback of tree partition width compared to treewidth is that a graph with large tree partition width does not necessarily contains a large substructure, whereas a graph of large treewidth is known to contain a large grid, as mentioned above (see Theorem 4.1 for a formal statement). For instance, the graph obtained by setting to k the multiplicity of every edge of a path on k edges (for some positive integer k) has tree partition width k, however its subgraphs are quite *poor* compared to those of a graph containing a large grid as a minor. Ding described in [DO96, Theorem 1.2] the subgraphs that one can expect in graphs of large tree partition width, as we mention in Subsection 5.3.3.

A decomposition that avoids this pitfall has been recently introduced by Wollan in [Wol15]. A *near-partition* of a set S is a collection of pairwise disjoint subsets $S_1, \ldots, S_k \subseteq S$ (for some $k \in \mathbb{N}$) such that $\bigcup_{i=1}^{k} S_i = S$. Observe that this definition allows a set of the family to be empty.

Definition 2.5 ([Wol15])**.** A *tree-cut decomposition* of a graph G is a pair $\mathcal{D} = (T, \mathcal{X})$ where T is a tree and $\mathcal{X} = \{X_t\}_{t \in V(T)}$ is a near-partition of $V(G)$.

A *rooted tree-cut decomposition* of a graph G is a triple $\mathcal{D} = (T, s, \mathcal{X})$ where (T, s) is a rooted tree and $(T, \mathcal{X})$ is a tree-cut decomposition of G.

As proved by Wollan in [Wol15], every graph of *large* tree-cut width contains a *large* wall as an immersion (cf. Theorem 4.16).

Let $\mathcal{D} = (T, s, \mathcal{X})$ be a rooted tree-cut decomposition of G. We set

$$G_t = G\left[\bigcup_{t \in \mathrm{desc}_{(T,s)}(t)} X_t\right].$$

The adhesion of a vertex t of T, that we write $\mathrm{adh}_{\mathcal{D}}(t)$, is the number of edges with exactly one endpoint in G_t. A vertex $t \in V(T)$ is *thin* if $\mathrm{adh}_{\mathcal{D}}(t) \leq 2$. We also say that $\mathcal{D}$ is *nice* if for every thin vertex $t \in V(T)$ we have

$$N(V(G_t)) \cap \bigcup_{b \text{ is a sibling of } t} V(G_b) = \emptyset.$$

In other words, there is no edge from a vertex of G_t to a vertex of G_b, for any sibling b of t, whenever t is thin. This notion can be seen as an analogue of nice tree-decompositions.

In [GKS15], Ganian et al. proved that every graph has a tree-cut decomposition of minimal width, that, moreover, is nice.

Proposition 2.1 ([GKS15]). *Every rooted tree-cut decomposition can be transformed into a nice one without increasing the width.*

2.6 Packing and covering

The main topic of Chapter 5 is the Erdős–Pósa property, which is a connection between invariants of packing and covering. Let us introduce the definitions on this topic.

Most of the definitions we give here have two variants: one is related to vertices and the other one is related to edges. In order to make the definitions more concise, we use symbols v and e in order to distinguish the vertex and the edge variants we of the properties/parameters that we are dealing with. For instance, if $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, and G is a graph, we set $A_{\mathsf{x}}(G) = V(G)$ if $\mathsf{x} = \mathsf{v}$ and $A_{\mathsf{x}} = E(G)$ if $\mathsf{x} = \mathsf{e}$. Similarly, x-*disjoint* stands for *vertex-disjoint* when $\mathsf{x} = \mathsf{v}$ and for *edge-disjoint* when $\mathsf{x} = \mathsf{e}$.

Packing and covering. Let $\mathcal{H}$ be a family of graphs and let $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$. An x-$\mathcal{H}$-*cover* of G is a set $C \subseteq A_{\mathsf{x}}(G)$ such that $G \setminus C$ does not contain any subgraph isomorphic to a member of $\mathcal{H}$. An x-$\mathcal{H}$-*packing* in G is a collection of x-disjoint subgraphs of G, each being isomorphic to some graph of $\mathcal{H}$.

We denote by $\mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(G)$ the maximum size of an x-$\mathcal{H}$-packing, which we call *packing number* (with respect to $\mathcal{H}$) and by $\mathsf{x}\text{-}\mathsf{cover}_{\mathcal{H}}(G)$ the minimum size of an x-$\mathcal{H}$-covering in G, also referred to as *covering number* (also defined relatively to $\mathcal{H}$).

There is an easy inequality between these two parameters.

Remark 2.2. For every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, for every graph class $\mathcal{H}$, and every graph G, the following holds:

$$\mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(G) \leq \mathsf{x}\text{-}\mathsf{cover}_{\mathcal{H}}(G).$$

Indeed, any x-$\mathcal{H}$-cover must contain at least one vertex (if $\mathsf{x} = \mathsf{v}$) or edge (if $\mathsf{x} = \mathsf{e}$) of each element of an x-$\mathcal{H}$-packing of maximum size.

The Erdős–Pósa property is concerned with the other direction, that is, bounding the covering number in terms of the packing number.

The Erdős–Pósa property.

Definition 2.6. Let $\mathcal{G}$ and $\mathcal{H}$ be two graph classes and let $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$. We say that $\mathcal{H}$ has the x-*Erdős–Pósa property* for $\mathcal{G}$ if there is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that the following holds:

$$\forall G \in \mathcal{G},\ \mathsf{x\text{-}cover}_{\mathcal{H}}(G) \leq f(\mathsf{x\text{-}pack}_{\mathcal{H}}(G)).$$

Any function satisfying the above inequality is called a *gap* of the x-Erdős–Pósa property of $\mathcal{H}$ for $\mathcal{G}$. When a class of graphs has the x-Erdős–Pósa-property for the class of finite graphs, we simply say that it has the x-*Erdős–Pósa-property*. We usually refer to $\mathcal{G}$ as the *host* graph class and by $\mathcal{H}$ as the *guest* graph class.

2.7 Approximation algorithms

A way to approach problems that are NP-hard is via approximation algorithms. The main idea is to trade accuracy for speed. As approximation algorithms appear very locally in this thesis, we only give the few required definitions.

In an *optimization problem*, one is typically given an object x from some set S and asked to compute a predefined function $f\colon S \to \mathbb{N}$ on x. Most of the classic problems on graph, such as Independent Set or Vertex Cover are optimization problems. Informally, an *approximation algorithm* for an optimization problem is an algorithm that computes an approximate solution for this problem. In order to give a guarantee on the performance of the algorithm, we use the following definition: for a constant $c \in \mathbb{R}$, $c > 1$, we say that an algorithm is a c-*approximation* for a given problem if on every input it returns a solution s such that the following holds:

$$1 \leq \frac{s}{\mathsf{OPT}} \leq c,$$

where OPT denotes the optimal solution for this input. In other words, s differs from the optimal solution by at most a constant factor. Such an algorithm is also called a *constant factor approximation*. If the solution output by the algorithm satisfies the following inequality:

$$1 \leq \frac{s}{\mathsf{OPT}} \leq f(\mathsf{OPT}),$$

for some function $f\colon \mathbb{N} \to \mathbb{R}$, we say that it is an $f(\mathsf{OPT})$-*approximation* for the considered problem. This relaxation allows to give a performance guarantee for problems where a constant factor approximation is unlikely. Last, if on every input the algorithm returns a solution s such that the following holds:

$$1 \leq \frac{s}{\mathsf{OPT}} \leq f(n),$$

for some function $f\colon \mathbb{N} \to \mathbb{R}$, we call it an $f(n)$-*approximation*.

Chapter 3

Graphs and well-quasi-ordering

In this chapter, we consider quasi-orders defined by the usual order relations on graphs. After presenting some results and techniques on well-quasi-ordering, we focus on the contraction and induced minor relations. For these relations, we identify all well-quasi-ordered subclasses, among those that are uniquely defined. This chapter contains material that previously appeared in the following articles:

- *Induced minors and well-quasi-ordering*, co-authored with Jarosław Błasiok, Marcin Kamiński, and Théophile Trunck, and presented at the Eighth European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2015, Bergen, Norway, 2015 [BKRT15];

- *Multigraphs without large bonds are well-quasi-ordered by contraction*, co-authored with Marcin Kamiński and Théophile Trunck, 2014, submitted [KRT14]; and

- *Well-quasi-ordering H-contraction-free graphs*, co-authored with Marcin Kamiński and Théophile Trunck, 2015, submitted [KRT16].

3.1 Preliminaries on well-quasi-orders

A *well-quasi-order* is a quasi-order which contains neither an infinite decreasing sequence nor an infinite collection of pairwise incomparable elements. This strengthening of the concept of well-order has been introduced in the 50's. Since then, a whole theory of well-quasi-orders has been developed and has led to surprising results and unsuspected developments.

Before going more into detail, let us present one of the most remarkable aspects of well-quasi-orders. Several objects of interest in graph theory are classes of graphs that are *downward closed*. That is, every graph that is smaller (wrt. a given order) than a graph in the class also belongs to the class. For instance, forests, planar graphs, more generally, graphs of genus at most g (for every fixed $g \in \mathbb{N}$), and graphs of treewidth at most k (for every fixed $k \in \mathbb{N}$) are downward closed wrt. the minor relation. On the opposite, the class of 3-colorable graphs is not downward closed wrt. the minor relation, as witnessed by the graph of Figure 3.1, which is 3-colorable but contains the non-3-colorable graph K_4 as a minor.

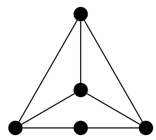


Figure 3.1: A 3-colorable graph that contains K_4 as minor.

For every class C of a quasi-order $(S, \preceq)$, we can ask the question : "can we *easily* characterize the elements of S that belong to C?". An approach on this question when C is downward closed is to consider the minimal elements of the complementary of C, when they exist. These elements are called *obstructions*, because every element x of S belongs to C if and only if there is no obstruction y such that $y \preceq x$. That way, obstructions provide a precise characterization of C. Here the well-quasi-orders come into play: if $(S, \preceq)$ is a well-quasi-order, then C has finitely many obstructions. These obstructions exist because $S \setminus C$ does not contain infinite decreasing sequences and they are finitely many as the set of obstructions is an antichain. In other words, there are elements $x_0, \ldots, x_c$ (for some $c \in \mathbb{N}$ depending on C) such that the following holds:

$$\forall x \in S,\ x \in C \iff \forall i \in [\![0, c]\!]\,,\ x_i \not\preceq x.$$

This property, that Pouzet refers to as a *finite basis property* in [Pou85], has the following algorithmic implication. If, for every $i \in [\![0, c]\!]$, there is a algorithm that can decide if $x_i \preceq y$ in a time that is polynomial in the size of y for every $y \in S$, then the membership of C can be decided in a time that is polynomial in the input size. As an illustration, let us consider the minor relation. Robertson and Seymour proved in their Graph Minors series [RS04] that this relation well-quasi-orders all (finite) graphs and they also gave an algorithm that, for every fixed graph H, decides if H is a minor of the input graph G in a time that is polynomial in the size of G [RS95]. As mentioned above, a consequence of these results is that, for every graph class that is downwards closed wrt. minors, there is an algorithm that answers in polynomial time whether the input graph belongs to the considered class. It is worth noting that these results are purely existential: they do not provide any way to construct the algorithm. Even the upper-bound on the complexity of these algorithms is existential, as it depends on the number of obstructions for the considered class.

Recall that a qoset (for quasi-ordered set) is a pair of a set and a relation that is reflexive and transitive. Also, for every qoset $(A, \preceq)$, we denote by $(A^\star, \preceq^\star)$ the qoset of finite sequences over A ordered by the subsequence relation (cf. Section 2.2). A cornerstone of the theory of well-quasi-orders is the following theorem, usually called "Higman's Lemma".

Theorem 3.1 (Higman's Lemma, [Hig52]). *If $(A, \preceq)$ is a wqo, then so is $(A^\star, \preceq^\star)$.*

It is noteworthy that Higman's Lemma appears in the proofs of several later results on well-quasi-ordering. In fact the main result of [Hig52] is more general than Theorem 3.1 but we will only use this form in this chapter. Given a qoset $(S, \preceq)$, the main question we are here concerned with is the following.

Question 3.1. Is $(S, \preceq)$ a well-quasi-order?

Let us now present some known results answering this question for various choices of S and $\preceq$ among graph classes and graph orderings. Unless otherwise specified, the word *graph* will in this chapter refer to simple graphs, i.e. graphs without loops or multiple edges. One of the most important well-quasi-ordering result on graphs is certainly the aforementioned Graph Minor Theorem by Robertson and Seymour.

Theorem 3.2 (Graph Minor Theorem, [RS04])**.** *The class of all graphs is well-quasi-ordered by the minor relation.*

The Graph Minor Theorem extends the earlier Kruskal Tree Theorem, which was concerned with trees.

Theorem 3.3 (Kruskal Tree Theorem, [Kru60], see also [NW63])**.** *The class of all trees is well-quasi-ordered by the topological minor relation.*

Robertson and Seymour later proved that the same also holds for the immersion relation.

Theorem 3.4 ([RS10])**.** *The class of all graphs is well-quasi-ordered by the immersion relation.*

However, among the usual containment relations on graphs (listed in Section 2.4), the minor relation and the immersion relation are the only relations known to be well-quasi-order in general (i.e. on the class of all graphs). Regarding other relations, either the problem is still open, or infinite antichains are known. Table 3.1 summarizes the status of Question 3.1 when S is the class of all graphs.

well-quasi-order	open	not a well-quasi-order
minors	strong immersions	subgraphs
immersions	induced immersions	induced subgraphs
		topological minors
		induced topological minors
		contractions
		induced immersions

Table 3.1: Status of Question 3.1 for the common orderings of graphs.

For all the qosets that are not well-quasi-orders, a natural question is to identify the well-quasi-ordered subclasses.

Question 3.2. For which $C \subseteq S$ is $(C, \preceq)$ a well-quasi-order?

Much attention has been brought to this question in the last decades. For instance, Fellows et al. proved in [FHR09] that graphs with bounded feedback-vertex-set are well-quasi-ordered by topological minors. Another result is that of Oum [Oum08] who proved that graphs of bounded rank-width are wqo by vertex-minors[1]. Other papers considering

[1]Vertex-minors is an other ordering of graphs, which we will not consider in this thesis.

this question include [Tho85,Dam90,Din92,Din98,Pet02,Din09,DRT10,HL14,AL14]. In order to approach Question 3.2, which is quite general, one may consider a fixed family of subclasses only. One way to do this is to look at graph classes defined by forbidden substructures.

Question 3.3. For which $x \in S$ is the set $\{y \in S,\ x \not\preceq y\}$ well-quasi-ordered by $\preceq$?

This line of research has been fruitful. Let us present some results in this direction, starting from the following theorem by Damaschke, that initiated the quest for answers to Question 3.3.

Theorem 3.5 ([Dam90, Theorem 4]). *Let H be a graph. The class of H-induced subgraph-free graphs is well-quasi-ordered by the induced subgraph relation iff $H \leq_{\mathrm{isg}} P_4$.*

Notice that that the above result is a full characterization of subclasses that are well-quasi-ordered by the induced subgraph relation, among those that are uniquely defined. Such a dichotomy has also been proved for other orderings. Damaschke has been followed by Ding who proved two years later the following non-induced counterpart of Theorem 3.5.

Theorem 3.6 ([Din92]). *Let H be a graph. The class of H-subgraph-free graphs is well-quasi-ordered by the subgraph relation iff $H \leq_{\mathrm{sg}} P_n$ for some $n \in \mathbb{N}$.*

More recently, Liu obtained a result of the same type for topological minors.

Theorem 3.7 ([HL14]). *Let H be a graph. The class of H-topological minor-free graphs is well-quasi-ordered by topological minors iff $H \leq_{\mathrm{tm}} R_n$ for some $n \in \mathbb{N}$, where R_n is the multigraph obtained by doubling every edge of a path on n edges.*

About the induced minor relation, a well-quasi-ordered class defined by excluding an induced minor has been identified by Thomas.

Theorem 3.8 ([Tho85]). *The class of K_4-(induced) minor-free graphs is well-quasi-ordered by the induced minor relation.*

Question 3.3 has also been considered in [KL11a, KL11b, Che11]. In this part, we provide an answer to this question for the following graph containment relations:

- the induced minor relation $\leq_{\mathrm{im}}$ on simple graphs, in Section 3.5;
- the contraction relation $\leq_{\mathrm{c}}$ on simple graphs, in Section 3.6;
- the contraction relation $\leq_{\mathrm{mc}}$ on multigraphs, in Section 3.7.

As mentioned above, none of these relations is a well-quasi order in general. Infinite antichains witnessing this fact will be given in Subsection 3.4.1. Let us introduce three graphs that play a major role in this chapter. They are depicted on Figure 3.2. The first one, $\hat{K}_4$, is obtained by adding a vertex of degree two to K_4. The second one, called the *gem*, is constructed by adding a dominating vertex to P_4. The last one is the *diamond* (or D_2), that one can obtain by removing an edge in K_4. In the forthcoming section we prove the following theorems.

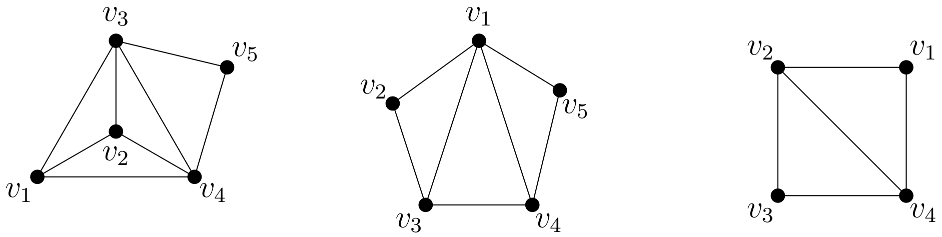


Figure 3.2: The graph $\hat{K}_4$, the gem, and the diamond (from left to right).

Theorem 3.9. *Let H be a graph. The class of H-induced minor-free simple graphs is wqo by $\leq_{\text{im}}$ iff $H \leq_{\text{im}} \hat{K}_4$ or $H \leq_{\text{im}} \text{Gem}$.*

Theorem 3.10. *Let H be a graph. The class of H-contraction-free simple graphs is wqo by $\leq_{\text{c}}$ iff $H \leq_{\text{c}} D_2$.*

Theorem 3.11. *Let $\mathcal{H}$ be a class of graphs. The class of $\mathcal{H}$-contraction-free multigraphs is wqo by $\leq_{\text{mc}}$ iff there is an integer n such that $\theta_{n'} \in \mathcal{H}$ and $\overline{K_{n'}} \in \mathcal{H}$ for every integer $n' \geq n$.*

Theorem 3.9 and Theorem 3.10 are the induced minor and contraction counterparts of Theorem 3.5. Besides, Theorem 3.9 completes Theorem 3.8 into a full dichotomy. We shall stress that Theorem 3.11 is more general, in the sense that it characterizes *all* the well-quasi-ordered classes defined by forbidding contractions, whereas Theorem 3.9 and Theorem 3.10 only deal with uniquely defined such classes. It could be interesting to extend Theorem 3.11 to induced minors and contractions in simple graphs, that is, to answer the following question.

Question 3.4. For which classes $\mathcal{H}$ is the class of $\mathcal{H}$-induced minor-free (resp. $\mathcal{H}$-contraction-free) graphs well-quasi-ordered by $\leq_{\text{im}}$ (resp. $\leq_{\text{c}}$)?

A first step towards this goal could be to look at classes defined by excluding two graphs.

Question 3.5. For which graphs H_1, H_2 is the class of $\{H_1, H_2\}$-induced minor-free (resp. $\{H_1, H_2\}$-contraction-free) graphs well-quasi-ordered by $\leq_{\text{im}}$ (resp. $\leq_{\text{c}}$)?

This question already received partial answers on the induced subgraph ordering, see e.g. [KL11a].

Another line of research when considering qosets that are not wqos is to consider infinite antichains. In his study of infinite antichains for the (induced) subgraph relation, Ding [Din09] introduced the concepts of *canonical antichain* and *fundamental antichain.* Let us restate the definition we gave in Chapter 2. An antichain $\mathcal{A}$ of a partial order $(\mathcal{S}, \preceq)$ is said to be *canonical* if it is such that every contraction-closed subclass $\mathcal{J}$ of $\mathcal{S}$ has an infinite antichain iff $\mathcal{J} \cap \mathcal{A}$ is infinite. If $\text{Incl}(\mathcal{A}) = \{x \in \mathcal{S},\ x \prec a \text{ for some } a \in \mathcal{A}\}$ has no infinite antichains, then $\mathcal{A}$ is a *fundamental antichain.* Note that canonical antichains can be used to characterize the $\preceq$-closed subclasses of a partial order $(\mathcal{S}, \preceq)$ and also to describe the variety of its antichains. This raises the following question.

Question 3.6. If $(S, \preceq)$ is not a wqo, does it have a canonical antichain?

As shown by the results below, the question of the presence or absence of a canonical antichain has been studied for several containment relations and graph classes.

Theorem 3.12 ([Din09]). *Under the subgraph relation, the class of finite graphs has a canonical antichain.*

Theorem 3.13 ([Din09]). *Under the induced subgraph relation, the class of finite graphs does not have a canonical antichain.*

Theorem 3.14 ([Din09]). *Under the induced subgraph relation, both the class of interval graphs and the class of bipartite permutation graphs have a canonical antichain.*

We give an answer to Question 3.6 for the contraction relation with the following result.

Theorem 3.15. *Under the contraction relation, the class of all graphs does not have a canonical antichain.*

The proof of Theorem 3.15 relies on the tools introduced in [Din09] that can be used to prove that a quasi-order does not have a canonical antichain. We also obtained a complete characterization of the canonical antichains of the multigraph contraction ordering.

Theorem 3.16. *Every antichain $\mathcal{A}$ of $\leq_{\mathrm{mc}}$ is canonical iff each of the following sets are finite:*

$$\mathcal{A}_\theta \setminus \mathcal{A}; \qquad \mathcal{A}_{\overline{K}} \setminus \mathcal{A};\ \textit{and} \qquad \mathcal{A} \setminus \{\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}\}.$$

In other words, an antichain $\mathcal{A}$ is canonical iff it contains all but finitely many graphs from $\mathcal{A}_\theta$, all but finitely many graphs from $\mathcal{A}_{\overline{K}}$, and a finite number of graphs that do not belong to $\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}$. Two straightforward consequences are that $\leq_{\mathrm{mc}}$ has infinite canonical antichains and the following result.

Corollary 3.1. *Every canonical antichain of $\leq_{\mathrm{mc}}$ is fundamental.*

3.2 Raising well-quasi-orders

As we explain in the forthcoming Section 3.3, a way to show that a given qoset $(A, \preceq_A)$ is a wqo is to build a wqo $(B, \preceq_B)$ from smaller qosets that are known to be well-quasi-ordered and then find a correspondence between the elements of A and B that *preserves* the well-quasi-orderness. In this section, we present different constructions to obtain well-quasi-orders. In particular, we recall that being well-quasi-ordered is preserved by several operations including union, Cartesian product, and application of a monotone function.

Let us start with an easy remark.

Remark 3.1. If $B \subseteq A$ and $(A, \preceq)$ is a wqo, then so is $(B, \preceq)$.

Indeed, any infinite antichain of $(B, \preceq)$ is an antichain of $(A, \preceq)$.

Union and product. Recall that, in the union of two qosets $(A, \preceq_A)$ and $(B, \preceq_B)$, two elements are comparable only if both belong to the same qoset, in which case they are ordered as in this qoset. In the Cartesian product of $(A, \preceq_A)$ and $(B, \preceq_B)$, the elements are pairs over $A \times B$ and are compared coordinate-wise. The formal definitions can be found in Section 2.2.

Remark 3.2 (union of wqos). If $(A, \preceq_A)$ and $(B, \preceq_B)$, are two wqos, then so is $(A \cup B, \preceq_A \cup \preceq_B)$.

In fact, for every infinite antichain S of $(A \cup B, \preceq_A \cup \preceq_B)$, there is an infinite subsequence of S whose all elements belong to one of A and B (otherwise S is finite). But then one of $(A, \preceq_A)$ and $(B, \preceq_B)$ has an infinite antichain, a contradiction to our initial assumption. Similarly, every finite union of wqos is a wqo.

Lemma 3.1 (Higman [Hig52])**.** *If $(A, \preceq_A)$ and $(B, \preceq_B)$ are wqo, then so is $(A \times B, \preceq_A \times \preceq_B)$.*

Sequences and subsets. Higman's Lemma (Theorem 3.1), which we mentioned in the previous section, states that finite sequences over a well-quasi-order $(A, \preceq)$ are well-quasi-ordered by $\preceq^\star$. It has the following easy corollary.

Corollary 3.2. *If $(A, \preceq)$ is a wqo, then so is $(\mathcal{P}^{<\omega}(A), \preceq^{\mathcal{P}})$.*

Surjective images. Recall that an epi is a function from one qoset to an other that is surjective and monotone. Epis have the following interesting property, which we will extensively use to show that some qosets are well-quasi-ordered.

Remark 3.3 (epi from a wqo). If the domain of an epi is wqo, then its codomain is also wqo.

Indeed, for any pair x, y of elements of the domain of an epi φ such that $\varphi(x)$ and $f(y)$ are incomparable, x and y are incomparable as well (by monotonicity of φ). Therefore, and as φ is surjective, any infinite antichain of the codomain of φ can be translated into an infinite antichain of its domain.

In order to apply Remark 3.3 one needs to show that the considered function is monotone. The aim of the next remark is to make this task easier when dealing with multivariate functions.

Remark 3.4 (componentwise monotonicity). Let $(A, \preceq_A)$, $(B, \preceq_B)$, and $(C, \preceq_C)$ be three qosets and let $f\colon (A \times B, \preceq_A \times \preceq_B) \to (C, \preceq_C)$ be a function. If we have both

$$\forall a \in A,\ \forall b, b' \in B,\ b \preceq_B b' \Rightarrow f(a, b) \preceq_C f(a, b') \tag{3.1}$$

$$\text{and } \forall a, a' \in A,\ \forall b \in B,\ a \preceq_A a' \Rightarrow f(a, b) \preceq_C f(a', b), \tag{3.2}$$

then f is monotone.

Indeed, let $(a, b), (a', b') \in A \times B$ be such that $(a, b) \preceq_A \times \preceq_B (a', b')$. By definition of the relation $\preceq_A \times \preceq_B$, we have both $a \preceq a'$ and $b \preceq b'$. From line (3.1) we get that $f(a, b) \preceq_C f(a, b')$ and from line (3.2) that $f(a, b') \preceq_C f(a', b')$, hence $f(a, b) \preceq_C f(a', b')$ by transitivity of $\preceq_C$. Thus f is monotone. Observe that this remark can be generalized to functions with more than two arguments.

Excluding elements. We will often deal with classes defined by forbidding an element, that is, subclasses of a qoset $(S, \preceq)$ of the form $\{y \in S,\ x \not\preceq y\}$ for some $x \in S$.

Remark 3.5. Let $(S, \preceq)$ be a qoset. For every $x, x' \in S$ such that $x' \preceq x$, we have $\{y \in S,\ x' \not\preceq y\} \subseteq \{y \in S,\ x \not\preceq y\}$. As a consequence, $(\{y \in S,\ x' \not\preceq y\}, \preceq)$ is a wqo whenever $(\{y \in S,\ x \not\preceq y\}, \preceq)$ is.

Labeled graphs. As defined in Section 2.3, a graph labeled by a qoset is a graph, the vertices of which are assigned subsets of this qoset. Informally speaking, labels will be used to encode connected graphs into labeled 2-connected graphs. Given a connected graph which is not 2-connected, we can pick an arbitrary block B, delete the rest of the graph, and label each vertex v of B by the subgraph it was attached to in the original graph if v was a cutvertex, and by $\emptyset$ otherwise. That way, contracting the label of a vertex v corresponds to contracting a subgraph. This intuition is formalized by the next result.

Lemma 3.2 ([FHR09]). *Let $\mathcal{G}$ be a class of graphs. If for any wqo $(S, \preceq)$ the class of $(S, \preceq)$-labeled 2-connected graphs of $\mathcal{G}$ is wqo by $\leq_{\mathrm{im}}$, then $(\mathcal{G}, \leq_{\mathrm{im}})$ is a wqo.*

This result can also be proved for multigraph contractions.

Lemma 3.3. *Let $\mathcal{G}$ be a class of connected multigraphs. If for any wqo $(S, \preceq)$ the class of $(S, \preceq)$-labeled 2-connected graphs of $\mathcal{G}$ is wqo by $\leq_{\mathrm{mc}}$, then $(\mathcal{G}, \leq_{\mathrm{mc}})$ is a wqo.*

Proof. This proof is very similar to the proof of Lemma 3.2 given in [FHR09]. We proceed by induction. Assuming that $(\mathcal{H}, \leq_{\mathrm{mc}})$ is not a wqo, we will reach a contradiction by showing that its rooted closure $(\mathcal{H}_r, \leq_{\mathrm{mc}})$ is a wqo.

Let $\langle G_i \rangle_{i \in \mathbb{N}}$ be a bad sequence in $\mathcal{H}_r$ such that, for every $i \in \mathbb{N}$, there is no $G \leq_{\mathrm{mc}} G_i$ such that a bad sequence starts with $G_0, \dots, G_{i-1}, G$ (a so-called *minimal bad sequence*). For every $i \in \mathbb{N}$, let A_i be the block of G_i which contains $\mathrm{root}(G_i)$. Let C_i the set of cutvertices of G_i that are included in A_i. For each cutvertex $c \in C_i$, let B_c^i the connected component in $G_i \setminus (V(A_i) \setminus C_i)$, and rooted at c (i.e., we set $\mathrm{root}(B_c^i) = c$). Note that we have $B_c^i \leq_{\mathrm{mc}} G_i$.

Let us denote by $\mathcal{B}$ the family of rooted graphs $\mathcal{B} = \{B_c^i\colon c \in C_i, i \in \mathbb{N}\}$. We will show that $(\mathcal{B}, \leq_{\mathrm{mc}})$ is a wqo. Let $\langle H_j \rangle_{j \in \mathbb{N}}$ be an infinite sequence in $\mathcal{B}$ and for every $j \in \mathbb{N}$ choose an $i = \varphi(j) \in \mathbb{N}$ for which $H_j \leq_{\mathrm{mc}} G_i$. Pick a j with smallest $\varphi(j)$ and consider the sequence $G_1, \dots, G_{\varphi(j)-1}, H_j, H_{j+1}, \dots$. By minimality of $\langle G_i \rangle_{i \in \mathbb{N}}$ and by our choice of j, since $H_j \leq_{\mathrm{mc}} G_{\varphi(j)}$ and $H_j \neq G_{\varphi(j)}$, this sequence is good so contains a good pair (G, G'). Now, if G is among the first $\varphi(j) - 1$ elements, then as $\langle G_i \rangle_{i \in \mathbb{N}}$ is bad we must have $G' = H_{j'}$ for some $j' \geq j$ and so we have $G_{i'} = G \leq_{\mathrm{mc}} G' = H_{j'} \leq_{\mathrm{mc}} G_{\varphi(j')}$, a contradiction. So there is a good pair in $\langle H_i \rangle_{i \geq j}$ and hence the infinite sequence $\langle H_j \rangle_{j \in \mathbb{N}}$ has a good pair, so $(\mathcal{B}, \leq_{\mathrm{mc}})$ is a wqo.

We will now find a good pair in $\langle G_i \rangle_{i \in \mathbb{N}}$ to show a contradiction. The idea is to label the graph family $\mathcal{A} = \{A_i\}_{i \in \mathbb{N}}$ so that each cutvertex c of a graph A_i gets labeled by their corresponding connected component B_c^i, and the roots are preserved under this labeling. More precisely, for each A_i we define a labeling σ_i that assigns to every vertex $v \in V(G_i)$ a label $\{(\sigma_i^1(v), \sigma_i^2(v))\}$ defined as follows:

- $\sigma_i^1(v) = 1$ if $v = \text{root}(G_i)$ and $\sigma_i^1(v) = 0$ otherwise;
- $\sigma_i^2(v) = B_v^i$ if $v \in C_i$ and $\sigma_i^2(v)$ is the one-vertex rooted graph otherwise.

The labeling σ of $\mathcal{A}$ is then $\{\sigma_i : i \in \mathbb{N}\}$. Let us define a quasi-ordering $\preceq$ on the set of labels Σ assigned by σ. For two labels $(s_a^1, s_a^2), (s_b^1, s_b^2) \in \Sigma$ we define $(s_a^1, s_a^2) \preceq (s_b^1, s_b^2)$ iff $s_a^1 = s_b^1$ and $s_a^2 \leq_{\text{mc}} s_b^2$. Note that in this situation, s_a^2 and s_b^2 are rooted graphs, so $\leq_{\text{mc}}$ compares rooted graphs. Observe that since $(\mathcal{B}, \leq_{\text{mc}})$ is wqo, then $(\Sigma, \preceq)$ is wqo. For every $i \in \mathbb{N}$, let A_i' be the $(\Sigma, \preceq)$-labeled rooted graph (A_i, σ_i). We now consider the infinite sequence $\langle A_i' \rangle_{i \in \mathbb{N}}$. By our initial assumption, $(\text{lab}_\Sigma(\mathcal{A}), \leq_{\text{mc}})$ is wqo (as $\mathcal{A}$ consists only in 2-connected graphs), so there is a good pair (A_i', A_j') in the sequence $\langle A_i' \rangle_{i \in \mathbb{N}}$.

To complete the proof, we will show that $A_i' \leq_{\text{mc}} A_j' \Rightarrow G_i \leq_{\text{mc}} G_j$. Let μ be a contraction model of A_i' in A_j'. Then for each cutvertex $c \in C_i$, $\mu(c)$ contains a vertex $d \in C_j$ with $B_c^i \leq_{\text{mc}} B_d^j$. Let μ_c denote a root-preserving contraction model of B_c^i onto B_d^i. We construct a model model g as follows:

$$\nu\colon \begin{cases} V(G_i) & \to & \mathcal{P}(V(G_j)) \\ v & \mapsto & \mu(v) \text{ if } v \in A_i \setminus C_i \\ v & \mapsto & \mu_c(v) \text{ if } v \in B_c^i \setminus C_i \\ v & \mapsto & \mu(v) \cup \mu_v(v) \text{ if } v \in C_i \end{cases}$$

We now prove that ν is a contraction model of G_i onto G_j. First note that by definition of μ and each μ_c, we have $\nu(u) \cap \nu(v) = \emptyset$, for any pair of distinct vertices u and v in G_i, and also every vertex of G_j is in the image of some vertex of G_i (items (M1) and (M7) in the definition of a contraction model). If $u \in C_i$, then $\mu(u)$ contains a vertex $v \in C_j$ for which $B_u^i \leq_{\text{mc}} B_v^j$ and v is also contained in $\mu_v(v)$ since μ_v preserves roots. Thus, $G_j[\nu(u)]$ is connected when $u \in C_i$ (point (M2)). This is obviously true when $u \notin C_i$ again by the definitions of μ and each μ_c. Moreover, the endpoints of every edge of G_i belong either both to A_i, or both to B_c^i, so point (M6) follows from the properties of μ and each μ_c. Finally, as the labeling σ ensures that $\text{root}(G_j) \in \nu(\text{root}(G_i))$, we establish that $G_i \leq_{\text{mc}} G_j$. So $\langle G_i \rangle_{i \in \mathbb{N}}$ has a good pair (G_i, G_j), a contradiction. □

Let us now summarize the tools introduced in this section.

Lemma 3.4 (Summary of Section 3.2)**.** *If $(A, \preceq_A)$ and $(B, \preceq_B)$ are wqos, then*

(i) $(C, \preceq_A)$ is a wqo for every $C \subseteq A$ (Remark 3.1);

(ii) $(A \cup B, \preceq_A \cup \preceq_B)$ is a wqo (Remark 3.2);

(iii) $(A \times B, \preceq_A \times \preceq_B)$ is a wqo (Lemma 3.1);

(iv) $(A^\star, \preceq^\star)$ is a wqo (Theorem 3.1);

(v) $(\mathcal{P}^{<\omega}(A), \preceq^{\mathcal{P}})$ is a wqo (Corollary 3.2);

(vi) $(C, \preceq_C)$ is a wqo, for every epi $f\colon (A, \preceq_A) \to (C, \preceq_C)$ (Remark 3.3);

(vii) *$(\{y \in C,\ x' \not\preceq_c y\}, \preceq_c)$ is a wqo whenever $(\{y \in C,\ x \not\preceq_C y\}, \preceq_C)$ is, for every x', x elements of a qoset $(C, \preceq_C)$ such that $x' \preceq x$ (Remark 3.5);*

(viii) *if for any wqo $(C, \preceq_C)$ the 2-connected graphs of a graph class $\mathcal{H}$ labeled by $(C, \preceq_C)$ are wqo by $\leq_{\mathrm{im}}$, then $(\mathcal{H}, \leq_{\mathrm{im}})$ is a wqo (Lemma 3.2);*

(ix) *if for any wqo $(C, \preceq_C)$ the 2-connected multigraphs of a multigraph class $\mathcal{H}$ labeled by $(C, \preceq_C)$ are wqo by $\leq_{\mathrm{mc}}$, then $(\mathcal{H}, \leq_{\mathrm{mc}})$ is a wqo (Lemma 3.3).*

In the next section we give the main lines of the proofs of Theorem 3.9, Theorem 3.10, and Theorem 3.11.

3.3 A high-level view of the proofs

The techniques that we use in the following sections mostly follow a general scheme. The purpose of this section is to provide an informal description of its steps. The general setting is the following: we are given a qoset $(S, \preceq)$ and the goal is to show that it is a wqo.

Step 1: Define a *compact representation* of the elements of S. More formally, we choose an injective function f from S to some other set S'. The intuition is that, usually, S is a subset of a wider set and its elements only span a *restricted area* of this set. Therefore we would like to *project* these elements into a smaller set, that will be easier to well-quasi-order afterwards. This representation can be seen as an encoding, as we will sometime break a graph into parts and represent is as a tuple containing these parts. It may be deduced from structural information about the graphs of S.

Step 2: Find a quasi-order relation $\preceq'$ on S' with good properties. This relation should be chosen such that whenever $f(x) \preceq' f(y)$ holds, we have $x \preceq y$, for every $x, y \in S$. We should stress that the choice of f must be done with this step in mind.

Step 3: The third and last step is to show that $(S', \preceq')$ is a wqo. This can be done by applying standard results on well-quasi-orders, if both S' and $\preceq'$ have been chosen carefully.

Finally, we obtain that f is a epi, the domain of which is a subset of the wqo $(S', \preceq')$. According to Remark 3.1 and Remark 3.3, we get that $(S, \preceq)$ is a wqo, as desired. As an illustration of this scheme, let us prove the following easy lemma.

Lemma 3.5. *Let $\mathcal{H}$ be a class of connected graphs and let $\mathcal{G}$ the class of graphs, the connected components of which belong to $\mathcal{H}$. If $(\mathcal{H}, \leq_{\mathrm{isg}})$ is a wqo, then so is $(\mathcal{G}, \leq_{\mathrm{isg}})$.*

Proof. Let us follow the steps detailed above.

Step 1: We represent every graph G of $\mathcal{G}$ as a sequence $f(G)$ listing its connected components taken in an arbitrary order. The function $f\colon \mathcal{G} \to \mathcal{H}^\star$ thus defined is clearly injective.

Step 2: We notice that for every $G, G' \in \mathcal{G}$, if $f(G) \leq_{\mathrm{isg}}{}^{\star} f(G')$ then $G \leq_{\mathrm{isg}} G'$. Therefore the quasi-order relation $\leq_{\mathrm{isg}}{}^{\star}$ suits our needs.

Step 3: We show that the codomain of f is a wqo. For this, we apply Higman's Lemma (Theorem 3.1) to $(\mathcal{H}, \leq_{\mathrm{isg}})$, which is a wqo by our initial assumption.

As explained above, this implies that $(\mathcal{G}, \leq_{\mathrm{isg}})$ is a wqo and we are done. □

Let us end this section with a simple but crucial observation in the study of the well-quasi-orderability of classes that are defined by forbidden structures (of any kind). If none of the elements of an infinite antichain A contains some element x, then excluding x does not give a well-quasi-order. Indeed, the class obtained still contains the infinite antichain A. Let us formally restate this observation.

Observation 3.1. Let $(S, \preceq)$ be a qoset and let $A \subseteq S$ be an infinite antichain wrt. $\preceq$. Let $x \in S$. If $\{y \in S,\ x \not\preceq y\}$ is well-quasi-ordered by $\preceq$, then all but finitely many elements of A are larger than x wrt. $\preceq$.

For this reason, we devote the next section to a study of infinite antichains for the orderings considered in this chapter.

3.4 The bestiary

It is worth noting that none of the orderings we consider in this chapter admits an infinite decreasing sequence. Indeed, for all these relations, the sum of the number of vertices and edges, which is a non-negative integer, is decreasing when considering smaller graphs. Hence, every decreasing sequence of graphs yields a decreasing sequence of positive integers of the same length, which is always finite. Therefore the only obstacle for these relations to be well-quasi-orders is the presence of an infinite antichain.

As noted in Observation 3.1, the study of infinite antichains may provide helpful information when proving dichotomy theorems. In this section, we present several antichains for the orderings we consider.

3.4.1 Antichains for induced minors

In 1985, Thomas [Tho85] presented an infinite sequence of planar graphs (also mentioned later in [RS93]) and proved that it is an antichain for induced minors. He showed that this relation does not well-quasi-order planar graphs. The elements of this antichain, called *alternating double wheels*, are constructed from an even cycle by adding two nonadjacent vertices and connecting one to one color class of the cycle and connecting the other vertex to the other color class (cf. Figure 3.3 for the three first such graphs). This infinite antichain shows that $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(K_5), \leq_{\mathrm{im}})$ is not a wqo since no alternating double wheel contains K_5 as (induced) minor. As a consequence, $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H), \leq_{\mathrm{im}})$ is not a wqo as soon as H contains K_5 as induced minor.

Therefore, in the quest for all graphs H such that $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H), \leq_{\mathrm{im}})$ is wqo, we can focus the cases where H is K_5-induced minor-free.

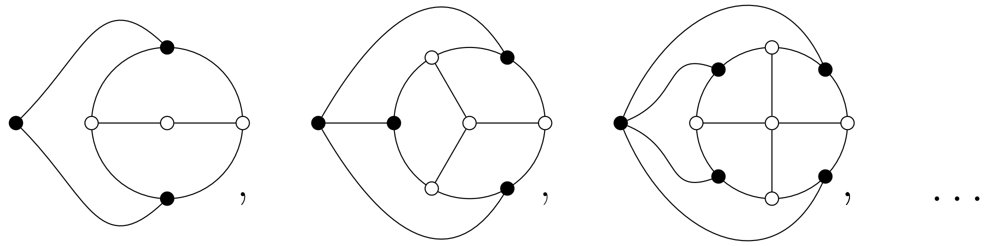


Figure 3.3: Thomas' alternating double wheels.

The infinite antichain $\mathcal{A}_M$ depicted in Figure 3.4 was introduced in [MNT88], where it is also proved that none of its members contains K_5^- as induced minor. Similarly as the above remark, it follows that if $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H), \leq_{\mathrm{im}})$ is a wqo then $K_5^- \not\leq_{\mathrm{im}} H$. Notice that graphs in this antichain have bounded maximum degree.

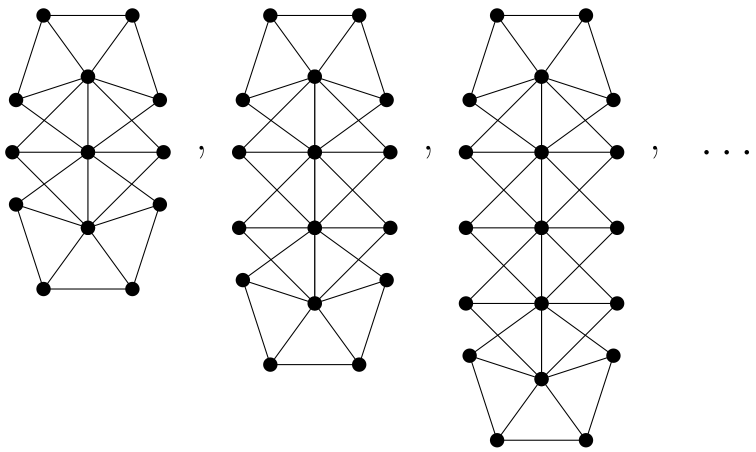


Figure 3.4: The infinite antichain $\mathcal{A}_M$

An *interval graph* is the intersection graph of segments of $\mathbb{R}$. A well-known property of interval graphs that we will use later is that they do not contains C_4 as induced minor. In order to show that interval graphs are not wqo by $\leq_{\mathrm{im}}$, Ding introduced in [Din98] an infinite sequence of graphs defined as follows. For every $n \in \mathbb{N}$, $n > 2$, let T_n be the set of closed intervals

- $[i, i]$ for i in $[\![-2n, -1]\!] \cup [\![1, 2n]\!]$;
- $[-2, 2]$, $[-4, 1]$, $[-2n+1, 2n]$, $[-2n+1, 2n-1]$;
- $[-2i+1, 2i+1]$ for i in $[\![1, n-2]\!]$;
- $[-2i, 2i-2]$ for i in $[\![3, n]\!]$.

Figure 3.5 depicts the intervals of T_6: the real axis (solid line) is folded up and an interval $[a, b]$ is represented by a dashed line between a and b.

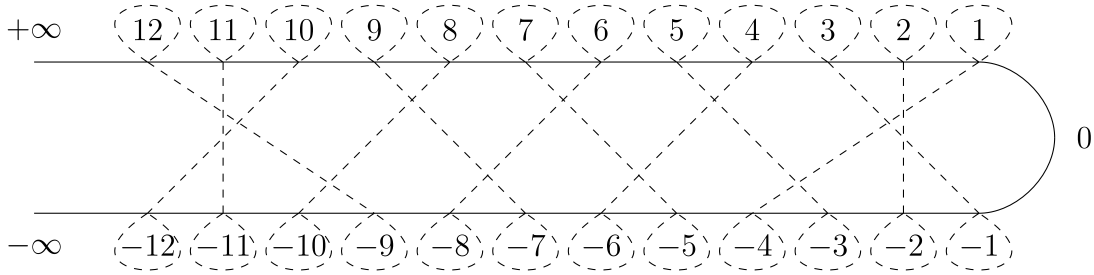


Figure 3.5: An illustration of the intervals in T_6.

For every $n \in \mathbb{N}$, $n > 2$, let A_n^D be the intersection graph of segments of T_n. Let $\mathcal{A}_D = \langle A_n^D \rangle_{n>2}$. Ding proved in [Din98] that $\mathcal{A}_D$ is an antichain for $\leq_{\text{im}}$, thus showing that interval graphs are not wqo by induced minors.

Let us now present two more infinite antichains. Let $A_{\overline{C}} = \langle \overline{C_n} \rangle_{n \geq 6}$ be the sequence of *antiholes* of order at least six, whose first elements are represented in Figure 3.6.

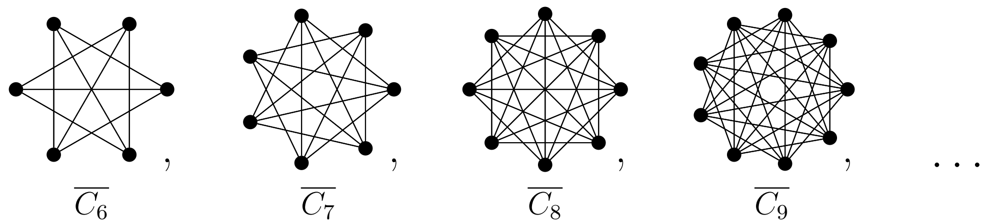


Figure 3.6: Antiholes antichain.

Lemma 3.6. *$\mathcal{A}_{\overline{C}}$ is an antichain.*

Proof. Let $\overline{\deg}$ be the function that maps a vertex v of a graph G to the value $|V(G)| - \deg(v)$. Remark that performing edge contractions and vertex deletions in a graph can only decrease the value of $\overline{\deg}$ of a given vertex. Also notice that in every graph G of $\mathcal{A}_{\overline{C}}$, every vertex v satisfies $\overline{\deg}(v) = 3$. An edge contraction in G yields a vertex v such that $\overline{\deg}(v) = 1$ whereas a vertex deletion gives a vertex v with $\overline{\deg}(v) = 2$. By an above remark, this value cannot be increased by further edge contractions of vertex deletions. Therefore there is no sequence of edge contractions and vertex deletions on G yielding an other graph of $\mathcal{A}_{\overline{C}}$. This proves that $\mathcal{A}_{\overline{C}}$ is an antichain wrt. $\leq_{\text{im}}$. $\square$

We will meet again the antichain $\mathcal{A}_{\overline{C}}$ in the proof of Theorem 3.9. Another infinite antichain which shares with $\mathcal{A}_M$ the properties of planarity and bounded maximum degree is the antichain of nested lozenges depicted in Figure 3.7. We will not go more into detail about it here as this antichain will not be used in our proofs.

3.4.2 Antichains for contractions

Let us first mention that since every contraction of a graph is also an induced minor, all the antichains for induced minors presented in Subsection 3.4.1 are also antichains for

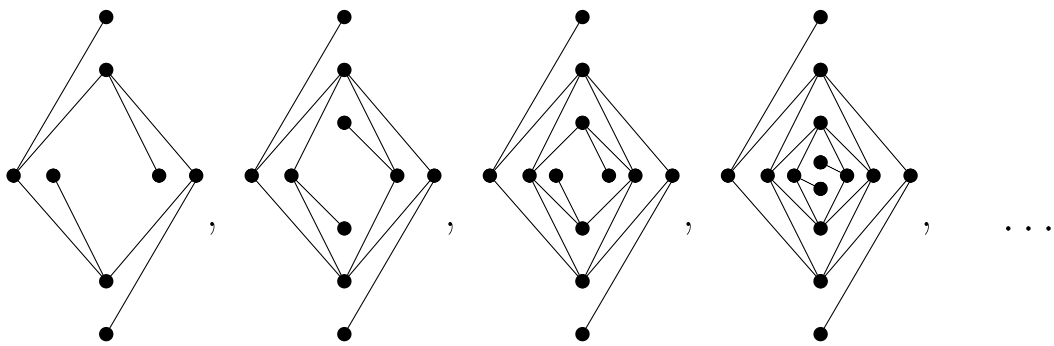


Figure 3.7: Nested lozenges.

the contraction relation. Not surprisingly there are more antichains for the contraction ordering. The most simple is certainly the class $\mathcal{A}_{\overline{K}} = \{\overline{K_i}\}_{i\in\mathbb{N}}$ of edgeless graphs. We present here one more infinite antichain for contractions, which is the class of complete bipartite graphs with one part of size two: $\mathcal{A}_K = \{K_{2,r},\ r \in \mathbb{N}_{\geq 2}\}$ (cf. Figure 3.8).

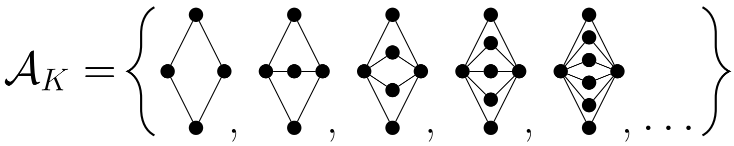


Figure 3.8: The antichain $\mathcal{A}_K$.

Lemma 3.7. *For every $(p, q), (p', q') \in \mathbb{N}_{\geq 2}$ such that $p \leq p'$ and $q < q'$, there is no contraction model of $K_{p,q}$ in $K_{p',q'}$.*

Proof. Let us assume for contradiction that there is a contraction model φ of $K_{p,q}$ in $K_{p',q'}$. As $K_{p',q'}$ has more vertices than $K_{p,q}$, there is a vertex v of $K_{p,q}$ such that $|\varphi(v)| \geq 2$. Observe that every subset of at least two vertices of $K_{p,q}$ that induced a connected subgraph is dominating. Indeed, such a subset must contain at least a vertex from each part of the bipartition. It follows from the definition of a contraction model that $K_{p',q'}$ has a dominating vertex, a contradiction. Therefore there is no contraction model of $K_{p,q}$ in $K_{p',q'}$. □

Corollary 3.3. *$\{K_{2,p},\ p \in \mathbb{N}_{\geq 2}\}$ is an antichain of $\leq_c$.*

If we allow multiple edges and consider the multigraph contraction ordering, then the sequence of multiedges $\mathcal{A}_\theta = \{\theta_i\}_{i\in\mathbb{N}^*}$ depicted in Figure 3.9 is also an infinite antichain.

As we will show in Subsection 3.7.3, every infinite antichain for multigraph contraction is mainly composed of $\mathcal{A}_{\overline{K}}$ and $\mathcal{A}_\theta$.

3.5 Induced minors and well-quasi-ordering

In this section, we prove the dichotomy Theorem 3.9. Our proof naturally has two parts: for different values of H, we need to show wqo of H-induced minor-free graphs or exhibit

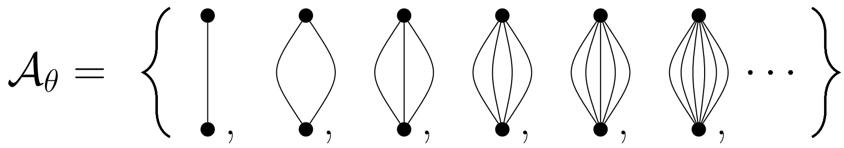


Figure 3.9: The multiedges antichain.

an H-induced minor-free antichain.

In order to achieve Step 1 presented in Section 3.3, we first prove two decomposition theorems. The following two theorems describe the structure of graphs with H forbidden as an induced minor, when H is $\hat{K}_4$ and the Gem, respectively.

Theorem 3.17 (Decomposition of $\hat{K}_4$-induced minor-free graphs)**.** *Let G be a 2-connected graph such that $\hat{K}_4 \not\leq_{\mathrm{im}} G$. Then:*

- *either $K_4 \not\leq_{\mathrm{im}} G$;*
- *or G is a subdivision of a graph among K_4, $K_{3,3}$, and the prism;*
- *or $V(G)$ has a partition (C, M) such that $G[C]$ is an induced cycle, $G[M]$ is a complete multipartite graph and every vertex of C is either adjacent in G to all vertices of M, or to none of them.*

Theorem 3.18 (Decomposition of Gem-induced minor-free graph)**.** *Let G be a 2-connected graph such that* Gem $\not\leq_{\mathrm{im}} G$*. Then G has a subset $X \subseteq V(G)$ of at most six vertices such that every connected component of $G \setminus X$ is either a cograph or a path whose internal vertices are of degree two in G.*

Using the two above structural results, we are able to show the well-quasi-ordering of the two classes with respect to induced minors by following the steps described in Section 3.3.

Theorem 3.19. *The class of $\hat{K}_4$-induced minor-free graphs is wqo by $\leq_{\mathrm{im}}$.*

Theorem 3.20. *The class of* Gem*-induced minor-free graphs is wqo by $\leq_{\mathrm{im}}$.*

Organization of the proof. Subsection 3.5.1 is devoted to the proof of Theorem 3.9, assuming Theorem 3.19 and Theorem 3.20, the proofs of which are respectively given in Subsection 3.5.2 and Subsection 3.5.3.

Let us first provide some definitions specific to this section. An induced subgraph of a graph G is said to be *basic in G* if it is either a cograph, or an induced path whose internal vertices are of degree two in G.

Complete multipartite graphs. A graph G is said to be *complete multipartite* if its vertex set can be partitioned into sets $V_1, \dots, V_k$ (for some positive integer k) in a way such that two vertices of G are adjacent iff they belong to different V_i's. The class of complete multipartite graphs is referred to as $\mathcal{K}_{\mathbb{N}^\star}$.

Containing K_4-subdivisions. A graph G contains K_4 as an induced minor if and only if G contains K_4-subdivision as a subgraph. This equivalence is highly specific to the graph K_4 and in general neither implication would be true. We will freely change between those two notions for containing K_4, depending on which one is more convenient in the given context.

A graph G will be said to contain a *proper K_4-subdivision*, if there is some vertex $v \in V(G)$, such that $G \setminus v$ contains a K_4-subdivision.

Cycle-multipartite. Given a graph G, a pair (C, R) of induced subgraphs of G is said to be a *cycle-multipartite decomposition* of G if the following conditions are satisfied:

(i) $(V(C), V(R))$ is a partition of $V(G)$;

(ii) C is a cycle and R is a complete multipartite graph;

(iii) $\forall u, v \in V(R),\ N_C(u) = N_C(v)$.

The class of graphs having cycle-multipartite decomposition is denoted by $\mathcal{W}$.

Cuts. In a graph G, a K_2-cut (resp. $\overline{K_2}$-cut) is a subset $S \subseteq V(G)$ such that $G - S$ is not connected and $G[S]$ is isomorphic to K_2 (resp. $\overline{K_2}$).

3.5.1 The dichotomy theorem

The purpose of this section is to prove Theorem 3.9, that is, to characterize all graphs H such that $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H), \leq_{\mathrm{im}})$ is a wqo. To this end, we will assume Theorem 3.19 and Theorem 3.20, which we will prove later, in Subsection 3.5.2 and Subsection 3.5.3 respectively.

The main ingredients of the proof are the infinite antichains presented in Subsection 3.4.1, together with Theorem 3.19 and Theorem 3.20. Infinite antichains will be used to discard every graph H that is not induced minor of all but finitely many elements of some infinite antichain. On the other hand, knowing that $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H), \leq_{\mathrm{im}})$ is a wqo gives that $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H'), \leq_{\mathrm{im}})$ is a wqo for every $H' \leq_{\mathrm{im}} H$, by the virtue of Remark 3.5.

In the statement of the following results, we assume that H is any graph.

Lemma 3.8. *If $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H), \leq_{\mathrm{im}})$ is a wqo then $\overline{H}$ is a linear forest.*

Proof. Let us show that $\mathrm{Excl}_{\leq_{\mathrm{im}}}(H)$ has an infinite antichain given that $\overline{H}$ is not a linear forest. In this case, $\overline{H}$ either has a vertex of degree at least 3 or it contains an induced cycle as induced subgraph.

First case: $\overline{H}$ has a vertex v of degree 3. Let x, y, z be three neighbors of v. In the graph $H[\{v, x, y, z\}]$, the vertex v is adjacent to none of x, y, z. In an antihole, every vertex has exactly two non-neighbors, so $H[\{v, x, y, z\}]$ is not an induced minor of any element of $\mathcal{A}_{\overline{C}}$. Therefore $\mathcal{A}_{\overline{C}} \subseteq \mathrm{Excl}_{\leq_{\mathrm{im}}}(H)$.

Second case: $\overline{H}$ contains an induced cycle as an induced subgraph. Let us first assume that for some integer $k \geq 6$ we have $\overline{C_k} \leq_{\mathrm{im}} H$. Now consider any $\overline{C_n}$ for $n > |H|$.

Clearly, we have $\overline{C_n} \not\leq_{\text{im}} H$. On the other hand if $H \leq_{\text{im}} \overline{C_n}$, then by the fact that $\overline{C_k} \leq_{\text{im}} H$ and transitivity, we would have $\overline{C_k} \leq_{\text{im}} \overline{C_n}$, which would yield a contradiction with the fact that $\mathcal{A}_{\overline{C}}$ is an antichain. Hence $\mathcal{A}_{\overline{C}} \cap \text{Excl}_{\leq_{\text{im}}}(H)$ contains all antiholes of size greater than $|H|$, and in particular is infinite as required. In the cases where $\overline{H}$ has a cycle of length 3,4, or 5, it is easy to check that no element of $\mathcal{A}_{\overline{C}}$ contains (respectively) three independent vertices, two independent edges, or an edge not adjacent to an other vertex (which is an induced subgraph of $\overline{C_5}$). □

Due to the interesting properties on $\overline{H}$ given by Lemma 3.8, we will be led below to work with this graph rather than with H. The following lemma presents step by step the properties that we can deduce on $\overline{H}$ by assuming that $\text{Excl}_{\leq_{\text{im}}}(H)$ is wqo by $\leq_{\text{im}}$.

Lemma 3.9. *If* $(\text{Excl}_{\leq_{\text{im}}}(H), \leq_{\text{im}})$ *is a wqo, then we have*

(R1) $\overline{H}$ *has at most 4 connected components;*

(R2) at most one connected component of $\overline{H}$ *is not a single vertex;*

(R3) the largest connected component of $\overline{H}$ *has at most 4 vertices;*

(R4) if $n = |V(H)|$ *and* $c = \mathbf{cc}(\overline{H})$ *then* $n \leq 7$ *and* $\overline{H} = (c-1)\cdot K_1 + P_{n-c+1}$*;*

(R5) if $\mathbf{cc}(\overline{H}) = 3$ *then* $|V(H)| \leq 5$.

(R6) if $\mathbf{cc}(\overline{H}) = 4$ *then* $|V(H)| \leq 4$.

Proof. Proof of item (R1). The infinite antichain $\mathcal{A}_M$ does not contain K_5 and (induced) minor, hence $K_5 \not\leq_{\text{im}} H$ and so $\overline{H}$ does not contain $5\cdot K_1$ as induced minor. Therefore it has at most 4 connected components.

Proof of items (R2) and (R3). The infinite antichain $\mathcal{A}_D$ does not contain C_4 as induced minor (as it is an interval graph), hence neither does H. Therefore $\overline{H}$ does not contain $2\cdot P_2$ as induced minor. This implies that $\overline{H}$ does not contain P_5 as induced minor and that given two connected components of $\overline{H}$ at least one must be of order one. As connected components of H are paths (by Lemma 3.8), the largest connected component of H has order at most 4.

Item (R4) follows from the above proofs and from the fact that $\overline{H}$ is a linear forest.

Proof of item (R5). Similarly as in the proof of item (R1), $\mathcal{A}_M$ does not contain K_5^- as induced minor so $\overline{K_5^-} = K_2 + 3\cdot K_1$ is not an induced minor of $\overline{H}$. If we assume that $\mathbf{cc}(\overline{H}) = 3$ and $|V(H)| \geq 6$, the largest component of $\overline{H}$ is a path on (a least) 4 vertices, so it contains $K_1 + K_2$ as induced subgraph. Together with the two other (single vertex) components, this gives an $K_2 + 3\cdot K_1$ induced minor, a contradiction.

Proof of item (R6). Let us assume that $\mathbf{cc}(\overline{H}) = 4$. If the largest connected component has more than one vertex, then $\overline{H}$ contains $K_2 + 3\cdot K_1$ as an induced minor, which is not possible (as in the proof of item (R5)). Therefore $\overline{H} = 4\cdot K_1$ and so $\left|V(\overline{H})\right| = 4$. □

We are now able to describe more precisely graphs H for which $(\text{Excl}_{\leq_{\text{im}}}(H), \leq_{\text{im}})$ could be a wqo. Let K_3^+ be the complement of $P_3 + K_1$ and let K_4^- be the complement of $K_2 + 2\cdot K_1$, which is also the graph obtained from K_4 by deleting an edge (sometimes referred as *diamond graph*).

Lemma 3.10. *If* $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H), \leq_{\mathrm{im}})$ *is a wqo, then* $H \leq_{\mathrm{im}} \hat{K}_4$ *or* $H \leq_{\mathrm{im}} \mathrm{Gem}$.

Proof. Using the information on $\overline{H}$ given by Lemma 3.9, we can build a table of possible graphs $\overline{H}$ depending on $\mathbf{cc}(\overline{H})$ and $|V(\overline{H})|$. Table 3.2 is such a table: each column corresponds for a number of connected components (between one and four according to item (R1)) and each line corresponds to an order (at most seven, by item (R4)). A grey cell means either that there is no such graph (for instance a graph with one vertex and two connected components), or that for all graphs $\overline{H}$ matching the number of connected components and the order associated with this cell, the qoset $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H), \leq_{\mathrm{im}})$ is not a wqo.

$\lvert V(H)\rvert \setminus \mathbf{cc}(\overline{H})$	1	2	3	4
1	K_1			
2	K_2	$2 \cdot K_1$		
3	P_3	$K_2 + K_1$	$3 \cdot K_1$	
4	P_4	$P_3 + K_1$	$K_2 + 2 \cdot K_1$	$4 \cdot K_1$
5	(R3)	$P_4 + K_1$	$P_3 + 2 \cdot K_1$	(R6)
6	(R3)	(R3)	(R5)	(R6)
7	(R3)	(R3)	(R5)	(R6)

Table 3.2: If $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H), \leq_{\mathrm{im}})$ is a wqo, then $\overline{H}$ belongs to this table.

