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Algebraic Legendrian varieties

PhD dissertation

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Author's declaration:
aware of legal responsibility I hereby declare that I have written this dissertation myself and all the contents of the dissertation have been obtained by legal means.

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Abstract

Real Legendrian subvarieties are classical objects of differential geometry and classical mechanics and they have been studied since antiquity (see [Arn74], [Sła91] and references therein). However, complex Legendrian subvarieties are much more rigid and have more exceptional properties. The most remarkable case is the Legendrian subvarieties of projective space and prior to the author's research only few smooth examples of these were known (see [Bry82], [LM04]). Strong restrictions on the topology of such varieties have been found and studied by Landsberg and Manivel ([LM04]).

The results of this thesis are two fold:

The first series of results is related to the automorphism group of any Legendrian subvariety in any projective contact manifold. The connected component of this group (under suitable minor assumptions) is completely determined by the sections of the distinguished line bundle on the contact manifold vanishing on the Legendrian variety. Moreover its action preserves the contact structure. The relation between the Lie algebra tangent to automorphisms and the sections is given by an explicit formula (see also [LeB95], [Bea99]). This extends the results of the author's MSc thesis [Buc03].

The second series of results is devoted to finding new examples of smooth Legendrian subvarieties of projective space. The examples known previously were some homogeneous spaces, many examples of curves and a family of surfaces birational to some K3 surfaces. The contribution of this thesis is in three steps: First we find an example of a smooth toric surface. Next we find a smooth quasihomogeneous Fano 8-fold that admits a Legendrian embedding. Finally, we realise that both of these are special cases of a very general construction: a general hyperplane section of a smooth Legendrian variety, after a suitable projection, is a smooth Legendrian variety of smaller dimension. By applying this result to known examples and decomposable Legendrian varieties, we construct infinitely many new examples in every dimension, with various Picard rank, canonical degree, Kodaira dimension and other invariants.

The original motivation for this research comes from a 50 year old problem of giving compact examples of quaternion-Kähler manifolds (see [Ber55], [LS94], [LeB95] and references therein). Also Legendrian varieties are related to some algebraic structures (see [Muk98], [LM01], [LM02]). A new potential application to classification of smooth varieties with smooth dual arises from this thesis.

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Legendrian variety, complex contact manifold, automorphism group;
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Streszczenie

Rzeczywiste rozmaitości legendrowskie stanowią standardowy przedmiot badań geometrii rózniczkowej oraz mechaniki klasycznej (zobacz [Arn74], [Sła91] oraz odnośniki tamże). W niniejszej pracy badamy ich geometro-algebraiczny odpowiednik: zespolone podrozmaitości legendrowskie zespolonych rozmaitości kontaktowych. W porównaniu z rzeczywistymi, zespolone są dużo bardziej sztywne i mają bardziej wyjątkowe własności. Najważniejszy przypadek to podrozmaitości legendrowskie w zespolonej przestrzeni rzutowej — przed badaniami autora znanych było jedynie kilka gładkich przykładów (zobacz [Bry82], [LM04]), a mocne ograniczenia dotyczące własności topologicznych takich rozmaitości zostały udowodnione przez Landsberga i Manivela [LM04].

Wyniki badań autora przedstawione w niniejszej pracy są dwojakie:

Pierwsza seria wyników jest rozszerzeniem pracy magisterskiej autora [Buc03] i dotyczy grupy automorfizmów dowolnej podrozmaitości legendrowskiej w dowolnej rzutowej rozmaitości kontaktowej. Spójna składowa jedności tej grupy (przy odpowiednich, mało istotnych założeniach) jest całkowicie wyznaczona przez te cięcia wyróżnionej wiązki liniowej na rozmaitosci kontaktowej, które znikają na ustalonej rozmaitości legendrowskiej. Co więcej, działanie tej składowej zachowuje strukturę kontaktową. Powyższy związek między cięciami a algebrą Liego grupy automorfizmów opisany jest konkretnie, przez zadany wzorem izomorfizm (zobacz także [LeB95], [Bea98]).

Pozostałe wyniki dotyczą znajdowania nowych przykładów gładkich podrozmaitości legendrowskich w przestrzeni rzutowej. Przykłady, które były znane wcześniej to pewne przestrzenie jednorodne, liczne przykłady krzywych i rodzina powierzchni biwymiernych z pewnymi powierzchniami K3. Wkład niniejszej pracy dzieli się na trzy części: Najpierw znajdujemy przykład gładkiej legendrowskiej torycznej powierzchni. Następnie znajdujemy przykład 8-wymiarowej gładkiej rozmaitości Fano. Na koniec pokazujemy, że obydwa te przykłady są szczególnymi przypadkami bardzo ogólnej konstrukcji: ogólne hiperpłaskie cięcie rozmaitości legendrowskiej, po odpowiednim zrzutowaniu, zadaje gładką rozmaitość legendrowską mniejszego wymiaru. Stosując wielokrotnie powyższe stwierdzenie do znanych przykładów oraz do rozkładalnych rozmaitości legendrowskich, otrzymujemy nieskończenie wiele nowych przykładów w każdym wymiarze. Przykłady te różnią się od siebie między innymi rangą grupy Picarda, stopniem dywizora kanonicznego oraz wymiarem Kodairy.

Inspiracją dla tej pracy jest 50-cio letni problem dotyczący skonstruowania zwartych przykładów rozmaitości kwaternionowo-kählerowskich (zobacz [Ber55], [LS94], [LeB95] oraz odnośniki tamże) oraz fakt, że rozmaitości legendrowskie są powiązane z pewnymi obiektami algebraicznymi (zobacz [Muk98], [LM01],

[LM02]). Konsekwencją tej pracy może być kolejne ich zastosowanie. Pokazujemy, że problem klasyfikacji gładkich rozmaitości o gładkiej rozmaitości dualnej jest równoważny klasyfikacji pewnych rozmaitości legendrowskich w przestrzeni rzutowej.

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Chapter I

Introduction

I.1 State of art

This thesis is devoted to study algebraic and geometric properties of Legendrian subvarieties. The main motivation for our research comes from the classification problem of contact Fano manifolds[1].

I.1.1 Contact Fano manifolds and quaternion-Kähler manifolds

Results of Demailly [Dem02] and Kebekus, Peternell, Sommese and Wiśniewski [KPSW00] prove that if Y is a complex projective contact manifold, then either Y is a Fano variety with second Betti number $b_2 = 1$ or Y is a projectivisation of the cotangent bundle to some projective manifold M.

The following conjecture would be an important classification result in algebraic geometry:

Conjecture I.1. *If Y^{2n+1} is a Fano complex contact manifold, then Y is a homogeneous variety which is the unique closed orbit of the adjoint action of a simple Lie group G on $\mathbb{P}(\mathfrak{g})$ (where the $\mathfrak{g}$ is the Lie algebra of G).*

The closed orbits appearing in the conjecture are called **adjoint varieties**.

This conjecture originated with a famous problem in Riemannian geometry. In 1955 Berger [Ber55] gave a list of all possible holonomy groups[2] of simply connected Riemannian manifolds. The existence problem for all the cases has

[1] A complex manifold Y^{2n+1} is called **a contact manifold** if there exists a rank $2n$ vector subbundle $F \subset TY$ of the tangent bundle, such that the map $F \otimes F \longrightarrow TY/F$ determined by the Lie bracket is nowhere degenerate (see chapter III for more details). A projective variety is **Fano** if its anticanonical bundle is ample.

[2] Given an m-dimensional Riemannian manifold M, **the holonomy group of** M is the subgroup of orthogonal group $\mathbf{O}(T_x M)$ generated by parallel translations along loops through x.

been solved locally. Compact non-homogeneous examples with most of the possible holonomy groups were constructed, for instance the two exceptional cases G_2 and $Spin_7$ were constructed by D. Joyce — see an excellent review on the subject [Joy00]. Since then all the cases from Berger's list have been illustrated with compact non-homogeneous examples, with the unique exception of the quaternion-Kähler manifolds[3]. Although there exist non-compact, non-homogeneous examples, it is conjectured that the compact quaternion-Kähler manifolds must be homogeneous, at least assuming positivity (see [LeB95] and references therein for an explanation of what positivity means and why it is reasonable to assume it).

Conjecture I.2 (LeBrun, Salamon). *Let M^{4n} be a positive quaternion-Kähler manifold. Then M is a homogeneous symmetric space (more precisely, it is one of the Wolf spaces — see [Wol65]).*

The relation between the two conjectures is given by the construction of a twistor space Y, an S^2-bundle of complex structures on tangent spaces to a quaternion-Kähler manifold M. If M is compact and has positive scalar curvature, then Y has a natural complex structure and is a contact Fano variety with a Kähler-Einstein metric. In particular, the twistor space of a Wolf space is an adjoint variety. Hence conjecture I.1 implies conjecture I.2. Conversely, LeBrun [LeB95] observed that if Y is a contact Fano manifold with Kähler-Einstein metric, then it is a twistor space of a quaternion-Kähler manifold.

A number of attempts have been undertaken to prove the above conjectures. They were proved in low dimension: for $n = 1$ by N. Hitchin [Hit81] and Y. Ye [Ye94], $n = 2$ by Y.S. Poon and S.M. Salamon [PS91] and S. Druel [Dru98] and conjecture I.2 for $n = 3$ by H.&R. Herrera [HH02]. Moreover A. Beauville, J. Wiśniewski, S. Kebekus, T. Peternell, A. Sommese, J.P. Demailly, C. LeBrun, J-M. Hwang and many other researchers have worked on this problem.

Let Y^{2n+1} be a contact Fano manifold not isomorphic to a projective space. Wiśniewski [Wiś00] and Kebekus [Keb01], [Keb05] have studied geometric properties of contact lines[4] and have proved that contact lines through a general point behave very much like ordinary lines in a projective space. Moreover the union of contact lines through any fixed point is a Legendrian subvariety[5] in Y. In addition, the variety X of tangent directions to such lines through a general point is a smooth Legendrian subvariety in $\mathbb{P}^{2n-1}$. If Y is one of the adjoint varieties, then X will be a homogeneous Legendrian subvariety called **a subadjoint variety** (see [LM04], [Muk98]). Proving that there is an embedding of Y into a projective space which maps contact lines to ordinary lines would imply conjecture I.1.

[3] A Riemannian $4n$-dimensional manifold M is called **quaternion-Kähler** if its holonomy group is a subgroup of $\mathbf{Sp}(1) \times \mathbf{Sp}(n)/\mathbb{Z}_2$.

[4] A rational curve $C \subset Y$ is **a contact line** if its intersection with the anticanonical bundle is minimal possible, i.e. equal to $n + 1$.

[5] A subvariety $X \subset Y$ is **Legendrian** if it is maximally F-integrable — see chapter III for the details.

Moreover it is proved by Hong [Hon00], that if X is homogeneous, then so is Y. Therefore contact lines and particularly the Legendrian varieties determined by them are important objects, useful in the study of conjecture I.1.

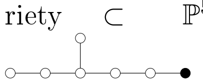
Lie group	Type	Contact manifold Y^{2n+1}	Legendrian variety X^{n-1}	Remarks
SL_{n+2}	A_{n+1}	$\mathbb{P}(T\mathbb{P}^{n+1})$	$\mathbb{P}^{n-1} \sqcup \mathbb{P}^{n-1} \subset \mathbb{P}^{2n-1}$	$b_2(Y) = 2$
Sp_{2n+2}	C_{n+1}	$\mathbb{P}^{2n+1}$	$\emptyset \subset \mathbb{P}^{2n-1}$	Y does not have any contact lines
SO_{n+4}	$B_{\frac{n+3}{2}}$ or $D_{\frac{n+4}{2}}$	$Gr_O(2, n+4)$	$\mathbb{P}^1 \times Q^{n-2} \subset \mathbb{P}^{2n-1}$	Y is the Grassmannian of projective lines on a quadric Q^{n+2}
	G_2	Grassmannian of special lines on Q^5	$\mathbb{P}^1 \subset \mathbb{P}^3$	X is the twisted cubic curve
	F_4	an F_4 variety	$Gr_L(3,6) \subset \mathbb{P}^{13}$	
	E_6	an E_6 variety	$Gr(3,6) \subset \mathbb{P}^{19}$	
	E_7	an E_7 variety	$\mathbb{S}_6 \subset \mathbb{P}^{31}$	X is the spinor variety
	E_8	an E_8 variety	the E_7 variety $\subset \mathbb{P}^{55}$ 	

Table I.1: Simple Lie groups together with the corresponding adjoint variety Y and its variety of tangent directions to contact lines: the subadjoint variety X (listed in details also in §I.1.2).

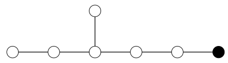
I.1.2 Legendrian subvarieties of projective space

Prior to the author's research the following were the only known examples of smooth Legendrian subvarieties of projective space (see [Bry82], [LM04]):

1) linear subspaces;

2) some homogeneous spaces called **subadjoint varieties** (see table I.1): the product of a line and a quadric $\mathbb{P}^1 \times Q^{n-2}$ and five exceptional cases:

   - twisted cubic curve $\mathbb{P}^1 \subset \mathbb{P}^3$,
   - Grassmannian $Gr_L(3,6) \subset \mathbb{P}^{13}$ of Lagrangian subspaces in $\mathbb{C}^6$,
   - full Grassmannian $Gr(3,6) \subset \mathbb{P}^{19}$,

- spinor variety $\mathbb{S}_6 \subset \mathbb{P}^{31}$ (i.e. the homogeneous $\mathbf{SO}(12)$-space parametrising the vector subspaces of dimension 6 contained in a non-degenerate quadratic cone in $\mathbb{C}^{12}$) and
- the 27-dimensional E_7-variety in $\mathbb{P}^{55}$ corresponding to the marked root: ;

3) every smooth projective curve admits a Legendrian embedding in $\mathbb{P}^3$ [Bry82];

4) a family of smooth surfaces birational to the Kummer $K3$-surfaces [LM04].

The subadjoint varieties are expected to be the only homogeneous Legendrian subvarieties in $\mathbb{P}^{2n-1}$ (a partial proof can be found in [LM04]) and they are the only symmetric Legendrian varieties. Also, they are the only smooth irreducible Legendrian varieties whose ideal is generated by quadratic polynomials (see [Buc06] or theorem III.5).

The subadjoint varieties are strongly related to the group they arise from. Landsberg and Manivel [LM02] use the subadjoint varieties to reprove the classification of simple Lie groups by means of projective geometry only. Also Mukai [Muk98] relates the symmetric Legendrian varieties with another algebraic structure: simple Jordan algebras. In [LM01] the authors give a uniform description of the exceptional cases (arising from F_4, E_6, E_7 and E_8).

I.2 Main results and structure of the thesis

The results of this thesis address two complementary problems regarding Legendrian varieties:

- write explicit restrictions on the properties of Legendrian varieties;
- give examples of smooth Legendrian varieties.

We contribute to the first problem by giving a very precise understanding of the embedded automorphism group of a Legendrian variety. The second problem is solved by proving that a general hyperplane section of a smooth Legendrian variety admits a Legendrian embedding.

In our masters thesis [Buc03], we prove that the quadratic part of the ideal of a Legendrian subvariety X of projective space $\mathbb{P}^{2n-1}$ produces a connected subgroup of projective automorphisms of X. In [Buc06] we improve this result by observing that this group is actually the maximal connected subgroup of automorphisms of the contact structure on $\mathbb{P}^{2n-1}$ preserving the Legendrian subvariety (see theorem III.5).

In the present dissertation we extend this result further. Firstly, we replace $\mathbb{P}^{2n-1}$ with an arbitrary contact manifold Y. Then the connected component of the subgroup of $\mathrm{Aut}(Y)$ that preserves both the contact structure and a given Legendrian subvariety $X \subset Y$, is completely determined by those sections of a distinguished line bundle L on Y that vanish on X. Secondly, we try to remove the assumption that the automorphisms preserve the contact structure. By applying the results of [LeB95] and [Keb01] on the uniqueness of contact structures we can deal with this problem for most projective contact Fano manifolds (see corollary III.25). The remaining cases are the projectivised cotangent bundles and the projective space. The first case is not very interesting, as all the Legendrian subvarieties are classified for these contact manifolds (see corollary III.19). On the other hand the case of projective space is the most important and interesting. It is described precisely in chapter IV. We prove there that a connected group of projective automorphisms that preserve a smooth Legendrian variety necessarily preserves the contact structure. We also give counterexamples to the analogous statement without assuming smoothness and provide some evidence that our counterexamples are the only possible ones.

Our methodology for finding new examples of smooth Legendrian subvarieties is the following. We pose questions of classification of smooth Legendrian varieties satisfying certain additional conditions. For instance, we assume that the variety is toric (see chapter V) or that it is contained in a specific F-cointegrable variety (see chapter VI). In this way we produce a few new smooth examples including a toric surface and a quasihomogeneous Fano 8-fold. Finally we prove that both examples are very close to subadjoint varieties — each of them is isomorphic to a hyperplane section of a subadjoint variety. We generalise this and prove that a general hyperplane section of a smooth Legendrian variety admits a Legendrian embedding into a smaller projective space.

Section I.3 is devoted to introducing our notation and presenting some elementary algebro-geometric facts.

Chapter II is a brief revision of symplectic geometry that will be used in our discussion of contact manifolds. Also some statements from [Buc06] are generalised to this context.

Chapter III contains an independent review of local geometry of contact manifolds, with emphasis on their infinitesimal automorphisms. There we compare (after [LeB95] and [Bea99]) two natural Lie algebra structures related to a contact manifold Y: the Lie bracket of vector fields and the Poisson bracket on the structure sheaf of the symplectisation of Y. We use this comparison to prove the first theorem on embedded automorphisms of Legendrian subvarieties. The theorem states that those automorphisms that preserve the contact structure are completely determined by the ideal of the variety.

In chapters IV–VII we turn our attention to Legendrian subvarieties of projective space.

In chapter IV we continue the topic of automorphisms of Legendrian varieties. We prove the second theorem on embedded automorphisms of Legendrian subvarieties, stating that under minor assumptions they must preserve the contact structure. The results of this chapter are published in [Buc07c].

In chapter V we illustrate, in the case of subvarieties of projective space, how to classify toric Legendrian subvarieties. We give the list of all smooth cases, which include a new example: the projective plane blown up in three linearly independent points. Also the results of that chapter are published in [Buc07c].

Chapter VI contains the classification of Legendrian varieties, which are contained in a specific F-cointegrable variety. Another new example arises in this way: the smooth quasihomogeneous 8-fold. Also we present two other variants of the construction, producing a smooth 5-fold and a smooth 14-fold. The contents of that chapter will be published as [Buc07b].

Finally chapter VII describes a Legendrian embedding of a hyperplane section of a Legendrian variety. Also a variant of an inverse construction (i.e. to describe a bigger Legendrian variety from a given one, such that a hyperplane section of the big one is the original one) is presented and is applied to Bryant's, Landsberg's and Manivel's examples of smooth Legendrian varieties. Parts of that chapter will be published as [Buc07a].

Appendix A revises the differential geometric properties of infinitesimal automorphisms that are necessary for chapter III, but can be expressed without any explicit reference to the contact structure.

I.2.1 Open problems

Keeping in mind the elegant results sketched in §I.1 and having many new examples of smooth Legendrian varieties (as well as families of such), several natural questions remain unanswered.

New contact manifolds?

Can we construct a new example of a contact manifold, whose variety of tangent directions to contact lines is one of the new Legendrian varieties (or is in the given family)? If conjecture I.1 is true, then the answer is negative. If the answer is negative, then what are the obstructions, i.e., what conditions should we require on the Legendrian variety to make the reconstruction of contact manifold possible?

Further applications to algebra?

Can the new Legendrian varieties be used in a similar manner as the subadjoint cases and will they prove themselves to be equally extraordinary varieties? The first tiny piece of evidence for this is explained in §VI.2.1. On the other hand, it is unlikely that such a big variety of examples can have analogous special properties.

Self-dual varieties?

Another problem we want to mention here is a classical question in algebraic geometry: what are the smooth subvarieties of projective space, whose dual variety[6] is also smooth? So far the only examples of these are the self-dual varieties. Thanks to L. Ein [Ein86], the classification of smooth self-dual varieties $Z \subset \mathbb{P}^m$ is known when $3 \operatorname{codim} Z \geq \dim Z$. In corollary VII.17 we prove that the problem of classifying smooth varieties with smooth dual can be expressed in terms of Legendrian varieties and possibly we can apply the techniques of Legendrian varieties to finish the classification.

Projectively and linearly normal Legendrian varieties?

We dare to conjecture:

Conjecture I.3. *Let $X \subset \mathbb{P}(V)$ be a smooth linearly normal[7] Legendrian variety. Then X is one of the subadjoint varieties.*

In view of theorems VII.1 and VII.10, the classification of linearly normal Legendrian varieties might be a necessary step towards a classification of Legendrian varieties.

Furthermore, the conjecture might also contribute to the proof of conjecture I.1. For instance assume conjecture I.3 holds and Y is a contact Fano manifold, for which the variety cut out by contact lines through a general point is normal. Then by applying the results of [Keb05] we get that the associated Legendrian variety $X \subset \mathbb{P}^{2n-1}$ is projectively normal[8] and by the conjecture and results of [Hon00] the manifold Y is an adjoint variety.

The author is able to prove conjecture I.3 if $\dim X = 1$, but this is not an elegant argument nor does it have important applications. We omit the proof here until we manage to improve the argument or to generalise it to higher dimensions.

I.3 Notation and elementary properties

In the present thesis we always work over the field of complex numbers $\mathbb{C}$.

I.3.1 Vector spaces and projectivisation

Let V be a vector space over $\mathbb{C}$. By $\mathbb{P}(V)$ we mean the naive projectivisation of V, i.e. the quotient $(V \backslash \{0\})/\mathbb{C}^*$.

[6] Given a subvariety $Z \subset \mathbb{P}(W)$, **the dual variety** $Z^* \subset \mathbb{P}(W^*)$ is the closure of the set of hyperplanes tangent to Z, see §VII.3 for details.

[7] A subvariety $X \subset \mathbb{P}^m$ is **linearly normal** if it is embedded by a complete linear system.

[8] A subvariety $X \subset \mathbb{P}^m$ is **projectively normal** if its affine cone is normal. If X is projectively normal, then it is also linearly normal by [Har77, ex. II.5.14(d)]

If $v \in V \setminus \{0\}$, then by $[v] \in \mathbb{P}(V)$ we denote the line spanned by v.

Analogously, if E is a vector bundle, by $\mathbb{P}(E)$ we denote the naive projectivisation of E. Let $s_0 \subset E$ be the zero section of E. If $v \in E \setminus s_0$, then by $[v] \in \mathbb{P}(E)$ we denote the line spanned by v in the appropriate fibre of E.

I.3.2 Bilinear forms and their matrices

Let V be a complex vector space of dimension m and f a bilinear form on V. Fix a basis $\mathcal{B}$ of V and let $M(f) = M(f, \mathcal{B})$ be the $m \times m$-matrix such that:

$$f(v, w) = v^T M(f) w,$$

where v and w are arbitrary column vectors of V. We say that $M(f)$ is **the matrix of f in the basis $\mathcal{B}$**.

In particular if ω is a symplectic form (see §II.1.1), $\dim V = 2n$ and $\mathcal{B}$ is a symplectic basis, then

$$J := M(\omega, \mathcal{B}) = \begin{bmatrix} 0 & \mathrm{Id}_n \\ -\mathrm{Id}_n & 0 \end{bmatrix}.$$

Moreover in such a case J is also the matrix of the linear map $\tilde{\omega}$:

$$\begin{aligned} \tilde{\omega} : \ V &\longrightarrow V^* \\ v &\longmapsto \omega(v, \cdot) \end{aligned}$$

in the basis $\mathcal{B}$ on V and the dual basis on V^*.

Similarly, if q is a quadratic form on V, then we denote by $M(q) = M(q, \mathcal{B})$ **the matrix of q in the basis $\mathcal{B}$**:

$$q(v) = v^T M(q) v.$$

I.3.3 Complex and algebraic manifolds

Our main concern is with complex projective manifolds and varieties. This is where two categories meet: complex algebraic varieties and analytic spaces (see [Gri74]). Since the author's origins lie in algebraic geometry, this thesis' intention is to study algebraic Legendrian varieties. However, for some statements there is no reason to limit to the algebraic case, so we state them also for the analytic situation.

So Y will be usually the ambient manifold (for example contact or symplectic manifold), either a complex manifold or smooth algebraic variety. Some statements are local for Y (in the analytic topology), hence it is enough to prove them for $Y \simeq D^{2n}$, where $D^{2n} \subset \mathbb{C}^n$ is a complex disc.

Our main interest is in $X \subset Y$, which will be either an analytic subspace (if Y is a complex manifold), or an algebraic subvariety (if Y is algebraic). For short, will always say $X \subset Y$ is a subvariety.

I.3.4 Vector bundles, sheaves and sections

Given an analytic space or algebraic variety Y, we denote by $\mathcal{O}_Y$ both the structure sheaf (consisting of either holomorphic or algebraic functions on Y in the appropriate analytic or Zariski topology) and the trivial line bundle. If $X \subset Y$ is a subvariety, then by $\mathcal{I}(X)$ we mean the sheaf of ideals in $\mathcal{O}_Y$ defining X.

Given a vector bundle E on Y we will use the same letter E for the sheaf of sections of E. To avoid confusion and too many brackets (for example $\mathcal{I}(X)(U)$) given an open subset $U \subset Y$ and a sheaf (or vector bundle) $\mathcal{F}$, we will write $H^0(U, \mathcal{F})$ rather than $\mathcal{F}(U)$ to mean the value of the sheaf at the open subset U (or sections of vector bundle). By $\mathcal{F}|_U$ we mean the sheaf (or vector bundle) restriction of $\mathcal{F}$ to the open subset U.

Where there can be no confusion, given a sheaf $\mathcal{F}$ which does not have any natural vector bundle structure we will write $s \in \mathcal{F}$ to mean:

$$\exists \text{ an open } U \subset Y \quad \text{with } s \in H^0(U, \mathcal{F}).$$

On the other hand, if E is a vector bundle, then by $v \in E$, we mean that v is a vector in the bundle.

Given a vector bundle E, we denote by E^* the dual vector bundle:

$$E^* := \mathcal{H}om(E, \mathcal{O}).$$

If $\theta \colon \mathcal{F} \longrightarrow \mathcal{G}$ is a map of sheaves or vector bundles and $s \in H^0(U, \mathcal{F})$, then by $\theta(s)$ we mean the image section of $\mathcal{G}$.

I.3.5 Derivatives

Given a complex manifold or smooth algebraic variety Y and a k-form $\theta \in H^0(U, \Omega^k Y)$ by $\mathrm{d}\theta$ we denote the exterior derivative of θ. This convention is also valid for 0-forms $f \in \mathcal{O}_Y = \Omega^0 Y$.

By TY we mean the tangent vector bundle. Nevertheless we keep in mind, that a vector field $\mu \in H^0(U, TY)$ can also be interpreted as a derivation $\mu : \mathcal{O}_Y \to \mathcal{O}_Y$. In particular, we can define the Lie bracket of two vector fields $\mu, \nu \in H^0(U, TY)$ as:

$$[\mu, \nu] = \nu\mu - \mu\nu.$$

This convention is in agreement with [Arn74].

Given a holomorphic or algebraic map $\phi : Y \longrightarrow Y'$, by $\mathrm{D}\phi$ we mean the derivative map:

$$\mathrm{D}\phi : TY \longrightarrow \phi^* TY'.$$

If $\theta \in H^0(U, \Omega^k Y)$ and $\mu \in H^0(U, TY)$, then by $\theta(\mu)$ we mean the contracted $(k-1)$-form. For example, if $\theta = \theta_1 \wedge \theta_2$ for 1-forms θ_i, then

$$\theta(\mu) = \theta_1(\mu)\theta_2 - \theta_2(\mu)\theta_1.$$

The reader should also refer to §A.2.2 for the convention on automorphisms and infinitesimal automorphisms.

I.3.6 Submersion onto image

We recall the standard fact, that every algebraic map is generically a submersion on the closure of the image.

Lemma I.4. *Let M and N be two algebraic varieties over an algebraically closed field of characteristic 0 and let $p : M \longrightarrow N$ be a map such that $N = \overline{p(M)}$. Then for a general $x \in M$, the derivative $D_x p : T_x M \longrightarrow T_{p(x)} N$ is surjective.*

Proof. See [Har77, thm III.10.6].

□

As a corollary, we prove an easy proposition about subvarieties of product manifolds.

Proposition I.5. *Let S_1 and S_2 be two smooth algebraic varieties and let $X \subset S_1 \times S_2$ be a closed irreducible subvariety. Let $X_i \subset S_i$ be the closure of the image of X under the projection π_i onto S_i. Assume that for a Zariski open dense subset of smooth points $U \subset X$ we have that the tangent bundle to X decomposes as $TX|_U = (TX \cap \pi_1^* TS_1)|_U \oplus (TX \cap \pi_2^* TS_2)|_U$ a sum of two vector bundles. Then $X = X_1 \times X_2$.*

Proof. Since X is irreducible, so is X_1 and X_2 and clearly $X \subset X_1 \times X_2$. So it is enough to prove that $\dim X_1 + \dim X_2 = \dim X = \dim U$. However, the maps $\mathrm{D}(\pi_i|_U)$ are surjective onto $TX \cap \pi_i^* TS_i$ and hence by lemma I.4:

$$\dim X_1 + \dim X_2 = \mathrm{rk}(TX \cap \pi_1^* TS_1)|_U + \mathrm{rk}(TX \cap \pi_2^* TS_2)|_U = \mathrm{rk}\, TX|_U = \dim X.$$

□

I.3.7 Line bundles and $\mathbb{C}^*$-bundles

Let Y be complex manifold or a smooth algebraic variety and let L be a line bundle on Y. By $\mathbf{L}^\bullet$ we denote the principal $\mathbb{C}^*$-bundle over Y obtained as the line bundle L^* with the zero section removed. Let π be the projection $\mathbf{L}^\bullet \longrightarrow Y$.

Let $\mathcal{R}_L$ be the sheaf of graded $\mathcal{O}_Y$-algebras $\bigoplus_{m \in \mathbb{Z}} L^m$ on Y. Given an open subset $U \subset Y$ the ring $\mathcal{R}_L(U)$ consists of all the algebraic functions on $\pi^{-1}(U)$, i.e. $\mathcal{R}_L = \pi_* \mathcal{O}_{\mathbf{L}^\bullet}$. Therefore

$$\mathbf{L}^\bullet = \mathrm{Spec}_Y \mathcal{R}_L.$$

Moreover, $H^0(U, L^m) \subset H^0(\pi^{-1}(U), \mathcal{O}_{\mathbf{L}^\bullet})$ is the set of homogeneous functions of weight m (see §A.1).

Lemma I.6. *Let Y be a smooth algebraic variety and let L be a line bundle on Y. Then* $\operatorname{Pic}(\mathbf{L}^\bullet) \simeq \operatorname{Pic}(Y)/\langle L \rangle$ *and the map* $\operatorname{Pic}(Y) \twoheadrightarrow \operatorname{Pic}(\mathbf{L}^\bullet)$ *is induced by the projection* $\pi : \mathbf{L}^\bullet \longrightarrow Y$.

Proof. The Picard group of the total space of L^* is isomorphic to $\operatorname{Pic} Y$ and the isomorphisms are given by the projection and the zero section $s_0 : Y \longrightarrow L^*$. Further, $s_0(Y)$ is a Cartier divisor linearly equivalent to any other rational section $s : Y \dashrightarrow L^*$. Therefore $s_0^*(s_0(Y)) = L^*$ and hence by [Har77, prop. 6.5(c)] the following sequence is exact:

$$\begin{array}{ccccccc} \mathbb{Z} & \longrightarrow & \operatorname{Pic} Y & \xrightarrow{\pi^*} & \operatorname{Pic} \ \mathbf{L}^\bullet & \longrightarrow & 0 \\ 1 & \longmapsto & [L^*] & & & & \end{array}$$

□

The relative tangent bundle, i.e. $\ker(D\pi : T\mathbf{L}^\bullet \longrightarrow \pi^* TY)$ is trivialised by the vector field $\mu_{\mathbb{C}^*}$ related to the action of $\mathbb{C}^*$ (see §A.2.2) and hence we have the short exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbf{L}^\bullet} \longrightarrow T\mathbf{L}^\bullet \longrightarrow \pi^* TY \longrightarrow 0.$$

In particular $K_{\mathbf{L}^\bullet} = \pi^* K_Y$.

I.3.8 Tangent cone

We recall the notion of the tangent cone and a few among many of its properties. For more details and the proofs we refer to [Har95, lecture 20] and [Mum99, III.§3,§4].

For an irreducible Noetherian scheme X over $\mathbb{C}$ and a closed point $x \in X$ we consider the local ring $\mathcal{O}_{X,x}$ and we let $\mathfrak{m}_x$ be the maximal ideal in $\mathcal{O}_{X,x}$. Let

$$R := \bigoplus_{i=0}^{\infty} \left(\mathfrak{m}_x^i / \mathfrak{m}_x^{i+1} \right),$$

where $\mathfrak{m}_x^0$ is just the whole of $\mathcal{O}_{X,x}$. Now we define *the tangent cone $TC_x X$ at x to X* to be $\operatorname{Spec} R$.

If X is a subscheme of an affine space $\mathbb{A}^m$ (which we will usually assume to be an affine piece of a projective space), the tangent cone at x to X can be understood as a subscheme of $\mathbb{A}^m$. Its equations can be derived from the ideal of X. For simplicity assume $x = 0 \in \mathbb{A}^m$ and then the polynomials defining $TC_0 X$ are the lowest degree homogeneous parts of the polynomials in the ideal of X.

Another interesting point-wise definition is that $v \in TC_0 X$ is a closed point if and only if there exists a holomorphic map φ_v from the disc $D_t := \{t \in \mathbb{C} \ :$

$|t| < \delta\}$ to X, such that $\varphi_v(0) = 0$ and the first non-zero coefficient in the Taylor expansion in t of $\varphi_v(t)$ is v, i.e.:

$$\begin{array}{rccl} \varphi_v : & D_t & \longrightarrow & X \\ & t & \mapsto & t^k v + t^{k+1} v_{k+1} + \ldots \end{array}$$

We list some of the properties of the tangent cone, which will be used freely in the proofs:

(1) The dimension of every component of $TC_x X$ is equal to the dimension of X.

(2) $TC_x X$ is naturally embedded in the Zariski tangent space to X at x and $TC_x X$ spans (as a scheme) the tangent space.

(3) X is regular at x if and only if $TC_x X$ is equal (as a scheme) to the tangent space.

Chapter II

Elementary symplectic geometry

We introduce some elementary facts from symplectic geometry, having in mind the needs of subsequent chapters. Most of this material is contained in (or can be easily deduced from) classical textbooks on symplectic geometry, such as [MS98], although we rewrite this over the ground field $\mathbb{C}$ rather than $\mathbb{R}$.

II.1 Linear symplectic geometry

In this section we study linear algebra of vector space, which has a symplectic form. Although it is elementary, it is very important for our considerations as it has threefold application: Firstly, the content of this section describes the local behaviour of symplectic manifolds (see §II.2), particularly the symplectisations of contact manifolds (see §III.2.1). Secondly, it describes very much of global geometry of projective space as a contact manifold (see III.12, but also look through chapters IV–VII). Finally, it explains the fibrewise behaviour of contact distribution (see §III.2).

II.1.1 Symplectic vector space

A symplectic form on a vector space V is a non-degenerate skew-symmetric bilinear form. So $\omega \in \bigwedge^2 V^*$ is a symplectic form if and only if

$$\forall v \in V \ \exists w \in V \ \text{ such that } \ \omega(v,w) \neq 0$$

or equivalently the map

$$\begin{array}{rccc} \tilde{\omega}: & V & \longrightarrow & V^* \\ & v & \longmapsto & \omega(v,\cdot) \end{array}$$

is an isomorphism.

If a vector space V has a symplectic form ω, we say that V (or (V,ω) if specifying the form is important) is **a symplectic vector space**. In such a case the dimension of V is even and there exists a basis $v_1,\dots,v_n,w_1,\dots,w_n$ (where

$n = \frac{1}{2}\dim V$) of V such that $\omega(v_i, w_i) = 1$, $\omega(v_i, v_j) = 0$ and $\omega(v_i, w_j) = 0$ for $i \neq j$. Such a basis is called **a symplectic basis**.

By $\omega^\vee$ we denote the corresponding symplectic form on V^*:

$$\omega^\vee := \left(\tilde{\omega}^{-1}\right)^* \omega.$$

Note that if $v_1, \ldots, v_n$, $w_1, \ldots, w_n$ is a symplectic basis of V and $x_1, \ldots, x_n$, $y_1, \ldots, y_n$ is the dual basis of V^*, then $x_1, \ldots, x_n, y_1, \ldots, y_n$ is a symplectic basis of V^*. In such a case $x_1, \ldots, x_n, y_1, \ldots, y_n$ are also called **symplectic coordinates on** V.

II.1.2 Isotropic, coisotropic, Lagrangian and symplectic subspaces

Assume V is a vector space of dimension $2n$ and ω is a symplectic form on V. Now suppose $W \subset V$ is a linear subspace. By $W^{\perp_\omega}$ we denote the ω perpendicular complement of W:

$$W^{\perp_\omega} := \{v \in V \mid \forall w \in W \quad \omega(v, w) = 0\}.$$

Denote by π the natural projection $V^* \to W^*$. We say that the subspace W is:

isotropic	$\Leftrightarrow$	$\omega\vert_W \equiv 0$	$\Leftrightarrow$	$W \subset W^{\perp_\omega}$	$\Leftrightarrow$	$\ker\pi$ is coisotropic;
coisotropic (or sometimes called **involutive**)	$\Leftrightarrow$	$\omega^\vee\vert_{\ker\pi} \equiv 0$	$\Leftrightarrow$	$W \supset W^{\perp_\omega}$	$\Leftrightarrow$	$\ker\pi$ is isotropic;
Lagrangian	$\Leftrightarrow$	W is isotropic or involutive and $\dim W = n = \frac{1}{2}\dim V$	$\Leftrightarrow$	$W = W^{\perp_\omega}$	$\Leftrightarrow$	$\ker\pi$ is Lagrangian;
symplectic	$\Leftrightarrow$	$\omega\vert_W$ is a symplectic form on W	$\Leftrightarrow$	$W \cap W^{\perp_\omega} = 0$	$\Leftrightarrow$	$\ker\pi$ is symplectic.

II.1.3 Symplectic reduction of vector space

With the assumptions as above let $W \subset V$ be any linear subspace and let $W' := W \cap W^{\perp_\omega}$. Define ω' to be the following bilinear form on $V' := W/W'$:

$$\text{for} \quad w_1, w_2 \in W \quad \text{let} \quad \omega'([w_1], [w_2]) := \omega(w_1, w_2).$$

Then (V', ω') is a symplectic vector space.

The particular case we are mostly interested in is when W is a hyperplane or more generally a coisotropic subspace.

Note the following elementary properties of this construction:

Proposition II.1. *For a subspace $L \subset V$ let L' be the image of $L \cap W$ in V'.*

(a) If L is isotropic (resp. coisotropic or Lagrangian) in V, then L' is isotropic (resp. coisotropic or Lagrangian) in V'.

(b) Conversely, if W is coisotropic, $L \subset W$ and L' is isotropic (resp. coisotropic or Lagrangian) in V', then L is isotropic (resp. coisotropic or Lagrangian) in V.

□

II.1.4 Symplectic automorphisms and weks-symplectic matrices

A linear automorphism ψ of a symplectic vector space (V, ω) is called a **symplectomorphism** if $\psi^*\omega = \omega$ i.e.:

$$\forall u, v \in V \quad \omega\big(\psi(u), \psi(v)\big) = \omega\big(u, v\big).$$

We denote by $\mathbf{Sp}(V)$ the group of all symplectomorphisms of V and by $\mathfrak{sp}(V)$ its Lie algebra:

$$\mathfrak{sp}(V) = \big\{g \in \mathrm{End}(V) \mid \forall u, v \in V \quad \omega\big(u, g(v)\big) + \omega\big(g(u), v\big) = 0\big\}.$$

A linear automorphism ψ of V is called a **conformal symplectomorphism** if $\psi^*\omega = c\omega$ for some constant $c \in \mathbb{C}^*$. We denote by $\mathbf{cSp}(V)$ the group of all conformal symplectomorphisms of V and by $\mathfrak{csp}(V)$ the tangent Lie algebra.

Fix a basis $\mathcal{B}$ of V and note that a matrix $g \in \mathfrak{gl}(V)$ is in the symplectic algebra $\mathfrak{sp}(V)$ if and only if

$$g^T J + Jg = 0$$

where $J := M(\omega, \mathcal{B})$. For the sake of chapter IV we also need to define a complementary linear subspace to $\mathfrak{sp}(V)$:

Definition. *A matrix $g \in \mathfrak{gl}(V)$ is* ***weks-symplectic***[1] *if and only if it satisfies the equation:*

$$g^T J - Jg = 0.$$

The vector space of all weks-symplectic matrices will be denoted by $\mathfrak{wsp}(V)$ (even though it is not a Lie subalgebra of $\mathfrak{gl}(V)$).

[1] A better name would be *skew-symplectic* or *anti-symplectic*, but these are reserved for some different notions.

We immediately see that a matrix is weks-symplectic if and only if it corresponds to a linear endomorphism g, such that for every $u, v \in V$:

$$\omega(gu, v) - \omega(u, gv) = 0. \tag{II.2}$$

This is a coordinate free way to describe $\mathfrak{wsp}(V)$.

Assume that our basis $\mathcal{B}$ is symplectic. In particular $J^2 = M(\omega, \mathcal{B})^2 = -\operatorname{Id}_{2n}$.

Remark II.3. *For a matrix $g \in \mathfrak{gl}(V)$ we have:*

(a) $g \in \mathfrak{sp}(V) \iff Jg$ *is a symmetric matrix;*

(b) $g \in \mathfrak{wsp}(V) \iff Jg$ *is a skew-symmetric matrix.*

□

Note that if $g \in \mathfrak{gl}(V)$, then we can write:

$$g = \frac{1}{2}(g + Jg^T J) + \frac{1}{2}(g - Jg^T J)$$

and the first component $g_+ := \frac{1}{2}(g + Jg^T J)$ is in $\mathfrak{sp}(V)$, while the second $g_- := \frac{1}{2}(g - Jg^T J)$ is in $\mathfrak{wsp}(V)$. Obviously, this decomposition corresponds to expressing the matrix Jg as a sum of symmetric and skew-symmetric matrices.

We list some properties of $\mathfrak{wsp}(V)$:

Proposition II.4. *Let $g, h \in \mathfrak{wsp}(V)$. The following properties are satisfied:*

(i) Write the additive Jordan decomposition for g:

$$g = g_s + g_n$$

where g_s is semisimple and g_n is nilpotent. Then both $g_s \in \mathfrak{wsp}(V)$ and $g_n \in \mathfrak{wsp}(V)$.

(ii) For $\lambda \in \mathbb{C}$, denote by V_λ the λ-eigenspace of g. For any $\lambda_1, \lambda_2 \in \mathbb{C}$ two different eigenvalues V_{λ_1} is ω-perpendicular to V_{λ_2}.

(iii) If g is semisimple, then each space V_λ is symplectic.

□

II.1.5 Standard symplectic structure on $W \oplus W^*$

Let W be any finite dimensional vector space. Set $V := W \oplus W^*$ and there is a canonical symplectic form on V:

$$\omega\big((v,\alpha),(w,\beta)\big) := \beta(v) - \alpha(w).$$

If $a_1, \ldots, a_n$ is any basis of W and $\lambda_1, \ldots, \lambda_n$ is the dual basis of W^*, then

$$a_1, \ldots, a_n, \lambda_1, \ldots, \lambda_n$$

is a symplectic basis of V. In particular, we have the natural embedding

$$\begin{aligned} \mathbf{GL}(W) &\hookrightarrow \mathbf{Sp}(V) \\ A &\mapsto A \oplus (A^{-1})^T. \end{aligned}$$

We note the following elementary lemma:

Lemma II.5. *Let $L \subset W$ be any linear subspace. Then $L \oplus \ker(W^* \to L^*) \subset V$ is a Lagrangian subspace.*

□

II.2 Symplectic manifolds and their subvarieties

Symplectic manifolds will serve us to understand some geometric and algebraic structures of the symplectisations of contact manifolds (see §III.2.1).

A complex manifold or a smooth complex algebraic variety Y is **a symplectic manifold** if there exists a global closed holomorphic 2-form $\omega \in H^0(\Omega^2 Y)$, $d\omega = 0$ which restricted to every fibre is a symplectic form on the tangent space. In other words, $\omega^{\wedge^n}$ is a nowhere vanishing section of $K_Y = \Omega^{2n} Y$. The form ω is called **a symplectic form on** Y.

Similarly as in the case of the vector space, the symplectic form determines an isomorphism:

$$\begin{array}{rccc} \tilde{\omega}: & TY & \xrightarrow{\simeq} & T^*Y \\ & v & \longmapsto & \omega(v, \cdot). \end{array}$$

The theory of compact (or projective) complex symplectic manifolds is well developed and has a lot of beautiful results (see for example [Leh04], [Huy03] and references therein). Yet here we will only use some non-compact examples as a tool for studying contact manifolds and we will only need a few of their basic properties. Also some extensions of the symplectic structure to the singularities of Y are studied, but we are interested only in the case where Y is smooth.

II.2.1 Lagrangian and other subvarieties of a symplectic manifold

Let (Y, ω) be a symplectic manifold. For a subvariety $X \subset Y$ we say X is respectively

1) **isotropic**,
2) **coisotropic**,
3) **Lagrangian**,

if and only if there exists an open dense subset U (equivalently, for any open dense subset U) of smooth points of X, such that for every $x \in U$ the tangent space $T_x X \subset T_x Y$ is respectively

1) isotropic,
2) coisotropic,
3) Lagrangian.

Or equivalently, for every $x \in U$ the conormal space $N^*_x X \subset T^*_x Y$ is respectively

1) coisotropic,
2) isotropic,
3) Lagrangian.

Note that a subvariety is Lagrangian if and only if it is isotropic (or coisotropic) and the dimension is equal to n.

II.2.2 Examples

The following examples are important for our considerations, as they will appear as symplectisations of projective contact manifolds (see §III.2.1).

The affine space

Our key example is the simplest possible: an affine space of even dimension. So assume (V, ω) is a symplectic vector space of dimension $2n$. Then take the affine space $\mathbb{A}^{2n}$ of the same dimension, whose tangent space at every point is V and globally $T\mathbb{A}^{2n} = \mathbb{A}^{2n} \times V$. Then ω trivially extends to the product and it is a symplectic form on $\mathbb{A}^{2n}$.

By an abuse of notation, we will denote the affine space by V as well (so in particular a 0 is fixed in the affine space and the action of $\mathbb{C}^*$ by homotheties is chosen). In this setup, the form ω is homogeneous of weight 2 (see §A.1).

Products

Assume Y_1 and Y_2 are two symplectic manifolds with symplectic forms ω_1 and ω_2 respectively. Clearly $Y_1 \times Y_2$ is a symplectic manifold with the symplectic form $p_1^*\omega_1 + p_2^*\omega_2$, where the p_i's are the appropriate projections.

Next, let $X_i \subset Y_i$ be two subvarieties. Both the X_i's are respectively

1) isotropic,

2) coisotropic,

3) Lagrangian,

if and only if the product $X_1 \times X_2 \subset Y_1 \times Y_2$ is respectively

1) isotropic

2) coisotropic,

3) Lagrangian.

Cotangent Bundle

Let M be a complex manifold or a smooth algebraic variety of dimension n. Set Y to be the total space of the cotangent vector bundle T^*M and let $p : Y \longrightarrow M$ be the projection map. If $x_1, \ldots, x_n$ are local coordinates on $U \subset M$, then $x_1, \ldots, x_n, y_1 = \mathrm{d}x_1, \ldots, y_n = \mathrm{d}x_n$ form the local coordinates on $Y|_U$. Then we can set:

$$\omega|_U := \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n \in H^0(U, \Omega^2 Y),$$

and these glue to a well defined symplectic form $\omega \in H^0(Y, \Omega^2 Y)$. This symplectic form is homogeneous of weight 1 with respect to the usual action on the cotangent spaces.