From Table 3.2 we can easily deduce Table 3.3 of the corresponding graphs.

$\lvert V(H)\rvert \setminus \mathbf{cc}(\overline{H})$	1	2	3	4
1	K_1			
2	$2 \cdot K_1$	K_2		
3	$K_2 + K_1$	P_3	K_3	
4	P_4	K_3^+	K_4^-	K_4
5	(R3)	Gem	$\hat{K}_4$	(R6)
6	(R3)	(R3)	(R5)	(R6)
7	(R3)	(R3)	(R5)	(R6)

Table 3.3: If $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(H), \leq_{\mathrm{im}})$ is a wqo, then H belongs to this table.

Remark that we have

- $K_1 \leq_{\mathrm{im}} 2 \cdot K_1 \leq_{\mathrm{im}} K_2 + K_1 \leq_{\mathrm{im}} P_4 \leq_{\mathrm{im}} \mathrm{Gem}$;
- $K_2 \leq_{\mathrm{im}} P_3 \leq_{\mathrm{im}} K_3^+ \leq_{\mathrm{im}} \mathrm{Gem}$;
- $K_3 \leq_{\mathrm{im}} K_4^- \leq_{\mathrm{im}} \hat{K}_4$; and
- $K_4 \leq_{\mathrm{im}} \hat{K}_4$.

This concludes the proof. □

We are now ready to give the proof of Theorem 3.9.

Proof of Theorem 3.9. If $H \not\leq_{\text{im}} \text{Gem}$ and $H \not\leq_{\text{im}} \hat{K}_4$, then by Lemma 3.10 $(\text{Excl}_{\leq_{\text{im}}}(H), \leq_{\text{im}})$ is not a wqo. On the other hand, by Theorem 3.19 and Theorem 3.20 we know that both $\text{Excl}_{\leq_{\text{im}}}(\hat{K}_4)$ and $\text{Excl}_{\leq_{\text{im}}}(\text{Gem})$ are wqo by $\leq_{\text{im}}$. Consequently, by Remark 3.5, $(\text{Excl}_{\leq_{\text{im}}}(H), \leq_{\text{im}})$ is wqo as soon as $H \leq_{\text{im}} \text{Gem}$ or $H \leq_{\text{im}} \hat{K}_4$. □

3.5.2 Graphs not containing $\hat{K}_4$

The main goal of this section is to provide a proof to Theorem 3.19. To this purpose, we first prove in Section 3.5.2 that the graphs of $\text{Excl}_{\leq_{\text{im}}}(\hat{K}_4)$ admit a simple structural decomposition. This structure is then used in Section 3.5.2 to show that graphs of $\text{Excl}_{\leq_{\text{im}}}(\hat{K}_4)$ are well-quasi-ordered by the relation $\leq_{\text{im}}$.

A decomposition theorem for $\text{Excl}_{\leq_{\text{im}}}(\hat{K}_4)$

The main topic of this section is the proof of Theorem 3.17. This theorem states that every graph in the class $\text{Excl}_{\leq_{\text{im}}}(\hat{K}_4)$, either does not have even K_4 as induced minor, or is a subdivision of some small graph, or has a cycle-multipartite decomposition. Most of the time, we show that some property P is not satisfied by graphs of $\text{Excl}_{\leq_{\text{im}}}(\hat{K}_4)$ by showing an induced minor model of $\hat{K}_4$ in graphs satisfying P. We first assume that G contains a proper K_4-subdivision, and we show in Lemma 3.20 how to deal with the other case.

Lemma 3.11. *If G contains as induced minor any graph H consisting of:*

- *a K_4-subdivision S;*
- *an extra vertex x linked by exactly two paths L_1 and L_2 to two distinct vertices $s_1, s_2 \in V(S)$, where the only common vertex of L_1 and L_2 is x;*
- *and possibly extra edges between the vertices of S, or between L_1 and L_2, or between the interior of the paths and S,*

then $\hat{K}_4 \leq_{\text{im}} G$.

Proof. Let us call $V = \{v_1, v_2, v_3, v_4\}$ the non-subdivision vertices of S, i.e. vertices corresponding to vertices of K_4. We present here a sequence of edge contractions in H leading to $\hat{K}_4$. Let us repeat the following procedure: as long as there is a path between two elements of $V \cup \{s_1, s_2, s\}$, internally disjoint with this set, contract the whole path to a single edge.

Once we can not apply this contraction any more, we end up with a graph that has two parts: the K_4-subdivision with at most 2 subdivisions (with vertex set $V \cup \{s_1, s_2\}$) and the vertex x, which is now only adjacent to s_1 and s_2.
First case: $s_1, s_2 \in V$. The graph H is isomorphic to $\hat{K}_4$: it is K_4 plus a vertex of degree two.
Second case: $s_1 \in V$ and $s_2 \notin V$ (and the symmetric case). As vertices of V are the only vertices of H that have degree 3 in S, s_2 is of degree 2 in S (it is introduced by

subdivision). The contraction of the edge between s_2 and one of its neighbors in S that is different to s_1 leads to first case.

Third case: $s_1, s_2 \notin V$. As in the second case, these two vertices have degree two in S. Since no two different edges of K_4 can have the same endpoints, the neighborhoods of s_1 and s_2 have at most one common vertex. Then for every $i \in \{1, 2\}$ there is a neighbor t_i of s_i that is not adjacent to s_{3-i}. Contracting the edges $\{s_1, t_1\}$ and $\{s_2, t_2\}$ leads to first case. □

Corollary 3.4 (Proof of Lemma 3.11)**.** *Let $G \in \mathrm{Excl}_{\leq_{\mathrm{im}}}(\hat{K}_4)$ be a 2-connected graph containing a proper K_4-subdivision. For every subdivision S of K_4 in G, and for every vertex $x \in V(G) \setminus V(S)$, $N_S(x) \geq 3$.*

Proof. Let S be a proper K_4-subdivision in G and $x \in V(G) \setminus V(S)$. Let L_1, L_2 be two shortest paths from x to S meeting only in x. Such paths exist by the 2-connectivity of G. Remark that if $|N_G(x) \cap V(S)| \leq 2$, then then graph induced by S, L_1, and L_2 satisfies conditions of Lemma 3.11. Therefore, $N_S(x) \geq 3$. □

Remark 3.6. For every two edges of K_4 there is a Hamiltonian cycle using these edges.

Remark 3.7. Three edges of K_4 are not contained into a same cycle iff they are incident with the same vertex.

Lemma 3.12. *Every 2-connected graph $G \in \mathrm{Excl}_{\leq_{\mathrm{im}}}(\hat{K}_4)$ containing a proper K_4-subdivision has a 3-wheel as subgraph.*

Proof. Let S be a minimum (proper) K_4-subdivision in G and let $x \in V(G) \setminus V(S)$. We define V as in the proof of Lemma 3.11 and we say that two neighbours of x in S are equivalent if they lie on the same path between two elements of V (intuitively they correspond to the same edge of K_4). By Corollary 3.4, we only have to consider the case $|N_S(x)| \geq 3$.

First of all, observe that if some three neighbors of x lie on a cycle of S, then we are done. Let us assume from now on, that there is no cycle of S containing three neighbors of x. This implies that no two neighbors of x are equivalent (by Remark 3.6), no neighbor of x belongs to V (by the same remark), and that $|N_S(x)| = 3$ (by Remark 3.7). Let us consider the induced minor H of $S + x$ obtained by contracting all edges not incident with two vertices of $V \cup N_S(x)$. By Remark 3.7 and since the three neighbors of x do not belong to a cycle, there is a vertex of $V(H) \setminus \{x\}$ adjacent to the three neighbors of x. Contracting two of the edges incident with this vertex merges two neighbors of x and the graph we obtain is a K_4-subdivision (corresponding to S) together with a vertex of degree 2 (corresponding to x). By Lemma 3.11, this would imply that $\hat{K}_4 \leq_{\mathrm{im}} G$, a contradiction. Therefore three neighbors of x lie on a cycle of S and this concludes the proof. □

Now we will deal with a graph G that satisfies conditions of Lemma 3.12; namely: G is a 2-connected graph, without $\hat{K}_4$ as an induced minor, containing a proper K_4-subdivision; C denotes the cycle of a minimum (in terms of number of vertices) 3-wheel in G, and R the graph induced by the remaining vertices.

Remark 3.8. As this 3-wheel is a subdivision of K_4 as subgraph of G, by Corollary 3.4, every vertex of R has at least three neighbors in C.

Remark 3.9. This 3-wheel contains no more vertices than the K_4-subdivision that we assumed to be contained in G. Therefore, every minimum K_4-subdivision of G (in terms of vertices) is a 3-wheel.

Lemma 3.13. *Let G be a 2-connected graph of $\mathrm{Excl}_{\leq_{\mathrm{im}}}(\hat{K}_4)$. Every minimum (in terms of number of vertices) 3-wheel W of cycle C and center r that is a subgraph of G is such that, if C is not an induced cycle in G,*

(i) the endpoints of every chord are both adjacent to some $u \in N_C(r)$;

(ii) every two distinct $v, w \in N_C(r) \setminus \{u\}$ are adjacent on C;

(iii) C has exactly one chord in G

(iv) $|N_C(r)| = 3$.

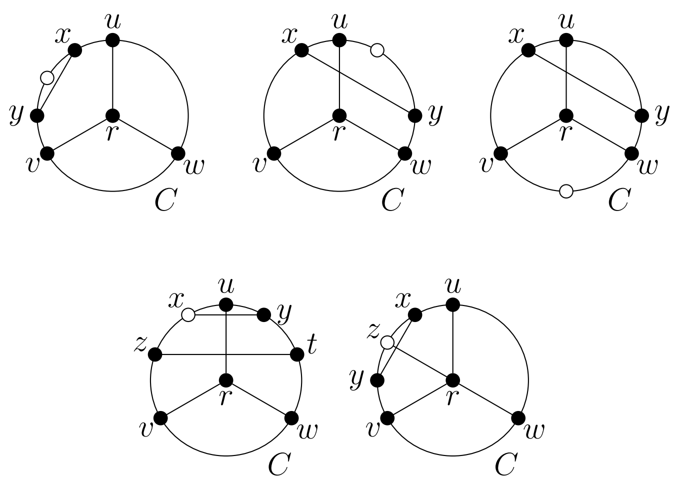


Figure 3.10: Forbidden configurations in the proof of Lemma 3.13.

Proof. Let $u, v, w \in N_C(r)$ be three distinct neighbours of r in C and let C_u be the path of C between v and w that does not contains the vertex u, and similarly for C_v and C_w. First of all, notice that no proper subgraph of W can be a subdivision of K_4, otherwise G would contain a graph smaller than W but meeting the same requirements, according to Lemma 3.12. Below we will show that when conditions (i)-(iv) are not fulfilled, W is not minimal, i.e. that when some vertices are deleted in W, it still contains a K_4-subdivision. Figure 3.10 illustrates such configurations, where white vertices can be deleted.

Let $\{x, y\}$ be a chord of C in G. Remark that the endpoints of a chord cannot belong to the same path C_l for any $l \in \{u, v, w\}$ without violating the minimality of S, as deleting vertices of C_l that are between x and y would still lead to a 3-wheel (first configuration of Figure 3.10). Therefore, x and y belongs to different C_l's, say without loss of generality that $x \in V(C_w)$ and $y \in V(C_v)$.

Let us prove that x and y must both be adjacent to u. By contradiction, we assume that, say, y and u are not adjacent. Let us consider the induced subgraph of W obtained by the deletion of the interior of the path yC_vu (containing at least one vertex). Notice that contracting each of the paths vC_uw, vC_wx, yC_vw and $(r,u) - uC_wx$ gives K_4, a contradiction (cf. the second configuration of Figure 3.10). The case where x and u are not adjacent is symmetric. This proves that every chord of C in G has endpoints adjacent to a same neighbor of r on C, that is (i).

Now, we show that the path C_u must be an edge. To see this, assume by contradiction that it has length at least 3. The subgraph of W induced by the six paths $(r,v)-vC_wx$, $\{r,u\}$, $(r,w) - wC_vy$, $\{x,y\}$, uC_wx and uC_vy does not contain the internal vertices of C_u, thus it is smaller than W (third configuration of Figure 3.10). However, it contains a subdivision of K_4 as subgraph, that can for instance be obtained by contracting each of these six paths to an edge. This contradicts our first remark, therefore the two neighbors v and w of r on the cycle are adjacent. This proves item (ii).

Let now assume that there C has a second chord $\{z,t\}$ in G. In the light of the previous remark, C_u is an edge, hence the only paths to which z and t can belong are the paths xC_wv and yC_vw and, according to our first remark, they do not both belong to the same of these two paths. Also, as $\{z,t\} \neq \{x,y\}$, one of z,t does not belong to $\{x,y\}$. We can thus assume without loss of generality that $z \in V(xC_wv)$, $t \in V(yC_vw)$ and $z \neq x$. This case is represented by the fourth configuration of Figure 3.10. Let us consider the cycle C' obtained by the concatenation of the paths zC_wv, C_u, wC_vt and (t,z) and the vertex r, which is connected to the cycle by the three paths (r,v), (r,w) and $(r,u) - uC_vt$ that only share the vertex r. This subgraph is smaller than H since it does not contain vertex x, but it is a subdivision of K_4 (three paths issued from the same vertex a and meeting the cycle C'). By one of the above remarks, this configuration is impossible and thus the chord (z,t) cannot exist. Hence we proved item (iii): C can have at most one chord in G.

Notice that we have $|N_C(r)| \geq 3$ since W is a 3-wheel. We now assume that $|N_C(r)| > 3$. Let u,v,w,z be four different neighbors of r such that z is a common neighbor of the endpoints of the chord x and y. This case is depicted in the fifth configuration of Figure 3.10. Then r has at least three neighbors (u, v, and w) on the cycle going through the edge $\{x,y\}$ and following C up to x without using the vertex z. As this contradicts the minimality of W, we have $|N_C(r)| = 3$, that is item (iv), and this concludes the proof. □

Corollary 3.5. *According to Remark 3.9, every minimum K_4-subdivision in G is a 3-wheel, so Lemma 3.13 is still true when replacing 3-wheel by K_4-subdivision in its statement.*

Lemma 3.14. *Every two non-adjacent vertices of R have the same neighborhood in C.*

Proof. By contradiction, we assume that there are two non-adjacent vertices $s,t \in V(R)$ and a vertex $u_1 \in V(C)$ such that $\{s,u_1\} \in E(G)$ but $\{t,u_1\} \notin E(G)$. By Remark 3.8, s and t have (at least) three neighbors in C. Let $U = \{u_1,u_2,u_3\}$ and $V = \{v_1,v_2,v_3\}$ be the respective neighbors of s and t. We consider the graph H induced in G by C and $\{s,t\}$ where we iteratively contracted every edge of S not incident with two vertices of $\{u_1, u_2, u_3, v_1, v_2, v_3\}$. This graph is a (non necessarily induced) cycle on at most 6 vertices, that we call C' plus the two non-adjacent vertices of degree at least three s and t. Remark that while two neighbors of s are adjacent and are not both neighbors of t, we can contract the edge between them and decrease the degree of s, without changing degree of t. If the degree of s reaches two by such means, then by Lemma 3.11, $\hat{K}_4 \leq_{\text{im}} H$, a contradiction. We can thus assume that every vertex of C' adjacent to a neighbor of s is a neighbor of t. This is also true when s and t are swapped since this argument can be applied to t too. This observation implies that $N_S(s) \cap N_S(t) = \emptyset$ (as u_1 is adjacent to s but not to t, none of its neighbors on C can be adjacent to s, and so on along the cycle) and that the neighbors of s and t are alternating on C'. Without loss of generality, we suppose that $C' = u_1v_1u_2v_2u_3v_3$. We consider now the five following sets of vertices of H : $M_1 = \{u_1\}, M_2 = \{s\}, M_3 = \{u_2, v_1\}, M_4 = \{v_2, u_3, v_3\}, M_5 = \{t\}$. They are depicted on Figure 3.11. Let $\mu \colon \hat{K}_4 \to \mathcal{P}^{<\omega}(V(H))$ be the function defined as follows: $\forall i \in [\![1, 5]\!]\,,\ \mu(v_i) = M_i$ (using the names of vertices of $\hat{K}_4$ defined on Figure 3.2). Now, remark that μ is an induced minor model of $\hat{K}_4$ in H: for every $i \in [\![1, 5]\!]$, the set M_i is connected, M_1, M_3, M_4 forms a cycle (using edges $\{u_1, v_1\}, \{u_2, v_2\}, \{v_3, u_1\}$ is this order), M_2 is adjacent to any of these three sets (by edges $\{s, u_1\}, \{s, u_2\}, \{s, u_3\}$) and the set M_5 is only adjacent to M_3 and M_4 (by edges $\{t, v_1\}, \{t, v_2\}$). Remark that the previous statement holds even when C' is not an induced cycle, as any possible chord of C' will be between vertices of the sets M_1, M_2, M_3, M_4 (as M_5 is reduced to $t \notin V(C')$) and these sets are already all pairwise adjacent. We assumed our initial graph to be $\hat{K}_4$-induced minor-free but we proved that it contains an induced minor model of $\hat{K}_4$: this is the contradiction we were looking for. $\square$

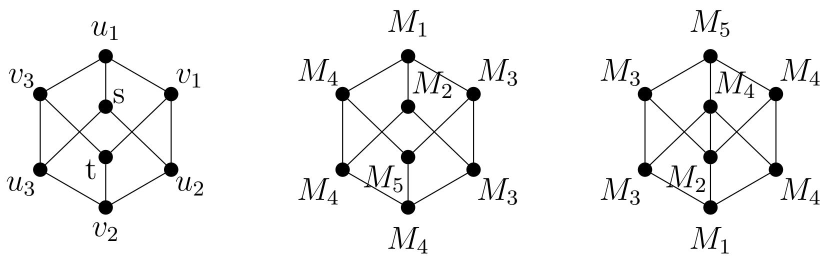


Figure 3.11: Graph H (left) used in Lemma 3.14 (middle) and in Lemma 3.15 (right).

Lemma 3.15. *Every two adjacent vertices of R have the same neighborhood in C.*

Proof. By contradiction, we assume that there are two adjacent vertices $s, t \in V(R)$ and $u_1 \in V(C)$ such that $\{s, u_1\} \in E(G)$ but $\{t, u_1\} \notin E(G)$. As this proof is very similar to the proof of Lemma 3.14, we define $u_1, u_2, u_3, v_1, v_2, v_3$, U, V, H, C' in the same way here.

First case: C' is an induced cycle. In this case, the graph H is the (induced) cycle $C' = u_1v_1u_2v_2u_3v_3$ plus the two adjacent vertices s and t. Let us define five vertex sets: $M_1 = \{v_2\}, M_2 = \{t\}, M_3 = \{u_3, v_3\}, M_4 = \{v_1, u_2, s\}, M_5 = \{u_1\}$. They are depicted on Figure 3.11. Now, remark that the function that sends the vertex of $\hat{K}_4$ labeled i

on Figure 3.2 to M_i is an induced minor model of $\hat{K}_4$ in H: every set M_i is connected, M_1, M_3, M_4 forms a cycle (using edges $\{v_2, u_3\}, \{u_3, s\}, \{u_2, v_2\}$ is this order), M_2 is adjacent to any of these three sets (by edges $\{t, v_2\}, \{t, v_3\}, \{t, v_1\}$), and the set M_5 is only adjacent to M_3 and M_4 (by edges $\{u_1, v_3\}, \{u_1, v_1\}$). As we assumed our initial graph to be $\hat{K}_4$-induced minor-free, this is a contradiction.
Second case: the cycle C' is not induced. By Lemma 3.13 and as C is supposed to be a minimal 3-wheel of G, the cycle C has only one chord. In this case, the graph H is the (induced) cycle $C' = u_1v_1u_2v_2u_3v_3$ plus the two adjacent vertices s and t and an edge e between two vertices of C'. In H, both $H \setminus \{s\}$ and $H \setminus \{t\}$ are minimal 3-wheels, sharing the same cycle C'. By applying Lemma 3.13 on these two 3-wheels, we obtain that the endpoints of e must both be adjacent to a vertex of u_1, u_2, u_3 (neighbor of s on C') and to a vertex of v_1, v_2, v_3 (neighbor of t on C'). Such a configuration is impossible. □

Lemma 3.16. *If C is not an induced cycle of G, then $|V(R)| = 1$.*

Proof. Let $r \in V(R)$ be the center of a minimum 3-wheel of cycle C. By contradiction, let us assume that R contains a vertex $s \neq r$. By Lemma 3.13, r has exactly three neighbors on C, one of which, that we call u, is adjacent to both endpoints of the only chord of C. Furthermore the two other neighbors or r, that we denote by $\{v, w\}$, are adjacent. According to Lemma 3.14 and Lemma 3.14, r and s are adjacent in G. There are now two different cases to consider depending whether $\{r, s\}$ is an edge or not. The case $\{r, s\} \in E(G)$ (respectively $\{r, s\} \notin E(G)$) is depicted on the right (respectively left) of Figure 3.12. Remark that if $\{r, s\} \in E(G)$, the graph G satisfies conditions of Lemma 3.11 (with $G[\{r, s, u, v, w\}]$ for S, $\{u, v\}$ for $\{s_1, s_2\}$ and x for x), so $G \leq_{\text{im}} \hat{K}_4$, what is contradictory. In the other hand, when $\{r, s\}$*not* $\in E(G)$, the induced subgraph $G[\{r, s, u, v, w\}]$ is isomorphic to $\hat{K}_4$, so we also have $G \leq_{\text{im}} \hat{K}_4$. Hence $|V(R)| < 2$. By definition of C and R, the subgraph R contains at least one vertex (that is the center of a 3-wheel of cycle C), thus $|V(R)| = 1$ as required.

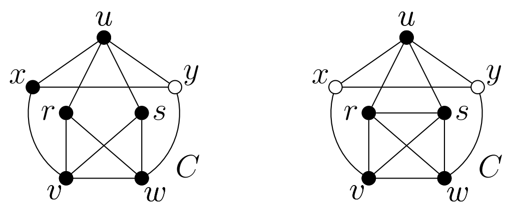


Figure 3.12: Two different cases in the proof of Lemma 3.16

□

Corollary 3.6. *If C is not an induced cycle of G, then G is a subdivision of the prism.*

Lemma 3.17. *R is complete multipartite.*

Proof. As a graph is complete multipartite iff it does not contain $K_1 + K_2$ as induced subgraph, we only need to show that the case where $R = K_1 + K_2$ is not possible. Consequently, let us assume that we are in this case, and let u, v, w be the vertices of R, $\{u, v\}$ being the only edge in R. As R is an induced subgraph of G, u and w are not adjacent in G neither. By Lemma 3.14, they have the same neighborhood on C. The same argument can be applied to v and w to show that $N_C(u) = N_C(v) = N_C(w)$. Let $x, y, z \in N_C(u)$ be three different vertices (they exist by Corollary 3.4) and let H be the graph obtained from G by contracting every edge of C that is not incident with two vertices of $\{x, y, z\}$. Such a graph is a triangle (obtained by contracting C) and the three vertices u, v, w each adjacent to every vertex of the triangle, as drawn in Lemma 3.17. Deleting vertex y gives a graph isomorphic to $\hat{K}_4$, as one can easily check (cf. Lemma 3.17). □

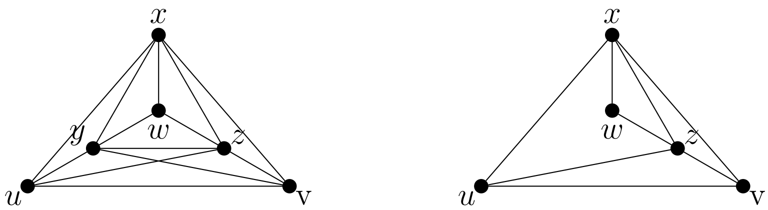


Figure 3.13: The graph H of Lemma 3.17 (left) and the graph obtained after deletion of y (right).

We now need to show that every graph containing a K_4-subdivision either has a proper K_4-subdivision, or fall in the possible cases of the statement of Theorem 3.17.

Lemma 3.18. *If G can be obtained by adding an edge between two vertices of a $K_{3,3}$-subdivision, then G has a proper K_4-subdivision.*

Proof. Let S be the spanning subgraph of G which is a $K_{3,3}$-subdivision. A *branch* of S is a maximal path, the internal vertices of which are of degree two. In S, non-subdivision vertices are connected by branches. Let us call a, b, c, x, y, z the non-subdivision vertices of S in a way such that there is neither a branch between any two vertices of $\{a, b, c\}$ nor between any two vertices of $\{x, y, z\}$ (intuitively $\{a, b, c\}$ and $\{x, y, z\}$ correspond to the two maximum independent sets of $K_{3,3}$). Observe that every $K_{3,3}$-subdivision contains a K_4-subdivision (but not a proper one). Let us now consider all the possible endpoints of the only edge e of $E(G) \setminus E(S)$.
First case: both endpoints of e belong to the same branch B of S. Let X be the set of internal vertices of the subpath of B starting at the one endpoint of e and ending at the other one. As G is a simple graph, $|X| \geq 1$. Then $G \setminus X$ has a K_4-subdivision (as it is a $K_{3,3}$-subdivision), which is a proper K_4-subdivision of G.
Second case: e is incident with two non-subdivision vertices. Observe that the case where e is incident with a vertex from $\{a, b, c\}$ and the other from $\{x, y, z\}$ is contained in the previous case. Let us assume without loss of generality that $e = \{a, b\}$. Then $G \setminus \{x\}$ has a K_4-subdivision. Indeed, if $B_{s,t}$ denotes the branch with endpoints the vertices s and t (for $(s, t) \in \{a, b, c\} \times \{x, y, z\}$), then the vertices of the paths $B_{b,y}$, $B_{b,z}$, $B_{c,z}$ and $B_{c,y}$ induce a cycle in G. The vertex a is then connected to this cycle with the paths $B_{a,y}$, $B_{a,z}$ and the edge e. Hence G has a proper K_4-subdivision, as required.
Third case: e is incident with two subdivision vertices. If the two endpoints of e belong to the same branch, then we are in the first case. Otherwise, we can easily reach the second

case as follows. If we contract all the edges on the path connecting the first endpoint of e to a vertex of $\{a, b, c\}$ and all the edges on the path connecting the second endpoint of e to a vertex of $\{x, y, z\}$, we get a K_4-subdivision (because we never contracted an edge incident with two non-subdivision vertices of S) plus the edge e which is now incident with two non-subdivision vertices. This concludes the proof. □

Lemma 3.19. *If G can be obtained by adding an edge between two vertices of a prism-subdivision, then G has a proper K_4-subdivision.*

Proof. Let S be a prism-subdivision in G and let $e \in E(G) \setminus E(S)$. We will use the concept of *branch* defined in the proof of Lemma 3.18, which is very similar to this one. Let us call a, b, c, x, y, z the non-subdivision vertices in a way such that there are branches between every pair of vertices of $\{a, b, c\}$ (respectively $\{x, y, z\}$) and between vertices of the pairs (a, x), (b, y), and (c, z). Intuitively $\{a, b, c\}$ and $\{x, y, z\}$ correspond to the two triangles of the prism. Let us consider the positions of the endpoints of e.
First case: both endpoints of e belong to the same branch of S. Since the prism contains a K_4 subdivision (but not a proper one), we can in this case find a smaller prism subdivision as in the first case of the proof of Lemma 3.18, and thus a proper K_4-subdivision.
Second case: e is incident with two non-subdivision vertices. Let us assume without loss of generality that $e = \{a, y\}$ (the cases $e \subseteq \{a, b, c\}$, $e \subseteq \{x, y, z\}$, and $e \in \{\{a, x\}, \{b, y\}, \{c, z\}\}$ are subcases of the first one). Then in $G \setminus \{x\}$, the paths $B_{a,b}$, $B_{b,z}$ and $B_{x,y}$ together with the edge e induces a cycle to which the vertex c is connected via the paths $B_{c,b}$, $B_{c,a}$, and $B_{c,y}$. Hence G has a proper K_4-subdivision.
Third case: e is incident with two branches between a,b, and c (and the symmetric case with branches between x, y, and z). Let us assume without loss of generality that e has one endpoint r among the interior vertices of $B_{a,c}$ and the other one s among the interior vertices of $B_{b,c}$. Then (r, s), $sB_{b,c}c$, and $cB_{a,c}r$ induce in $G \setminus \{x\}$ a cycle to which the vertex b is connected via the paths $bB_{b,c}s$, $bB_{a,b}a$ together with $aB_{a,c}r$, and $B_{b,z}$ together with $B_{z,y}$ and $B_{y,b}$. Again, G has a proper K_4-subdivision.
Fourth case: e is incident with two branches, the one connected to a vertex in $\{a, b, c\}$ and the other one connected to a vertex in $\{x, y, z\}$. In this case, by contracting the edges of the first branch that are between the endpoint of e and a vertex of $\{a, b, c\}$ and similarly with the second branch and a vertex of $\{x, y, z\}$ gives a graph with a prism-subdivision plus an edge between two non-subdivision vertices (that is, first case), exactly as in the proof of Lemma 3.18. □

Lemma 3.20. *If graph G contains a K_4-subdivision, then either G has a proper K_4-subdivision, or G is a wheel, or a subdivision of one of the following graphs: K_4, $K_{3,3}$, and the prism.*

Proof. Looking for a contradiction, let G be a counterexample with the minimum number of vertices and, subject to that, the minimum number of edges. Let S be a K_4-subdivision in G and let $e \in E(G) \setminus E(S)$. Observe that since G has no proper K_4-subdivision, S is a spanning subgraph of G. Also, e is well defined as we assume that G is not a K_4-subdivision. Notice that since the minimum degree of K_4 is 3, contracting an edge incident with a vertex of degree 2 in G would yield a smaller counterexample. Therefore G has minimum degree at least 3. Let $G' = G \setminus \{e\}$. This graph clearly contains S. By minimality of G, the graph G' is either a wheel, or a subdivision of a graph among K_4, $K_{3,3}$, and the prism. Observe that G' cannot have a proper K_4-subdivision because it would also be a proper K_4-subdivision in G.

First case: G' is a wheel. Let C be the cycle of the wheel and let r be its center. Obviously, in G the edge e has not r as endpoint otherwise G would also be a wheel. Therefore e is incident with two vertices of C. Let P and P' be the two subpaths of C whose endpoints are then endpoints of e. Observe that none of P and P' contains more than two neighbors or r. Indeed, if, say, P contained at least three neighbors of r, then the subgraph of G induced by the vertices of P, e, and r would contain a K_4-subdivision, hence contradicting the fact that G has no proper K_4-subdivision.

Therefore G is the cycle C with exactly one chord, e, and the vertex r which has at most 4 neighbors on C. Because G has maximum degree at least 3, it has at most 7 vertices. If r has three neighbors on C, then necessarily P contains one of them and P' the other two (or the other way around). We can easily check in this case that G is a subdivision of the prism. If r has four neighbors on C, the interior of P and P' must each contain two of them according to the above remarks. The deletion of any neighbor of r in this graph yields a K_4-subdivision of non-subdivision vertices r and the remaining neighbors. Observe that both cases contradict the assumptions made on G.

Second case: G' is a subdivision of K_4, or $K_{3,3}$, or the prism. If G' is a subdivision of $K_{3,3}$ or of the prism, then the result follows by Lemma 3.18 and Lemma 3.19. Let us now assume that G' is a subdivision of K_4 and let us consider branches of this subdivision as defined in the proof of Lemma 3.18. If e has endpoints in the same branch, then as in the first cases of the aforementioned lemmas we can find in G' a K_4-subdivision with fewer vertices and thus a proper K_4-subdivision in G. In the case where the endpoints of e belong to the interior of two different branches, then it is easy to see that G is a prism-subdivision. Let $\{x, y, z, t\}$ be the non-subdivision vertices of the K_4-subdivision. Finally, let us assume that the one endpoint of e is a non-subdivision vertex, say x, and the other one, that we call u, is a subdivision vertex of a branch, say $B_{y,z}$ (using the same notation as in the proof of Lemma 3.18). If X is set set of interior vertices of one of $B_{x,y}$, $B_{x,z}$, or $B_{x,t}$, then the graph $G \setminus X$ has a K_4-subdivision of non-subdivision vertices x, u, z, t, x, y, y, t or x, y, u, z respectively. In this case G has a proper K_4-subdivision. If none of $B_{x,y}$, $B_{x,z}$, and $B_{x,t}$ has internal vertices, then G is a wheel of center x.

In all the possible cases we reached the contradiction we were looking for. This concludes the proof. □

We are now ready to prove Theorem 3.17.

Proof of Theorem 3.17. Let $G \in \mathrm{Excl}_{\leq_{\mathrm{im}}}(\hat{K}_4)$ be a 2-connected graph. If G does not contain a K_4-subdivision, then the theorem is trivially true for G. If graph G contains a K_4-subdivision but not a proper one, from Lemma 3.20 we get that G is a subdivision of one of K_4, $K_{3,3}$, or the prism, in which case the theorem holds, or that G is a wheel, which has a trivial cycle-multipartite decomposition, with the center being the multipartite part.

Finally, let us assume that G contains a proper K_4-subdivision. By Lemma 3.12, G contains a 3-wheel. Let C be the cycle of a minimum 3-wheel in G and R the subgraph of G induced by $V(G) \setminus V(C)$. According to Corollary 3.6, if C is not an induced cycle, then G is a subdivision of the prism. When C is induced, then by Lemma 3.17, R is complete. Furthermore, vertices of R have the same neighborhood on C, as proved in Lemma 3.14 and Lemma 3.15. Therefore, (C, R) is a cycle-multipartite decomposition of G and we are done. $\square$

From a decomposition theorem to well-quasi-ordering

This section is devoted to the proof of Theorem 3.19.

The proof relies on the two following lemmas which are proved in the next subsections.

Lemma 3.21. *For every (unlabeled) graph G and every wqo $(S, \preceq)$, the class of $(S, \preceq)$-labeled G-subdivisions is well-quasi-ordered by the contraction relation.*

Lemma 3.22. *For every wqo $(S, \preceq)$, the class of $(S, \preceq)$-labeled graphs having a cycle-multipartite decomposition is well-quasi-ordered by induced minors.*

Proof of Theorem 3.19. The class of graphs not containing K_4 as minor (or, equivalently, as induced minor) has been shown to be well-quasi-ordered by induced minors in [Tho85], cf. Theorem 3.8. According to Remark 3.2, we can then restrict our attention to graphs of $\mathrm{Excl}_{\leq_{\mathrm{im}}}(\hat{K}_4)$ that contain K_4 as minor. As some of these graphs might not be 2-connected, we use Lemma 3.2: it is enough to show that for every wqo $(S, \preceq)$, the class of $(S, \preceq)$-labeled 2-connected graphs containing K_4 as minor are wqo by induced minors. By Theorem 3.17, this class can be divided into two subclasses:

- (2-connected) subdivisions of a graph among K_4, $K_{3,3}$, and the prism;
- graphs having a cycle-multipartite decomposition.

Lemma 3.21 and Lemma 3.22 handle these two cases, hence by Remark 3.2 the class of $(S, \preceq)$-labeled 2-connected graphs containing K_4 as minor are wqo by induced minors for every wqo $(S, \preceq)$. This concludes the proof. $\square$

The following subsections contains the proofs of Lemma 3.21 and Lemma 3.22, that follow the steps described in Section 3.3.

Well-quasi-ordering subdivisions

Let $\mathcal{OP}$ denote the class of paths whose endpoints are distinguished, i.e. one end is said to be the beginning and the other one the end. In the sequel, $\mathrm{fst}(P)$ denotes the first vertex of the path P and $\mathrm{lst}(P)$ its last vertex. We extend the relation $\leq_{\mathrm{im}}$ to $\mathcal{OP}$ as follows: for every $G, H \in \mathcal{OP}$, $G \leq_{\mathrm{im}} H$ if there in an induced minor model μ of G in H such that $\mathrm{fst}(H) \in \mu(\mathrm{fst}(G))$ and $\mathrm{lst}(H) \in \mu(\mathrm{lst}(G))$, and similarly for $\leq_{\mathrm{c}}$.

Lemma 3.23. *If the qoset $(Q, \preceq_Q)$ is a wqo, then the qoset $(\mathrm{lab}_{(Q,\preceq_Q)}(\mathcal{OP}), \leq_{\mathrm{c}})$ also is a wqo.*

Proof. Let us take $(S, \preceq)$ to be $(\mathcal{P}^{<\omega}(Q), \preceq_Q^{\mathcal{P}})$. By Corollary 3.2 $(S, \preceq)$ is a wqo. We consider the function $f\colon (S^\star, \preceq^\star) \to (\mathrm{lab}_{(S,\preceq)}(\mathcal{OP}), \leq_{\mathrm{c}})$ that, given a sequence $\langle s_1, \dots, s_k\rangle \in S^\star$ of elements of S, returns the path P on k vertices whose i-th vertex v_i is labeled by s_i for every $i \in [\![1, l]\!]$ and where $\mathrm{fst}(P) = v_1$ and $\mathrm{lst}(P) = v_k$. The image of this function is clearly $\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})$ and by Theorem 3.1 its domain is well-quasi-ordered by $\preceq^\star$. By the virtue of Remark 3.3, it is thus enough to show that $f\colon (S^\star, \preceq^\star) \to (\mathrm{lab}_{(S,\preceq)}(\mathcal{OP}), \leq_{\mathrm{c}})$ is monotone in order to prove that $(\mathrm{lab}_{(S,\preceq)}(\mathcal{OP}), \leq_{\mathrm{c}})$ is wqo.

Let $R = \langle r_1, \dots, r_k\rangle, S = \langle s_1, \dots, s_l\rangle \in S^\star$, be two sequences such that $R \preceq^\star S$ and let us show that $f(R) \leq_{\mathrm{c}} f(S)$. We will use the following notation: $f(R)$ is the path $v_1 \dots v_k$ labeled by λ_R and similarly for $f(S)$, $u_1 \dots u_l$ and λ_S. Let $\varphi\colon [\![1, k]\!] \to [\![1, l]\!]$ be an increasing function such that $\forall i \in [\![1, k]\!]$, we have $r_i \preceq s_{\varphi(i)}$ (such a function exists since $R \preceq^\star S$). Let us consider the path obtained from $f(S)$ by, for every $i \notin \{\varphi(j)\}_{j \in [\![1,k]\!]}$, contracting the label of u_i to the empty set and then dissolving u_i. Remark that this graph is a path on k vertices $p_1 p_2 \dots p_k$ such that $\forall i \in [\![1, k]\!], \lambda_R(v_i) = r_i \preceq \lambda_S(p_i) = s_{\varphi(i)}$. Furthermore, this path is a contraction of $f(S)$ where either $u_1 = p_1$ (respectively $u_l = p_k$) or this vertex has been contracted to p_1 (respectively p_k), hence $f(R) \leq_{\mathrm{c}} f(S)$, as desired. ☐

Proof of Lemma 3.21. Let G be a non labeled graph, let $(S, \preceq)$ be a wqo and let $\mathcal{G}$ be the class of all $(S, \preceq)$-labeled G-subdivisions. We set $m = |E(G)|$. Let us show that $(\mathcal{G}, \leq_{\mathrm{c}})$ is a wqo. First, we arbitrarily choose an orientation to every edge of G and an enumeration $e_1, \dots, e_m$ of these edges. We now consider the function f that, given a tuple $(Q_1, \dots, Q_m)$ of m paths of $\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})$, returns the graph constructed from G by, for every $i \in [\![1, m]\!]$, replacing the edge e_i by the path Q_i, while respecting the orientation, i.e. the first (respectively last) vertex of Q_i goes to the first (respectively last) vertex of e_i. By Lemma 3.1 on Cartesian products of wqos and since $(\mathrm{lab}_{(S,\preceq)}(\mathcal{OP}), \leq_{\mathrm{c}})$ is a wqo (Lemma 3.23), the domain $\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^m$ of f is well-quasi-ordered by $\leq_{\mathrm{c}}{}^m$. Notice that every element of the codomain of f is an G-subdivision (by definitions of f), and moreover that f is surjective on $\mathcal{G}$: for every $(S, \preceq)$-labeled G-subdivision H we can construct a tuple $(Q_1, \dots, Q_m)$ of m paths of $\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})$, such that $f(Q_1, \dots, Q_m) = H$.

In order to show that $(\mathcal{G}, \leq_{\mathrm{c}})$ is a wqo, it is enough to prove that $f\colon (\mathrm{lab}_{(S,\preceq)}(\mathcal{OP}), \leq_{\mathrm{c}}{}^m) \to (\mathcal{G}, \leq_{\mathrm{c}})$ is an epi, as explained in Remark 3.3, that is, to prove that for every two tuples $\mathcal{Q}, \mathcal{R} \in \mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^m$ such that $\mathcal{Q} \leq_{\mathrm{c}}{}^m \mathcal{R}$, we have $f(\mathcal{Q}) \leq_{\mathrm{c}} f(\mathcal{R})$. According to Remark 3.4, we only need to care, for every $i \in [\![1, m]\!]$, of the case where $\mathcal{Q}$ and $\mathcal{R}$ only differs by the i-th coordinate. It is at this point of the proof important to remark the symmetry of the definition of f: since the different coordinates any element of the domain of f are playing the same role, we only have to deal with the case where $\mathcal{Q}$ and $\mathcal{R}$ differs by one (fixed) coordinate, say the first one. Therefore, let us consider two tuples $\mathcal{Q} = (Q, Q_2, \dots, Q_m)$ and $\mathcal{R} = (R, Q_2, \dots, Q_m)$ of $\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^m$ such that $\mathcal{Q} \leq_{\mathrm{c}}{}^m \mathcal{R}$, i.e. satisfying $Q \leq_{\mathrm{c}} R$. Let $\mu\colon V(Q) \to \mathcal{P}^{<\omega}(V(R))$ be a contraction model of Q in R and let $\mu' \colon V(f(\mathcal{Q})) \to \mathcal{P}^{<\omega}(V(f(\mathcal{Q})))$ be the trivial contraction model of $f(\mathcal{Q}) \setminus V(Q)$ in itself defined by $\forall u \in V(f(\mathcal{Q})) \setminus V(Q),\ \mu'(u) = \{u\}$. We now consider the function $\nu \colon V(f(\mathcal{Q})) \to \mathcal{P}^{<\omega}(V(f(\mathcal{R})))$ defined as follows:

$$\nu \colon \begin{cases} V(f(\mathcal{Q})) & \to & \mathcal{P}^{<\omega}(V(f(\mathcal{R}))) \\ u & \mapsto & \mu(u) \quad \text{if } u \in V(Q) \\ u & \mapsto & \mu'(u) \quad \text{otherwise.} \end{cases}$$

Let us show that ν is a contraction model of $f(\mathcal{Q})$ in $f(\mathcal{R})$. First, notice that since both μ and μ' are contraction models, ν inherits some of their properties: for every $u \in V(f(\mathcal{Q}))$, the induced subgraph $V(f(\mathcal{R}))[u]$, is connected and $\lambda_{f(\mathcal{Q})}(u) \subseteq \bigcup_{u \in \mu(u)} \lambda_{f(\mathcal{R})}(u)$. For the same reason, we have:

$$\begin{aligned} \bigcup_{u \in V(f(\mathcal{Q}))} \nu(u) &= \bigcup_{u \in V(Q)} \mu(u) \cup \bigcup_{u \in V(f(\mathcal{Q}) \setminus V(Q))} \mu'(u) \\ &= V(R) \cup V(f(\mathcal{Q})) \setminus V(Q) \\ &= V(f(\mathcal{R})). \end{aligned}$$

Let us now consider two distinct vertices u and v of $f(\mathcal{Q})$.
First case: u and v both belong to the same set among $V(Q)$ and $V(f(\mathcal{Q})) \setminus V(Q)$. In this case $\nu(u)$ and $\nu(v)$ are disjoint and they are adjacent iff $\{u, v\} \in E(f(\mathcal{Q}))$ since both μ and μ' are contraction models.
Second case: $u \in V(Q)$ and $v \in V(f(\mathcal{Q})) \setminus V(Q)$ (or the symmetric case). As in the previous case, $\nu(u)$ and $\nu(v)$ are disjoint. Assume that $\{u, v\}$ is an edge of $f(\mathcal{Q})$. Notice that we necessarily have either $u = \operatorname{fst}(Q)$ and $v \in N_{f(\mathcal{Q}) \setminus V(Q)}(\operatorname{fst}(Q))$, or $u = \operatorname{lst}(Q)$ and $v \in N_{f(\mathcal{Q}) \setminus V(Q)}(\operatorname{lst}(Q))$. Let us assume, without loss of generality, that we are in the first of these two subcases. By definition of $f(\mathcal{R})$, $\{\operatorname{fst}(R), v\}$ is an edge. Since μ is a contraction model, we then also have $\operatorname{fst}(R) \in \nu(u)$ and therefore $\nu(v)$ and $\nu(u)$ are adjacent in $f(\mathcal{R})$.

We just proved that ν is an induced minor model of $f(\mathcal{Q})$ in $f(\mathcal{R})$. As explained above, this is enough in order to show that f is monotone with regard to ${\leq_{\mathrm{c}}}^m, \leq_{\mathrm{c}}$. Hence $(\mathcal{G}, \leq_{\mathrm{c}})$ is a wqo and this concludes the proof. □

Well-quasi-ordering cycle-multipartite decompositions

In this section, we show that graphs having a cycle-multipartite decomposition are well-quasi-ordered by induced minors.

Lemma 3.24. *If $(Q, \preceq_Q)$ is wqo then the class of $(Q, \preceq_Q)$-labeled independent sets is wqo by the induced subgraph relation.*

Proof. We will again define $(S, \preceq) := (\mathcal{P}^{<\omega}(Q), \preceq_Q^{\mathcal{P}}$, and observe that it is a wqo.

The function f that maps every sequence $\langle x_1, \dots, x_k \rangle$ (for some positive integer k) of elements of S to the $(S, \preceq)$-labeled independent set on vertex set $\{v_1, \dots, v_k\}$ where v_i have label x_i for every $i \in [\![1, k]\!]$ has clearly the class of $(S, \preceq)$-labeled independent sets as codomain. Let us show that f is an epi. Let $X = \langle x_1, \dots, x_k \rangle, Y = \langle x_1, \dots, x_l \rangle \in$

$S^\star$ be two sequences such that $X \preceq^\star Y$. By definition of the relation $\preceq^\star$, there is an increasing function $\varphi\colon [\![1,k]\!] \to [\![1,l]\!]$ such that $\forall i \in [\![1,k]\!],\ x_i \preceq y_{\varphi i}$. Therefore the function $\mu\colon V(f(X)) \to V(f(Y))$ that maps the vertex v_i of $f(X)$ to the singleton $\{v_{\varphi(i)}\}$ of $f(Y)$ is an induced subgraph model of $f(X)$ in $f(Y)$ and this proves the monotonicity of f with regard to $\preceq^\star, \leq_{\mathrm{im}}$. By the virtue of Remark 3.3 and since $(S^\star, \preceq^\star)$ is a wqo, we get that the class of $(S,\preceq)$-labeled independent sets is wqo by the induced subgraph relation. □

Corollary 3.7. *With a very similar proof, we can also show that if $(S,\preceq)$ is wqo then the class of $(S,\preceq)$-labeled cliques is wqo by the induced subgraph relation.*

Corollary 3.8. *If a class of $(S,\preceq)$-labeled graphs $(\mathcal{G},\leq_{\mathrm{im}})$ is wqo, then so is its closure by finite disjoint union (respectively join).*

Proof. Let $\mathcal{U}$ be the closure of $(\mathcal{G},\leq_{\mathrm{im}})$ by disjoint union. Remark that every graph of $\mathcal{U}$ can be partitioned in a family of pairwise non-adjacent graphs of $\mathcal{G}$. Therefore we can define a function mapping every $\mathcal{G}$-labeled independent set to the graph of $\mathcal{U}$ obtained from G by replacing each vertex by its label (which is an $(S,\preceq)$-labeled graph). It is easy to check that this function is an epi of $(\mathcal{G},\leq_{\mathrm{im}}) \to (\mathcal{U},\leq_{\mathrm{im}})$. Together with Remark 3.3 and Lemma 3.24, this yields the desired result. □

Corollary 3.9. *If $(S,\preceq)$ is a wqo then the class of $(S,\preceq)$-labeled complete multipartite graphs are wqo by the induced subgraph relation.*

Proof of Lemma 3.22. We consider the function $f\colon (\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^\star \times \mathrm{lab}_{(S,\preceq)}(\mathcal{K}_{\mathbb{N}^\star}), \leq_{\mathrm{c}}{}^\star \times \leq_{\mathrm{isg}}) \to (\mathrm{lab}_{(S,\preceq)}(\mathcal{W}), \leq_{\mathrm{im}})$ that, given a sequence $[R_0, \dots, R_{k-1}] \in \mathrm{lab}_{(S,\preceq)}(\mathcal{OP})$ of $(S,\preceq)$-labeled paths of $\mathcal{OP}$ and a $(S,\preceq)$-labeled complete multipartite graph K, returns the graph constructed as follows.

1. consider the disjoint union of K and the paths of $\{R_i\}_{i\in[\![0,k-1]\!]}$ and call v_i the vertex obtained by identifying the two vertices $\mathrm{lst}(R_i)$ and $\mathrm{fst}(R_{(i+1) \bmod k})$, for every $i \in [\![0,k-1]\!]$ (informally, this graph is the disjoint union of K and the cycle built by putting R_i's end-to-end);
2. for every element v of $\{v_i\}_{i\in[\![0,k-1]\!]}$, add all possible edges between v and the vertices of K.

Remark that the codomain of f is $\mathcal{W}$. Indeed, every element of the image of f has a cycle-multipartite decomposition (by construction) and conversely, if $G \in \mathcal{W}$ is of cycle-multipartite decomposition (C,K), one can construct a sequence of $R_0, \dots R_k$ of subpaths of C meeting only on endpoints and whose interior vertices are of degree two such that $G = f(R_, \dots R_{k-1}, K)$. Let us show that the domain of f is well-quasi-ordered by $\leq_{\mathrm{c}}{}^\star \times \leq_{\mathrm{isg}}$. We proved in Lemma 3.23 that $(\mathrm{lab}_{(S,\preceq)}(\mathcal{OP}), \leq_{\mathrm{c}})$ is a wqo and Corollary 3.9 shows that $(\mathcal{K}_{\mathbb{N}^\star}, \leq_{\mathrm{isg}})$ is a wqo, so by applying Theorem 3.1 we get first that $(\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^\star, \leq_{\mathrm{c}}{}^\star)$ is a wqo, and then by Lemma 3.1 together with Corollary 3.9 that $(\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^\star \times \mathcal{K}_{\mathbb{N}^\star}, \leq_{\mathrm{c}}{}^\star \times \leq_{\mathrm{isg}})$ is a wqo.

According to Remark 3.3, it is enough to show that the function

$$f\colon (\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^{\star} \times \mathrm{lab}_{(S,\preceq)}(\mathcal{K}_{\mathbb{N}^{\star}}), \leq_{\mathrm{c}}{}^{\star} \times \leq_{\mathrm{isg}}) \to (\mathcal{W}, \leq_{\mathrm{im}})$$

is an epi in order to prove that $(\mathrm{lab}_{(S,\preceq)}(\mathcal{W}), \leq_{\mathrm{isg}})$ is wqo. We show the monotonicity of f in two steps: the first by showing

$$\forall R \in \mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^{\star},\ \forall H, H' \in \mathrm{lab}_{(S,\preceq)}(\mathcal{K}_{\mathbb{N}^{\star}}),\ H \leq_{\mathrm{isg}} H' \Rightarrow f(R,H) \leq_{\mathrm{im}} f(R,H')$$

and the second by proving

$$\forall Q, R \in \mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^{\star},\ \forall H \in \mathrm{lab}_{(S,\preceq)}(\mathcal{K}_{\mathbb{N}^{\star}}),\ Q \leq_{\mathrm{c}}{}^{\star} R \Rightarrow f(Q,H) \leq_{\mathrm{im}} f(R,H).$$

According to Remark 3.4, the desired result follows from these two assertions.
First step. Let $R = \langle R_o, \dots, R_{k-1}\rangle \in \mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^{\star}$ and $H, H' \in \mathrm{lab}_{(S,\preceq)}(\mathcal{K}_{\mathbb{N}^{\star}})$ be such that $H \leq_{\mathrm{isg}} H'$. We therefore have $(R,H) \leq_{\mathrm{c}}{}^{\star} \times \leq_{\mathrm{isg}} (R,H')$. Let us show that $f(R,H) \leq_{\mathrm{im}} f(Q,H')$. Since $H \leq_{\mathrm{isg}} H'$, there is a subset $A \subseteq V(H')$ such that $H' \backslash A = H$. Let C denote the cycle obtained from the disjoint union of $\{R_i\}_{i \in [\![0,k-1]\!]}$ by the identification of the two vertices $\mathrm{lst}(R_i)$ and $\mathrm{fst}(R_{(i+1) \bmod k})$, for every $i \in [\![0,k-1]\!]$, where we call v_i the vertex resulting from this identification. Let us consider the graph $f(Q,H)\backslash A$: to construct this graph we started with the disjoint union of C and H, then added all possible edges between v_i and $V(H)$ for every $i \in [\![0,k-1]\!]$, and at last deleted the vertices of $A \setminus H$. Remark that this graph is isomorphic to $f(R,H)$ and therefore $f(R,H) \leq_{\mathrm{im}} f(R,H')$, as desired.
Second step. Let $Q = \langle Q_0, \dots, Q_{k-1}\rangle$ and $R = \langle R_0, \dots, R_{l-1}\rangle$ be two elements of $\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})^{\star}$ such that $Q \leq_{\mathrm{c}}{}^{\star} R$ and let $H \in \mathrm{lab}_{(S,\preceq)}(\mathcal{K}_{\mathbb{N}^{\star}})$. We thus have $(Q,H) \leq_{\mathrm{c}}{}^{\star} \times \leq_{\mathrm{isg}} (R,H)$. Let us show that $f(Q,H) \leq_{\mathrm{im}} f(R,H)$. By definition of the relation $\leq_{\mathrm{c}}{}^{\star}$, there is an increasing function $\varphi\colon [\![0,k-1]\!] \to [\![0,l-1]\!]$ such that

$$\forall i \in [\![0,k-1]\!],\ Q_i \leq_{\mathrm{c}} R_{\varphi(i)}.$$

For every $i \in [\![0,k-1]\!]$, let $\mu_i\colon V(Q_i) \to \mathcal{P}^{<\omega}(R_{\varphi(i)})$ be a contraction model of Q_i in R_i. Recall that since Q_i and $R_{\varphi(i)}$ are oriented paths, the contraction sending $R_{\varphi(i)}$ on Q_i preserves endpoints. We now consider the function μ defined as follows

$$\mu \left\{ \begin{array}{rcll} V(f(Q,H)) & \to & \mathcal{P}^{<\omega}(V(f(R,H))) & \\ x & \to & \{x\} & \text{if } x \in V(H) \\ \mathrm{lst}(Q_i) & \to & \mu_i(\mathrm{lst}(Q_i)) \cup \bigcup_{j=\varphi(i)+1}^{\varphi(i+1)-1} V(R_j) \setminus \{\mathrm{fst}(R_{\varphi(i+1)})\} & \\ x & \to & \mu_i(x) \subseteq R_{\varphi(i)} & \text{if } x \in Q_i \setminus \{\mathrm{lst}(Q_i)\}. \end{array} \right.$$

We will show that μ is an induced minor model of $f(Q,H)$ is $f(R,H)$. First at all, remark that every element of the image of f induces in $f(R,H)$ a connected subgraph:

- either $x \in V(H)$ and $\mu(x)$ is a singleton;
- or $x \in \mathrm{lst}(Q_i) \setminus \{\mathrm{lst}(Q_i)\}$ and $f(R,H)[\mu(x)]$ is connected since $\mu_i(x) = \mu(x)$ is an induced minor model;

- or $x = \mathrm{lst}(Q_i)$ and $\mu_i(\mathrm{lst}(Q_i)) \cup \bigcup_{j=\varphi(i)+1}^{\varphi(i+1)-1} V(R_j) \setminus \{\mathrm{fst}(R_{\varphi(i+1)})\}$ induces a in $f(R, H)$ a connected subgraph because $f(R, H)[\mu_i(\mathrm{lst}(Q_i))]$ is connected and the other vertices are consecutive on the cycle.

Let us now show that adjacencies are preserved by μ. Let u, v be two distinct vertices of $f(Q, H)$. If $u, v \in H$, then $\mu(u)$ and $\mu(v)$ are adjacent in $f(R, H)$ iff u and v are in $f(Q, H)$, as $\mu(u) = \{u\}$ and $\mu(v) = \{v\}$ (informally, the "H-part" of $f(R, H)$ is not changed by the model). If $u, v \in Q$, observe that u and v are adjacent in $f(Q, H)$ iff they belong to the same path of $\{Q_i\}_{i \in [\![0,k-1]\!]}$. Thus in this case, the property that u and v are adjacent in $f(Q, H)$ iff $\mu(u)$ is adjacent to $\mu(v)$ in $f(R, H)$ is given by the fact that $\{\mu_i\}_{i \in [\![0,k-1]\!]}$ are contraction models.

If $u \in Q$ and $v \in H$, then $\{u, v\} \in f(Q, H)$ (resp. $\mu(u)$ is adjacent to $\mu(v)$ in $f(R, H)$) iff u is an endpoint of a path of $\{Q_i\}_{i \in [\![0,k-1]\!]}$ (resp. $\mu(u)$ contains an endpoint of a path of $\{R_i\}_{i \in [\![0,l-1]\!]}$), by definition of f. As the contraction relation on oriented paths of $\mathcal{OP}$ is required to contract endpoints to endpoints, the image $\mu(u)$ must contain the endpoint of a path of $\{R_i\}_{i \in [\![0,l-1]\!]}$ iff u is the endpoint of a path of $\{Q_i\}_{i \in [\![0,k-1]\!]}$. Therefore u and v are adjacent in $f(Q, H)$ iff $\mu(u)$ is adjacent to $\mu(v)$ in $f(R, H)$, as required. We finally proved that f is monotone with regard to $\leq_c^\star \times \leq_{\mathrm{isg}}, \leq_{\mathrm{im}}$. This was the only missing step in order to prove that $(\mathrm{lab}_{(S,\preceq)}(\mathcal{W}), \leq_{\mathrm{isg}})$ is a wqo. $\square$

3.5.3 Graphs not containing Gem

The purpose of this section to give a proof to Theorem 3.20. This will be done by first proving a decomposition theorem for graphs of $\mathrm{Excl}_{\leq_{\mathrm{im}}}(\mathrm{Gem})$, and then using this theorem to prove that $(\mathrm{Excl}_{\leq_{\mathrm{im}}}(\mathrm{Gem}), \leq_{\mathrm{im}})$ is a wqo.

A Decomposition theorem for $\mathrm{Excl}_{\leq_{\mathrm{im}}}(\mathrm{Gem})$

This section is devoted to the proof of Theorem 3.18, which is split in several lemmas. In the sequel, G is a 2-connected graph of $\mathrm{Excl}_{\leq_{\mathrm{im}}}(\mathrm{Gem})$. When G is 3-connected, we will rely on the following result originally proved by Ponomarenko.

Proposition 3.1 ([Pon91]). *Every 3-connected* Gem*-induced minor-free graph is either a cograph or has an induced subgraph S isomorphic to P_4, such that every connected component of $G \setminus S$ is a cograph.*

Therefore we will here focus on the case where G is 2-connected but not 3-connected. A *rooted diamond* is a graph which can be constructed from a rooted C_4 by adding a chord incident with exactly one endpoint of the root (cf. Figure 3.14).

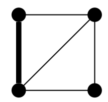


Figure 3.14: A rooted diamond, the root being the thick edge.

Lemma 3.25. *Let $S = \{v_1, v_2\}$ be a cutset in G and let C be a component of $G \setminus S$. Let H be the graph $G[V(C) \cup \{v_1, v_2\}]$ rooted at $\{v_1, v_2\}$. If C' has a rooted diamond as an induced minor, then $G \leq_{\text{im}} \text{Gem}$.*

Proof. Let C' be a component of $G \setminus S$ other than C and let G' be the graph obtained from G by:

1. applying the necessary operations (contractions and vertex deletions) to transform $G[V(C) \cup \{v_1, v_2\}]$ into a rooted diamond;

2. deleting every vertex not belonging to $V(C) \cup V(C') \cup \{v_1, v_2\}$;

3. contracting C' to a single vertex.

The graph G' is then a rooted diamond and a vertex adjacent to both endpoints of its root, that is, G' is isomorphic to Gem. □

Let us now characterize these 2-connected graphs avoiding rooted diamonds.

Lemma 3.26. *Let G be a graph rooted at $\{u, v\} \in E(G)$. If $\{u, v\}$ is not a cut of G and G does not contain a rooted diamond as an induced minor, then either G is an induced cycle or both u and v are dominating in G.*

Proof. Assuming that u is not dominating and G is not an induced cycle, let us prove that G contains a rooted diamond as induced minor. Let $w \in V(G)$ be a vertex such that $\{u, w\} \notin E(G)$. Such a vertex always exists given that u is not dominating. Let C be a shortest cycle using the edge $\{u, v\}$ and the vertex w (which exists since G is 2-connected), let P_u be the subpath of C linking u to w without meeting v and similarly let P_v be the subpath of C linking v to w without meeting u. By the choice of C, both P_u and P_v are induced paths. Notice that if there is an edge connecting a vertex of $P_u \setminus \{w\}$ to vertex of $P_v \setminus \{w\}$, then G contains a rooted diamond. Therefore we can now assume that C is an induced cycle.

Recall that we initially assumed that G is not an induced cycle. Therefore G contains a vertex not belonging to C. Let G' be the graph obtained from G by contracting to one vertex x any connected component of $G \setminus C$ and deleting all the other components. Obviously we have $G' \leq_{\text{im}} G$. Let us show that G' contains a rooted diamond as induced minor.

Remark that the neighborhood of x, which is of size at least two (as G is 2-connected), is not equal to $\{u, v\}$, otherwise $\{u, v\}$ would be a cut in G. Now contract in G' all the edges of $C \setminus \{u, v\}$ except three in a way such that $|N(x)| \geq 2$ and $N(x) \neq \{u, v\}$. Let G'' be the obtained graph, which consists of a cycle of length four rooted at $\{u, v\}$ and a vertex x adjacent to at least two vertices of this cycle. We shall here recall that since this cycle is a contraction of the induced cycle C, it is induced too. If x is adjacent (among others) to two vertices at distance two on this cycle, then, by contracting the edge between x and one of these vertices, we get a rooted diamond. The remaining case is when x is only adjacent to the vertices of the cycle which are not u and v. The contraction of the edge between v and one of these vertices gives a rooted diamond, and this concludes the proof. □

Remark 3.10. In a Gem-induced minor-free graph G, every induced subgraph H dominated by a vertex $v \in V(G) \setminus V(H)$ is a cograph.

Indeed, assuming that H is not a cograph, let P be a path on four vertices which is subgraph of H. Then $G[V(P) \cup \{v\}]$ is isomorphic to Gem, a contradiction.

Lemma 3.27. *If G has a K_2-cut $S = \{v_1, v_2\}$, then every connected component of $G \setminus S$ is basic in G.*

Proof. By Lemma 3.25, for every connected component C of $G \setminus S$ we know that the graph $G[V(C) \cup S]$ rooted at $\{u, v\}$ contains no rooted diamond. By the virtue of Lemma 3.26, this graph either is an induced cycle or has a dominating vertex among u and v. In the first case, C is a path whose all internal vertices are of degree two in G, hence H is basic. If one of u and v is dominating, then C is a cograph according to Remark 3.10. Therefore in both cases C is basic in G. □

Let us now focus on 2-connected graphs with a $\overline{K_2}$-cut, which is the last case in our characterization theorem.