Since for $m \in M$, $x \in T_m^*M$ we have $T_{(m,x)}Y = T_mM \oplus T_m^*M$ this example of symplectic manifold, generalises the standard symplectic structure on $W \oplus W^*$ (see §II.1.5). The following example generalises lemma II.5:

Example II.6. *Let $Z \subset M$ be any subvariety. Define $\hat{Z}^\# \subset Y$ to be **the conormal variety to** Z, i.e. the closure of the union of conormal spaces to smooth points of Z:*

$$\hat{Z}^\# := \overline{N^* Z_0 / M}.$$

Then $\hat{Z}^\#$ is a Lagrangian subvariety in Y.

Proof. Let $z \in Z$ be a smooth point and let $x \in N_z^* Z_0/M$. Then one can choose appropriate local coordinates on M around z and an appropriate local trivialisation of the cotangent bundle T^*M, such that:

$$T_x \hat{Z}^\# = T_z Z \oplus N_z^* Z_0/M \subset T_z M \oplus T_z^* M.$$

This is a Lagrangian subspace by lemma II.5.

□

Lemma II.7. *Conversely, assume M is a smooth algebraic variety and Y is the total space of T^*M. Moreover assume $X \subset Y$ is an irreducible closed Lagrangian subvariety invariant under the $\mathbb{C}^*$-action on Y. If $Z = \overline{p(X)}$, then $X = \hat{Z}^{\#}$.*

Proof. Let $x \in X$ be a general point and let $z := p(x)$. So x is a point in $T^*_z M$ and

$$T_x Y = T_z M \oplus T^*_z M.$$

Since X is $\mathbb{C}^*$-invariant, under the above identification

$$(0, x) \in T_x X \subset T_x Y.$$

We want to prove that $(0, x) \in N^*_z Z/M$ and this will follow if we prove $T_x X \cap T^*_z M = N^*_z Z/M$.

By lemma I.4 the map $\mathrm{D}p : T_x X \longrightarrow T_z Z$ is surjective, so

$$T_x X + T^*_z M = T_z Z \oplus T^*_z M.$$

Since X is Lagrangian, we also have the dual equality:

$$\begin{aligned} T_x X \cap T^*_z M &= (T_x X)^{\perp_\omega} \cap (T^*_z M)^{\perp_\omega} \\ &= (T_x X + T^*_z M)^{\perp_\omega} \\ &= (T_z Z \oplus T^*_z M)^{\perp_\omega} \\ &= N^*_z Z/M. \end{aligned}$$

Hence $T_x X \cap T^*_z M = N^*_z Z/M$ as claimed and therefore $x \in N^*_z Z/M$. Since x was a general point of X and both X and Z were irreducible, we have $X \subset \hat{Z}^{\#}$ and by dimension counting $X = \hat{Z}^{\#}$.

□

Adjoint and coadjoint orbits

Let G be a semisimple complex Lie group and consider the coadjoint action on the dual of its Lie algebra $\mathfrak{g}^*$. Let Y be an orbit of this action. The tangent space at $\xi \in Y$ is naturally isomorphic to $\mathfrak{g}/Z(\xi)$, where

$$Z(\xi) = \{v \in \mathfrak{g} \mid \forall w \in \mathfrak{g} \;\; \xi([v, w]) = 0\}.$$

Here $[v, w]$ denotes the Lie algebra operation in $\mathfrak{g}$. For $v, w \in \mathfrak{g}$ let $[v]$ and $[w]$ be the corresponding vector fields on Y determined by v and w. We define:

$$\omega_\xi([v], [w]) := \xi([v, w]).$$

Then ω is a symplectic form on Y, which is called the Kostant-Kirillov form, see for instance [Bea98, (2.1)].

Now assume G is simple and Y is invariant under homotheties (for instance Y is the unique minimal nonzero orbit — see [Bea98, prop. 2.2 and prop. 2.6]). Then the actions of G and $\mathbb{C}^*$ commute (because G acts on $\mathfrak{g}^*$ by linear automorphisms, $\mathbb{C}^*$ via homotheties and every linear map commutes with a homothety). Therefore the vector fields of the form $[v]$ for some $v \in \mathfrak{g}$ are homogeneous of weight 0 and hence:

$$(\lambda_t^*\omega)_\xi([v],[w]) = \omega_{\lambda_t(\xi)}([v],[w]) = t\xi([v,w]) = t\omega_\xi([v],[w]).$$

i.e. ω is homogeneous of weight 1.

We can identify $\mathfrak{g}^*$ and $\mathfrak{g}$ by Killing form (see [Hum75]), so equally well we can consider adjoint orbits. Therefore if Y is as above, then it is isomorphic to a $\mathbb{C}^*$-bundle over an adjoint variety (see §I.1.1). More precisely Y is a symplectisation (see §III.2.1) of the adjoint variety.

Open subsets

Let (Y,ω) be a symplectic manifold and let U be an open subset. Then $(U,\omega|_U)$ is again a symplectic manifold.

II.3 Poisson bracket

The Poisson bracket is an important algebraic structure of a symplectic manifold. In corollary III.14 we observe that given a contact manifold and its symplectisation, the Poisson bracket descents from the symplectisation to a bracket on a specific sheaf of rings on the contact manifold. Moreover, this descended structure is strictly related to the automorphisms of the contact manifold (see theorem III.15).

Let (Y,ω) be a symplectic manifold and let $\mathcal{O}_Y$ be the sheaf of holomorphic (or algebraic) functions on Y. Given $f,g \in H^0(U,\mathcal{O}_Y)$ let $\xi_g \in H^0(U,TY)$ be the unique vector field such that $\omega(\xi_g) = \mathrm{d}g$. Then we set:

$$\{f,g\} := \mathrm{d}f(\xi_g),$$

or equivalently:

$$\{f,g\}(x) := \omega_x^\vee(\mathrm{d}g_x, \mathrm{d}f_x).$$

The bilinear skew-symmetric map $\{\cdot,\cdot\} : \mathcal{O}_Y \times \mathcal{O}_Y \to \mathcal{O}_Y$ is called **the Poisson bracket**.

Lemma II.8. *The Poisson bracket satisfies the Jacobi identity and therefore makes $\mathcal{O}_Y$ into a sheaf of Lie algebras. The compatibility between the Poisson bracket and the standard ring multiplication on $\mathcal{O}_Y(U)$ is given by the following Leibniz rule:*

$$\{fg, h\} = f\{g, h\} + g\{f, h\}.$$

Proof. See for example [Arn74, §40] — the proof is identical to the real case. □

The Poisson bracket is determined by the symplectic form and moreover it is defined locally. Hence we have the following property:

Proposition II.9. *Assume (Y, ω) and (Y', ω') are two symplectic manifolds of dimension $2n$. Assume moreover, that we have a finite covering map:*

$$\psi : Y \longrightarrow Y'$$

such that $\psi^*\omega' = \omega$. Then the Poisson structures are compatible: for $f, g \in \mathcal{O}_{Y'}$ we have:

$$\psi^*\{f, g\} = \{\psi^* f, \psi^* g\}.$$

□

Theorem II.10. *Assume Y is a symplectic manifold.*

(i) Suppose $X \subset Y$ is a coisotropic subvariety. Then the sheaf of ideals $\mathcal{I}(X) \subset \mathcal{O}_Y$ is a subalgebra with respect to the Poisson bracket.

(ii) Conversely, suppose $X \subset Y$ is a closed, generically reduced subscheme and that $\mathcal{I}(X)$ is preserved by the Poisson bracket. Then the corresponding variety X_{red} is coisotropic.

Versions of the theorem can be found in [Cou95, chapter 11, prop. 2.4] and in [Buc06, thm 4.2]. We follow more or less the proof from [Buc06]:

Proof. Let X_0 be the locus of smooth points of X. We must show that $\omega^\vee|_{N^*X_0/Y} \equiv 0$ if and only if $\mathcal{I}(X)$ is a Lie subalgebra sheaf in $\mathcal{O}_Y$.

Suppose that $x \in X_0$ is any point, $U \subset Y$ is an open neighbourhood of x and that $f, g \in H^0(U, \mathcal{I}(X))$ are some functions vanishing on X. Then $\mathrm{d}f_x, \mathrm{d}g_x \in N^*X_0/Y$.

If $\omega^\vee|_{N^*X_0/Y} \equiv 0$, then

$$\{f, g\}(x) = \omega_x^\vee(\mathrm{d}g_x, \mathrm{d}f_x) = 0,$$

i.e. $\{f, g\}|_{X_0} = 0$, so extending the equality to the closure of X_0 we get

$$\{f, g\} \in H^0(U, \mathcal{I}(X)).$$

Hence $\mathcal{I}(X)$ is a Lie subalgebra.

Conversely, if $\mathcal{I}(X)$ is a Lie subalgebra, then

$$\omega^{\vee}(\mathrm{d}g_x, \mathrm{d}f_x) = \{f, g\}(x) = 0.$$

Since the map

$$\begin{array}{ccc} H^0(U, \mathcal{I}(X)) & \longrightarrow & N^*_x X_0/Y \\ f & \longmapsto & \mathrm{d}f_x \end{array}$$

is an epimorphism of vector spaces for each $x \in X_0$ and for U sufficiently small, we have $\omega|_{N^*X_0/Y} \equiv 0$.

□

II.3.1 Properties of Poisson bracket

In our considerations on contact manifolds and their various subvarieties we will need the three lemmas that are explained in this subsection. These lemmas refer to proposition II.10 — we have seen that there is a relation between coisotropic varieties and Lie subalgebras of $\mathcal{O}_Y$ that are ideals under the standard ring multiplication.

The first lemma claims that to test if an ideal is a subalgebra it is enough to test it on an appropriate open cover of Y.

Lemma II.11. *Let Y be a symplectic manifold and let $\mathcal{I} \lhd \mathcal{O}_Y$ be a coherent sheaf of ideals. In such a case $\mathcal{I}$ is preserved by the Poisson bracket if and only if there exists an open cover $\{U_i\}$ of Y such that for each i:*

- *if $V \subset U_i$ is another open subset, then the functions in $H^0(V, \mathcal{O}_Y)$ are determined by the functions in $H^0(U_i, \mathcal{O}_Y)$ — this means that if Y is algebraic variety (respectively, analytic space), then the elements of $H^0(V, \mathcal{O}_Y)$ can all be written as quotients (respectively, Taylor series) of elements of $H^0(U_i, \mathcal{O}_Y)$; such property holds for instance if U_i is affine or if U_i is biholomorphic to a disk $D^{4n} \subset \mathbb{C}^{2n}$ or it is biholomorphic to $D^{4n-2} \times \mathbb{C}^*$;*

- *and the ideal $H^0(U_i, \mathcal{I}) \lhd H^0(U_i, \mathcal{O}_Y)$ is preserved by the Poisson bracket.*

Proof. One implication is obvious, while the other follows from the Leibniz rule (see lemma II.8) and from elementary properties of coherent sheaves.

□

The second lemma asserts that for an isotropic subvariety X, only functions constant on X can preserve $\mathcal{I}(X)$ by Poisson multiplication.

Lemma II.12. *Assume Y is a symplectic manifold, X is a closed irreducible isotropic subvariety. Let $h \in H^0(Y, \mathcal{O}_Y)$ be any function such that*

$$\Big\{h|_U, H^0\big(U, \mathcal{I}(X)\big)\Big\} \subset H^0\big(U, \mathcal{I}(X)\big) \text{ for any open subset } U \subset Y.$$

Then h is constant on X.

Proof. Choose an arbitrary $x \in X_0$, a small enough open neighbourhood $U \subset Y$ of x, and take any $f \in H^0\big(U, \mathcal{I}(X)\big)$.

Since $\{h|_U, f\} \in H^0\big(U, \mathcal{I}(X)\big)$:

$$0 = \{h|_U, f\}(x) = \omega(\mathrm{d}f_x, \mathrm{d}h_x),$$

and since U can be taken so small that $\big\{\mathrm{d}f_x \mid f \in H^0\big(U, \mathcal{I}(X)\big)\big\}$ span the conormal space we have:

$$\mathrm{d}h_x \in (N^*_x X/Y)^{\perp_\omega} \overset{\text{since } X \text{ is isotropic}}{\subset} N^*_x X/Y.$$

So $\mathrm{d}h$ vanishes on TX_0 and hence h is constant on X.

□

Lemma II.13. *Assume Y is a symplectic manifold, X is a closed irreducible isotropic subvariety and $S \subset X$ is a closed subvariety. If $\{\mathcal{I}(S), \mathcal{I}(X)\} \subset \mathcal{I}(S)$, then either S is contained in the singular locus of X or X is Lagrangian and $S = X$.*

Proof. The proof goes along the lines of the proof of [Buc06, thm 5.8]. Suppose S is not contained in the singular locus of X, so that a general point $s \in S$ is a smooth point of both X and S. Let $U \subset Y$ be an open neighbourhood of s. Then for all $f \in H^0(U, \mathcal{I}(S))$ and $g \in H^0(U, \mathcal{I}(X))$

$$0 = \{f, g\}(s) = \omega(\mathrm{d}f_s, \mathrm{d}g_s), \tag{II.14}$$

so

$$\begin{aligned} N^*_s X/Y &= \operatorname{span}\big\{(\mathrm{d}g)_s \mid g \in H^0\big(U, \mathcal{I}(X)\big)\big\} \\ &\subseteq (N^*_s S/Y)^{\perp_\omega} && \text{by (II.14)} \\ &\subseteq (N^*_s X/Y)^{\perp_\omega} \\ &\subseteq N^*_s X/Y && \text{because } X \text{ is isotropic.} \end{aligned}$$

Therefore we have all inclusions becoming equalities and in particular $\operatorname{codim} S = \operatorname{codim} X$ and hence $S = X$. Moreover $(N^*_s X/Y)^{\perp_\omega} = N^*_s X/Y$, where s is a general point of X, so X is Lagrangian.

□

II.3.2 Homogeneous symplectic form

Lemma II.15. *Assume (Y, ω) is a symplectic manifold with a $\mathbb{C}^*$-action and that ω is homogeneous. Let $U \subset Y$ be a $\mathbb{C}^*$-invariant open subset and let $f, g \in H^0(U, \mathcal{O}_Y)$ be some homogeneous functions. Then $\{f, g\}$ is homogeneous of weight* $\mathrm{wt}(f) + \mathrm{wt}(g) - \mathrm{wt}(\omega)$.

Proof. Let $\xi_g \in H^0(U, TY)$ be such a vector field, that $\omega(\xi_g) = \mathrm{d}g$. By lemma A.1(i) we have $\mathrm{wt}(\xi_g) = \mathrm{wt}(g) - \mathrm{wt}(\omega)$ and since $\{f, g\} = (\mathrm{d}f)(\xi_g)$, the claim follows from lemma A.1(i)&(iii).

□

II.3.3 Example: Veronese map of degree 2

The following example is important for our considerations, as it proves that for the contact manifold $\mathbb{P}^{2n-1}$, we can equally well consider the Poisson structure on $\bigoplus_{i\in\mathbb{N}} \mathrm{Sym}^i \mathbb{C}^{2n}$ (as we do in [Buc03] and [Buc06]) and the Poisson structure on $\bigoplus_{i\in 2\mathbb{N}} \mathrm{Sym}^i \mathbb{C}^{2n}$; as naturally will arise from the point of view of contact manifolds — see §III.2.1. Also this example will be used to illustrate that every contact structure on $\mathbb{P}^{2n-1}$ comes from a symplectic structure on $\mathbb{C}^{2n}$.

Let (V, ω) be a symplectic vector space. We let

$$\mathbb{C}[V] = \mathbb{C}[x_1, \dots x_{2n}] = \bigoplus_{i\in\mathbb{N}} \mathrm{Sym}^i V^*$$

be the coordinate ring of V. Also consider

$$\mathcal{S} := \mathbb{C}[V]^{even} = \bigoplus_{i\in 2\mathbb{N}} \mathrm{Sym}^i V^*$$

and let $Y' := \mathrm{Spec}\, \mathcal{S} \setminus \{0\}$. Then we have the following $\mathbb{Z}_2$ covering map:

$$\psi : V \setminus \{0\} \longrightarrow Y',$$

which is the restriction of the map induced by $\mathcal{S} \hookrightarrow \mathbb{C}[V]$. This is the underlying map of the second Veronese embedding of $\mathbb{P}(V)$. In the language of §I.3.7, we have $Y' = \big(\mathcal{O}_{\mathbb{P}(V)}(2)\big)^\bullet$ and $V \setminus \{0\} = \big(\mathcal{O}_{\mathbb{P}(V)}(1)\big)^\bullet$.

The symplectic form ω is $\mathbb{Z}_2$ invariant:

$$\omega(-v, -w) = \omega(v, w),$$

hence it descents to a symplectic form ω' on Y', making Y' a symplectic manifold, such that:

$$\psi^*\omega' = \omega.$$

The natural gradings on $\mathbb{C}[V]$ and on $\mathcal{S}$ induce the actions of $\mathbb{C}^*$ on $V \setminus \{0\}$ and on Y' (note that the action on Y' is not faithful, its kernel is $\mathbb{Z}_2$) and ψ is equivariant with respect to these actions.

Corollary II.16. *With the setup as above, the form ω' is homogeneous of weight* 2 *with respect to the* $\mathbb{C}^*$*-action described above, so it is of weight* 1 *with respect to the faithful action of* $\mathbb{C}^*/\mathbb{Z}_2 \simeq \mathbb{C}^*$*. Conversely, if* ω' *is a homogeneous symplectic form on* Y' *of weight* 2*, then* $\psi^*\omega'$ *is a constant symplectic form on* $V \setminus \{0\}$*.*

Proof. This follows from lemma A.1(ii) and the characterisation of constant forms on an affine space in §A.1.

□

Corollary II.17. *The Poisson bracket on* $\mathcal{S}$ *induced by* ω' *is the restriction of the Poisson bracket on* $\mathbb{C}[V]$ *induced by* ω*.*

Proof. This follows immediately from proposition II.9.

□

We note that Y' is the minimal adjoint orbit (see §II.2.2) for the simple group $\mathbf{Sp}_{2n}$. This simple Lie group and its minimal adjoint orbit have quite exceptional behaviour (see table I.1) and it is worth explain this in more detail.

Chapter III

Contact geometry

A projective space seems to be the most standard example of a projective variety. Yet, as a contact manifold, the projective space of odd dimension is the most exceptional among exceptional examples. As a consequence, the study of its Legendrian subvarieties is quite complicated and very interesting. We start our considerations by introducing this case. Further we generalise to the other contact manifolds.

III.1 Projective space as a contact manifold

Let (V, ω) be a symplectic vector space and let $\mathbb{P}(V)$ be its naive projectivisation. Then for every $[v] \in \mathbb{P}(V)$ the tangent space $T_{[v]}\mathbb{P}(V)$ is naturally isomorphic to the quotient $V/[v]$. Let $F = F_{\mathbb{P}(V)} \subset T\mathbb{P}(V)$ be a corank 1 vector subbundle defined fibrewise:

$$F_{[v]} := \left([v]^{\perp_\omega}\right)/[v].$$

Also let L be the quotient line bundle, so that we have the following short exact sequence:

$$0 \longrightarrow F \longrightarrow T\mathbb{P}(V) \xrightarrow{\theta} L \longrightarrow 0.$$

We say that F (respectively θ) is **the contact distribution** (respectively **the contact form**) **associated with the symplectic form** ω.

By §II.1.3 the vector space F_p carries a natural symplectic structure ω_{F_p}. By proposition A.2 (i) $\mathrm{d}\theta$ gives a well defined twisted 2-form on F:

$$\mathrm{d}\theta := \bigwedge^2 F \longrightarrow L.$$

Proposition III.1. *With an appropriate choice of local trivialisation of L, for every $p \in \mathbb{P}(V)$ one has $\omega_{F_p} = (\mathrm{d}\theta)_p$. In particular $\mathrm{d}\theta$ is nowhere degenerate and it determines an isomorphism:*

$$F \simeq F^* \otimes L.$$

Moreover $L \simeq \mathcal{O}_{\mathbb{P}(V)}(2)$.

Proof. See also [LeB95, Ex. 2.1].

Let $x_1 \dots, x_n, y_1, \dots, y_n$ be some symplectic coordinates on V. Then the ω-perpendicular space to $(a_1, \dots, a_n, b_1, \dots, b_n)$ is given by the equation

$$b_1 x_1 + \ldots + b_n x_n - a_1 y_1 - \ldots - a_n y_n = 0.$$

We look for a twisted 1-form θ on $\mathbb{P}(V)$ whose kernel at each point is exactly as above. This is for instance satisfied by

$$\theta = \frac{1}{2}(-y_1 \mathrm{d}x_1 - \ldots - y_n \mathrm{d}x_n + x_1 \mathrm{d}y_1 + \ldots + x_n \mathrm{d}y_n).$$

The ambiguity is only in the choice of the scalar coefficient — we choose $\frac{1}{2}$ in order to acquire the right formula for $\mathrm{d}\theta$. Choose an affine piece $U \subset \mathbb{P}(V)$, say where $x_1 = 1$. On U we have

$$\theta|_U = \frac{1}{2}(-y_2 \mathrm{d}x_2 - \ldots - y_n \mathrm{d}x_n + \mathrm{d}y_1 + x_2 \mathrm{d}y_2 + \ldots + x_n \mathrm{d}y_n)$$

and then:

$$\mathrm{d}\theta|_U = \mathrm{d}x_2 \wedge \mathrm{d}y_2 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

On the other hand, fixing $p \in U$, $p = [1, a_2, \dots a_n, b_1, \dots b_n]$:

$$F_p = \Big\{(x_1, \dots, x_n, y_1, \dots, y_n) \in V \mid b_1 x_1 + b_2 x_2 + \dots b_n x_n - \\ - y_1 - a_2 y_2 - \ldots - a_n y_n = 0\Big\} \Big/ [p].$$

Therefore F is the image under the projection $V \to V/[p]$ of:

$$\hat{F}_p := \Big\{(0, x_2, \dots, x_n,\ a_2 y_2 + \ldots + a_n y_n - b_2 x_2 - \ldots - b_n x_n, y_2, \dots y_n) \in V\Big\}$$

and

$$\omega|_{\hat{F}_p} = \mathrm{d}x_2 \wedge \mathrm{d}y_2 + \ldots + \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

To see that $L \simeq \mathcal{O}_{\mathbb{P}(V)}(2)$ take a section of $T\mathbb{P}(V)$, for instance $x_1 \frac{\partial}{\partial x_1}$. Then

$$\theta\left(x_1 \frac{\partial}{\partial x_1}\right) = -x_1 y_1$$

is a section of L and hence $L \simeq \mathcal{O}_{\mathbb{P}(V)}(2)$.

□

III.1.1 Legendrian subvarieties of projective space

Assume (V, ω) is a symplectic vector space of dimension $2n$.

In our works [Buc03], [Buc06], [Buc07c], [Buc07b] and [Buc07a] we find convenient to use the following definition:

Definition. *We say that a subvariety $X \subset \mathbb{P}(V)$ is **Legendrian** if the affine cone $\hat{X} \subset V$ is a Lagrangian subvariety (see §II.2.1).*

Yet the original definition is formulated in a slightly different, but equivalent manner:

Proposition III.2. *Let $X \subset \mathbb{P}(V)$ be a subvariety. The following conditions are equivalent:*

- *X is Legendrian;*
- *X is $F_{\mathbb{P}(V)}$-integrable and it is of pure dimension $n-1$.;*

Proof. If X is $F_{\mathbb{P}(V)}$-integrable, then X is Legendrian by propositions III.1 and A.2(iv). The other implication is obvious.