Corollary 3.10. *If G has a $\overline{K_2}$-cut S such that $G \setminus S$ contains more than two connected components, then every connected component of $G \setminus S$ is basic in G.*

Proof. It follows directly from Lemma 3.27. Indeed, if the connected components of $G \setminus S$ are $C_1, C_2, \dots C_k$, let us contract C_1 to an edge between the two vertices of S. The obtained graph fulfills the assumptions of Lemma 3.27: S is a K_2-cut. Therefore each of the components $C_2, \dots, C_k$ is basic in G. Applying the same argument with C_2 instead of C_1 yields that C_1 is basic in G as well. □

Lemma 3.28. *Let $S = \{u, v\}$ be a $\overline{K_2}$-cut, such that and $G \setminus S$ has only two connected components H_1 and H_2. Then G contains a cycle C as induced subgraph such that every connected component of $G \setminus C$ is basic in G.*

Proof. For every $i \in \{1, 2\}$, let Q_i be a shortest path linking u to v in $G[V(H_i) \cup \{u, v\}]$. Notice that the cycle $C = G[V(Q_1) \cup V(Q_2)]$ is then an induced cycle. For contradiction, let us assume that some connected component J of $G[V \setminus C]$ is not basic in G. By symmetry, we can assume that $J \subset H_1$.

Notice that since G is 2-connected, J has at least two distinct neighbors x, y on C. Let G' be the graph obtained from $G[V(H_1) \cup V(C) \cup V(H_2)]$ by contracting Q_1 to an edge between u and v in a way such that x is not contracted to y (that is, x is contracted to one of u, v and y to the other one). In G', $\{u, v\}$ is a K_2-cut, therefore by Lemma 3.27, every connected component of $G \setminus S$ is basic in G'. As this consequence holds for every choice of J and G' is an induced minor of G, we eventually get that every connected component of $G \setminus C$ is basic in G. □

In the sequel, S, u, v and C follow the definitions of the statement of Lemma 3.28. In order to be more accurate on how the connected components of $G \setminus C$ are connected to C, we will prove the following lemma according to which most of the vertices of C have degree 2 in G. Let us assume that

Remark 3.11. Every connected component J of $G \setminus C$ has at least two and at most three neighbours on C.

Indeed, it has at least two neighbours on C because G is 2-connected. Besides if J has at least four neighbours on C, then contracting in $G[V(C) \cup V(J)]$ the component J to a single vertex, deleting a vertex of C not belonging to $N(J)$ (which exists since J belongs to only one of the components of $G \setminus S$) and then contracting every edge incident with a vertex of degree two would yield Gem.

Lemma 3.29. *If C has at least one vertex of degree two, then for every distinct connected components J_1 and J_2 of $G \setminus C$ we have $N_C(J_1) \subseteq N_C(J_2)$ or $N_C(J_2) \subseteq N_C(J_1)$.*

Proof. Let us assume, for contradiction, that the claim is not true and let G be a minimal counterexample with respect to induced minors. In such a case both J_1 and J_2 are single vertices (say j_1 and j_2 respectively) and they are the only connected components of $G \setminus C$. We now would like to argue that any such minimal counterexample must contain as induced minor one of graphs presented on Figure 3.15 (where thick edges represent the cycle C). This would conclude the proof as each of these graphs contains Gem as induced minor, as shown in Figure 3.16.

First of all, in such a minimal counterexample there is only one vertex in C of degree 2, let us call it c. We will consider all the ways that the vertices j_1 and j_2 can be connected to the neighbors of c and show that in every such case we can contract our graph to one of the graphs on Figure 3.15. According to Remark 3.11, each of j_1 and j_2 will have either two or three neighbors on C.
First case: both j_1 and j_2 are connected with both neighbours of c. As $N(j_1) \nsubseteq N(j_2)$ and $N(j_2) \nsubseteq N(j_1)$, each of j_1, j_2 has a neighbor which is not adjacent to the other. But since j_1 and j_2 can have at most three neighbors, the neighborhood of j_1 and j_2 is now completely characterized. The leftmost part of Figure 3.15 presents the only possible graph for this case.

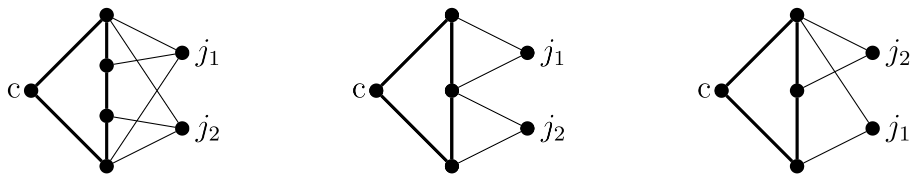


Figure 3.15: Induced minor-minimal counterexamples in Lemma 3.29.

Second case: j_1 is connected with exactly one of neighbours of c and j_2 is connected with the other one. In this case, as each of j_1, j_2 has at least two neighbors on C, contracting all the edges of C whose both endpoints are at distance at least two from c gives the graph depicted in the center of Figure 3.15.
Third case: j_1 is connected with both neighbours of c, and j_2 is connected with at most one of them. In this case, as long as C has more than 4 edges, we can contract an edge of C to find a smaller counterexample. Precisely, if there are at least 4 edges, there are two edges e_1, e_2 in C within distance exactly 1 to c and those two do not share an endpoint. Moreover j_2 has a neighbour s in $C \setminus N(c)$, say x, which is not a neighbour

of j_1. Now one of the edges e_1, e_2 is not incident to s and contracting this edge would yield a smaller counterexample.

Therefore, we only have to care about the case where C has exactly 4 edges and this case is exactly the graph represented on the right of Figure 3.15.

We have considered all possible induced minor-minimal counterexamples (up to symmetry between j_1 and j_2). For each of these cases, which are presented on Figure 3.15, we will now give an induced minor model of Gem, which proves that they all contain Gem as induced minor. For each graph of depicted on Figure 3.16 we consider the model mapping the vertex v_i of Gem to the set of vertices labeled M_i. It is easy to check that each of these sets induces a connected subgraph and that the adjacencies between two sets correspond to the ones between the corresponding vertices of Gem. This concludes the proof. □

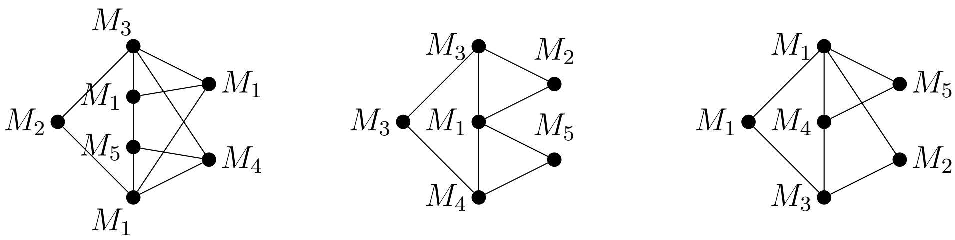


Figure 3.16: Models of Gem in graphs from Figure 3.15.

Corollary 3.11. *If C has at least one vertex of degree two, then it has at most three vertices of degree greater than two.*

Proof. Notice that the set of vertices of C that have degree greater then two is exactly the union of $N_C(J)$ over all connected components J of $G \setminus C$. We just saw in Lemma 3.29 that for every two connected components of $G \setminus C$, the neighborhood on C of one is contained in the neighborhood on C of the other and that these neighborhoods have size at most three. Therefore their union have size at most three as well. □

Corollary 3.12. *Every connected component of $G \setminus C$ is basic and C has at most six vertices of degree greater than two.*

Proof. Remark that contracting H_1 to a single vertex h in G gives a graph G' and a cycle C' (contraction of C) such that every connected component of $G' \setminus C'$ is basic and C' has at least one vertex of degree 2, h. By Corollary 3.11 C' has at most three vertices of degree greater than two. Notice that these vertices belong to $G' \setminus h$ which is isomorphic to $G \setminus H_1$. Hence $G \setminus H_1$ has at most three vertices of degree greater than two. Applying the same argument with H_2 instead of H_1 we get the desired result. □

Now we are ready to prove main decomposition theorem for Gem-induced minor-free graphs.

Proof of Theorem 3.18. Recall that we are looking for a subset X of $V(G)$ of size at most 6 such that each component of $G \setminus X$ is basic in G.

If G is 3-connected, by Proposition 3.1 it is either a cograph, or has a subset X of four vertices such that every connected component of $G \setminus X$ is a cograph. Let us now assume that G is not 3-connected.

In the case where G has a K_2-cut S, or if G has a $\overline{K_2}$-cut S such that $G \setminus S$ has more than two connected components, then according to Lemma 3.27 and Corollary 3.10 respectively, S satisfies the required properties. In the remaining case, by Corollary 3.12 G has a cycle C such that every connected component of $G \setminus C$ is basic in G and which has at most six vertices of degree more than two in G. Let X be the set containing those vertices of degree more than two. Observe that every connected component of $G \setminus X$ is either a connected component of $G \setminus C$ (hence it is basic) or a part of C, i.e. a path whose internal vertices are of degree two in G (which is basic as well). As $|X| \leq 6$, X satisfies the desired properties. $\square$

Well-quasi-ordering of labelled cographs

We were able to show that structure of 2-connected Gem-induced minor-free graphs is essentially very simple, with building blocks being cographs and long induced paths. To conclude that labelled 2-connected Gem-induced minor-free graphs are wqo by induced minor relation we will need the fact that the building blocks, in particular labelled cographs, are themselves well-quasi-ordered by the induced minor relation.

This following result has been proven by Damaschke in [Dam90] in the unlabelled case. The proof for the labelled case follows the same general approach. We present below the sketch of the proof.

Let us denote $\mathcal{C}$ to be a class of all cographs.

Theorem 3.21. *For any wqo $(Q, \preceq_Q)$, the class $\mathrm{lab}_{(Q,\preceq_Q)}(\mathcal{C})$ is wqo with respect to $\leq_{\mathrm{isg}}$.*

Proof. Let us define as usual $(S, \preceq)$ to be $(\mathcal{P}^{<\omega} Q, \preceq^{\mathcal{P}})$. Define $(S^+, \preceq^+)$ such that S^+ is disjoint union of S and $\{0, 1\}$; the order $\preceq^+$ is such that $\preceq^+$ is just $\preceq$ when restricted to S, but $0, 1$ and elements of S are incomparable.

By virtue of Remark 3.2 and Corollary 3.2, we know that $(S^+, \preceq^+)$ is wqo. By the labelled version of the famous Kruskal theorem (see [Kru60]), the class of all finite trees labelled by $(S^+, \preceq^+)$ is wqo, with respect to a topological minor relation. In particular, we can consider class $\mathcal{T}$ of finite trees, labelled by $(S^+, \preceq^+)$, such that all internal nodes have labels $\{0, 1\}$, and all leaves has label from S. We will consider this class again with the ordering by a labelled topological minor relation. As it is a subclass of a wqo, the class $\mathcal{T}$ itself is also wqo. We will now provide a epi $\phi : \mathcal{T} \to \mathrm{lab}_{(Q,\preceq_Q)}(\mathcal{C})$, and we will conclude by Remark 3.3, that $\mathrm{lab}_{(Q,\preceq_Q)}(\mathcal{C})$ is wqo.

The function ϕ is defined as follows: given a labelled tree T, if the whole tree is only a single leaf, it produces a graph with a single vertex, and with the same label as the one the leaf has in T. If the tree is larger than a single vertex, it has root r with label s, and subtrees $T_1, T_2, \ldots, T_k$, all rooted at some children of r. Then $\phi(T)$ is defined as disjoint union of $\phi(T_i)$ if label s were 0, or join of $\phi(T_i)$ of label of s were 1. It is well-known, that every cograph has such a presentation, i.e. that function ϕ indeed is surjective. The tree T which is mapped G by ϕ is usually called *cotree* of G.

Now we only need to prove that ϕ is monotone. Indeed, consider two labelled trees $T_1, T_2 \in \mathcal{T}$, such that $T_1 \leq T_2$, and let $i : V(T_1) \mapsto V(T_2)$ to be an embedding of T_1 in T_2 as a topological minor, such that $\lambda_{T_1}(u) \preceq^+ \lambda_{T_2}(i(u))$. In particular, by the second property, we conclude that i maps leaves of T_1 to leaves of T_2. Therefore we can consider a corresponding mapping $\tilde{i}$ from $V(\phi(T_1))$ to $V(\phi(T_1))$. Clearly it is injective, and has the property that $\lambda_{\phi(T_1)}(u) \preceq \lambda_{\phi(T_2)}(\tilde{i}(u))$. To show that it defines an induced minor model of $\phi(T_1)$ as an induced subgraph of $\phi(T_2)$, we only need to prove that $\tilde{i}(u)$ and $\tilde{i}(v)$ are connected by an edge iff u and v are connected by an edge.

Remark that two vertices $u, v \in V(\phi(T))$ are connected by an edge in $\phi(T)$ iff the label of a lowest common ancestor of the corresponding leaves in T is 1. But i was an embedding of T_1 in T_2 as a topological minor, so in particular lowest common ancestor of $i(u)$ and $i(v)$ is the same as an image of lowest common ancestor of u, v. Moreover, by the definition of the order S^+, embedding i preserves exactly labels of internal nodes. Hence ϕ is indeed monotone and this concludes the proof of the theorem. □

Well-quasi-ordering Gem-induced minor-free graphs

In this section we will give a proof of Theorem 3.20.

We define $\mathcal{B}$ as the class of graphs which are disjoint unions of induced paths and cographs, and $\mathcal{C}$ as the class of cographs.

Lemma 3.30. *Let $k \in \mathbb{N}$, let $(S, \preceq)$ be a wqo and let $\mathcal{G}$ be the class of $(S, \preceq)$-labelled graphs such that the removal of at most k vertices yields a graph of $\mathcal{B}$. Then $(\mathcal{G}, \leq_{\mathrm{im}})$ is a wqo.*

Proof. Let k be fixed.

For every $G \in \mathcal{G}$, let X_G be a set of at most six vertices of G such that $G \setminus X \in \mathcal{B}$.

For every graph H on at most six vertices, let $\mathcal{G}_H = \{G \in \mathcal{G},\ G[X_G] = H\}$. Observe that this gives a partition of $\mathcal{G}$ into a finite number of subclasses. By the virtue of Remark 3.2, we only need to focus on one of these classes. For the sake of simplicity, we assume that H has exactly k vertices $\{v_1, \dots v_k\}$.

Informally, our goal is now to define a function f which constructs a graph of $\mathcal{G}_H$, given an encoding in terms of graphs of $\mathcal{B}$. We will then show that f is an epi.

Le f be the function whose domain is the quasi-order

$$(\mathcal{D}, \preceq_{\mathcal{D}}) = (S, \preceq)^k \times \left(\mathcal{P}^{<\omega}(\mathrm{lab}_{(S,\preceq)}(\mathcal{OP})), \leq_{\mathrm{c}}{}^{\star}\right)^{\binom{k}{2}} \times (\mathrm{lab}_{(S,\preceq)\times(2^{[\![1,k]\!]},=)}(\mathcal{C}), \leq_{\mathrm{isg}})$$

(where $\preceq_{\mathcal{D}} = \preceq^k \times (\leq_{\mathrm{c}}{}^{\star})^{\binom{k}{2}} \times \leq_{\mathrm{isg}}$) and which, given a tuple

$$\left((s_i)_{i\in[\![1,k]\!]}, (L_{i,j})_{i,j\in[\![1,k]\!], i<j}, J\right)$$

such that

- $(s_i)_{i\in[\![1,k]\!]} \in S^k$ is a tuple of k labels from S,

- $(L_{i,j})_{i,j \in [\![1,k]\!], i<j}$ is a tuple of $\binom{k}{2}$ subsets of $(S, \preceq)$-labeled oriented paths, and
- $J \in \mathrm{lab}_{(S,\preceq)\times(2^{[\![1,k]\!]},=)}(\mathcal{C})$, is a $(S, \preceq) \times (2^{[\![1,k]\!]}, =)$-labeled cograph,

returns the graph constructed as follows, starting from H:

1. label s_i the vertex v_i, for every $i \in [\![1, k]\!]$;
2. for every $i, j \in [\![1, k]\!]^2$, $i < j$, and for every path $L \in L_{i,j}$, add a copy of L to the current graph, connect v_i to $\mathrm{fst}(L)$ and v_j to $\mathrm{lst}(L)$;
3. add to the current graph a copy of the underlying graph of J and, for every vertex labeled $(s, \{e_1, \dots e_l\})$ (for some $l \in [\![1, k]\!]$), give the label s to the corresponding vertex in the current graph and make it adjacent to vertices $v_{e_1}, \dots, v_{d_l}$.

By construction, the codomain of f is included in $\mathrm{lab}_{(S,\preceq)}(\mathcal{G}_H)$. Let us now show that f is surjective on $\mathrm{lab}_{(S,\preceq)}(\mathcal{G}_H)$. Let $G \in \mathrm{lab}_{(S,\preceq)}(\mathcal{G}_H)$ and let us consider the connected components of $G \setminus X_G$. Let J be the disjoint union of all such components that are cographs. Note that J is a cograph as well. For every vertex v of J of label s, we relabel v with the label $(s, \{e_1, \dots e_l\})$, where $\{e_1, \dots e_l\}$ are all the integers $i \in [\![1, k]\!]$ such that v is adjacent to v_i. For every $i, j \in [\![1, k]\!]$, $i < j$, let $L_{i,j}$ be the set of paths of $G \setminus X_H$ which are neighbors in G of v_i and v_j, to which we give the following orientation: the first vertex of such a path is the one which is adjacent to v_i and its last vertex is the one adjacent to v_j. Last, let s_i be the label of v_i for every $i \in [\![1, k]\!]$. Then it is clear that G is isomorphic to $f(\{s_i\}_{i \in [\![1,k]\!]}, \{L_{i,j}\}_{i,j \in [\![1,k]\!], i<j}, J)$. Consequently f is surjective on $\mathrm{lab}_{(S,\preceq)}(\mathcal{G}_H)$.

Our current goal is now, in order to show that $f\colon (\mathcal{B}, \preceq_{\mathcal{B}}) \to (\mathrm{lab}_{(S,\preceq)}(\mathcal{G}_H), \leq_{\mathrm{im}})$ is an epi, is to prove that it is monotone. Let A, B be two elements of

$$S^k \times \left(\mathcal{P}^{<\omega}(\mathrm{lab}_{(S,\preceq)}(\mathcal{OP}))\right)^{\binom{k}{2}} \times \mathrm{lab}_{(S,\preceq)\times(2^{[\![1,k]\!]},=)}(\mathcal{C})$$

such that $A \preceq_{\mathcal{D}} B$. Let us show that $f(A) \leq_{\mathrm{im}} f(B)$. According to Remark 3.4, it is enough to focus on the cases where A and B differ by only one coordinate.

First case: A and B differ by the i-th coordinate, for $i \in [\![1, k]\!]$. Let s_A (resp. s_B) be the value of the i-th coordinate of A (resp. of B). According to the definition of f, the graphs $f(A)$ and $f(B)$ differ only by the label of vertex v_i: this label is s_A in $f(A)$ whereas it equals s_B in $f(B)$. But since we have $s_A \preceq s_B$ (as $A \leq_{\mathcal{B}} B$), we get $f(A) \leq_{\mathrm{im}} f(B)$.

Second case: A and B differ by the last coordinate. Let J_A (resp. J_B) be the value of the last coordinate of A (resp. of B). As previously, $A \preceq_{\mathcal{B}} B$ gives $J_A \leq_{\mathrm{isg}} J_B$, therefore we can obtain J_A by removing vertices of J_B and contracting labels. As the adjacencies of vertices of J_A and J_B to the rest of $f(A)$ and $f(B)$ (respectively) depend only on the label of their vertices, the same deletion and contraction operations in $f(B)$ give $f(A)$, hence $f(A) \leq_{\mathrm{im}} f(B)$.

Third case: A and B differ by the i-th coordinate, for some $i \in \left[\!\left[k + 1, k + \binom{k}{2}\right]\!\right]$. Let L_A (resp. L_B) be the value of this coordinate in A (resp. in B). As previously again, $A \preceq_{\mathcal{B}} B$ gives $L_A \leq_{\mathrm{c}} L_B$, consequently we can obtain L_A by contracting edges of J_B and contracting labels. Since the contraction relation on $\mathcal{OP}$ requires that endpoints (beginning and end of a path) are preserved, the same contraction operations in $f(B)$ give $f(A)$, thus we again get $f(A) \leq_{\text{im}} f(B)$.

We just proved that f is monotone, therefore it is an epi. By Remark 3.3, it is enough to show that $(\mathcal{B}, \preceq_B)$ is a wqo in order to prove that $(\text{lab}_{(S,\preceq)}(\mathcal{G}_H), \leq_{\text{im}})$ is a wqo.

Notice that $(\mathcal{B}, \preceq_B)$ is a Cartesian product of wqos and of the set of finite subsets of a wqo. Indeed, we assumed that $(S, \preceq)$ is a wqo. Furthermore, we proved in Lemma 3.23 that for every wqo $(S, \preceq)$, the quasi-order $(\text{lab}_{(S,\preceq)}(\mathcal{OP}), \leq_{\text{c}})$ is a wqo, and hence so is $(\mathcal{P}^{<\omega}(\text{lab}_{(S,\preceq)}(\mathcal{OP})), \leq_{\text{c}}{}^{\star})$ (cf. Corollary 3.2). Last, we proved in Theorem 3.21 that the class of cographs labeled by a wqo is well-quasi-ordered by the induced subgraph relation. Therefore, $(\mathcal{B}, \preceq_B)$ is a wqo, which concludes the proof. $\square$

Proof of Theorem 3.20. According to Lemma 3.2, it is enough to prove that for every wqo $(S, \preceq)$, the class of $(S, \preceq)$-labeled 2-connected graphs which does not contain Gem as induced minor is well-quasi-ordered by induced minors. By Theorem 3.18, these graphs can be turned into a disjoint union of paths and cographs by the deletion of at most six vertices. A consequence of Lemma 3.30 (for $k = 6$), these graphs are well-quasi-ordered by induced minors and we are done. $\square$

3.6 Contractions and well-quasi-ordering

This section is devoted to the proof of Theorem 3.10.

3.6.1 On graphs with no diamond

In this section we show that graphs in $\text{Excl}_{\leq_{\text{c}}}(D_2)$ have a simple structure. More precisely, we prove the following lemma.

Lemma 3.31. *Graphs of* $\text{Excl}_{\leq_{\text{c}}}(D_2)$ *are exactly the connected clique-cactus graphs.*

The proof of Lemma 3.31 will be given after a few lemmas. If C is a cycle of a graph G and $\{u, v\}, \{u', v'\} \subseteq V(C)$, we say that $\{u, v\}$ and $\{u', v'\}$ are *crossing in* C if u, v, u', v' are distinct and are appearing in this order on the cycle.

Lemma 3.32. *Let G be a graph and let C be a cycle in G. If C has at least one chord and one non-chord in G, then it has one chord and one non-chord that are crossing in C.*

Proof. Let $\{x, x'\}$ be a non-chord of C in G and let P and Q be the connected components of $C \setminus \{x, x'\}$ which obviously are paths. Let us assume that every chord of C in G has both endpoints either in P or in Q (otherwise we are done) and let $\{y, y'\}$ be a chord of C in G, the endpoints of which belong, say, to P. Let z be a vertex of the subpath of P delimited by y and y' such that $z \notin \{y, y'\}$ and let z' be a vertex of Q. If $\{z, z'\}$ is a chord of C in G, then $\{x, x'\}$ and $\{z, z'\}$ are satisfying the required property. Otherwise, $\{z, z'\}$ is a non-chord and now $\{y, y'\}$ and $\{z, z'\}$ are crossing. $\square$

Lemma 3.33. *Let $G \in \mathrm{Excl}_{\leq_c}(D_2)$. Every cycle of G is either an induced cycle, or it induces a clique in G.*

Proof. Let G be a graph of $\mathrm{Excl}_{\leq_c}(D_2)$ and let C be a cycle of G. For contradiction, let us assume that C has at least one chord $\{u, u'\}$ and one non-chord $\{v, v'\}$. According to Lemma 3.32 we can assume without loss of generality that they are crossing in C. Let P and Q be the two connected components of $C \setminus \{v, v'\}$. In the graph $G[C]$, contracting P to a single vertex x and Q to y yields a graph on four vertices v, v', x, y such that

1. v, x, v', y lie on the cycle in this order;
2. $\{v, v'\} \notin E(G)$; and
3. $\{x, y\} \in E(G)$ (as $\{u, u'\}$ connects the subgraphs that are respectively contracted to x and y).

The obtained graph is D_2, a contradiction. Therefore C has either no chords or no non-chords in G. It is clear that in the first case C is an induced cycle of G and that in the second case it induces a clique. □

Lemma 3.34. *Let $G \in \mathrm{Excl}_{\leq_c}(D_2)$ be a 2-connected graph. Then G is either a cycle, or a clique.*

Proof. We assume that $|V(G)| > 1$, otherwise the result is trivial. Let C be a longest cycle of G. By Lemma 3.33 the cycle C is either an induced cycle or it induces a clique in G. Let us treat these two cases separately. For contradiction, we assume that $V(G) \setminus V(C)$ is not empty and we call $H_1, \dots, H_t$ the connected components of $G \setminus C$, for some $t \in \mathbb{N}_{\geq 1}$. Let us consider the graph G' where H_i, which is connected, has been contracted to a single vertex h_i, for every $i \in [\![1, t]\!]$. Observe that G' is 2-connected, given that G is 2-connected. Also, $G' \in \mathrm{Excl}_{\leq_c}(D_2)$.

First case: C induces a clique in G'. Notice that C is then a maximal clique. Let $u = h_1$. As C is maximal, there is a vertex $v \in V(C)$ such that $\{u, v\} \notin E(G')$. Let x and y be two neighbors of u on C (they exist since G' is 2-connected). These vertices define two subpaths of C. Let R be the longest of these paths that contains v. Observe that in this case, R has at least three vertices. The union of $\{u, x\}$, $\{u, y\}$ and R is a cycle of G' that we call C'. According to Lemma 3.33, this cycle is either induced or it induces a clique. As $\{u, v\} \notin E(G')$, C' cannot induce a clique in G. On the other hand, C is not an induced cycle as every pair of vertices of R are adjacent (and $|V(R)| \geq 3$ as mentioned earlier). We reached the contradiction we were looking for.

Second case: C is an induced cycle and has at least 4 vertices. Let $i \in [\![1, t]\!]$. As G' is 2-connected, h_i has at least two neighbors on C: let x and y be two of them.

Claim 3.1. x and y are not adjacent.

Proof. Let us assume that $\{x, y\} \in E(G')$. Let C' be the cycle obtained from C by replacing the edge $\{x, y\}$ by the path xh_iy. This cycle is not induced as x, y are not adjacent in C' whereas $\{x, y\} \in E(G)$. It does not induce a clique neither since x is not adjacent with the other neighbor of y on C (which is not x as we assume that

C has at least 4 vertices). This contradicts Lemma 3.33 and therefore proves that $\{x, y\} \notin E(G)$. $\square$

Every pair of distinct vertices of the cycle C defines two subpaths of C meeting only at these vertices. Let u and v be two vertices of C such that h_i has at least one neighbor in the interior of each of the subpaths of C defined by u and v, that we will respectively call P and Q. Such vertices exists, as a consequence of Claim 3.1.

Let us consider the contraction H of G' obtained by contracting the interior path of P (respectively Q) to a single vertex w_P (respectively w_Q) and then by contracting the edge connecting h_i to w_P. This edge exists by definition of u and v. Then uw_Pvw_Q is a cycle of H where $\{w_P, w_Q\}$ is a chord (because we contracted to w_P the vertex h_1 which was adjacent to both w_P and w_Q) and $\{u, v\}$ is a non-chord (as they were non-adjacent vertices of the induced cycle C and that noting has been contracted to them). According to Lemma 3.33, the graph H contains D_2 as contraction. As H is a contraction of G, then $D_2 \leq_c G$, a contradiction.

In both cases we reached a contradiction, therefore $V(G) \setminus V(C)$ is empty: G is a clique or an induced cycle. $\square$

We are now ready to prove Lemma 3.31.

Proof of Lemma 3.31. The fact that a graph of $\mathrm{Excl}_{\leq_c}(D_2)$ is a clique-cactus is a straightforward corollary of Lemma 3.34. It is easy to see that a cactus graph does not contain D_2 as contraction by noticing that D_2 is a contraction of a graph if and only if it is a contraction of one of its 2-connected components. As D_2 is neither a contraction of a cycle, nor of a clique, we get the desired result. $\square$

3.6.2 Well-quasi-ordering clique-cactus graphs

We proved in the previous section that graphs of $\mathrm{Excl}_{\leq_c}(D_2)$ are exactly the connected clique-cactus graphs. This section contains the last part of the proof of Theorem 3.10, which is the following lemma.

Lemma 3.35. *Connected clique-cactus graphs are well-quasi-ordered by $\leq_c$.*

In this section we deal with rooted graphs. Let us denote by $\mathcal{C}$ the class of rooted connected clique-cactus graphs. In this class, two isomorphic graphs with a different root are seen as different. It is clear that proving that $(\mathcal{C}, \leq_c)$ is a wqo implies Lemma 3.35. This is what we will do.

Building blocks. Let us define three graph constructors $\mathsf{stick}\colon \mathcal{C}^\star \to \mathcal{C}$, $\mathsf{cycle}\colon \mathcal{C}^\star \to \mathcal{C}$, and $\mathsf{clique}\colon \mathcal{C}^\star \to \mathcal{C}$. Given a sequence $\langle G_0, \dots, G_{p-1}\rangle \in \mathcal{C}^\star$ (for some $p \in \mathbb{N}$), if U denote the union of the graphs $G_1, \dots, G_{p-1}$, then we define;

- $\mathsf{stick}(G_0, \dots, G_{p-1})$ is the graph obtained from U by identifying the vertices
$$\mathrm{root}(G_0), \dots, \mathrm{root}(G_{p-1})$$
;

- $\mathsf{cycle}(G_0, \dots, G_{p-1})$ is the graph obtained from U by adding the edges

$$\{\mathrm{root}(G_i), \mathrm{root}(G_{(i+1) \mod p})\}$$

for every $i \in [\![0, p-1]\!]$; and

- $\mathsf{clique}(G_0, \dots, G_{p-1})$ is the graph obtained from U by adding the edges

$$\{\mathrm{root}(G_i), \mathrm{root}(G_j)\}$$

for every distinct $i, j \in [\![0, p-1]\!]$.

The root of $\mathsf{stick}(G_0, \dots, G_{p-1})$, $\mathsf{cycle}(G_0, \dots, G_{p-1})$ and $\mathsf{clique}(G_0, \dots, G_{p-1})$ is the vertex that is the root of G_0. These constructors will allow us to encode graphs of $\mathcal{C}$ into sequences.

We will now decompose graphs of $\mathcal{C}$ along blocks.

For every block B of a graph G, let $\mathsf{dec}_B(G)$ denote the collection of all the graphs H that can be constructed from some connected component C of $G \setminus V(B)$ by adding a new vertex v adjacent to the vertices of C that are adjacent to a vertex of B in G and setting $\mathrm{root}(H) = v$.

Observe that, as soon as $\mathrm{root}(G) \in V(B)$, every graph of $\mathsf{dec}_B(G)$ is a proper contraction of G. Let $\mathsf{dec}(G)$ denote the union of the sets $\mathsf{dec}_B(G)$ for every block B of G containing the root of G. The following observation is a consequence of Lemma 3.31.

Observation 3.2. For every graph $G \in \mathcal{C}$ there is a (non necessarily unique) sequence $\langle \mathcal{G}_0, \dots, \mathcal{G}_{p-1} \rangle \in \mathsf{dec}(G)^\star$ (for some $p \in \mathbb{N}$) such that either

$$G = \mathsf{cycle}(\mathsf{stick}(\mathcal{G}_0), \dots, \mathsf{stick}(\mathcal{G}_{p-1})), \text{ or}$$
$$G = \mathsf{clique}(\mathsf{stick}(\mathcal{G}_0), \dots, \mathsf{stick}(\mathcal{G}_{p-1})).$$

From encodings to well-quasi-ordering. The following lemma will allow us to work on sequences in order to show that two graphs are comparable.

Lemma 3.36. *Let $\sigma, \tau \in \mathcal{C}^\star$. If $\sigma \leq_{\mathrm{c}}^\star \tau$, then*

(i) $\mathsf{cycle}(\sigma) \leq_{\mathrm{c}} \mathsf{cycle}(\tau)$*;*

(ii) $\mathsf{clique}(\sigma) \leq_{\mathrm{c}} \mathsf{clique}(\tau)$*; and*

(iii) $\mathsf{stick}(\sigma) \leq_{\mathrm{c}} \mathsf{stick}(\tau)$*.*

Proof. Let $\sigma = \langle H_1, \dots, H_p \rangle$ and $\tau = \langle G_1, \dots, G_q \rangle$ (for some positive integers p, q) and let $H = \mathsf{cycle}(\sigma)$ and $G = \mathsf{cycle}(\tau)$. For the sake of readability we will refer to H_i's (respectively G_i's) either as elements of σ (respectively τ) or as subgraphs of H (respectively G).

If $\sigma \leq_{\mathrm{c}}^\star \tau$, then there is, by definition of $\leq_{\mathrm{c}}^\star$, an increasing function $\varphi\colon [\![1, p]\!] \to [\![1, q]\!]$ such that $\forall i \in [\![1, p]\!],\ H_i \leq_{\mathrm{c}} G_\varphi(i)$. Therefore there is a sequence of edge contractions transforming $G_{\varphi(i)}$ into H_i for every $i \in [\![1, p]\!]$. Let us perform the following operations on G:

1. for every $j \in [\![1, q]\!] \setminus \{\varphi(i),\ i \in [\![1, p]\!]\}$ we contract the subgraph G_j to a single vertex v_j and we then contract some edge incident with v_j;
2. for every $i \in [\![1, p]\!]$ we contract the subgraph G_i in order to obtain the subgraph $H_\varphi(i)$.

Observe that after step 1., we obtain the graph $\mathsf{cycle}(\tau^-)$, where τ^- can be obtained from τ be deleting elements of indices in $[\![1, q]\!] \setminus \{\varphi(i),\ i \in [\![1, p]\!]\}$. Intuitively, we contracted the graphs that do not appear in H and removed their attachment point from the cycle. Then we replace in step 2. every graph of τ^- by its corresponding contraction of σ. Therefore the graph obtained at the end is $\mathsf{cycle}(\sigma)$, that is H, as required.

The cases (ii) and (iii) are very similar: H can be obtained from G by following the same operations as above. □

Proof of Lemma 3.35. Let us assume by contradiction that $(\mathcal{C}, \leq_c)$ is not a wqo. All decreasing sequences of this quasi-order are finite (as each contraction decreases the number of edges by one), therefore $(\mathcal{C}, \leq_c)$ contains an infinite antichain. Let us consider a minimal antichain $\{A_i\}_{i\in\mathbb{N}}$ of $(\mathcal{C}, \leq_c)$. Let $\mathcal{B} = \bigcup_{i\in\mathbb{N}} \mathsf{dec}(A_i)$ and let us show that $(\mathcal{B}, \leq_c)$ is a wqo. For contradiction, let us assume that it is not a wqo and let $\{B_i\}_{i\in\mathbb{N}}$ be a minimal antichain of this quasi-order.

By definition of $\mathcal{B}$, for every $H \in \mathcal{B}$ there is an integer $i \in \mathbb{N}$ such that $H \leq_c A_i$ (for instance, an integer i such that $H \in \mathsf{dec}(A_i)$). Therefore for every $i \in \mathbb{N}$ there is an integer $\pi(i)$ such that $B_i \leq_c A_{\pi(i)}$. Let $k \in \mathbb{N}$ be the integer where π is minimum. Then the following sequence

$$\mathcal{A} = A_0, \ldots, A_{\pi(k)-1}, B_k, B_{k+1}, \ldots$$

is an infinite antichain of $(\mathcal{C}, \leq_c)$. Indeed, as both $\{A_i\}_{i\in\mathbb{N}}$ and $\{B_i\}_{i\in\mathbb{N}}$ are antichains, every pair of comparable graphs of $\mathcal{A}$ involves one graph of $\{A_i\}_{i\in[\![1,\pi(k)-1]\!]}$ and one graph of $\{B_i\}_{i\in\mathbb{N}_{\geq k}}$. Let us assume that for some $i \in [\![0, \pi(k) - 1]\!]$ and $j \in \mathbb{N}_{\geq k}$ we have $A_i \leq B_j$. Then $A_i \leq B_j \leq A_{\pi(i)}$, a contradiction with the fact that $\{A_i\}_{i\in\mathbb{N}}$ is an antichain. The case $B_j \leq A_i$ is not possible by the choice of k. This proves that $(\mathcal{B}, \leq_c)$ is a wqo. According to Theorem 3.1, $(\mathcal{B}^\star, \leq_c{}^\star)$ is also a wqo. Let $\mathcal{B}' = \{\mathsf{stick}(\sigma),\ \sigma \in \mathcal{B}^\star\}$. Item (iii) of Lemma 3.36 implies that any antichain in $(\mathcal{B}', \leq_c)$ can be translated into an antichain of the same length in $(\mathcal{B}^\star, \leq_c{}^\star)$, hence $(\mathcal{B}', \leq_c)$ is a wqo. By the same argument (now using items (i) and (ii) of Lemma 3.36), we deduce that the quasi-orders

$$(\{\mathsf{cycle}(\sigma),\ \sigma \in \mathcal{B}'^\star\}, \leq_c) \quad \text{and} \quad (\{\mathsf{clique}(\sigma),\ \sigma \in \mathcal{B}'^\star\}, \leq_c)$$

are well-quasi-orders. Therefore $\mathcal{U} = \{\mathsf{cycle}(\sigma),\ \sigma \in \mathcal{B}'^\star\} \cup \{\mathsf{clique}(\sigma),\ \sigma \in \mathcal{B}'^\star\}$ is well-quasi-ordered by $\leq_c$, as a consequence of Remark 3.2. According to Observation 3.2, we have $\{A_i\}_{i\in\mathbb{N}} \subseteq \mathcal{U}$. This contradicts the fact that $\{A_i\}_{i\in\mathbb{N}}$ is an infinite antichain. Therefore $(\mathcal{C}, \leq_c)$ is a wqo and we are done. □

3.6.3 The dichotomy

The next observations will allow us to give the the proof of Theorem 3.10, using the results obtained in the previous section. Recall that our goal is to characterize the graphs H such that $(\mathrm{Excl}_{\leq_c}(H), \leq_c)$ is a wqo.

Observation 3.3. For every $p \in \mathbb{N}_{\geq 1}$, contracting one edge in D_p gives either D_{p-1}, or $K_{1,p}$, depending on which edge is contracted.

As we want to identify graphs H such that $(\mathrm{Excl}_{\leq_c}(H), \leq_c)$ is a wqo, we must consider every graph H such that $\mathcal{A} \cap \mathrm{Excl}_{\leq_c}(H)$ is finite, for every antichain $\mathcal{A}$. A first step towards this goal is the following observation.

Lemma 3.37. *Let $p \in \mathbb{N}_{\geq 2}$. If $H \leq_c K_{2,p}$ then $H \in \mathcal{D} \cup \mathcal{S}$.*

Proof. Given that $H \leq_c K_{2,p}$, there is a sequence of contractions transforming $K_{2,p}$ into H. If this sequence contains only one contraction, then it is straightforward that $H = D_{p-1}$. Therefore in the other cases H is a contraction of D_{p-1}. We get the result from Observation 3.3 and the observation that every contraction of a graph of $\mathcal{S}$ belongs to $\mathcal{S}$. □

Observation 3.4. For every positive integers p, q such that $p < q$, we have $D_p \leq_c K_{2,q}$.

Indeed, if F is a collection of $q - p$ edges that are pairwise not incident with the same vertex of degree 2 of $K_{2,p}$, then it is easy to check that contracting F in $K_{2,p}$ yields D_p. An immediate consequence of Observation 3.4 is that $\mathcal{A}_K \cap \mathrm{Excl}_{\leq_c}(D_p)$ is finite for every positive integer p.

From the fact that every graph of $\mathcal{D} \cup \mathcal{S}$ is a contraction of D_p for some positive integer p, Observation 3.4 gives.

Observation 3.5. If $(\mathrm{Excl}_{\leq_c}(H), \leq)$ is a wqo, then $H \leq_c D_p$ for some $p \in \mathbb{N}_{\geq 1}$

However we will need an other antichain in order to find more properties that H must satisfy. Let us consider the antichain $\mathcal{A}_{\overline{C}}$ of antiholes, that has been presented in Subsection 3.4.1.

Again, we look at graphs H such that $\mathrm{Excl}_{\leq_c}(H) \cap \mathcal{A}_{\overline{C}}$ is finite. As a wqo must contain none of $\mathcal{A}_K$ and $\mathcal{A}_{\overline{C}}$, it is enough to consider graphs such that $\mathrm{Excl}_{\leq_c}(H) \cap \mathcal{A}_{\overline{C}}$ is finite among those for which $\mathrm{Excl}_{\leq_c}(H) \cap \mathcal{A}_K$ is finite.

Lemma 3.38. *If $p \geq 3$ then $\mathrm{Excl}_{\leq_c}(D_p) \cap \mathcal{A}_{\overline{C}}$ is infinite.*

Proof. For every $p \geq 3$, then graph D_p has independence number at least 3. Let $q > p$. As contracting edges can only decrease the independence number, there is no sequence of contractions transforming $\overline{C_q}$ (which has independent number 2) to D_p, for every integer $q > p$. Therefore $\overline{C_q} \in \mathrm{Excl}_{\leq_c}(D_p)$, for every integer $q > p$. □

Corollary 3.13. *If $(\mathrm{Excl}_{\leq_c}(H), \leq_c)$ is a wqo, then $H \leq_c D_2$.*

We are now ready to give the proof of Theorem 3.10.

Proof of Theorem 3.10. Let H be a graph such that $\mathrm{Excl}_{\leq_c}(H)$ is a wqo. Then $H \leq_c D_2$, by Corollary 3.13. On the other hand, if $H \leq_c D_2$ then $\mathrm{Excl}_{\leq_c}(H) \subseteq \mathrm{Excl}_{\leq_c}(D_2)$. Observe that every antichain (respectively decreasing sequence) of $(\mathrm{Excl}_{\leq_c}(H), \leq_c)$ is an antichain (respectively a decreasing sequence) of $(\mathrm{Excl}_{\leq_c}(D_2), \leq_c)$. As a consequence of Lemma 3.35 we get that $(\mathrm{Excl}_{\leq_c}(H), \leq_c)$ is a wqo and we are done. □

3.6.4 Canonical antichains and contractions

In this section, we will use the following result of Ding in order to prove Theorem 3.15.

Lemma 3.39 ([Din09]). *Let $(S, \preceq)$ be a quasi-order, let $\{A_i\}_{i\in\mathbb{N}}$ be a sequence of elements of S and let $\{\mathcal{W}_i\}_{i\in\mathbb{N}}$ be a sequence of sequences of elements of S. If we have*

(i) $\{A_i\}_{i\in\mathbb{N}}$ is a fundamental infinite antichain; and

(ii) for every $i \in \mathbb{N}$, $\mathcal{W}_i$ is a fundamental infinite antichain; and

(iii) for every $i \in \mathbb{N}$ and every $H \in \mathcal{W}_i$, $A_i \preceq H$,

then $(S, \preceq)$ does not have a canonical antichain.

We will now define some sequences of graphs and show that they satisfy the properties of Lemma 3.39.

For every $p, q \in \mathbb{N}$, let $W_{p,q}$ be the graph obtained by adding two non-adjacent dominating vertices to the disjoint union of $\overline{K_p}$ and $K_{2,q}$. These two vertices are called *poles*, and the two vertices corresponding to the part of $K_{2,q}$ of size 2 are called *semipoles*. Observe that the other vertices either have degree two (in which case they are adjacent to the two poles, only), or have degree four (and they are adjacent to both poles and both semipoles).

Lemma 3.40. *For every $p, p', q, q' \in \mathbb{N}_{\geq 3}$, there is no contraction model of $W_{p,q}$ in $W_{p',q'}$ if $(p, q) \neq (p', q')$.*

Proof. Let us assume that there is a contraction model φ of $W_{p,q}$ in $W_{p',q'}$. Let u be a vertex of $W_{p,q}$ of degree two. By definition of a contraction model, its image by φ must be a subset of degree 2. In $W_{p',q'}$, the connected subsets of degree 2 are either of the form $\{v\}$, or $V(W_{p',q'}) \setminus \{v\}$, where $v \in V(W_{p',q'})$ has degree 2. As $W_{p,q}$ has more than two vertices, the only possible form for $\varphi(u)$ is $\{v\}$ for some vertex $v \in V(W_{p',q'})$ of degree 2. Therefore we have $p \leq p'$. The same argument applied to vertices of degree 4 yields $q \leq q'$. Let us now consider poles and semipoles.

Let u be a pole. Observe that according to the above remarks, $\varphi(u)$ must be adjacent to vertices of degree two, so it should contain a pole of $W_{p,q}$. If $\varphi(u)$ contains in addition a vertex of degree 2 or 4 of $W_{p,q}$, then $\varphi(u)$ is dominating. This is not possible since u is not dominating, therefore $\varphi(u) = \{v\}$ for some pole v of $W_{p',q'}$. Let us now assume that u is a semipole of $W_{p,q}$. As previously, the above remarks imply that $\varphi(u)$ is adjacent to vertices of degree 4 of $W_{p',q'}$. Hence $\varphi(u)$ contains a semipole of $W_{p',q'}$ (it cannot contain a pole as both belong to the image of poles of $W_{p,q}$). Therefore each semipole of $W_{p,q}$ is sent to a subset of $V(W_{p',q'})$ containing a semipole. Observe that $\varphi(u)$ cannot contain a vertex of degree two otherwise it would not be connected. Besides, it cannot contain a vertex of degree 4 otherwise it would be adjacent to the image by φ of the other semipole of $W_{p,q}$. Consequently $\varphi(u)$ contains a semipole of $W_{p',q'}$ and no other vertex. We proved that for every $u \in V(W_{p,q})$, the set $\varphi(u)$ is a singleton. Therefore $|V(W_{p,q})| = |V(W_{p',q'})|$. Given that $p \leq p'$ and $q \leq q'$ (as proved above), this is possible only if $p = p'$ and $q = q'$. This concludes the proof. □

Corollary 3.14. *$\{W_{p,q}\}_{p,q\geq 3}$ is an antichain for $\leq_{\mathrm{c}}$.*

For every $i \in \mathbb{N}_{\geq 3}$, let $\mathcal{W}_i = \{W_{i,q}\}_{q\in\mathbb{N}_{\geq 3}}$.

Lemma 3.41. *For every $p, q \in \mathbb{N}_{\geq 3}$, $K_{2,p+1} \leq_{\mathrm{c}} W_{p,q}$.*

Proof. Let S be the set of vertices of degree 4 and semipoles of $W_{p,q}$ (i.e. the vertices of the copy of $K_{2,q}$ used in the construction of $W_{p,q}$). These vertices induced a connected subgraph as every vertex of degree 4 is adjacent to both semipoles. It is not hard to see that contracting S to a single vertex yields $K_{2,p+1}$. □

Observation 3.6. Let $p, q \in \mathbb{N}_{\geq 3}$. There is no path with containing four independent vertices in $W_{p,q}$.

Corollary 3.15. *For every $p, q \in \mathbb{N}_{\geq 3}$, the graph $W_{p,q}$ does not contain the gem as induced minor.*

Corollary 3.16. *No graph of* $\mathrm{Incl}_{\leq_{\mathrm{c}}}(\mathcal{W}_i)$ *contains the gem as induced minor, for every $i \in \mathbb{N}_{\geq 3}$.*

The following observation will allow us to use Theorem 3.20, which deals with induced minors.

Observation 3.7. Let H and G be two graphs. If both of them have a dominating vertex, then $H \leq_{\mathrm{c}} G \iff H \leq_{\mathrm{im}} G$.

The following corollary is a direct consequence of Theorem 3.20, Observation 3.7 and Corollary 3.16.

Corollary 3.17. *Graphs of* $\mathrm{Incl}_{\leq_{\mathrm{c}}}(\mathcal{W}_i)$ *with a dominating vertex are wqo by $\leq_{\mathrm{c}}$, for every $i \in \mathbb{N}_{\geq 3}$.*

Lemma 3.42. *$\mathcal{W}_i$ is a fundamental antichain, for every $i \in \mathbb{N}_{\geq 3}$.*

Proof. Let $i \in \mathbb{N}_{\geq 3}$. We need to show that $(\mathrm{Incl}_{\leq_{\mathrm{c}}}(\mathcal{W}_i), \leq_{\mathrm{c}})$ is a wqo. Let us call *inner edge* every edge of $W_{p,q}$ that is not incident with a pole, for every $p, q \in \mathbb{N}_{\geq 3}$. Observe that if a graph $H \in \mathrm{Incl}_{\leq_{\mathrm{c}}}(\mathcal{W}_i)$ has been obtained by contracting at least one edge incident with a pole, then H has a dominating vertex. According to Corollary 3.17, these graphs are wqo by $\leq_{\mathrm{c}}$, therefore we will here consider graphs of $\mathrm{Incl}_{\leq_{\mathrm{c}}}(\mathcal{W}_i)$ that have been obtained by only contracting inner edges. We call $\mathcal{I}$ this class.

We first show that $\mathcal{I}$ is the union of the two following classes:

- the class $\mathcal{I}_0$ of graphs that can be obtained by adding two non-adjacent dominating vertices to $\overline{K_i} + D_q$ for some $q \in \mathbb{N}_{\geq 0}$; and
- the class $\mathcal{I}_1$ of graphs that can be obtained by adding two non-adjacent dominating vertices to $\overline{K_i} + S_q$ for some $q \in \mathbb{N}_{\geq 0}$.

Again we use the notion of *poles* to denote the two dominating vertices added to construct graphs of $\mathcal{I}_0$ and $\mathcal{I}_1$. A *semipole* is either a dominating vertex of D_q (when dealing with graphs of $\mathcal{I}_0$), or the dominating vertex of S_q (when dealing with graphs of $\mathcal{I}_1$).

Contracting an inner edge in $W_{i,q}$ clearly yields a graph of $\mathcal{I}_0$. Now, observe that any further contraction of an edge connecting a vertex of degree 4 to a semipole gives a graph of $\mathcal{I}_0$ again. If, on the other hand, we contract the edge connecting the two semipoles, then we get a graph of $\mathcal{I}_1$. On a graph of $\mathcal{I}_1$, contracting an edges of the star (used in the construction of this graph) still gives a graph of $\mathcal{I}_1$. Therefore $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$.

Let us assume that $\mathcal{I}$ is not wqo by $\leq_c$. Therefore it has an infinite antichain. As $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1$, one of $\mathcal{I}_0$ and $\mathcal{I}_1$ (at least) has an infinite antichain. Let $\mathcal{A}$ be such an infinite antichain.

We now look at vertices of graphs of $\mathcal{A}$ that are neither poles, nor semipoles, nor have degree 2. These vertices are the vertices of degree 2 of the copy of D_q or the vertices of degree one of the copy of S_q used in the construction of the graphs of $\mathcal{A}$ (depending whether $\mathcal{A} \subseteq \mathcal{I}_0$ or $\mathcal{A} \subseteq \mathcal{I}_1$). We call them *inner vertices.*

Let A and A' be two graphs of $\mathcal{A}$ such that A has less inner vertices than A'. These graphs exist since the elements of $\mathcal{A}$ are distinct. Let q be the number of inner vertices of A and q' the one of A'.

In both cases $\mathcal{A} \subseteq \mathcal{I}_0$ and $\mathcal{A} \subseteq \mathcal{I}_1$ we can obtain A from A' by contracting $q' - q$ inner vertices of A' to a semipole. This contradicts the fact that $\mathcal{A}$ is an antichain. Therefore $(\mathcal{I}, \leq_c)$ is a wqo. This implies that $\mathcal{W}_i$ is fundamental, as required. □

We are now ready to prove Theorem 3.15.

Proof of Theorem 3.15. Let $A_i = K_{2,i+1}$ for every $i \in \mathbb{N}_{\geq 3}$. According to Lemma 3.42 and Lemma 3.41, these sequences of graphs satisfy the requirements of Lemma 3.39. Therefore there is no canonical antichain for the contraction relation. □

3.7 Multigraph contractions and well-quasi-ordering

In this section, we prove Theorem 3.11 and Theorem 3.16. Let us introduce some definitions and present the intermediate results that we use. First of all, as opposite to the previous sections of this chapter, we deal here with multigraphs: multiple edges are allowed, but not loops.

A *bond* is a minimal non-empty edge cut, i.e. a minimal set of edges whose removal increases the number of connected components (cf. Figure 3.17).

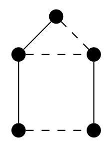


Figure 3.17: A bond of size 3 (dashed edges) in the house graph.

For every $p, k \in \mathbb{N}$, let $\mathcal{G}_{p,k}$ be the class of graphs having at most p connected components and not containing a bond of order more than k. What we prove in this section following.

Theorem 3.22. *For every $p, k \in \mathbb{N}$, the class $\mathcal{G}_{p,k}$ is well-quasi-ordered by $\leq_{\mathrm{mc}}$.*

It is easy to see that Theorem 3.11 is a consequence of Theorem 3.22. Remark that a graph has a bond of order k iff it contains θ_k as contraction, and that it has p connected components iff it can be contracted to $\overline{K_p}$. Theorem 3.22 is proven in Subsection 3.7.2 and results on canonical antichains appear in Subsection 3.7.3.

Let us now introduce the notation and definitions we will use in this section. As we have to handle many objects with several indices, we find more convenient to use the dot notation $A.b$, informally meaning "object b related to object A". For every $i \in \{2, 3\}$ we denote by $\mathcal{H}_k^{(i)}$ the class of all i-connected graphs in a class $\mathcal{H}$.

2-rooted graphs. We define a *2-rooted graph* in a very similar way as a rooted graph is defined. A 2-rooted graph is a triple (G, r, s) where G is a graph and r and s are two distinct vertices of G. Given two 2-rooted graphs $(G, r, s), (H, r', s')$, we say that (H, r', s') is a contraction of (G, r, s), what we denote by $(H, r', s') \leq_{\mathrm{mc}} (G, r, s)$, if there is a contraction model μ of H in G such that $r' \in \mu(r)$ and $s' \in \mu(s')$. For the sake of simplicity, we sometimes denote by G the 2-rooted graph (G, r, s) and refer to its first (respectively second) root by $G.r$ (respectively $G.s$). For every rooted graph G, we define $\mathrm{root}(G) = \{G.r, G.s\}$. A 2-rooted graph G is *edge-rooted* if $\{G.r, G.s\} \in E(G)$.

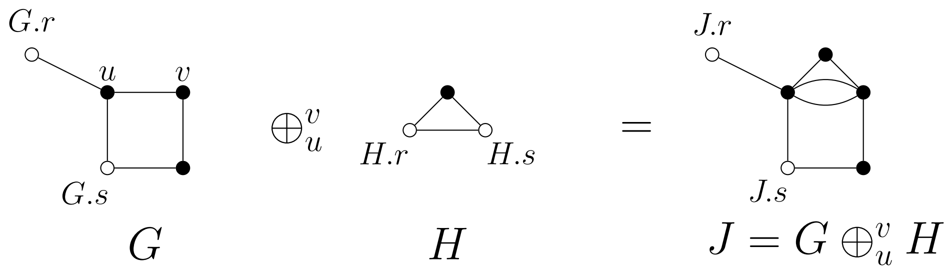


Figure 3.18: Attaching H to vertices (u, v) of G (roots are the white vertices).

The operation of *attaching* a 2-rooted graph H on the pair of vertices (u, v) of graph G, denoted $G \oplus_u^v H$, yields the graph rooted in $(G.r, G.s)$ obtained by identifying u with $H.r$ and v with $H.s$ in the disjoint union of G and H (see Figure 3.18 for an illustration). If both G and H are $(\Sigma, \preceq)$-labeled (for some qoset $(\Sigma, \preceq)$), then the labeling function λ of the graph $G \oplus_u^v H$ is defined as follows:

$$\lambda\colon \begin{cases} V(G \oplus_u^v H) & \to & \mathcal{P}(\Sigma) \\ \qquad w & \mapsto & G.\lambda(w) \quad \text{if } w \in V(G) \setminus \{u, v\} \\ \qquad w & \mapsto & H.\lambda(w) \quad \text{if } w \in V(H) \setminus \{H.r, H.s\} \\ \qquad w & \mapsto & G.\lambda(w) \cup H.\lambda(w) \quad \text{otherwise, i.e. when } w \in \{u, v\}. \end{cases}$$

Now we state several results that we will use. The first one is a decomposition theorem for 2-connected graphs by Tutte.

Proposition 3.2 ([Tut61], see also [Die05, Exercise 20 of Chapter 12]). *Every 2-connected simple graph has a tree-decomposition $(T, \mathcal{X})$ such that $|X_t \cap X_{t'}| = 2$ for every edge $\{t, t'\} \in T$ and all torsos are either 3-connected or a cycle.*

Proposition 3.3 ([OOT93]). *For every $k \in \mathbb{N}$ there is a positive integer ζ_k such that every 3-connected simple graph of order at least ζ_k contains a wheel of order k or a $K_{3,k}$ as minor.*

3.7.1 Gluing graphs

This section is devoted to building larger wqos from smaller ones in classes of labeled graphs that are rooted by two vertices. Step by step, we will construct wqos that will be directly used in the proof of the main result, as in Step 3 of the general scheme of Section 3.3. Labels will be used to reduce the study of (unlabeled) graphs to the case of 2-connected graphs with labels (by the virtue of Lemma 3.3), whereas roots enable us to construct graphs using the operation $\oplus$. In this section, $(\Sigma, \preceq)$ be any qoset.

Lemma 3.43. *Let H, H', G, G' be four $(\Sigma, \preceq)$-labeled 2-rooted graphs. If $H \leq_{\mathrm{mc}} H'$ and $G \leq_{\mathrm{mc}}{}^{\mu} G'$, then for every distinct u, v in $V(G)$ and $u' \in \mu(u)$, $v' \in \mu(v)$ we have*

$$G \oplus_u^v H \leq_{\mathrm{mc}} G' \oplus_{u'}^{v'} H'.$$

Proof. Let $\mu_H \colon V(H) \to \mathcal{P}(V(H'))$ (respectively $\mu_G \colon V(G) \to \mathcal{P}(V(G'))$) be a contraction model of H in H' (respectively of G in G'). We consider the following function:

$$\nu \colon \begin{cases} V(G \oplus_u^v H) & \to & \mathcal{P}(V(G' \oplus_{u'}^{v'} H')) & \\ v & \mapsto & \mu_H(v) & \text{if } v \in H \setminus \mathrm{root}(H) \\ v & \mapsto & \mu_G(v) & \text{if } v \in G \setminus \{u, v\} \\ v & \mapsto & \mu_H(v) \cup \mu_G(v) & \text{otherwise.} \end{cases}$$

Let us check that ν is a contraction model of $G \oplus_u^v H$ in $G' \oplus_{u'}^{v'} H'$. First, observe that for every $x \in V(G \oplus_u^v H)$, the subgraph induced in $G' \oplus_{u'}^{v'} H'$ by $\nu(x)$ is connected (M2): either $\nu(x) = \mu_H(x)$ or $\nu(x) = \mu_G(x)$ (and in these cases it follows from the fact that μ_H and μ_G are models) or $\nu(x) = \mu_H(x) \cup \mu_G(x)$ (if $x \in \{u, v\}$) and $(G' \oplus_{u'}^{v'} H')[\nu(x)]$ is connected because both $\mu_H(x)$ and $\mu_G(x)$ induce a connected subgraph and both contain the root of H'. Furthermore, the images through ν of two distinct vertices are always disjoint (M1), and every vertex of $G' \oplus_{u'}^{v'} H'$ belongs to the image of a vertex (M7), again because μ_H and μ_G are models. Let us now show point (M6). For every distinct $x, y \in V(G \oplus_u^v H)$,

- either $x, y \in V(H)$ and $\{x, y\} \neq \mathrm{root}(H)$ and then

  $$\mathrm{mult}_{G \oplus_u^v H}(x, y) = \sum_{(x', y') \in \nu(x) \times \nu(y)} \mathrm{mult}_{G' \oplus_u^v H'}(x', y')$$

  as μ_H is a contraction model (and symmetrically for the case $x, y \in V(G)$ and $\{x, y\} \neq \{u, v\}$);

- or $x \in V(H) \setminus \mathrm{root}(H)$ and $y \in V(G) \setminus \{u, v\}$: there are no edges between x and y because every edge of $G \oplus_u^v H$ is either an edge of H or an edge of G, neither between $\nu(x)$ and $\nu(y)$ since $\nu(x) \subseteq V(H) \setminus \mathrm{root}(H)$ and $\nu(y) \subseteq V(G) \setminus \{u, v\}$, therefore we get

$$\mathrm{mult}_{G \oplus_u^v H}(x, y) = \sum_{(x', y') \in \nu(x) \times \nu(y)} \mathrm{mult}_{G' \oplus_u^v H'}(x', y') = 0;$$

- or $\{x, y\} = \{u, v\} = \mathrm{root}(H)$:

$$\begin{aligned}
\mathrm{mult}_{G \oplus_u^v H}(x, y) &= \mathrm{mult}_G(x, y) + \mathrm{mult}_H(x, y) \quad \text{(by definition of } \oplus\text{)} \\
&= \sum_{(x', y') \in \mu_G(x) \times \mu_G(y)} \mathrm{mult}_{G'}(x', y') \\
&\quad + \sum_{(x', y') \in \mu_H(x) \times \mu_H(y)} \mathrm{mult}_{H'}(x', y') \\
&= \sum_{(x', y') \in \nu(x) \times \nu(y)} \mathrm{mult}_{G' \oplus_u^v H'}(x', y').
\end{aligned}$$

Besides, as a consequence that μ_G is root-preserving, ν also has this property. Last, let us check that ν is label-preserving. Let $x \in V(G \oplus_u^v H)$. If $x \notin \{u, v\}$, then $(G \oplus_u^v H).\lambda(x) = G.\lambda(x)$ or $(G \oplus_u^v H).\lambda(x) = H.\lambda(x)$ (depending whether $x \in V(G) \setminus \{u, v\}$ or $x \in H \setminus \mathrm{root}(H)$) and in these cases labels are preserved, since μ_G and μ_H are label-preserving. If $x \in \{u, v\}$, then, as μ_G and μ_H are label-preserving we have:

$$\begin{aligned}
(G \oplus_u^v H).\lambda(x) &= G.\lambda(x) \cup H.\lambda(y) \\
&\preceq^\star \bigcup_{x' \in \mu_G(x)} G'.\lambda(x') \cup \bigcup_{x' \in \mu_H(x)} H'.\lambda(x') \\
&\preceq^\star \bigcup_{x' \in \nu(x)} (G' \oplus_u^v H').\lambda(x')
\end{aligned}$$

and thus ν is label-preserving as well. We just proved that ν is a contraction model of $G \oplus_u^v H$ in $G' \oplus_u^v H'$. Consequently, $G \oplus_u^v H \leq_{\mathrm{mc}} G' \oplus_u^v H'$, as desired. □

Corollary 3.18. *Let $l \in \mathbb{N}^*$, let J be a $(\Sigma, \preceq)$-labeled 2-rooted graph and $\langle (u_i, v_i) \rangle_{i \in [\![1,l]\!]}$ be a sequence of pairs of distinct vertices of J. Let $\mathcal{H}$ be a class of $(\Sigma, \preceq)$-labeled 2-rooted graphs, $\langle G_1, \ldots, G_l \rangle, \langle H_1, \ldots, H_l \rangle \in \mathcal{H}^l$ and let G (respectively H) be the graph constructed by attaching G_i (respectively H_i) to the vertices (u_i, v_i) of J, for every $i \in [\![1, l]\!]$. If $\langle H_1, \ldots, H_l \rangle \leq_{\mathrm{mc}}{}^l \langle G_1, \ldots, G_l, \rangle$ then $H \leq_{\mathrm{mc}} G$.*

Proof. By induction on l. The case $l = 1$ follows from Lemma 3.43. If $l \geq 2$, then, let G' (respectively H') be the graph constructed by attaching G_i (respectively H_i) to the vertices (u_i, v_i) of J, for every $i \in [\![1, l-1]\!]$. By induction hypothesis, we have $H' \leq_{\mathrm{mc}} G'$. Since H (respectively G) is isomorphic to $H' \oplus_{u_l}^{v_l} H_l$ (respectively $G' \oplus_{u_l}^{v_l} G_l$) and $H_l \leq_{\mathrm{mc}} G_l$, by Lemma 3.43, we have $H \leq_{\mathrm{mc}} G$ as desired. □

Lemma 3.44. *Let $\mathcal{H}$ be a family of $(\Sigma, \preceq)$-labeled 2-rooted connected graphs, let J be a $(\Sigma, \preceq)$-labeled 2-rooted graph, and let $\mathcal{H}_J$ be the class of $(\Sigma, \preceq)$-labeled 2-rooted graphs that can be constructed by attaching a graph $H \in \mathcal{H}$ to (u, v) for every $u, v \in V(J)$. If $(\mathcal{H}, \leq_{\mathrm{mc}})$ is a wqo, then so is $(\mathcal{H}_J, \leq_{\mathrm{mc}})$.*

Proof. Let $(u_1, v_1), \ldots, (u_l, v_l)$ be an enumeration of all the pairs of distinct vertices of J. In this proof, we will design an epi that constructs graphs of $\mathcal{H}_{\mathcal{J}}$ from a tuple of l graphs of $\mathcal{H}$. Let $f \colon (\mathcal{H}^l, {\leq_{\mathrm{mc}}}^l) \to (\mathcal{H}_J, \leq_{\mathrm{mc}})$ be the function that, given a tuple $(H_1, \ldots, H_l)$ of l graphs of $\mathcal{H}$, returns the graph constructed from J attaching H_i to (u_i, v_i) for every $i \in [\![1, l]\!]$. This function is clearly surjective. Let us show that it is monotone.