□

III.1.2 Decomposable and degenerate Legendrian subvarieties

Definition. *Let V_1 and V_2 be two symplectic vector spaces and let $X_1 \subset \mathbb{P}(V_1)$ and $X_2 \subset \mathbb{P}(V_2)$ be two Legendrian subvarieties. Now assume $V := V_1 \oplus V_2$ and $X := X_1 * X_2 \subset \mathbb{P}(V)$, i.e. X is the join of X_1 and X_2 meaning the union of all lines from X_1 to X_2. Now, clearly, the affine cone $\hat{X}$ is the product $\hat{X}_1 \times \hat{X}_2$ (where $\hat{X}_i$ is the affine cone of X_i), so by §II.2.2 X is Legendrian. In such a situation we say that X is a **decomposable Legendrian subvariety**. We say that a Legendrian subvariety in $\mathbb{P}(V)$ is **indecomposable**, if it is not of that form for any non-trivial symplectic decomposition $V = V_1 \oplus V_2$.*

The indecomposable Legendrian subvarieties have more consistent description of their projective automorphisms group (see chapter IV). On the other hand, decomposable Legendrian varieties (which usually themselves are badly singular) will provide some very interesting families of examples of smooth Legendrian varieties (see chapter VII).

We say a subvariety of projective space is **degenerate** if it is contained in some hyperplane. Otherwise, we say it is **non-degenerate**. The following easy proposition in some versions is well known. The presented version comes from [Buc06, thm 3.4] but see also [LM04, prop. 17 (1)] or [Buc03, tw. 3.16].

Proposition III.3. *Let $X \subset \mathbb{P}(V)$ be a Legendrian subvariety. Then the following conditions are equivalent:*

(i) *X is degenerate.*

(ii) *There exists a symplectic linear subspace $W' \subset V$ of codimension 2, such that $X' = \mathbb{P}(W') \cap X$ is a Legendrian subvariety in $\mathbb{P}(W')$ and X is a cone over X'.*

(iii) *X is a cone over some variety X'.*

In particular degenerate Legendrian subvarieties are decomposable.

□

We also quote [LM04, prop. 17 (2)]:

Proposition III.4. *Let $X \subset \mathbb{P}(V)$ be a smooth Legendrian variety. If X is non-degenerate, then the tangent variety $\tau(X) \subset \mathbb{P}(V)$ and the dual variety*[1] *$X^* \subset \mathbb{P}(V^*)$ are hypersurfaces isomorphic via $\tilde{\omega} : V \to V^*$.*

We note that original formulation in [LM04] omits the smoothness assumption. Otherwise, the decomposable Legendrian varieties are counterexamples. In the proof the authors freely interchange the tangent variety $\tau(X)$ (which by definition is the union of the limits of secants through two points approaching a third fixed point) and the closure of the union of embedded tangent spaces at smooth points. These are the same for X smooth. The tangent variety $\tau(X)$ is indeed a hypersurface in the secant variety $\sigma(X)$ which for a non-degenerate Legendrian variety is $\mathbb{P}(V)$. The closure of the embedded tangent spaces at smooth points is indeed isomorphic to X^*. The mistake does not influence any other result of the paper, but the reader should be careful in applying the proposition.

III.1.3 Quadrics

In [Buc06] we prove:

Theorem III.5. *Let $X \subset \mathbb{P}(V)$ be a Legendrian subvariety. Consider the following map ρ:*

$$H^0(\mathcal{O}_{\mathbb{P}(V)}(2)) \simeq \operatorname{Sym}^2 V^* \ni \ q = \big(x \mapsto x^T M(q) x\big) \stackrel{\rho}{\longmapsto} 2J \cdot M(q) \ \in \mathfrak{sp}(V).$$

where $M(q)$ is the matrix of q and $J = M(\omega)$. Let $\widetilde{\mathcal{I}}(X)_2 \subset \operatorname{Sym}^2 V^*$ be the vector space of quadrics containing X. Then:

[1]Given a subvariety $Z \subset \mathbb{P}(V)$, **the dual variety** is the closure of the set of hyperplanes tangent to Z, see §VII.3 for details.

- *$\rho(\widetilde{\mathcal{I}}(X)_2)$ is a Lie subalgebra of $\mathfrak{sp}(V)$ tangent to a closed subgroup*
$$\overline{\exp\Big(\rho\big(\widetilde{\mathcal{I}}(X)_2\big)\Big)} < \mathbf{Sp}(V).$$
- *We have the natural action of $\mathbf{Sp}(V)$ on $\mathbb{P}(V)$. The group $\overline{\exp\Big(\rho\big(\widetilde{\mathcal{I}}(X)_2\big)\Big)}$ is the maximal connected subgroup in $\mathbf{Sp}(V)$ which under this action preserves $X \subset \mathbb{P}(V)$.*

Proof. See [Buc06, cor. 4.4, cor. 5.5, lem. 5.6].

□

We skip the proof because in §III.3 we generalise this theorem to Legendrian subvarieties of an arbitrary contact manifold. In chapter IV we prove that for smooth X the group $\overline{\exp\Big(\rho\big(\widetilde{\mathcal{I}}(X)_2\big)\Big)}$ is maximal also in $\mathbb{P}\mathbf{GL}(V)$.

III.2 Contact manifolds

Definition. *Let Y be a complex manifold or smooth algebraic variety and fix a short exact sequence*
$$0 \longrightarrow F \longrightarrow TY \stackrel{\theta}{\longrightarrow} L \longrightarrow 0$$
*where $F \subset TY$ is a corank 1 subbundle of the tangent bundle. We say that Y is **a contact manifold** if the twisted 2-form*
$$\mathrm{d}\theta : \bigwedge^2 F \longrightarrow L$$
*(see proposition A.2(i)) is nowhere degenerate, so that $\mathrm{d}\theta_y$ is a symplectic form on F_y for every $y \in Y$. In such a case F is called **the contact distribution** on Y and θ is **the contact form** on Y.*

Example III.6. *By proposition III.1, the projective space with the contact distribution associated with a symplectic form is a contact manifold.*

The following properties are standard, well known (see for instance [Bea98]):

Proposition III.7. *We have the following properties of contact manifold Y:*

(i) The dimension of Y is odd.

(ii) Let $U \subset Y$ be an open subset, let $\mu_F \in H^0(U, F)$ be any section and let $\phi_{\mu_F} : F|_U \to L|_U$ be a map of sheaves:
$$\forall \nu \in H^0(U,F) \quad \phi_{\mu_F}(\nu) := \theta\big([\mu_F, \nu]\big).$$
Then ϕ_{μ_F} is a map of $\mathcal{O}_U$-modules and the assignment $\mu_F \mapsto \phi_{\mu_F}$ is an isomorphism of $\mathcal{O}_Y$-modules:
$$F \simeq F^* \otimes L.$$

(iii) The canonical divisor K_Y is isomorphic to $L^{\otimes(-n-1)}$. In particular Y is a Fano variety if and only if L is ample.

Proof. We only prove (ii), the other parts follow easily. Map ϕ_{μ_F} is a map of $\mathcal{O}_U$-modules by A.2(iii). By A.2(ii) we have equality:

$$\phi_{\mu_F}(\nu) = \mathrm{d}\theta(\mu_F, \nu).$$

Since $\mathrm{d}\theta$ is non-degenerate, it follows that $\mu_F \mapsto \phi_{\mu_F}$ is indeed an isomorphism.

□

III.2.1 Symplectisation

The following construction is standard — see for instance [Arn74], [KPSW00], [Bea98].

Let $\mathbf{L}^\bullet$ be the principal $\mathbb{C}^*$-bundle as in §I.3.7. In §A.2.3 and §A.2.4 we study in detail the properties of $\mathbf{L}^\bullet$ and an extension of the twisted form θ to $\mathbf{L}^\bullet$. We have an equivalence between contact structures on Y and symplectic homogeneous weight 1 structures on $\mathbf{L}^\bullet$:

Theorem III.8. *Let Y be a complex manifold or smooth algebraic variety with a line bundle L and the principal $\mathbb{C}^*$-bundle $\mathbf{L}^\bullet$ as in §I.3.7.*

- *If $\theta : TY \longrightarrow L$ is a contact form, then $\mathrm{d}\theta^\bullet$ (see §A.2.4) is a homogeneous symplectic form on $\mathbf{L}^\bullet$ of weight 1.*
- *Conversely, assume ω is a symplectic form on $\mathbf{L}^\bullet$, which is homogeneous of weight 1. Then there exists a unique contact form $\theta : TY \longrightarrow L$ on Y, such that $\omega = \mathrm{d}\theta^\bullet$.*

Proof. See proposition A.18.

□

If (Y, F) is a contact manifold, then the symplectic manifold $(\mathbf{L}^\bullet, \mathrm{d}\theta^\bullet)$ from the theorem is called **the symplectisation of** (Y, F).

Using the theorem and §II.2.2 we have following examples of contact manifolds:

Example III.9. *Let G be a simple group and let Y be the closed orbit in $\mathbb{P}(\mathfrak{g})$. Then Y is a contact manifold (compare with conjecture I.1).*

Example III.10. *If $Y \simeq \mathbb{P}(T^*M)$, then let $L = \mathcal{O}_{\mathbb{P}(T^*M)}(1)$ and hence $\mathbf{L}^\bullet \simeq T^*M \backslash s_0$, where s_0 is the zero section and Y is a contact manifold.*

Example III.11. *If Y is a contact Fano manifold, then*

$$Y \simeq \operatorname{Proj}\left(\bigoplus_{m\in\mathbb{N}} H^0(Y, L^m)\right),$$
$$\mathbf{L}^\bullet \simeq \operatorname{Spec}\left(\bigoplus_{m\in\mathbb{N}} H^0(Y, L^m)\right) \setminus \{0\}$$

where 0 is the point corresponding to the maximal ideal $\bigoplus_{m\geq 1} H^0(Y, L^m)$ (see [Gro61, §2.3]).

□

Example III.12. *If $Y \simeq \mathbb{P}(V)$, then by proposition III.7(iii) we have $L \simeq \mathcal{O}_{\mathbb{P}(V)}(2)$. Therefore $V\backslash\{0\}$ is a 2 to 1 unramified cover of $\mathbf{L}^\bullet$, see §II.3.3. In particular, from theorem III.8 and corollary II.16 we conclude that every contact structure on $\mathbb{P}(V)$ is associated to some constant symplectic form ω on V (see §III.1).*

By [KPSW00] combined with [Dem02] every contact projective manifold Y is either isomorphic to $\mathbb{P}(T^*M)$ or it is Fano with $b_2 = 1$. In the second case by proposition III.7(iii) and the Kobayashi-Ochiai characterisation of projective space [KO73] either $Y \simeq \mathbb{P}(V)$ or $\operatorname{Pic} Y = \mathbb{Z}[L]$.

III.2.2 Contact automorphisms

Automorphisms of contact manifolds preserving the contact structure were also studied by LeBrun [LeB95] and Beauville [Bea98]. We use their methods to state slightly more general results about infinitesimal automorphisms. In the end we globalise the automorphisms for projective contact manifolds.

Let Y be a contact manifold and let $\pi : \mathbf{L}^\bullet \longrightarrow Y$ be the symplectisation as in §III.2.1. Also let $\mathcal{R}_L$ be as in §I.3.7.

Example III.13. *If Y is a contact Fano manifold, then*

$$H^0(Y, \mathcal{R}_L) = H^0(\mathbf{L}^\bullet, \mathcal{O}_{\mathbf{L}^\bullet}) = \left(\bigoplus_{m\in\mathbb{N}} H^0(Y, L^m)\right).$$

Since $Y = \operatorname{Proj}\big(H^0(Y, \mathcal{R}_L)\big)$ (see example III.11), all the structure of Y as well as its global and local behaviour is determined by this ring of global sections. Hence in this case whatever is stated below for the sheaf $\mathcal{R}_L$ can be deduced from the analogous statement about $H^0(Y, \mathcal{R}_L)$ only.

Corollary III.14.

(i) *Let $f, g \in \mathcal{O}_{\mathbf{L}^\bullet}$ be two functions on $\mathbf{L}^\bullet$ homogeneous with respect to the action of $\mathbb{C}^*$. Then $\{f, g\}$ is also homogeneous and* $\operatorname{wt}\{f,g\} = \operatorname{wt} f + \operatorname{wt} g - 1$

(ii) *The Poisson bracket descends to $\mathcal{R}_L$ and determines a bilinear map:*

$$H^0(U, L^{m_1}) \times H^0(U, L^{m_2}) \longrightarrow H^0(U, L^{m_1+m_2-1}).$$

Proof. This follows from corollary II.15. See also [LeB95, rem. 2.3].

□

We will refer to the Lie algebra structure on $\mathcal{R}_L$ defined above also as Poisson structure and denote the bracket by $\{\cdot,\cdot\}$. For $s \in H^0(U, L)$ let $\tilde{s}$ be the corresponding element in $H^0\left(\pi^{-1}(U), \mathbf{L}^\bullet\right) = \mathcal{R}_L(U)$.

By corollary III.14 the invertible sheaf L has a Lie algebra structure and it is crucial for our considerations, that it is isomorphic to the sheaf $\mathfrak{aut}_F^{\inf}$ of infinitesimal automorphisms of Y preserving F (see §A.2.3 for more details):

$$\mathfrak{aut}_F^{\inf}(U) := \left\{\mu \in H^0(U, TY) \mid [\mu, F] \subset F\right\}.$$

Theorem III.15. *Let Y be a contact manifold, F be the contact distribution, θ be the contact form and let $U \subset Y$ be an open subset. Using the notation of §A.2 we have:*

1) *$TY = \mathfrak{aut}_F^{\inf} \oplus F$ as sheaves of Abelian groups.*

2) *The map of sheaves $\theta|_{\mathfrak{aut}_F^{\inf}} : \mathfrak{aut}_F^{\inf} \longrightarrow L$ maps isomorphically the Lie algebra structure of $\mathfrak{aut}_F^{\inf}$ onto the Lie algebra structure of L given by the Poisson bracket.*

3) *The following two Lie algebra representations of $\mathfrak{aut}_F^{\inf}$ on $\mathcal{O}_{\mathbf{L}^\bullet}$ are equal:*

- *The induced representation of $\mathfrak{aut}_F^{\inf}$ on $\mathbf{L}^\bullet$ (see §A.2.3).*
- *The representation induced by the adjoint representation:*

$$\mu \in \mathfrak{aut}_F^{\inf}(U), f \in H^0(U, \mathcal{O}_{\mathbf{L}^\bullet}) \implies \mu.f := \left\{\widetilde{\theta(\mu)}, f\right\}.$$

Proof. The following proof of 1) is taken from [Bea98, prop. 1.1], but see also [LeB95, prop. 2.1].

To prove 1), take any $\mu \in H^0(U, TY)$ and consider the map of sheaves:

$$\begin{array}{rcl} F|_U & \longrightarrow & L|_U \\ \nu & \longmapsto & \theta\big([\mu, \nu]\big). \end{array}$$

By proposition A.2(iii) the above map is a map of $\mathcal{O}_Y|_U$-modules, hence it is an element of $H^0(U, F^* \otimes L)$. Let μ_F be the corresponding element of $H^0(U, F)$ (see proposition III.7(ii)). By the definition of the isomorphism $F^* \otimes L \simeq F$, we have

$$\theta\left([\mu_F, \nu]\right) = \theta\left([\mu, \nu]\right)$$

for every $\nu \in F|_U$, hence $[\mu - \mu_F, \nu] \in F|_U$. Therefore $\mu - \mu_F \in \mathfrak{aut}_F^{\inf}(U)$, so

$$\mu = \mu_F + (\mu - \mu_F)$$

gives the required splitting.

For 2) see also [Bea98, prop. 1.6] and [LeB95, rem. 2.3]. By 1), the map $\theta|_{\mathfrak{aut}_F^{\inf}}$ is an isomorphism of sheaves of Abelian groups. So it is enough to prove that $\theta|_{\mathfrak{aut}_F^{\inf}}$ preserves the Lie algebra structures. For every $\mu, \nu \in \mathfrak{aut}_F^{\inf}(U)$ denote by $\breve{\mu}$ and $\breve{\nu}$ the induced infinitesimal automorphisms of $\mathbf{L}^\bullet$ (see §A.2.3). We have:

$$\begin{aligned}
\left\{\widetilde{\theta(\mu)}, \widetilde{\theta(\nu)}\right\} &= (\mathrm{d}\theta^\bullet)^\vee \left(\mathrm{d}\left(\widetilde{\theta(\nu)}\right), \mathrm{d}\left(\widetilde{\theta(\mu)}\right)\right) \\
&= (\mathrm{d}\theta^\bullet)^\vee \big(\mathrm{d}\theta^\bullet(\breve{\nu}), \mathrm{d}\theta^\bullet(\breve{\mu})\big) && \text{by prop. A.21} \\
&= \mathrm{d}\theta^\bullet\left(\breve{\nu}, \breve{\mu}\right) \\
&= \widetilde{\theta([\mu, \nu])} && \text{by cor. A.22.}
\end{aligned}$$

Hence $\theta|_{\mathfrak{aut}_F^{\inf}}$ preserves the Lie algebra structures.

Part 3) is local and since both representations satisfy the Leibniz rule (see equation (A.14) and lemma II.8), it is enough to check the equality for multiplicative generators of $\mathcal{O}_{\mathbf{L}^\bullet}$. Locally, these might be taken for instance as sections of L and so 3) follows from 2).

□

We underline, that $\mathfrak{aut}_F^{\inf}$, as a subsheaf of TY is not a $\mathcal{O}_Y$-submodule (see §A.2.3). So in particular the obtained splitting of the short exact sequence of sheaves of Abelian groups

$$0 \longrightarrow F \longrightarrow TY \xrightarrow{\theta} L \longrightarrow 0$$

is not a splitting of vector bundles.

Turning to global situation assume Y is projective and let $\mathrm{Aut}(Y)$, $\mathrm{Aut}_F(Y)$ and $\mathfrak{aut}(Y)$, $\mathfrak{aut}_F(Y)$ denote, respectively, the group of automorphisms of Y, the group of automorphisms of Y preserving the contact structure and their Lie algebras.

LeBrun [LeB95] and Kebekus [Keb01] observed that in the case of projective contact Fano manifolds with Picard group generated by L, the global sections of L are isomorphic to $\mathfrak{aut}(Y)$:

Corollary III.16. *Let Y be a projective contact manifold with contact distribution F.*

(i) Then θ maps isomorphically $\mathfrak{aut}_F(Y)$ onto $H^0(Y, L)$.

(ii) If moreover Y is Fano with $\operatorname{Pic}(Y) = \mathbb{Z}[L]$, then $\operatorname{Aut}(Y) = \operatorname{Aut}_F(Y)$ and hence the Lie algebra $H^0(Y, L)$ is naturally isomorphic to $\mathfrak{aut}(Y)$.

Proof. By corollary A.10 we have $\mathfrak{aut}_F(Y) = \mathfrak{aut}_F^{\inf}(Y)$, so (i) follows from theorem III.15 2).

On the other hand (ii) follows from [Keb01, cor. 4.5].

□

III.3 Legendrian subvarieties in contact manifold

Definition. *Let Y be a complex contact manifold with a contact distribution F. A subvariety $X \subset Y$ is **Legendrian** if X is F-integrable (i.e., $TX \subset F$) and $2 \dim X + 1 = \dim Y$ (i.e., X has maximal possible dimension).*

If $Y \simeq \mathbb{P}^{2n+1}$, then the above definition agrees with the definition in §III.1.1 by proposition III.2. In general, we have analogous properties with V replaced by $\mathbf{L}^\bullet$:

Proposition III.17. *Let Y be a contact manifold with a contact distribution $F \subset TY$ and with its symplectisation $\pi : \mathbf{L}^\bullet \to Y$. Assume $X \subset Y$ is a subvariety. Then:*

(a) X is F-integrable if and only if $\pi^{-1}(X) \subset \mathbf{L}^\bullet$ is isotropic.

(b) X is Legendrian if and only if $\pi^{-1}(X) \subset \mathbf{L}^\bullet$ is Lagrangian.

Proof. Part (a) follows from lemma A.19 and part (b) follows from (a).

□

In the case of subvarieties of a symplectic manifold, we have three important types of subvarieties (isotropic, Legendrian and coisotropic). Also for subvarieties of contact manifold in addition to F-integrable and Legendrian subvarieties, it is useful to consider the subvarieties corresponding to the coisotropic case:

Definition. *In the setup of proposition III.17, we say that X is F-**cointegrable** if $\pi^{-1}(X) \subset \mathbf{L}^\bullet$ is coisotropic.*

Example III.18. *Assume $\widetilde{X} \subset \mathbf{L}^\bullet$ is irreducible and Lagrangian and let X be the closure of $\pi(X) \subset Y$. Then X is F-cointegrable. If moreover $\widetilde{X}$ is not $\mathbb{C}^*$-invariant, then $\dim X = \frac{1}{2}(\dim Y + 1)$.*

Corollary III.19. *If $Y = \mathbb{P}(T^*M)$ for some smooth algebraic variety M and X is an algebraic Legendrian subvariety, then X is the conormal variety $Z^{\#}$ to some algebraic subvariety $Z \subset M$.*

Proof. It follows from proposition III.17, example III.10 and lemma II.7.

□

Let $\mathcal{R}_L = \pi_* \mathcal{O}_{\mathbf{L}^\bullet}$ be the sheaf of rings on Y defined in I.3.7. For a subvariety $X \subset Y$, let $\widetilde{\mathcal{I}}(X) \lhd \mathcal{R}_L$ be the sheaf of ideals generated by those local sections of L^m that vanish on X. Then:

$$\pi_* \mathcal{I}\left(\pi^{-1}(X)\right) = \widetilde{\mathcal{I}}(X) \tag{III.20}$$

where $\mathcal{I}(\pi^{-1}(X)) \lhd \mathcal{O}_{\mathbf{L}^\bullet}$ is the ideal sheaf of $\pi^{-1}(X)$. In this context, the meaning of lemma II.11 is the following:

Lemma III.21. *With the notation as above, let $\mathcal{I} \lhd \mathcal{O}_{\mathbf{L}^\bullet}$ be a coherent sheaf of ideals. Then $\mathcal{I}$ is preserved by the Poisson bracket on $\mathcal{O}_{\mathbf{L}^\bullet}$ if and only if $\pi_*\mathcal{I}$ is preserved by the Poisson bracket on $\mathcal{R}_L$.*

□

Hence we get the description of F-cointegrable subvarieties in terms of the Poisson bracket on $\mathcal{R}_L$:

Proposition III.22. *With the assumptions as above, a subvariety $X \subset Y$ is F-cointegrable if and only if $\widetilde{\mathcal{I}}(X)$ is preserved by the Poisson bracket on $\mathcal{R}_L$.*

Proof. The proposition combines equation (III.20), theorem II.10 and lemma III.21.

□

Given a subvariety $X \subset Y$, we define $\mathfrak{aut}_F^{\inf}(\cdot, X)$ to be the sheaf of Lie algebras of those infinitesimal automorphisms of Y, which preserve X and contact distribution F (see also §A.2.3):

$$\mathfrak{aut}_F^{\inf}(U, X) := \Big\{\mu \in H^0(U, TY) \mid [\mu, F] \subset F \text{ and } \forall f \in \mathcal{I}(X)|_U \ \ (\mathrm{d}f)(\mu) \in \mathcal{I}(X)|_U\Big\}$$

Further, let $\widetilde{\mathcal{I}}(X)_1 \subset L$ be the degree 1 part of the sheaf of homogeneous ideals $\widetilde{\mathcal{I}}(X)$. Since L is a line bundle with the action of $\mathfrak{aut}_F^{\inf}$ (see §A.2.3), choosing a local trivialisation and using the gluing property of sheaves we can replace $\mathcal{I}(X)$ in the definition of $\mathfrak{aut}_F^{\inf}(\cdot, X)$ with $\widetilde{\mathcal{I}}(X)_1$:

$$\mathfrak{aut}_F^{\inf}(U, X) = \Big\{\mu \in H^0(U, TY) \mid [\mu, F] \subset F \text{ and } \mu.\widetilde{\mathcal{I}}(X)_1|_U \subset \widetilde{\mathcal{I}}(X)_1|_U\Big\} \tag{III.23}$$

where . denotes the induced action of $\mathfrak{aut}_F^{\inf}$ on L described in §A.2.3.

The following theorem establishes a connection between the infinitesimal automorphisms of a Legendrian variety and its ideal:

Theorem III.24. *Let Y be a contact manifold with a contact distribution F and let $\theta : TY \to L$ be the quotient map. Also let $U \subset Y$ be an open subset. Assume $X \subset Y$ is an irreducible subvariety.*

A. If X is F-integrable, then $\theta\left(\mathfrak{aut}_F^{\inf}(U,X)\right) \subset H^0\left(U, \widetilde{\mathcal{I}}(X)_1\right)$.