Let $(G_1, \ldots, G_l), (H_1, \ldots, H_l) \in \mathcal{H}^l$ be two tuples such that the following holds:

$$(H_1, \ldots, H_l) \, {\leq_{\mathrm{mc}}}^l (G_1, \ldots, G_l).$$

According to Remark 3.4, it is enough to deal with the cases where these two sequences differ only in one coordinate. Since all parameters of f play a similar role, we only look at the case where $H_1 \leq_{\mathrm{mc}} G_1$ and $\forall i \in [\![2, l]\!]$, $H_i = G_i$. Let J' be the graph obtained from J by attaching G_i to (u_i, v_i), for every $i \in [\![2, l]\!]$. Remark that $f(H_1, \ldots, H_l)$ (respectively $f(G_1, \ldots, G_l)$) can be obtained by attaching H_1 (respectively G_1) to (u_1, v_1) in J'. By Lemma 3.43 and since $H_1 \leq_{\mathrm{mc}} G_1$, we have $J \oplus_{u_1}^{v_1} H_1 \leq_{\mathrm{mc}} J \oplus_{u_1}^{v_1} G_1$ and thus $f(H_1, \ldots, H_l) \leq_{\mathrm{mc}} f(G_1, \ldots, G_l)$. Consequently, f is monotone and surjective: f is an epi. In order to show that $\mathcal{H}_J$ is a wqo, it suffices to prove that the domain of f is a wqo (cf. Remark 3.3). As a finite Cartesian product of wqos, $(\mathcal{H}^l, {\leq_{\mathrm{mc}}}^l)$ is a wqo by Lemma 3.1. This concludes the proof. $\square$

Lemma 3.45. *Let $\mathcal{H}$ be a family of $(\Sigma, \preceq)$-labeled 2-rooted connected graphs and let $\mathcal{H}_\circ$ be the class of $(\Sigma, \preceq)$-labeled graphs that can be constructed from a cycle by attaching a graph of $\mathcal{H}$ to either (u, v) or (v, u) for every edge $\{u, v\}$, after deleting the edge $\{u, v\}$. If $(\mathcal{H}, \leq_{\mathrm{mc}})$ is a wqo, then so is $(\mathcal{H}_\circ, \leq_{\mathrm{mc}})$.*

Proof. Again, this proof relies on the property of epimorphisms to send wqos on wqos: we will present an epi that maps sequences of graphs of $(\mathcal{H}, \leq_{\mathrm{mc}})$ to graphs of $(\mathcal{H}_\circ, \leq_{\mathrm{mc}})$. Let $\mathcal{H}' = \mathcal{H} \cup \{(H, s, r),\ (H, r, s) \in \mathcal{H}\}$, i.e. $\mathcal{H}'$ contains graphs of $\mathcal{H}$ with the roots possibly swapped. As the union of two wqos, $(\mathcal{H}', \leq_{\mathrm{mc}})$ is a wqo (Remark 3.2). We consider the function $f \colon (\mathcal{H}'^\star, {\leq_{\mathrm{mc}}}^\star) \to (\mathcal{H}_\circ, \leq_{\mathrm{mc}})$ that, given a sequence $\langle H_1, \ldots, H_k \rangle$ of graphs of $(\mathcal{H}', \leq_{\mathrm{mc}})$ (for some integer $k \geq 2$), returns the graph obtained from the cycle on vertices $v_0, \ldots, v_{k-1}$ (in this order) by deleting the edge $\{v_i, v_{(i+1) \bmod k}\}$ and attaching H_i to $(v_i, v_{(i+1) \bmod k})$, for all $i \in [\![1, k]\!]$. Observe that by definition of $\mathcal{H}_\circ$ and $\mathcal{H}'$, the function f is surjective. We now show that f is monotone. Let $G = \langle G_0, \ldots, G_{k-1} \rangle$ and $H = \langle H_0, \ldots, H_{l-1} \rangle \in \mathcal{H}'^\star$ be two sequences such that $G \, {\leq_{\mathrm{mc}}}^\star H$. For the sake of readability, we will refer to the vertices of $f(G)$ (respectively $f(H)$) and to the graphs of G (respectively H) by the same names. By definition of the relation ${\leq_{\mathrm{mc}}}^\star$, there is an increasing function $\rho \colon [\![0, k-1]\!] \to [\![0, l-1]\!]$ such that for every $i \in [\![0, k-1]\!]$, we have $G_i \leq_{\mathrm{mc}} H_{\rho(i)}$.

A crucial remark here is that since the graphs of $\mathcal{H}'$ are connected, each of them can be contracted to an edge between its two roots. Therefore, for every graph H_i of the sequence H (for some $i \in \llbracket 0, l-1 \rrbracket$) we can first contract H_i to an edge in $f(H)$ and then contract this edge. That way we obtain a graph similar to $f(H)$ except that H_i has been deleted and its roots merged: this is the graph $f(\langle H_0, \dots, H_{i-1}, H_{i+1}, \dots, H_{l-1}\rangle)$. By applying this operation on every subgraph of $f(H)$ belonging to $\{H_i,\ i \in \llbracket 1, l \rrbracket \setminus \rho(\llbracket 0, k \rrbracket)\}$, we obtain the graph $f(\langle H_{\rho(i)} \rangle_{i \in \llbracket 1, k \rrbracket})$ and we thus have $f(\langle H_{\rho(i)} \rangle_{i \in \llbracket 1, k \rrbracket}) \leq_{\mathrm{mc}} f(H)$. Now, recall that the function ρ is such that for every $i \in \llbracket 0, k-1 \rrbracket$, we have $G_i \leq_{\mathrm{mc}} H_{\rho(i)}$. Furthermore, the graphs $f(G)$ and $f(\langle H_{\rho(i)} \rangle_{i \in \llbracket 1, k \rrbracket})$ are both constructed by attaching graphs to the same graph (a cycle on k vertices). By Corollary 3.18, we therefore have $f(G) \leq_{\mathrm{mc}} f(\langle H_{\rho(i)} \rangle_{i \in \llbracket 1, k \rrbracket})$, hence $f(G) \leq_{\mathrm{mc}} f(H)$ by transitivity of $\leq_{\mathrm{mc}}$. We just proved that f is an epi. The domain of f is a wqo (as a set of finite sequences from a wqo, cf. Theorem 3.1), so its codomain $(\mathcal{H}_\circ, \leq_{\mathrm{mc}})$ is a wqo as well according to Remark 3.3 and this concludes the proof. □

Lemma 3.46. *Let $k \in \mathbb{N}$ and let $\mathcal{H}$ be a class of 2-rooted graphs, none of which having more than k edges between the two roots. Let $\mathcal{H}^-$ be the class of graphs of $\mathcal{H}$ where all edges between the two roots have been removed. If $(\mathcal{H}, \leq_{\mathrm{mc}})$ is a wqo, then so is $(\mathcal{H}^-, \leq_{\mathrm{mc}})$.*

Proof. Let us assume that $(\mathcal{H}, \leq_{\mathrm{mc}})$ is a wqo. For every $i \in \llbracket 0, k \rrbracket$, let $\mathcal{H}_i$ be the subclass of graphs of $\mathcal{H}$ having exactly i edges between the two roots. Each class $\mathcal{H}_i$ ($i \in \llbracket 0, k \rrbracket$) is a subclass of $\mathcal{H}$ which is well-quasi-ordered by $\leq_{\mathrm{mc}}$, therefore it is well-quasi-ordered by $\leq_{\mathrm{mc}}$ as well. Let f be the function that, given a 2-rooted graph G, returns a copy of G where all edges between the roots have been deleted. The rest of the proof draws upon the following remark.

Remark 3.12. Let G, H be two edge-rooted graphs where the edge between the roots has the same multiplicity. Then $H \leq_{\mathrm{mc}} G \Leftrightarrow f(H) \leq_{\mathrm{mc}} f(G)$ (every model of H in G is also a contraction model of $f(H)$ in $f(G)$, and vice-versa).

Let $i \in \llbracket 0, k \rrbracket$, let $\mathcal{H}_i^- = \{f(H),\ H \in \mathcal{H}_i\}$, and let $\langle f(G_i) \rangle_{i \in \mathbb{N}}$ be an infinite sequence of $\mathcal{H}_i^-$. By an observation above, $(\mathcal{H}_i, \leq_{\mathrm{mc}})$ is a wqo, hence $\langle G_i \rangle_{i \in \mathbb{N}}$ has a good pair (G_i, G_j) (with $i, j \in \mathbb{N},\ i < j$). According to Remark 3.12, $(f(G_i), f(G_j))$ is a good pair of $\langle f(G_i) \rangle_{i \in \mathbb{N}}$. Every infinite sequence of $(\mathcal{H}_i^-, \leq_{\mathrm{mc}})$ has a good pair, therefore this qoset is a wqo. Remark that $(\mathcal{H}^-, \leq_{\mathrm{mc}})$ is the union of the $k+1$ wqos $\{(\mathcal{H}_i^-, \leq_{\mathrm{mc}})\}_{i \in \llbracket 0, k \rrbracket}$, therefore it is a wqo as well (cf. Remark 3.2) and this concludes the proof. □

Proposition 3.2 provides an interesting description of the structure of 2-connected simple graphs. The two following easy lemmas show that it can easily be adapted to multigraphs.

Lemma 3.47. *Let G be a graph and let G' be its underlying simple graph. The graph G is 2-connected iff G' is 2-connected or $G = \theta_k$ for some integer $k \geq 2$.*

Proof. It is clear that G is 2-connected whenever G' is. Let us now assume that G is 2-connected but G' is not, and let $u, v \in V(G')$ be two distinct vertices of G' such that there is no pair of internally disjoint paths from u to v in G'. Since G is 2-connected, there are two internally disjoint paths P and Q in G linking u to v. Remark that if P and Q are edge-disjoint, then the corresponding paths in G' are internally disjoint and link u

to v, a contradiction with the choice of these two vertices. Therefore P and Q share an edge (which has multiplicity at least two). Since these paths are internally disjoint, their ends must be the ends of the edge that they share: $\{u, v\}$ is an edge with multiplicity at least two. Removing the edge $\{u, v\}$ in G yields two connected components, one, G_u, containing u and the other, G_v, containing v. Since every path from vertices of G_u to vertices of G_v in G contains u, the graph G_u contains only the vertex u (otherwise G is not 2-connected) and by symmetry $V(G_v) = \{v\}$. Therefore $G = \theta_k$, for some integer $k \geq 2$, as required. □

Lemma 3.48 (extension of Proposition 3.2 to graphs)**.** *Every 2-connected graph has a a tree-decomposition $(T, \mathcal{X})$ such that $|X_t \cap X_{t'}| = 2$ for every edge $\{t, t'\} \in T$ and where every torso is either 3-connected or a cycle.*

Proof. Let G be a 2-connected graph and G' be its underlying simple graph. If G' is 2-connected, then by Proposition 3.2 it has a tree-decomposition $(T, \mathcal{X})$ such that $|X_t \cap X_{t'}| = 2$ for every edge $\{t, t'\} \in T$ and where every torso is either 3-connected, or a cycle. Noticing that $(T, \mathcal{X})$ is also a tree-decomposition of G concludes this case. If G' is not 2-connected, then by Lemma 3.47 we have $G = \theta_k$ for some integer $k \geq 2$. If $k = 2$ the graph G is a cycle, and if $k > 2$ it is 3-connected, therefore it has a trivial tree-decomposition with one bag, which satisfies the properties required in the statement of the lemma. □

We call a tree decomposition as in Lemma 3.48 a *Tutte decomposition*.

3.7.2 Well-quasi-ordering graphs without big bonds

The main result is proved in three steps. First, we show that for every $k \in \mathbb{N}$, the class of labeled 2-connected graphs of $\mathcal{G}_{1,k}$ is well-quasi-ordered by $\leq_{\mathrm{mc}}$. Then, we use Lemma 3.3 to extend this result to all graphs of $\mathcal{G}_{1,k}$, i.e. all connected graphs not containing a bond of size more than k. The result for disconnected graphs then follows by the application of Lemma 3.5.

Lemma 3.49. *For every $k \in \mathbb{N}$, and for every wqo $(\Sigma, \preceq)$, the qoset*

$$(\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}^{(2)}_{1,k}), \leq_{\mathrm{mc}})$$

is a wqo.

Proof. Let $k \in \mathbb{N}$, and let $(\Sigma, \preceq)$ be a wqo. By contradiction, let us assume that $(\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}^{(2)}_{1,k}), \leq_{\mathrm{mc}})$ is not a wqo. We consider the edge-rooted closure $\mathcal{H}$ of $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}^{(2)}_{1,k})$, i.e. the class of all edge-rooted graphs whose underlying non-rooted graphs belongs to $\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}^{(2)}_{1,k})$. Clearly, $(\mathcal{H}, \leq_{\mathrm{mc}})$ is not a wqo, as a consequence of our initial assumption. We will show that this leads to a contradiction.

Let $\{A_i\}_{i \in \mathbb{N}}$ be an infinite minimal (wrt. $\leq_{\mathrm{mc}}$) bad sequence of $(\mathcal{H}, \leq_{\mathrm{mc}})$: for every $i \in \mathbb{N}$, A_i is a minimal graph (wrt. $\leq_{\mathrm{mc}}$) such that there is an infinite bad sequence starting with $A_0, \ldots, A_i$. For every $i \in \mathbb{N}$, A_i has a Tutte decomposition (Lemma 3.48) which has a bag containing the endpoints of the edge $\{A_i.r, A_i.s\}$ (because it is a tree decomposition). Let $A_i.X$ be the torso of some (arbitrarily chosen) bag in such a decomposition which contains $A_i.r$ and $A_i.s$.

For every edge $x, y \in V(A_i.X)$, let $A_i.V_{x,y}$ be the vertex set of the (unique) block which contains both x and y in the graph obtained from A_i by deleting vertices $V(A_i.X) \setminus \{x, y\}$ and adding the edge $\{x, y\}$ with multiplicity 2.

Let us consider graphs obtained by contracting all the edges of A_i that does not have both endpoints in $A_i.V_{x,y}$ in a way such that $A_i.r$ gets contracted to x and $A_i.s$ gets contracted to y. Remark that for fixed i and (x, y), these graphs differ only by the multiplicity of the edge between the two roots x and y. For every $i \in \mathbb{N}$ and $x, y \in V(A_i.X)$, we denote by $A_i.C_{x,y}$ an arbitrarily chosen such graph. Eventually, we set $A_i.\mathcal{C} = \{A_i.C_{x,y},\ x, y \in V(A_i.X)\}$. Remark that every graph of $A_i.\mathcal{C}$ belongs to $\mathcal{G}^{(2)}_{1,k}$ and is a contraction of A_i.

Claim 3.2. $\mathcal{C} = \cup_{i\in\mathbb{N}} A_i.\mathcal{C}$ is wqo by $\leq_{\mathrm{mc}}$.

Proof. By contradiction, assume that $(\mathcal{C}, \leq_{\mathrm{mc}})$ has an infinite bad sequence $\{B_i\}_{i\in\mathbb{N}}$. By definition of $\mathcal{C}$, for every $i \in \mathbb{N}$ there is a $j = \varphi(i) \in \mathbb{N}$ such that $B_i \leq_{\mathrm{mc}} A_j$. Let $i_0 \in \mathbb{N}$ be an integer with $\varphi(i_0)$ minimum. Let us consider the following infinite sequence:

$$A_0, \dots, A_{\varphi(i_0)-1}, B_{i_0}, B_{i_0+1}, \dots.$$

Remark that this sequence cannot have a good pair of the form $A_i \leq_{\mathrm{mc}} A_j$, $0 \leq i < j < \varphi(i_0)$ (respectively $B_i \leq_{\mathrm{mc}} B_j$, $i_0 \leq i < j$) since $\{A_i\}_{i\in\mathbb{N}}$ (respectively $\{B_i\}_{i\in\mathbb{N}}$) is an antichain. Let us assume that there is a good pair of the form $A_i \leq_{\mathrm{mc}} B_j$, for some $i \in [\![0, \varphi(i_0) - 1]\!]$, $j \geq i_0$. Then we have $A_i \leq_{\mathrm{mc}} B_j \leq_{\mathrm{mc}} A_{\varphi(j)}$. By the choice of i_0 we have $\varphi(i_0) \leq \varphi(j)$, hence $i < \varphi(j)$ so $(A_i, A_{\varphi(j)})$ is a good pair of $\{A_i\}_{i\in\mathbb{N}}$, a contradiction. Therefore, this sequence is an infinite bad sequence of $(\mathcal{H}, \leq_{\mathrm{mc}})$ and we have $B_{i_0} \leq_{\mathrm{mc}} A_{\varphi(i_0)}$ and $B_{i_0} \neq A_{\varphi(i_0)}$. This contradicts the minimality of $\{A_i\}_{i\in\mathbb{N}}$, therefore $(\mathcal{C}, \leq_{\mathrm{mc}})$ is a wqo. □

Let $\mathcal{C}^-$ be the class of 2-rooted graphs obtained from graphs of $\mathcal{C}$ by deleting the edge between the roots. We set $\mathcal{C}^+ = \{H \oplus^{H.s}_{H.r} \theta_i,\ i \in [\![0, k]\!],\ H \in \mathcal{C}^-\}$. In other words C^+ is the class of graphs that can be constructed by possibly replacing the edge at the root of a graph of $\mathcal{C}$ by an edge of multiplicity i, for any $i \in [\![1, k]\!]$.

Remark 3.13. It follows from Lemma 3.46 that both $(\mathcal{C}^-, \leq_{\mathrm{mc}})$ and $(\mathcal{C}^+, \leq_{\mathrm{mc}})$ are wqos.

Notice that for every $i \in \mathbb{N}$ and $\{x, y\} \in E(A_i.X)$, the graph $A_i[A_i.V_{x,y}]$ rooted in (x, y) belongs to $\mathcal{C}^+$. As explained thereafter, this property enables us to see A_i as a graph built from graphs of $\mathcal{C}^+$.

According to Lemma 3.48, for every $i \in \mathbb{N}$, the graph $A_i.X$ (which is the torso of a bag of a Tutte decomposition) is either a 3-connected graph (and thus $|V(A_i.X)| < \zeta_k$ by Proposition 3.3), or a cycle (of any length). Therefore we can partition $\{A_i\}_{i\in\mathbb{N}}$ into at most ζ_k subsequences depending on the *type* of $A_i.X$, where this type can be either "cycle" or one type for each possible value of $|V(A_i.X)|$ when $A_i.X$ is 3-connected. Let us show that each of these subsequences is finite.

First case: $\{A_i\}_{i\in\mathbb{N}}$ has an infinite subsequence $\{D_i\}_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$, $D_i.X$ is a cycle. Then each graph of $\{D_i\}_{i\in\mathbb{N}}$ can be constructed by attaching a graph of the wqo $(\mathcal{C}^+, \leq_{\mathrm{mc}})$ to each edge of a cycle after deleting this edge. By Lemma 3.45, these graphs are wqo by $\leq_{\mathrm{mc}}$, a contradiction.

Second case: for some positive integer $n < \zeta_k$, $\{A_i\}_{i\in\mathbb{N}}$ has an infinite subsequence $\{D_i\}_{i\in\mathbb{N}}$ such that for every $i \in \mathbb{N}$, $|V(D_i.X)| = n$. Then every graph of $\{D_i\}_{i\in\mathbb{N}}$ can be constructed by attaching a graph of the wqo $(\mathcal{C}^+, \leq_{\mathrm{mc}})$ to each pair of distinct vertices of $\overline{K}_n$. By Lemma 3.44, $\{D_i\}_{i\in\mathbb{N}}$ has a good pair, which is contradictory since it is a bad sequence.

We just proved that $\{A_i\}_{i\in\mathbb{N}}$ can be partitioned into a finite number of subsequences each of which is finite. Hence $\{A_i\}_{i\in\mathbb{N}}$ is finite as well, a contradiction. Therefore our initial assumption is false and $(\mathrm{lab}_{(\Sigma,\preceq)}(\mathcal{G}^{(2)}_{1,k}), \leq_{\mathrm{mc}})$ is a wqo. $\square$

Corollary 3.19. *For every $k \in \mathbb{N}$, the class $\mathcal{G}_{1,k}$ is well-quasi-ordered by $\leq_{\mathrm{mc}}$.*

Proof. According to Lemma 3.49, for every wqo $(\Sigma, \preceq)$, the class of Σ-labeled 2-connected graphs of $\mathcal{G}$ are wqo by $\leq_{\mathrm{mc}}$. By Lemma 3.3, this implies that $(\mathcal{G}_{1,k}, \leq_{\mathrm{mc}})$ is a wqo and we are done. $\square$

The proof of Theorem 3.22 now follows from the combination of Corollary 3.19 and Lemma 3.5.

3.7.3 Canonical antichains and multigraph contractions

This section is devoted to the proof of the two results related to canonical antichains of $\leq_{\mathrm{mc}}$.

Remark 3.14. Every canonical antichain of $\leq_{\mathrm{mc}}$ is infinite.

Proof of Theorem 3.16. "$\Rightarrow$": Let $\mathcal{A}$ be a canonical antichain of $\leq_{\mathrm{mc}}$ and let us assume for contradiction that $\mathcal{B} = \mathcal{A}_\theta \setminus \mathcal{A}$ (respectively $\mathcal{B} = \mathcal{A}_{\overline{K}} \setminus \mathcal{A}$) is infinite. Let $\mathcal{B}^+$ be the closure of $\mathcal{B}$ and remark that $\mathcal{B}^+ = \mathcal{B} \cup \{K_1\}$ (respectively $\mathcal{B}^+ = \mathcal{B}$). Then the contraction-closed class $\mathcal{B}^+$ has finite intersection with $\mathcal{A}$ whereas it contains the infinite antichain $\mathcal{B}$. This is a contradiction with the fact that A is canonical, hence both $\mathcal{A}_\theta \setminus \mathcal{A}$ and $\mathcal{A}_{\overline{K}} \setminus \mathcal{A}$ are finite.

Let us now assume that $\mathcal{C} = \mathcal{A} \setminus \{\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}\}$ is infinite and let $\mathcal{C}^+$ be the closure of $\mathcal{C}$. Being a subset of an antichain, $\mathcal{C}$ is an antichain as well and consequently $\mathcal{C}^+$ is a contraction-closed class that is not well-quasi-ordered. By Theorem 3.11, $\mathcal{C}^+$ contains infinitely many elements of $\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}$. Notice that besides being infinite, $\mathcal{C}^+ \cap (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}})$ is also disjoint from $\mathcal{A} \cap (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}})$, otherwise $\mathcal{A}$ would contain an element from $\mathcal{C}$ contractible to an element of $\mathcal{A} \cap (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}})$. But then one of $\mathcal{A}_\theta \setminus \mathcal{A}$ and $\mathcal{A}_{\overline{K}} \setminus \mathcal{A}$ is infinite, a contradiction with our previous conclusion. Therefore $\mathcal{C}$ is finite.

"$\Leftarrow$": Let $\mathcal{A}$ be an antichain such that each of $\mathcal{A}_\theta \setminus \mathcal{A}$, $\mathcal{A}_{\overline{K}} \setminus \mathcal{A}$, and $\mathcal{A} \setminus \{\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}\}$ is finite, and let us show that $\mathcal{A}$ is canonical. Let $\mathcal{F}$ be a contraction-closed class of $\mathfrak{G}$. If $\mathcal{F} \cap \mathcal{A}$ is infinite, then $\mathcal{F}$ trivially contains the infinite antichain $\mathcal{F} \cap \mathcal{A}$. On the other hand, if $\mathcal{F} \cap \mathcal{A}$ is finite then by Theorem 3.11 the class $\mathcal{F}$ is well-quasi-ordered, hence by definition it does not contain an infinite antichain. Consequently, $\mathcal{A}$ is canonical, as required. $\square$

Proof of Corollary 3.1. Let $\mathcal{A}$ be a canonical antichain of $\leq_{\mathrm{mc}}$. Observe that we have the following:

$$\mathrm{Incl}_{\leq_{\mathrm{mc}}}(\mathcal{A}) = \mathrm{Incl}_{\leq_{\mathrm{mc}}}(\mathcal{A} \cap \mathcal{A}_\theta) \cup \mathrm{Incl}_{\leq_{\mathrm{mc}}}(\mathcal{A} \cap \mathcal{A}_{\overline{K}}) \cup \mathrm{Incl}_{\leq_{\mathrm{mc}}}(\mathcal{A} \setminus (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}})).$$

Now, is is easy to notice that:

- $\mathrm{Incl}_{\leq_{\mathrm{mc}}}(\mathcal{A} \cap \mathcal{A}_\theta) \subseteq \mathrm{Incl}_{\leq_{\mathrm{mc}}}(\mathcal{A}_\theta) = \{K_1\}$;
- $\mathrm{Incl}_{\leq_{\mathrm{mc}}}(\mathcal{A} \cap \mathcal{A}_{\overline{K}}) \subseteq \mathrm{Incl}_{\leq_{\mathrm{mc}}}(\mathcal{A}_{\overline{K}}) = \emptyset$;
- $\mathrm{Incl}_{\leq_{\mathrm{mc}}}(\mathcal{A} \setminus (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}}))$ is finite, because $\mathcal{A} \setminus (\mathcal{A}_\theta \cup \mathcal{A}_{\overline{K}})$ is finite by Theorem 3.16 and since $\mathcal{A}$ is canonical.

Therefore, $\mathrm{Incl}_{\leq_{\mathrm{mc}}}(\mathcal{A})$ is finite as well and hence cannot contain an infinite antichain; this proves that $\mathcal{A}$ is fundamental. □

Chapter 4

Exclusion theorems

This chapter contains material that previously appeared in the following articles:

- *Low polynomial exclusion of planar graph patterns*, co-authored with Dimitrios M. Thilikos, to appear in Journal of Graph Theory, 2015 [RT15];
- *An edge variant of the Erdős-Pósa property*, co-authored with Ignasi Sau and Dimitrios M. Thilikos, to appear in Discrete Mathematics, Volume 339, Issue 8, 2016 [RST16];
- *Minors in graphs of large θ_r-girth*, co-authored with Dimitris Chatzidimitriou, Ignasi Sau, and Dimitrios M. Thilikos, 2015, submitted [CRST15b].
- *Packing and covering immersion models of planar subcubic graphs*, co-authored with Archontia Giannopoulou, O-joung Kwon, and Dimitrios M. Thilikos, presented at the 42nd International Workshop on Graph-Theoretic Concepts in Computer Science, WG 2016, Istanbul, Turkey, 2016 [GKRT16];

One of the most celebrated results from the Graph Minors series of Robertson and Seymour is the following result, also known as the *Grid Exclusion Theorem*.

Theorem 4.1 (Grid Exclusion Theorem, [RS86])**.** *There exists a function $f\colon \mathbb{N} \to \mathbb{N}$ such that, for every integer h, every graph that does not contain a minor isomorphic to the $h \times h$-grid has treewidth at most $f(h)$.*

This result is an *exclusion theorem*: it relates a graph parameter (in this case the treewidth) with the absence of a pattern as a substructure. In this chapter, we present exclusion theorems related to the parameters of girth (Section 4.1), treewidth (Section 4.3), maximum degree (Section 4.2), and tree-cut width (Section 4.4).

Beside their combinatorial value, exclusion theorems has proven useful in order to obtain Erdős–Pósa-type results. This aspect will be illustrated in Chapter 5.

4.1 Clique majors in graphs of large θ_r-girth

In this section, we introduce the concept of θ_r-girth of a graph and show that graphs of sufficiently large minimum degree contain clique-majors whose order is an exponential function of their girth, extending a result of [KO03]. We also show that the minimum degree can be replaced by some connectivity measurement.

4.1.1 The quest for large clique-majors

A classic result in graph theory asserts that if a graph has minimum degree $ck\sqrt{\log k}$, then it can be transformed to a complete graph of at least k vertices by applying edge contractions (i.e., it contains a *k-clique minor*). This result has been proved by Kostochka in [Kos84] and Thomason in [Tho83a] and a precise estimation of the constant c has been given by Thomason in [Tho01a]. For recent results related to conditions that force a clique minor see [Mar04, JW13, DHJ+13, FKO09, KO04a].

Recall that the *girth* of a graph G is the minimum length of a cycle in G. Interestingly, graphs of large minimum degree contain clique-minors whose order is an *exponential* function of their girth. In particular, it follows by the main result of Kühn and Osthus in [KO03] that there is a constant c such that, if a graph has minimum degree $d \geq 3$ and girth z, then it contains as a minor a clique of size k, where

$$k \geq \frac{d^{cz}}{\sqrt{z \cdot \log d}}.$$

In this section we provide conditions, alternative to the above one, that can force the existence of a clique-minor whose size is exponential.

H-girth. We say that a graph H is a *minor* of a graph G, if H can be obtained from G by using the operations vertex-removal, edge-removal, and edge-contraction. An *H-model* in G is a subgraph of G that contains H as a minor. Given two graphs G and H, we define the *H-girth* of G as the minimum number of edges of an H-model in G. If G does not contain H as a minor, we will say that its H-girth is equal to infinity. For every $r \in \mathbb{N}$, recall that θ_r denote the graph with two vertices and r parallel edges. Clearly, the girth of a graph is its θ_2-girth and, for every $r_1 \leq r_2$, the θ_{r_1}-girth of a graph is at most its θ_{r_2}-girth.

Our first result is the following extension of the result of Kühn and Osthus in [KO03] for the case of θ_r-girth.

Theorem 4.2. *There is a constant $c > 0$ such that, for every $r \geq 2$, $d \geq 3r$, and $z \geq r$, if a graph has minimum degree d and θ_r-girth at least z, then it contains as a minor a clique of size k, where*

$$k \geq \frac{(\frac{d}{r})^{\frac{cz}{r}}}{\sqrt{\frac{z}{r} \cdot \log d}}.$$

In the formula above, a lower bound to the minimum degree as a function of r is necessary. An easy computation shows that when applying Theorem 4.2 for $r = 2$, we can get the aforementioned formula of Kühn and Osthus, where the constant in the exponent is one fourth of the constant of Theorem 4.2.

Our second finding is that this degree condition can be replaced by some "loose connectivity" requirement.

Loose connectivity. For two integers $\alpha, \beta \in \mathbb{N}$, a graph G is called (α, β)*-loosely connected* if for every $A, B \subseteq V(G)$ such that $V(G) = A \cup B$ and G has no edge between $A \setminus B$ and $B \setminus A$, we have that $|A \cap B| < \beta \Rightarrow \min(|A \setminus B|, |B \setminus A|) \leq \alpha$. Intuitively, this means that a *small* separator (i.e., on less than β vertices) cannot "split" the graph into two *large* parts (that is, with more than α vertices each).

Our second result indicates that the requirement on the minimum degree in Theorem 4.2 can be replaced by the loose connectivity condition as follows.

Theorem 4.3. *There is a constant $c > 0$ such that, for every $r \geq 2$, $z > r$, and $\alpha \geq 1$, it holds that if a graph has more than $(\alpha + 1) \cdot (2r - 1)$ vertices, is $(\alpha, 2r - 1)$-loosely connected, and has θ_r-girth at least z, then it contains as a minor a clique of size k where*

$$k \geq \frac{2^{c \cdot \frac{z}{r\alpha}}}{\sqrt{r}}.$$

Both Theorem 4.2 and Theorem 4.3 are derived from two more general results, namely Theorem 4.5 and Theorem 4.4, respectively. Theorem 4.5 asserts that graphs with large θ_r-girth and sufficiently large minimum degree contain as a minor a graph whose minimum degree is exponential in the girth. Theorem 4.4 replaces the minimum degree condition with the absence of sufficiently large "edge-protrusions", that are roughly tree-like structured subgraphs with small boundary to the rest of the graph (see Subsection 4.1.2 for the detailed definitions).

Organisation of the section. The main notions used in this section are defined in Subsection 4.1.2. Then, we show in Subsection 4.1.3 that the proofs of Theorem 4.2 and Theorem 4.3 can be derived from Theorem 4.5 and Theorem 4.4, which are proved in Subsection 4.1.4.

4.1.2 Definitions specific to this section

In this section, when giving the running time of an algorithm involving some graph G, we agree that $n = |V(G)|$ and $m = |E(G)|$.

In order to decompose graphs along edge cuts, we introduce the following edge-counterpart of the notion of *(vertex-)protrusion* used in [BFL+09a, BFL+09b] (among others). A subset $Y \subseteq V(G)$ is a *t-edge-protrusion* of G with *extension* w (for some positive integer w) if the graph $G[Y \cup N_G(Y)]$ has a rooted tree-partition $\mathcal{D} = (T, s, \mathcal{X})$ of width at most t and such that $N_G(Y) = X_s$ and $|V(T)| \geq w$. The protrusion Y is said to be *connected* whenever $Y \cup N_G(Y)$ induces a connected subgraph in G.

Distance-decompositions. A *distance-decomposition* of a connected graph G is a rooted tree-partition $\mathcal{D} = (T, s, \mathcal{X})$ of G, where the following additional requirements are met (see also [YBdFT99]):

(i) X_s contains only one vertex, we shall call it u, refered to as the *origin* of $\mathcal{D}$;

(ii) for every $t \in V(T)$ and every $x \in X_t$, $\mathbf{dist}_G(x, u) = \mathbf{dist}_T(t, s)$;

(iii) for every $t \in V(T)$, the graph $G_t = G\left[\bigcup_{t' \in \mathrm{desc}_{(T,s)}(t)} X_{t'}\right]$ is connected; and

(iv) if C is the set of children of a vertex $t \in V(T)$, then the graphs $\{G_{t'}\}_{t' \in C}$ are the connected components of $G_t \setminus X_t$.

An example of distance-decomposition is given in Figure 4.1. For every vertex u of a graph on m edges, a distance-decomposition $(T, s, \mathcal{X})$ with origin u can be constructed in $O(m)$ steps by breadth-first search.

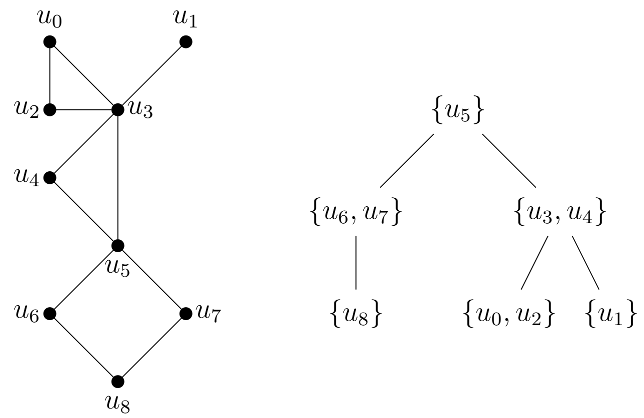


Figure 4.1: A graph (left) and a distance-decomposition with origin u_5 of it (right).

For every $t \in V(T) \setminus \{s\}$, we define $E^{(t)}$ as the set of edges that have one endpoint in X_t and the other in $X_{\mathbf{p}(t)}$.

Let P be a path in G and $\mathcal{D} = (T, s, \mathcal{X})$ a distance-decomposition of P. We say that P is a *straight path* if the heights, in (T, s), of the indices of the bags in $\mathcal{D}$ that contain vertices of P are pairwise distinct. Obviously, in that case, the sequence of the heights of the bags that contain each subsequent vertex of the path is strictly monotone.

Grouped partitions. Let G be a connected graph and let $d \in \mathbb{N}$. A *d-grouped partition* of G is a partition $\mathcal{R} = \{R_1, \ldots, R_l\}$ of $V(G)$ (for some positive integer l) such that for each $i \in \{1, \ldots, l\}$, the graph $G[R_i]$ is connected and there is a vertex $s_i \in R_i$ with the following properties:

(i) $\mathbf{ecc}_{G[R_i]}(s_i) \leq 2d$ and

(ii) for each edge $e = \{x, y\} \in E(G)$ where $x \in R_i$ and $y \in R_j$ for some distinct integers $i, j \in \{1, \ldots, l\}$, it holds that $\mathbf{dist}_G(x, s_i) \geq d$ and $\mathbf{dist}_G(y, s_j) \geq d$.

A set $S = \{s_1, \ldots, s_l\}$ as above is a *set of centers* of $\mathcal{R}$ where s_i is the *center* of R_i for $i \in \{1, \ldots, l\}$.

Given a graph G, we define a *d-scattered set* W of G as follows:

- $W \subseteq V(G)$ and
- $\forall u, v \in W$, $\mathbf{dist}_G(u, v) > d$.

If W is inclusion-maximal, it will be called a *maximal d-scattered set* of G.

Frontiers and ports. Let G be a graph, let $\mathcal{R} = \{R_1, \ldots, R_l\}$ be a d-grouped partition of G, and let $S = \{s_1, \ldots, s_l\}$ be a set of centers of $\mathcal{R}$. For every $i \in [\![1, l]\!]$, we denote by $\mathcal{D}_i = (T_i, s_i, \mathcal{X}_i)$ the unique distance-decomposition with origin s_i of the graph $G[R_i]$ where $\mathcal{X}_i = \{X_t^i\}_{t \in V(T_i)}$. For every $i \in [\![1, l]\!]$ and every $h \in [\![0, \mathbf{ecc}_{T_i}(s_i)]\!]$, we denote by I_i^h the vertices of (T_i, s_i) that are at distance h from s_i, and we set $I_i^{<h} = \bigcup_{h'=0}^{h-1} I_{h'}^s$ and $I_i^{\geq h} = \bigcup_{h'=h}^{\mathbf{ecc}_{T_i}(s_i)} I_i^{h'}$. We also set

$$V_i^h = \bigcup_{t \in I_i^h} X_t^i, \qquad V_i^{<h} = \bigcup_{t \in I_i^{<h}} X_t^i, \text{ and } \qquad V_i^{\geq h} = \bigcup_{t \in I_i^{\geq h}} X_t^i.$$

The *vertex-frontier* F_i of R_i is the set of vertices in V_i^{d-1} that are connected in G to a vertex $x \in V(G) \setminus R_i$ via a path, the internal vertices of which belong to $V_i^{\geq d}$. The *node-frontier* of T_i is

$$N_i \quad = \quad \{t \in V(T_i) \mid F_i \cap X_t \neq \emptyset\}. \tag{4.1}$$

A vertex $t \in I_i^{\geq d-1}$ is called a *port* of T_i if X_t^i contains some vertex that is adjacent in G to a vertex of $V(G) \setminus R_i$.

4.1.3 Finding small θ_r-models

Two intermediate results

The main results of this section are the following.

Theorem 4.4. *There exists an algorithm that, with input three positive integers r, w, z and an n-vertex graph G, outputs one of the following:*

- *a θ_r-model in G with at most z edges,*
- *a connected $(2r-2)$-edge-protrusion Y of G with extension more than w, or*
- *an H-model in G for some graph H where $\delta(H) \geq \frac{1}{r-1} 2^{\frac{z-5r}{4r(2w+1)}}$,*

in $O_r(m)$ steps.

Theorem 4.5. *There exists an algorithm that, with input three integers r, δ, z, where $r \geq 2$, $\delta \geq 3r$, and $z \geq r$ and an n-vertex graph G, outputs one one the following:*

- *a θ_r-model in G with at most z edges,*
- *a vertex v of G of degree less than δ, or*
- *an H-model in G for some graph H where $\delta(H) \geq \frac{\delta-2r+3}{r-1} \cdot \lfloor \frac{\delta}{r-1} - 1 \rfloor^{\frac{z-r}{4r}}$,*

in $O_r(m)$ steps.

The results of Chandran and Subramanian in [CS05] imply that if G has girth at least z and mimumum degree at least δ, then $\mathbf{tw}(G) \geq \delta^{c \cdot z}$, for some constant c. As in the third condition of Theorem 4.5 it holds that $\mathbf{tw}(G) \geq \mathbf{tw}(H) \geq \delta(H)$, Theorem 4.5 can also be seen as a qualitative extension of the results of [CS05].

The above two results will be used to prove Theorem 4.2 and Theorem 4.3. We will also need the following result of Kostochka [Kos84].

Proposition 4.1 ([Kos84], see also [Tho83a, Tho01a]). *There exists a constant $c_K \in \mathbb{R}$ such that for every $d \in \mathbb{N}$, every graph of average degree at least d contains a clique of order k as a minor, for some integer k satisfying*

$$k \geq c_K \cdot \frac{d}{\sqrt{\log d}}.$$

Proof of Theorem 4.2. Observe that since G has no θ_r-model with at most z edges and G has minimum degree $\delta \geq 3r$, a call to the algorithm of Theorem 4.5 on (r, δ, z, G) should return an H-model in G, for some graph H where $\delta(H) \geq \frac{\delta - 2r + 3}{r-1} \cdot \lfloor \frac{\delta}{r-1} - 1 \rfloor^{\frac{z-r}{4r}} =: d$. It is not hard to check that there is a positive constant $c' \in \mathbb{R}$ such that

$$c_K \cdot \frac{d}{\sqrt{\log d}} \geq \frac{(\frac{\delta}{r})^{c' \cdot \frac{z}{r}}}{\sqrt{\frac{z}{r} \cdot \log \delta}}.$$

Hence by Proposition 4.1, G has a clique of the desired order as a minor. $\square$

Proof of Theorem 4.3. As in the proof of Theorem 4.2, the properties that G enjoys will force a minor of large minimum degree. Let us call the algorithm of Theorem 4.4 on $(r, 3\alpha, z, G)$. We assumed that G has no θ_r-model with z edges or less, hence the output of the algorithm cannot be such a model. Let us now assume that the algorithm outputs a $(2r-2)$-edge-protrusion Y of extension more than 3α, and let $(T, s, \mathcal{X})$ be a rooted tree-partition of Y of width at most $2r-2$ such that $N_G(Y) = X_s$ and $|V(T)| > 3\alpha$. It is known that every tree of order n has a vertex, the removal of which partitions the tree into components of size at most $n/2$ each. Hence, there is a vertex $v \in V(T)$ and a partition (Z, Z') of $V(T) \setminus \{v\}$ such that:

- both $Z \cup \{v\}$ and $Z' \cup \{v\}$ induce connected subtrees of T;
- $\frac{1}{3}|V(T)| \leq |Z|, |Z'| \leq \frac{2}{3}|V(T)|$; and
- $X_s \subseteq Z$ or $v = s$.

Let $A = Z' \cup \{X_v\}$ and $B = V(G) \setminus Z'$. Notice that $V(G) = A \cup B$ and that no edge of G lies between A and B. As $A \cap B = X_v$, we have $|A \cap B| < 2r - 1$. Last, $Z' \subseteq A \setminus B$ and $Z \subseteq B \setminus A$ give that $|A \setminus B|, |B \setminus A| \geq \alpha$. The existence of A and B contradicts the fact that G is $(\alpha, 2r-1)$-loosely connected. Thus G has no $(2r-2)$-edge-protrusion Y of extension more than 3α.

A consequence of this observation is that the only possible output of the algorithm mentioned above is an H-model in G for some graph H, where

$$\delta(H) \geq \frac{1}{r-1} \cdot 2^{\frac{z-5r}{4r(6\alpha+1)}} =: d.$$

As in the proof of Theorem 4.2, it suffices to remark that there is a positive constant $c'' \in \mathbb{R}$ such that

$$c_K \cdot \frac{d}{\sqrt{\log d}} \geq \frac{2^{c'' \cdot \frac{z}{r\alpha}}}{\sqrt{r}}$$

in order to conclude the proof. □

4.1.4 The proofs of Theorem 4.4 and Theorem 4.5

Preliminary results

Before proving Theorem 4.4 and Theorem 4.5 (in Section 4.1.4 and Section 4.1.4, respectively) we need some preliminary results. Let us start we some definitions.

Let (T, s) be a rooted tree and let N be a subset of its leaves. We say that a vertex u of T is *N-critical* if either it belongs to $N \cup \{s\}$ or there are at least two vertices in N that are descendants of two distinct children of u. An *N-unimportant* path in T is a path with at least 2 vertices, with exactly two N-critical vertices, which are its endpoints (see Figure 4.2 for a picture). Notice that an N-unimportant path in T cannot have an internal vertex that belongs to some other N-unimportant path. Also, among the two endpoints of an N-unimportant path there is always one which is a descendant of the other. As we see in the proof of the following lemma, N-unimportant paths are the maximal paths with internal vertices of degree 2 that appear if we repeatedly delete leaves that do not belong to $N \cup \{s\}$.

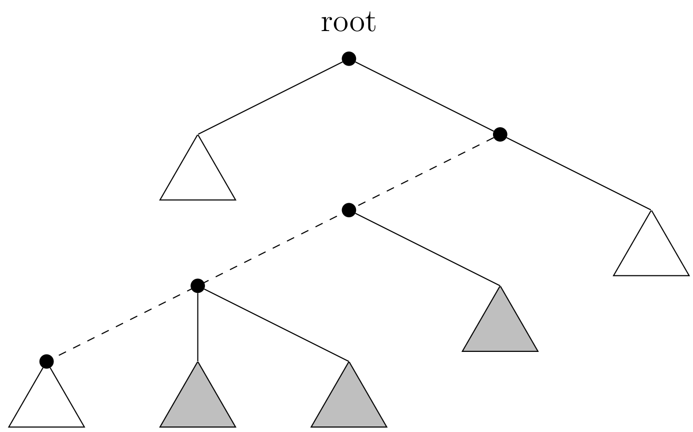


Figure 4.2: An unimportant path (dashed) in a tree. Gray subtrees are those without vertices from N.

Lemma 4.1. *Let $d, k \in \mathbb{N}$, $k \geq 1$. Let (T, s) be a rooted tree and let N be a set of leaves of (T, s), each of which is at distance at least than d from s. If for some integer k, every N-unimportant path in T has length at most k, then $|N| \geq 2^{d/k}$.*

Proof. We consider the subtree T' of T obtained by repeatedly deleting leaves that do not belong to $N \cup \{s\}$. By construction, every leaf of (T', s) belongs to N, hence our goal is then to show that (T', s) has many leaves. Notice that in (T', s), every vertex of degree at least 3 is N-critical. Therefore, the N-unimportant paths of (T', s) are the maximal paths, the internal vertices of which have degree two. By contracting each of these paths into an edge, we obtain a tree T'' where every internal vertex has degree at least 3. Observe that every edge on a root-leaf path of T'' is originated from the contraction of a path on at most k edges, as we assume that every N-unimportant path in T has length at most k. We deduce that T'' has height at least d/k, hence it has at least $2^{d/k}$ leaves. Consequently, T' has at least $2^{d/k}$ leaves, and then $|N| \geq 2^{d/k}$. □

Recall that if $(T, s, \mathcal{X})$ is a distance-decomposition of a graph and $t \in V(T) \setminus \{s\}$, $E^{(t)}$ denotes as the set of edges that have one endpoint in X_t and the other in $X_{\mathbf{p}(t)}$.

Lemma 4.2. *Let G be an n-vertex graph, let r be a positive integer, let $\mathcal{D} = (T, s, \mathcal{X})$ be a distance-decomposition of G, and let $d > 1$ be the height of (T, s). Then either G contains a θ_r-model with at most $2 \cdot r \cdot d$ edges or for every vertex $i \in V(T) \setminus s$, it holds that $|E^{(i)}| \leq r - 1$. Moreover there exists an algorithm that, in $O_r(m)$ steps, either finds such a model, or asserts that $|E^{(i)}| \leq r - 1$ for every $i \in V(T) \setminus s$.*

Proof. We consider the non-trivial case where $r \geq 2$. Suppose that there exists a node t of (T, s) such that $|E^{(t)}| \geq r$. Clearly, such a t can be found in $O(m)$ steps. We will prove that G contains a θ_r-model. Let k be the height of t in T.

We need first the following claim.

Claim 4.1. Given a non-empty proper subset U of X_t, we can find in G_t a path of length at most $2k$ from a vertex of U to a vertex of $X_t \setminus U$, in $O(m)$ steps.

Proof of Claim 4.1. We can compute a shortest path P from a vertex of U to a vertex of $X_t \setminus U$, in $O(m)$ steps using a BFS. Let us show that P has length at most $2k$. Let $u \in U$ and $v \in X_t \setminus U$ be the endpoints of P, and let w be a vertex of P of the lowest possible height h $(0 \leq h \leq k)$. Then it holds that $\mathbf{dist}_{G_t}(v, u) = \mathbf{dist}_{G_t}(U, v)$. We examine the non-trivial case where P has more than one edge. By minimality of P we have $w \notin X_t$.

Our next step is to prove that if P has more than one edge, then both the subpaths of P from u to w and from v to w are straight. Suppose now, without loss of generality, that the subpath from u to w is not straight and let z be the first vertex of it (starting from u) which is contained in a bag of height greater than or equal to the height of the bag of its predecessor in P. By definition of a distance-decomposition (in particular items (ii) and (iii)), there is at least one vertex $x \in X_t$ which is connected by a straight path P' to z in G. Then there are two possibilities:

- either $x \in U$, and then the union of the path P' and the portion of P between z and v is a path that is shorter than P;
- or $x \in X_t \setminus U$, and in this case the union of the path P' and the portion of P between u and z is a path that is shorter than P.

As, in both cases, the occurring paths contradict the construction of P, we conclude that both the subpath of P from u to w and the one from v to w are straight. This implies that P has length at most $2 \cdot (k - h) \leq 2 \cdot k$ and the claim follows. ◇

Our next step is to construct a vertex set U and a set of paths $\mathcal{P}$ as follows. We set $\mathcal{P} = \emptyset$, $U = \emptyset$, and we start by adding in U an arbitrarily chosen vertex $u \in X_t$. Using the procedure of Claim 4.1, we repeatedly find a path from a vertex of U to a vertex of $X_t \setminus U$, add this second vertex to U and the path to $\mathcal{P}$, until there are at least r edges in $E^{(t)}$ that have endpoints in U.

The construction of U requires at most r repetitions of the procedure of Claim 4.1, and therefore $O(r \cdot m)$ steps in total. Clearly $|U| \leq r$, hence $|\mathcal{P}| \leq r - 1$. Besides, every path in $\mathcal{P}$ has length at most $2k$ according to Claim 4.1. Notice now that $\pmb{\bigcup}\mathcal{P}$ is a connected subgraph of G_t with at most $2k \cdot (r-1)$ edges.

As there are at least r edges in $E^{(t)}$ with endpoints in U we may consider a subset F of them where $|F| = r$. Since $\mathcal{D}$ is a distance-decomposition (by item (ii) of the definition), each edge $e \in F$ is connected to the origin by a path of length $d - k - 1$ whose edges do not belong to G_t. Let $\mathcal{P}'$ be the collection of these paths. Clearly, the paths in $\mathcal{P}'$ contain, in total, at most $r \cdot (d - k - 1)$ edges.

If we now contract in G all edges in $\mathcal{P}$ and all edges in $\mathcal{P}'$, except those in F, and then remove all edges not in F, we obtain a graph isomorphic to θ_r. Therefore we found in G a θ_r-model with at most

$$\begin{aligned} r \cdot (d-k-1) + 2 \cdot k \cdot (r-1) + r &\leq r \cdot (d-k-1) + 2 \cdot k \cdot r + r \\ &= r \cdot (d+k) \\ &\leq 2 \cdot r \cdot d && \text{(since } d \geq k\text{)} \end{aligned}$$

edges in $O(r \cdot m)$ steps. □

The following result is a direct consequence of Lemma 4.2 and item (ii) of the definition of a distance-decomposition.

Corollary 4.1. *Let G be an n-vertex graph, let r be a positive integer, let $\mathcal{D} = (T, s, \mathcal{X})$ be a distance-decomposition of G, and let $d > 1$ be the height of (T, s). If some bag of $\mathcal{D}$ contains at least r vertices, then G contains a θ_r-model with at most $2 \cdot r \cdot d$ edges, which can be found in $O_r(m)$ steps.* □

The remaining lemmata will be related to grouped partitions.

Lemma 4.3. *For every positive integer d and every connected graph G there is a d-grouped partition of G that can be constructed in $O(m)$ steps.*

Proof. If $\mathbf{diam}(G) \leq 2d$, then $\{V(G)\}$ is a d-grouped partition of G. Otherwise, let $R = \{s_1, \ldots, s_l\}$ be a maximal $2d$-scattered set in G. This set can be constructed in $O(m)$ steps by breadth-first search. The sets $\{R_i\}_{i \in [\![1,l]\!]}$ are constructed by the following procedure:

1. Set $k = 0$ and $R_i^0 = \{s_i\}$ for every $i \in [\![1, l]\!]$;
2. For every $i \in [\![1, l]\!]$, every $v \in R_i^k$ and every $u \in N_G(v)$, if u has not been considered so far, add u to R_i^{k+1};

3. If $k < 2d$, increment k by 1 and go to step 2;

4. Let $R_i = \bigcup_{k=0}^{2d} R_i^k$ for every $i \in [\![1, l]\!]$.

Let $\mathcal{R} = \{R_i\}_{i \in [\![1,l]\!]}$. By construction, each set R_i induces a connected graph in G. It remains to prove that $\mathcal{R}$ is a partition of $V(G)$ and that it has the desired properties.

Notice that in the above construction if a vertex is assigned to the set R_i, then it is not assigned to R_j, for every distinct integers $i, j \in [\![1, l]\!]$. Let $v \in V(G)$ be a vertex that does not belong to R_i for any $i \in [\![1, l]\!]$ after the procedure is completed. Then for every $i \in [\![1, l]\!]$ we have $\mathbf{dist}_G(v, s_i) > 2d$ and $v \notin R$, which contradicts the maximality of R. Therefore $\mathcal{R}$ is a partition of $V(G)$.

Since for each vertex v in R_i it holds that $\mathbf{dist}_G(v, s_i) \leq 2d$, $\mathcal{R}$ obviously satisfies property (i) of the definition.

For property (ii) of the definition, let $e = \{x, y\}$ be an edge in G such that $x \in R_i$, $y \in R_j$, for some distinct integers $i, j \in [\![1, l]\!]$. Towards a contradiction, we assume without loss of generality that $\mathbf{dist}_G(x, s_i) < d$. This means that during the construction of R_i, the vertex x was added to the set R_i^k for some $k \leq d - 1$. Also, since the vertex y is adjacent to x but was added to R_j^l for some $l \leq 2d$ instead of R_i^{k+1}, it follows that $l \leq k + 1$, which means that $\mathbf{dist}_G(y, s_j) \leq k + 1$. Hence $\mathbf{dist}_G(s_i, s_j) \leq \mathbf{dist}_G(s_i, x) + \mathbf{dist}_G(x, y) + \mathbf{dist}_G(y, s_j) \leq k + 1 + k + 1 \leq 2d$ again is not possible since R is a $2d$-scattered set.

Finally, in the procedure above, each edge of the graph is encountered at most once, hence the whole algorithm will take at most $O(m)$ time. This concludes the proof of the lemma. $\square$

Lemma 4.4. *Let G be a graph, let $\mathcal{R} = \{R_1, \ldots, R_l\}$ be a d-grouped partition of G, and let s_i be a center of R_i, for every $i \in [\![1, l]\!]$. If for some distinct $i, j \in [\![1, l]\!]$, G has at least r edges from vertices in R_i to vertices in R_j then $G[R_i \cup R_j]$ contains a θ_r-model with at most $4 \cdot r \cdot d + r$ edges, which can be found in $O_r(m)$ steps.*

Proof. Suppose that for some $i \in [\![1, l]\!]$, G has a set F of at least r edges from vertices in R_i to vertices in R_j. Let $R_i' \subseteq R_i$ and $R_j' \subseteq R_j$ be the sets of the endpoints of those edges. Since $\mathcal{R}$ is a d-grouped partition of G, it holds that, for each $x \in R_i'$ and $y \in R_j'$, $\mathbf{dist}_G(x, s_i) \leq 2d$ and $\mathbf{dist}_G(y, s_j) \leq 2d$. That directly implies that for every $h \in \{i, j\}$, there is a collection $\mathcal{P}_h$ of r paths, each of length at most $2d$ and not necessarily disjoint, in $G[R_h]$ connecting s_h with each vertex in R_h', which we can find in $O_r(m)$ steps. It is now easy to observe that the graph Q, obtained from $\pmb{\bigcup}\mathcal{P}_i \cup \pmb{\bigcup}\mathcal{P}_j$ by adding all edges of F, is the union of r paths between s_i and s_j, each containing at most $4 \cdot d + 1$ edges. Therefore, Q is a model of θ_r with at most $4 \cdot r \cdot d + r$ edges, as required. As mentioned earlier the construction of $\mathcal{P}_i$ and $\mathcal{P}_j$ takes $O_r(m)$ steps. $\square$

Lemma 4.5. *Let G be a graph, let $\mathcal{R} = \{R_1, \ldots, R_l\}$ be a d-grouped partition of G, and let $S = \{s_1, \ldots, s_l\}$ be a set of centers of $\mathcal{R}$. For every $i \in [\![1, l]\!]$, let $\mathcal{D}_i = (T_i, r_i, \mathcal{X}_i)$ be the distance-decomposition with origin s_i of the graph $G[R_i]$. If for some $i \in [\![1, l]\!]$ and $w \in \mathbb{N}$, the tree T_i, with node-frontier N_i, has an N_i-unimportant path of length at least $2(w + 1)$, then G has a connected $(2r - 2)$-edge-protrusion Y with extension more than w, which can be constructed in $O_r(m)$ steps.*

Proof. Let $P = t_0 \dots t_p$ be a N_i-unimportant path of length $p \geq 2(w+1)$ in T_i. We assume without loss of generality that $t_p \in \mathrm{desc}_{(T_i,r_i)}(t_0)$. Due to the definition of distance-decompositions, the vertices in $X^i_{t_0}$ or $X^i_{t_p}$ form a vertex-separator of G. Let $Z \subseteq E(G)$ be the set containing all edges between $X^i_{t_0}$ and $X^i_{t_1}$ and all edges between $X^i_{t_{p-1}}$ and $X^i_{t_p}$ in G. Clearly, Z is an edge-separator of G with at most $2r-2$ edges. Let T'_i be the subtree of T_i that we obtain if we remove the descendants of t_p and any vertex that is not a descendant of t_1. Let $Y = \bigcup_{t \in V(T'_i) \setminus \{t_0,t_p\}} X^i_t$. In other words, Y consists of the vertices in the bags of T'_i excluding X^i_i and X^i_j. Obviously, $N_G(Y) = X_{t_0} \cup X_{t_p}$.

We will now construct a rooted tree-partition $\mathcal{F} = (T_\mathcal{F}, r_\mathcal{F}, \mathcal{X}_\mathcal{F})$ of $G[Y \cup N_G(Y)]$ of width at most $2r-2$ and such that $|V(T_\mathcal{F})| > w$. Let $T_\mathcal{F}$ be the tree obtained from T'_h by identifying, for every $j \in [\![0, \lfloor (p-1)/2 \rfloor]\!]$, the vertex t_j with the vertex t_{p-j}. If multiple edges are created during this identification, we replace them with simple ones. We also delete loops that may be created. Let us define the elements of $\mathcal{X}^\mathcal{F} = \{X^\mathcal{F}_t\}_{t \in V(T_F)}$ as follows. If $t \in V(T_F)$ is the result of the identification of t_j and t_{p-j} for some $j \in [\![0, \lfloor (p-1)/2 \rfloor]\!]$, then we set $X^\mathcal{F}_t = X_{t_j} \cup X_{t_{p-j}}$. On the other hand, if $t \in V(T_F)$ is a vertex of T'_i that has not been identified with some other vertex, then $X^\mathcal{F}_t = X_t$. The construction of $\mathcal{F}$ is completed by setting $r_\mathcal{F}$ to be the result of the identification of t_0 and t_p, the endpoints of P.

It is easy to verify that $\mathcal{F}$ is a rooted tree-partition of $G[Y \cup N_G(Y)]$ of width at most $2r-2$. Notice also that the identification of the antipodal vertices of the path P creates a path in $T_\mathcal{F}$ of length $\lfloor (p-1)/2 \rfloor$. This implies that the extension of $\mathcal{F}$ is at least $\lfloor (p-1)/2 \rfloor \geq w+1$. Besides, all the operations performed to construct $\mathcal{F}$ can be implemented in $O_r(m)$ steps. This completes the proof. $\square$

We conclude this section with two easy lemmata related to ports and frontiers.

Lemma 4.6. *Let G be a graph, let $\mathcal{R} = \{R_1, \dots, R_l\}$ be a d-grouped partition of G, and let $S = \{s_1, \dots, s_l\}$ be a set of centers of $\mathcal{R}$. For every $i \in [\![1, l]\!]$, let $\mathcal{D}_i = (T_i, r_i, \mathcal{X}_i)$ be the distance-decomposition with origin s_i of the graph $G[R_i]$, and let N_i be the node-frontier of T_i. Then, for every $i \in [\![1, l]\!]$, there are at least $|N_i|$ ports in T_i.*

Proof. Let $i \in [\![1, l]\!]$. We will show that every vertex in the node-frontier of T_i has a descendant which is a port. For every vertex $t \in N_i \subseteq V(T_i)$, there is, by definition, a path from t to a vertex in $G \setminus R_i$, the internal vertices of which belong to $V^{\geq d}_i$. Let v be the last vertex of this path (starting from t) which belongs to R_i and let $t' \in V(T)$ be the vertex such that $v \in X^i_{t'}$. Then t' is a port of T_i. Observe that t' cannot be the descendant of any other vertex of N_i. Therefore there are at least $|N_i|$ ports in T_i. $\square$

Corollary 4.2. *Let G be a graph, let $\mathcal{R} = \{R_1, \dots, R_l\}$ be a d-grouped partition of G, and let $S = \{s_1, \dots, s_l\}$ be a set of centers of $\mathcal{R}$. For every $i \in [\![1, l]\!]$, let $\mathcal{D}_i = (T_i, r_i, \mathcal{X}_i)$ be the distance-decomposition with origin s_i of the graph $G[R_i]$, and let N_i be the node-frontier of T_i. If for some integer k, every N_i-unimportant path in T_i has length at most k, then T_h contains at least $2^{d/k}$ ports.* $\square$

Proof. Let $i \in [\![1, l]\!]$. From Lemma 4.6, it is enough to prove that $|N_i| \geq 2^{d/k}$. Then the result follows by applying Lemma 4.1 for (T_i, s_i), d, N_i, and k. $\square$

Proof of Theorem 4.4

Proof. Let $d = \frac{z-r}{4r}$. According to Lemma 4.3, we can construct in $O(m)$ steps a d-grouped partition $\mathcal{R} = \{R_1, \ldots, R_l\}$ of $V(G)$, with a set of centers $S = \{s_1, \ldots, s_l\}$, and also, for every $i \in [\![1, l]\!]$, the distance-decompositions $\mathcal{D}_i = (T_i, r_i, \mathcal{X}_i)$ with origins s_i of the graphs $G[R_i]$. For every $i \in [\![1, l]\!]$, we use the notation $\mathcal{X}_i = \{X^i_t\}_{t \in V(T_i)}$ and denote by N_i the node-frontiers of T_i.