B. If X is F-cointegrable, then $\theta\left(\mathfrak{aut}_F^{\inf}(U,X)\right) \supset H^0\left(U, \widetilde{\mathcal{I}}(X)_1\right)$.

C. If X is Legendrian, then $\theta\left(\mathfrak{aut}_F^{\inf}(U,X)\right) = H^0\left(U, \widetilde{\mathcal{I}}(X)_1\right)$.

Proof. In the case of A, choose arbitrary $\mu \in \mathfrak{aut}_F^{\inf}(U,X)$. We must prove that $\theta(\mu) \in H^0(U, \widetilde{\mathcal{I}}_1(X))$ or, equivalently, that

$$\widetilde{\theta(\mu)} \in H^0\Big(\pi^{-1}(U), \mathcal{I}\big(\pi^{-1}(X)\big)\Big)$$

(recall that for a section $s \in H^0(U, L)$ by $\tilde{s}$ we denote the corresponding element in $H^0(\pi^{-1}(U), \mathcal{O}_{\mathbf{L}^\bullet})$).

By (III.23) the action of μ preserves $\widetilde{\mathcal{I}}(X)|_U$ and hence also $\mathcal{I}\big(\pi^{-1}(X)\big)|_{\pi^{-1}(U)}$. By theorem III.15 3) this means that

$$\Big\{\widetilde{\theta(\mu)},\ \mathcal{I}\big(\pi^{-1}(X)\big)|_{\pi^{-1}(U)}\Big\} \quad \subset \quad \mathcal{I}\big(\pi^{-1}(X)\big)|_{\pi^{-1}(U)}.$$

Moreover $\pi^{-1}(X)$ is isotropic by proposition III.17.

By lemma II.12 function $\widetilde{\theta(\mu)}$ is constant on $\pi^{-1}(X)$. But $\widetilde{\theta(\mu)}$ is also a $\mathbb{C}^*$-homogeneous function of weight 1, so it must vanish on $\pi^{-1}(X)$. Therefore $\widetilde{\theta(\mu)} \in H^0\Big(\pi^{-1}(U), \mathcal{I}\big(\pi^{-1}(X)\big)\Big)$ as claimed.

To prove B let $\mu \in \mathfrak{aut}_F^{\inf}(U)$ be an infinitesimal automorphism such that $\theta(\mu) \in \widetilde{\mathcal{I}}(X)_1$. By proposition III.22

$$\Big\{\theta(\mu), \widetilde{\mathcal{I}}(X)\Big\} \subset \widetilde{\mathcal{I}}(X)$$

so by theorem III.15 3) we have

$$\mu.\widetilde{\mathcal{I}}(X) \subset \widetilde{\mathcal{I}}(X)$$

(where . denotes the induced representation of $\mathfrak{aut}_F^{\inf}$ on $\mathbf{L}^\bullet$, see §A.2.3). Hence by equation (III.23) the infinitesimal automorphism μ is contained in $\mathfrak{aut}_F^{\inf}(U,X)$ and $H^0\left(U, \widetilde{\mathcal{I}}(X)_1\right) \subset \theta\left(\mathfrak{aut}_F^{\inf}(U,X)\right)$ as claimed.

Part C is an immediate consequence of A and B.

□

The following corollary says that in the case when Y is projective also the global automorphisms of a Legendrian subvariety can be understood in terms of the ideal of the variety. In particular, in (i), we generalise [Buc03, wn. 4.3] or [Buc06, cor. 5.5 & lem. 5.6].

Corollary III.25. *Let Y be a projective contact manifold, let F be the contact distribution and let X be a Legendrian subvariety. Let $\mathfrak{aut}(Y,X)$ (resp. $\mathfrak{aut}_F(Y,X)$) be the Lie algebra of group of automorphisms of Y preserving X (resp. preserving X and F). Then:*

(i) $\theta\big(\mathfrak{aut}_F(Y,X)\big) = H^0\left(Y, \widetilde{\mathcal{I}}(X)_1\right)$;

(ii) If in addition $\operatorname{Pic} Y = \mathbb{Z}[L]$, *then* $\theta\left(\mathfrak{aut}(Y,X)\right) = H^0\left(Y, \widetilde{\mathcal{I}}(X)_1\right)$.

Proof. It follows from corollary III.16 and theorem III.24C.

□

In chapter IV we discuss the extension of corollary III.25(ii) to $Y \simeq \mathbb{P}^{2n+1}$.

The following corollary generalises [Buc06, thm 5.8]:

Corollary III.26. *If Y is a projective contact manifold and $X \subset Y$ is an irreducible Legendrian subvariety such that $\widetilde{\mathcal{I}}(X)$ is generated by $H^0\big(Y, \widetilde{\mathcal{I}}(X)_1\big)$, then $\operatorname{Aut}_F(Y,X)$ acts transitively on the smooth locus of X. In particular, if X is in addition smooth, then X is a homogeneous space.*

Proof. If $S \subset X, S \neq X$ is a closed subvariety invariant under the action of $\operatorname{Aut}_F(Y,X)$, then by theorem III.15 3) and by corollary III.25(i):

$$\forall f \in H^0\left(Y, \widetilde{\mathcal{I}}(X)_1\right) \quad \left\{\widetilde{\mathcal{I}}(S), f\right\} \subset \widetilde{\mathcal{I}}(S).$$

Hence by the Leibniz rule and since $\widetilde{\mathcal{I}}(X)$ is generated by $H^0\big(Y, \widetilde{\mathcal{I}}(X)_1\big)$, we have:

$$\left\{\mathcal{I}\left(\pi^{-1}(S)\right), \mathcal{I}\left(\pi^{-1}(X)\right)\right\} \subset \mathcal{I}\left(\pi^{-1}(S)\right).$$

So by lemma II.13, variety S is contained in the singular locus of X.

Now let $O \subset X$ be an orbit of a smooth point under the action of $\operatorname{Aut}_F(Y,X)$. Then the closure $\overline{O}$ is not contained in the singular locus so by above it must be equal to all of X. Moreover $\overline{O} \setminus O$ is a closed subset invariant under the action and not equal to X, so it is contained in the singular locus. So O is the whole smooth locus of X.

□

We conclude this chapter by underlining that, unfortunately, the above results are proved only for automorphisms of Y, that preserve Legendrian subvariety X, not simply for automorphisms of X.

Chapter IV

Projective automorphisms of a Legendrian variety

The content of this chapter is published in [Buc07c].

We are interested in the following conjecture:

Conjecture IV.1. *Let $X \subset \mathbb{P}^{2n-1}$ be an irreducible indecomposable Legendrian subvariety and let $G < \mathbb{P}\mathbf{GL}_{2n}$ be a connected subgroup of linear automorphisms preserving X. Then G is contained in the image of the natural map $\mathbf{Sp}_{2n} \to \mathbb{P}\mathbf{GL}_{2n}$.*

It is quite natural to believe, that if a linear map preserves a form on a big number of linear subspaces, then it actually preserves the form (at least up to scalar). With this approach, [JJ04, cor. 6.4] proved the conjecture in the case where the image of X under the Gauss map is non-degenerate in the Grassmannian of Lagrangian subspaces in $\mathbb{C}^{2n}$. Unfortunately, this is not enough - for example $\mathbb{P}^1 \times Q_1 \subset \mathbb{P}^5$ has a degenerate image under the Gauss map and this is one of the simplest examples of smooth Legendrian subvarieties.

In §IV.2 we prove:

Theorem IV.2. *If $X \subset \mathbb{P}^{2n-1}$ is a smooth Legendrian subvariety which is not a linear subspace and $G < \mathbb{P}\mathbf{GL}_{2n}$ is a connected subgroup preserving X, then G is contained in the image of the natural map $\mathbf{Sp}_{2n} \to \mathbb{P}\mathbf{GL}_{2n}$.*

This theorem, combined with corollary III.25 gives us a good understanding of the group of projective automorphisms of a smooth Legendrian subvariety in $\mathbb{P}^{2n-1}$.

IV.1 Discussion of assumptions

One obvious remark is that homotheties act trivially on $\mathbb{P}(V)$, but in general are not symplectic on V. Therefore, it is more convenient to think of conformal symplectomorphisms (see §II.1.4).

It is clear, that if we hope for a positive answer to the question whether a projective automorphism of a Legendrian subvariety necessarily preserves the contact structure, then we must assume that our Legendrian variety is non-degenerate.

Another natural assumption is that X is irreducible — one can also easily produce a counterexample if we skip this assumption. Yet still this is not enough.

Let $X = X_1 * X_2 \subset \mathbb{P}(V_1 \oplus V_2)$ be a decomposable Legendrian variety. Then we can act via $\lambda_1 \operatorname{Id}_{V_1}$ on V_1 and via $\lambda_2 \operatorname{Id}_{V_2}$ on V_2 - such an action will preserve X and in general it is not conformal symplectic. This explains why the assumptions of our conjecture IV.1 are necessary.

IV.2 Preservation of contact structure

Let $X' \subset \mathbb{P}(V)$ be an irreducible, indecomposable Legendrian subvariety, let X be the affine cone over X' and X_0 be the smooth locus of X. Assume that G is the maximal connected subgroup in $\mathbf{GL}_{2n}$ preserving X. Let $\mathfrak{g} < \mathfrak{gl}_{2n}$ be the Lie algebra tangent to G. To prove the conjecture it would be enough to show that $\mathfrak{g}$ is contained in the Lie algebra $\mathfrak{csp}_{2n}$ tangent to conformal symplectomorphisms, i.e. the Lie algebra spanned by $\mathfrak{sp}_{2n}$ and the identity matrix Id_{2n}.

Recall from §II.1.4 the notion of weks-symplectic matrices.

Theorem IV.3. *With the above notation the following properties hold:*

I. *The underlying vector space of $\mathfrak{g}$ decomposes into symplectic and weks-symplectic part:*
$$\mathfrak{g} = \big(\mathfrak{g} \cap \mathfrak{sp}(V)\big) \oplus \big(\mathfrak{g} \cap \mathfrak{wsp}(V)\big).$$

II. *If $g \in \mathfrak{g} \cap \mathfrak{wsp}(V)$, then g preserves every tangent space to X:*
$$\forall x \in X_0 \quad g(T_x X) \subset T_x X$$
and hence also
$$\forall t \in \mathbb{C} \quad \forall x \in X_0 \qquad T_{\exp(tg)(x)} X = \exp(tg)(T_x X) = T_x X.$$

III. *If $g \in \mathfrak{g} \cap \mathfrak{wsp}(V)$ is semisimple, then $g = \lambda \operatorname{Id}$ for some $\lambda \in \mathbb{C}$.*

IV. *Assume $0 \neq g \in \mathfrak{g} \cap \mathfrak{wsp}(V)$ is nilpotent and let $m \geq 1$ be an integer such that $g^{m+1} = 0$ and $g^m \neq 0$. Then $g^m(X)$ is always non-zero and is contained in the singular locus of X. In particular, X' is singular.*

In what follows we prove the four parts of theorem IV.3.

I. Decomposition into symplectic and weks-symplectic part

Proof. Take $g \in \mathfrak{g}$ to be an arbitrary element. Then for every $x \in X_0$ one has

$$g(x) \in T_x X$$

and therefore

$$0 = \omega\big(g(x), x\big) = x^T g^T J x = \frac{1}{2} x^T \left(g^T J - J g\right) x.$$

Hence the quadratic polynomial $f(x) := x^T(g^T J - Jg)x$ is identically zero on X and hence it is in the ideal of X. Therefore by maximality of G and theorem III.5 the map $J\left(g^T J - Jg\right)$ is also in $\mathfrak{g}$. However,

$$J\left(g^T J - Jg\right) = Jg^T J + g,$$

so $Jg^T J \in \mathfrak{g}$ and both symplectic and weks-symplectic components g_+ and g_- are in $\mathfrak{g}$.

□

From the point of view of the conjecture, the symplectic part is fine. We would only need to prove that $g_- = \lambda \operatorname{Id}$. So from now on we assume $g = g_- \in \mathfrak{wsp}(V)$.

II. Action on tangent space

Proof. Let $\gamma_t := \exp(tg)$ for $t \in \mathbb{C}$. Then $\gamma_t \in G$ and hence it acts on X. Choose a point $x \in X_0$ and two tangent vectors in the same tangent space $u, v \in T_x X$. Then clearly also $\gamma_t(u)$ and $\gamma_t(v)$ are contained in one tangent space, namely $T_{\gamma_t(x)} X$. Hence:

$$\begin{aligned} 0 =& \omega\left(\gamma_t(u), \gamma_t(v)\right) \\ =& \omega\big((\operatorname{Id}_{2n} + tg + \ldots)u, (\operatorname{Id}_{2n} + tg + \ldots)v\big) \\ =& \omega(u,v) + t\big(\omega(gu,v) + \omega(u,gv)\big) + t^2(\ldots). \end{aligned}$$

In particular the part of the expression linear in t vanishes, hence $\omega(gu,v) + \omega(u,gv) = 0$. Combining this with equation (II.2) we get that:

$$\omega(gu,v) = \omega(u,gv) = 0.$$

However, this implies that $gu \in (T_x X)^{\perp_\omega} = T_x X$. Therefore g preserves the tangent space at every smooth point of X and hence also γ_t preserves that space.

□

III. Semisimple part

Since G is an algebraic subgroup in $\mathbf{GL}(V)$, hence $\mathfrak{g}$ has the natural Jordan decomposition inherited from $\mathfrak{gl}(V)$, i.e. if we write the Jordan decomposition for $g = g_s + g_n$, then $g_s, g_n \in \mathfrak{g}$ (see [Hum75, thm 15.3(b)]). Therefore by proposition II.4(i), proving that for $g \in \mathfrak{g} \cap \mathfrak{wsp}(V)$ we have $g_s = \lambda \operatorname{Id}_{2n}$ and $g_n = 0$ would be enough to establish the conjecture.

Here we deal with the semisimple part.

Proof. Argue by contradiction. Let V_1 be an arbitrary eigenspace of g and let V_2 be the sum of the other eigenspaces. If $g \neq \lambda \operatorname{Id}_{2n}$, then both V_1 and V_2 are non-zero and by proposition II.4(ii) and (iii) they are ω-perpendicular, complementary symplectic subspaces of V. Let $x \in X_0$ be any point. Since g preserves $T_x X$ by part II it follows that $T_x X = (T_x X \cap V_1) \oplus (T_x X \cap V_2)$. But then both $(T_x X \cap V_i) \subset V_i$ are Lagrangian subspaces, hence have constant (independent of x) dimensions. Hence $T_x X_0 = (T_x X_0 \cap V_1) \oplus (T_x X_0 \cap V_2)$ is a sum of two vector bundles and from proposition I.5 we get that X is a product of two Lagrangian subvarieties, contradicting our assumption that X' is indecomposable.

□

IV. Nilpotent part — X' is singular

Lemma IV.4. *Assume $X' \subset \mathbb{P}(V)$ is any closed subvariety preserved by the action of $\exp(tg)$ for some nilpotent endomorphism $g \in \mathfrak{gl}(V)$. If v is a point of the affine cone over X' and m is an integer such that $g^{m+1}(v) = 0$ and $g^m(v) \neq 0$, then $[g^m(v)] \in \mathbb{P}(V)$ is in X'.*

Proof. Point $[g^m(v)] \in \mathbb{P}(V)$ is just the limit of $[\exp(tg)(v)]$ as t goes to ∞.

□

Lemma IV.5. *Assume $g \in \mathfrak{gl}(V)$ is nilpotent and $g^{m+1} = 0$, $g^m \neq 0$ for an integer $m \geq 1$. Let $X \subset V$ be an affine cone over some irreducible projective subvariety in $\mathbb{P}(V)$, which is preserved by the action of $\exp(tg)$, but is not contained in the set of the fixed points. Assume that this action preserves the tangent space $T_x X$ at every smooth point x of X. If there exists a non-zero vector in V which is a smooth point of X contained in $g^m(X)$, then X is a linear subspace.*

Proof. *Step 0 - notation.* We let Y to be the closure of $g^m(X)$, so in particular Y is irreducible. By lemma IV.4, we know that $Y \subset X$. Let y be a general point of Y. Then by our assumptions y is a smooth point of both X and Y.

Next denote by

$$W_y := (g^m)^{-1}(\mathbb{C}^* y).$$

You can think of W_y as union of those lines in V (or points in the projective space $\mathbb{P}(V)$), which under the action of $\exp(tg)$ converge to the line spanned by

y (or $[y]$)[1] as t goes to ∞ . We also note that the closure $\overline{W_y}$ is a linear subspace spanned by an arbitrary element $v \in W_y$ and $\ker g^m$.

Also we let $F_y := W_y \cap X$, so that:

$$F_y := (g^m|_X)^{-1}(\mathbb{C}^* y).$$

Finally, v from now on will always denote an arbitrary point of F_y.

Step 1 - tangent space to X at points of F_y. Since y is a smooth point of X also F_y consists of smooth points of X. This is because the set of singular points is closed and $\exp(tg)$ invariant. By our assumptions $\exp(tg)$ preserves every tangent space to X and thus for every $v \in F_y$ we have:

$$T_v X = T_{\frac{1}{t^m}\exp(tg)(v)} X = T_{\lim_{t\to\infty}\left(\frac{1}{t^m}\exp(tg)(v)\right)} = T_y X.$$

So the tangent space to X is constant over the F_y and in particular $F_y \subset T_y X$.

Step 2 - dimensions of Y and F_y. From the definitions of Y and y and by step 1 we get that for any point $v \in F_y$:

$$T_y Y \;=\; \operatorname{im}(g^m|_{T_v X}) = \operatorname{im}(g^m|_{T_y X}).$$

Hence $\dim Y = \dim T_y Y = \operatorname{rk}(g^m|_{T_y X})$.

Since y was a general point of Y, we have that:

$$\dim Y + \dim F_y = \dim X + 1.$$

So $\dim F_y = \dim \ker(g^m|_{T_y X}) + 1$.

Step 3 - the closure of F_y is a linear subspace. From the definition of F_y and step 1 we know that $F_y \subset T_y X \cap W_y$ and

$$T_y X \cap \overline{W_y} = T_y X \cap \operatorname{span}\{v, \ker g^m\} = \operatorname{span}\{v, \ker(g^m|_{T_y X})\}.$$

Hence $\dim F_y = \dim T_y X \cap W_y$, so the closure of F_y is exactly $T_y X \cap \overline{W_y}$ and clearly this closure is contained in X. In particular $\ker(g^m|_{T_y X}) \subset X$.

Step 4 - Y is contained in $\ker(g^m|_{T_y X})$. Let Z be $X \cap \ker g^m$. By step 3 we know that $\ker(g^m|_{T_y X}) \subset Z$. Now we calculate the local dimension of Z at y:

$$\dim \ker(g^m|_{T_y X}) \le \dim_y Z \le \dim T_y Z \le \dim(T_y X \cap \ker g^m) = \dim \ker(g^m|_{T_y X}).$$

Since the first and the last entries are identical, we must have all equalities. In particular the local dimension of Z at y is equal to the dimension of the tangent space to Z at y. So y is a smooth point of Z and therefore there is a

[1] This statement is not perfectly precise, though it is true on an open dense subset. There are some other lines, which converge to $[y]$, namely those generated by $v \in \ker g^m$, but $g^k(v) = \lambda y$ for some $k < m$. We are not interested in those points.

unique component of Z passing through y, namely the linear space $\ker(g^m|_{T_yX})$. Since Y is contained in Z (because $\operatorname{im} g^m \subset \ker g^m$) and $y \in Y$, we must have $Y \subset \ker(g^m|_{T_yX})$.

Step 5 - vary y. Recall, that by step 1 the tangent space to X is the same all over F_y. So also it is the same on every smooth point of X, which falls into the closure of F_y. But by step 4, Y is a subset of $\ker(g^m|_{T_yX})$, which is in the closure of F_y by step 3. So the tangent space to X is the same for an open subset of points in Y. Now apply again step 1 for different y's in this open subset and we get that X has constant tangent space on a dense open subset of X. This is possible if and only if X is a linear subspace, which completes the proof of the lemma.

□

Now part IV of the theorem follows easily:

Proof. By the assumptions of the theorem X is not contained in any hyperplane, so in particular X is not contained in $\ker g^m$. So by lemma IV.4 the image $g^m(X)$ contains points other than 0. Next by lemma IV.5 and part II of the theorem, since X cannot be a linear subspace, there can be no smooth points of X in $g^m(X)$.

□

Smooth case

We conclude that parts I, III and IV of theorem IV.3 together with proposition II.4(i) and [Hum75, thm. 15.3(b)] imply theorem IV.2. We only note that a smooth Legendrian subvariety is either a linear subspace or it is indecomposable.

IV.3 Some comments

Conjecture IV.1 is now reduced to the following special case not covered by theorem IV.3:

Conjecture IV.6. *Let $X' \subset \mathbb{P}(V)$ be an irreducible Legendrian subvariety. Let $g \in \mathfrak{wsp}(V)$ be a nilpotent endomorphism and m be an integer such that $g^m \neq 0$ and $g^{m+1} = 0$. Assume that the action of $\exp(tg)$ preserves X'. Assume moreover, that X' is singular at points of the image of the rational map $g^m(X')$. Then X' is decomposable.*

We also note the improved relation between projective automorphisms of a Legendrian subvariety and quadratic equations satisfied by its points:

Corollary IV.7. *Let $X \subset \mathbb{P}(V)$ be an irreducible Legendrian subvariety for which conjecture IV.1 holds (for example X is smooth). If $G < \mathbb{P}\mathbf{GL}(V)$ is the* *maximal subgroup preserving* X*, then* $\dim G = \dim \mathcal{I}_2(X)$*, where* $\mathcal{I}_2(X)$ *is the space of homogeneous quadratic polynomials vanishing on* X*.*

Proof. It follows immediately from the statement of the conjecture and theorem III.5.

□

Finally, it is important to note, that theorem IV.3 part III does not imply that every torus acting on an indecomposable, but singular Legendrian variety X' is contained in the image of $\mathbf{Sp}(V)$. It only says that the intersection of such a torus with the weks-symplectic part is always finite. Therefore if there is a non-trivial torus acting on X', there is also some non-trivial connected subgroup of $\mathbf{Sp}(V)$ acting on X' and also some quadratic equations in the ideal of X'.

Chapter V

Toric Legendrian subvarieties in projective space

The content of this chapter is published in [Buc07c].

We apply theorem IV.2 to classify smooth toric Legendrian subvarieties. We choose appropriate coordinates to reduce this problem to some combinatorics (for surface case — see §V.2) and some elementary geometry of convex bodies (for higher dimensions — see §V.3). Eventually we get:

Theorem V.1. *Every smooth toric Legendrian subvariety in a projective space is isomorphic to one of the following:*

- *a linear subspace,*
- $\mathbb{P}^1 \times Q_1 \subset \mathbb{P}^5$,
- $\mathbb{P}^1 \times Q_2 \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$
- *or* $\mathbb{P}^2$ *blown up in three non-colinear points.*

For proofs see corollaries V.7 and V.11. The linear subspace is not really interesting, the products $\mathbb{P}^1 \times Q_1$ and $\mathbb{P}^1 \times Q_2$ are well known (see §I.1.2). The last case of blow up was an original example of [Buc07c].

V.1 Classification of toric Legendrian varieties

Within this chapter X is a toric subvariety of dimension $n-1$ in a projective space of dimension $2n-1$. We assume it is embedded torically, so that the action of $T := (\mathbb{C}^*)^{n-1}$ on X extends to an action on the whole $\mathbb{P}^{2n-1}$, but we do not assume that the embedding is projectively normal. The notation is based on [Stu97] though we also use techniques of [Oda88]. We would like to understand when X can be Legendrian with respect to some contact structure on $\mathbb{P}^{2n-1}$ and in particular, when it can be a smooth toric Legendrian variety.

There are two reasons for considering non projectively normal toric varieties here. The first one is that the new example we find is not projectively normal. The second one is the conjecture [Stu97, conj. 2.9], which says that a smooth, toric, projectively normal variety is defined by quadrics. We do not expect to produce a counterexample to this conjecture and on the other hand all smooth Legendrian varieties defined by quadrics are known to be just the subadjoint varieties (see [Buc06, thm.5.11]).

In addition we assume that either X is smooth or at least the following condition is satisfied:

($\star$) The action of the torus T on $\mathbb{P}^{2n-1}$ preserves the standard contact structure on $\mathbb{P}^{2n-1}$. In other words, the image of $T \to \mathbb{P}\mathbf{GL}_{2n}$ is contained in the image of $\mathbf{Sp}_{2n} \to \mathbb{P}\mathbf{GL}_{2n}$.

In the case where X is smooth, the ($\star$) condition is always satisfied by theorem IV.2. But for some statements below we do not need non-singularity, so we only assume ($\star$).

Theorem V.2. *Let $X \subset \mathbb{P}^{2n-1}$ be a toric (in the above sense) non-degenerate Legendrian subvariety satisfying ($\star$). Then there exists a choice of symplectic coordinates on V and coprime integers $a_0 \geq a_1 \geq \ldots \geq a_{n-1} > 0$ such that X is the closure of the image of the following map:*

$$T \ni (t_1, \ldots, t_{n-1}) \mapsto [-a_0 t_1^{a_1} t_2^{a_2} \ldots t_{n-1}^{a_{n-1}},\ \ a_1 t_1^{a_0},\ a_2 t_2^{a_0},\ \ldots,\ a_{n-1} t_{n-1}^{a_0},$$
$$t_1^{-a_1} t_2^{-a_2}, \ldots, t_{n-1}^{-a_{n-1}},\ \ t_1^{-a_0},\ t_2^{-a_0},\ \ldots,\ t_{n-1}^{-a_0}] \in \mathbb{P}^{2n-1}.$$

In other words, X is the closure of the orbit of a point

$$[-a_0, a_1, a_2, \ldots, a_{n-1}, 1, 1, \ldots 1] \in \mathbb{P}^{2n-1}$$

under the torus action with weights

$$w_0 := (a_1, a_2, \ldots, a_{n-1}),$$
$$w_1 := (a_0, 0, \ldots, 0),\ w_2 := (0, a_0, 0, \ldots, 0),\ \ldots,\ w_{n-1} := (0, \ldots, 0, a_0)$$
$$\text{and } -w_0, -w_1, \ldots, -w_{n-1}.$$

Moreover every such X is a non-degenerate toric Legendrian subvariety.

We are aware that for many choices of the a_i's from the theorem, the action of the torus on X (and on $\mathbb{P}^{2n-1}$) is not faithful, so that for such examples a better choice of coordinates could be made. However, we are willing to pay the price of taking a quotient of T to get a uniform description. One advantage of the description given in the theorem is that a part of it is almost independent of the choice of the a_i's. This part is the $(n-1)$-dimensional "octahedron" $\operatorname{conv}\{w_1, \ldots w_{n-1}, -w_1, \ldots - w_{n-1}\} \subset \mathbb{Z}^{n-1} \otimes \mathbb{R}$.

Proof. Assume X is Legendrian with respect to a symplectic form ω, that X is non-degenerate, that the torus T acts on $\mathbb{P}^{2n-1}$ preserving X and satisfies $(\star)$. Replacing if necessary T by some covering we may assume that $T \to \mathbb{P}\mathbf{GL}_{2n}$ factorises through a maximal torus $T_{\mathbf{Sp}_{2n}} \subset \mathbf{Sp}_{2n}$:

$$T \to T_{\mathbf{Sp}_{2n}} \subset \mathbf{Sp}_{2n} \to \mathbb{P}\mathbf{GL}_{2n}.$$

This implies, that for an appropriate symplectic basis the variety X is the closure of the image of the map $T \to \mathbb{P}^{2n-1}$ given by:

$$T \ni t \mapsto [x_0 t^{w_0}, x_1 t^{w_1} \ldots, x_{n-1} t^{w_{n-1}}, t^{-w_0}, t^{-w_1} \ldots, t^{-w_{n-1}}] \in \mathbb{P}^{2n-1}$$

where $x_i \in \mathbb{C}$, $w_i \in \mathbb{Z}^{n-1}$ and for $v = (v_1, \ldots v_{n-1}) \in \mathbb{Z}^{n-1}$ we let $t^v := t_1^{v_1} \ldots t_{n-1}^{v_{n-1}}$. This means that X is the closure of the T-orbit of the point[1] $[x_0, \ldots x_{n-1}, 1, \ldots, 1]$ where T acts with weights $w_0, \ldots w_{n-1}, -w_0, \ldots, -w_{n-1}$.

Since X is non-degenerate, the weights are pairwise different. Also the weights are not contained in any hyperplane in $\mathbb{Z}^{n-1} \otimes \mathbb{R}$, because the dimension of T is equal to the dimension of X and we assume X has an open orbit of the T-action. So there exists exactly one (up to scalar) linear relation:

$$-a_0 w_0 + a_1 w_1 + \ldots + a_{n-1} w_{n-1} = 0.$$

We assume that the a_i's are coprime integers. Permuting coordinates appropriately we can assume that $|a_0| \geq |a_1| \geq \ldots \geq |a_{n-1}| \geq 0$. After a symplectic change of coordinates, we can assume without loss of generality that all the a_i's are non negative by exchanging w_i with $-w_i$ (and x_i with $-\frac{1}{x_i}$) if necessary. Clearly not all the a_i's are zero so in particular $a_0 > 0$ and hence

$$w_0 = \frac{a_1 w_1 + \ldots + a_{n-1} w_{n-1}}{a_0}.$$

Therefore, if we set $e_i := \frac{w_i}{a_0}$ for $i \in \{1, \ldots, n-1\}$, the points e_i form a basis of a lattice M containing all w_i's. The lattice M might be finer than the one generated by the w_i's. Replacing again T by a finite cover, we can assume that the action of T is expressible in the terms of weights in M. Then:

$$\begin{aligned} w_0 &= a_1 e_1 + \ldots + a_{n-1} e_{n-1}, \\ w_1 &= a_0 e_1, \\ &\vdots, \\ w_{n-1} &= a_0 e_{n-1}. \end{aligned}$$

[1] Note that usually one assumes that this point is just [1,…,1]. In our case we would have to consider non-symplectic coordinates. We prefer to deal with a point with more complicated coordinates.

It remains to prove three things: that $a_{n-1} > 0$, that the x_i's might be chosen as in the statement of the theorem and finally that every such variety is actually Legendrian. We will do all three together.

The torus acts symplectically on the projective space, thus the tangent spaces to the affine cone are Lagrangian if and only if just one tangent space at a point of the open orbit is Lagrangian. So take the point $[x_0, \dots x_{n-1}, 1, \dots, 1]$. The affine tangent space is spanned by the following vectors:

$$\begin{aligned}
v :=& (x_0, \quad x_1, \quad x_2, \dots, \quad x_{n-1}, \quad 1, \quad 1, \quad 1, \dots, \quad 1),\\
u_1 :=& (x_0 a_1, x_1 a_0, \quad 0, \dots, \quad 0, \quad -a_1, -a_0, \quad 0, \dots, \quad 0),\\
u_2 :=& (x_0 a_2, \quad 0, x_2 a_0, \dots, \quad 0, \quad -a_2, \quad 0, -a_0, \dots, \quad 0),\\
&\vdots\\
u_{n-1} :=& (x_0 a_{n-1}, \quad 0, \quad 0, \dots, x_{n-1} a_0, \; -a_{n-1}, \quad 0, \quad 0, \dots, -a_0).
\end{aligned}$$

Now the products are following:

$$\begin{aligned}
\omega(u_i, u_j) &= 0;\\
\omega(u_i, v) &= 2(x_0 a_i + x_i a_0).
\end{aligned}$$

Therefore the linear space spanned by v and the u_i's is Lagrangian if and only if:

$$x_i = -x_0 \frac{a_i}{a_0}.$$

In particular, since $x_i \neq 0$, the a_i cannot be zero either. After another conformal symplectic base change, we can assume that $x_0 = -a_0$ and then $x_i = a_i$. On the other hand, the above equation is satisfied for the variety in the theorem. Hence the theorem is proved.

□

Our next goal is to determine for which values of the a_i's the variety X is smooth. The curve case is not interesting at all and also very easy, so we start from $n = 3$, i.e. Legendrian surfaces.

V.2 Smooth toric Legendrian surfaces

We are interested in knowing when the toric projective surface with weights of torus action

$$\begin{aligned}
w_0 :=&(a_1, a_2), & w_1 :=&(a_0, 0), & w_2 :=&(0, a_0),\\
-w_0 =&(-a_1, -a_2), & -w_1 =&(-a_0, 0), & -w_2 =&(0, -a_0)
\end{aligned}$$

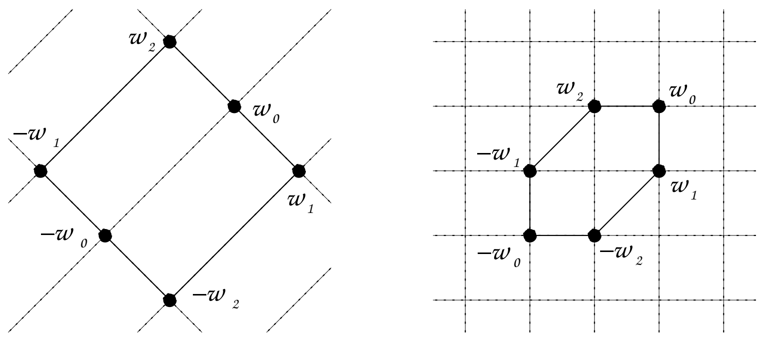


Figure V.1: The two examples of weights giving smooth toric Legendrian surfaces.

is smooth. Our assumptions on the a_i's are following:

$$a_0 \geq a_1 \geq a_2 > 0 \tag{V.3}$$

and a_0, a_1, a_2 are coprime integers.

Example V.4. *Let $a_0 = 2$ and $a_1 = a_2 = 1$ (see figure V.1). Then X is the product of $\mathbb{P}^1$ and a quadric plane curve Q_1.*

Example V.5. *Let $a_0 = a_1 = a_2 = 1$ (see figure V.1). Although the embedding is not projectively normal (we lack the weight $(0,0)$ in the middle), the image is smooth anyway. Then X is the blow up of $\mathbb{P}^2$ in three non-colinear points.*

We will prove there is no other smooth example.

We must consider two cases (see figure V.2): either $a_0 > a_1 + a_2$ (which means that w_0 is in the interior of the square $\operatorname{conv}\{w_1, w_2, -w_1, -w_2\}$) or $a_0 \leq a_1 + a_2$ (so that w_0 is outside or on the border of the square).

Geometrically, case $a_0 > a_1 + a_2$ means, that the normalisation of X is $\mathbb{P}^1 \times \mathbb{P}^1$. It is just an easy explicit verification that X is not smooth with these additional weights in the interior.

In the other case, for a vertex v of the polytope

$$\operatorname{conv}\{w_0, w_1, w_2, -w_0, -w_1, -w_2\},$$

we define the sublattice M_v to have the origin at v and to be generated by

$$\{w_0 - v, w_1 - v, w_2 - v, -w_0 - v, -w_1 - v, -w_2 - v\}.$$

Since X is smooth, for every vertex v the vectors of the edges meeting at v must form a basis of M_v (compare with [Stu97, prop.2.4 & lemma 2.2]). In particular, if $v = -w_2$ (it is immediate from inequalities (V.3) that v is indeed a vertex), then $w_2 - (-w_2) = (0, 2a_0)$ can be expressed as an integer combination

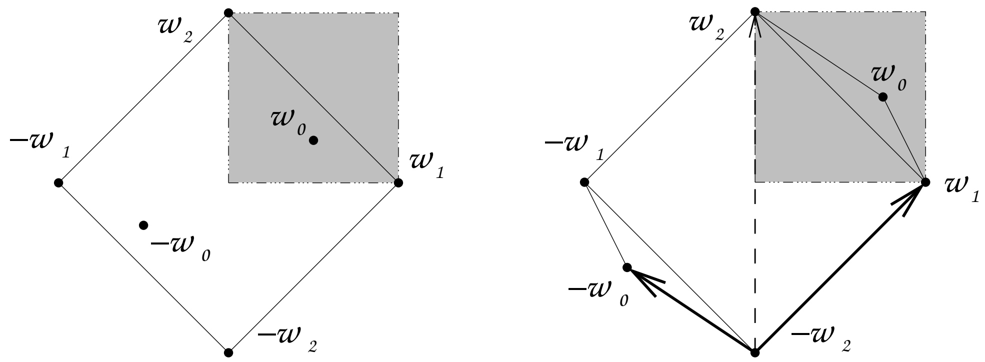


Figure V.2: Due to the inequalities $a_0 \geq a_1 > 0$ and $a_0 \geq a_2 > 0$, the weight w_0 is located somewhere in the grey square. The two cases we consider are if w_0 is also inside the square $\operatorname{conv}\{w_1, w_2, -w_1, -w_2\}$ (left figure) or it is outside (right figure). In the second case, a necessary condition to get a smooth variety, is that the two bold vectors generate a lattice containing all the weights. In particular the dashed vector can be obtained as an integer combination of the bold ones.

of $w_1 + w_2 = (a_0, a_0)$ and $-w_0 + w_2 = (-a_1, a_0 - a_2)$ (see the right hand side of figure V.2). So write:

$$(0, 2a_0) = k(a_0, a_0) + l(-a_1, a_0 - a_2) \tag{V.6}$$

for some integers k and l. It is obvious that k and l must be strictly positive, since w_2 is in the cone generated by $w_1 + w_2$ and $-w_0 + w_2$ with the vertex at $-w_2$. But then (since $a_0 - a_2 \geq 0$) from equation (V.6) on the second coordinate we get that either $k = 1$ or $k = 2$.

If $k = 1$, then we easily get that:

$$\begin{cases} a_0 = l a_1 \\ a_0 = a_1 + a_2. \end{cases}$$

Hence $(l-1)a_1 = a_2$ and by inequalities (V.3) we get $l = 2$ and therefore (since the a_i's are coprime) $(a_0, a_1, a_2) = (2, 1, 1)$, which is example V.4.

On the other hand, if $k = 2$, then

$$a_0 = a_2$$

and hence by inequalities (V.3) and since the a_i's are coprime, we get $(a_0, a_1, a_2) = (1, 1, 1)$, which is example V.5.

Corollary V.7. *If $X \subset \mathbb{P}^5$ is smooth toric Legendrian surface, then it is either $\mathbb{P}^1 \times Q_1$ or $\mathbb{P}^2$ blown up in three non-colinear points or plane $\mathbb{P}^2 \subset \mathbb{P}^5$.*

□

V.3 Higher dimensional toric Legendrian varieties

In this section we assume that $n \geq 4$. By means of the geometry of convex bodies we will prove there is only one smooth toric non-degenerate Legendrian variety in dimension $n-1=3$ and no more in higher dimensions. We use theorem V.2 so that we have a toric variety with weights:

$$\begin{aligned} &w_0 := (a_1, a_2, \ldots, a_{n-1}),\\ &w_1 := (a_0, 0, \ldots 0),\\ &\vdots\\ &w_{n-1} := (0, \ldots 0, a_0),\\ &-w_0, -w_1, \ldots, -w_{n-1} \end{aligned}$$

where the a_i's are coprime positive integers with $a_0 \geq a_1 \geq \ldots \geq a_{n-1}$.

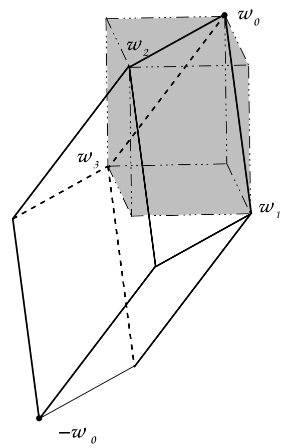


Figure V.3: The smooth example in dimension 3: $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$.

Example V.8. *Let $n = 4$ and $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$. Then the related toric variety is $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ (see figure V.3).*

Further, let A be the polytope defined by the weights:

$$A := \operatorname{conv}\{w_0, w_1, \ldots, w_{n-1}, -w_0, -w_1, \ldots, -w_{n-1}\} \subset \mathbb{Z}^{n-1} \otimes \mathbb{R}.$$

Lemma V.9. *Let $I, J \subset \{1, \ldots, n-1\}$ be two complementary subsets of indexes.*

(a) Assume $i_1, i_2 \in I$ and $i_1 \neq i_2$. If

$$\left| \sum_{i \in I} a_i - \sum_{j \in J} a_j \right| < a_0,$$

then the interval (w_{i_1}, w_{i_2}) is an edge of A.

(b) Assume $k \in I$ and $l \in J$. If

$$\sum_{i \in I} a_i - \sum_{j \in J} a_j > a_0,$$

then both intervals (w_0, w_k) and $(w_0, -w_l)$ are edges of A.

(c) If $k, l \in \{1, \dots, n-1\}$ and $k \neq l$, then $(w_k, -w_l)$ is an edge of A.

Proof. Fix $\epsilon > 0$ small enough, set $\alpha := \sum_{i \in I} a_i - \sum_{j \in J} a_j$ and define the following hyperplanes in $\mathbb{Z}^{n-1} \otimes \mathbb{R}$:

$$H_a := \left\{ \sum_{i \in I} x_i - (1-\epsilon) \sum_{j \in J} x_j = a_0 \right\},$$

$$H_b := \left\{ (a_0 - a_k) \left(\sum_{i \in I} x_i - \sum_{j \in J} x_j - \alpha \right) + (\alpha - a_0)(x_k - a_k) = 0 \right\},$$

$$H'_b := \left\{ (a_0 + a_l) \left(\sum_{i \in I} x_i - \sum_{j \in J} x_j - \alpha \right) + (\alpha - a_0)(x_l + a_l) = 0 \right\}$$

$$\text{and } H_c := \{ x_k - x_l = a_0 \}.$$

Assuming the inequality of (a), $H_a \cap A$ is equal to $\operatorname{conv}\{w_i \mid i \in I\}$ and the rest of A lies on one side of H_a. So H_a is a supporting hyperplane for the face $\operatorname{conv}\{w_i \mid i \in I\}$, which is a simplex of dimension $(\#I - 1)$ and therefore all its edges are also edges of A as claimed in (a).

Next assume that the inequality of (b) holds. Then H_b (respectively H'_b) is a supporting hyperplane for the edge (w_0, w_k) (respectively $(w_0, -w_l)$).

Similarly, in the case of (c), H_c is a supporting hyperplane for $\{w_k, -w_l\}$.

□

Theorem V.10. *Let $X \subset \mathbb{P}^{2n-1}$ be a toric non-degenerate Legendrian variety of dimension $n-1$ satisfying $(\star)$ (see page 54). If $n \geq 4$ and normalisation of X has at most quotient singularities, then $n = 4$ and $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Since the normalisation of X has at most quotient singularities, it follows that the polytope A is simple, i.e. every vertex has exactly $n-1$ edges (see [Ful93] or [Oda88, §2.4, p. 102]). We will prove this is impossible, unless $n = 4$ and $(a_0, a_1, a_2, a_3) = (1, 1, 1, 1)$.

If $w_0 \in B := \operatorname{conv}\{w_1, \dots, w_{n-1}, -w_1, \dots - w_{n-1}\}$, then A is just equal to B and clearly in such a case every vertex of A has $2(n-2)$ edges. Hence more than $n-1$ for $n \geq 4$.

Thus from now on we can assume that $a_1 + \ldots + a_{n-1} > a_0$. So by lemma V.9(b), (w_0, w_i) is an edge for every $i \in \{1, \ldots, n-1\}$.

Choose any $j \in \{1, \ldots, n-1\}$ and set $I := \{1, \ldots, j-1, j+1, \ldots, n-1\}$.

If either

$$\left| \left(\sum_{i \in I} a_i \right) - a_j \right| < a_0 \quad \text{or}$$

$$\left(\sum_{i \in I} a_i \right) - a_j > a_0,$$

then using lemma V.9 we can count the edges at either w_i or w_0 and see that there is always more than $n-1$ of them. We note that $a_j - \left(\sum_{i \in I} a_i\right) \geq a_0$ never happens due to our assumptions on the a_i's.

Therefore the remaining case to consider is

$$\left(\sum_{i \in I} a_i \right) - a_j = a_0,$$

where the equality holds for every $j \in \{1, \ldots, n-1\}$. This implies:

$$a_1 = a_2 = \ldots = a_{n-1} = \frac{1}{n-3} a_0.$$

Since the a_i's are positive integers and coprime, we must have

$$(a_0, a_1, \ldots, a_{n-1}) = (n-3, 1, \ldots, 1)$$

which is exactly example V.8 for $n = 4$. Otherwise, if $n \geq 5$ we can take $J := \{j_1, j_2\}$ for any two different $j_1, j_2 \in \{1, \ldots, n-1\}$ and set I to be the complement of J. Then $\#I \geq 2$ and by lemma V.9(a) and (c) there are too many edges at the w_i's.

□

Corollary V.11. *If $X \subset \mathbb{P}^{2n-1}$ is a smooth toric Legendrian subvariety and $n \geq 4$, then it is either a linear subspace or $n = 4$ and $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.*

□

Chapter VI

Examples of quasihomogeneous Legendrian varieties

The content of this chapter is published in [Buc07b].

We construct a family of examples of Legendrian subvarieties in projective spaces. Although most of them are singular, a new example of a smooth Legendrian variety in dimension 8 is in this family. The 8-fold has interesting properties: it is a compactification of the special linear group, a Fano manifold of index 5 and Picard number 1 (see theorem VI.4(b)). Also we show how this construction generalises to give new smooth examples in dimensions 5 and 14 (see §VI.2.1).

In §VI.1 we introduce the notation for this chapter. In §VI.2 we formulate the results and make some comments on possible generalisations. In §VI.3 we study the structure of a group action related to the problem. In §VI.4 we finally prove the results.

VI.1 Notation and definitions

For this chapter we fix an integer $m \geq 2$.

Vector space V

Let V be a vector space over complex numbers $\mathbb{C}$ of dimension $2m^2$, which we interpret as a space of pairs of $m \times m$ matrices. The coordinates are: a_{ij} and b_{ij} for $i, j \in \{1, \dots m\}$. By A we denote the matrix (a_{ij}) and similarly for B and (b_{ij}).

Given two $m \times m$ matrices A and B, by (A, B) we denote the point of the vector space V, while by $[A, B]$ we denote the point of the projective space $\mathbb{P}(V)$.

Sometimes, we will represent some linear maps $V \longrightarrow V$ and some 2-linear forms $V \otimes V \longrightarrow \mathbb{C}$ as $2m^2 \times 2m^2$ matrices. In such a case we will assume the coordinates on V are given in the lexicographical order:

$$a_{11}, \dots, a_{1m}, a_{21}, \dots, a_{mm}, b_{11}, \dots, b_{1m}, b_{21}, \dots, b_{mm}.$$

Symplectic form ω

On V we consider the standard symplectic form

$$\omega\big((A,B),(A',B')\big) := \sum_{i,j}(a_{ij}b'_{ij} - a'_{ij}b_{ij}) = \operatorname{tr}\left(A(B')^T - A'B^T\right). \qquad \text{(VI.1)}$$

Further we set J to be the matrix of ω:

$$J := M(\omega) = \begin{bmatrix} 0 & \mathrm{Id}_{m^2} \\ -\mathrm{Id}_{m^2} & 0 \end{bmatrix}.$$

Varieties Y, $X_{\mathrm{inv}}(m)$ and $X_{\mathrm{deg}}(m,k)$

We consider the following subvariety of $\mathbb{P}(V)$:

$$Y := \left\{[A,B] \in \mathbb{P}(V) \mid AB^T = B^TA = \lambda^2 \,\mathrm{Id}_m \text{ for some } \lambda \in \mathbb{C}\right\}. \qquad \text{(VI.2)}$$

The square at λ seems to be irrelevant here, but it slightly simplifies the notation in the proofs of theorem VI.4(b) and proposition VI.10(ii). Although it is not essential for the content of this chapter, we note that Y is F-cointegrable.

Further we define two types of subvarieties of Y:

$$X_{\mathrm{inv}}(m) := \overline{\left\{ \left[g, (g^{-1})^T\right] \in \mathbb{P}(V) \mid \det g = 1 \right\}},$$
$$X_{\mathrm{deg}}(m,k) := \left\{[A,B] \in \mathbb{P}(V) \mid AB^T = B^TA = 0,\ \operatorname{rk} A \le k,\ \operatorname{rk} B \le m-k\right\},$$

where $k \in 0, 1, \dots m$. The varieties $X_{\mathrm{deg}}(m,k)$ have been also studied by [Str82] and [MT99]. $X_{\mathrm{inv}}(m)$ (especially $X_{\mathrm{inv}}(3)$) is the main object of this chapter.