By applying the algorithm of Lemma 4.4, in $O_r(m)$ steps, we either find a θ_r-model in G with at most $z = 4 \cdot r \cdot d + r$ edges or we know that for every two distinct $i, j \in [\![1, l]\!]$ there are at most $r - 1$ edges of G with one endpoint in R_i and one in R_j.

Similarly, by applying the algorithm of Lemma 4.2, in $O_r(m)$ steps we either find a θ_r-model in G with at most $2 \cdot r \cdot d \leq z$ edges or we know that for every $i \in [\![1, k]\!]$ and every $t \in V(T_i)$, the bag X^i_t contains at most $r - 1$ vertices.

Using the algorithm of Lemma 4.5, in $O_r(m)$ steps we either find a $(2r-2)$-edge-protrusion with extension more than w, or we know that for every $i \in [\![1, l]\!]$, all N_i-unimportant paths of T_i have length at most $2w + 1$.

We may now assume that none of the above algorithms provided a θ_r-model with z edges, or a $(2r-2)$-edge-protrusion.

From Corollary 4.2, for every $i \in [\![1, l]\!]$ the tree T_i contains at least $2^{\frac{d-1}{2w+1}} = 2^{\frac{z-5r}{4r\cdot(2w+1)}}$ ports, which by definition means that there are at least $2^{\frac{z-5r}{4r\cdot(2w+1)}}$ edges in G with one endpoint in R_i and the other in $V(G) \setminus R_i$. By Lemma 4.4, for every distinct integers $i, j \in [\![1, l]\!]$ there are at most $r - 1$ edges with one endpoint in R_i and the other in R_j. As a consequence of the two previous implications, for every $i \in [\![1, l]\!]$ there is a set $Z_i \subseteq [\![1, l]\!] \setminus \{i\}$, where $|Z_i| \geq \frac{1}{r-1} 2^{\frac{z-5r}{4r(2w+1)}}$, such that for every $j \in Z_i$ there exists an edge with one endpoint in R_i and the other in R_j. Consequently, if we now contract all edges in $G[R_i]$ for every $i \in [\![1, l]\!]$, the resulting graph H is a minor of G of minimum degree at least $\frac{1}{r-1} 2^{\frac{z-5r}{4r(2w+1)}}$. Therefore, we output G, which is an H-model, as required in this case. $\square$

Proof of Theorem 4.5

Proof. The proof is quite similar to the one of Theorem 4.4. If G contains a vertex v of degree less than δ, we can easily find it in $O_r(m)$ steps. Hence, from now on we can assume that every vertex has degree at least δ.

Let $d = \frac{z-r}{4r}$. From Lemma 4.3, in $O(m)$ steps, we can construct a d-grouped partition $\mathcal{R} = \{R_1, \ldots, R_l\}$ of G, with a set of centers $S = \{s_1, \ldots, s_l\}$, and also the distance-decomposition $\mathcal{D}_i = (T_i, r_i, \mathcal{X}_i)$ with origins s_i of the graphs $G[R_i]$, for every $i \in [\![1, l]\!]$. We use again the notation $\mathcal{X}_i = \{X^i_t\}_{t \in V(T_i)}$.

As in the proof of Theorem 4.4, in $O_r(m)$ steps, we can either find a θ_r-model in G with at most $z = 4 \cdot r \cdot d + r$ edges or we know that for every distinct integers $i, j \in [l]$ there are at most $r - 1$ edges of G with one endpoint in R_i and one in R_j (*cf.* Lemma 4.4).

Using Corollary 4.1, we can in $O_r(m)$ steps either find a θ_r-model in G with at most z edges or we know that every bag of $\mathcal{D}_i$ has less than r vertices, for every $i \in [\![1, l]\!]$. Let $i \in [\![1, l]\!]$ and let $u \in R_i$ be a vertex at distance less than d from s_i. As u has degree at least $3r$, it must have neighbors in at least 3 different bags of $\mathcal{D}_i$, apart from the one containing it. This means that every vertex in T_i of distance less than d from r_i has degree at least $\lfloor\frac{\delta}{r-1}\rfloor \geq 3$ and therefore T_i has at least $\lfloor\frac{\delta}{r-1} - 1\rfloor^d$ leaves. Notice also that if t is a leaf of T_i, then each vertex in X_t^i can have at most $r-1$ neighbors in $X_{\mathbf{p}(t)}^i$ and at most $r-2$ neighbors in X_t^i. Therefore there are at least $\delta-(r-1)-(r-2) = \delta-2r+3$ edges in G with one endpoint in X_t^i and the other in $V(G) \setminus R_i$. This means that for every $i \in [\![1,l]\!]$ there are at least $(\delta-2r+3)\cdot\lfloor\frac{\delta}{r-1}-1\rfloor^d$ edges with one endpoint in R_i and the other $V(G) \setminus R_i$.

Similarly to the proof of Theorem 4.4, we deduce that, for each $i \in [\![1,l]\!]$, there is a set $Z_i \subseteq [\![1,l]\!] \setminus \{i\}$ where $|Z_i| \geq \frac{\delta-2r+3}{r-1}\cdot\lfloor\frac{\delta}{r-1}-1\rfloor^d$ such that, for every $j \in Z_i$, there exists an edge with one endpoint in R_i and the other in R_j. This implies the existence of an H-model in G for some H with $\delta(H) \geq \frac{\delta-2r+3}{r-1}\cdot\lfloor\frac{\delta}{r-1}-1\rfloor^{\frac{z-r}{4r}}$. We then output G, which, in this case, is an H-model. $\square$

4.1.5 Concluding remarks

In this section, we introduced the concept of H-girth and proved that for every $r \in \mathbb{N}_{\geq 2}$, a *large* θ_r-girth forces an exponentially large clique minor. This extends the results of Kühn and Osthus related to the usual notion of girth. We also gave a variant of our result where the minimum degree is replaced by a connectivity measure. As an application of our result, we will in Subsection 4.3.6 optimally improve (up to a constant factor) the upper-bound on the treewidth of graphs excluding $k\cdot\theta_r$ as a minor. A first question is whether our lower-bound on the clique minor size can be improved.

Let us now state more general questions spawned by this work. A natural line of research is to investigate the H-girth parameter for different instantiations of H. An interesting problem in this direction could be to characterize the graphs H for which our results (Theorem 4.2 and Theorem 4.3) can be extended.

From its definition, the H-girth is related to the minor relation. An other direction of research would be to extend the parameter of H-girth to other containment relations. One could consider, for a fixed graph H, the minimum size of an induced subgraph that can be contracted to H, or the minimum size of a subdivision of H in a graph. The first one of these parameters is related to induced minors and the second one to topological minors.

As the usual notion of girth appears in various contexts in graph theory, we wonder for which graphs H the results related to girth can be extended to the H-girth or to the two aforementioned variants.

4.2 Degree and θ_r-packings

In this section we show how lower bounds on the degree of the vertices of a graph can be used to prove the existence of a packing of θ_r-majors.

4.2.1 On maximal degree and edge-disjoint packings

The main result of this section is the following. It relates the maximum degree of a 2-connected graph with the presence of an edge-disjoint union of θ_r-majors.

Lemma 4.7. *Let $k > 0, r > 0$ be two integers, and let G be a 2-connected graph with $\Delta(G) \geq 2kr$. Then G contains has a subgraph that is the disjoint union of k edge-disjoint $\mathcal{M}(\theta_r)$-subgraphs.*

For the purpose of the proof, we deal with graphs in which some vertices are marked. If G is a graph and $m: V(G) \to \{0,1\}$ is a function, we say that (G, m) is a graph *marked* by m. A vertex v of G such that $m(v) = 1$ is said to be *marked*. We denote by μ the function that, given a graph, returns its number of marked vertices. We now define an r-good partition. Given a positive integer r, a marked tree (T, m) is said to have an *r-good partition of root v* if there is a pair $((T_1, m_1), (T_2, m_2))$ of marked trees such that:

(i) T_1 and T_2 are subtrees of T such that $(E(T_1), E(T_2))$ is a partition of $E(T)$;

(ii) $r \leq \mu\left((T_1, m_1)\right) \leq 2r$;

(iii) $v \in V(T_2)$; and

(iv) every vertex that is marked in (T, m) is either marked in (T_1, m_1) or marked in (T_2, m_2), but not in both. In other words, for every $u \in V(T)$,

- if $v \in V(T_1) \cap V(T_2)$ then $m(v) = 1 \Leftrightarrow m_1(v) = 1$ or $m_2(v) = 1$ but not both;
- otherwise, let $i \in \{1, 2\}$ be the integer such that $v \in V(T_i)$. Then we have $m(v) = m_i(v)$.

We remark that because of (iv), $\mu(T) = \mu(T_1) + \mu(T_2)$. If for every $v \in V(T)$, (T, m) has an r-good partition of root v, then T is said to have an *r-good partition*.

Lemma 4.8. *For every integer $r > 0$ and every marked tree (T, m), if $\mu(T) \geq 2r$ then (T, m) has an r-*good partition.

Proof. Let $r > 0$ be an integer. We prove this lemma by induction on the size of the tree.
Base case: $|V(T)| = 0$. Since $2r \geq 2 > |V(T)|$, T does not have $2r$ marked vertices and we are done.
Induction step: Assume that for every integer $n' < n$, every marked tree (T', m') on n' vertices and satisfying $\mu((T', m')) \geq 2r$ has an r-good partition (induction hypothesis).

Let us prove that every marked tree on n vertices has a r-good partition if it has at least $2r$ marked vertices. Let (T, m) be a tree on n vertices and let v be a vertex of T. We assume that $\mu((T, m)) \geq 2r$. We distinguish two cases.

- $\mu((T, m)) = 2r$:

  Let $T_1 = T$, let $m_1 = m$, let $T_2 = (\{v\}, \emptyset)$, and let $m_2: V(T_2) \to \{0, 1\}$ be the function equal to 0 on every vertex of T_2. Remark that $(E(T_1), E(T_2)) = (E(T), \emptyset)$

is a partition of $E(T)$, T_2 contains v, and as (T, m) contains (exactly) $2r$ marked vertices, so does (T_1, m). Consequently $((T_1, m_1), (T_2, m_2))$ is an r-good partition of (T, m).

- $\mu((T, m)) > 2r$:

  We distinguish different cases depending on the degree of the root v in T.

Case 1: $\deg(v) = 1$.

Let u be the neighbor of v in T, let $T' = T \setminus \{v\}$, and $m' = m_{|V(T')}$. Remark that $\mu((T', m')) \geq 2r$ and $|V(T')| = |V(T)| - 1$. By induction hypothesis, (T', m') has an r-good partition $((T'_1, m_1), (T'_2, m_1))$ of root u. We extend it to T by setting $T_1 = T'_1$ and $T_2 = (V(T'_2) \cup \{v\}, E(T'_2) \cup \{v, u\})$. Notice that T_2 contains v. As the subtree T'_1 contains at least r and at most $2r$ marked vertices (induction hypothesis), so does T_1. Also, remark that $(E(T_1), E(T_2))$ is a partition of $E(T)$ and that since $u \in T'_2$, the graph T_2 is connected. Therefore the pair (T_1, T_2) is an r-good partition of T.

Case 2: $\deg(v) = d > 1$.

Let $u_1, \ldots, u_d$ be the neighbors of v in T and for every $i \in [\![1, d]\!]$, let C_i be the connected component of $T \setminus \{v\}$ that contains u_i. We also define, for every $i \in [\![1, d]\!]$, the restricted marking function $w_i = m_{|V(C_i)}$.

Subcase (a): there exists $i \in [\![1, d]\!]$ such that $\mu((C_i, w_i)) > 2r$.
Let $T' = (V(C_i) \cup \{v\}, E(C_i) \cup \{u, v\})$ and let $m' = m_{|V(T')}$. Remark that $|V(T')| < |V(T)|$ and $\mu((T', m')) > 2r$. According to the induction hypothesis, (T', m') has an r-good partition $((T_1, m_1), (T_2, m_2))$ of root u_i. Similarly as before, we can extend it into an r-good partition $((T_1, m_1), (T_2, m_2))$ of (T, m). This is done by setting:

$$
\begin{aligned}
T_1 &= T'_1, \\
m_1 &= m'_1, \\
T_2 &= (V(T'_2) \cup \{v\}, E(T'_2) \cup \{v, u_i\}), \text{ and} \\
m_2 &: \left\{ \begin{array}{l} v \mapsto 0 \\ u \in V(T_2) \setminus \{v\} \mapsto m'_2(u) \end{array} \right. .
\end{aligned}
$$

Subcase (b): there exists $i \in [\![1, d]\!]$ such that $r \leq \mu((C_i, w_i)) \leq 2r$.
Let $T_1 = C_i$ and $T_2 = T[E(T) \setminus E(T_1)]$. In this case, $(E(T_1), E(T_2))$ is a partition of $E(T)$ and T_2 is connected since it contains v, the vertex which is adjacent to the C_j's. Thus, if we set $m_1 = m_{|V(T_1)}$ and $m_2 = m_{|V(T_2)}$, $((T_1, m_1), (T_2, m_2))$ is an r-good partition of (T, m).

Subcase (c): for all $i \in [\![1, d]\!]$, $\mu((C_i, w_i)) < r$.

Let $j = \min\left\{j \in [\![2,d]\!],\ \sum_{i=1}^{j} \mu((C_i,w_i)) \geq r\right\}$. We set:

$$\begin{aligned}
T_1 &= (\cup_{i\in[\![1,j]\!]} V(C_i) \cup \{v\}, \cup_{i\in[\![1,j]\!]}(E(C_i) \cup \{v,u_i\})),\\
m_1 &: \begin{cases} v \mapsto 0 \\ u \in VT_1 \setminus \{v\} \mapsto m(u) \end{cases},\\
T_2 &= T[E(T) \setminus E(T_1)], \text{ and}\\
m_2 &= m_{|V(T_2)}.
\end{aligned}$$

By definition of j, $\mu((T_1,m_1)) \geq r$ and as for every $i \in [\![1,d]\!]$, $\mu((C_i,w_i)) < r$ we also have $\mu((T_1,m_1)) < 2r$. As before, the pair $((T_1,m_1),(T_2,m_2))$ is an r-good partition of (T,m).

In conclusion, we proved by induction that for every integer r, every tree having at least $2r$ marked vertices has an r-good partition. $\square$

We are now ready to give the proof of Lemma 4.7.

Proof of Lemma 4.7. As G is 2-connected, the removal of a vertex v of maximum degree gives a connected graph. Let T be a minimal tree of $G \setminus \{v\}$ spanning the neighborhood $N_G(v)$ of v. We mark the vertices of T that are elements of $N_G(v)$: this gives the marking function m for T. Let us prove by induction on k that (T,m) has k edge-disjoint marked subtrees $(T_1,m_1),\ldots,(T_k,m_k)$, each containing at least r marked vertices. If we do so, then we are done because $\{\{v\},T_i\}_{i\in[\![1,k]\!]}$ is a collection of k edge-disjoint θ_r models. In fact, as for every $i \in [\![1,k]\!]$, T_i contains $r' \geq r$ vertices adjacent to v in G, contracting the edges of T_i in $G[\{v\} \cup V(T_i)]$ gives the graph $\theta_{r'}$. Let $r > 0$ be an integer.
Base case $k = 1$: Clear.
Induction step $k > 1$: Assume that for every $k' < k$, every tree with at least $2k'r$ vertices marked has k' edge-disjoint subtrees, each with at least r marked vertices. Let (T,m) be a marked tree such that $\mu((T,m)) \geq 2kr$. According to Lemma 4.8, (T,m) has an r-good partition $((T_1,m_1),(T'_1,m'_1))$ such that $r \leq \mu((T_1,m_1)) \leq 2r$ and $\mu((T'_1,m'_1)) = \mu((T,m)) - \mu((T_1,m_1)) \geq 2(k-1)r$. By induction hypothesis, (T'_1,m'_1) has $k-1$ edge-disjoint marked subtrees $(T_2,m_2),\ldots,(T_k,m_k)$ each containing at least r marked vertices. Remark that as all these trees are subgraphs of T'_1, which is edge-disjoint from T_1 in T, they are edge-disjoint from T_1 as well. Consequently, (T_1,m_1), $(T_2,m_2),\ldots,$ (T_k,m_k) is the family of edge-disjoint subtrees we were looking for. $\square$

4.2.2 On the minimum degree and vertex-disjoint packings

In this subsection we show that every graph of large minimum degree contains $k \cdot \theta_r$ as minor, which can be found in polynomial time. Our proof relies on the following result.

Theorem 4.6 (Theorem 12 of [BTV07]). *Given $k, r \in \mathbb{N}_{\geq 1}$ and an input graph G such that $\delta(G) \geq k(r+1)-1$, a partition $(V_1,\ldots,V_k)$ of $V(G)$ satisfying $\forall i \in [\![1,k]\!]$, $\delta(G[V_i]) \geq r$ can be found in $O(n^c)$ steps, for some $c \in \mathbb{N}$.*

Theorem 4.6 is the algorithmic version of the next older result.

Theorem 4.7 ([Sti96, Corollary 3]). *For every $k, r \in \mathbb{N}_{\geq 1}$, every graph G with $\delta(G) \geq k(r+1)-1$ has a partition $(V_1, \dots, V_k)$ of its vertex set satisfying $\delta(G[V_i]) \geq r$ for every $i \in [\![1, k]\!]$.*

Lemma 4.9. *There is an algorithm that, given $k, r \in \mathbb{N}_{\geq 1}$ and a graph G with $\delta(G) \geq kr$, returns a e-$\mathcal{M}(\theta_r)$-packing of G of size k, in $O(m)$ steps.*

Proof. Starting from any vertex u, we grow a maximal path P in G by iteratively adding to P a vertex that is adjacent to the previously added vertex but does not belong to P. Since $\delta(G) \geq kr$, any such path will have length at least $kr+1$. At the end, all the neighbors of the last vertex v of P belong to P (otherwise P could be extended). Since v has degree at least kr, v has at least kr neighbors in P. Let $w_0, \dots, w_{kr-1}$ be an enumeration of the kr first neighbors of v in the order given by P, starting from u. For every $i \in [\![0, k-1]\!]$, let S_i be the subgraph of G induced by v and the subpath of P starting at w_{ir} and ending at $w_{(i+1)r-1}$. Observe that for every $i \in [\![0, k-1]\!]$, S_i contains a θ_r-major and that the intersection of every pair of graphs from $\{S_i\}_{i \in [\![0,k-1]\!]}$ is $\{v\}$. Hence P contains a e-$\mathcal{M}(\theta_r)$-packing of G of size k, as desired. Every edge of G is considered at most once in this algorithm, yielding to a running time of $O(m)$ steps. ☐

Corollary 4.3. *There is an algorithm that, given $r \in \mathbb{N}_{\geq 1}$ and a graph G with $\delta(G) \geq r$, returns a θ_r-major of G in $O(m)$-steps.*

Observe that the previous lemma only deals with edge-disjoint packings. An analogue of Lemma 4.9 for vertex-disjoint packings can be proved using Theorem 4.6 to the price of a worse time complexity.

Lemma 4.10. *There is an algorithm that, given $k, r \in \mathbb{N}_{\geq 1}$ and a graph G with $\delta(G) \geq k(r+1)-1$, outputs a v-$\mathcal{M}(\theta_r)$-packing of G of size k in $O(n^c + m)$ steps, where c is the constant of Theorem 4.6.*

Proof. After applying the algorithm of Theorem 4.6 on G to obtain in $O(n^c)$-time k graphs $G[V_1], \dots, G[V_k]$, we extract a θ_r-major from each of them using Corollary 4.3. ☐

4.3 Treewidth and excluded majors

In this section, we show upper-bounds on the treewidth of graphs not containing a major of some fixed pattern, among which: the wheel, the double wheel, any graph of pathwidth at most 2, the yurt graph, and the disjoint union of copies of the graph θ_r.

In [RST94a], Robertson, Seymour, and Thomas proved that every planar graph is a minor of a large enough grid.

Lemma 4.11 ([RST94a, (1.5)]). *For every positive integer h, the $h \times h$-grid contains as a minor any planar graph H satifsying $2|V(H)| + 4|E(H)| \leq h$.*

Together with Theorem 4.1, the above lemma implies the next result.

Theorem 4.8. *There is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that, for every for every planar graph H on h vertices, every graph G that does not contain a minor isomorphic to H has treewidth at most $f(h)$.*

The original proof of Theorem 4.1 in [RS86] does not provide any explicit estimation for the function f. Later, in [RST94a], Robertson, Seymour, and Thomas proved the same result for $f(h) = 2^{O(h^5)}$, while a less complicated proof appeared in [DJGT99a]. The bound $f(h) \leq h-2$ was also obtained in [BRST91] in the case where H is required to be a forest. Theorem 4.8 has several applications in algorithms and a lot of research has been devoted to optimizing the function f in general or for specific instantiations of H (see [RST94b, DJGT99b]).

For a long time, whether Theorem 4.8 can be proved for a polynomial f was an open problem. In [RST94a], an $\Omega(h^2 \cdot \log h)$ lower bound was provided for the best possible estimation of f and was also conjectured that the optimal estimation should not be far away from this lower bound. In fact, a more precise variant of the same conjecture was given by Demaine, Hajiaghayi, and Kawarabayashi in [DHK09] where they conjectured that Theorem 4.8 holds for $f(h) = O(h^3)$. The bounds of [RST94a] were then improved by Kawarabayashi and Kobayashi [iKK12], where they show that Theorem 4.8 holds for $f(h) = 2^{O(h \cdot \log h)}$. The same bounds were obtained by Leaf and Seymour [LS15]. Until recently, this was the best known estimation of the function f.

In a breakthrough result [CC13b], Chekuri and Chuzhoy proved that Theorem 4.1 holds for $f(h) = O(h^{228})$. Chuzhoy recently improved this bound.

Theorem 4.9 ([Chu16], see also [CC13b, Chu15])**.** *There exists a function $f_1 \colon \mathbb{N} \to \mathbb{N}$ with $f_1(h) = O(h^{19} \operatorname{polylog} h)$ such that, for every integer h, every graph that does not contain a minor isomorphic to the $h \times h$-grid has treewidth at most $f(h)$.*

The remaining open question is whether the degree of this polynomial bound can be substantially reduced in general. In this direction, one may still consider restrictions either on the graph G or on the graph H that yield a low polynomial dependence between the treewidth and the size of the excluded minor. In the first direction, Demaine and Hajiaghayi proved in [DH08] that, when G is restricted to belong to some graph class excluding some fixed graph as a minor, then Theorem 4.8 (optimally) holds for $f(h) = O(h)$. Similar results have been proved by Fomin, Saurabh, and Lokshtanov, in [FLS12], for the case where G is either a unit disk graph or a map graph that does not contain a clique as a subgraph.

In a second direction, one may consider H to be some specific planar graph and find a good upper bound for the treewidth of the graphs that exclude it as a minor. More generally, we can consider a parametrized class of planar graphs $\mathcal{H}_k$ where each graph in $\mathcal{H}_k$ has size bounded by a polynomial in k, and prove that the following fragment of Theorem 4.8 holds for some low degree polynomial function $f \colon \mathbb{N} \to \mathbb{N}$:

$$\forall k \geq 0 \; \forall H \in \mathcal{H}_k, \; if\, H \not\leq_{\mathrm{m}} G \text{ then } \mathbf{tw}(G) \leq f(k). \tag{4.2}$$

The question can be stated as follows: find pairs $(\mathcal{H}_k, g(k))$ for which (4.2) holds for some $f(k) = O(g(k))$, where $\mathcal{H}_k$ is as general as possible and g is as small as possible (and certainly polynomial). It is known, for example, that (4.2) holds for the pair $(\{C_k\}, k)$, where C_k is the cycle or a path of k vertices (see e.g. [Bod93, FL94]), and for the pair $(\{K_{2,k}\}, k)$, (see [BvLTT97b]). Two more results in the same direction that appeared in the last decade are the following: according to the result of Birmelé, Bondy, and Reed in [BBR07a], (4.2) holds for the pair $(\mathcal{P}_k, k^2)$ where $\mathcal{P}_k$ contains all minors of $K_2 \times C_k$ (we denote by $K_2 \times C_k$ the Cartesian product of K_2 and the cycle of k vertices, also known as the k-*prism*). Finally, one of the consequences of the recent results of Leaf and Seymour in [LS15], implies that (4.2) holds for the pair $(\mathcal{F}_k, k)$, where $\mathcal{F}_k$ contains every graph on k vertices that contains a vertex that meets all its cycles.

Results presented in this section. In this section we provide new exclusion theorems by proving that (4.2) holds for the pairs:

- $(\mathcal{H}_k^0, k^2)$, where $\mathcal{H}_k^0$ contains all simple graphs H on k vertices and of pathwidth at most 2;
- $(\mathcal{H}_k^1, k)$, where $\mathcal{H}_k^1$ contains all minors of a wheel on $k+1$ vertices – see Figure 4.3;
- $(\mathcal{H}_k^2, k^2 \log^2 n)$, where $\mathcal{H}_k^2$ contains all minors of a double wheel on $k+2$ vertices – see Figure 4.3;
- $(\mathcal{H}_k^3, k^4)$, where $\mathcal{H}_k^3$ contains all minors of the yurt graph on $2k+1$ vertices (i.e. the graph obtained it we take a $(2 \times k)$-grid and add a new vertex adjacent with all the vertices of its "upper layer" – see Figure 4.6); and
- $(\mathcal{H}_{k,r}^4, k \log k)$, where $\mathcal{H}_{k,r}^4$ contains all minors of the graph $k \cdot \theta_r$.

Notice that none of the classes $\mathcal{H}_k^1$, $\mathcal{H}_k^2$, $\mathcal{H}_k^3$, and $\mathcal{H}_k^4$ is minor comparable with the classes $\mathcal{P}_k$ and $\mathcal{F}_k$ treated in [BBR07a] and [LS15], whereas $\mathcal{H}_{k,r}^3 \subsetneq \mathcal{P}_{kr}$. Moreover, $\mathcal{H}_k^1 \subsetneq \mathcal{H}_k^2 \subsetneq \mathcal{H}_k^3$, while $\mathcal{H}_{k,r}^3 \subsetneq \mathcal{H}_k^0$. The above results are presented thereafter in detail, without the O-notation.

Theorem 4.10. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geq 36k - 2$, then G contains a wheel of order k as minor.*

Theorem 4.11. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geq 12(8k\log(8k) + 2)^2 - 4$, then G contains a double wheel of order at least k as minor.*

Theorem 4.12. *Let G be a graph, let H be a simple graph on k vertices such that $\mathbf{pw}(H) \leq 2$. If $\mathbf{tw}(G) \geq 3(k-2)^2 - 1$ then G contains H as a minor.*

Theorem 4.12 can be extended to the setting of graphs that are not simple as follows.

Corollary 4.4. *Let G be a graph, let H be a graph such that $\mathbf{pw}(H) \leq 2$ and let $k = |V(H)| + |E(H)|$. If $\mathbf{tw}(G) \geq 3(k-2)^2 - 1$ then G contains H as a minor.*

Theorem 4.13. *Let $k > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geq 6k^4 - 24k^3 + 48k^2 - 48k + 23$, then G contains the yurt graph of order k as minor.*

Theorem 4.14. *Let $k > 0$ and $r \geq 2$ be two integers and let G be a graph. If $\mathbf{tw}(G) \geq 2^{6r} \cdot k \cdot \log(k+1)$, then G contains $k \cdot \theta_r$ as a minor.*

The aforementioned results will we proved in the forthcoming sections, in this order. The proofs of the four first results use as a departure point the results of Leaf and Seymour in [LS15], whereas the last one uses the results of Geelen, Gerards, Robertson, and Whittle on the excluded minors for the matroids of branch-width k [GGRW03] together with Theorem 4.4. In Subsection 4.3.1, we introduce notions that we will use and we prove two lemmas, for later use. We give in Subsection 4.3.7 lower bounds on the best function f of (4.2) that one can expect for the classes studied here and we discuss the tightness of our results.

4.3.1 Our tools

This section contains some definitions, as well as two lemmas that will be useful later.

Definition 4.1 (linked set)**.** Let G be a graph and $S \subseteq V(G)$. The set S is said to be *linked* in G if for every two subsets X_1, X_2 of S (not necessarily disjoint) such that $|X_1| = |X_2|$, there is a set Q of $|X_1|$ (vertex-)disjoint paths between X_1 and X_2 in G whose length is not one (but can be null) and whose endpoints only are in S.

Definition 4.2 (left-contains, [LS15])**.** Let H be a graph on r vertices, G a graph and (A, B) a separation of order r in G. We say that (A, B) *left-contains* H if $G[A]$ contains a minor model $\mathcal{M}$ of H such that $\forall M \in \mathcal{M},\ |M \cap (A \cap B)| = 1$

In this section, we denote by B_h the complete binary tree of height h, for every integer $h > 0$.

Proposition 4.2 ([LS15, (4.3)])**.** *Let $k > 0$ be an integer, let F be a forest on k vertices and let G be a graph. If $\mathbf{tw}(G) \geq \frac{3}{2}k - 1$, then G has a separation (A, B) of order k such that*

- *$G[B \setminus A]$ is connected;*
- *$A \cap B$ is linked in $G[B]$;*
- *(A, B) left-contains F.*

Proposition 4.3 (Erdős–Szekeres Theorem, [ES87])**.** *Let k and ℓ be two positive integers. Then any sequence of $(\ell - 1)(k - 1) + 1$ distinct integers contains either an increasing subsequence of length k or a decreasing subsequence of length ℓ.*

Lemma 4.12. *For every tree T, $|V(T)| \leq \frac{|L(T)| \cdot \mathbf{diam}(T)}{2} + 1$.*

Proof. Root T to a vertex $r \in V(T)$ that is halfway of a longest path of T. For each leaf $x \in L(T)$, we know that $|V(xT\mathring{r})| \leq \left\lfloor \frac{\mathbf{diam}(T)}{2} \right\rfloor$. Observe that $V(T) = \{r\} \cup \bigcup_{x \in L(T)} V(xT\mathring{r})$. Therefore,

$$|V(T)| \leq \sum_{x \in L(T)} |V(xT\mathring{r})| + 1$$
$$|V(T)| \leq |L(T)| \cdot \left\lfloor \frac{\mathbf{diam}(T)}{2} \right\rfloor + 1.$$

Notice that equality holds for the subdivided star (obtained from $K_{1,n}$ by subdividing k times every edge, for some $n, k \in \mathbb{N}$). $\square$

Definition 4.3 (The set $\Lambda(T)$)**.** Let T be a tree. We denote by $\Lambda(T)$ the set containing every graph obtained as follows: take the disjoint union of T, a path P where $|V(P)| \geq \sqrt{|L(T)|}$, and an extra vertex v_{new}, and add edges such that

(i) there is an edge between v_{new} and every vertex of P;

(ii) there are $|V(P)|$ disjoint edges between P and $L(T)$;

(iii) there are no more edges than the edges of T and P and the edges mentioned in (i) and (ii).

Lemma 4.13. *Let $n \geq 1$ be an integer, T be a tree on n vertices an let G be a graph. If $\mathbf{tw}(G) \geq 3n - 1$, then $H \leq_{\text{m}} G$ for some $H \in \Lambda(T)$.*

Proof. Let n, T, and G be as in the statement of the lemma. Let l be the number of leaves of T, and let J be a path on l vertices. We consider the disjoint union of J and T.

The graph G has treewidth at least $\frac{3}{2}(n + l) - 1$, then by Proposition 4.2, G has a separation (A, B) of order $n + l$ such that

(i) $G[B \setminus A]$ is connected;

(ii) $A \cap B$ is linked in $G[B]$;

(iii) (A, B) left-contains the graph $J \cup T$.

Let $(\mathcal{M}, \varphi)$ be the a minor model of $J \cup T$ in $G[A]$ that witnesses (iii). We call the vertices of $A \cap B$ that belong to $\varphi(v)$ for some $v \in V(J)$ the J-part, and vertices that belong to $\varphi(v)$ for some $v \in L(T)$ forms the $L(T)$-part. Notice that two distinct vertices of the J-part (resp. $L(T)$-part) will be contracted to distinct vertices by the minor model.

Let $\mathcal{P}$ a set of l disjoint paths with the one endpoint in the J-part and the other in the $L(T)$-part, and whose interior belongs to $B \setminus A$. The existence of such paths is given by (ii). For each $P \in \mathcal{P}$, we arbitrarily choose a vertex v_P of the interior of P, that is, $v_P \in V(P) \setminus A$. By (i), $G[B \setminus A]$ is connected: let Y be a smallest tree spanning the vertices $\{v_P\}_{P \in \mathcal{P}}$. Let $s = \sqrt{|L(T)|}$, and let Y^* be the tree obtained from Y by dissolving every vertex of degree two that is not v_P for some $P \in \mathcal{P}$. We are now facing two possible situations.

Case 1: Y^* has a path of length s. Let Q be the path of Y corresponding to a path of lenght s in Y^* and let S be the set of vertices $u \in V(Q)$ that are not of degree two or that are v_P for some $P \in \mathcal{P}$. Observe that from every $u \in S$, there is a path J_u to the $L(T)$-part and a path J'_u to the J-part. Indeed, if $u = v_P$ for some $P \in \mathcal{P}$, then u is a vertex of P linking (by definition) a vertex of the $L(T)$-part to a vertex of the J-part. Otherwise, u is of degree at least 3 in Y and every leaf of the subtrees of $Y \setminus Q$ (at least one of which is adjacent to u), is a v_P for some $P \in \mathcal{P}$ (by minimality of Y), so is connected to the $L(T)$-part and the J-part as explained above. Furthermore, for every two distinct $u, v \in S$, the aforementioned path are disjoint.

Let us now summarize. $G[\cup_{v \in V(J)} \varphi(v)]$ is a connected subgraph of G, which is connected by the s disjoint paths $J'_{uu \in S}$ to the path Y. All the endpoints of the paths $J'_{uu \in S}$ on Y are connected by s disjoint paths $J_{uu \in S}$ to the $L(T)$-part, which correspond to the leaves in a minor model of T. Therefore this graph contains a member of $\Lambda(T)$ as a minor, as required.

Case 2: $\mathbf{diam}(Y^*) < s$. From Lemma 4.12, $|L(Y)| = |L(Y^*)| \geq s$. Observe that $L(Y) \subseteq \{v_P\}_{P \in \mathcal{P}}$ (this follows by the minimality of Y). Let $S = V(Y) \setminus L(Y)$. We consider the minor of G obtained by contracting, for every $P \in \mathcal{P}$ such that $v_P \in L(Y)$, every edge of the subpath connecting the J-part to a leaf of Y. In this graph, S induces a connected subgraph adjacent to at least s distinct vertices of the J-part. All these s vertices of the J-part are connected by s disjoint paths to distinct vertices of the $L(T)$-part. Thus this contains a member of $\Lambda(T)$ as a minor, and so do G.

□

The proof of Theorem 4.14 requires additional definitions, that we introduce in the corresponding section as they are not used in the proofs of the other results.

4.3.2 Excluding a wheel with a linear bound on treewidth

In this section we prove Theorem 4.10. For every integer $r \geq r$, we denote by W_r the wheel of order r where every vertex of the cycle is adjacent to the center. An example is given in Figure 4.3.

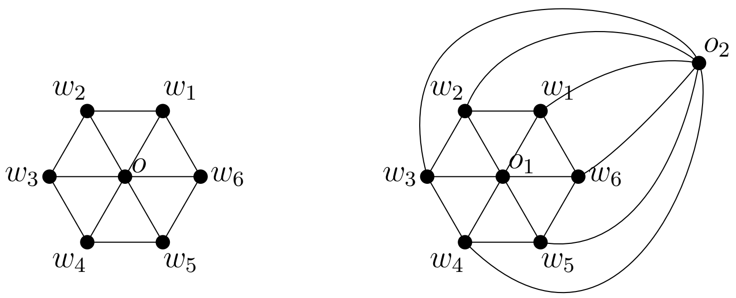


Figure 4.3: A wheel of order six (left) and a double wheel of order 6 (right).

Lemma 4.14. *Let $h > 2$ be an integer. Let G be a graph obtained from the union of the tree $T = B_h$ and a path P by adding the edges $\{l, \psi(l)\} \in E(G)$ for every $l \in L(T)$, where $\psi\colon L(T) \to V(P)$ is a bijection. Then G contains a wheel of order $2^{h-2}+1$ as a minor.*

Proof. Let h, ψ, T, $P = p_1 \dots p_{2^h}$ and G be as above. Let r be the root of T.

In the arguments to follow, if $t \in V(T)$, we denote by T_t the subtree of T rooted at t (i.e. the subtree of T whose vertices are all the vertices $t' \in V(T)$ such that the path $t'Tr$ contains t).

We consider the vertices $u = \psi^{-1}(p_1) \in L(T)$ and $v = \psi^{-1}(p_{2^h}) \in L(T)$ and $w = \mathbf{lca}\, T(u, v) \in V(T) \setminus L(T)$.

Let τ be a largest subtree of T which is disjoint from uTv. Let $L_\tau = L(\tau) \cap L(T)$ and let $Q = \psi(L_\tau) \subseteq P$. It is not hard to see that G contains $W_{|Q|+1}$ as a minor. Indeed, the paths P and uTv together with the edges $\{p_1, u\}$ and $\{p_{2h}, v\}$ form a cycle in G. Besides, the tree τ, which is disjoint from this cycle, has at least $|Q| + 1$ vertices that are adjacent to distinct vertices of P: $|Q|$ of them are the elements of Q, and the other one is the (only) vertex of τ adjacent to uTv (which exists by maximality of τ). In the subgraph of G induced by $V(P) \cup V(uTv) \cup V(\tau)$, contracting τ to a vertex gives a vertex adjacent to at least $|Q| + 1$ vertices of a (non necessarily induced) cycle, a graph containing $W_{|Q|+1}$ as subgraph.

Depending on G, $|Q|$ may take different values. However, we show that it is never less than 2^{h-2}. Remember, $|Q|$ is the number of leaves that a largest subtree of T that is disjoint from uTv shares with T. The root r of T has two children r_1 and r_2, inducing two subtrees T_{r_1} and T_{r_2} of T. Recall, $w = \mathbf{lca}\, T(u, v)$.

Case 1. $w \neq r$. As $w \neq r$, w is a vertex of one of $\{T_{r_1}, T_{r_2}\}$, say T_{r_1}, which contains also u and v, and thus the path uTv. The other subtree T_{r_2} is then disjoint from uTv, it has height $h-1$ and is complete so it has 2^{h-1} leaves. Consequently, in this case $|Q| \geq 2^{h-1}$.

Case 2. $w = r$. In this case, the path uTv contains r (and $r \neq u$, $r \neq v$ as u and v are leaves) so u and v are not in the same subtree of $\{T_{r_1}, T_{r_2}\}$ and uTv contains the two edges $\{r, r_1\}$ and $\{r, r_2\}$. For every $i \in \{1, 2\}$, we denote by $r_{i,1}$ and $r_{i,2}$ the two children of r_i in T. We assume without loss of generality that $u \in V(T_{r_{1,1}})$ and $v \in V(T_{r_{2,1}})$ (if not, we just rename the r_i's ans $r_{i,j}$'s). Notice that the path uTv is the concatenation of the paths $uT_{r_1}r_1$, r_1Tr_2, $r_2T_{r_2}v$. Since the tree $T_{r_{1,2}}$ is disjoint from uTv, is complete and is of height $h - 2$, it has 2^{h-2} leaves. Therefore we have $|Q| \geq 2^{h-2}$.

In both cases, $|Q| \geq 2^{h-2}$ and according to what we proved before, G contains a minor model of $\mathrm{W}_{|Q|+1}$. As every wheel contains as a minor every smaller wheel, we proved that G contains a wheel of order at least 2^{h-2}. $\square$

We are now ready to prove Theorem 4.10.

Proof of Theorem 4.10. Let $k > 0$ be an integer, G be a graph such that $\mathbf{tw}(G) \geq 36k - 2$, and let $h = \lceil \log 4k \rceil$. Since every wheel contains a minor model of every smaller wheel, we have

$$\begin{aligned} \mathrm{W}_k &\leq_{\mathrm{m}} \mathrm{W}_{2^{\lceil \log k \rceil}+1} \\ &\leq_{\mathrm{m}} \mathrm{W}_{2^{\lceil (\log 4k)-2 \rceil}+1} \\ &\leq_{\mathrm{m}} \mathrm{W}_{2^{h-2}+1} \end{aligned}$$

Therefore, if we prove that G contains a $\mathrm{W}_{2^{h-2}+1}$-minor model, then we are done because the minor relation is transitive. Let Y_h^- be the graph of the following form: the disjoint union of the complete binary tree B_h of height h with leaves set Y_L and of the path Y_P on 2^h vertices, and let $\mathcal{Y}_h$ be the set of graphs of the same form, but with the extra edges $\{\{l,\phi(l)\}\}_{l\in Y_L}$, where $\phi : Y_L \to V(Y_P)$ is a bijection. As we proved in Lemma 4.14 that every graph in $\mathcal{Y}_h$ contains the wheel of order $2^{h-2}+1$ as minor, showing that G contains an element of $\mathcal{Y}_h$ as minor suffices to prove this lemma. That is what we will do.

From our initial assumption, we deduce the following.

$$\begin{aligned}
\mathbf{tw}(G) &\geq 36k - \frac{5}{2}\\
&\geq \frac{3}{2}(3\cdot 2^{\log 8k}-1)-1\\
&\geq \frac{3}{2}(3\cdot 2^{\lfloor \log 4k\rfloor+1}-1)-1\\
\mathbf{tw}(G) &\geq \frac{3}{2}(3\cdot 2^{h}-1)-1
\end{aligned}$$

According to Proposition 4.2, G has a separation (A,B) of order $3\cdot 2^h-1$ such that

(i) $G[B\setminus A]$ is connected;

(ii) $A\cap B$ is linked in $G[B]$;

(iii) (A,B) left-contains the graph Y_h^-.

By definition of *left-contains*, $G[A]$ contains a minor model $(\mathcal{M}^-,\varphi^-)$ of Y_h^- and every element of $\mathcal{M}^-$ contains exactly one element of $A\cap B$. For every $x\in A\cap B$, we denote by M_x^- the element of $\mathcal{M}^-$ that contains x. Let L (resp. R) be the subset of $A\cap B$ of vertices that belong to an element of M related to the leaves of B_h in Y_h^- (resp. to the path P). We remark that these sets are both of cardinality 2^h.

Since $A\cap B$ is linked in $G[B]$ (see (ii)), there is a set $\mathcal{P}$ of 2^h disjoint paths between the vertices of L and the elements of R. Let $\psi\colon L\to V(P)$ be the function that match each element l of L with the (unique) element of R it is linked to by a path (that we call $\mathcal{P}_l$) of $\mathcal{P}$. Observe that ψ is a bijection. We set

$$\begin{aligned}
\forall l\in L,\ M_l &= M_l^-\cup V(l\mathcal{P}_l\mathring{\psi}(l))\\
\forall r\in (A\cap B)\setminus L,\ M_r &= M_r^-
\end{aligned}$$

$$\mathcal{M} = \bigcup_{x\in A\cup B} M_x.$$

Let us show that $\mathcal{M}$ allows us to define a minor model of some $H\in\mathcal{Y}_h$. Let us consider the following mapping.

$$\varphi\colon \left\{\begin{array}{ccc} V(Y_h^-) & \to & \mathcal{M}\\ x & \mapsto & M_x\end{array}\right.$$

We claim that $(\mathcal{M}, \varphi)$ is a minor model of H for some $H \in \mathcal{Y}_h$. This is a consequence of the following remarks.

Remark 4.1. Every element of $\mathcal{M}$ is either an element of $\mathcal{M}^-$, or the union of a element M of $\mathcal{M}^-$ and of the vertices of a path that start in M, thus every element of $\mathcal{M}$ induces a connected subgraph of G.

Remark 4.2. The paths of $\mathcal{P}$ are all disjoint and are disjoint from the elements of $\mathcal{M}^-$. Every interior of path of $\mathcal{P}$ is in but one element of $\mathcal{M}$, therefore the elements of $\mathcal{M}$ are disjoint.

Remark 4.3. The elements $\{m_l\}_{l \in L}$ are in bijection with the elements of $\{m_r\}_{r \in R}$ (thanks to the function ψ) and every two vertices $l \in L$ and $\psi(l) \in R$ are such that there is an edge between m_l and $m_{\psi(l)}$ (by definition of $\mathcal{M}^+$).

We just proved that $(\mathcal{M}, \varphi)$ is a minor model of a graph of $\mathcal{Y}_h$ in G. Finally, we apply Lemma 4.14 to find a minor model of the wheel of order $2^{h-2} + 1 = 2^{\lceil \log k \rceil - 2} + 1 \geq k$ in G and this concludes the proof. □

4.3.3 Excluding a double wheel with a $O(k \log k)^2$ bound on treewidth

This section is devoted to the proof of Theorem 4.11. Recall that for every integer $n \geq 3$, a double wheel of order n is obtained from a cycle of order n by adding two non-adjacent, each connected to at least three vertices of the cycle. We denote by W_n^2 the double wheel of order n where the two extra vertices are adjacent to every vertex of the cycle.

Lemma 4.15. *Let G be a graph and $h > 0$ be an integer. If $tw(G) \geq 6 \cdot 2^h - 4$, then G contains as minor a double wheel of order at least $\frac{2^{\frac{h}{2}}-2}{2h-3}$.*

Proof. Let h and G be as above. Observe that $\mathbf{tw}(G) \geq 3(2^{h+1} - 1) - 1$. As the binary tree $T = B_h$ has $2^{h+1} - 1$ vertices, G contains a graph $H \in \Lambda(B_h)$ as minor (by Lemma 4.13). Let us show that any graph $H \in \Lambda(B_h)$ contains a double wheel of order at least $\frac{2^{\frac{h}{2}}-2}{2h-3}$ as minor.

Let P be the path of length at least $2^{\frac{h}{2}}$ in the definition of H. Let L be the set, of size at least $2^{\frac{h}{2}}$, of the leaves of T that are adjacent to P in H. Such a set exists by definition of $\Lambda(B_h)$. We also define u (resp. u') as the vertex of $L(T)$ that is adjacent to one end of P (resp. to the other end of P) and $Q = uTu'$.

As T is a binary tree of height h, Q has at most $2h - 1$ vertices. Each vertex of Q is of degree at most 3 in T except the two ends which are of degree 1. Consequently, $T \setminus Q$ has at most $2h - 3$ connected components that are subtrees of T. Notice that every vertex of the $2^{\frac{h}{2}}$ elements of L is either a leaf of one of these $2h - 3$ subtrees, or one of the two ends of Q. By the pigeonhole principle, one of these subtrees, which we call T_1, has at least $\frac{2^{\frac{h}{2}}-2}{2h-3}$ leaves that are elements of L.

Let M_{o_1} be the set of vertices of this subtree T_1. We also set $M_{o_2} = \{v_{\text{new}}\}$ (cf. Definition 4.3 for a definition of v_{new}). Let us consider the cycle C made by the concatenation of the paths uPu' and $u'Tu$ in H.

By definition of M_{o_1}, there are at least $\frac{2^{\frac{h}{2}}-2}{2h-3}$ vertices of C adjacent to vertices of M_{o_1}. Let $J = \{j_1, \dots, j_{|J|}\}$ be the set of such vertices of C, in the same order as they appear in C (we then have $|J| \geq \frac{2^{\frac{h}{2}}-2}{2h-3}$).

We arbitrarily choose an orientation of C and define the sets of vertices $M_1, M_2, \dots, M_{|J|}$ as follows.

$$\begin{aligned} \forall i \in [\![1, |J|-1]\!],\ M_i &= V(j_i C \mathring{j}_{i+1}) \\ M_{|J|} &= V(j_{|J|} C \mathring{j}_1) \end{aligned}$$

Let $\mathcal{M} = \{M_1, \dots, M_{|J|}, M_{o_1}, M_{o_2}\}$ and $\psi\colon V(\mathrm{W}^2_{|J|}) \to \mathcal{M}$ be the function defined by

$$\begin{aligned} \forall i \in [\![1, |J|]\!],\ \psi(w_i) &= M_i \\ \psi(o_1) &= M_{o_1} \\ \psi(o_2) &= M_{o_2} \end{aligned}$$

Notice that ψ maps the vertices of $\mathrm{W}^2_{|J|}$ to connected subgraphs of H such that $\forall (v, w) \in E(\mathrm{W}^2_{|J|})$, there is a vertex of $\psi(v)$ adjacent in H to a vertex of $\psi(w)$. Therefore, $(\mathcal{M}, \psi)$ is a $\mathrm{W}^2_{|J|}$-minor model in H.

Since $|J| \geq \frac{2^{\frac{h}{2}}-2}{2h-3}$, H contains a double wheel of order at least $\frac{2^{\frac{h}{2}}-2}{2h-3}$, which is what we wanted to show. ☐

Corollary 4.5. *Let $l > 0$ be an integer and G be a graph. If $\mathbf{tw}(G) \geq 12l - 4$ then G contains a double wheel of order at least $\frac{\sqrt{l}-2}{2\log l - 5}$ as minor.*

Proof. Let l and G be as above. First remark that

$$\lceil \log l \rceil - 1 \leq \log l \leq \lceil \log l \rceil \tag{4.3}$$

Our initial assumption on $\mathbf{tw}(G)$ gives the following.

$$\begin{aligned} \mathbf{tw}(G) &\geq 12l - 4 \\ &\geq 6 \cdot 2^{\log(2l)} - 4 \\ &\geq 6 \cdot 2^{\log l + 1} - 4 \\ &\geq 6 \cdot 2^{\lceil \log l \rceil} - 4 && \text{by (4.3)} \end{aligned}$$

By Lemma 4.15, G contains a double wheel of order at least

$$\begin{aligned} q &= \frac{2^{\frac{\lceil \log l \rceil}{2}} - 2}{2\lceil \log l \rceil - 3} \\ &\geq \frac{2^{\frac{1}{2}\log l} - 2}{2(\log l - 1) - 3} && \text{by (4.3)} \\ &\geq \frac{\sqrt{l} - 2}{2\log l - 5} \end{aligned}$$

Therefore, G contains a double wheel of order $q \geq \frac{\sqrt{l}-2}{2\log l - 5}$, as required. ☐

We can now deduce Theorem 4.11 from Corollary 4.5.

Proof of Theorem 4.11. Applying Corollary 4.5 for $l = (8k\log(8k)+2)^2$ yields that G contains a double wheel of order at least

$$\begin{aligned} q &\geq \frac{\sqrt{l}-2}{2\log l - 5} \\ &\geq \frac{8k\log(8k)}{4\log(8k\log(8k)+2)-5} \\ &\geq \frac{8k\log(8k)}{4\log(8k\log(8k))-1} \\ &\geq \frac{8k\log(8k)}{4(\log(8k)+\log\log(8k))-1} \\ &\geq \frac{8k\log(8k)}{8\log(8k)-1} \\ &\geq k \end{aligned}$$

Consequently G contains a double wheel of order $q \geq k$ and we are done. □

4.3.4 Excluding a graph of pathwidth at most 2 with a quadratic bound on treewidth

This sections contains the proofs of Theorem 4.12 and Corollary 4.4. We define the graph Ξ_r as the graph of the following form (see Figure 4.4).

$$\begin{cases} V(G) = \{x_0,\ldots,x_{r-1},y_0,\ldots,y_{r-1},z_0,\ldots,z_{r-1}\} \\ E(G) = \{\{x_i,x_{i+1}\},\{z_i,z_{i+1}\}\}_{i\in[\![1,r-1]\!]} \cup \{\{x_i,y_i\},\{y_i,z_i\}\}_{i\in[\![0,r-1]\!]} \end{cases}$$

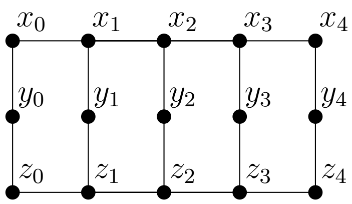


Figure 4.4: The graph Ξ_5.

Graphs of pathwidth 2 in Ξ_r

Instead of proving that a treewidth quadratic in $|V(H)|+|E(H)|$ forces an H-minor for every graph H of pathwidth at most 2, we prove that a treewidth quadratic in r forces an Ξ_r-minor and then that every graph H of pathwidth at most 2 with $|V(H)|+|E(H)| \leq r$ is a minor of Ξ_r. Recall that every graph has an optimal path decomposition which is nice. Let us state some observations on path decompositions.

Remark 4.4. Let G be a graph and let $(p_1p_2\dots p_k, \mathcal{X})$, $\mathcal{X} = \{X_{p_i}\}_{i\in[\![1,k]\!]}$ be a nice (non necessarily optimal) path decomposition of G. Let w be the width of this decomposition.

For every $i \in [\![2, k-1]\!]$, if p_i is a forget node, $|X_{p_i}| \le w-1$ and p_{i+1} is an introduce node, then by setting

$$\begin{aligned} X'_{p_i} &= X_{p_{i-1}} \cup X_{p_{i+1}} \\ \forall j \in [\![1,k]\!]\,,\ j \neq i,\ X'_{p_j} &= X_{p_j} \\ \mathcal{X}' &= \left\{X'_{p_j}\right\}_{j\in[\![1,k]\!]} \end{aligned}$$

we create from $(p_1p_2\dots p_k, \mathcal{X}')$ a valid path decomposition of G, where p_i is now an introduce node and p_{i+1} a forget node. Observe that $\left|X'_{p_i}\right| \le |X_{p_i}|+2 = w+1$ Therefore the new path decomposition has the same width as the original one. Note that the condition $|X_{p_i}| \le w-1$ holds, for instance, when p_{i-1} is required to be a forget node too (for $i \in [\![3, k-1]\!]$).

Remark 4.5. Let G be a graph and $P = (p_1p_2\dots p_k, \mathcal{X})$ be a nice path decomposition of G. For every $i \in [\![1, k]\!]$, the path $p_1 \dots p_i$ contains at most as many forget nodes as introduce nodes and the difference between these two numbers is at most $w + 1$ where w is the width of P.

Lemma 4.16. *Let G be a graph on n vertices . Then G has an optimal path decomposition P such that*

(i) every bag of P has size $\mathbf{pw}(G) + 1$;

(ii) every two adjacent bags differs by exactly one element, i.e. for every two adjacent vertices u and v of P, $|X_u \setminus X_v| = |X_v \setminus X_u| = 1$.

Proof. Let $P = (p_1p_2\dots p_{2k}, \mathcal{X})$ with $\mathcal{X} = \{X_{p_i}\}_{i\in[\![1,2k]\!]}$ be a nice optimal path decomposition of G with as many introduce nodes (resp. forget nodes) as there are vertices in G.

Let $s = \mathbf{pw}(G) + 1$. According to Remark 4.4 and Remark 4.5, P can be modified into a path decomposition of G of the same width and such that

(a) the s first vertices of P are introduce nodes and p_{s+1} is a forget node;

(b) the s last vertices of P are forget nodes and p_{2k-s} is an introduce node;

(c) for every $i \in [\![s, 2k-s]\!]$, p_i and p_{i+1} are nodes of different type.

In the arguments to follow, we assume that P satisfies this property.

Remark 4.6. Introduce nodes all have bags of cardinality s.

Remark 4.7. For every $i \in [\![0, k-s]\!]$, the node p_{s+2i} is an introduce node and the node p_{s+2i+1} is a forget node, which implies $X_{p_{s+2i}} \subsetneq X_{p_{s+2i+1}}$. Also note that for every $i \in [\![1, s-1]\!]$, $X_{p_i} \subsetneq X_{p_s}$ and for every $i \in [\![2k-s+1, 2k]\!]$, $X_{p_i} \subsetneq X_{p_{2k-s}}$.

Intuitively, every bag X that is included in one of its adjacent bags X' contains no more information than what X' already contains, so we will just remove it.

We thus define $P' = p_s p_{s+2} \dots p_{s+2i} \dots p_{2k-s}$ (a path made of all introduce nodes of P). Clearly, P and P' have the same width and as we deleted only redundant nodes, P' is still a valid path decomposition of G.

Since every two adjacent nodes of P' were introduce nodes separated by a forget node in P, they only differ by one element. According to Remark 4.6 and since every node of P' was an introduce node in P, every bag of P' have size $\mathbf{pw}(G)+1$. Consequently, P' is an optimal path decomposition that satisfies the conditions of the lemma statement. □

Remark 4.8. The path decomposition of Lemma 4.16 has length $V(G) - \mathbf{pw}(G)$.

Proof. Let $(P, \mathcal{X})$ be such a path decomposition. Remember that the first node of P has a bag of size $\mathbf{pw}(G) + 1$ and that every two adjacent nodes of P have bags which differs by exactly one element. Since every vertex of G is in a bag of P, in addition to the first bag containing $\mathbf{pw}(G)+1$ vertices of G, P must have $V(G) - \mathbf{pw}(G) - 1$ other bags in order to contain all vertices of G. Therefore P has length $V(G) - \mathbf{pw}(G)$. □

A proof of a slightly weaker version of the following lemma previously appeared [Pro89].

Lemma 4.17. *For every simple graph G on n vertices and of pathwidth at most 2, there is a minor model of G in Ξ_{n-1}.*

Proof. Let G be as in the statement of the lemma. We assume that $\mathbf{pw}(G) = 2$ (if this is not the case we add edges to G in order to obtain a graph of pathwidth 2 which contains G as a minor). Let $r = V(G) - \mathbf{pw}(G) = n - 2$.

Let $P = (p_1 \dots p_r, \{X_{p_1}, \dots, X_{p_r}\})$ be an optimal path decomposition of G satisfying the properties of Lemma 4.16, of length r. Such decomposition exists according to Lemma 4.16 and Remark 4.8).

Using this decomposition, we will now define a labeling λ of the vertices of Ξ_{r+1}. When dealing with the vertices of Ξ_{r+1} we will use the notations given in the definition of this graph. Let $\lambda\colon V(\Xi_{r+1}) \to V(G)$ be the function defined as follows:

(a) $\lambda(x_0)$ and $\lambda(y_0)$ are both equal to one (arbitrarily chosen) element of the set $X_{p_1} \cap X_{p_2}$;

(b) $\lambda(z_0)$ is equal to the only element of the set $X_{p_1} \cap X_{p_2} \setminus \{\lambda(x_1)\}$;

(c) $\forall i \in [\![2, r]\!]$, $\lambda(y_i) = X_{p_i} \setminus X_{p_{i-1}}$ and we consider two cases:

Case 1: $X_{p_{i-1}} \cap X_{p_i} = X_{p_i} \cap X_{p_{i+1}}$
$\lambda(x_i) = \lambda(x_{i-1})$ and $\lambda(z_i) = \lambda(z_{i-1})$;

Case 2: $X_{p_{i-1}} \cap X_{p_i} \neq X_{p_i} \cap X_{p_{i+1}}$
if $X_{p_{i-1}} \cap X_{p_i} \cap X_{p_{i+1}} = \lambda(x_{i-1})$,
then $\lambda(x_i) = \lambda(x_{i-1})$ and $\lambda(z_i) = X_{p_i} \setminus X_{p_{i-1}}$;
else $\lambda(x_i) = X_{p_i} \setminus X_{p_{i-1}}$ and $\lambda(z_i) = \lambda(z_{i-1})$.

Thanks to this labeling, we are now able to present a minor model of G in Ξ_{r+1} :

$$\forall v \in V(G),\ M_v = \{u \in V(\Xi_{r+1}),\ \lambda(u) = v\}$$
$$\mathcal{M} = \{M_v\}_{v \in V(G)}$$

$$\varphi\colon \begin{cases} V(G) & \to & \mathcal{M} \\ u & \mapsto & M_u \end{cases}$$

To show that $(\mathcal{M}, \varphi)$ is a G-minor model in Ξ_{r+1}, we now check if it matches the definition of a minor model.

By definition, every element of $\mathcal{M}$ is a subset of $V(\Xi_{r+1})$. To show that every element of $\mathcal{M}$ induces a connected subgraph in G, it suffices to show that nodes of Ξ_{r+1} which have the same label induces a connected subgraph in G (by construction of the elements of $\mathcal{M}$). This can easily be seen by remarking that for every $i \in [\![2, r]\!]$, every vertex y_i of Ξ_{r+1} gets a new label and that every vertex x_i (resp. z_i) of Ξ_{r+1} receive either the same label as y_i, or the same label as x_{i-1} (resp. z_{i-1}).

Let us show that this labeling ensure that if two vertices u and v of G are in the same bag of P, there are two adjacent vertices of Ξ_{r+1} that respectively gets labels u and v. Let u, v be two vertices of G which are in the same bag of P. Let i be such that X_i is the first bag of P (with respect to the subscripts of the bags of P) which contains both u and v. The case $i = 1$ is trivial so we assume that $i > 1$. We also assume without loss of generality that $X_i \setminus X_{i-1} = \{v\}$, what gives $\lambda(y_i) = v$. Depending on in what case we are, either either $\lambda(x_i) = u$ (c1) or $\lambda(z_i) = u$ ((c1) and (c2)). In both cases, u and v are the labels of two adjacent nodes of Ξ_{r+1}. By construction of the elements of $\mathcal{M}$, this implies that if $\{u, v\} \in E(G)$, then there are vertices $u' \in \varphi(u)$ and $v' \in \varphi(v)$ such that $\{u', v'\} \in E(\Xi_{r+1})$.

Therefore, $(\mathcal{M}, \varphi)$ is a G-minor model in Ξ_{n-1}, what we wanted to find. □

Observe that Lemma 4.17 can be straighforwardly extended to the setting of graphs that are not simple. Indeed, given a graph G, one can subdivide once every edge in order to obtain a simple graph G' that contains G as a minor, and then apply Lemma 4.17 on G', that satisfies $|V(G')| = |V(G)| + |E(G)|$.

Corollary 4.6. *For every graph G of pathwidth at most 2, there is a minor model of G in Ξ_n, where $n = |V(G)| + |E(G)| - 1$.*

Exclusion of Ξ_r

Lemma 4.18. *For any graph, if $\mathbf{tw}(G) \geq 3\ell - 1$ then G contains as minor the following graph: a path $P = p_1 \dots p_{2\ell}$ of length 2ℓ and a family Q of ℓ paths of length 2 such that every vertex of P is the end of exactly one path of Q and every path of Q has one end in $p_1 \dots p_l$ (the first half of P) and the other end in $p_{l+1} \dots p_{2l}$ (the second half of P) (see Figure 4.5).*

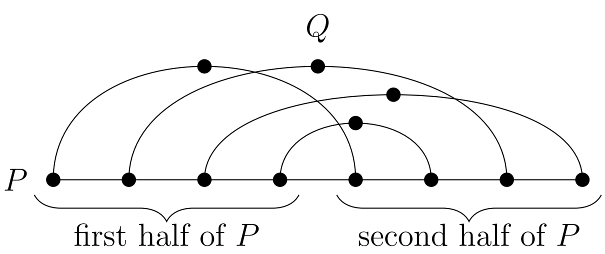


Figure 4.5: Example for Lemma 4.18.

Proof. Let $\ell > 0$ be an integer and G be a graph of treewidth at least $3\ell - 1$. According to Proposition 4.2, G has a separation (A, B) of order 2ℓ such that

(i) $G[B \setminus A]$ is connected;

(ii) $A \cap B$ is linked in $G[B]$;

(iii) (A, B) left-contains a path $P = p_1 \dots p_{2\ell}$ of length 2ℓ.

Let $(\mathcal{M}, \varphi)$ be a minor model of P in $G[A]$, with $\mathcal{M} = \{M_1, \dots, M_{2\ell}\}$. We assume without loss of generality that φ maps p_i to M_i for every $i \in [\![1, 2\ell]\!]$.

As $A \cap B$ is linked in $G[B]$, there is a set Q of ℓ disjoint paths in $G[B]$ of length at least 2 and such that every path $q \in Q$ has one end in $(A \cap B) \cap \bigcup_{i \in [\![1,\ell]\!]} M_i$, the other end in $(A \cap B) \cap \bigcup_{i \in [\![\ell+1,2\ell]\!]} M_i$ and its internal vertices are not in $A \cap B$.