Automorphisms ψ_μ

For any $\mu \in \mathbb{C}^*$ we let ψ_μ be the following linear automorphism of V:

$$\psi_\mu\big((A,B)\big) := (\mu A, \mu^{-1}B).$$

Also the induced automorphism of $\mathbb{P}(V)$ will be written in the same way:

$$\psi_\mu\big([A,B]\big) := [\mu A, \mu^{-1}B].$$

Groups G and $\widetilde{G}$, Lie algebra $\mathfrak{g}$ and their representation

We set $\widetilde{G} := \mathbf{GL}_m \times \mathbf{GL}_m$ and let it act on V by:

$$(g,h) \in \widetilde{G},\ \ g,h \in \mathbf{GL}_m,\ \ (A,B) \in V$$
$$(g,h) \cdot (A,B) := (g^T A h, g^{-1} B (h^{-1})^T).$$

This action preserves the symplectic form ω.

We will mostly consider the restricted action of $G := \mathbf{SL}_m \times \mathbf{SL}_m < \widetilde{G}$.

We also set $\mathfrak{g} := \mathfrak{sl}_m \times \mathfrak{sl}_m$ to be the Lie algebra of G and we have the tangent action of $\mathfrak{g}$ on V:

$$(g,h) \cdot (A,B) = (g^T A + Ah, -gB - Bh^T).$$

Though we denote the action of the groups G, $\widetilde{G}$ and the Lie algebra $\mathfrak{g}$ by the same $\cdot$ we hope it will not lead to any confusion. Also the induced action of G and $\widetilde{G}$ on $\mathbb{P}(V)$ will be denoted by $\cdot$.

Orbits $\mathcal{INV}^m$ and $\mathcal{DEG}^m_{k,l}$

We define the following sets:

$$\mathcal{INV}^m := \left\{ \left[g, \left(g^{-1}\right)^T\right] \in \mathbb{P}(V) \mid \det g = 1 \right\},$$
$$\mathcal{DEG}^m_{k,l} := \left\{ [A,B] \in \mathbb{P}(V) \mid AB^T = B^T A = 0,\ \operatorname{rk} A = k,\ \operatorname{rk} B = l \right\},$$

so that $X_{\text{inv}}(m) = \overline{\mathcal{INV}^m}$ and $X_{\text{deg}}(m,k) = \overline{\mathcal{DEG}^m_{k,m-k}}$.

Clearly, if $k + l > m$, then $\mathcal{DEG}^m_{k,l}$ is empty, so whenever we are considering $\mathcal{DEG}^m_{k,l}$ we will assume $k + l \leq m$.

Elementary matrices E_{ij} and points p_1 and p_2

Let E_{ij} be the elementary $m \times m$ matrix with unit in the i^{th} row and the j^{th} column and zeroes elsewhere.

We distinguish two points $p_1 \in \mathcal{DEG}^m_{1,0}$ and $p_2 \in \mathcal{DEG}^m_{0,1}$:

$$p_1 := [E_{mm}, 0] \quad \text{and} \quad p_2 := [0, E_{mm}]$$

These points will be usually chosen as nice representatives of the closed orbits $\mathcal{DEG}^m_{1,0}$ and $\mathcal{DEG}^m_{0,1}$.

Submatrices - extracting rows and columns

Assume A is an $m \times m$ matrix and I, J are two sets of indices of cardinality k and l respectively:

$$I := \{i_1, i_2, \ldots, i_k \mid 1 \le i_1 < i_2 < \ldots < i_k \le m\},$$
$$J := \{j_1, j_2, \ldots, j_l \mid 1 \le j_1 < j_2 < \ldots < j_l \le m\}.$$

Then we denote by $A_{I,J}$ the $(m-k)\times(m-l)$ submatrix of A obtained by removing rows of indices I and columns of indices J. Also for a set of indices I we denote by I' the set of $m-k$ indices complementary to I.

We will also use a simplified version of the above notation when we remove only a single column and single row: A_{ij} denotes the $(m-1)\times(m-1)$ submatrix of A obtained by removing i-th row and j-th column, i.e. $A_{ij} = A_{\{i\},\{j\}}$

Also in the simplest situation where we remove only the last row and the last column, we write A_m, so that $A_m = A_{mm} = A_{\{m\},\{m\}}$.

VI.2 Main results

In this chapter we give a classification[1] of Legendrian subvarieties in $\mathbb{P}(V)$ that are contained in Y.

Theorem VI.3. *Let projective space $\mathbb{P}(V)$, varieties Y, $X_{\mathrm{inv}}(m)$, $X_{\mathrm{deg}}(m,k)$ and automorphisms ψ_μ be defined as in §VI.1. Assume $X \subset \mathbb{P}(V)$ is an irreducible subvariety. Then X is Legendrian and contained in Y if and only if X is one of the following varieties:*

1. *$X = \psi_\mu(X_{\mathrm{inv}}(m))$ for some $\mu \in \mathbb{C}^*$ or*
2. *$X = X_{\mathrm{deg}}(m,k)$ for some $k \in \{0, 1, \ldots m\}$.*

The idea of the proof of theorem VI.3 is based on the observation that every Legendrian subvariety that is contained in Y must be invariant under the action of the group G. This is explained in §VI.3. A proof of the theorem is presented in §VI.4.1.

Also we analyse which of the above varieties appearing in 1. and 2. are smooth:

Theorem VI.4. *With the definition of $X_{\mathrm{inv}}(m)$ as in §VI.1, the family $X_{\mathrm{inv}}(m)$ contains the following varieties:*

(a) $X_{\mathrm{inv}}(2)$ is a linear subspace.

[1]This problem was suggested by Sung Ho Wang.

(b) $X_{\mathrm{inv}}(3)$ *is smooth, its Picard group is generated by a hyperplane section. Moreover* $X_{\mathrm{inv}}(3)$ *is a compactification of* $\mathbf{SL}_3$ *and it is isomorphic to a hyperplane section of Grassmannian* $Gr(3,6)$*. The connected component of* $\mathrm{Aut}(X_{\mathrm{inv}}(3))$ *is equal to* $G = \mathbf{SL}_3 \times \mathbf{SL}_3$ *and* $X_{\mathrm{inv}}(3)$ *is not a homogeneous space.*

(c) $X_{\mathrm{inv}}(4)$ *is the 15 dimensional spinor variety* $\mathbb{S}_6$*.*

(d) For $m \geq 5$*, the variety* $X_{\mathrm{inv}}(m)$ *is singular.*

A proof of the theorem is explained in §VI.4.3.

Variety $X_{\mathrm{inv}}(3)$ is an original example of [Buc07b]. Also it is the first described example of a smooth non-homogeneous Legendrian variety of dimension bigger than 2 (see §I.1.2). This example is very close to a homogeneous one, namely it is isomorphic to a hyperplane section of $Gr(3,6)$, a well known subadjoint variety. So a natural question arises, whether general hyperplane sections of other Legendrian varieties admit a Legendrian embedding. The answer is yes and we explain it (as well as many conclusions from this surprisingly simple observation) in chapter VII.

Theorem VI.5. *With the definition of* $X_{\mathrm{deg}}(m)$ *as in §VI.1, variety* $X_{\mathrm{deg}}(m,k)$ *is smooth if and only if* $k = 0$ *,* $k = m$ *or* $(m,k) = (2,1)$*. In the first two cases,* $X_{\mathrm{deg}}(m,0)$ *and* $X_{\mathrm{deg}}(m,m)$ *are linear spaces, while* $X_{\mathrm{deg}}(2,1) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$*.*

A proof of the theorem is presented in §VI.4.2.

VI.2.1 Generalisation: Representation theory and further examples

The interpretation of theorem VI.4 (b) and (c) can be following: We take the exceptional Legendrian variety $Gr(3,6)$, slice it with a linear section and we get a description, which generalised to matrices of bigger size, gives the bigger exceptional Legendrian variety $\mathbb{S}_6$. A similar connection can be established between other exceptional Legendrian varieties (see §I.1.2).

For instance, assume that V^{sym} is a vector space of dimension $2\binom{m+1}{2}$, which we interpret as the space of pairs of $m \times m$ symmetric matrices A, B. Now in $\mathbb{P}(V^{sym})$ consider the subvariety $X^{sym}_{\mathrm{inv}}(m)$, which is the closure of the following set:

$$\{[A, A^{-1}] \in \mathbb{P}(V^{sym}) | A = A^T \text{ and } \det A = 1\}.$$

Theorem VI.6. *All the varieties* $X^{sym}_{\mathrm{inv}}(m)$ *are Legendrian and we have:*

(a) $X^{sym}_{\mathrm{inv}}(2)$ *is a linear subspace.*

(b) $X^{sym}_{\mathrm{inv}}(3)$ *is smooth and it is isomorphic to a hyperplane section of Lagrangian Grassmannian* $Gr_L(3,6)$*.*

(c) $X^{sym}_{\mathrm{inv}}(4)$ *is smooth and it is Grassmannian variety* $Gr(3,6)$*.*

(d) For $m \geq 5$*, the variety* $X^{sym}_{\mathrm{inv}}(m)$ *is singular.*

The proof is exactly as the proof of theorem VI.4.

Similarly, we can take V^{skew} to be a vector space of dimension $2\binom{2m}{2}$, which we interpret as the space of pairs of $2m \times 2m$ skew-symmetric matrices A, B. Now in $\mathbb{P}(V^{skew})$ consider subvariety $X^{skew}_{\mathrm{inv}}(m)$, which is the closure of the following set:

$$\{[A, -A^{-1}] \in \mathbb{P}(V^{skew}) | A = -A^T \text{ and } \operatorname{Pfaff} A = 1\}.$$

Theorem VI.7. *All the varieties* $X^{skew}_{\mathrm{inv}}(m)$ *are Legendrian and we have:*

(a) $X^{skew}_{\mathrm{inv}}(2)$ *is a linear subspace.*

(b) $X^{skew}_{\mathrm{inv}}(3)$ *is smooth and it is isomorphic to a hyperplane section of the spinor variety* $\mathbb{S}_6$*.*

(c) $X^{skew}_{\mathrm{inv}}(4)$ *is smooth and it is the 27 dimensional* E_7 *variety.*

(d) For $m \geq 5$*, the variety* $X^{skew}_{\mathrm{inv}}(m)$ *is singular.*

Here the only difference is that we replace the determinants by the Pfaffians of the appropriate submatrices and also for the previous cases we will be picking some diagonal matrices as nice representatives. Since there is no non-zero skew-symmetric diagonal matrix, we must modify our calculations a little bit, but there is essentially no difference in the technique.

Prior to [Buc07b] neither $X^{sym}_{\mathrm{inv}}(3)$ nor $X^{skew}_{\mathrm{inv}}(3)$ have been identified as smooth Legendrian subvarieties.

Therefore we have established a connection between the subadjoint varieties of the 4 exceptional groups F_4, E_6, E_7 and E_8. A similar connection was obtained by [LM02].

We note that $m \times m$ symmetric matrices, $m \times m$ matrices and $2m \times 2m$ skew-symmetric matrices naturally correspond to $m \times m$ Hermitian matrices with coefficients in $\mathbb{F} \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathbb{F}$ is the field of, respectively, real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ (see [LM01] and references therein). An algebraic relation (analogous to parts (c) of theorems VI.4 VI.6 and VI.7) between Lie algebras of types E_6, E_7 and E_8 and 4×4 Hermitian matrices with coefficients in $\mathbb{F} \otimes_{\mathbb{R}} \mathbb{C}$ is described in [BK94].

VI.3 G-action and its orbits

Recall the definition of Y in §VI.1.

The following polynomials are in the homogeneous ideal of Y (the indices i, j below run through $\{1, \ldots, m\}$, k is a summation index):

$$\sum_{k=1}^{m} a_{ik} b_{ik} - \sum_{k=1}^{m} a_{1k} b_{1k}, \tag{VI.8a}$$

$$\sum_{k=1}^{m} a_{ik} b_{jk} \quad \text{for } i \neq j, \tag{VI.8b}$$

$$\sum_{k=1}^{m} a_{ki} b_{ki} - \sum_{k=1}^{m} a_{k1} b_{k1}, \tag{VI.8c}$$

$$\sum_{k=1}^{m} a_{ki} b_{kj} \quad \text{for } i \neq j. \tag{VI.8d}$$

These equations simply come from eliminating λ from the defining equation of Y — see equation (VI.2).

For the statement and proof of the following proposition, recall our notation of §VI.1.

Proposition VI.9. *Let $X \subset \mathbb{P}(V)$ be a Legendrian subvariety. If X is contained in Y, then X is preserved by the induced action of G on $\mathbb{P}(V)$.*

Proof. Let $\widetilde{\mathcal{I}}(X)_2$ be as in the theorem III.5 and define $\widetilde{\mathcal{I}}(Y)_2$ analogously. Clearly $\widetilde{\mathcal{I}}(Y)_2 \subset \widetilde{\mathcal{I}}(X)_2$. Also let ρ be the map described in theorem III.5. By theorem III.5 it is enough to show that $\mathfrak{g} \subset \rho\left(\widetilde{\mathcal{I}}(Y)_2\right)$ or that the images of the quadrics (VI.8a)–(VI.8d) under ρ generate $\mathfrak{g}$.

We write out the details of the proof only for $m = 2$. There is no difference between this case and the general one, except for the complexity of notation.

Let us take the quadric

$$q_{ij} := \sum_{k=1}^{m} a_{ik} b_{jk} = a_{i1} b_{j1} + a_{i2} b_{j2}$$

for any $i, j \in \{1, \ldots, m\} = \{1, 2\}$. Also let Q_{ij} be the $2m^2 \times 2m^2$ symmetric matrix corresponding to q_{ij}. For instance:

$$Q_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Choose an arbitrary $(A, B) \in V$ and at the moment we want to think of it as of a single vertical $2m^2$-vector: $(A, B) = [a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22}]^T$, so that the following multiplication makes sense:

$$\rho(q_{12}) = 2J \cdot Q_{12} \cdot (A, B) =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{21} \\ a_{22} \\ b_{11} \\ b_{12} \\ b_{21} \\ b_{22} \end{bmatrix} =$$

$$= \begin{bmatrix} 0 \\ 0 \\ a_{11} \\ a_{12} \\ -b_{21} \\ -b_{22} \\ 0 \\ 0 \end{bmatrix} \overset{\text{back to the matrix notation}}{=} \left(\begin{bmatrix} 0 & 0 \\ a_{11} & a_{12} \end{bmatrix}, \begin{bmatrix} -b_{21} & -b_{22} \\ 0 & 0 \end{bmatrix} \right) =$$

$$= \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, -\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \right) = (E_{12}^T A, \ -E_{12} B).$$

Similar calculations show that:

$$2J \cdot Q_{ij} \cdot (A, B) \;=\; (E_{ij}^T A,\; -E_{ij} B).$$

Next in the ideal of Y we have the following quadrics: q_{ij} for $i \neq j$ (see (VI.8b)) and $q_{ii} - q_{11}$ (see (VI.8a)). By taking images under ρ of the linear combinations of those quadrics we can get an arbitrary traceless matrix $g \in \mathfrak{sl}_m$ acting on V in the following way:

$$g \cdot (A, B) = (g^T A, -gB).$$

Exponentiate this action of $\mathfrak{sl}_m$ to get the action of $\mathbf{SL}_m$:

$$g \cdot (A, B) = (g^T A, g^{-1} B).$$

This proves that the action of subgroup $\mathbf{SL}_m \times 0 < G = \mathbf{SL}_m \times \mathbf{SL}_m$ preserves X as claimed in the proposition. The action of the other component $0 \times \mathbf{SL}_m$ is calculated in the same way, but using quadrics (VI.8c)–(VI.8d).

□

VI.3.1 Invariant subsets

Here we want to decompose Y into a union of some G-invariant subsets, most of which are orbits.

Proposition VI.10.

(i) The sets $\mathcal{INV}^m$, $\psi_\mu(\mathcal{INV}^m)$ and $\mathcal{DEG}^m_{k,l}$ are G-invariant and they are all contained in Y.

(ii) Y is equal to the union of all $\psi_\mu(\mathcal{INV}^m)$ (for $\mu \in \mathbb{C}^*$) and all $\mathcal{DEG}^m_{k,l}$ (for integers $k, l \geq 0$, $k + l \leq m$).

(iii) Every $\psi_\mu(\mathcal{INV}^m)$ is an orbit of the action of G. If m is odd, then $\mathcal{INV}^m$ is isomorphic (as algebraic variety) to $\mathbf{SL}_m$. Otherwise if m is even, then $\mathcal{INV}^m$ is isomorphic to $(\mathbf{SL}_m/\mathbb{Z}_2)$. In both cases

$$\dim \psi_\mu(\mathcal{INV}^m) = \dim \mathcal{INV}^m = m^2 - 1.$$

Proof. The proof of part (i) is an explicit verification from the definitions in §VI.1.

To prove part (ii), assume $[A, B]$ is a point of Y, so $AB^T = B^T A = \lambda^2 \operatorname{Id}_m$. First assume that the ranks of both matrices are maximal:

$$\operatorname{rk} A = \operatorname{rk} B = m.$$

Then λ must be non-zero and $B = \lambda^2 (A^{-1})^T$. Let $d := (\det A)^{-\frac{1}{m}}$ so that

$$\det(dA) = 1$$

and let $\mu := \frac{1}{d\lambda}$. Then we have:

$$\begin{aligned}[A,B] &= \left[A, \lambda^2 \left(A^{-1}\right)^T\right] = \left[\frac{dA}{d\lambda}, d\lambda \left((dA)^{-1}\right)^T\right] = \\ &= \left[\mu(dA), \mu^{-1}\left((dA)^{-1}\right)^T\right] = \psi_\mu\left(\left[(dA), \left((dA)^{-1}\right)^T\right]\right).\end{aligned}$$

Therefore $[A,B] \in \psi_\mu(\mathcal{INV}^m)$.

Next, if either of the ranks is not maximal:

$$\operatorname{rk} A < m \text{ or } \operatorname{rk} B < m,$$

then by (VI.2) we must have $AB^T = B^T A = 0$. So $[A,B] \in \mathcal{DEG}^m_{k,l}$ for $k = \operatorname{rk} A$ and $l = \operatorname{rk} B$.

Now we prove (iii). The action of G commutes with ψ_μ:

$$(g,h) \cdot \psi_\mu\big([A,B]\big) = \psi_\mu\big((g,h)\cdot [A,B]\big).$$

So to prove $\psi_\mu(\mathcal{INV}^m)$ is an orbit it is enough to prove that $\mathcal{INV}^m$ is an orbit, which follows from the definitions of the action and $\mathcal{INV}^m$.

We have the following epimorphic map:

$$\begin{array}{rcl}\mathbf{SL}_m & \longrightarrow & \mathcal{INV}^m \\ g & \longmapsto & [g, (g^{-1})^T].\end{array}$$

If $[g_1, (g_1^{-1})^T] = [g_2, (g_2^{-1})^T]$, then we must have $g_1 = \alpha g_2$ and $g_1 = \alpha^{-1} g_2$ for some $\alpha \in \mathbb{C}^*$. Hence $\alpha^2 = 1$ and $g_1 = \pm g_2$. If m is odd and $g_1 \in \mathbf{SL}_m$, then $-g_1 \notin \mathbf{SL}_m$ so $g_1 = g_2$. So $\mathcal{INV}^m$ is either isomorphic to $\mathbf{SL}_m$ or to $\mathbf{SL}_m/\mathbb{Z}_2$ as stated.

□

From proposition VI.10(ii) we conclude that $X_{\text{inv}}(m)$ is an equivariant compactification of $\mathbf{SL}_m$ (if m is odd) or $\mathbf{SL}_m/\mathbb{Z}_2$ (if m is even). See [Tim03] and references therein for the theory of equivariant compactifications. In the setup of [Tim03, §8], this is the compactification corresponding to the representation $W \oplus W^*$, where W is the standard representation of $\mathbf{SL}_m$. Therefore some properties of $X_{\text{inv}}(m)$ could also be read from the general description of group compactifications.

Proposition VI.11.

(i) The dimension of $\mathcal{DEG}^m_{k,l}$ is $(k+l)(2m-k-l)-1$. In particular, if $k+l = m$, then the dimension is equal to $m^2 - 1$.

(ii) $\mathcal{DEG}^m_{k,l}$ *is an orbit of the action of* G*, unless* m *is even and* $k = l = \frac{1}{2}m$*.*

(iii) If $m \geq 3$*, then there are exactly two closed orbits of the action of* G*:* $\mathcal{DEG}^m_{1,0}$ *and* $\mathcal{DEG}^m_{0,1}$*.*

Proof. Part (i) follows from [Str82, prop 2.10].

For part (ii) let $[A, B] \in \mathcal{DEG}^m_{k,l}$ be any point. By Gaussian elimination and elementary linear algebra, we can prove that there exists $(g, h) \in G$ such that $[A', B'] := (g, h) \cdot [A, B]$ is a pair of diagonal matrices. Moreover, if $k + l < m$, then we can choose g and h such that:

$$A' := \operatorname{diag}(\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{l}, \underbrace{0, \dots, 0}_{m-k-l}),$$

$$B' := \operatorname{diag}(\underbrace{0, \dots, 0}_{k}, \underbrace{1, \dots, 1}_{l}, \underbrace{0, \dots, 0}_{m-k-l}).$$

Hence $\mathcal{DEG}^m_{k,l} = G \cdot [A', B']$ and this finishes the proof in the case $k + l < m$.

So assume $k + l = m$. Then we can choose (g, h) such that:

$$A' := \operatorname{diag}(\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{l}),$$

$$B' := \operatorname{diag}(\underbrace{0, \dots, 0}_{k}, \underbrace{d, \dots, d}_{l}),$$

for some $d \in \mathbb{C}^*$. If $k \neq l$, then set $e := d^{\frac{1}{l-k}}$ and let

$$g' := \operatorname{diag}(\underbrace{e^l, \dots, e^l}_{k}, \underbrace{e^{-k}, \dots, e^{-k}}_{l}).$$

Clearly $\det(g') = 1$ and:

$$(g', \mathrm{Id}_m) \cdot [A', B'] = \left[\operatorname{diag}(\underbrace{e^l, \dots, e^l}_{k}, \underbrace{0, \dots, 0}_{l}), \operatorname{diag}(\underbrace{0, \dots, 0}_{k}, \underbrace{de^k, \dots, de^k}_{l}) \right]$$

where

$$de^k = d^{1+\frac{k}{l-k}} = d^{\frac{l}{l-k}} = e^l.$$

So rescaling we get:

$$(g', \mathrm{Id}_m) \cdot [A', B'] = \left[\operatorname{diag}(\underbrace{1, \dots, 1}_{k}, \underbrace{0, \dots, 0}_{l}), \operatorname{diag}(\underbrace{0, \dots, 0}_{k}, \underbrace{1, \dots, 1}_{l}) \right]$$

and this finishes the proof of (ii).

For part (iii), denote by W_1 (respectively, W_2) the standard representation of the first (respectively, the second) component of $G = \mathbf{SL}_m \times \mathbf{SL}_m$. Then our representation V is isomorphic to $(W_1 \otimes W_2) \oplus (W_1^* \otimes W_2^*)$. For $m \geq 3$ the representation W_i is not isomorphic to W_i^* and therefore V is a union of two irreducible non-isomorphic representations, so there are exactly two closed orbits of this action on $\mathbb{P}(V)$. These orbits are simply $\mathcal{DEG}^m_{1,0}$ and $\mathcal{DEG}^m_{0,1}$.

□

VI.3.2 Action of $\widetilde{G}$

The action of $\widetilde{G}$ extends the action of G, but it does not preserve $X_{\text{inv}}(m)$. So we will only consider the action of $\widetilde{G}$ when speaking of $X_{\deg}(m,k)$.

We have properties analogous to proposition VI.11 (ii) and (iii) but with no exceptional cases:

Proposition VI.12.

(i) Every $\mathcal{DEG}^m_{k,l}$ is an orbit of the action of $\widetilde{G}$.