Let G' be the graph obtained from $G[\left(\bigcup_{q \in Q} V(q)\right) \cup \left(\bigcup_{M \in \mathcal{M}} M\right)]$ after the following operations.

1. iteratively contract the edges of every path of Q until it reaches a length of 2. The paths of Q have length at least 2, so this is always possible.

2. for every $i \in [\![1, 2\ell]\!]$, contract M_i to a single vertex. The elements of a minor model are connected (by definition) thus this operation can always be performed.

As one can easily check, the graph G' is the graph we were looking for and it has been obtained by contracting some edges of a subgraph of G, therefore $G' \leq_{\mathrm{m}} G$. □

We can now prove Theorem 4.12.

Proof of Theorem 4.12. Let G, H and h be as in the statement of the Lemma. According to Lemma 4.17, every simple graph F on n vertices and of pathwidth at most two is a minor of Ξ_{n-1}. Therefore, in order to show that $H \leq_{\mathrm{m}} G$ it is enough to prove that $\Xi_{k-1} \leq_{\mathrm{m}} G$. This is what we will do.

According to Lemma 4.18, G contains as minor two paths $P = p_1 \dots p_{(k-2)^2}$ and $R = r_1 \dots r_{k-2)^2}$ and a family Q of $(k-2)^2$ paths of length 2 such that every vertex of P or R is the end of exactly one path of Q and every path of Q has one end in P and the other end in R. For every $p \in P$, we denote by $\varphi(p)$ the (unique) vertex of R to which p is linked to by a path of Q. Observe that φ is a bijection. By Proposition 4.3, there is a subsequence $P' = (p'_1, p'_2, \dots, p'_{k-1})$ of the vertices of P such that the vertices $\varphi(p'_1), \varphi(p'_2), \dots, \varphi(p'_{k-1})$ appear in R either in this order or in the reverse order. Let $R' = (\varphi(p'_1), \varphi(p'_2), \dots, \varphi(p'_{k-1}))$ and Q' be the set of inner vertices of the paths from p'_i to $\varphi(p'_i)$ for all $i \in [\![1, k-1]\!]$.

Iteratively contracting in G the edges of P (resp. R) which have at most one end in P' (resp. in R') and removing the vertices that are not in P', R' or Q' gives the graph Ξ_{k-1}. The operations used to obtain it are vertices and edge deletions, and edge contractions, thus Ξ_{k-1} is a minor of G. This concludes the proof. □

Observe that Theorem 4.12 can be extended to the setting of graphs that are not simple as we did for Corollary 4.6, what gives Corollary 4.4.

4.3.5 Excluding a yurt graph with a $O(k^4)$ bound on treewidth

In this section we prove Theorem 4.13. For every positive integer n, we denote by Y_n the Yurt graph of order n (as a reminder, see Figure 4.6).

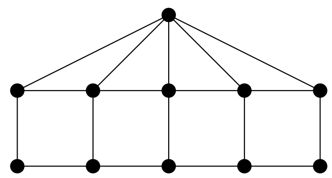


Figure 4.6: The yurt graph of order 5.

For every $r > 0$, we define the *comb of order r* as the tree made from the path $p_1 p_2 \dots p_r$ and the extra vertices $v_1, v_2, \dots, v_r$ by adding an edge between p_i and v_i for every $i \in [\![1, r]\!]$.

By using Lemma 4.13 we can immediately prove Theorem 4.13.

Proof. Let $k > 0$ be an integer and G be a graph such that $\mathbf{tw}(G) \geq 6k^4 - 24k^3 + 48k^2 - 48k + 23$. Let C be the comb with $l = k^4 - 4k^3 + 8k^2 - 8k + 4$ teeth. As $\mathbf{tw}(G) \geq 3|V(C)| - 1$, G contains as a minor some graph of $\Lambda(C)$ by Lemma 4.13.

Let us prove that every graph of $\Lambda(C)$ contains the yurt graph of order k. Let H be a graph of $\Lambda(C)$. We respectively call T, P and o the tree, path and extra vertex of $\Lambda(C)$. Let F be the subset of edges between P and the leaves of T

Let $L = l_0, \dots, l_{k^2-2k+2}$ (resp. $Q = q_0, \dots, q_{k^2-2k+2}$) be the leaves of T (resp. of P)that are the end of an edge of F We assume without loss of generality that they appears in this order.

According to Proposition 4.3, there is a subsequence Q' of Q of length k such that the corresponding vertices L' of L appear in the same order. As one can easily see, this graph contains the yurt of order k and we are done. □

4.3.6 Excluding a union of k disjoint copies of θ_r with a $O(k \log k)$ bound on treewidth

This section is devoted to the proof of Theorem 4.14. Before we present the proof, we need to introduce some definitions and related results.

Preliminaries

Let G be a graph and G_1, G_2 two non-empty subgraphs of G. We say that (G_1, G_2) is a *separation* of G if:

- $V(G_1) \cup V(G_2) = V(G)$; and
- $(E(G_1), E(G_2))$ is a partition of $E(G)$.

Let G be a graph. Given a set $E \subseteq E(G)$, we define V_E as the set of all endpoints of the edges in E. Given a partition (E_1, E_2) of $E(G)$ we define $\delta(E_1, E_2) = |V_{E_1} \cap V_{E_2}|$.

A *cut* $C = (X, Y)$ of G is a partition of $V(G)$ into two subsets X and Y. We define the *cut-set* of C as $E_C = \{\{x, y\} \in E(G) \mid x \in X \text{ and } y \in Y\}$ and call $|E_C|$ the *order* of the cut. Also, given a graph G, we denote by $\sigma(G)$ the number of connected components of G.

The branchwidth of a graph. A *branch-decomposition* of a graph G is a pair (T, τ) where T is a ternary tree and τ a bijection from the edges of G to the leaves of T. Deleting any edge e of T partitions the leaves of T into two sets, and thus the edges of G into two subsets E_1^e and E_2^e. The *width* of a branch-decomposition (T, τ) is equal to $\max_{e \in E(T)}\{\delta(E_1^e, E_2^e)\}$. The *branchwidth* of a graph G, denoted $\mathbf{bw}(G)$, is defined as the minimum width over all branch-decompositions of G.

The branchwidth of a matroid. We assume that the reader is familiar with the basic notions of matroid theory. We will use the standard notation from Oxley's book [Oxl92]. The branchwidth of a matroid is defined very similarly to that of a graph. Let $\mathcal{M}$ be a matroid with finite ground set $E(\mathcal{M})$ and rank function r. The *order* of a non-trivial partition (E_1, E_2) of $E(\mathcal{M})$ is defined as $\lambda(E_1, E_2) = r(E_1) + r(E_2) - r(E) + 1$. A *branch-decomposition* of a matroid $\mathcal{M}$ is a pair (T, μ) where T is a ternary tree and μ is a bijection from the elements of $E(\mathcal{M})$ to the leaves of T. Deleting any edge e of T partitions the leaves of T into two sets, and thus the elements of $E(\mathcal{M})$ into two subsets E_1^e and E_2^e. The *width* of a branch-decomposition (T, μ) is equal to $\max_{e \in E(T)}\{\lambda(E_1^e, E_2^e)\}$. The *branchwidth* of a matroid $\mathcal{M}$, denoted $\mathbf{bw}(\mathcal{M})$, is again defined as the minimum width over all branch-decompositions of $\mathcal{M}$. The *cycle matroid of a graph* G denoted $\mathcal{M}_G$, has ground set $E(\mathcal{M}_G) = E(G)$ and the cycles of G as the cycles of $\mathcal{M}_G$. Let G be a graph, $\mathcal{M}_G$ its cycle matroid and (G_1, G_2) a separation of G. Then clearly $(E(G_1), E(G_2))$ is a partition of $E(\mathcal{M}_G)$, but to avoid confusion we will henceforth denote it (E_1, E_2) and we will call it *the partition of* $\mathcal{M}_G$ *that corresponds to the separation* (G_1, G_2) *of* G. Observe that the order of this partition is:

$$\lambda(E_1, E_2) = \delta(E(G_1), E(G_2)) - \sigma(G_1) - \sigma(G_2) + \sigma(G) + 1. \qquad (\star)$$

Minor obstructions. Let $\mathcal{G}$ be a graph class. We denote by $\mathbf{obs}(\mathcal{G})$ the set of all minor-minimal graphs H such that $H \notin \mathcal{G}$ and we will call it *the minor obstruction set for* $\mathcal{G}$. Clearly, if $\mathcal{G}$ is closed under minors, the minor obstruction set for $\mathcal{G}$ provides a complete characterization for $\mathcal{G}$: a graph G belongs in $\mathcal{G}$ if and only if none of the graphs in $\mathbf{obs}(\mathcal{G})$ is a minor of G.

Given a class of matroids $\mathbf{M}$, *the minor obstruction set for* $\mathbf{M}$, denoted by $\mathbf{obs}(\mathbf{M})$, is defined very similarly to its graph-counterpart: it is simply the set of all minor-minimal matroids $\mathcal{M}$ such that $\mathcal{M} \notin \mathbf{M}$.

We will need the following results.

Proposition 4.4 ([RS91, Theorem 5.1]). *Let G be a graph of branchwidth at least 2. Then, $\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \lfloor \frac{3}{2} \mathbf{bw}(G) \rfloor$.*

Proposition 4.5 ([BvLTT97a]). *Let $r \in \mathbb{N}_{\geq 1}$ and let G be a graph. If $\mathbf{bw}(G) \geq 2r+1$, then G contains a θ_r-model.*

Proposition 4.6 ([HJ07, Theorem 4]). *Let G be a graph that contains a cycle and $\mathcal{M}_G$ be its cycle matroid. Then, $\mathbf{bw}(G) = \mathbf{bw}(\mathcal{M}_G)$.*

Proposition 4.7 ([GGRW03, Lemma 4.1]). *Let a matroid $\mathcal{M}$ be a minor obstruction for the class of matroids of branchwidth at most k and let $g(n) = (6^{n-1} - 1)/5$. Then, for every partition (X, Y) of $\mathcal{M}$ with $\lambda(X, Y) \leq k$, either $|X| \leq g(\lambda(X, Y))$ or $|Y| \leq g(\lambda(X, Y))$.*

The following observations are also crucial.

Observation 4.1. Let $\mathcal{G}$ be a graph class that is closed under minors and let $\mathcal{M}_{\mathcal{G}} = \{\mathcal{M}_G \mid G \in \mathcal{G}\}$. $\mathcal{G}$ is minor closed if and only if $\mathcal{M}_{\mathcal{G}}$ is minor closed. Moreover, for every $H \in \mathbf{obs}(\mathcal{G})$ it holds that $\mathcal{M}_H \in \mathbf{obs}(\mathcal{M}_{\mathcal{G}})$.

Observation 4.2. There is a $c \in \mathbb{R}_{\geq 2}$, such that for any integer $k \geq r \geq 2$, if $g(n) = (6^{n-1} - 1)/5$, then $\frac{1}{r-1} 2^{\frac{c^r \log k - 5r}{4r(2g(2r-2)+1)}} \geq k(r+1) - 1$. Moreover, this holds for $c = 2^6 \log_r \frac{2}{3}$.

Now we are ready to prove the main result of this section.

Proof of Theorem 4.14

For every $r \in \mathbb{N}$, we define $f(r) = \frac{2}{3} 2^{6r}$. By Proposition 4.4, it is enough to prove that if $\mathbf{bw}(G) \geq f(r) \cdot k \cdot \log(k+1)$, then G contains $k \cdot \theta_r$ as a minor. To prove this we use induction on k.

The case where $k = 1$ follows from Proposition 4.5 and the fact that $f(r) \geq 2r+1$. We now examine the case where $k > 1$, assuming that the proposition holds for smaller values of k. As $\mathbf{bw}(G) \geq f(r) \cdot k \cdot \log(k+1)$, G contains a minor obstruction for the class of graphs of branchwidth at most $f(r) \cdot k \cdot \log(k+1) - 1$.

Claim 4.2. Any $(2r-2)$-edge-protrusion of G has extension at most $g(2r-2)$.

Proof of Claim 4.2. Let $C = (X, Y)$ be a cut in G of order at most $2r-2$ and let G_X be the subgraph of G with $V(G_X) = X \cup N_G(X)$ and let $E(G_X) = E(G[X]) \cup E_C$. Clearly the pair $(G_X, G[Y])$ is a separation of G. Let $\mathcal{M}_G$ be the cycle matroid of G and

(E_X, E_Y) be the partition of $\mathcal{M}_G$ that corresponds to the aforementioned separation. By Proposition 4.6, $\mathbf{bw}(\mathcal{M}_G) = \mathbf{bw}(G) \geq f(r) \cdot k \cdot \log(k+1)$. Therefore, by Observation 4.1, $\mathcal{M}_G$ is a minor obstruction for the class of matroids of branchwidth $f(r) \cdot k \cdot \log(k+1) - 1$. We set $\lambda = \lambda(E_X, E_Y)$. From $(\star)$, we have:

$$\begin{aligned}
\lambda &= r(E_X) + r(E_Y) - r(\mathcal{M}_G) + 1 \\
&= \delta(E(G_X), E(G[Y])) - \sigma(G_X) - \sigma(G[Y]) + \sigma(G) + 1 \\
&\leq \delta(E(G_X), E(G[Y])) \\
&\leq |E_C| = 2r - 2 \\
&\leq f(r) \cdot k \cdot \log(k+1) - 1.
\end{aligned}$$

Thus, by Proposition 4.7, either $|E_X| \leq g(\lambda)$ or $|E_Y| \leq g(\lambda)$. Since g is non-decreasing, either $|E(G_X)| \leq g(2r-2)$ or $|E(G[Y])| \leq g(2r-2)$. This directly implies that for any $(2r-2)$-edge-protrusion Z of G, $G[Z \cup N_G(Z)]$ has at most $g(2r-2)$ edges. Therefore Z's extension is also at most $g(2r-2)$ and the claim follows. ◇

Combining the above claim, Observation 4.2, and Theorem 4.4, we infer that either G contains a θ_r-model M with at most $f(r) \cdot \log k$ edges, or it contains a minor with minimum degree at least $\frac{1}{r-1} \cdot 2^{\frac{f(r) \log k - 5r}{4r(2g(2r-2)+1)}} \geq k(r+1) - 1$. If the second case is true, then by Lemma 4.10 G contains $k \cdot \theta_r$ as a minor, which proves the inductive step. We now consider the first case. Because M is 2-connected, we obtain that $|V(M)| \leq |E(M)|$. Therefore, $|V(M)| \leq |E(M)| \leq f(r) \cdot \log k$ and we can bound the treewidth of the graph $G' = G \setminus V(M)$ as follows:

$$\begin{aligned}
\mathbf{tw}(G') &\geq \mathbf{tw}(G) - |V(M)| \\
&\geq f(r) \cdot k \cdot \log(k+1) - f(r) \cdot \log k \\
&\geq f(r) \cdot k \cdot \log k - f(r) \cdot \log k \\
&= f(r) \cdot (k-1) \cdot \log k.
\end{aligned}$$

Then, from the induction hypothesis, G' contains a $(k-1) \cdot \theta_r$-model M' and obviously $M \cup M'$ is a $k \cdot \theta_r$-model in G, which concludes our proof. □

Theorem 4.14 implies that for every fixed r, it holds that every graph excluding $k \cdot \theta_r$ as a minor has treewidth $O(k \cdot \log k)$. We give in Subsection 4.3.7 a lemma indicating that this bound is tight up to the constants hidden in the O-notation.

4.3.7 Lower bounds

An natural question is whether the aforementioned results for the classes $\{\mathcal{H}_k^i\}_{i \in [\![1,4]\!]}$ are tight. In this section we provide lower bounds on the best function f that one can obtain in (4.2) for these classes.

Lemma 4.19. *There is a sequence $(G_k)_{k \in \mathbb{N}_{\geq 3}}$ of graphs such that:*

- *G_k does not contain W_k^2 as a minor, for every $k \in \mathbb{N}_{\geq 3}$; and*

- $\mathbf{tw}(G_k) = \Omega(k)$.

Proof. The sequence $(K_{k-1})_{k\in\mathbb{N}_{\geq 3}}$ satisfies the above properties. $\square$

As $\mathrm{W}_k \leq_{\mathrm{m}} \mathrm{W}_k^2$ for every $k \in \mathbb{N}_{\geq 3}$, Lemma 4.19 implies that (4.2) does not hold for the classes $\mathcal{H}_k^1$ and $\mathcal{H}_k^2$ with a function f such that $f(k) = o(k)$.

Corollary 4.7. *The bound given in Theorem 4.10 is tight up to a constant factor.*

Lemma 4.20. *There is a sequence $(G_i)_{i\in\mathbb{N}_{\geq 1}}$ of graphs and an increasing sequence $(k_i)_{i\in\mathbb{N}_{\geq 1}}$ of integers such that:*

- *G_i does not contain $k_i \cdot K_3$ as a minor, for every $i \in \mathbb{N}_{\geq 1}$; and*
- *$\mathbf{tw}(G_i) = \Omega(k_i \log k_i)$.*

Proof. According to [Mor94, Theorem 5.13], there is an infinite familly $\{G_i\}_{i\in\mathbb{N}}$ of 3-regular Ramanujan graphs G_i such that $i \mapsto |G_i|$ is an increasing function. Furthermore, for every $i \in \mathbb{N}$, the graph G_i has girth at least $\frac{2}{3}\log|V(G_i)|$ ([Mor94, Theorem 5.13]) and satisfies $\mathbf{tw}(G_i) = \Omega(|V(G_i)|)$ (see [BEM+04, Corollary 1]). For every $i \in \mathbb{N}$, let k_i be the minimum integer such that $|V(G_i)| < k_i \cdot \frac{2}{3}\log|V(G_i)|$. Observe that $(k_i)_{i\in\mathbb{N}}$ is increasing. Notice that $|V(G_i)| = \Omega(k_i \cdot \log k_i)$, and thus $\mathbf{tw}(G_i) = \Omega(k_i \cdot \log k_i)$. We will show that G_i does not contain k_i vertex-disjoint cycles, which implies that $k_i \cdot \theta_r$ is not a minor of G_i, for every $r \in \mathbb{N}_{\geq 2}$. Suppose for contradiction that G_i contains k_i vertex-disjoint cycles. As the girth of G_i is at least $\frac{2}{3}\log|V(G_i)|$, each of these cycles has at least $\frac{2}{3}\log|V(G_i)|$ vertices. Therefore G should contain at least $k \cdot \frac{2}{3}\log|V(G_i)|$ vertices. This implies that $|V(G)| \geq k \cdot \frac{2}{3}\log|V(G_i)| > |V(G_i)|$, a contradiction. Therefore $(k_i)_{i\in\mathbb{N}}$ and $(G_i)_{i\in\mathbb{N}}$ satisfy the required properties. $\square$

As every graph of $\mathcal{H}_{2k}^0 \cup \mathcal{H}_{2k}^3 \cup \mathcal{H}_{k,r}^4$ contains $k \cdot K_3$ as a minor (for every $r \in \mathbb{N}_{\geq 2}$), we deduce that (4.2) does not hold for the classes $\mathcal{H}_k^0$, $\mathcal{H}_k^3$, and $\mathcal{H}_k^4$ with a function f such that $f(k) = o(k \log k)$.

Corollary 4.8. *For every fixed $r \in \mathbb{N}_{\geq 2}$, the bound given in Theorem 4.14 is tight up to a constant factor.*

4.4 Immersions of planar subcubic graphs in graphs of large tree-cut width

This section contains the proof of the following result.

Theorem 4.15. *There is a function $f : \mathbb{N} \to \mathbb{N}$ where $f(h) = O(h^{29}\operatorname{polylog}(h))$ such that for every planar subcubic graph H with h edges and every graph G, if $\mathbf{tcw}(G) \geq f(h)$ then H is an immersion of G.*

This theorem is a consequence of Theorem 4.9 and the following exclusion theorem for walls. We present it here as we will use it in Chapter 5.

Theorem 4.16 ([Wol15, Theorem 7]). *Let G be a graph and $r \geq 1$ be a positive integer. Let f be a function for which Theorem 4.1 (the grid exclusion theorem) holds. If G has tree-cut width at least $4r^{10}f(r)$, then G admits an immersion of the r-wall.*

We will also use the following ingredient.

Lemma 4.21 ([Kan96]). *Every simple planar subcubic graph of n vertices is a topological minor of the $\lfloor \frac{n}{2} \rfloor$-grid.*

According to Theorem 4.16, graphs of large enough tree-cut width contain a large wall as an immersion. Therefore we only need to show that every planar subcubic graph is an immersion of a (large enough) wall in order to prove Theorem 4.15. For this we use a supergraph of the wall defined as follows from the grid W_k. Let $E = \{e,\ e \in E(P_j^{(v)}) \cap E(P_i^{(h)}), j \in \{1, 2, \ldots, k\}, i \in \{1, 2, \ldots, k+1\}\}$. We obtain $\widehat{W}_k$ by W_k by adding a second copy of every edge in E. (For an example, see Figure 4.7.)

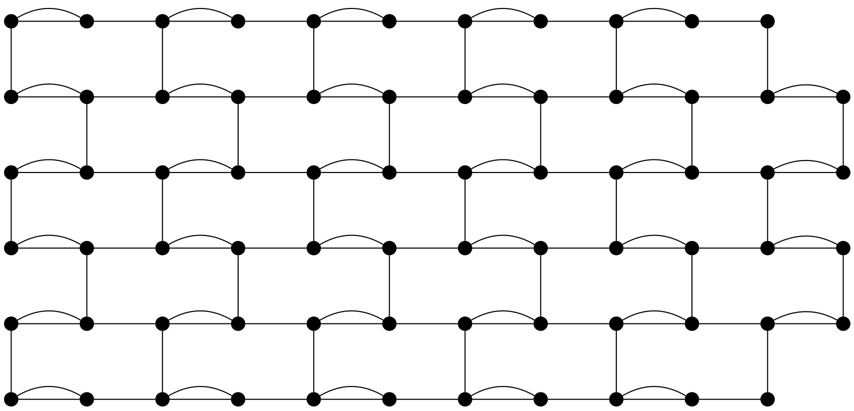


Figure 4.7: The graph $\widehat{W}_5$.

The next observation is a formal statement of what is depicted on Figure 4.8: $\widehat{W}_n$ contains Γ_n as a strong immersion. Branch vertices are depicted by white nodes and horizontal (respectively vertical) paths use the color green (respectively red).

Observation 4.3. Let $k \geq 2$ be an integer. If we define ϕ and ψ with domains $V(\Gamma_n)$ and $E(\Gamma_n)$, respectively, as follows:

$$\begin{aligned}
\phi((i,j)) &= (i, 2j-1) \\
\psi(\{(i,j),(i,j+1)\}) &= (i,2j-1)(i,2j)(i,2(j+1)-1) \\
\psi(\{(i,j),(i+1,j)\}) &= (i,2j-1)(i+1,2j-1) \text{ for odd } i \\
\psi(\{(i,j),(i+1,j)\}) &= (i,2j-1)(i,2j)(i+1,2j)(i+1,2j-1) \text{ for even } i,
\end{aligned}$$

then (ϕ, ψ) is a Γ_k-strong-immersion model in $\widehat{W}_k$ (where we assume that Γ_k has vertex set $[\![1,k]\!]^2$).

The next result is mentioned in [Tho88] but not proof is provided.

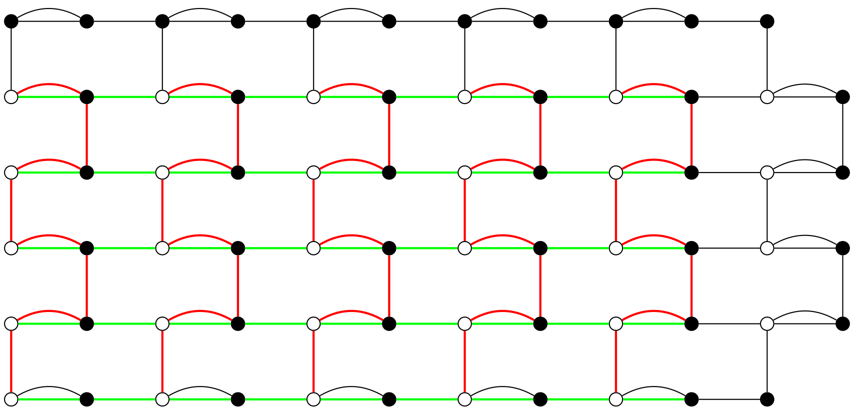


Figure 4.8: Finding Γ_5 as a strong immersion in $\widehat{W}_5$.

Lemma 4.22. *Every planar subcubic graph on n-vertices is a topological minor of the wall W_n.*

Proof. Let H be a graph on n vertices. The proof goes as follows: we first construct a topological expansion H' of H that is a simple graph. Then we prove that H' is a strong-immersion of $\widehat{W}_n$ and obtain the following ordering:

uniformize (topological)model formalism

$$H \leq_{\text{tm}} H' \leq_{\text{tm}} \Gamma_n \leq_{\text{sim}} \widehat{W}_n. \tag{4.4}$$

Finally, we construct a topological model of H' in $\widehat{W}_n$. The expansion of this model is simple, hence it will be a subgraph of W_n, as required.

Let H be a planar subcubic graph and let H' be the simple subcubic planar graph obtained from H by subdividing all but one edges of every multiedge. Notice that the first inequality of equation (4.4) is satisfied. Let us count how many vertices are added during the construction of H'. As H is subcubic, among the edges incident to a given vertex, at most two are being subdivided. That way we count each subdivided edge twice (once for each of its endpoints), hence we get:

$$|V(H')| \leq 2|V(H)|.$$

According to Lemma 4.21, H' is a topological minor of Γ_n: this gives the second inequality of the equation. Observation 4.3 gives the third inequality.

Let (ϕ_1, ψ_1) be an H'-topological model in Γ_n and let (ϕ_2, ψ_2) be the Γ_n-strong-immersion model in $\widehat{W}_n$ given by Observation 4.3. These two models can be used to construct an H'-strong immersion model (ϕ, ψ) in $\widehat{W}_n$, as the composition of (ϕ_1, ψ_1) and (ϕ_2, ψ_2): for every $v \in V(H')$, $\phi(v) = \phi_2(\phi_1(v))$ and for every $e \in E(H')$, $\psi(e)$ is the concatenation of the paths obtained by applying ψ_2 to the edges of the path $\psi_1(e)$ (taken in the same order as they appear in this path). Observe that this model satisfies the following properties:

- the expansion of (ϕ, ψ) is a subgraph of the expansion of (ϕ_2, ψ_2); and

- the branch vertices of (ϕ, ψ) are branch vertices of (ϕ_2, ψ_2).

We provide the following diagram to recall the roles of the different models we use (topological models are indicated by double arrows and strong immersion models by simple ones).

$$\begin{tikzcd} H' \arrow[r, Rightarrow, "(\phi_1,\psi_1)"] \arrow[dr, "(\phi,\psi)"'] & \Gamma_n \arrow[d, "(\phi_2,\psi_2)"] \\ & \widehat{W}_n \end{tikzcd}$$

Let us show the following claim.

Claim 4.3. Let $e, f \in E(H')$. If v is an internal vertex of both $\psi(e)$ and $\psi(f)$, then these paths also share an endpoint, which is adjacent to v.

If $\psi(e)$ and $\psi(f)$ share an internal vertex v, there are two edges $a \in \psi_1(e)$ and $b \in \psi_1(f)$ such that both $\psi_2(a)$ and $\psi_2(b)$ contain v. By definition of (ϕ_2, ψ_2), such a situation occurs only if $a = \{(i, j), (i + 1, j)\}$ (for even i) and $b = \{(i, j), (i, j + 1)\}$ or $b = \{(i+1, j), (i+1, j+1)\}$, for some even $i \in [\![1, n]\!]$ and some $j \in [\![1, n]\!]$ (see Figure 4.8). Observe that in both cases a and b share an endpoint. As (ϕ_1, ψ_1) is a topological minor model, $\psi_1(e)$ and $\psi_1(f)$ may meet on endpoints only. Therefore the common endpoint of a and b is an endpoint of both $\psi_1(e)$ and $\psi_1(f)$, hence $\psi(e)$ and $\psi(f)$ have a common endpoint. This proves the first part of the claim. The second part is now clear from the definition of (ϕ_2, ψ_2), as we know that the paths $\psi_1(e)$ and $\psi_1(f)$ start from the same vertex, one with a "vertical" edge, the other with a "horizontal" edge (see Figure 4.8).$\diamond$

If (ϕ, ψ), which is a strong immersion model, is a topological model, then we can directly jump to the next step. Otherwise, according to Claim 4.3, there are two edges $e = \{u, v\}$, $f = \{u, w\}$ of H and vertices $x, y \in \widehat{W}_n$ such that $\psi(e)$ and $\psi(f)$ both start with $x = \phi(u)$ followed by y. Hence $\{x, y\}$ is a double edge of $\widehat{W}_n$. As (ϕ, ψ) is a strong immersion model of a subcubic graph, x has degree at most three in the expansion of (ϕ, ψ) We can therefore modify (ϕ, ψ) as follows: we set $\phi(u) = x$ and we shorten $\psi(\{u, v\})$ and $\psi(\{u, w\})$ by removing the edge $\{x, y\}$ from each of them. In the case where there is a third vertex $t \in V(H) \setminus \{v, w\}$ adjacent to u, we also extend the path $\psi(\{t, u\})$ by adding the edge $\{x, y\}$. See Figure 4.9 for an example.

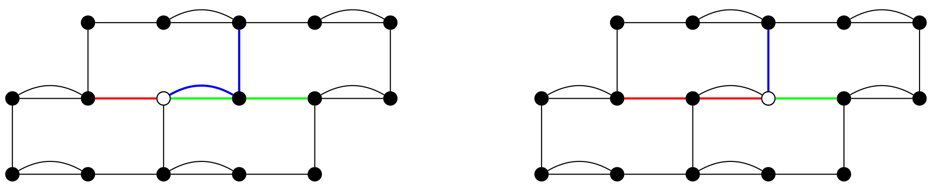


Figure 4.9: Swapping branch vertices.

It is easy to see that by applying these changes we still get an H'-strong immersion model, with less crossings of certifying paths. By repeatedly applying these steps we eventually obtain a H'-topological model in $\widehat{W}_n^+$. Notice that its expansion is a simple graph, as H' is a simple graph. Therefore this expansion is also a subgraph of W_n. We proved that H' is a topological minor of W_n. It follows that the same holds for H and we are done. $\square$

The proof of Theorem 4.15 is now staighforward: given that $\mathbf{tcw}(G) \geq 4h^{10} f_1(h)$ (recall that f_1 is the function of the Grid Exclusion Theorem, Theorem 4.9), G contains W_h as an immersion (Theorem 4.16). As we just proved with Lemma 4.22, every planar subcubic graph on h vertices is an immersion of W_n, so it is an immersion of G. Also, $4h^{10} f_1(h) = O(h^{29}\,\mathrm{polylog}(h))$.

Chapter 5

The Erdős-Pósa property

This chapter contains material that previously appeared in the following articles:

- *Polynomial gap extensions of the Erdős-Pósa Theorem*, co-authored with Dimitrios M. Thilikos, and presented at the Seventh European Conference on Combinatorics, Graph Theory and Applications, EuroComb 2013, Pisa, Italy, 2013 [RT13];

- *An edge variant of the Erdős-Pósa property*, co-authored with Ignasi Sau and Dimitrios M. Thilikos, to appear in Discrete Mathematics, Volume 339, Issue 8, 2016 [RST16];

- *An $O(\log \mathrm{OPT})$-approximation for covering/packing minor models of θ_r*, co-authored with Dimitris Chatzidimitriou, Ignasi Sau, and Dimitrios M. Thilikos, presented in Approximation and Online Algorithms: 13th International Workshop, WAOA 2015, Patras, Greece, 2015 [CRST15a];

- *Packing and covering immersion models of planar subcubic graphs*, co-authored with Archontia Giannopoulou, O-joung Kwon, and Dimitrios M. Thilikos, to be presented at the 42nd International Workshop on Graph-Theoretic Concepts in Computer Science, WG 2016, Istanbul, Turkey, 2016 [GKRT16];

- *Recent techniques and results on the Erdős-Pósa property*, co-authored with Dimitrios M. Thilikos, 2016, submitted [RT16].

5.1 Introduction

A considerable part of Combinatorics has been developed around min-max theorems. Min-max theorems usually identify dualities between certain objects in graphs, hypergraphs, and other combinatorial structures. The target is to prove that the absence of the primal object implies the presence of the dual one and vice versa.

A classic example of such a duality is Menger's theorem: the primal concept is the existence of k internally disjoint paths between two vertex sets S and T of a graph G, while the dual concept is a collection of k vertices that intersect all (S, T)-paths. Another example is Kőnig's theorem where the primal notion is the existence of a matching of k

vertices in a bipartite graph and the dual one is the existence of a vertex cover of size k. It is also known that, in case of general graphs, this duality becomes an approximate one, i.e., a vertex cover of size $2k$. In both aforementioned examples, the duality is between the notions of *packing* and *covering* of a collection $\mathcal{C}$ of combinatorial objects of a graph. In Menger's theorem $\mathcal{C}$ consists of all (S, T)-paths of G while in Kőnig's theorem $\mathcal{C}$ is the set of all edges of G. That way, both aforementioned min-max theorems can be stated, for some class of graphs $\mathcal{G}$ (called *host* class) and some gap function $f : \mathbb{N} \to \mathbb{N}$, as follows:

> *For every graph G in $\mathcal{G}$, either G contains k-vertex disjoint objects in $\mathcal{C}$ or it contains $f(k)$ vertices intersecting all objects in $\mathcal{C}$ that appear in G.*

Clearly, for the case of Menger's theorem the host class is the class of all graphs while in the case of Kőnig's theorem the host class is restricted to the class of bipartite graphs. In both cases the derived duality is an *exact* one in the sense that f is the identity function. However, this is not the case if we want to extend the duality of Kőnig's theorem in the case of all graphs, where we can consider $f : k \mapsto 2k$ (i.e., we have an *approximate* duality).

One of the most celebrated results about packing/covering dualities was obtained by Paul Erdős and Lajos Pósa in 1965 where the object to cover and pack was the set of all cycles of G [EP65]. In this case the host class contains all graphs, while $f : k \mapsto O(k \cdot \log k)$. Moreover, Erdős and Pósa proved that this gap is optimal in the sense that it cannot be improved to a function $f : k \mapsto o(k \cdot \log k)$. This result motivated a long line of research for min-max dualities that are not necessarily exact or approximate. Since then, a multitude of results on Erdős–Pósa properties have appeared for several combinatorial objects, including extensions to digraphs [LY78, Sey96, RRST96, HM13, GT10], rooted graphs [KKK12, PW12, Joo14, BJS14], labeled graphs [KW05], signed graphs [HNP06, ADG04], hypergraphs [Alo02, Bou13, BT15], matroids [GK09], and other combinatorial structures [GL69] (see [Ree97] for a survey on this topic). Also it is worth stressing that Erdős–Pósa dualities have been useful in more applied domains. For example, in bioinformatics where they where useful for upper-bounding the number of fixed-points of a boolean networks [Ara08, ADG04, ARS16].

The chapter is organized as follows. We first describe some recent techniques for proving Erdős–Pósa properties, mainly based on techniques related to tree-like decompositions of graphs (Subsection 5.2.1 and Subsection 5.2.2). We focused our presentation to the description of general frameworks that, we believe, might be useful for further investigations. We then present several results related to the Erdős-Pósa property. Lastly, in Subsection 5.4.2, we provide an extensive update of results on Erdős–Pósa properties, reflecting the current progress on this vibrant area of graph theory. Most of the notions used in this chapter are defined in Section 2.6.

5.2 General techniques for proving Erdős-Pósa type results

In this section we present general tools for proving Erdős–Pósa type results.

5.2.1 Erdős–Pósa from graph decompositions

Let $\mathcal{H}$ be a graph class, $\mathbf{p}$ be a graph parameter, and $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$. We say that a function $f\colon \mathbb{N} \to \mathbb{N}$ is a *ceiling* for the triple $(\mathbf{p}, \mathcal{H}, \mathsf{x})$ if for every graph G, $\mathbf{p}(G) \leq f(\mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(G))$. Intuitively, there is a ceiling for the triple $(\mathbf{p}, \mathcal{H}, \mathsf{x})$ if a large value of $\mathbf{p}$ on a graph forces a large x-packing of elements of $\mathcal{H}$. Notice that every ceiling for $(\mathbf{p}, \mathcal{H}, \mathsf{v})$ is a ceiling for $(\mathbf{p}, \mathcal{H}, \mathsf{e})$, since a vertex-disjoint packing is a special case of an edge-disjoint packing.

Given a graph parameter $\mathbf{p}$ and an integer k, we denote

$$\mathcal{G}_{\mathbf{p}\leq k} = \{G,\ \mathbf{p}(G) \leq k\}.$$

Theorem 5.1. *Let $\mathcal{H}$ be a class of graphs, $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, $\mathbf{p}$ be a graph parameter, let $f\colon \mathbb{N} \to \mathbb{N}$ be a function and let $h_r\colon \mathbb{N} \to \mathbb{N}$ be a function, for every $r \in \mathbb{N}$. Suppose that the following two conditions hold:*

A. *f is a ceiling for the triple $(\mathbf{p}, \mathcal{H}, \mathsf{x})$;*

B. *for every $r \in \mathbb{N}$, $\mathcal{H}$ has the x-Erdős–Pósa property for $\mathcal{G}_{\mathbf{p}\leq r}$ with gap h_r;*

then $\mathcal{H}$ has the x-Erdős–Pósa property with gap $k \mapsto h_{f(k)}(k)$.

Proof. Let G be a graph and let $k = \mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(G)$. We have $\mathbf{p}(G) \leq f(k)$, by definition of a ceiling. Therefore, $G \in \mathcal{G}_{\mathbf{p}\leq r}$, and thus $\mathsf{x}\text{-}\mathsf{cover}_{\mathcal{H}}(G) \leq h_{f(k)}(k)$. □

Theorem 5.1 will be used as a master theorem for the results of this section.

Vertex version and tree decompositions

In a breakthrough paper [CC13a], Chekuri and Chuzhoy proved that every graph of large treewidth can be partitioned into several subgraphs of large treewidth, with a polynomial dependency between the treewidth of the original graph, the one of the subgraphs, and the number of subgraphs. In fact they proved the two next results.

Theorem 5.2 ([CC13a, Theorem 1.1]). *There is a non-decreasing function $f_2\colon \mathbb{R} \to \mathbb{R}$ with $f_2(t) = \operatorname{polylog} t$ such that, for every graph G and every positive integers h and p, if*

$$hp^2 \leq \frac{\mathbf{tw}(G)}{f_2(\mathbf{tw}(G))},$$

then there is a partition $G_1, \ldots, G_h$ of G into vertex-disjoint subgraphs such that $\mathbf{tw}(G_i) \geq p$ for every $i \in [\![1, h]\!]$.

Theorem 5.3 ([CC13a, Theorem 1.2]). *There is a non-decreasing function $f_3\colon \mathbb{R} \to \mathbb{R}$ with $f_3(t) = \operatorname{polylog} t$ such that, for every graph G and every positive integers h and p, if*

$$h^3p \leq \frac{\mathbf{tw}(G)}{f_3(\mathbf{tw}(G))},$$

then there is a partition $G_1, \ldots, G_h$ of G into vertex-disjoint subgraphs such that $\mathbf{tw}(G_i) \geq p$ for every $i \in [\![1, h]\!]$.

There results have been used to obtain ceilings. In fact, [CC13a] also contains the following result.

Lemma 5.1 ([CC13a, from the proof of Theorem 5.4])**.** *If r is an integer and $\mathcal{H}$ is a class of graphs such that every graph of treewidth at least r contains an $\mathcal{H}$-subgraph, then there is a ceiling f_4 for $(\mathbf{tw}, \mathcal{H}, \mathsf{v})$ such that $f_4(k) = kr^2 \operatorname{polylog}(kr)$.*

Let us now see the role of ceilings with respect to the Erdős–Pósa property. A function $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *superadditive* if $f(x) + f(y) \leq f(x + y)$ for every pair x, y of positive reals. The following argument has been first used in [FST11] (see also [CC13a, RST16, CRST15a]).

Lemma 5.2. *Let $\mathcal{H}$ be a family of connected graphs. If f is a superadditive ceiling for $(\mathbf{tw}, \mathcal{H}, \mathsf{v})$ then $\mathcal{H}$ has the* v*-Erdős–Pósa property with gap $k \mapsto 5 \cdot f(k) \log(k + 1)$.*

Proof. Let us show the following for every integer k: for every graph G, if $\mathsf{v\text{-}pack}_{\mathcal{H}}(G) = k$ then $\mathsf{v\text{-}cover}_{\mathcal{H}} \leq 5f(k)\log(k+1)$. The proof is by induction on k. The base case $k = 0$ is trivial. Let $k > 0$, and let us assume that the above statement holds for every positive integer $k' < k$ (induction hypothesis).

Let G be a graph such that $\mathsf{v\text{-}pack}_{\mathcal{H}}(G) = k$. We will rely on the following claim.

Claim 5.1*.* There is a separation (A, B) of order $\mathbf{tw}(G) + 1$ of G such that

$$k/3 \leq \mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B]) \leq 2k/3.$$

Proof. It is known that every graph G has an nice tree decomposition with width $\mathbf{tw}(G)$ [Klo94a]. We therefore can assume that $(T, r, (X_t)_{t \in V(T)})$ is a nice tree decomposition of G of optimal width. We define

$$G_t = G\left[\bigcup_{s \in \operatorname{desc}_{(T,r)}(t)} X_s\right] \quad \text{and} \quad G_t^- = G_t \setminus X_t.$$

Let t be a vertex of T at minimal distance from a leaf subject to the requirement $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_t) > 2k/3$. Such a vertex exists, as $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_r) = k$. Observe that t is either a forget node, or a join node. Indeed, for every base node u we have $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_u) = 0$. Moreover, every introduce node u with child v satisfies $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_u) = \mathsf{v\text{-}pack}_{\mathcal{H}}(G_v)$, since $G_u^- = G_v^-$.

First case: t is a forget node with child u. We set $A = V(G_u)$ and $B = V(G) \setminus V(G_u^-)$.

Second case: t is a join node with children u_1, u_2. We set $A = V(G_{u_i})$ and $B = V(G) \setminus V(G_{u_i}^-)$, where u_i is a child of t such that $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_{u_i}) \geq k/3$. Such child exist because $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_t) = \mathsf{v\text{-}pack}_{\mathcal{H}}(G_{u_1}) + \mathsf{v\text{-}pack}_{\mathcal{H}}(G_{u_2})$ (as t is a join node) and $\mathsf{v\text{-}pack}_{\mathcal{H}}(G_t) > 2k/3$, by definition of t.

It is clear that in both cases (A, B) is a separation of order $\mathbf{tw}(G)+1$. The inequality $\mathsf{v\text{-}pack}_{\mathcal{H}}(G[A]) \leq 2k/3$ follows from the definition of t. In the first case, we have

$$\begin{aligned}
\mathsf{v\text{-}pack}_{\mathcal{H}}(G[A]) &= \mathsf{v\text{-}pack}_{\mathcal{H}}(G_u) \\
&\geq \mathsf{v\text{-}pack}_{\mathcal{H}}(G_t) - 1 \qquad \text{(as } t \text{ is a forget node)} \\
&> \frac{2k}{3} - 1 \\
&\geq \frac{k}{3}.
\end{aligned}$$

In the second case, the choice of i ensures that $k/3 \leq \mathsf{v\text{-}pack}_{\mathcal{H}}(G[A])$. □

Observe that $\mathbf{tw}(G) \leq f(k)$, by definition of f. According to Claim 5.1, there is a separation (A, B) of order $\mathbf{tw}(G)+1$ in G such that $k/3 \leq \mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B]) \leq 2k/3$. Let $k_A = \mathsf{v\text{-}pack}_{\mathcal{H}}(G[A \setminus B])$ and $k_B = \mathsf{v\text{-}pack}_{\mathcal{H}}(G[B \setminus A])$. It follows that $k_A, k_B \leq \lfloor 2k/3 \rfloor$.

We then have

$$\begin{aligned}
\mathsf{v\text{-}cover}_{\mathcal{H}}(G) &\leq \mathsf{v\text{-}cover}_{\mathcal{H}}(G[A \setminus B]) + \mathsf{v\text{-}cover}_{\mathcal{H}}(G[B \setminus A]) + |A \cap B| \\
&\leq \mathsf{v\text{-}cover}_{\mathcal{H}}(G[A \setminus B]) + \mathsf{v\text{-}cover}_{\mathcal{H}}(G[B \setminus A]) + f(k) + 1 \\
&\leq 5f(k_A)\log(k_A + 1) + 5f(k_B)\log(k_B + 1) + f(k) + 1.
\end{aligned}$$

Notice that in the case where $k = 1$, we get $k_A = k_B = 0$ and we have $\mathsf{v\text{-}cover}_{\mathcal{H}}(G) \leq f(k) \leq 3\, f(k)\log(k+1)$. Therefore we now assume $k \geq 2$. We can deduce from $k_A, k_B \leq \left\lfloor \frac{2}{3}k \right\rfloor$ that $k_A + 1 \leq \frac{3}{4}(k+1)$ and $k_B + 1 \leq \frac{3}{4}(k+1)$.

$$\begin{aligned}
\mathsf{v\text{-}cover}_{\mathcal{H}}(G) &\leq 5 \cdot (f(k_A) + f(k_B)) \log\left(\frac{3(k+1)}{4}\right) + f(k) + 1 \\
&\leq 5 \cdot f(k) \log\left(\frac{3(k+1)}{4}\right) + f(k) + 1 \qquad \text{(superadditivity of } f\text{)} \\
&\leq 5 \cdot f(k)\log(k+1) - 3 \cdot \log(4/3) f(k) + 2f(k) \\
&\leq 5 \cdot f(k)\log(k+1).
\end{aligned}$$

□

Corollary 5.1 (see also [CC13a] and [CC13b]). *For every connected planar graph H, the class $\mathcal{M}(H)$ has the* v*-Erdős–Pósa property with gap $O(k \cdot h^{38} \cdot \operatorname{polylog}(kh))$, where $h = V(H) + 2E(H)$.*

For every connected planar graph H, Corollary 5.1 provides a gap for $\mathcal{M}(H)$ that is polynomial in both k and $h = V(H) + 2E(H)$. In Subsection 5.3.1 we will apply Lemma 5.1 to majors of specific planar graphs in order to obtain better gaps.

Notice that the proof of Lemma 5.2 strongly relies on the fact that H is connected. The non-connected case requires some more ideas that are originating from [RS86] (also used for forests in [FJW13a]). We expose them hereafter. We will need the two next lemmas.

Lemma 5.3 ([RS86]). *Let q, k be two positive integers, let T be a tree and let $\mathcal{A}_1, \dots, \mathcal{A}_q$ be families of subtrees of T. Assume that for every $i \in [\![1, q]\!]$, there are kq elements of $\mathcal{A}_i$ that are pairwise vertex-disjoint. Then for every $i \in [\![1, q]\!]$, there are k elements $T_1^i, \dots, T_k^i$ of $\mathcal{A}_i$ such that*

$$T_1^1, \dots T_k^1, T_1^2, \dots T_k^2, \dots, T_1^q, \dots T_k^q$$

are all pairwise vertex-disjoint.

The next lemma is the Erdős–Pósa property of subtrees of a tree. It can be obtained from the fact that subtrees of a tree have the Helly property.

Lemma 5.4 (see [GL69]). *Let T be a tree and let $\mathcal{A}$ be a collection of subtrees of T. For every positive integer k, either T has (at least) k vertex disjoint subtrees that belong to $\mathcal{A}$, or T has a subset X of less than k vertices such that no subtree of $T \setminus X$ belongs to $\mathcal{A}$.*

We are now ready to deal with disconnected patterns.

Lemma 5.5 ([RS86]). *Let w be a positive integer and let H be a graph on q connected components. $\mathcal{M}(H)$ has the* v-*Erdős–Pósa property on the class of graphs of treewidth at most w with gap $k \mapsto (w-1)(kq-1)$.*

Proof. Let k be a positive integer. We want to show that either $\mathsf{v\text{-}pack}_{\mathcal{M}(H)}(G) \geq k$ or $\mathsf{v\text{-}cover}_{\mathcal{M}(H)}(G) \leq (w-1)(kq-1)$. Let $H_1, \dots, H_q$ be the connected components of H. Let $(T, \mathcal{X})$ be a tree-decomposition of G of width w. For every subgraph F of G, we denote by $T(F)$ the subgraph of T induced by the bags containing vertices of F. Notice that $T(F)$ is connected if F is connected.

For every $i \in [\![1, q]\!]$, we let $\mathcal{H}_i$ be the class of subgraphs of G that are isomorphic to a graph in $\mathcal{M}(H_i)$ and we consider the class $\mathcal{T}_i = \{T(F), F \in \mathcal{H}_i\}$.

If for every $i \in [\![1, q]\!]$, $\mathcal{T}_i$ contains kq vertex-disjoint trees, then according to Lemma 5.3 there are pairwise vertex-disjoint trees $\{T_j^i\}_{i \in [\![1,q]\!],\ j \in [\![1,k]\!]}$. Observe that for every two subgraphs F, F' of G, if $T(F)$ and $T(F')$ are vertex-disjoint, then so are F and F'. Therefore G has pairwise vertex-disjoint subgraphs $\{F_j^i\}_{i \in [\![1,q]\!],\ j \in [\![1,k]\!]}$ such that F_j^i is isomorphic to an element of $\mathcal{H}_i$ for every $i \in [\![1, q]\!]$ and $j \in [\![1, k]\!]$. This proves that in this case, $\mathsf{v\text{-}pack}_{\mathcal{M}(H)}(G) \geq k$.

We therefore now assume that the above condition does not hold, namely there is an index $i \in [\![1, q]\!]$ such that $\mathcal{T}_i$ contains less than kq vertex-disjoint trees. Lemma 5.5 implies the existence of a subset X with $|X| \leq kq-1$ such that $T \setminus X$ is free from subtrees isomorphic to a member of $\mathcal{T}_i$. Let Y denote the union of the bags indexed by vertices in X. Observe that $|Y| \leq (w-1)|X| \leq (w-1)(kq-1)$. The choice of Y ensures that $G \setminus Y$ has no subgraph isomorphic to a member of $\mathcal{H}_i$. Hence $\mathsf{v\text{-}cover}_{\mathcal{M}(H_i)} \leq (w-1)(kq-1)$. We deduce $\mathsf{v\text{-}cover}_{\mathcal{M}(H)} \leq (w-1)(kq-1)$. □

Corollary 5.2. *For every planar graph H with q connected components, the class $\mathcal{M}(H)$ has the Erdős–Pósa property with gap $O(qk^2 \cdot h^{38} \cdot \mathrm{polylog}(kh))$, where $h = V(H) + 2E(H)$.*

Edge version and tree partitions

In the edge variant of the Erdős–Pósa properties we use tree-partition width as a possible edge-analogue of treewidth.

Let $\mathcal{H}$ be a class of graphs. We define $\tilde{\mathcal{H}}$ as the set of all the *subgraph minimal elements of* $\mathcal{H}$, i.e.,

$$\tilde{\mathcal{H}} = \{H,\ H \in \mathcal{H} \text{ and none of the subgraphs of } H \text{ belongs to } \mathcal{H}\}.$$

We define $\Delta(\mathcal{H})$ as the maximum number of edges incident to a vertex in a graph of $\mathcal{H}$ (counting multiple edges). We also set $\tilde{\Delta}(\mathcal{H}) = \Delta(\tilde{\mathcal{H}})$.

Observation 5.1. For every graph H of h edges, it holds that $\tilde{\Delta}(\mathcal{M}(H)) \leq h, \tilde{\Delta}(\mathcal{T}(H)) \leq h, \tilde{\Delta}(\mathcal{I}(H)) \leq 2h$.

Lemma 5.6. *Let $\mathcal{H}$ be a class of connected non-trivial graphs where $\tilde{\Delta}(\mathcal{H}) \leq d$. Then for every $r \in \mathbb{N}$, $\mathcal{H}$ has the* e-*Erdős–Pósa-property on $\mathcal{G}_{\mathbf{tpw}\leq r}$ with gap $g_r(k) = k \cdot r \cdot (dr+1)$.*

Proof. Let $r \in \mathbb{N}$. We will show the following for every $k \in \mathbb{N}$: for every graph $G \in \mathcal{G}_{\mathbf{tpw}\leq r}$, if $\mathsf{e\text{-}pack}_{\mathcal{H}}(G) = k$ then $\mathsf{e\text{-}cover}_{\mathcal{H}}(G) \leq g_r(k)$.

We proceed by induction. The base case $k = 0$ is trivial. We thus assume that $k > 0$ and that the above statement holds for every positive integer $k' < k$ (induction hypothesis).

Let $G \in \mathcal{G}_{\mathbf{tpw}\leq r}$ be a graph such that $\mathsf{e\text{-}pack}_{\mathcal{H}}(G) = k$. We assume that G is connected, as otherwise we can treat each connected component separately.

Let $(T, s, \{X_t\}_{t\in V(T)})$ be an optimal rooted tree partition decomposition of G. We define $G_t = G\left[\bigcup_{u\in \mathrm{desc}_{(T,s)}(t)} X_u\right]$. For every edge $\{u, v\}$ of T we denote by $E_{\{u,v\}}$ the edges of G with the one endpoint in X_u and the other one in X_v. Let t be a vertex of T of minimum distance from a leaf, subject to $\mathsf{e\text{-}pack}_{\mathcal{H}}(G_t) > 0$.

Let M be a subgraph-minimal subgraph of G_t that is isomorphic to some member of $\mathcal{H}$ and let $t_1, \ldots, t_p$ be the children of t such that $V(G_{t_i}) \cap V(M) \neq \emptyset$ for every $i \in [\![1, p]\!]$. By minimality of M, it has no vertex with more than $\tilde{\Delta}(\mathcal{H}) \leq d$ incident edges. As $|X_t| \leq r$, we deduce that $p \leq rd$.

Let $C = E(X_t) \cup \bigcup_{i=1}^{p} E_{\{t,u_i\}}$. Notice that $|C| \leq r + dr^2$. Let us consider then graph $G' = G \setminus C$. Let M' is a subgraph of G' that is isomorphic to some member of $\mathcal{H}$. By minimality of t, $\mathsf{e\text{-}pack}_{\mathcal{H}}(G_{t_i}) = 0$, for every $i \in [\![1, p]\!]$. Therefore, if M' contained an edge $e \in E(G_{t_i})$ (for some $i \in [\![1, p]\!]$), it would also contain an edge of $E(G) \setminus E(G_{t_i})$. Since every graph of $\mathcal{H}$ is connected, M' would also need to contain some edge of E_{t,u_i} in order to be connected to edges of $E(G) \setminus E(G_{t_i})$. However $E(G') \cap E_{t,u_i} = \emptyset$. We deduce that for every subgraph M' of G' that is isomorphic to some member of $\mathcal{H}$, we have $E(M') \cap E(M) = \emptyset$. It follows that every e-$\mathcal{H}$-packing in G' is edge-disjoint with M.

Hence $\mathsf{e\text{-}pack}_{\mathcal{H}}(G') < k$, as otherwise a packing of size k in G' would, together with M, yield a packing of size $k + 1$ in G whereas $\mathsf{e\text{-}pack}_{\mathcal{H}}(G) = k$. By applying the induction hypothesis on G', there is a subset $D \subseteq E(G')$ such that $\mathsf{e\text{-}pack}_{\mathcal{H}}(G' \setminus D) = 0$ and moreover $|D| \leq g_r(k - 1)$. It is easy to see that $C \cup D$ is an e-$\mathcal{H}$-cover of G. Furthermore $|C \cup D| \leq r(dr + 1) + g_r(k - 1) = g_r(k)$, as required. □

In this section we presented decomposition-based techniques aiming to prove the edge-Erdős-Pósa property, and towards this purpose we used tree-partition decompositions as a possible edge-counterpart to tree-decompositions, that are used in the vertex case. Let us briefly mention that there are other tree-like decompositions that deserve attention. For instance, tree-cut decompositions can be succesfully used to deal with immersion expansions, as we will see in Subsection 5.3.4. Applications of the techniques described in this section will be presented in Section 5.3.

5.2.2 Erdős–Pósa from girth

In this section, we give a proof of the Erdős–Pósa Theorem that highlights a technique for proving more general Erdős–Pósa type results. The technique can be informally summarized as follow. We prove that either G contains a *small* cycle or that it can be reduced to a smaller graph with the same packing and covering number. We then apply induction on either the graph where a small cycle has been deleted (in the first case), or on the reduced graph (in the second case). This technique has been successfully applied in [FJW13a, CRST15a], for instance. Let us first recall the following result.

Lemma 5.7 ([Tho83b], see also [Die05, Theorem 7.4.2])**.** *There is a constant $c \in \mathbb{R}$, such that every graph of minimum degree at least 3 and girth at least $c \log q$ contains K_q as a minor, for every $q \in \mathbb{N}_{\geq 1}$.*

A direct consequence of this result is the following trichotomy.

Corollary 5.3. *For every graph G and every integer $q > 1$, one of the following holds:*

(i) G has a cycle on at most $c \log q$ vertices;

(ii) G has a vertex of degree at most 2;

(iii) G contains K_q as a minor.

We now prove the lemma that implies the classic Erdős–Pósa Theorem both for the vertex and its edge version. Recall that $A_{\mathsf{x}}(G)$ denotes $V(G)$ or $E(G)$, depending if $\mathsf{x} = \mathsf{v}$ or $\mathsf{x} = \mathsf{e}$.

Lemma 5.8. *For every $q \in \mathbb{N}^+$ and every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, the set $\mathcal{M}(\theta_2)$ has the x-Erdős–Pósa property for the class of K_q-minor-free with gap $O(k \cdot \log q)$.*

Proof. We will prove that for every non-negative integer k and every K_q-minor-free graph G, either G has k x-disjoint cycles, or G has a subset $X \subseteq A_{\mathsf{x}}(G)$ of size at most $ck \log q$ such that $G \setminus X$ is a forest. We proceed by induction on the pair (k, G), with the well-founded order defined by $(k', G') \leq (k, G) \iff (k' \leq k$ and $|A_{\mathsf{x}}(G')| \leq |A_{\mathsf{x}}(G)|)$, for all graphs G, G' and non-negative integers k, k'.

The base cases corresponding to $k = 0$ or $|A_{\mathsf{x}}(G)| = 0$ are trivial. Let us now assume that $k \geq 1$, $|A_{\mathsf{x}}(G)| \geq 1$, and that the lemma holds for every pair (k', G') such that $(k', G') \leq (k, G)$.

According to Corollary 5.3, either G_i has a cycle C on at most $c \log q$ vertices, or it has a vertex v of degree at most two, or it contains K_q as a minor. The last case is not possible, as we require G to be K_q-minor-free.

Whenever the first case applies, we set $G' = G \setminus A_{\mathsf{x}}(C)$ and we consider the pair $(k-1, G')$. If G' contains $k-1$ x-disjoint cycles, then G contains k x-disjoint cycles obtained by adding C to those of G' and we are done. Otherwise, the induction hypothesis implies the existence of a subset $X' \subseteq A_{\mathsf{x}}(G')$ with $|X| \leq c(k-1)\log q$ such that $G' \setminus X'$ is a forest. Then by definition of C, $X = X' \cup A_{\mathsf{x}}(C)$ has size at most $c \log q$ and $G \setminus X$ is a forest, as required.

In the second case, we delete v if it is isolated and we contract an edge e incident with it otherwise. Notice that since we cannot apply the first case, this contraction does not decrease the number of cycles in G. Also, we can assume without loss of generality that v (respectively e) is not part of a minimum x-cover of cycles in G, as any vertex adjacent to v (respectively edge incident with e) covers all the cycles covered by v (respectively e). Therefore the obtained graph G' satisfies $\mathsf{x\text{-}pack}_{\mathcal{H}}(G')$ and $\mathsf{x\text{-}cover}_{\mathcal{H}}(G) = \mathsf{x\text{-}cover}_{\mathcal{H}}(G')$. It is not hard to see that $A_{\mathsf{x}}(G') < A_{\mathsf{x}}(G)$. Therefore we can apply the induction hypothesis on G' and obtain the desired result on G', that immediately translates to G by the above remarks. □

By setting $q = 3k$ and observing that every graph containing K_{3k} as a minor also contains k vertex-disjoint cycles (hence also edge-disjoint), Lemma 5.8 yields the vertex and edge versions of the classic Erdős–Pósa theorem as a corollary.

The technique presented in this section has been used to show the following result.

Theorem 5.4 ([FJW13a]). *For every forest H, $\mathcal{M}(H)$ has the* v*-Erdős–Pósa property with gap $O_H(k)$.*

Other applications will be presented in Section 5.3. To extend the idea of Lemma 5.8 in order to prove that some graph class $\mathcal{H}$ has the x-Erdős–Pósa property with gap $f\colon \mathbb{N} \to \mathbb{N}$, one should show that for every positive integer k and every graph G with $\mathsf{x\text{-}pack}_{\mathcal{H}}(G) \leq k$,

- either there is a graph G' with $\mathsf{x\text{-}pack}_{\mathcal{H}}(G')$ and $\mathsf{x\text{-}cover}_{\mathcal{H}}(G) = \mathsf{x\text{-}cover}_{\mathcal{H}}(G')$ and such that $|G'| + \|G'\| < |G| + \|G\|$ (reduction case);
- or G has a subgraph isomorphic to a member of $\mathcal{H}$ on at most $f(k)/k$ vertices/edges (progress case).

An application of the aforementioned techniques to these cases is given in Section 5.3.

5.3 Applications to selected classes of graphs

In this section, we show classes of graphs where the techniques presented in Section 5.2 can be applied to yield Erdős–Pósa type results. The classes we will consider are the following:

- majors of wheels, yurts, and graphs of pathwidth at most two in Subsection 5.3.1;

- majors of θ_r in Subsection 5.3.2;
- immersion models of connected planar subcubic graphs in Subsection 5.3.4;

5.3.1 Wheels, yurts, and graphs of pathwidth at most two

The gap provided by Corollary 5.1 relies on general exclusion theorems for majors of planar graphs. In this section we provide an application of the results of Section 5.2.1 to decrease the contribution of H, for specific planar patterns where a better exclusion theorem is known. The classes we will consider are the majors of the follwing graphs: wheels, double wheels, graphs of pathwidth at most 2, and yurt graphs; and we will use the exclusion theorems introduced in Section 4.3.

Theorem 5.5. *The class $\mathcal{M}(H)$ has the* v*-Erdős–Pósa-property with gap:*

(i) $k \mapsto kh^2 \operatorname{polylog}(kh)$ *if H is a wheel of order h;*

(ii) $k \mapsto kh^4 \operatorname{polylog}(kh)$ *if H is a graph of pathwidth at most two with $|V(H)| + |E(H)| = h$ or a double wheel of order h;*

(iii) $k \mapsto kh^8 \operatorname{polylog}(kh)$ *if H is a yurt graph of order h.*

Proof. Let us prove (i). We denote by W_h the wheel of order h, for every positive integer h. According to Theorem 4.10, for every positive integer h, every graph of treewidth at least $36h - 2$ contains a major of W_h. Lemma 5.1 then implies that there is a ceiling f_h for $(\mathbf{tw}, \mathcal{M}(W_h), \mathsf{v})$, with $f_h(k) = k(36h - 2)^2 \operatorname{polylog}(kh)$. Thanks to Lemma 5.2, this ceiling yields a gap $g_h(k) = 5k(36h-2)^2 \operatorname{polylog}(kh) \cdot \log(k+1)$ and we are done. The proofs of (ii) and (iii) follow the very same path, using exclusion theorems Theorem 4.11, Theorem 4.12, and Theorem 4.13. □

5.3.2 Pumpkins

A way to extend the classic Erdős-Pósa Theorem is to consider generalizations of the class of cycles. A class that attracted some attention in this direction is $\mathcal{M}(\theta_r)$ for $r \in \mathbb{N}$, which is the class of cycles when $r = 2$. For instance, Fomin et al. gave in [FST11] the following extension.

Theorem 5.6 ([FST11]). *There is a function $f_r(k) = O(k^2r^2)$ such that for every $r \in \mathbb{N}$, $\mathcal{M}(\theta_r)$ has the* v*-Erdős–Pósa property with gap f_r.*

In an unpublished manuscript [FJS13], Fiorini et al. proved that the classic Erdős–Pósa Theorem can be extended to $\mathcal{M}(\theta_r)$ (instead of cycles) without increasing the order of magnitude of the gap, $O(k \log k)$.

Theorem 5.7 ([FJS13], see also [CRST15a]). *There is a function $f_r(k) = O_r(k \log k)$ such that for every $r \in \mathbb{N}$, $\mathcal{M}(\theta_r)$ has the* v*-Erdős–Pósa property with gap f_r.*

We present in this section the following result.

Theorem 5.8. *There is a function* f_r *with* $f_r(k) = O(k^2 r^2 \operatorname{polylog} kr)$ *and* $f_r(k) = O(k^4 r^2 \operatorname{polylog} kr)$*, such that for every* $r \in \mathbb{N}$*,* $\mathcal{M}(\theta_r)$ *has the* e*-Erdős–Pósa property with gap* f_r*.*

Theorem 5.8 is an edge-analogue if Theorem 5.8. The bound on the gap is worse in terms of k but it indicated the contribution of r, which is polynomial.

We will in Chapter 6 prove the follwing result, that completes the symmetry between the vertex and edge settings.

Theorem 5.9. *There is a function* $f_r(k) = O_r(k \log k)$ *such that for every* $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$ *and every* $r \in \mathbb{N}$*,* $\mathcal{M}(\theta_r)$ *has the* x*-Erdős–Pósa property with gap* f_r*.*

Let us here prove Theorem 5.8. Recall that we proved in Section 4.2 that every 2-connected graph that has a vertex of degree at least $2kr$ has a subgraph that is an edge-disjoint union of k $\mathcal{M}(\theta_r)$-subgraphs. Let us show the next result.

Lemma 5.9. *Let* $r \in \mathbb{N}$*. If* $\mathcal{M}(\theta_r)$ *has the* v*-Erdős–Pósa property with a gap* f *that is superadditive, then it has the* e*-Erdős–Pósa property with gap* $k \mapsto 2kr \cdot f(k)$*.*

We first need some intermediate lemmas in order to be able to use Theorem 5.1.

Lemma 5.10. *Let* $d \in \mathbb{N}$*. The class* $\mathcal{M}(\theta_r)$ *has the* e*-Erdős–Pósa property for the class of graphs of maximum degree at most* d *with gap* $k \mapsto d \cdot f(k)$*, where* f *is the (vertex-) gap provided by Theorem 5.7.*

Proof. Let G be a graph satisfying $\Delta(G) \leq d$ and let k be a positive integer. If G contains k vertex-disjoint $\mathcal{M}(\theta_r)$-subgraphs, these subgraphs are in particular edge-disjoint hence $\mathsf{e\text{-}pack}_{\mathcal{M}(\theta_r)}(G) \geq k$. On the other hand, if $\mathsf{v\text{-}pack}_{\mathcal{M}(\theta_r)}(G) < k$, according to Theorem 5.7 there is a subset $X \subseteq V(G)$ such that $|X| \leq f(k)$ and $G \setminus X$ does not contain any $\mathcal{M}(\theta_r)$-subgraph. Let Y be the set of edges incident to the vertices in X. As $\Delta(G) \leq d$ we have $|Y| \leq d \cdot f(k)$. Notice that any $\mathcal{M}(\theta_r)$-subgraph of $G \setminus Y$ does not contain a vertex from X, hence the existence of such a subgraph would contradict the definition of X. This proves that $\mathsf{e\text{-}cover}_{\mathcal{M}(\theta_r)} \leq d \cdot f(k)$ and we are done. □

Lemma 5.11. *If a class of 2-connected graphs* $\mathcal{H}$ *has the* e*-Erdős–Pósa property for 2-connected graphs with a superadditive gap, then it has the* e*-Erdős–Pósa property (for all graphs) with the same gap.*

Proof. Let f be the gap mentioned in the statement of the lemma. Let G be a graph and let $G_1, \dots, G_t$ be its 2-connected components, for some positive integer t. Let k be an integer and let $p_i = \mathsf{e\text{-}pack}_{\mathcal{H}}(G_i)$, for every $i \in [\![1, t]\!]$. If $p_1 + \dots + p_t \geq k$, then $\mathsf{e\text{-}pack}_{\mathcal{H}}(G) \geq k$, as two subgraphs in two distinct 2-connected components of G are always edge-disjoint. On the other hand, let us assume that $p_1 + \dots + p_t < k$. We have $\mathsf{e\text{-}cover}_{\mathcal{H}}(G_i) \leq f(p_i)$ for every $i \in [\![1, t]\!]$. Since no $\mathcal{H}$-subgraph of G contains edges from two or more distinct 2-connected components, we can cover $\mathcal{H}$-subgraphs in each

2-connected component in order to cover them in the graph:

$$\begin{aligned}
\text{e-cover}_{\mathcal{H}}(G) &\leq \sum_{i=1}^{t} \text{e-cover}_{\mathcal{H}}(G_i) \\
&\leq \sum_{i=1}^{t} f(p_i) \\
&\leq f\left(\sum_{i=1}^{t} p_i\right) && (f \text{ is superadditive}) \\
&\leq f(k-1).
\end{aligned}$$

□

We can now prove Lemma 5.9.

Proof of Lemma 5.9. By applying Theorem 5.1 to Lemma 4.7 and Lemma 5.10, we obtain the result for 2-connected graphs. Lemma 5.11 then allows us to extend it to graphs that are not 2-connected. □

We also need the following result.

Theorem 5.10 ([BvLTT97b, Theorem 14]). *For every $r \in \mathbb{N}$, every graph of treewidth at least $2r-1$ contains a $K_{2,r}$ major.*

Let us now prove Lemma 5.9.

Proof of Lemma 5.9. Lemma 5.1 applied to $\mathcal{M}(\theta_r)$ using Theorem 5.10 yields a gap $f_r(k) = O(kr^2 \operatorname{polylog} kr)$ for the vertex-Erdős-Pósa property of $\mathcal{M}(\theta_r)$. Then, Lemma 5.9 yields a gap $2kr \cdot f_r(k) = O(k^2r^3 \operatorname{polylog} kr)$ for the edge version, as required. A gap of order of magnitude $O(k^4r^2 \operatorname{polylog} kr)$ can be obtained the same way using Theorem 5.3 instead of Lemma 5.1. □

5.3.3 Double pumpkins

For every $r, r' \in \mathbb{N}$, we denote by $\theta_{r,r'}$ the graph obtained by identifying one vertex of θ_r with one vertex of $\theta_{r'}$. In this section, we use the tools provided by Section 5.2 to prove that $\mathcal{M}(\theta_{r,r'})$ have the edge-Erdős–Pósa-property for simple graphs.

Theorem 5.11. *For every $r, r' \in \mathbb{N}$, $\mathcal{M}(\theta_{r,r'})$ has the edge-Erdős–Pósa property for simple graphs.*

We must note that at the time of writing, the graphs $\{\theta_{r,r'}\}_{r,r' \in \mathbb{N}}$ are the only graphs on at least three vertices, the majors of which are known to have the edge-Erdős–Pósa property. This fact must be compared to the results on the vertex variant where the graphs H for which $\mathcal{M}(H)$ has the vertex-Erdős–Pósa property have been completely characterized: they are exactly the planar graphs [RS86, Theorem 8.2]. This raises the following question.

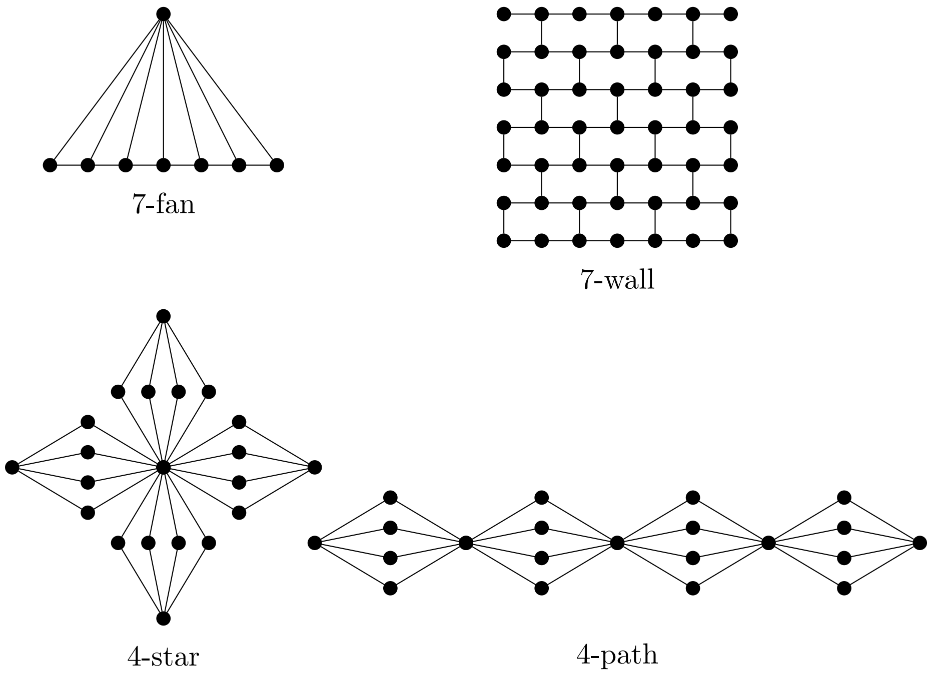


Figure 5.1: Unavoidable patterns of graphs of large tree-partition width.

Question 5.1. What are the graphs, the majors of which have the edge-Erdős–Pósa property?

Lemma 5.23 provides a partial answer to Question 5.1: all these graphs are planar. Prior to the proof of Theorem 5.11, we need to introduce a result of Ding et al. [DO96], the statement of which requires additional definitions.

Walls, fans, paths, and stars. Wall have been defined in Subsection 2.3.2. As a reminder, the 7-wall is depicted in Figure 5.1. The *n-fan* is the graph obtained by adding a dominating vertex to a path on n vertices. A collection of paths is said to be *independent* if two paths of the collection never share interior vertices. The *n-star* is the graph obtained by replacing every edge of $K_{1,n}$ with n independent paths of two edges. The *n-path* is the graph obtained by replacing every edge of an n-edge path with n independent paths of two edges. Examples of these graphs are depicted in Figure 5.1. The *wall number* (resp. *fan number*, *star number*, and *path number*) of a graph G is defined as the largest integer k such that G contains a subdivision of a k-wall (resp. of a k-fan, of a k-star, of a k-path), or infinity is no such integer exists. Let $\gamma(G)$ denote the maximum of the wall number, fan number, star number, and path number of a graph G.

Ding et al. gave in [DO96] the following characterization of classes of graphs of bounded tree-partition width in terms of excluded topological minors.

Theorem 5.12 ([DO96])**.** *There is a function $f_5\colon \mathbb{N} \to \mathbb{N}$ such that every simple graph G satisfies* $\mathbf{tpw}(G) \leq f_5(\gamma(G))$.

In other words, for every integer k, every simple graph of large enough tree-partition width contains a subdivision of one of the following graphs: the k-wall, the k-fan, the k-path, or the k-star.

Notice that for every $r, r' \in \mathbb{N}$, $r' \leq r$, the graph $\theta_{r,r'}$ is a minor of the following graphs: the r-path, the r-star, the $(r+r'+1)$-fan, and the r-wall (for $r \geq 6$). Hence, every simple graph of large enough tree-partition width contains a $\theta_{r,r'}$-major. This can easily be generalized to edge-disjoint packings, as follows.

Lemma 5.12. *For every $r, r', k \in \mathbb{N}$, every graph G satisfying $\gamma(G) \geq k(r+r'+7)$ contains an e-$\mathcal{M}(\theta_{r,r'})$-packing of size k.*

Using Theorem 5.12, we get the following corollary.

Corollary 5.4. *For every $r, r', k \in \mathbb{N}$, every simple graph G satisfying $\mathbf{tpw}(G) \geq f_5(k(r+r'+7))$ contains an e-$\mathcal{M}(\theta_{r,r'})$-packing of size k.*

In other words, Corollary 5.4 provides a ceiling for the triple $(\mathbf{tpw}, \mathcal{M}(\theta_{r,r'}, \mathsf{e})$ in simple graphs. The proof of Theorem 5.11 now follows by a straighforward application of Theorem 5.1 to Corollary 5.4 and Lemma 5.6.

5.3.4 Planar subcubic graphs

In this section, we apply the tools of Section 5.2 to immersion expansions of planar subcubic graphs. The result we prove is the following.

Theorem 5.13. *For every connected planar subcubic graph H on $h > 0$ edges and every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, the class $\mathcal{I}(H)$ has the x-Erdős–Pósa property with a gap that is polynomial in both h and the packing number.*

We will show in Section 5.4.1 that neither the planarity requirement, not the subcubicity can be droped.

The main tools of our proof are the graph invariants of tree-cut width and tree-partition width. Our proof uses the exclusion result Theorem 4.15 which implies that, for every fixed planar subcubic graph H and every positive integer k, every graph of large enough tree-cut width contains k vertex-disjoint immersion expansions of H. This allows us to focus on graphs of bounded tree-cut width. By applying suitable reductions, we finally reduce the problem to graphs of bounded tree partition width (Lemma 5.13). The result then follows by the application of Theorem 5.1.

From tree-cut decompositions to tree-partitions

The purpose of this section is to reduce the proof of Theorem 5.13 (when $\mathsf{x} = \mathsf{e}$) for host graphs with bounded tree-cut width to the case where host graphs have bounded tree-partition width. In particular, we prove the following lemma.

Lemma 5.13. *For every connected graph G, and every connected graph H with at least one edge, there is a graph G' and a graph H' such that*

- $\mathbf{tpw}(G') \leq (\mathbf{tcw}(G)+1)^2/2$,

- $\mathsf{e\text{-}pack}_{\mathcal{I}(H')}(G') \leq \mathsf{e\text{-}pack}_{\mathcal{I}(H)}(G)$, *and*
- $\mathsf{e\text{-}cover}_{\mathcal{I}(H)}(G) \leq \mathsf{e\text{-}cover}_{\mathcal{I}(H')}(G')$.

Observe that, with the notation of Lemma 5.13, if we prove that

$$\mathsf{e\text{-}cover}_{\mathcal{I}(H')}(G') \leq f(\mathsf{e\text{-}pack}_{\mathcal{I}(H')}(G'))$$

for some function $f\colon \mathbb{N} \to \mathbb{N}$, then it immediately implies

$$\mathsf{e\text{-}cover}_{\mathcal{I}(H)}(G) \leq f(\mathsf{e\text{-}pack}_{\mathcal{I}(H)}(G)).$$

For every graph G, we define G^+ as the graph obtained if, for every vertex v, we add two new vertices v' and v'' and the edges $\{v', v''\}$ (of multiplicity 2), $\{v, v'\}$ and $\{v, v''\}$ (both of multiplicity 1). Observe that for every G, we have $\mathrm{m}\delta(G^+) \geq 3$. We also define G^* as the graph obtained by adding, for every vertex v, the new vertices $v'_1, \ldots, v'_{\mathrm{mdeg}(v)}$ and $v''_1, \ldots, v''_{\mathrm{mdeg}(v)}$ and the edges $\{v'_i, v''_i\}$ (of multiplicity 2), $\{v, v'_i\}$, and $\{v, v''_i\}$ (both of multiplicity 1), for every $i \in [\![1, \deg(v)]\!]$. If v is a vertex of G, then we denote by $Z_{v,i}$ the subgraph $G^*[\{v, v'_i, v''_i\}]$ for every $i \in [\![1, \mathrm{mdeg}_G(v)]\!]$.

Observation 5.2. Let H and G be two graphs, and let (ϕ, ψ) be an H-immersion model in G. Then for every vertex x of G, we have $\mathrm{mdeg}_H(x) \leq \mathrm{mdeg}_G(\phi(x))$.

Our first aim is to prove the following three lemmata.

Lemma 5.14. *Let G be a graph, let H be a connected graph with at least one edge and let G' be a subdivision of G^*. Then we have*

- $\mathsf{e\text{-}pack}_{\mathcal{I}(H^+)}(G^*) = \mathsf{e\text{-}pack}_{\mathcal{I}(H^+)}(G')$ *and*
- $\mathsf{e\text{-}cover}_{\mathcal{I}(H^+)}(G^*) = \mathsf{e\text{-}cover}_{\mathcal{I}(H^+)}(G')$.

Proof. We denote by S the set of subdivision vertices added during the construction of G' from G^+. As G' is a subdivision of G^*, we have $\mathsf{e\text{-}pack}_{\mathcal{I}(H^+)}(G') \geq \mathsf{e\text{-}pack}_{\mathcal{I}(H^+)}(G^*)$ and $\mathsf{e\text{-}cover}_{\mathcal{I}(H^+)}(G') \geq \mathsf{e\text{-}cover}_{\mathcal{I}(H^+)}(G^*)$.

As a consequence of Observation 5.2 and the fact that $\mathrm{m}\delta(H^+) \geq 3$, if M is an H^+-immersion expansion in G' then no branch vertex of M belongs to S. Indeed, every vertex of S has multidegree 2 in G'. Therefore, by dissolving in M the vertices of S that belong to $V(M)$, we obtain an H^+-immersion expansion in G^*. It follows that $\mathsf{e\text{-}pack}_{\mathcal{I}(H^+)}(G^*) \geq \mathsf{e\text{-}pack}_{\mathcal{I}(H^+)}(G')$, hence $\mathsf{e\text{-}pack}_{\mathcal{I}(H^+)}(G^*) = \mathsf{e\text{-}pack}_{\mathcal{I}(H^+)}(G')$.

On the other hand, let X be an H^+-cover of G^* and let X' be a set of edges constructed by taking, for every $e \in X$, an edge of the path of G' connecting the endpoints of e that has been created by subdividing e. Assume that X' is not an H^+-cover of G'. According to the remark above, this implies that X is not an H^+-cover of G^*, a contradiction. Hence X' is an H^+-cover of G' and thus $\mathsf{e\text{-}cover}_{\mathcal{I}(H^+)}(G^*) = \mathsf{e\text{-}cover}_{\mathcal{I}(H^+)}(G')$. □

Lemma 5.15. *For every two graphs H and G such that H is connected and has at least one edge, we have $\mathsf{e\text{-}pack}_{\mathcal{I}(H^+)}(G^*) \leq \mathsf{e\text{-}pack}_{\mathcal{I}(H)}(G)$.*

Proof. In G^* (respectively H^+), we say that a vertex is *original* if it belongs to $V(G)$ (respectively $V(H)$). Let (ϕ, ψ) be an H^+-immersion model in G^*.

We first show that if u is an original vertex of H^+, then $\phi(u)$ is an original vertex of G^*. By contradiction, let us assume that $\phi(u)$ is not original, for some original vertex u of H^+. Then $\phi(u) = v_i'$ or $\phi(u) = v_i''$, for some $v \in V(G)$ and $i \in [\![1, \mathrm{mdeg}_G(v)]\!]$.

Observe that since H is connected and has at least one edge, every vertex of H^+ has degree at least three: let x, y, and z be the endpoints of three multiedges incident with u. Then $\psi(\{u, x\})$, $\psi(\{u, x\})$, and $\psi(\{u, x\})$ are edge-disjoint paths connecting $\phi(u)$ to three distinct vertices. This is not possible because there is an edge cut of size two, $\{\{v, v_i'\}, \{v, v_i''\}\}$, separating the two vertices v_i' and v_i'' (among which is $\phi(u)$) from the rest of the graph. Consequently, if $u \in V(H^+)$ is original, then $\phi(u)$ is original.

Let us now consider an edge $\{u, v\} \in E(H)$. By the above remark, $\phi(u)$ and $\phi(v)$ are original vertices of G^*. It is easy to see that $\psi(\{u, v\})$ contains only original vertices of G^*. Indeed, if this path contained a non-original vertex w' or w'' for some original vertex w of $V(G^*)$, it would use w twice in order to reach u and v, what is not allowed. Therefore, from the definition of H^+, the pair $(\phi|_{V(H)}, \psi|_{E(H)})$ is an H-immersion model of G.

We proved that every H^+-immersion-expansion of G^* contains an H-immersion-expansion that belongs to the subgraph G of G^*. Consequently every H^+-packing of G^* contains an H-packing of the same size that belongs to G, and the desired inequality follows. $\square$

Lemma 5.16. *For every two graphs H and G such that H is connected and has at least one edge, we have* $\mathsf{e\text{-}cover}_{\mathcal{I}(H)}(G) \leq \mathsf{e\text{-}cover}_{\mathcal{I}(H^+)}(G^*)$.

Proof. Similarly to the proof of Lemma 5.15, we say that an edge of G^* is *original* if it belongs to $E(G)$. Let $X \subseteq E(G^*)$ be a minimum cover of H^+-immersion expansions in G^*.

First case: all the edges in X are original. In this case, X is an H-cover of G as well. Indeed, if $G \setminus X$ contains an H-immersion expansion M, then $G^* \setminus X$ contains M^* that, in turn, contains H^+. Hence in this case, $\mathsf{e\text{-}cover}_{\mathcal{I}(H)}(G) \leq \mathsf{e\text{-}cover}_{\mathcal{I}(H^+)}(G^*)$.

Second case: there is an edge $e \in X$ that is not original. Let v be the original vertex of G^* such that either $e \in Z_{v,l}$ for some $l \in [\![1, \mathrm{mdeg}_G(v)]\!]$. Let us first show the following claim.

Claim: For every $i \in [\![1, \mathrm{mdeg}_G(v)]\!]$, there is an edge of $Z_{v,i}$ that belongs to X.

Proof of claim: Looking for a contradiction, let us assume that we have $E(Z_{v,i}) \cap X = \emptyset$, for some $i \in [\![1, \mathrm{mdeg}_G(v)]\!]$. Clearly $i \neq l$. By minimality of X, the graph $G \setminus (X \setminus \{e\})$ contains an H^+-immersion expansion M that uses e. Observe that $M' = M \setminus E(Z_{v,l}) \cup E(Z_{v,i})$ contains an H^+-immersion expansion (since $Z_{v,l}$ and $Z_{v,i}$ are isomorphic). Hence, M' is a subgraph of $G \setminus (X \setminus \{e\})$ that contains an H^+-immersion expansion. This is not possible as X is a cover, so we reach the contradiction we were looking for and the claim holds. $\diamond$

We build a set Y as follows. For every edge $f \in X$, if f is original then we add to Y. Otherwise, if v_f is the (original) vertex of G^* such that $e \in E(Z_{v_f,i})$ for some $i \in [\![1, \mathrm{mdeg}_G(v_f)]\!]$, then we add to Y all edges that are incident to v_f.

The above claim ensures that when a non-original edge f of X is encountered, then X contains an edge in each of $Z_{v_f,1}, \ldots, Z_{v_f,\mathrm{mdeg}_G(v_f)}$. Therefore, the same set of edges, of size $\mathrm{mdeg}_G(v_f)$, will be added to Y when encountering an other edge from $Z_{v_f,1}, \ldots, Z_{v_f,\mathrm{mdeg}_G(v_f)}$. Consequently, $|X| = |Y|$.

Let us not show that Y is an H^+-cover of G^*. Suppose that there exists an H^+-immersion expansion M in $G^* \setminus Y$. Observe that since H is connected and has at least one edge, M does not belong to $\bigcup_{i\in\{1,\ldots,\mathrm{mdeg}_G(u)\}} Z_{u,i}$, for every original vertex u of G^*. Let

$$Z = \bigcup_{u\in V(G)} \bigcup_{i\in\{1,\ldots,\mathrm{mdeg}_G(u)\}} E(Z_{u,i})$$

Then M is a subgraph of $G \setminus (Y \cup Z)$. As $X \subseteq Y \cup Z$, this contradicts the fact that X is a cover. Therefore, Y is an H^+-cover. Moreover all the edges in Y are original. As this situation is treated by the first case above, we are done. $\square$

If $(T, \{X_t\}_{t\in V(T)})$ is a tree-cut decomposition of a graph G and $t \in V(T)$, we say than an edge of G *crosses* the bag X_t if its endpoints belongs to bags X_{t_1} and X_{t_2}, for some $t_1, t_2 \in V(T)$ such that t belongs to the interior of the (unique) path of T connecting t_1 to t_2. We are now ready to prove Lemma 5.13.

of Lemma 5.13. Let $k = \mathbf{tcw}(G)$. We examine the nontrivial case where G is not a tree, i.e., $\mathbf{tcw}(G) \geq 2$. Let us consider the graph G^*. We claim that $\mathbf{tcw}(G^*) = \mathbf{tcw}(G)$. Indeed, starting from an optimal tree-cut decomposition of G, we can, for every vertex v of G and for every $i \in [\![1, \mathrm{mdeg}_G(v)]\!]$, create a bag that is a children of the one of v and contains $\{v_i', v_i''\}$. According to the definition of G^*, this creates a tree-cut decomposition $\mathcal{D} = ((T, s), \{X_t\}_{t\in V(T)})$ of G^*. Observe that for every vertex x that we introduced to the tree of the decomposition during this process, $\mathrm{adh}_{\mathcal{D}}(x) = 2$ and the corresponding bag has size two. This proves that $\mathbf{tcw}(G^*) \leq \max(\mathbf{tcw}(G), 2) = \mathbf{tcw}(G)$. As G is a subgraph of G^*, we obtain $\mathbf{tcw}(G) \leq \mathbf{tcw}(G^*)$ and the proof of the claim is complete.

According to Proposition 2.1, we can assume that G^* has a nice rooted tree-cut decomposition of width $\leq k$. For notational simplicity we again denote it by $\mathcal{D} = ((T, s), \{X_t\}_{t\in V(T)})$ and, obviously, we can also assume that all leaves of T correspond to non-empty bags.

Our next step is to transform the rooted tree-cut decomposition $\mathcal{D}$ into a rooted tree-partition $\mathcal{D}' = ((T, s), \{X_t'\}_{t\in V(T)})$ of a subdivision G' of G^*. Notice that the only differences between two decompositions are that, in a tree-cut decomposition, empty bags are allowed as well as edges connecting vertices of bags corresponding to non-adjacent vertices of T.

We proceed as follows: if X is a bag crossed by edges, we subdivide every edge crossing X and add the obtained subdivision vertex to X. By repeating this process we decrease at each step the number of bags crossed by edges, that eventually reaches zero. Let G' be the obtained graph and observe that G' is a subdivision of G. As G is connected, the obtained rooted tree-cut decomposition $\mathcal{D}' = ((T, s), \{X_t'\}_{t\in V(T)})$ is a rooted tree partition of G'.

Notice that the adhesion of any bag of T in $\mathcal{D}$ is the same as in $\mathcal{D}'$. However, the bags of $\mathcal{D}'$ may grow during the construction of G'. Let t be a vertex of T and let $\{t_1, \ldots, t_m\}$ be the set of children of t. We claim that $|X'_t| \leq (k+1)^2/2$.

Let E_t be the set of edges crossing X_t in G. Let H_t be the torso of $\mathcal{D}$ at t, and let $H'_t = H_t \setminus X_t$. Observe that $|E_t|$ is the same as the number of edges in H'_t. Let z_p be the vertex of H'_t corresponding to the parent of t, and similarly for each $i \in \{1, \ldots, m\}$ let z_i be the vertex of H'_t corresponding to the child t_i of t. Notice that if t_i is a thin child of t, then z_i can be adjacent to only z_p as $\mathcal{D}$ is a nice rooted tree-cut decomposition. Thus the sum of the number of incident edges with z_i in H'_t for all thin children t_i of t is at most $\mathrm{adh}_{\mathcal{D}}(t) \leq k$. On the other hand, if t_i is a bold child of t, then z_i has at least 3 neighbors in H_t, and thus it is contained in the 3-center of (H_t, X_t). Thus, the number of all bold children of t is bounded by $k - |X_t|$. Since each vertex in H'_t is incident with at most k edges, the total number of edges in H'_t is at most $(k - |X_t| + 1)k/2 + k$. As $|E(H'_t)| = |E_t| = |X'_t \setminus X_t|$, it implies that $|X'_t| \leq |X_t| + k \cdot (k - |X_t| + 2)/2 \leq \max\{2k, k(k+2)/2\} \leq (k+1)^2/2$. We conclude that G' has a rooted tree-partition of width at most $(\mathbf{tcw}(G)+1)^2/2$.

Recall that G' is a subdivision of G^*. By the virtue of Lemma 5.16, Lemma 5.15, and Lemma 5.14, we obtain that $\mathsf{e\text{-}pack}_{\mathcal{I}(H^+)}(G') \leq \mathsf{e\text{-}pack}_{\mathcal{I}(H)}(G)$ and $\mathsf{e\text{-}cover}_{\mathcal{I}(H)}(G) \leq \mathsf{e\text{-}cover}_{\mathcal{I}(H^+)}(G')$. Hence G' satisfies the desired properties. □

Therefore we get the following lemma.

Lemma 5.17. *Let H be a graph on h edges, let r be an integer and let G be a graph such that $\mathbf{tcw}(G) \leq r$. Then $\mathsf{e\text{-}pack}_{\mathcal{I}(H)}(G) \leq \mathsf{e\text{-}pack}_{\mathcal{I}(H)}(G) \cdot h(r+1)^4$.*

Proof. Let G' and H' be the graphs given by Lemma 5.13. As $\mathbf{tcw}(G) \leq r$ we have $\mathbf{tpw}(G') \leq (r+1)^2/2$. Applying Lemma 5.6, we get:

$$\mathsf{e\text{-}pack}_{\mathcal{I}(H')}(G') \leq \mathsf{e\text{-}pack}_{\mathcal{I}(H')}(G') \cdot h(r+1)^4.$$

Then Lemma 5.13 provides the desired inequality (cf. the remark following it). □

The edge version of Theorem 5.13 now follows from the application of Theorem 5.1, using the ceiling provided by Theorem 4.15 together with Lemma 5.17.

The vertex case

To prove the vertex version of Theorem 5.13 is a much easier task. For this, we follow the same methodology by using the graph parameter of treewidth instead of tree-cut width, and topological minors instead of immersions. We use the following vertex-counterpart of Theorem 4.15.

Lemma 5.18. *For every $h \in \mathbb{N}$ there is a function $f\colon \mathbb{N} \to \mathbb{N}$ with $f(k) = (h \cdot k)^{O(1)}$ such that, for every planar subcubic graph H with $|V(H)| + |E(H)| = h$, f is a ceiling for $(\mathbf{tw}, \mathcal{I}(H), \mathsf{v})$.*

Proof. A equivalent statement of Theorem 4.9 is that every graph of treewidth at least $f_1(k)$ contains a subdivision of a wall of height and width $\Omega(k)$ as a subgraph, for every $k \in \mathbb{N}$. According to Lemma 4.22, every planar subcubic graph H is an immersion of the wall of width and height h, where $h = |V(H)| + |E(H)|$. Therefore the wall of width and height kh contains k-edge-disjoint $\mathcal{I}(H)$-subgraphs. We deduce that every graph of treewidth at least $f_1(c \cdot hk)$ contains k vertex-disjoint $\mathcal{I}(H)$-subgraphs, for some constant c not depending on H. We note that f_1 is polynomial and this concludes the proof. $\square$

The vertex version of Theorem 5.13 follows by the application of Theorem 5.1 to Lemma 5.18 and Lemma 5.2.

5.4 Summary of results

5.4.1 Results in terms of containment relations

For every partial order $\preceq$ on graphs, and for every graph H, let

$$\mathcal{G}_{\preceq}(H) = \{G,\ H \preceq G\}.$$

For instance, $\mathcal{G}_{\leq_{\mathrm{m}}}(H)$ is the class of all graphs containing H as a minor. For every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, we define

$$\mathcal{EP}^{\mathsf{x}}_{\preceq} \quad = \quad \{H,\ \mathcal{G}_{\preceq}(H) \text{ has the } \mathsf{x}\text{-Erdős–Pósa property}\}$$

A general question on Erdős–Pósa properties is to characterize $\mathcal{EP}^{\mathsf{x}}_{\preceq}$ for several containment relations. In this section we mainly provide some negative results about this problem. We start with the following easy observation.

Lemma 5.19. *If $\preceq$ is the subgraph or the induced subgraph relation, $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, and H is a non-trivial graph, then $\mathcal{G}_{\preceq}(H)$ has the x-Erdős–Pósa property, with gap $f : k \mapsto k \cdot |A_{\mathsf{x}}(G)|$. In other words, $\mathcal{EP}^{\mathsf{x}}_{\preceq}$ is the set of all graphs.*

Proof. Let H and G be two graphs and let $k = \mathsf{x}\text{-}\mathsf{pack}_{\mathcal{G}_{\preceq}(H)}(G)$. Let $M_1, \dots, M_k$ be a v-$\mathcal{G}_{\preceq}(H)$-packing (respectively e-$\mathcal{G}_{\preceq}(H)$-packing) of size k with the minimal number of vertices (respectively edges). Observe that in this case, $|M_i| = |H|$ (respectively $\|M_i\| = \|H\|$) for every $i \in [\![1, k]\!]$. Let $X = \bigcup_{i=1}^{k} V(M_i)$ (respectively $X = \bigcup_{i=1}^{k} E(M_i)$). As the packing we consider is of size k, the graph $G \setminus X$ does not have any subgraph isomorphic to a member of $\mathcal{G}_{\preceq}(H)$. Hence X is an v-$\mathcal{G}_{\preceq}(H)$-cover (respectively e-$\mathcal{G}_{\preceq}(H)$-cover), and besides we have $|X| = k \cdot |H|$ (respectively $|X| = k \cdot \|H\|$). $\square$

Notice that in case $\mathsf{x} = \mathsf{v}$, it is not necessary to demand that H is non-trivial in the statement of Lemma 5.19.

Some negative results

Let us now state several negative results on the Erdős–Pósa property of classes related to topological minors.

In the proofs below, we use the notion of *Euler genus* of a graph G. The *Euler genus* of a non-orientable surface Σ is equal to the non-orientable genus $\tilde{g}(\Sigma)$ (or the crosscap number). The *Euler genus* of an orientable surface Σ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ. We refer to the book of Mohar and Thomassen [MT01] for more details on graph embeddings. The *Euler genus* of a graph G is the minimum integer γ such that G can be embedded on a surface of the Euler genus γ.

Lemma 5.20. *Let H be a non-planar graph. Then $\mathcal{T}(H)$ does not have the* v*-Erdős–Pósa property.*

Proof. For every integers $k > 0$ and d, we denote by $\Gamma_{d,k}$ the graph obtained from a grid of width dk and height $d + k - 1$ by adding k vertices $a_1, \dots, a_k$ (that we call *apices*) and connecting a_1 to the d first vertices on the first row of the grid (starting from the left), a_2 to the d next vertices, and so on. For every $i \in [\![0, d-1]\!]$, the set of vertices at indices $\{ik + j,\ j \in [\![0, k-1]\!]\}$ on the last row of $\Gamma_{d,k}$ is called the *i-th port* of $\Gamma_{d,k}$. We will refer to the vertex at index $ik + j$ of the last row as the j-th vertex of the i-th port. See Figure 5.2 for a drawing of $\Gamma_{4,3}$. On this drawing, the ports are $U_0, \dots, U_3$.

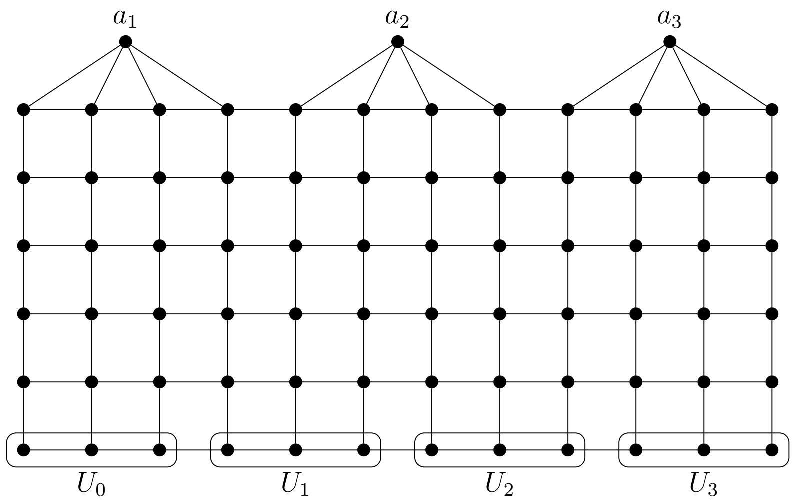


Figure 5.2: The gadget $\Gamma_{4,3}$ used in Lemma 5.20.

Let k be a positive integer. For every vertex v of H, we arbitrarily choose an ordering of its neighbors and we denote by $\sigma_v(u)$ the rank of u in this ordering (ranging from 0 to $\deg(v) - 1$), for every neighbor u of v. We also let F_v be a copy of the graph $\Gamma_{\deg(v),k}$.

The graph G_k can be constructed from the disjoint union of the graphs of $\{F_v,\ v \in V(H)\}$ by adding, for every pair u, v of adjacent vertices, the edge connecting the i-th vertex of the $\sigma_v(u)$-th port of F_v to the i-th vertex of the $\sigma_u(v)$-th port of F_u, for every $i \in [\![0, k-1]\!]$. Informally, we connect the vertices of the $\sigma_v(u)$-th port of F_v to the vertices of the $\sigma_u(v)$-th port of F_u using "parallel" edges.

It can be easily checked that the Euler genera of G_k and H are equal. As the Euler genus of the disjoint union of two copies of H is larger than the one of H (see [BHK62]), we get that $\mathsf{v\text{-}pack}_{\mathcal{T}(H)}(G) < 2$. On the other hand, our construction ensures that $\mathsf{v\text{-}pack}_{\mathcal{T}(H)}(G) \geq 1$.

Let us now show that for every subset $X \subseteq V(G_k)$ with $|X| < k$ we have $\mathsf{v\text{-}pack}_{\mathcal{T}(H)}(G \setminus X) \geq 1$. This would complete the proof, since $\{G_k,\ k \in \mathbb{N}_{\geq 1}\}$ would be an infinite family of graphs that have no v-$\mathcal{T}(H)$-packings of size 2 but where a minimum v-$\mathcal{T}(H)$-cover can be arbitrarily large.

Let u and v be two adjacent vertices of H, and let $d = \deg(v)$. For For every $i \in [\![0, k-1]\!]$, let C_i denote the vertices that are

- either in the same column of F_u as the i-th vertex of the $\sigma_u(v)$-th port of F_u;
- or in the same column of F_v as the i-th vertex of the $\sigma_v(u)$-th port of F_v.

The family $\{C_i,\ i \in [\![1, k]\!]\}$ contains k vertex disjoint elements, therefore at least one of the does not contain any vertex from X (as $|X| < k$). Therefore, for every edge $\{u, v\}$ of H there is an edge $f(\{u, v\})$ between a vertex x of the $\sigma_u(v)$-th port of F_u and a vertex y of the $\sigma_v(u)$-th port of F_v such that no vertex of the same column as x in F_u (respectively y in F_v) belong to X. Using the same argument we can show that for every vertex $v \in V(H)$ there is an apex a such that the columns of F_v adjacent to a are free of vertices of X. Also we know that at least d rows do not contain vertices from X, as the grid of F_v has height $d + k - 1$. Therefore F_v contains as a subgraph a grid S_v such that:

1. an apex a is adjacent to d vertices of the first row of S_v;
2. for every vertex u adjacent to v, the edge $f(\{u, v\})$ shares one vertex the last row of S_v;
3. no vertex of the last row of S_v belong to two edges $f(\{u, v\})$ and $f(\{u', v\})$ for some distinct neighbors u, u' of v;
4. S_v has height and width at least d;
5. S_v does not contain any vertex of X.

We deduce that $F_v \setminus X$ contains d paths $P_0, \ldots, P_{d-1}$ that have only the apex a as common vertex and such that P_i connects a to an endpoint of $f(\{v, u_i\})$, where u_i is the neighbor of v of rank i, for every $i \in [\![0, d-1]\!]$. It is now easy to see that the graph

$$G_k\left[\bigcup v \in V(H) \bigcup_{i=0}^{\deg_H(v)-1} V(P_i^v)\right]$$

contains a subdivision of H that does not contain any vertex of X. This concludes the proof. $\square$

The proof of Lemma 5.20 can be adapted to the setting of the edge-Erdős–Pósa property under the additional requirement that the pattern is subcubic.

Lemma 5.21. *Let H be a subcubic non-planar graph. Then $\mathcal{T}(H)$ does not have the* e*-Erdős–Pósa property.*

Proof. Let k be a positive integer. We use the same construction of G_k as in the proof of Lemma 5.20 with the following modifications: each vertex v of degree $d \geq 4$ of G_k is replaced by a subcubic tree, the leaves of which are the neighbors of v. Let us call G'_k the graph we obtain. It is not hard to see that the genera of G'_k and G_k are equal. Moreover, as G'_k is subcubic, every e-$\mathcal{T}(H)$-packing is also an v-$\mathcal{T}(H)$-packing. We then obtain as previously that $\mathsf{e\text{-}pack}_{\mathcal{T}(H)}(G'_k) = 1$. The arguments to show that $\mathsf{e\text{-}cover}_{\mathcal{T}(H)}(G'_k) \geq k$ are identical to the ones used in the proof of Lemma 5.20. □

In fact, Lemma 5.20 and Lemma 5.21 can be used to prove that more general classes do not have the Erdős–Pósa property, as follows. As we will see in Corollary 5.5 and Corollary 5.6, the conditions of Lemma 5.22 already encompass several well-studied classes.

Lemma 5.22. *Let $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, let H be a non-planar graph and let $\mathcal{H}$ be a class of graphs such that:*

(i) $\mathcal{T}(H) \subseteq \mathcal{H}$; and

(ii) H is graph of minimum Euler genus in $\mathcal{H}$;

(iii) if $\mathsf{x} = \mathsf{e}$, then H is subcubic.

Then $\mathcal{H}$ does not have the x*-Erdős–Pósa property.*

Proof. Let k be a positive integer. We again consider the constructions of G_k and G'_k used in the proofs of Lemma 5.20 and Lemma 5.21. Let J_k be G_k if $\mathsf{x} = \mathsf{v}$ and $J_k = G'_k$ if $\mathsf{x} = \mathsf{e}$. Let show that $\mathsf{v\text{-}pack}_{\mathcal{H}}(J_k) = 1$. For this, let us assume that there is an x-$\mathcal{T}(H)$-packing $F_1, \dots, F_p$, for some $p \in \mathbb{N}_{\geq 2}$ in J_k. It is crucial to note that in both the cases $\mathsf{x} = \mathsf{v}$ and $\mathsf{x} = \mathsf{v}$, the subgraphs $F_1, \dots, F_p$ are vertex-disjoint. In fact, when $\mathsf{x} = \mathsf{v}$, this follows from the definition of a v-$\mathcal{T}(H)$-packing, and if $\mathsf{x} = \mathsf{e}$ it is because G'_k is subcubic. Then we have:

$$\begin{aligned}
\gamma(J_k) &\geq \gamma(F_1 \cup \dots \cup F_p) && \\
&= \sum_{i=1}^{p} \gamma(F_i) && \text{(see [BHK62])} \\
&\geq p \cdot \gamma(H) && \text{(by minimality of } H) \\
\gamma(J_k) &> \gamma(H) && \text{(contradiction).}
\end{aligned}$$

We reached a contradiction, hence $\mathsf{v\text{-}pack}_{\mathcal{H}}(J_k) = 1$. On the other hand,

$$\mathsf{v\text{-}cover}_{\mathcal{H}}(J_k) \geq \mathsf{v\text{-}cover}_{\mathcal{T}(H)}(J_k) \geq k.$$

The last inequality can be found in the proof of Lemma 5.20 or Lemma 5.21 (depending if $\mathsf{x} = \mathsf{v}$ or $\mathsf{x} = \mathsf{e}$). This concludes the proof. □

Corollary 5.5. *For every non-planar graph H, none of $\mathcal{I}(H)$ and $\mathcal{M}(H)$ have the* v*-Erdős–Pósa property.*

Corollary 5.6. *For every subcubic non-planar graph H, none of $\mathcal{I}(H)$ and $\mathcal{M}(H)$ have the* e*-Erdős–Pósa property.*

Corollary 5.6 can be strengthened by dropping the degree condition on H when considering minor models of H, as follows.

Lemma 5.23. *For every non-planar graph H, $\mathcal{M}(H)$ does not have the* e*-Erdős–Pósa property.*

Proof. Let k be a positive integer. Again we use the graph G'_k constructed as in Lemma 5.21. We modify it by replacing every apex a by a subcubic tree, the leaves of which are the neighbors of a. Let G''_k denote the graph that we obtain. Observe that G''_k is subcubic. Therefore, using the same argument as in the proof of Lemma 5.21 we can show that $\mathsf{e}\text{-}\mathsf{pack}_{\mathcal{M}(H)}(G) = 1$. In the sequel we use the terminology of the proof of Lemma 5.20. Let F''_v denote the graph obtained from F_v by replacing every vertex u of degree at least 4 by a subcubic tree, the leaves of which are the neighbors of u, for every $v \in V(H)$. The proof that $\mathsf{e}\text{-}\mathsf{cover}_{\mathcal{M}(H)}(G) \geq k$ goes as in the proof of Lemma 5.20, except that we obtain, for every $v \in V(H)$, that $F''_v \setminus X$ contains a tree, the leaves of which are endpoints of $f(\{v, u_i\})$ for $i \in [\![0, d-1]\!]$ (instead of paths connecting an apex to endpoints of $f(\{v, u_i\})$). Fortunately this is enough to guarantee that $G''_k \setminus X$ contains H as a minor, and we are done. □

Thomassen in [Tho88] provided an example of a tree such that $H \notin \mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{tm}}}$ (the same graph does not belong to $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{tm}}}$ neither). Inspired by this construction we give another such graph that, additionally, is biconnected. This graph H is depicted in Figure 5.3. To see that $H \notin \mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{tm}}}$ and $H \notin \mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{tm}}}$, consider as host graph G the graph in Figure 5.4. This graph consists of a main body that is a wall of height 3 and three triples of graphs attached at its upper, leftmost, and lower paths. Each of these triples consists of three copies of some of the 3-connected components of H. Notice that G does not contain more than one H-immersion expansion. However, in order to cover all H-immersion expansions of G one needs to remove at least 3 edges/vertices. By increasing the heigh of the wall of G, we may increase the minimum size of an $\mathcal{I}(H)$-vertex/edge cover while no $\mathcal{I}(H)$-vertex/edge packing of size greater than 1 will appear. It is easy to modify H so to make it 3-connected: just add a new vertex and make it adjacent with the tree vertices of degree 4. The resulting graph H' remains planar. The same arguments, applied to an easy modification of the host graph, can prove that H' is not a graph in $\mathcal{H}^{\mathsf{v}}$ or $\mathcal{H}^{\mathsf{e}}$.

Lemma 5.24. *There is a 2-connected (respectively 3-connected) planar graph that belongs to none of $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{tm}}}$, $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{tm}}}$, $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{imm}}}$, and $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{imm}}}$.*

Let us now summarize results related to the most common containment relations.

Subgraphs and induced subgraphs: $\mathcal{EP}^{\mathsf{x}}_{\preceq}$ is the class of all graphs, both for $\preceq$ being the subgraph and induced subgraph relation, for every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$ (Lemma 5.19).

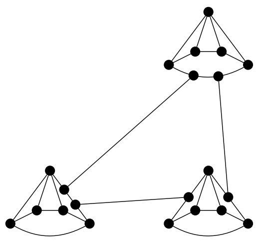


Figure 5.3: A biconnected graph H for which $\mathcal{I}(H)$ does not have the v/e-E&P property.

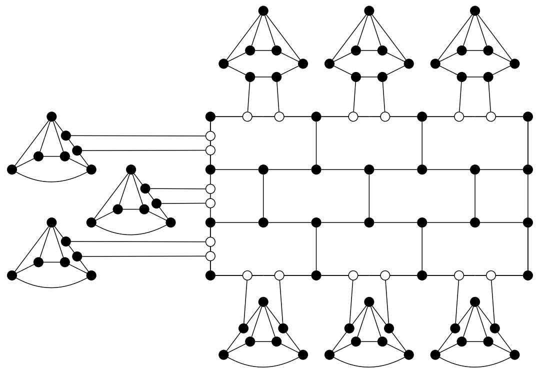


Figure 5.4: The host graph G.

Minors: $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{m}}}$ is the class of planar graphs [RS86, Theorem 8.2]. About the edge version, Theorem 5.9, Theorem 5.8, and Theorem 5.11 imply that $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{m}}}$ includes the class $\{\theta_r\}_{r\in\mathbb{N}_{\geq 1}} \cup \{\theta_{r,r'}\}_{r,r'\in\mathbb{N}}$, and we show in Lemma 5.23 that $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{m}}}$ is a subclass of planar graphs.

Topological Minors: $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{tm}}}$ has been characterized in [LPW14]. There are trees, 2-connected and 3-connected graphs that belongs to none of $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{tm}}}$ and $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{tm}}}$ ([Tho88] and Lemma 5.24). The class $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{tm}}}$ does not contain any non-planar graph (Lemma 5.20) and $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{tm}}}$ does not contain any non-planar subcubic graph (Lemma 5.21).

Immersions: As proved in Subsection 5.3.4, $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{imm}}}$ contains all planar subcubic graphs and $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{imm}}}$ contains all non-trivial, connected, planar subcubic graphs. Moreover, $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{imm}}}$ does not contain any non-planar graph (Corollary 5.5) and $\mathcal{EP}^{\mathsf{e}}_{\leq_{\mathrm{imm}}}$ does not contain any subcubic non-planar graph (Corollary 5.6). On the other hand there is a 3-connected planar graph that belongs to none of $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{imm}}}$ and $\mathcal{EP}^{\mathsf{v}}_{\leq_{\mathrm{imm}}}$ Lemma 5.24.

5.4.2 Results in terms of graph classes

We provide a series of tables presenting known results on the Erdős–Pósa property of some graph classes, sorted depending on the pattern. Results related to other structures (matroids, hypergraphs, geometry), to directed graphs or to fractional versions are not mentioned here.

A dash in the "gap" column means that the authors did not explicitly provided a gap function (even though one may be computable from the proof).

The fourth column refers to the type of packing/cover, cf. Subsection 5.4.3. Most (but not all) of the notation used in this section is defined in Subsection 5.4.3.

Positive results

Acyclic patterns

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[Kőn31]	K_2	bipartite	v	k
[Men27]	S-T-paths	any	v/e	k
[Grü38]	directed S-T-paths	any	v/e	k
[FJW13a], Theorem 5.4	$\mathcal{M}(H)$, H forest	any	v	$O_{\mathcal{H}}(k)$

Triangles

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[Tuz90]	triangles	planar graphs	e	$2k$
		G with $\|\|G\|\| \geq 7\|G\|^2/16$	e	$2k$
		tripartite graphs	e	7k/3
[Kri95]	triangles	$\mathcal{T}(K_{3,3})$-free graphs	e	$2k$
[HK88]	triangles	tripartite graphs	e	$1.956k$
[Hax99]	triangles	any	e	$(3 - \frac{3}{23})k$
[ALBT11]	triangles	odd-wheel-free graphs	e	$2k$
		4-colorable graphs		
[HKT11]	triangles	K_4-free planar graphs	e	$3k/2$
		K_4-free flat graphs		

Cycles

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[EP65]	cycles	any	v	$(4 + o(1))k \log k$
[Sim67]	cycles	any	v	$\left(\frac{1}{2} + o(1)\right) k \log k$
[Die05]	cycles	any	e	$(2 + o(1))k \log k$
[DZ02]	cycles	$\mathcal{G}_1$, weighted	w	k
[DXZ03]	cycles	$\mathcal{G}_2$	v	k
[KLL02b]	cycles	planar graphs	v	$5k$
		outerplanar graphs	v	$2k$
[MYZ13]	cycles	planar graphs	v	$3k$
			e	$4k - 1$

Cycles with length constraints

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[Ree99]	odd cycles	planar graphs	v	superexponential
[FHRV05]	odd cycles	planar graphs	v	$10k$
			e	$2k$
[Tho01b]	odd cycles	$2^{3^{9k}}$-connected graphs	v	$2k-2$
[RR01]	odd cycles	$576k$-connected graphs	v	$2k-2$
[KR09]	odd cycles	$24k$-connected graphs	v	$2k-2$
[Ree99]	odd cycles	k-near bipartite graphs	v	–
[KN07]	odd cycles	embeddable in an orientable surface of Euler genus g	v/e	–
[BR00]	odd cycles	any	e	–
[KV04]	odd cycles	planar graphs	e	$2k$
[KK12]	odd cycles	4-edge-connected graphs	e	$2^{2^{O(k \log k)}}$
[Ree97]	odd cycles	any	$\mathsf{v}_{1/2}$	–
[KW05]	non-zero cycles	$(15k/2)$-connected group-labeled graphs	v	$2k-2$
[BBR07b]	$\mathcal{C}_{\geq t}$	any	v	$(13+o_t(1))tk^2$
[FH14]	$\mathcal{C}_{\geq t}$	any	v	$(6t+4+o_t(1))k \log k$
[MNŠW16]	$\mathcal{C}_{\geq t}$	any	v	$6kt+(10+o(1))k \log k$

Extensions of cycles

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[Sim67]	dumb-bells	any	v	$(4000+o(1))k \log k$
[FLM+13]	$\mathcal{M}(\theta_t)$	any	v	$O(t^2k^2)$
[FJS13]	$\mathcal{M}(\theta_t)$	any	v	$O_t(k \log k)$
[RST16]	$\mathcal{M}(\theta_t)$	any	e	$O(k^2t^2 \operatorname{polylog} kt)$ $O(k^4t^2 \operatorname{polylog} kt)$
[CRST15a], Th. 5.9	$\mathcal{M}(\theta_t)$	any	v/e	$O_t(k \log k)$

Minor models

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[RS86], Lemma 5.5	$\mathcal{M}(H)$, H planar	any	v	–
		$\{G,\ \mathbf{tw}(G) \leq t\}$	v	$(t-1)(k\,\mathbf{cc}(H)-1)$
[DKW10]	$\mathcal{M}(K_t)$	$O(kt)$-connected graphs	v	–
[FST11]	$\mathcal{M}(H)$, H planar connected	K_q-minor free	v	$O_{h,q}(k)$
[RT13]	$\mathcal{M}(H)$, $\mathbf{pw}(H) \leq 2$ and H connected	any	v	$2^{O(\vert H\vert^2)} \cdot k^2 \log k$
[CC13a], [CC13b], Cor. 5.1	$\mathcal{M}(H)$, H planar connected	any	v	$O(\vert H\vert^{O(1)} \cdot k \operatorname{polylog} k)$
[CRST15a], Lemma 5.6, Th. 5.11	$\mathcal{M}(H)$, H connected	$\{G,\ \mathbf{tpw}(G) \leq t\}$	v/e	$O_{H,t}(k)$
	$\mathcal{M}(\theta_{t,t'})$	simple graphs	e	–

Topological minor models

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[Tho88]	$\mathcal{T}_{(0 \bmod t)}(H)$, H planar subcubic	any	v	–
[LPW14]	$\mathcal{T}(H), H \in \mathcal{L}$	any	v	–

Immersion expansions

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[Liu15]	$\mathcal{I}(H)$	4-edge-connected	e	–
[GKRT16], Th. 5.13 Lemma 5.17	$\mathcal{I}(H)$, H planar subcubic connected non-trivial	any	e	$(\vert H\vert k)^{O(1)}$
	$\mathcal{I}(H)$, H connected non-trivial	$\{G, \mathbf{tpw}(G) \leq t\}$ $\{G, \mathbf{tcw}(G) \leq t\}$	e	$\Vert H\Vert \cdot t^2 \cdot k$
[Liu15]	$\mathcal{I}_{1/2}(H)$	any	$\mathsf{e}_{1/2}$	–

Patterns with prescribed vertices

Let us first present the two settings of Erdős–Pósa problems with prescribed vertices that we want to deal with here. The first type is when the guest class consists of fixed subgraphs of the host graph. For instance, one can consider a family $\mathcal{F}$ of (non necessarily disjoint) subtrees of a tree T, and compare the maximum number of disjoint elements in $\mathcal{F}$ with the minimum number of vertices/edges of T meeting all elements of $\mathcal{F}$. We will refer to these guest classes by words indicating that we are dealing with substructures (like “subtrees”). We stress that in this setting, the host class is allowed to contain one subgraph F of the host graph, but not one other subgraph F' even if F and F' are isomorphic.

In order to introduce the second type of problem, we need the following definition. Let $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$. If $\mathcal{H}$ is a class of graphs, G is a graph and $S \subseteq A_{\mathsf{x}}(G)$, then a S-$\mathcal{H}$-subgraph of G is a subgraph of G isomorphic to some member of $\mathcal{H}$ and that contain one edge/vertex of S. We are now interested in comparing, for every graph G and every $S \subseteq A_{\mathsf{x}}(G)$, the maximum number of S-$\mathcal{H}$-subgraph of G with the minimum number of elements of $A_{\mathsf{x}}(G)$ that meet all S-$\mathcal{H}$-subgraphs of G. We refer to these problems by prefixing the guest class with an "S" (like in "S-cycles"). A generalization of this type of problem has been introduced in [KM15]: instead of one set S, one considers a collection $Z = \{Z_1, Z_2, Z_3\}$ of three subsets of $A_{\mathsf{x}}(G)$ and a Z-subgraph is required to intersect at least two sets of Z.

For every positive integer t, a t-path is a disjoint union of t paths, and a t-subpath of a t-path G is a subgraph that has a connected intersection with every connected component of G. The concept of t-forests and t-subforests is defined similarly.

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[HS58]	subpaths	paths	v	k
[GL69]	t-subpaths	t-paths	v	$O(k^{t!})$
	subgraphs H with $\mathbf{cc}(H) \leq t$	paths	v	–
	t-subforests	t-forests	v	–
[GL69]	subtrees of a tree	trees	v	k
[Kai97]	t-subpaths	t-paths	v	$(t^2 - t + 1)k$
[Alo98]	t-subpaths	t-paths	v	$2t^2k$
[Alo02]	subgraphs H with $\mathbf{cc}(H) \leq t$	trees	v	$2t^2k$
	subgraphs H with $\mathbf{cc}(H) \leq t\}$	$\{G,\ \mathbf{tw}(G) \leq w\}$	v	$2(w+1)t^2k$
[KiKM11]	S-cycles	any	v	$O(k^2 \log k)$
[PW12]	S-cycles	any	v/e	$O(k \log k)$
[BJS14]	S-cycles $\cap\, \mathcal{C}_{\geq t}$	any	v	$O(tk \log k)$
[Joo14]	odd S-cycles	$50k$-connected graphs	v	$O(k)$
[KM15]	Z-$\mathcal{M}(H)$, H planar	any	v	–

Classes with bounded parameters

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[Tho88]	any family of connected graphs	$\{G,\ \mathbf{tw}(G) \leq t\}$	v	$k(t+1)$
[FJW13a]	$\{H,\ \mathbf{pw}(H) \geq t\}$	any	v	$O_t(k)$
[CRST15a]	any finite family of connected graphs	$\{G,\ \mathbf{tpw}(G) \leq t\}$	v/e	$O_{\mathcal{H},t}(k)$

Negative results

The next table present classes of patterns that do not have the Erdős–Pósa property for some class of hosts, as well as classes that do not have the Erdős–Pósa property for a certain gap function. It indicates to which extend the results of the table of Section 5.4.2 are best possible.

Negative results on cycles

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
[Tuz90]	triangles	all graphs	e	$< 2k$
[EP65]	cycles	all graphs	v	$o(k \log k)$
[KLL02b]	cycles	planar graphs	v	$< 2k$
[MYZ13]	cycles	planar graphs	e	$ck - 1,\ c < 4$
[DL88]	odd cycles	all graphs	v	any
[Ree99]	odd cycles	all graphs	e	any
[Tho01b]	odd cycles	planar graphs	v	$< 2k - 2$
[KV04]	odd cycles	planar graphs	e	$< 2k$
[PW12]	S-cycles	any	v	$o(k \log k)$
[FH14]	$\mathcal{C}_{\geq t}$	all graphs	v	$o(k \log k)$, t fixed
				$o(t)$, k fixed
[MNŠW16]	$\mathcal{C}_{\geq t}$	all graphs	v	$< (k-1)t$
				$< \frac{(k-1)\log k}{8}$

Negative results on patterns related to containment relations

Ref.	Guest class $\mathcal{H}$	Host class $\mathcal{G}$	T.	Gap
from [EP65]	$\mathcal{M}(H)$, H has a cycle	all graphs	v	$o(k \log k)$
[RS86]	$\mathcal{M}(H)$, H non-planar	all graphs	v	any
Lemma 5.23	$\mathcal{M}(H)$, H non-planar	all graphs	e	any
Lemma 5.20	$\mathcal{T}(H)$, H non-planar	all graphs	v	any
[Tho88]	$\mathcal{T}(H)$, for infinitely many trees H with $\Delta(H) = 4$	planar graphs	e	any
Lemma 5.21	$\mathcal{T}(H)$, H non-planar subcubic	all graphs	e	any
copying [Tho88]	$\mathcal{I}(H)$, for infinitely many trees H with $\Delta(H) = 4$	planar graphs	e	any
Cor. 5.5	$\mathcal{I}(H)$, H non-planar	all graphs	v	any
Cor. 5.6	$\mathcal{I}(H)$, H non-planar subcubic	all graphs	e	any
[GKRT16]	$\mathcal{I}(H)$, for some 3-connected H with $\Delta(H) = 4$	planar graphs	e	any
[Liu15]	$\mathcal{I}(H)$, for every H	3-edge-connected graphs	e	any

5.4.3 Notation used in Subsection 5.4.2

In this section we recall or introduce (some of) the notations that are used in Subsection 5.4.2.

Types of packings/covers. The fourth column gives the type of the packings/covers the current line is about. The character `v` (respectively `e`) refers to vertex-disjoint (respectively edge-disjoint) packings and vertex (respectively edge) covers. We write `v/e` when the mentioned result holds for both the vertex and the edge version. $\mathtt{e}_{1/2}$ corresponds to half-integral packings (i.e. each edge can be used at most twice) and edge covers. Finally, `w` stands for vertex covers and packings where every vertex v of the host graph can be used at most $w(v)$ times by every packing, where w is a function mapping reals to the vertices of the host graph.

Classes of graphs. We denote by $\mathcal{I}_{1/2}(H)$ the class of all graphs containing H as a half-integral[1] immersion. For every $t \in \mathbb{N}$, $\mathcal{T}_{(0 \bmod t)}(H)$ denotes the class of subdivisions of H where every edge is subdivided 0 mod t times. The class of cycles of length at least t is referred to as $\mathcal{C}_{\geq t}$. A *dumb-bell* is a graph obtained by connecting two cycles by a (non-trivial) path. We denote by $\mathcal{L}$ a graph class defined in the manuscript [LPW14]. For every positive integer k with, we say that a graph is *k-near bipartite* if every set X of vertices contains a stable of size at least $|X|/2 - k$. A graph is *flat* if every edge belongs to at most two triangles.

Definitions from [DZ02, DXZ03]. An odd ring is a graph obtained from an odd cycle by replacing every edge $\{u, v\}$ by either a triangle, or three triangles uab and ucd together with the edges $\{b, c\}$ and $\{a, d\}$. We denote by $\mathcal{G}_1$ the class of graphs with no induced subdivision of the following: $K_{2,3}$, a wheel, or an odd ring. We denote by $\mathcal{G}_2$ the class of graphs with no induced subdivision of the following: $K_{3,3}$, a wheel, or an odd ring.

[1] A graph H is a *half-integral immersion* of a graph G is H is an immersion of the graph obtained by G after duplicating the multiplicity of all its edges.

Chapter 6

From the Erdős–Pósa property to approximation

In this chapter we show how the combinatorial connection between invariants of packing and covering provided by the Erdős–Pósa property can be used to design approximation algorithms. We focus on packing and covering graphs from $\mathcal{M}(\theta_r)$ for any $r \in \mathbb{N}$, in both the vertex and edge setting. Drawing upon combinatorial results presented in Chapter 4, we give an algorithmic proof that $\mathcal{M}(\theta_r)$ has the vertex- and edge-Erdős–Pósa property with gap $O(k \log k)$, which is optimal. Using the algorithmic machinery of our proofs we introduce a unified approach for the design of an $O(\log \mathsf{OPT})$-approximation algorithm for $\mathsf{v\text{-}pack}_{\theta_r}$, $\mathsf{v\text{-}cover}_{\theta_r}$, $\mathsf{e\text{-}pack}_{\theta_r}$, and $\mathsf{e\text{-}cover}_{\theta_r}$ that runs in $O(n \cdot \log(n) \cdot m)$ steps.

This chapter contains material that previously appeared in the following article: *An $O(\log \mathsf{OPT})$-approximation for covering/packing minor models of θ_r*, co-authored with Dimitris Chatzidimitriou, Ignasi Sau, and Dimitrios M. Thilikos, presented in Approximation and Online Algorithms: 13th International Workshop, WAOA 2015, Patras, Greece, 2015 [CRST15a].

6.1 Introduction

From the algorithmic point of view, the computation of $\mathsf{x\text{-}pack}_{\mathcal{H}}$ (for $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$) corresponds to the general family of graph packing problems, while the computation of $\mathsf{x\text{-}cover}_{\mathcal{H}}$ belongs to the general family of graph modification problems where the modification operation is the removal of vertices/edges (depending on whether $\mathsf{x} = \mathsf{v}$ or $\mathsf{x} = \mathsf{e}$). Interestingly, particular instantiations of $\mathcal{H} = \mathcal{M}(\theta_r)$ generate known, well studied, NP-hard problems. For instance, asking whether $\mathsf{v\text{-}cover}_{\mathcal{M}(\theta_r)} \leq k$ generates Vertex Cover for $r = 1$, Feedback Vertex Set for $r = 2$, and Diamond Hitting Set for $r = 3$ [FJP10, FLMS12b]. Moreover, asking whether $\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(G) \geq k$ corresponds to Vertex Cycle Packing [BTY11, KLL02a] and Edge Cycle Packing [ACR03, KNS+07] when $\mathsf{x} = \mathsf{v}$ and $\mathsf{x} = \mathsf{e}$, respectively. Finally, asking whether $|E(G)| - \mathsf{e\text{-}cover}_{\mathcal{M}(\theta_r)}(G) \leq k$ corresponds to the Maximum Cactus Subgraph[1]. All

[1]The Maximum Cactus Subgraph problem asks, given a graph G and an integer k, whether G contains a subgraph with k edges where no two cycles share an edge. It can be reduced to the Vertex

parameters keep being NP-complete to compute because the aforementioned base cases can be reduced to the general one by replacing each edge by one of multiplicity $r-1$.

From the approximation point of view, it was proven in [FLMS12b] that, when H is a planar graph, there is a randomized polynomial $O(1)$-approximation algorithm for $\mathsf{v\text{-}cover}_{\mathcal{M}(H)}$. For the cases of $\mathsf{v\text{-}cover}_{\mathcal{M}(\theta_r)}$ and $\mathsf{v\text{-}pack}_{\mathcal{M}(\theta_r)}$, $O(\log n)$-approximations are known for every $r \geq 1$ because of [JPS+11, JPS+14] (see also [SV05]). Moreover, $\mathsf{v\text{-}cover}_{\mathcal{M}(\theta_3)}$ admits a deterministic 9-approximation [FJP10]. About the edge variant, it is known, from [KNY05], that there is a polynomial $O(\sqrt{\log n})$-approximation algorithm for $\mathsf{e\text{-}pack}_{\mathcal{M}(\theta_2)}(G)$. Notice also that it is trivial to compute $\mathsf{e\text{-}cover}_{\mathcal{M}(\theta_1)}(G)$ in polynomial time. However, to our knowledge, nothing is known about the computation of $\mathsf{e\text{-}cover}_{\mathcal{M}(\theta_r)}(G)$ for $r \geq 3$.

In this section we introduce a unified approach for the study of the combinatorial interconnections and the approximability of the parameters $\mathsf{v\text{-}cover}_{\mathcal{M}(\theta_r)}$, $\mathsf{e\text{-}cover}_{\mathcal{M}(\theta_r)}$, $\mathsf{v\text{-}pack}_{\mathcal{M}(\theta_r)}$, and $\mathsf{e\text{-}pack}_{\mathcal{M}(\theta_r)}$. Our main combinatorial result is the following.

Theorem 6.1. *For every $r \in \mathbb{N}_{\geq 2}$ and every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$ the graph class $\mathcal{M}(\theta_r)$ has the* x-EP-property *with (optimal) gap function $f(k) = O(k \cdot \log k)$.*

Our proof is unified and treats simultaneously the covering and the packing parameters for both the vertex and the edge cases. This verifies the optimal combinatorial bound for the case where $\mathsf{x} = \mathsf{v}$ [FJS13] and optimally improves (in terms of k) the bound in given in Subsection 5.3.2 (which appeared in [RST16]) for the case where $\mathsf{x} = \mathsf{e}$. In this section, when giving the running time of an algorithm with input some graph G, we agree that $n = |V(G)|$ and $m = |E(G)|$. Based on the proof of Theorem 6.1, we prove the following algorithmic result.

Theorem 6.2. *For every $r \in \mathbb{N}_{\geq 2}$ and every $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, there exists an $O(n \cdot \log(n) \cdot m)$-step algorithm that, given a graph G, outputs an $O(\log \mathsf{OPT})$-approximation for $\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}$ and $\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}$.*

Theorem 6.2 improves the results in [JPS+11, JPS+14] for the cases of $\mathsf{v\text{-}cover}_{\mathcal{M}(\theta_r)}$ and $\mathsf{v\text{-}pack}_{\mathcal{M}(\theta_r)}$ and is, to our knowledge, the first approximation algorithm for $\mathsf{e\text{-}cover}_{\mathcal{M}(\theta_r)}$ and $\mathsf{e\text{-}pack}_{\mathcal{M}(\theta_r)}$ for $r \geq 3$.

Overview of the proof. Our proofs are based on the notion of *partitioned protrusion* that, roughly, is a tree-structured subgraph of G with small boundary to the rest of G (see Subsection 6.2.2 for the precise definition). Partitioned protrusions, that we met in Section 4.1 under the name of *edge-protrusions*, can be seen as the edge-analogue of the notion of protrusions introduced in [BFL+09a] (see also [BFL+09b]). Our approach makes strong use of the main result of Section 4.1, that is equivalently stated as Theorem 4.4 in this section. According to this result, there exists a polynomial algorithm that, given a graph G and an integer k as an input, outputs one of the following:

1. a collection of k edge/vertex disjoint θ_r-majors of G,

Cycle Packing problem on cubic graphs which, in turn, can be proved to be NP-complete using a simple variant of the NP-completeness proof of Exact Cover by 2-Sets [Gol15].

2. a θ_r-major J with $O(\log k)$ edges, or

3. a large partitioned protrusion of G.

Our approximation algorithm does the following for each $k \leq |V(G)|$. If the first case of the above combinatorial result applies on G, we can safely output a packing of k θ_r-majors in G. In the second case, we make some progress as we may remove the vertices/edges of J from G and then set $k := k - 1$. In order to deal with the third case, we prove that in a graph G with a sufficiently large partitioned protrusion, we can either find some θ_r-major with $O(\log k)$ edges (which is the same as in the second case), or we can replace it by a smaller graph where both $\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}$ and $\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}$ remain invariant (Lemma 6.1). The proof that such a reduction is possible is given in Section 6.3 and is based on a suitable dynamic programming encoding of partial packings and coverings that is designed to work on partitioned protrusions.

Notice that the "essential step" in the above procedure is the second case that reduces the packing number of the current graph by 1 to the price of reducing the covering number by $O(\log k)$. This is the main argument (previously used in [FJW13b]) that supports the claimed $O(\log \mathsf{OPT})$-approximation algorithm (Theorem 6.2) and the corresponding Erdős–Pósa relations in Theorem 6.1.

Organization of the chapter. In Section 6.2 we provide all concepts and notation that we use in our proofs. Section 6.3 contains the proof of Lemma 6.1, which is the main technical part of the paper. The presentation and analysis of our approximation algorithm is done in Section 6.4, where Theorem 6.1 and Theorem 6.2 are proven.

6.2 Definitions specific to this section

6.2.1 Basic definitions

If $\mathcal{H}$ is a finite collection of graphs, we set $n(\mathcal{H}) = \sum_{H \in \mathcal{H}} n(H)$, $m(\mathcal{H}) = \sum_{H \in \mathcal{H}} m(H)$.

Given a graph H and a graph J that are both subgraphs of the same graph G, we define the subgraph $H \cap_G J$ of G as the graph $(V(H) \cap V(J), E(H) \cap E(J))$.

Given a graph G and a set $S \subseteq V(G)$, such that all vertices in S have degree 2 in G, we define $\mathsf{diss}(G, S)$ as the graph obtained from G after we dissolve in it all vertices in S.

Topological minors. Whereas the results presented in this chapter deal with majors, the tools that we provide in Section 6.3 are expressed in the setting of subdivisions. We show in Section 6.4 how this more general setting can be applied to majors. If G is a graph and $\mathcal{H}$ is a finite collection of connected graphs, recall than an *$\mathcal{H}$-subdivision* in G is a subgraph M of G that is a subdivision of a graph, denoted by $\hat{M}$ in this chapter, that is isomorphic to a member of $\mathcal{H}$. Clearly, the vertices of $\hat{M}$ are vertices of G and its edges correspond to paths in G between their endpoints such that internal vertices of a path do not appear in any other path. We refer to the vertices of $\hat{M}$ in G as the *branch vertices* of the $\mathcal{H}$-subdivision M, whereas internal vertices of the paths between branch vertices are called *subdivision vertices* of M.

6.2.2 Boundaried graphs

Informally, a boundaried graph is used to represent a graph that has been obtained by "dissecting" a larger graph along some of its edges, where the boundary vertices correspond to edges that have been cut. In this section we formally define boundaried graphs and related notions. We also give a notion of equivalence that is a cornerstone of our algorithms.

Boundaried graphs A *boundaried graph* $\mathbf{G} = (G, B, \lambda)$ is a triple consisting of a graph G, a set B of vertices of degree one (called *boundary*), and a bijection λ from B to a subset of $\mathbb{N}_{\geq 1}$. The edges with at least one endpoint in B are called *boundary edges*. We define $E^{\mathsf{s}}(G)$ as the subset of $E(G)$ of boundary edges. We stress that instead of $\mathbb{N}_{\geq 1}$ we could choose any other set of symbols to label the vertices of B. We denote the set of labels of $\mathbf{G}$ by $\Lambda(\mathbf{G}) = \lambda(B)$. Given a collection $\mathcal{H}$ of graphs, we say that a $\mathbf{G}$ is $\mathcal{H}$-free if $G \setminus B$ is $\mathcal{H}$-free.

Two boundaried graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ are *compatible* if $\Lambda(\mathbf{G}_1) = \Lambda(\mathbf{G}_2)$. Let now $\mathbf{G}_1 = (G_1, B_1, \lambda_1)$ and $\mathbf{G}_2 = (G_2, B_2, \lambda_2)$ be two compatible boundaried graphs. We define the graph $\mathbf{G}_1 \oplus \mathbf{G}_2$ as the graph obtained by first taking the disjoint union of G_1 and G_2, then, for every $i \in \Lambda(\mathbf{G}_1)$, identifying $\lambda_1^{-1}(i)$ with $\lambda_2^{-1}(i)$, and finally dissolving all resulting identified vertices. Suppose that e is an edge of $G = \mathbf{G}_1 \oplus \mathbf{G}_2$ that was created after dissolving the vertex resulting from the identification of a vertex v_1 in B_1 and a vertex v_2 in B_2 and that e_i is the boundary edge of G_i that has v_i as endpoint, for $i = 1, 2$. Then we say that e is the *heir* of e_i in G, for $i = 1, 2$, and we denote this by $\mathsf{heir}_G(e_i)$. For $i \in \{1, 2\}$, if $S \subseteq E(G_i)$, then

$$\mathsf{heir}_G(S) = (E(G_i) \cap S) \cup \{\mathsf{heir}_G(e) \mid e \in E^{\mathsf{s}}(G_i) \cap S\}.$$

For reasons of notational consistency, if $V \subseteq V(G_i)$, we denote $\mathsf{heir}_G(V) = V$.

Figure 6.1 shows the result of the operation $\oplus$ on two graphs. Boundaries are drawn in gray and their labels are written next to them. The graphs $\mathbf{G}_1$ and $\mathbf{G}_2$ on this picture are compatible as $\Lambda(\mathbf{G}_1) = \Lambda(\mathbf{G}_2) = \{0, 1, 2, 3\}$.

For every $t \in \mathbb{N}_{\geq 1}$, we denote by $\mathcal{B}_t$ all boundaried graphs whose boundary is labeled by numbers in $[\![1, t]\!]$. Given a boundaried graph $\mathbf{G} = (G, B, \lambda)$ and a subset S of $V(G)$ such that all vertices in S have degree 2 in G, we define $\mathsf{diss}(\mathbf{G}, S)$ as the graph $\hat{\mathbf{G}} = (\hat{G}, B, \lambda)$ where $\hat{G} = \mathsf{diss}(G, S)$.

Let W be a graph and S be a non-empty subset of $V(W)$. An *S-splitting* of W is a pair $(\mathbf{G}_S, \mathbf{G}_{S^{\mathsf{c}}})$ consisting of two boundaried graphs $\mathbf{G}_S = (G_S, B_S, \lambda_S)$ and $\mathbf{G}_{S^{\mathsf{c}}} = (G_{S^{\mathsf{s}}}, B_{S^{\mathsf{s}}}, \lambda_{S^{\mathsf{s}}})$ that can be obtained as follows: First, let W^+ be the graph obtained by subdividing in W every edge with one endpoint in S and the other in $V(W) \setminus S$ and let B be the set of created vertices. Let λ be any bijection from B to a subset of $\mathbb{N}_{\geq 1}$. Then $G_S = W^+[S \cup B]$, $G_{S^{\mathsf{c}}} = W^+ \setminus S$, $B_S = B_{S^{\mathsf{c}}} = B$, and $\lambda_S = \lambda_{S^{\mathsf{c}}} = \lambda$. Notice that there are infinite such pairs, depending on the numbers that are assigned to the boundaries of

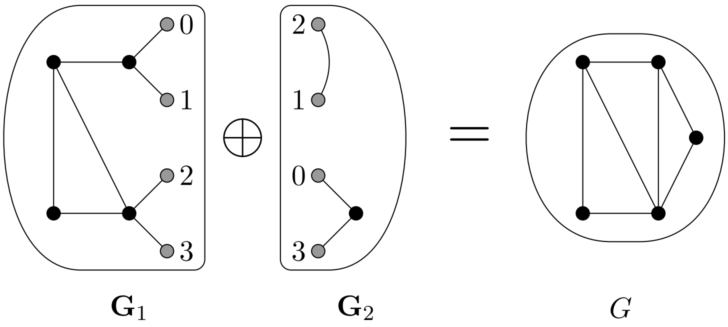


Figure 6.1: Gluing graphs together: $G = \mathbf{G}_1 \oplus \mathbf{G}_2$.

$\mathbf{G}_S$ and $\mathbf{G}_{S^c}$. Moreover, keep in mind that all the boundary edges of G_S are non-loop edges with exactly one endpoint in B and the same holds for the boundary edges of G_{S^s}. An example of a splitting is given in Figure 6.2, where boundaries are depicted by gray vertices.

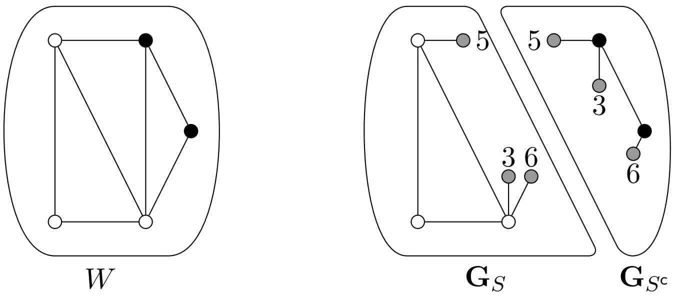


Figure 6.2: Cutting a graph: $(\mathbf{G}_S, \mathbf{G}_{S^c})$ is an S-splitting of W, where S consists of all the white vertices.

We say that $\mathbf{G}' = (G', B', \lambda')$ is a *boundaried subgraph* of $\mathbf{G} = (G, B, \lambda)$ if G' is a subgraph of G, $B' \subseteq B$ and $\lambda' = \lambda_{|B'}$. On the other hand, $\mathbf{G}$ is a subgraph of a (non-boundaried) graph H if $G = \mathbf{H}_S$ for some S-splitting $(\mathbf{H}_S, \mathbf{H}_{S^c})$, where $S \subseteq V(H)$.

If H is a graph, G is a subgraph of H, and $\mathbf{F} = (F, B, \lambda)$ is a boundaried subgraph of H, we define $G \cap_H \mathbf{F}$ as follows. Let $S = V(G) \cap (V(F) \setminus B)$ and let G^+ be the graph obtained by subdividing once every edge of G that has one endpoint in S and the other in $V(G) \setminus S$. We call B' the set of created vertices and let $G' = G^+[S \cup B']$. Then G' is a subgraph of F where $B' \subseteq B$. For every $v \in B'$, we set $\lambda'(v) = \lambda(v)$, which is allowed according to the previous remark. Then $G \cap_H \mathbf{F} = (G', B', \lambda')$. Observe that $G \cap_H \mathbf{F}$ is an S-splitting of G.

Given two boundaried graphs $\mathbf{G}' = (G', B', \lambda')$ and $\mathbf{G} = (G, B, \lambda)$, we say that they are *isomorphic* if there is an isomorphism from G' to G that respects the labelings of B and B', i.e., maps every vertex $x \in B'$ to $\lambda^{-1}(\lambda'(x)) \in B$. Given a boundaried graph $\mathbf{G} = (G, B, \lambda)$, we denote $n(\mathbf{G}) = n(G) - |B|$ and $m(\mathbf{G}) = m(G)$.

Given a boundaried graph $\mathbf{G} = (G, B, \lambda)$ and an $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, we set $A_{\mathsf{x}}(\mathbf{G}) = V(G) \setminus B$ or $A_{\mathsf{x}}(\mathbf{G}) = E(G)$, depending on whether $\mathsf{x} = \mathsf{v}$ or $\mathsf{x} = \mathsf{e}$.

Partial structures. Let $\mathcal{F}$ be a finite family of connected graphs. A boundaried subgraph $\mathbf{J}$ of a boundaried graph $\mathbf{G}$ is a *partial $\mathcal{F}$-subdivision* if there is a boundaried graph $\mathbf{H}$ which is compatible with $\mathbf{G}$ and a boundaried subgraph $\mathbf{J}'$ of $\mathbf{H}$ which is compatible with $\mathbf{J}$ such that $\mathbf{J} \oplus \mathbf{J}'$ is an $\mathcal{F}$-subdivision of $\mathbf{G} \oplus \mathbf{H}$. Intuitively, this means that $\mathbf{J}$ can be extended into an $\mathcal{F}$-subdivision in some larger graph. In this case, the $\mathcal{F}$-subdivision $\mathbf{J} \oplus \mathbf{J}'$ is said to be an *extension* of $\mathbf{J}$.

Similarly, for every $p \in \mathbb{N}_{\geq 1}$, a collection of boundaried subgraphs $\mathcal{J} = \{\mathbf{J}_1, \ldots, \mathbf{J}_p\}$ of a graph $\mathbf{G}$ is a *partial* x-*$\mathcal{F}$-packing* if there is a boundaried graph $\mathbf{H}$ which is compatible with $\mathbf{G}$ and a collection of boundaried subgraphs $\{\mathbf{J}'_1, \ldots, \mathbf{J}'_p\}$ of $\mathbf{H}$ such that $\{\mathbf{J}_1 \oplus \mathbf{J}'_1, \ldots, \mathbf{J}_p \oplus \mathbf{J}'_p\}$ is an x-$\mathcal{F}$-packing of $\mathbf{G} \oplus \mathbf{H}$. The obtained packing is said to be an *extension* of the partial packing $\mathcal{J}$. A partial packing is *$\mathcal{T}(\mathcal{F})$-free* if none of its members is an F-subdivision for some $F \in \mathcal{F}$. Observe that since every graph in $\mathcal{F}$ is connected, every partial subdivision of an $\mathcal{T}(\mathcal{F})$-free partial packing in $\mathbf{G}$ must contain at least one boundary vertex of $\mathbf{G}$.

Partitions and protrusions. In order to decompose graphs along edge cuts, we introduce the following edge-counterpart of the notion of *(vertex) protrusion* introduced in [BFL+09a, BFL+09b].

Given a rooted tree-partition $\mathcal{D} = (T, s, \mathcal{X})$ of G and a vertex $i \in V(T)$, we define

$$T_i = T[\mathrm{desc}_{T,s}(i)], \quad V_i = \bigcup_{h \in V(T_i)} X_h, \quad \text{and} \quad G_i = G[V_i].$$

Let W be a graph and $t \in \mathbb{N}_{\geq 1}$. A pair $\mathbf{P} = (\mathbf{G}, \mathcal{D})$ is a *t-partitioned protrusion* of W if there exists an $S \subseteq V(W)$ such that

- $\mathbf{G} = (G, B, \lambda)$ is a boundaried graph where $\mathbf{G} \in \mathcal{B}_t$ and $\mathbf{G} = \mathbf{G}_S$ for some S-splitting $(\mathbf{G}_S, \mathbf{G}_{S^c})$ of W and
- $\mathcal{D} = (T, s, \{X_u\}_{u \in V(T)})$ is a rooted tree-partition of $G \setminus B$ of width at most t, where X_s is the set of neighbors in G of the vertices in B.

Given a family $\mathcal{F}$ of graphs, we say that a t-partitioned protrusion $(\mathbf{G}, \mathcal{D})$ of a graph W is *$\mathcal{F}$-free* if $\mathbf{G}$ is $\mathcal{F}$-free. For every vertex $u \in V(T)$, we also define the *t-partitioned protrusion* $\mathbf{P}_u$ *of* W as a pair $\mathbf{P}_u = (\mathbf{G}_u, \mathcal{D}_u)$, where $\mathcal{D}_u = (T_u, u, \{X_v\}_{v \in V_u})$ and $\mathbf{G}_u = \mathbf{G}_{V_u}$ for some V_u-splitting $(\mathbf{G}_{V_u}, \mathbf{G}_{V_u^c})$ of W. We choose the labeling function of $\mathbf{G}_s$ so that it is the same as the one of $\mathbf{G}$, i.e., $\mathbf{G}_s = \mathbf{G}$. Notice that the labelings of all other $\mathbf{G}_u$'s are arbitrary. For every $u \in V(T)$ we define

$$\mathcal{G}_u = \{\mathbf{G}_l\}_{l \in \mathrm{children}_{(T,s)}(u)}.$$

6.2.3 Encodings, signatures, and folios

In this section we introduce tools that we will use to sort boundaried graphs depending on the subdivisions that are realizable inside.

Encodings. Let $\mathcal{H}$ be a family of graphs, let $t \in \mathbb{N}_{\geq 1}$, and let $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$. If $\mathbf{G} = (G, B, \lambda) \in \mathcal{B}_t$ is a boundaried graph and $S \subseteq A_{\mathsf{x}}(\mathbf{G})$, we define $\mathsf{pp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S)$ (pp as the initials of *partial packings*) as the collection of all sets $\{(\mathbf{J}_1, L_1), \ldots, (\mathbf{J}_\sigma, L_\sigma)\}$ such that

(i) $\{\mathbf{J}_1, \ldots, \mathbf{J}_\sigma\}$ is a partial x-$\mathcal{T}(\mathcal{H})$-packing of $G \setminus S$ of size σ and

(ii) $L_i = V(\hat{M}_i) \cap V(G)$, where M_i is an extension of J_i, for every $i \in [\![1, \sigma]\!]$.

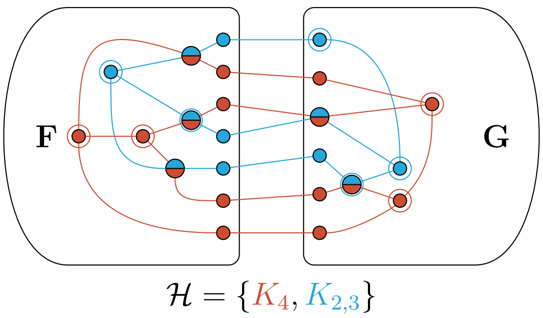


Figure 6.3: An e-$\mathcal{T}(H)$-packing in $\mathbf{G}$. Branch vertices are circled.

In other words, L_i contains branch vertices of the partial $\mathcal{H}$-subdivision $\mathbf{J}_i$ for every $i \in [\![1, \sigma]\!]$ (see Figure 6.3 and Figure 6.4). The set $\mathsf{pp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S)$ encodes all different restrictions in $\mathbf{G}$ of partial x-$\mathcal{H}$-packings that avoid the set S. Given a boundaried graph $\mathbf{G} = (G, B, \lambda)$ and a set $L \subseteq V(G)$ such that every vertex of $V(G) \setminus L$ has degree 2 in G, we define $\kappa(\mathbf{G}, L)$ as the boundaried graph obtained from $\mathbf{G}$ by dissolving every vertex of $V(G) \setminus L$, i.e., $\kappa(\mathbf{G}, L) = (\mathsf{diss}(\mathbf{G}, V(G) \setminus L), B, \lambda)$. In the definition of κ we assume that the boundary vertices of $\kappa(\mathbf{G}, L)$ remain the same as in $\mathbf{G}$ while the other vertices are treated as new vertices (see Figure 6.5).

This allows us to introduce the following notation aimed at representing, intuitively, the essential part of each partial packing:

$$\begin{aligned} \mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S) \;=\; & \{\hat{\mathcal{J}} = \{\hat{\mathbf{J}}_1, \ldots, \hat{\mathbf{J}}_\sigma\} = \{\kappa(\mathbf{J}_1, L_1), \ldots, \kappa(\mathbf{J}_\sigma, L_\sigma)\} \mid \\ & \{(\mathbf{J}_1, L_1), \ldots, (\mathbf{J}_\sigma, L_\sigma)\} \in \mathsf{pp}_{\mathcal{H}}(\mathbf{G}, S)\} \end{aligned}$$

(here, cpp is mnemonic for *compressed partial packings*).

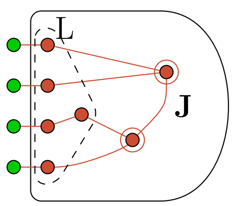


Figure 6.4: A partial subdivision from the packing of Figure 6.3, where L is the set of subdivision vertices.

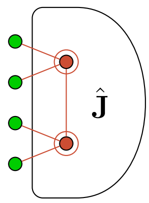


Figure 6.5: The *compression* of the partial packing of Figure 6.4: $\hat{\mathbf{J}} = \kappa(\mathbf{J}, L)$.

Isomorphisms. If $\mathbf{G} = (G, B, \lambda)$ and $\mathbf{G}' = (G', B', \lambda')$ are two compatible boundaried graphs in $\mathcal{B}_t$, $S \in V(G)$, and $S' \in V(G')$, we say that a member $\hat{\mathcal{J}}$ of $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S)$ and a member $\hat{\mathcal{J}}'$ of $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}', S')$ are *isomorphic* if there is a bijection between them such that paired elements are isomorphic. We also say that $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S)$ and $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}', S')$ are *isomorphic* if there is a bijection between them such that paired elements are isomorphic.

We now come to the point where we can define, for every boundaried graph, a signature encoding all the possible partial packings that can be realized in this graph.

Signatures and folios. For every $y \in \mathbb{N}$, we set

$$\mathsf{sig}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, y) \quad = \quad \{\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S),\ S \subseteq A_{\mathsf{x}}(G),\ |S| = y\}$$

and, given two compatible boundaried graphs $\mathbf{G}, \mathbf{G}' \in \mathcal{B}_t$ and a $y \in \mathbb{N}$, we say that $\mathsf{sig}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, y)$ and $\mathsf{sig}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}', y)$ are *isomorphic* if there is a bijection between them such that paired elements are isomorphic.

Finally, for $\rho \in \mathbb{N}$, we set

$$\mathsf{folio}_{\mathcal{H},\rho}(\mathbf{G}) \quad = \quad (\mathsf{sig}^{\mathsf{v}}_{\mathcal{H}}(\mathbf{G}, 0), \ldots, \mathsf{sig}^{\mathsf{v}}_{\mathcal{H}}(\mathbf{G}, \rho), \mathsf{sig}^{\mathsf{e}}_{\mathcal{H}}(\mathbf{G}, 0), \ldots, \mathsf{sig}^{\mathsf{e}}_{\mathcal{H}}(\mathbf{G}, \rho)).$$

Given two $\mathbf{G}, \mathbf{G}' \in \mathcal{B}_t$, a $\rho \in \mathbb{N}$, and a finite collection of connected graphs $\mathcal{H}$, we say that $\mathbf{G} \simeq_{\mathcal{H},\rho} \mathbf{G}'$ if $\mathbf{G}$ and $\mathbf{G}'$ are compatible, neither $\mathbf{G}$ nor $\mathbf{G}'$ contains an $\mathcal{H}$-subdivision, and the elements of $\mathsf{folio}_{\mathcal{H},\rho}(\mathbf{G})$ and $\mathsf{folio}_{\mathcal{H},\rho}(\mathbf{G}')$ are position-wise isomorphic.

6.3 The reduction

Let $\mathcal{H}$ be a finite set of connected graphs. In this section we show that one can, in linear time, either find a *small* $\mathcal{H}$-subdivision in a t-partitionned protrusion, or *reduce* it so that the parameters of packing and covering (wrt. $\mathcal{H}$-subdivisions) remain unchanged. More formally, the purpose of this section is to prove the following lemma.

Lemma 6.1. *There exists a function $f_6 \colon \mathbb{N}^2 \to \mathbb{N}$ and an algorithm that, given a positive integer t, a finite collection $\mathcal{H}$ of connected graphs where $h = m(\mathcal{H})$, and a t-partitioned protrusion $\mathbf{P} = (\mathbf{G}, (T, s, \mathcal{X}))$ of a graph W with $n(\mathbf{G}) > f_6(h, t)$, outputs either*

- *an $\mathcal{H}$-subdivision of W with at most $f_6(h, t)$ edges, or*

- *a graph W' such that*

$$\begin{aligned}\textsf{x-pack}_{\mathcal{T}(\mathcal{H})}(W') &= \textsf{x-pack}_{\mathcal{T}(\mathcal{H})}(W),\\ \textsf{x-cover}_{\mathcal{T}(\mathcal{H})}(W') &= \textsf{x-cover}_{\mathcal{T}(\mathcal{H})}(W), \text{ and}\\ n(W') &< n(W).\end{aligned}$$

Furthermore, this algorithm runs in time $O_{t,h}(n(T))$.

Before giving the proof of Lemma 6.1, we need to prove several intermediate results. In the sequel, unless stated otherwise, we assume that $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, $t \in \mathbb{N}_{\geq 1}$ and that $\mathcal{H}$ is a finite collection of connected graphs. We set $h = m(\mathcal{H})$.

Lemma 6.2. *There are two functions $f_7 \colon \mathbb{N}^2 \to \mathbb{N}$ and $f_8 \colon \mathbb{N}^2 \to \mathbb{N}$ such that, for every graph W and every t-partitionned protrusion $(\mathbf{G}, (T, s, \mathcal{X}))$ of W, if $\mathcal{P}$ is an $\mathcal{H}$-free partial x-$\mathcal{H}$-packing in $\mathbf{G}$ then:*

(a) The partial subdivisions of graphs of $\mathcal{H}$ that are contained in $\mathcal{P}$ have in total at most $f_7(h,t)$ branch vertices.

(b) $\mathcal{P}$ intersects at most $f_8(h,t)$ graphs of $\mathcal{G}_s$.

Proof. Proof of (a). First, note that $\mathcal{P}$ has cardinality at most t. Indeed, since every element of $\mathcal{P}$ is a partial subdivision (because the packing is $\mathcal{T}(\mathcal{H})$-free) of a connected graph, it contains a boundary edge of $\mathbf{G}$ (which by definition has degree one). Also, two distinct partial subdivisions in $\mathcal{P}$ are (at least) edge-disjoint. Finaly, each of these partial subdivisions contains at most $\max_{H\in\mathcal{H}} n(H) \leq h$ branch vertices. Consequently, the number of branch vertices of graphs of $\mathcal{H}$ induced by the elements of $\mathcal{P}$ in $\mathbf{G}$ is at most $t \cdot h$. Hence the function $f_7(h,t) := t \cdot h$ upper-bounds the amount of branch vertices each $\mathcal{T}(\mathcal{H})$-free partial packing can contain.

Proof of (b). Let ζ be the maximum multiplicity of an edge in a graph of $\mathcal{H}$. Because of (a), $\mathcal{P}$ has at most $f_7(h,t)$ branch vertices of graphs of $\mathcal{H}$, so at most $f_7(h,t)$ graphs of $\mathcal{G}_s$ may contain such vertices. Besides, $\mathcal{P}$ might also contain paths free of branch vertices linking pairs of branch vertices. Since there are at most $(f_7(h,t))^2$ such pairs and no pair will need to be connected with more than $\zeta \leq h$ distinct paths, it follows that at most $(f_7(h,t))^2 \cdot h$ graphs of $\mathcal{G}_s$ contain vertices from these paths. Therefore, the elements of $\mathcal{P}$ intersects all together at most $f_7(h,t) + (f_7(h,t))^2 \cdot h =: f_8(h,t)$ graphs of $\mathcal{G}_s$. ☐

Lemma 6.3. *The size of the image of the function $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}$, when its domain is restricted to*

$$\{(\mathbf{G}, S),\ \mathbf{G} \text{ is } \mathcal{T}(\mathcal{H})\text{-free and } S \subseteq A_{\mathsf{x}}(\mathbf{G})\},$$

is upper-bounded by a function of h and t.

Proof. Let $\mathbf{G} \in \mathcal{B}_t$ be $\mathcal{T}(\mathcal{H})$-free and let $S \subseteq A_{\mathsf{x}}(\mathcal{G})$. By Lemma 6.2(a), every $\mathcal{T}(\mathcal{H})$-free partial x-$\mathcal{H}$-packing in $\mathbf{G}$ contains at most $f_7(h,t)$ branch vertices. A partial packing may in addition use at most t boundary vertices. Let $\mathcal{C}_{h,t}$ be the class of all boundaried graphs of $\mathcal{B}_t$ on at most $f_7(h,t)+t$ vertices. Clearly the size of this class is a function depending on h and t only. Recall that the elements of the set $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S)$ are obtained from partial x-$\mathcal{H}$-packings by dissolving internal vertices of the paths linking branch vertices, hence every element of $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S)$ is a boundaried graph of $\mathcal{B}_t$ having at most $f_7(h,t)+t$ vertices. Therefore, for any $\mathcal{T}(\mathcal{H})$-free boundaried graph $\mathbf{G} \in \mathcal{B}_t$ and subset $S \subseteq A_{\mathsf{x}}(\mathbf{G})$, we have $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S) \subseteq \mathcal{C}_{h,t}$. As a consequence, the image of the function $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}$ when restricted to $\mathcal{T}(\mathcal{H})$-free boundaried graphs $\mathbf{G} \in \mathcal{B}_t$ (and subsets $S \subseteq A_{\mathsf{x}}(\mathbf{G})$) is a subset of the power set of $\mathcal{C}_{h,t}$, so its size is upper-bounded by a function that depends only on h and t. □

Corollary 6.1. *There is a function $f_9\colon \mathbb{N}^2 \to \mathbb{N}$ such that the relation $\simeq_{\mathcal{H},t}$ partitions $\mathcal{T}(\mathcal{H})$-free boundaried graphs of $\mathcal{B}_t$ into at most $f_9(h,t)$ equivalence classes.*

The following follows directly from the definition of $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}$.

Remark 6.1. Let $\mathbf{F}, \mathbf{G} \in \mathcal{B}_t$ be two compatible boundaried graphs and let $k \in \mathbb{N}$. The following are equivalent:

- $\mathbf{F} \oplus \mathbf{G}$ has an x-$\mathcal{T}(H)$-packing of size k;
- there is a $\hat{\mathcal{J}} \in \mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, \emptyset)$ such that $\mathbf{F} \oplus \mathbf{U}\hat{\mathcal{J}}$ has an x-$\mathcal{T}(H)$-packing of size k.

The choice of the definition of the relation $\simeq$ is justified by the following lemma. Informally speaking, it states that we can replace a t-partitioned protrusion of a graph with any other $\simeq_{\mathcal{H},t}$-equivalent t-partitioned protrusion without changing the covering and packing number of the graph. The reduction algorithm that we give after this lemma relies on this powerful property.

Lemma 6.4 (protrusion replacement)**.** *Let $\mathbf{F}, \mathbf{G}, \mathbf{G}' \in \mathcal{B}_t$ be three compatible boundaried graphs such that $\mathbf{G} \simeq_{\mathcal{H},t} \mathbf{G}'$. For every $k \in \mathbb{N}$, we have:*

(i) there is an x-$\mathcal{T}(H)$-packing of size k in $\mathbf{F} \oplus \mathbf{G}$ iff there is one in $\mathbf{F} \oplus \mathbf{G}'$; and

(ii) there is an x-$\mathcal{T}(H)$-cover of size k in $\mathbf{F} \oplus \mathbf{G}$ iff there is one in $\mathbf{F} \oplus \mathbf{G}'$.

Proof. Proof of (i), "⇒". Let $\mathcal{M}$ be an x-$\mathcal{T}(\mathcal{H})$-packing of size at least k in $\mathbf{F} \oplus \mathbf{G}$, whose set of branch vertices is L. We define

$$\begin{aligned}
\mathbf{J}_{\mathbf{F}} &= (\mathbf{U}\mathcal{M}) \cap_{\mathbf{F}\oplus\mathbf{G}} \mathbf{F}, \\
\mathbf{J}_{\mathbf{G}} &= (\mathbf{U}\mathcal{M}) \cap_{\mathbf{F}\oplus\mathbf{G}} \mathbf{G}, \textit{and} \\
\hat{\mathbf{J}}_{\mathbf{G}} &= \bigcup_{M \in \mathcal{M}} \kappa(M \cap_{\mathbf{F}\oplus\mathbf{G}} \mathbf{G}, L \cap V(G)).
\end{aligned}$$

Note that $\hat{\mathbf{J}}_{\mathbf{G}} \in \mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, \emptyset)$ and that $\mathbf{F} \oplus \hat{\mathbf{J}}_{\mathbf{G}}$ has an x-$\mathcal{T}(\mathcal{H})$-packing of size at least k (cf. Remark 6.1). By definition of $\simeq$, there is a bijection ψ between $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, \emptyset)$ and $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}', \emptyset)$. Let $\hat{\mathbf{J}}'_{\mathbf{G}}$ be the image of $\hat{\mathbf{J}}_{\mathbf{G}}$ by ψ. Since $\hat{\mathbf{J}}'_{\mathbf{G}}$ and $\hat{\mathbf{J}}_{\mathbf{G}}$ are isomorphic, $\mathbf{F} \oplus \hat{\mathbf{J}}'_{\mathbf{G}}$ also has an x-$\mathcal{T}(\mathcal{H})$-packing of size at least k. By Remark 6.1, this implies that such a packing exists in $\mathbf{F} \oplus \mathbf{G}'$ as well. The direction "⇐" is symmetric as $\mathbf{G}$ and $\mathbf{G}'$ play the same role.

Proof of (ii), "$\Rightarrow$". Let $C \subseteq A_{\mathsf{x}}(\mathbf{F} \oplus \mathbf{G})$ be a minimum x-$\mathcal{H}$-covering of $\mathbf{F} \oplus \mathbf{G}$ of size at most k. Let $S = C \cap A_{\mathsf{x}}(\mathbf{G})$. Since we assume that $\mathbf{G}$ is $\mathcal{T}(\mathcal{H})$-free and that C is minimum, we can also assume that $|S| \leq t$ (otherwise we could get a smaller covering by taking the t boundary vertices/edges of $\mathbf{G}$). By our assumption that $\mathbf{G} \simeq_{\mathcal{H},t} \mathbf{G}'$, there is an isomorphism between $\mathsf{sig}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, |S|)$ and $\mathsf{sig}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}', |S|)$. Let $S' \subseteq A_{\mathsf{x}}(\mathbf{G}')$ be a set such that $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S)$ is sent to $\mathsf{cpp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}', S')$ by this isomorphism. Then observe that every partial packing $\mathcal{J}'$ of $\mathbf{G}' \setminus S'$, such that $(\mathbf{F} \setminus C) \oplus (\bigcup \mathcal{J}')$ has an $\mathcal{H}$-subdivision, can be translated into a partial packing $\mathcal{J}$ of $\mathbf{G} \setminus S$ such that $(\mathbf{F} \setminus C) \oplus (\bigcup \mathcal{J})$ also has such a subdivision, in the same way as in the proof of (i) above. As C is a cover, this would lead to contradiction. Therefore $\mathsf{pp}^{\mathsf{x}}_{\mathcal{H}}(\mathbf{G}, S)$ does not contain such a partial packing. As a consequence, $C \cap A_{\mathsf{x}}(\mathbf{F}) \cup S'$ is a covering of $\mathbf{F} \oplus \mathbf{G}'$ of size at most k. As in the previous case, the proof of direction "$\Leftarrow$" comes from the symmetry in the statement. $\square$

Lemma 6.4 can be rewritten as follows.

Corollary 6.2. *Under the assumptions of Lemma 6.4, we have* $\mathsf{x\text{-}pack}(\mathbf{F} \oplus \mathbf{G}) = \mathsf{x\text{-}pack}(\mathbf{F} \oplus \mathbf{G}')$ *and* $\mathsf{x\text{-}cover}(\mathbf{F} \oplus \mathbf{G}) = \mathsf{x\text{-}cover}(\mathbf{F} \oplus \mathbf{G}')$.

Recall that $f_9(h,t)$ denotes the number of equivalence classes of $\simeq_{\mathcal{H},t}$ among boundaried graphs of $\mathcal{B}_t$. For every $h,t \in \mathbb{N}$, let $f_{10}(h,t) = f_9(h,t) \cdot f_8(h,t)$ and let and $f_6(h,t) = 2ht^3 \cdot (f_{10}(h,t))^{f_9(h,t)+1}$. Let us give some intuition about these definitions. The first remark is an application of the pigeonhole principle.

Remark 6.2. In a collection of more than $f_{10}(h,t)$ $\mathcal{T}(\mathcal{H})$-free boundaried graphs of $\mathcal{B}_t$, there is one that is equivalent (w.r.t. $\simeq_{\mathcal{H},t}$) to $f_8(h,t)$ other graphs of the collection.

Lemma 6.5. *If $(T, s, \mathcal{X})$ is a rooted tree-partition of a graph G with the following properties:*

- *$(T, \mathcal{X})$ has width at most t;*
- *T has height at most $f_9(h,t)$; and*
- *T has degree at most $f_{10}(h,t) + 1$,*

then G has at most $f_6(h,t)$ vertices, and every $\mathcal{H}$-subdivision of G has at most $f_6(h,t)$ edges.

Proof. Indeed, the above assumptions imply that T has at most $(f_{10}(h,t))^{f_9(h,t)+1}$ vertices. Every bag of $(T, \mathcal{X})$ contains at most t vertices of G, therefore G has at most $(f_{10}(h,t))^{f_9(h,t)+1} \cdot t \leq f_6(h,t)$ vertices. Also, every bag induces a subgraph with at most $t(t-1)/2$ multiedges (i.e. without counting multiplicities), and for every edge f of T we have $|E_f| \leq t$, hence every bag contributes for at most $t^2 + t$ multiedges. Therefore G has at most $(t+t^2)\,(f_{10}(h,t))^{f_9(h,t)+1}$ multiedges. Now, observe that a multiedge of G is used at most h times by an $\mathcal{H}$-subdivision, since every path connecting two branch vertices of a subdivision uses a given multiedge at most once. We deduce that an $\mathcal{H}$-subdivision of G contains at most $h \cdot (t+t^2)\,(f_{10}(h,t))^{f_9(h,t)+1} \leq f_6(h,t)$ edges. $\square$

The two next lemmas are the main tools used in the proof of Lemma 6.1. Under different conditions, they provide either a small subdivision, or a reduced graph. Lemma 6.6 considers the case where a vertex of T has high degree, whereas Lemma 6.7 deals with the situation where T has a long path.

Lemma 6.6. *There is an algorithm that, given a t-partitioned protrusion $\mathbf{P} = (\mathbf{G}, (T, s, \mathcal{X}))$, and a vertex $u \in V(T)$ with more than $f_{10}(h,t)$ children such that for every $v \in \mathrm{children}_{(T,s)}(u)$, we have $m(\mathbf{G}_v) \leq f_6(h,t)$, outputs either*

- *an $\mathcal{H}$-subdivision with at most $f_6(h,t)$ edges, or*
- *a graph W' such that*

$$\begin{aligned} \text{x-pack}_{\mathcal{T}(\mathcal{H})}(W') &= \text{x-pack}_{\mathcal{T}(\mathcal{H})}(W), \\ \text{x-cover}_{\mathcal{T}(\mathcal{H})}(W') &= \text{x-cover}_{\mathcal{T}(\mathcal{H})}(W), \text{ and} \\ n(W') &< n(W). \end{aligned}$$

Moreover, this algorithm runs in $O_{h,t}(1)$ steps.

Proof. If $\mathbf{G}_v$ contains an $\mathcal{H}$-subdivision for some child v of u, then this subdivision has at most $f_6(h,t)$ edges and we are done. Therefore we now consider the case where $\mathbf{G}_v$ is $\mathcal{H}$-free for every child v of u. This allows us to consider the folios of these boundaried graphs.

As u has more than $f_{10}(h,t)$ children, it contains a collection of $d = f_8(h,t) + 1$ children $v_1, \ldots, v_d$, such that $\mathbf{G}_{v_1} \simeq_{\mathcal{H},t} \mathbf{G}_{v_i}$ for every $i \in [\![2, d]\!]$ (by Remark 6.2). Since every x-$\mathcal{T}(\mathcal{H})$-packing of W will intersect at most $f_8(h,t)$ bags of children of u (by Lemma 6.2(b)), we can safely delete one of the $f_8(h,t) + 1$ equivalent subgraphs mentioned above.

We use the following procedure in order to identify such a subgraph to delete or a small $\mathcal{H}$-subdivision;

1. let A be an array of $f_9(h,t)$ counters initialized to 0, each corresponding to a distinct equivalence class of $\simeq_{\mathcal{H},t}$;
2. pick a vertex $v \in \mathrm{children}_{(T,s)}(u)$ that has not been considered yet;
3. if G_v contains an $\mathcal{H}$-subdivision M, then return M and exit;
4. otherwise, increment the counter of A corresponding to the equivalence class of $\mathbf{G}_v$ by one;
5. if this counter reaches $d + 1$, return v, otherwise go back to line 2.

Notice that the subdivision returned in line 3 has size at most $f_6(h,t)$ as mentioned above, and that the vertex returned in line 5 has the desired property. The relation $\simeq_{\mathcal{H},t}$ has at most $f_9(h,t)$ equivalence classes (Corollary 6.1), thus the main loop will be run at most $f_9(h,t) \cdot f_8(h,t) + 1$ times (by the pigeonhole principle). Eventually, lines 3 and 4 can be performed in $O_{h,t}(1)$-time given that G_v has size bounded by a function of h and t.

In the end, we return $W' = W \setminus V(G_v)$ if the algorithm above outputs v and M otherwise. $\square$

Lemma 6.7. *There is an algorithm that, given a t-partitioned protrusion $\mathbf{P} = (\mathbf{G}, (T, s, \mathcal{X}))$ of a graph W and a vertex $u \in V(T)$ of height $f_9(h,t)$ in (T,s) such that T_u has maximum degree at most $f_{10}(h,t)+1$, outputs either*

- *an $\mathcal{H}$-subdivision with at most $f_6(h,t)$ edges, or*
- *a graph W' such that*

$$\begin{aligned} \text{x-pack}_{\mathcal{T}(\mathcal{H})}(W') &= \text{x-pack}_{\mathcal{T}(\mathcal{H})}(W), \\ \text{x-cover}_{\mathcal{T}(\mathcal{H})}(W') &= \text{x-cover}_{\mathcal{T}(\mathcal{H})}(W), \quad \textit{and} \\ n(W') &< n(W). \end{aligned}$$

Moreover, this algorithm runs in $O_{h,t}(1)$-time.

Proof. As in the proof of Lemma 6.5, we use the fact that every $\mathcal{H}$-subdivision of $\mathbf{G}_u$ uses every multiedge at most h times. A consequence is that the boundaried subgraph of $\mathbf{G}_u$ obtained by setting the multiplicity of every multiedge e to $\min(\text{mult}(e), h)$ contains an $\mathcal{H}$-subdivision iff $\mathbf{G}_u$ does. As the number of vertices and edges of this subgraph is bounded by a function of h and t, we can therefore check in $O_{h,t}(1)$-time if $\mathbf{G}_u$ contains an $\mathcal{H}$-subdivision. If one is found, it has at most $f_6(h,t)$ edges (Lemma 6.5) and we are done.

Let us now consider the case where $\mathbf{G}_u$ is $\mathcal{T}(\mathcal{H})$-free. By definition of vertex u, there is a path on $f_9(h,t)+1$ vertices from a leaf of T_u to u. Let us arbitrarily choose, for every vertex v of this path, a V_v-splitting $(\mathbf{G}_{G_v}, \mathbf{G}_{G_v^c})$ of $\mathbf{G}$. By definition of $f_9(h,t)$ (the number of equivalence classes in $\simeq_{\mathcal{H},t}$ in $\mathcal{B}_t$), there are two distinct vertices v, w on this path such that $\mathbf{G}_v \simeq_{\mathcal{H},t} \mathbf{G}_w$. As mentioned above, the number edges of $\mathbf{G}_u$ is bounded by a function of h and t, hence finding these two vertices can be done in $O_{h,t}(1)$-time. Let us assume without loss of generality that s is closer to v than w. Let $\mathbf{H}$ be the boundaried graph such that $W = \mathbf{H} \oplus \mathbf{G}_{G_v}$ and let $W' = \mathbf{H} \oplus \mathbf{G}_{G_w}$. By Corollary 6.2, we have $\text{x-pack}_{\mathcal{T}(\mathcal{H})}(W') = \text{x-pack}_{\mathcal{T}(\mathcal{H})}(W)$ and $\text{x-cover}_{\mathcal{T}(\mathcal{H})}(W') = \text{x-cover}_{\mathcal{T}(\mathcal{H})}(W)$. Furthermore, the graph W' is clearly smaller than W. $\square$

We are now ready to prove Lemma 6.1.

Proof of Lemma 6.1. Let us consider the following procedure:

1. by a DFS on (T,s), compute the height of each vertex of T and find (if it exists) a vertex v of degree more than $f_{10}(h,t)+1$ and height at most $f_9(h,t)-1$ that has minimum height;
2. if such a vertex v is found, then apply the algorithm of Lemma 6.6 on $\mathbf{P}$ and v, and return the obtained result;

3. otherwise, find a vertex u of height exactly $f_9(h,t)$ in (T,s) and then apply the algorithm of Lemma 6.7 on $\mathbf{P}$ and (T_u,u) and return the obtained result.

Observe that since $n(G) > f_6(h,t)$, Lemma 6.5 implies that either T has diameter more than $f_9(h,t)$, or it contains a vertex of degree more than $f_{10}(h,t)+1$. Therefore, the vertex u of line 3 always exists in the case where no vertex of high degree is found in line 1. The correctness of this algorithm follows from Lemma 6.6 and Lemma 6.7. The DFS done on line 1 takes time $O(n(T))$ and the rest of the algorithm takes time $O_{h,t}(1)$ according to the aforementioned lemmas. □

6.4 Approximation meets the Erdős–Pósa property

We show in this section an application of the results of Section 6.3. Given a graph H, we denote by $\mathsf{ex}(H)$ the set of all graphs that contain H as a minor, and, subject to this condition, are minimal for the topological minor relation. Whereas Lemma 6.1 is stated in terms of subdivisions, we translate to the setting of majors using the following remark.

Remark 6.3. Recall that $\mathcal{M}(H)$ is defined as the set of subgraph-mininal graphs containing H as a minor. For every graph H, the following holds:

$$\mathcal{M}(H) = \mathcal{T}(\mathsf{ex}(H)).$$

Also, the size of $\mathsf{ex}(H)$ is upper-bounded by some function of $m(H)$.

Applied to Lemma 6.1, the above remark yields the next corollary.

Corollary 6.3. *There is an algorithm that, given a positive integer t, a finite collection $\mathcal{H}$ of connected graphs where $h = m(\mathcal{H})$, and a t-partitioned protrusion $\mathbf{P} = (\mathbf{G},(T,s,\mathcal{X}))$ of a graph W with $n(\mathbf{G}) > f_6(h,t)$, outputs either*

- *an $\mathcal{H}$-major of W with at most $f_6(h,t)$ edges, or*
- *a graph W' such that*

$$\begin{aligned} \mathsf{x\text{-}pack}_{\mathcal{M}(\mathcal{H})}(W') &= \mathsf{x\text{-}pack}_{M(\mathcal{H})}(W), \\ \mathsf{x\text{-}cover}_{\mathcal{M}(\mathcal{H})}(W') &= \mathsf{x\text{-}cover}_{\mathcal{M}(\mathcal{H})}(W), \ \textit{and} \\ n(W') &< n(W). \end{aligned}$$

Furthermore, this algorithm runs in time $O_{t,h}(n(T))$.

Let us restate the main result of Section 4.1.3, that we will use in the subsequent proofs.

Theorem 6.3 (Restatement of Theorem 4.4)**.** *There is an algorithm that, with input three positive integers r, w, z and a graph W, outputs one of the following:*

- *a θ_r-major of W with at most z edges,*

- *a $(2r-2)$-partitioned protrusion $(\mathbf{G}, \mathcal{D})$ of W, where $\mathbf{G} = (G, B, \lambda)$ and such that G is a connected graph and $n(\mathbf{G}) > w$, or*

- *an H-major of W for some graph H with $\delta(H) \geq \frac{1}{r-1} 2^{\frac{z-5r}{4r(2w+1)}}$,*

in $O_r(m)$ steps.

6.4.1 Reduce or progress

The proof of the next lemma combines Theorem 4.4 and Corollary 6.3.

Lemma 6.8 (reduce or progress). *There is an algorithm that, with input $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, $r \in \mathbb{N}_{\geq 2}$, $k \in \mathbb{N}$ and an n-vertex graph W, outputs one of the following:*

- *a θ_r-major of W with at most $O_r(\log k)$ edges;*

- *a graph W' where*

$$\begin{aligned}\mathsf{x}\text{-}\mathsf{cover}_{\mathcal{M}(\theta_r)}(W') &= \mathsf{x}\text{-}\mathsf{cover}_{\mathcal{M}(\theta_r)}(W),\\ \mathsf{x}\text{-}\mathsf{pack}_{\mathcal{M}(\theta_r)}(W') &= \mathsf{x}\text{-}\mathsf{pack}_{\mathcal{M}(\theta_r)}(W), \ \ \textit{and}\\ n(W') &< n(W); \textit{or}\end{aligned}$$

- *an H-major in W, for some graph H with $\delta(H) \geq k(r+1)$,*

in $O_r(m)$ steps.

Proof. We set $t = 2r-2$, $w = f_6(h,t)$, $z = 2r(w-1)\log(k(r+1)(r-1)) + 5r$, and $h = m(\mathcal{M}(\theta_r))$. Observe that $z = O_r(\log k)$ and $h, t, w = O_r(1)$. Also observe that our choice for variable z ensures that $2^{\frac{z-5r}{2r(w-1)}}/(r-1) = k(r+1)$.

By applying the algorithm of Theorem 4.4 to r, w, z, and W, we obtain in $O_r(m(W))$-time either:

First case: a θ_r-major in W of at most z edges,

Second case: a $(2r-2)$-edge-protrusion Y of W with extension $> w$, or

Third case: an H-major M in W, for some graph H with $\delta(H) \geq k(r+1)$.

In the first case, we return the obtained θ_r-major.

In the second case, by applying the algorithm of Corollary 6.3 on Y, we get in $O(n(W))$-time either a θ_r-major of W on at most $w = O_r(1)$ vertices, or a graph W' where, for $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, $\mathsf{x}\text{-}\mathsf{cover}_{\mathcal{H}}(W') = \mathsf{x}\text{-}\mathsf{cover}_{\mathcal{H}}(W)$, $\mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(W') = \mathsf{x}\text{-}\mathsf{pack}_{\mathcal{H}}(W)$ and $n(W') < n(W)$.

In the third case, we return the major M.

In each of the above cases, we get after $O(m)$ steps either a major of a graph with *large* minimum degree, or a *small* θ_r-major in W, or an *equivalent* graph that has less vertices. $\square$

It might not be clear yet to what purpose the major of a graph of degree more than $k(r+1)$ output by the algorithm of Lemma 6.8 can be used. Recall that we presented in Subsection 4.2.2 an algorithm that finds a large packing of θ_r-majors in a graph of large minimum degree. Applying this algorithm to the graph output by Lemma 6.8 gives the desired packing.

6.4.2 Approximation algorithms

Theorem 6.1 is a direct combinatorial consequence of the following.

Theorem 6.4. *There is a function $f_{11}\colon \mathbb{N} \to \mathbb{N}$ and an algorithm that, with input $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, $r \in \mathbb{N}_{\geq 2}$, $k \in \mathbb{N}$, and an n-vertex graph W, outputs either a x-$\mathcal{M}(\theta_r)$-packing of W of size k or an x-$\mathcal{M}(\theta_r)$-covering of W of size at most $f_{11}(r) \cdot k \cdot \log k$. Moreover, this algorithm runs in $O(n \cdot m)$ steps if $\mathsf{x} = \mathsf{e}$ and in $O(n^c + n \cdot m)$ steps if $\mathsf{x} = \mathsf{v}$, where c is the constant from Theorem 4.6.*

Proof. Let $f_{11}\colon \mathbb{N} \to \mathbb{N}$ be a function such that each θ_r-major output by the algorithm of Lemma 6.8 has size at most $f_{11}(r) \cdot \log k$. We consider the following procedure:

1. $G := W$; $P := \emptyset$;
2. apply the algorithm of Lemma 6.8 on (x, r, k, G):

   Progress: if the output is a θ_r-major M, let $G := G \setminus A_{\mathsf{x}}(M)$ and $P = P \cup \{M\}$;

   Win: if the output is a H-major M in W for some graph H with $\delta(H) \geq k(r+1)$, apply the algorithm of Lemma 4.9 (if $\mathsf{x} = \mathsf{e}$) or the one of Lemma 4.10 (if $\mathsf{x} = \mathsf{v}$) to H to obtain an x-$\mathcal{M}(\theta_r)$-packing of size k in H; using M, translate this packing into an x-$\mathcal{M}(\theta_r)$-packing of size k in W and return this new packing;

   Reduce: otherwise, the output is a graph G': let $G := G'$;
3. if $|P| = k$ then return P which is an x-$\mathcal{M}(\theta_r)$-packing of size k in W;
4. if $n(W) = 0$ then return P which is in this case a a x-$\mathcal{M}(\theta_r)$-covering of size at most $f_{11}(r) \log k$ of W;
5. Otherwise, go back to line 2.

This algorithm clearly returns the desired result. Furthermore, the loop is executed at most $n(W)$ times and each call to the algorithm of Lemma 6.8 takes $O(m(W))$ steps. When the algorithm reaches the "Win" case (which can happen at most once), the calls to the algorithm of Lemma 4.9 (if $\mathsf{x} = \mathsf{e}$) or the one of Lemma 4.10 (if $\mathsf{x} = \mathsf{v}$), respectively, take $O(m(H))$ and $O\left((n(H))^c\right)$ steps. Therefore, in total, this algorithm terminates in $O(n \cdot m)$ steps if $\mathsf{x} = \mathsf{e}$ and in $O\left(n^c + n \cdot m\right)$ steps if $\mathsf{x} = \mathsf{v}$. $\square$

Observe that if the algorithm of Theorem 6.4 reaches the "Win" case, then the input graph is known to contain an x-$\mathcal{M}(\theta_r)$-packing of size at least k. As a consequence, if we are only interested in the existence of a packing or covering, the call to the algorithm of Lemma 4.9 or Lemma 4.10 is not necessary. This gives a faster algorithm for the existential version of Theorem 6.4.

Corollary 6.4. *There is an algorithm that, with input* $\mathsf{x} \in \{\mathsf{v}, \mathsf{e}\}$, $r \in \mathbb{N}_{\geq 2}$, $k \in \mathbb{N}$, *and a graph* W, *outputs 0 only if* W *has an* x-$\mathcal{M}(\theta_r)$*-packing of size* k *or 1 only if* W *has an* x-$\mathcal{M}(\theta_r)$*-covering of size at most* $f_{11}(r) \cdot k \cdot \log k$. *Furthermore this algorithm runs in* $O(n \cdot m)$ *steps.*

Notice that there may be graphs where both the outputs 0 and 1 of the algorithm of Corollary 6.4 are valid. We now conclude this section with the proof of Theorem 6.2.

Proof of Theorem 6.2. Let us call A the algorithm of Corollary 6.4. Let $k_0 \in [\![1, n(W)]\!]$ be an integer such that $\mathsf{A}(\mathsf{x}, r, k_0, W) = 1$ and $\mathsf{A}(\mathsf{x}, r, k_0 - 1, W) = 0$, and let us show that the value $k_0 \log k_0$ is an $O(\log \mathsf{OPT})$-approximation of both $\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W)$ and $\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W)$.

First, notice that for every $k > \mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W)$, the value returned by $\mathsf{A}(\mathsf{x}, r, k, W)$ is 1. Symmetrically, for every k such that $k \log k < \mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W)$, the value of $\mathsf{A}(\mathsf{x}, r, k, W)$ is 0. Therefore, the value k_0 is such that:

$$k_0 - 1 \leq \mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W) \text{ and}$$
$$\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W) \leq k_0 \log k_0.$$

As every minimal covering must contain at least one vertex or edge (depending on whether $\mathsf{x} = \mathsf{v}$ or $\mathsf{x} = \mathsf{e}$) of each model of a maximal packing, $\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W) \leq \mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W)$, hence we have the following two equations:

$$\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W) \leq k_0 \log k_0 \leq (\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W) + 1) \log(\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W) + 1) \tag{6.1}$$
$$\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W) \leq k_0 \log k_0 \leq (\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W) + 1) \log(\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W) + 1). \tag{6.2}$$

Dividing (6.1) by $\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W)$ and (6.2) by $\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W)$, we get:

$$\begin{aligned} 1 \leq \frac{k_0 \log k_0}{\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W)} &\leq \log(\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W) + 1) + \frac{\log \mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W)}{\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W)} \\ &\leq c \cdot \log \mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W) \text{ for some } c \in \mathbb{N}_{\geq 1}, \text{ and} \end{aligned}$$

$$\begin{aligned} 1 \leq \frac{k_0 \log k_0}{\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W)} &\leq \log(\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W) + 1) + \frac{\log \mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W)}{\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W)} \\ &\leq c' \cdot \log \mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W) \text{ for some } c' \in \mathbb{N}_{\geq 1}. \end{aligned}$$

Therefore the value $k_0 \log k_0$ is both an $O(\log \mathsf{OPT})$-approximation of $\mathsf{x\text{-}pack}_{\mathcal{M}(\theta_r)}(W)$ and $\mathsf{x\text{-}cover}_{\mathcal{M}(\theta_r)}(W)$. The value k_0 can be found by performing a binary search in the interval $[\![1, n]\!]$, with $O(\log n)$ calls to Algorithm A. Hence, our approximation algorithm runs in $O(n \cdot \log(n) \cdot m)$ steps. □

Notice that all our results are strongly exploiting Lemma 6.1 that holds for every finite collection $\mathcal{H}$ of connected graphs. Actually, what is missing in order to have an overall generalization of all of our results, is an extension of Theorem 4.4 where $\mathcal{M}(\theta_r)$ is replaced by any finite collection $\mathcal{H}$ of connected planar graphs. This is an an interesting combinatorial problem even for particular instantiations of $\mathcal{H}$.

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