(ii) For every m there are exactly two closed orbits of the action of $\widetilde{G}$: $\mathcal{DEG}^m_{1,0}$ and $\mathcal{DEG}^m_{0,1}$.

Proof. This is exactly as the proof of proposition VI.11 (ii) and (iii).

□

VI.4 Legendrian varieties in Y

In this section we prove the main results of the chapter.

VI.4.1 Classification

We start by proving the theorem VI.3.

Proof. First assume X is Legendrian and contained in Y. If X contains a point $[A,B]$ where both A and B are invertible, then by proposition VI.9 it must contain the orbit of $[A,B]$, which by proposition VI.10(ii) and (iii) is equal to $\psi_\mu(\mathcal{INV}^m)$ for some $\mu \in \mathbb{C}^*$. But the dimension of X is m^2-1 which is exactly the dimension of $\psi_\mu(\mathcal{INV}^m)$ (see proposition VI.10(iii)), so

$$X = \overline{\psi_\mu(\mathcal{INV}^m)} = \psi_\mu(X_{\text{inv}}(m)).$$

On the other hand, if X does not contain any point $[A, B]$ where both A and B are invertible, then in fact X is contained in the locus $Y_0 := \{[A, B] : AB^T = B^T A = 0\}$. This locus is just the union of all $\mathcal{DEG}^m_{k,l}$ and its irreducible components are the closures of $\mathcal{DEG}^m_{k,m-k}$, which are exactly $X_{\deg}(m, k)$. So in particular every irreducible component has dimension $m^2 - 1$ (see proposition VI.11(i)) and hence X must be one of these components.

Therefore it remains to show that all these varieties are Legendrian.

The fact that $X_{\deg}(m, k)$ is a Legendrian variety follows from [Str82, pp524–525]. Strickland proves there that the affine cone over $X_{\deg}(m, k)$ (or $W(k, m-k)$ in the notation of [Str82]) is the closure of a conormal bundle. Conormal bundles are classical examples of Lagrangian varieties (see example II.6).

Since ψ_μ preserves the symplectic form ω, it is enough to prove that $X_{\mathrm{inv}}(m)$ is Legendrian.

The group G acts symplectically on V and the action has an open orbit on $X_{\mathrm{inv}}(m)$ — see proposition VI.10 (iii). Thus the tangent spaces to the affine cone over $X_{\mathrm{inv}}(m)$ are Lagrangian if and only if just one tangent space at a point of the open orbit is Lagrangian.

So we take $[A, B] := [\mathrm{Id}_m, \mathrm{Id}_m]$. Now the affine tangent space to $X_{\mathrm{inv}}(m)$ at $[\mathrm{Id}_m, \mathrm{Id}_m]$ is the linear subspace of V spanned by $(\mathrm{Id}_m, \mathrm{Id}_m)$ and the image of the tangent action of the Lie algebra $\mathfrak{g}$. We must prove that for every four traceless matrices g, h, g', h' we have:

$$\omega\big((g,h)\cdot(\mathrm{Id}_m,\mathrm{Id}_m),\ (g',h')\cdot(\mathrm{Id}_m,\mathrm{Id}_m)\big) = 0 \quad \text{and} \tag{VI.13a}$$

$$\omega\big((\mathrm{Id}_m,\mathrm{Id}_m),\ (g,h)\cdot(\mathrm{Id}_m,\mathrm{Id}_m)\big) = 0. \tag{VI.13b}$$

Equality (VI.13a) is true without the assumption on the trace of the matrices:

$$\begin{aligned}
&\omega\big((g,h)\cdot(\mathrm{Id}_m,\mathrm{Id}_m),\ (g',h')\cdot(\mathrm{Id}_m,\mathrm{Id}_m)\big)\\
&\quad= \omega\Big(\big(g^T+h,\ -(g+h^T)\big),\quad \big((g')^T+h',\ -(g'+(h')^T)\big)\Big)\\
&\quad\overset{\text{by (VI.1)}}{=} \operatorname{tr}\Big(-\big(g^T+h\big)\big((g')^T+h'\big) + \big(g+h^T\big)\big(g'+(h')^T\big)\Big) = 0.
\end{aligned}$$

For equality (VI.13b) we calculate:

$$\begin{aligned}
&\omega\big((\mathrm{Id}_m,\mathrm{Id}_m),\ (g,h)\cdot(\mathrm{Id}_m,\mathrm{Id}_m)\big)\\
&\quad= \omega\Big((\mathrm{Id}_m,\mathrm{Id}_m),\quad \big(g^T+h,\ -(g+h^T)\big)\Big)\\
&\quad\overset{\text{by (VI.1)}}{=} -\operatorname{tr}(g^T+h) - \operatorname{tr}(g+h^T) = 0.
\end{aligned}$$

Hence we have proved that the closure of $\mathcal{INV}^m$ is Legendrian.

□

VI.4.2 Degenerate matrices

By [Str82, prop. 1.3] the ideal of $X_{\deg}(m,k)$ is generated by the coefficients of AB^T, the coefficients of $B^T A$, the $(k+1)\times(k+1)$-minors of A and the $(m-k+1)\times(m-k+1)$-minors of B. In short, we will say that the equations of $X_{\deg}(m,k)$ are given by:

$$AB^T = 0,\ B^T A = 0, \quad \operatorname{rk}(A) \le k,\ \operatorname{rk}(B) \le m-k. \tag{VI.14}$$

Lemma VI.15. *Assume $m \ge 2$ and $1 \le k \le m-1$. Then:*

(i) The tangent cone to $X_{\deg}(m,k)$ at p_1 is a product of a linear space of dimension $(2m-2)$ and the affine cone over $X_{\deg}(m-1,k-1)$.

(i') The tangent cone to $X_{\deg}(m,k)$ at p_2 is a product of a linear space of dimension $(2m-2)$ and the affine cone of $X_{\deg}(m-1,k)$.

(ii) $X_{\deg}(m,k)$ is smooth at p_1 if and only if $k=1$.

(ii') $X_{\deg}(m,k)$ is smooth at p_2 if and only if $k=m-1$.

Proof. We only prove (i) and (ii), while (i') and (ii') follow in the same way by exchanging a_{ij} and b_{ij}. Consider equations (VI.14) of $X_{\deg}(m,k)$ restricted to the affine neighbourhood of p_1 obtained by substituting $a_{mm}=1$. Taking the lowest degree part of these equations we get some of the equations of the tangent cone at p_1 (recall our convention on the notation of submatrices — see §VI.1):

$$b_{im} = b_{mi} = 0,\ A_m B_m^T = 0,\ B_m^T A_m = 0,$$

$$\operatorname{rk} A_m \le k-1, \operatorname{rk} B_m \le m-k.$$

These equations define the product of the linear subspace $A_m = B_m = 0, b_{im} = b_{mi} = 0$ and the affine cone over $X_{\deg}(m-1,k-1)$ embedded in the set of those pairs of matrices, whose last row and column are zero: $a_{im} = a_{mi} = 0, b_{im} = b_{mi} = 0$. So the variety defined by those equations is irreducible and its dimension is equal to $(m-1)^2 + 2m - 2 = m^2 - 1 = \dim X_{\deg}(m,k)$. Since this contains the tangent cone we are interested in and by §I.3.8(1), they must coincide as claimed in (i).

Next (ii) follows immediately, since for $k=1$ the equations above reduce to

$$b_{im} = b_{mi} = 0, \quad \text{and } A_m = 0$$

and hence the tangent cone is just the tangent space, so p_1 is a smooth point of $X_{\deg}(m,1)$. Conversely, if $k>1$, then $X_{\deg}(m-1,k-1)$ is not a linear space, so by (i) the tangent cone is not a linear space either and X is singular at p_1 — see §I.3.8(3).

□

Now we can prove theorem VI.5:

Proof. It is obvious from the definition of $X_{\mathrm{deg}}(m,k)$, that $X_{\mathrm{deg}}(m,0) = \{A = 0\}$ and $X_{\mathrm{deg}}(m,m) = \{B = 0\}$, so these are indeed linear spaces.

Therefore assume $1 \le k \le m-1$. But $X_{\mathrm{deg}}(m,k)$ is $\widetilde{G}$ invariant (see proposition VI.12(i)) and so is its singular locus S. Hence $X_{\mathrm{deg}}(m,k)$ is singular if and only if S contains a closed orbit of $\widetilde{G}$.

So $X_{\mathrm{deg}}(m,k)$ is smooth, if and only if it is smooth at both p_1 and p_2 (see proposition VI.12(ii)), which (by lemma (ii) and (ii')) holds if and only if $k = 1$ and $m = 2$.

To finish the proof, it remains to verify what kind of variety is $X_{\mathrm{deg}}(2,1)$. Consider the following map:

$$\begin{aligned} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 &\longrightarrow \mathbb{P}(V) \simeq \mathbb{P}^7 \\ [\mu_1,\mu_2],[\nu_1,\nu_2],[\xi_1,\xi_2] &\longmapsto \left[\xi_1 \begin{pmatrix} \mu_1\nu_1 & \mu_1\nu_2 \\ \mu_2\nu_1 & \mu_2\nu_2 \end{pmatrix}, \xi_2 \begin{pmatrix} \mu_2\nu_2 & -\mu_2\nu_1 \\ -\mu_1\nu_2 & \mu_1\nu_1 \end{pmatrix}\right] \end{aligned}$$

Clearly this is the Segre embedding in appropriate coordinates. The image of this embedding is contained in $X_{\mathrm{deg}}(2,1)$ (see equation (VI.14)) and since dimension of $X_{\mathrm{deg}}(2,1)$ is equal to the dimension of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we conclude the above map gives an isomorphism of $X_{\mathrm{deg}}(2,1)$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

□

VI.4.3 Invertible matrices

We wish to determine some of the equations of $X_{\mathrm{inv}}(m)$. Clearly the equations of Y (see (VI.8)) are quadratic equations of $X_{\mathrm{inv}}(m)$. To find other equations, we recall that

$$X_{\mathrm{inv}}(m) := \overline{\left\{ \left[g, (g^{-1})^T\right] \in \mathbb{P}(V) \mid \det g = 1 \right\}}.$$

However, for a matrix g with determinant 1 we know that the entries of $(g^{-1})^T$ consist of the appropriate minors (up to sign) of g. Therefore we get many inhomogeneous equations satisfied by every pair $\left(g, (g^{-1})^T\right) \in V$ (recall our convention on the notation of submatrices — see §VI.1):

$$\det(A_{ij}) = (-1)^{i+j} b_{ij} \quad \text{and} \quad a_{kl} = (-1)^{k+l} \det(B_{kl})$$

To make them homogeneous, multiply two such equations appropriately:

$$\det(A_{ij}) a_{kl} = (-1)^{i+j+k+l} b_{ij} \det(B_{kl}). \tag{VI.16}$$

These are degree m equations, which are satisfied by the points of $X_{\mathrm{inv}}(m)$ and we state the following theorem:

Theorem VI.17. *Let $m = 3$. Then the quadratic equations* (VI.8a)–(VI.8d) *and the cubic equations* (VI.16) *generate the ideal of $X_{\mathrm{inv}}(3)$. Moreover $X_{\mathrm{inv}}(3)$ is smooth.*

Proof. It is enough to prove that the scheme X defined by equations (VI.8a)–(VI.8d) and (VI.16) is smooth, because the reduced subscheme of X coincides with $X_{\mathrm{inv}}(3)$.

The scheme X is G invariant, hence as in the proof of theorem VI.5 and by proposition VI.11(iii) it is enough to verify smoothness at p_1 and p_2. Since we have the additional symmetry here (exchanging a_{ij}'s with b_{ij}'s) it is enough to verify the smoothness at p_1.

Now we calculate the tangent space to X at p_1 by taking linear parts of the equations evaluated at $a_{33} = 1$. From (VI.8) we get that

$$b_{31} = b_{32} = b_{33} = b_{23} = b_{13} = 0.$$

Now from equations (VI.16) for $k = l = 3$ and $i, j \neq 3$ we get the following evaluated equations:

$$a_{i'j'} - a_{i'3}a_{3j'} = \pm b_{ij}B_{33}$$

(where i' is either 1 or 2, which ever is different than i and analogously for j') so the linear part is just $a_{i'j'} = 0$. Hence by varying i and j we can get

$$a_{11} = a_{12} = a_{21} = a_{22} = 0.$$

Therefore the tangent space has codimension at least 9, which is exactly the codimension of $X_{\mathrm{inv}}(3)$ — see VI.10(iii). Hence X is smooth (in particular reduced) and $X = X_{\mathrm{inv}}(3)$.

□

To describe $X_{\mathrm{inv}}(m)$ for $m > 3$ we must find more equations.

There is a more general version of the above property of an inverse of a matrix with determinant 1, which is less commonly known.

Proposition VI.18.

(i) Assume A is a $m \times m$ matrix of determinant 1 and I, J are two sets of indices, both of cardinality k (again recall our convention on indices and submatrices — see §VI.1). Denote by $B := (A^{-1})^T$. Then the appropriate minors are equal (up to sign):

$$\det A_{I,J} = (-1)^{\Sigma I + \Sigma J} \det B_{I',J'}.$$

(ii) A coordinate free way to express these equalities is following: Assume W is a vector space of dimension m, f is a linear automorphism of W and $k \in \{0, \ldots, m\}$. Let $\bigwedge^k f$ be the induced automorphism of $\bigwedge^k W$. If $\bigwedge^m f = \mathrm{Id}_{\bigwedge^m W}$, then:

$$\bigwedge^{m-k} f = \bigwedge^k \left(\bigwedge^{m-1} f\right).$$

(iii) Consider the induced action of G on the polynomials on V. Then the vector space spanned by the set of equations of (i) for a fixed k is G invariant.

Proof. Part (ii) follows immediately from (i), since if A is a matrix of f, then the terms of the matrices of the maps $\bigwedge^{m-k} f$ and $\bigwedge^k(\bigwedge^{m-1} f)$ are exactly the appropriate minors of A and B.

Part (iii) follows easily from (ii).

As for (i), we only sketch the proof, leaving the details to the reader. Firstly, reduce to the case when I and J are just $\{1, \dots k\}$ and the determinant of A is possibly ± 1 (which is where the sign shows up in the equality). Secondly if both determinants $\det A_{I,J}$ and $\det B_{I',J'}$ are zero, then the equality is clearly satisfied. Otherwise assume for example $\det A_{I,J} \neq 0$. Then performing the appropriate row and column operations we can change $A_{I,J}$ into a diagonal matrix, $A_{I',J}$ and $A_{I,J'}$ into the zero matrices and all these operations can be done without changing $B_{I',J'}$ nor $\det A_{I,J}$. Then the statement follows easily.

□

In particular we get:

Corollary VI.19. *Assume k, I and J are as in proposition VI.18(i).*

(a) If m is even and $k = \frac{1}{2}m$, then the equation

$$\det A_{I,J} = (-1)^{\Sigma I + \Sigma J} \det B_{I',J'}$$

is homogeneous of degree $\frac{1}{2}m$ and it is satisfied by points of $X_{\text{inv}}(m)$.

(b) If $0 \leq k < \frac{1}{2}m$ and $l = m - 2k$, then

$$(\det A_{I,J})^2 = (\det B_{I',J'})^2 \cdot (a_{11}b_{11} + \ldots + a_{1m}b_{1m})^l$$

is a homogeneous equation of degree $2(m-k)$ satisfied by points of $X_{\text{inv}}(m)$.

Proof. Clearly both equations are homogeneous. If $\det A = 1$ and $B = (A^{-1})^T$, then the following equations are satisfied:

$$\det A_{I,J} = (-1)^{\Sigma I + \Sigma J} \det B_{I',J'}, \tag{VI.20}$$

$$1 = (a_{11}b_{11} + \ldots a_{1m}b_{1m})^l \tag{VI.21}$$

(equation (VI.20) follows from proposition VI.18(i) and (VI.21) follows from $AB^T = \mathrm{Id}_m$). Equation in (b) is just (VI.20) squared multiplied side-wise by (VI.21).

So both equations in (a) and (b) are satisfied by every pair $\left(A, (A^{-1})^T\right)$ and by homogeneity also by $\left(\lambda A, \lambda (A^{-1})^T\right)$. Hence (a) and (b) hold on an open dense subset of $X_{\text{inv}}(m)$, so also on whole $X_{\text{inv}}(m)$.

□

We know enough equations of $X_{\text{inv}}(m)$ to prove the theorem VI.4:

Case $m = 2$ **— linear subspace**

Proof. To prove (a) just take the linear equations from corollary VI.19(a) for $k = 1$:

$$a_{ij} = \pm b_{i'j'},$$

where $\{i, i'\} = \{j, j'\} = \{1, 2\}$.

$\square$

Case $m = 3$ **— hyperplane section of** $Gr(3, 6)$

Proof. For (b), $X_{\mathrm{inv}}(3)$ is smooth by theorem VI.17 and it is a compactification of $\mathcal{INV}^3 \simeq \mathbf{SL}_3$ by proposition VI.10(i) and (iii).

Picard group of $X_{\mathrm{inv}}(3)$**.** The complement of the open orbit

$$D := X_{\mathrm{inv}}(3) \backslash \mathcal{INV}^3$$

must be a union of some orbits of G, each of them must have dimension smaller than $\dim \mathcal{INV}^3 = 8$. So by propositions VI.10(ii), (iii), VI.11 (i) and (ii) the only candidates are $\mathcal{DEG}^3_{1,1}$, $\mathcal{DEG}^3_{0,1}$ and $\mathcal{DEG}^3_{1,0}$. We claim they are all contained in $X_{\mathrm{inv}}(3)$. It is enough to prove that $\mathcal{DEG}^3_{1,1} \subset X_{\mathrm{inv}}(3)$, since the other orbits are in the closure of $\mathcal{DEG}^3_{1,1}$. Take the curve in $X_{\mathrm{inv}}(3)$ parametrised by:

$$\left[\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix} \right].$$

For $t = 0$ the curve meets $\mathcal{DEG}^3_{1,1}$, which finishes the proof of the claim.

Since $\dim \mathcal{DEG}^3_{1,1} = 7$ (see proposition VI.11(i)), D is a prime divisor. We have $\mathrm{Pic}(\mathbf{SL}_3) = 0$ and by [Har77, prop. II.6.5(c)] the Picard group of $X_{\mathrm{inv}}(3)$ is isomorphic to $\mathbb{Z}$ with the ample generator $[D]$.

Next we check that D is linearly equivalent (as a divisor on $X_{\mathrm{inv}}(3)$) to a hyperplane section H of $X_{\mathrm{inv}}(3)$. Since we already know that $\mathrm{Pic}(X_{\mathrm{inv}}(3)) = \mathbb{Z} \cdot [D]$, we must have $H \overset{lin}{\sim} kD$ for some positive integer k. But there are lines contained in $X_{\mathrm{inv}}(3)$ (for example those contained in $\mathcal{DEG}^3_{1,0} \simeq \mathbb{P}^2 \times \mathbb{P}^2)^2$. So let $L \subset X_{\mathrm{inv}}(3)$ be any line and we intersect:

$$D \cdot L = \frac{1}{k} H \cdot L = \frac{1}{k}.$$

But the result must be an integer, so $k = 1$ as claimed.

[2]Actually, the reader could also easily find explicitly some lines (or even planes) which intersect the open orbit and conclude that $X_{\mathrm{inv}}(3)$ is covered by lines.

Complete embedding. Since D itself is definitely not a hyperplane section of $X_{\mathrm{inv}}(3)$, the conclusion is that the Legendrian embedding of $X_{\mathrm{inv}}(3)$ is not given by a complete linear system. The natural guess for a better embedding is the following:

$$X' := \overline{\left\{ \left[1, g, \bigwedge^2 g\right] \in \mathbb{P}^{18} = \mathbb{P}(\mathbb{C} \oplus V) \mid \det g = 1 \right\}},$$

(we note that $\bigwedge^2 g = (g^{-1})^T$ for g with $\det g = 1$) and one can verify that the projection from the point $[1, 0, 0] \in \mathbb{P}^{18}$ restricted to X' gives an isomorphism with $X_{\mathrm{inv}}(3)$.

The Grassmannian $Gr(3, 6)$ in its Plücker embedding can be described as the closure of:

$$\left\{ \left[1, g, \bigwedge^2 g, \bigwedge^3 g\right] \in \mathbb{P}^{19} = \mathbb{P}(\mathbb{C} \oplus V \oplus \mathbb{C}) \mid g \in M_{3\times 3} \right\}$$

and we immediately identify X' as the section $H := \left\{\bigwedge^3 g = 1\right\}$ of the Grassmannian.

Though it is not essential, we note that $H^1(\mathcal{O}_{Gr(3,6)}) = 0$ (see Kodaira vanishing theorem [Laz04, thm 4.2.1]; alternatively, it follows from the fact that $b_1 = 0$ for Grassmannians) and hence the above embedding of $X_{\mathrm{inv}}(3)$ is given by the complete linear system.

Automorphism group. It remains to calculate $\mathrm{Aut}\left(X_{\mathrm{inv}}(3)\right)^0$ — the connected component of the automorphism group.

The tangent Lie algebra of the group of automorphisms of a complex projective manifold is equal to the global sections of the tangent bundle, see theorem A.7. A vector field on $X_{\mathrm{inv}}(3)$ is also a section of $TGr(3, 6)|_{X_{\mathrm{inv}}(3)}$ and we have the following short exact sequence:

$$0 \longrightarrow TGr(3, 6)(-1) \longrightarrow TGr(3, 6) \longrightarrow TGr(3, 6)|_{X_{\mathrm{inv}}(3)} \longrightarrow 0$$

The homogeneous vector bundle $TGr(3, 6)(-1)$ is isomorphic to $U^* \otimes Q \otimes \bigwedge^3 U$, where U is the universal subbundle in $Gr(3, 6)\times\mathbb{C}^6$ and Q is the universal quotient bundle. This bundle corresponds to an irreducible module of the parabolic subgroup in $\mathbf{SL}_6$. Calculating explicitly its highest weight and applying Bott formula [Ott95] we get that $H^1\big(TGr(3, 6)(-1)\big) = 0$. Hence every section of $TX_{\mathrm{inv}}(3)$ extends to a section of $TGr(3, 6)$. In other words, if $P < \mathrm{Aut}(Gr(3, 6)) \simeq \mathbb{P}\mathbf{GL}_6$ is the subgroup preserving $X_{\mathrm{inv}}(3) \subset Gr(3, 6)$, then the restriction map

$$P \longrightarrow \mathrm{Aut}\left(X_{\mathrm{inv}}(3)\right)^0$$

is epimorphic.

The action of $\mathbf{SL}_6$ on $\bigwedge^3 \mathbb{C}^6$ preserves the natural symplectic form ω':

$$\omega' : \bigwedge^2 \left(\bigwedge^3 \mathbb{C}^6\right) \longrightarrow \bigwedge^6 \mathbb{C}^6 \simeq \mathbb{C}.$$

Since the action of P on $\mathbb{P}\left(\bigwedge^3 \mathbb{C}^6\right)$ preserves the hyperplane H containing $X_{\mathrm{inv}}(3)$, it must also preserve $H^{\perp_{\omega'}}$, i.e. P preserves $[1,0,0,1] \in \mathbb{P}^{19} = \mathbb{P}(\mathbb{C} \oplus V \oplus \mathbb{C})$. Therefore P acts on the quotient $H/(H^{\perp_{\omega'}}) = V$ and hence the restriction map factorises:

$$P \longrightarrow \mathrm{Aut}(\mathbb{P}(V), X_{\mathrm{inv}}(3))^0 \twoheadrightarrow \mathrm{Aut}(X_{\mathrm{inv}}(3))^0.$$

By theorem IV.2, group $\mathrm{Aut}(\mathbb{P}(V), X_{\mathrm{inv}}(3))^0$ is contained in the image of $\mathbf{Sp}(V) \to \mathbb{P}\mathbf{GL}(V)$, so by theorem III.5, proposition VI.9 and theorem VI.17

$$\mathrm{Aut}\left(\mathbb{P}(V), X_{\mathrm{inv}}(3)\right)^0 = G.$$

In particular $X_{\mathrm{inv}}(3)$ cannot be homogeneous as it contains more than one orbit of the connected component of automorphism group.

□

We note that the fact that $X_{\mathrm{inv}}(3)$ is not homogeneous can be also proved without calculating the automorphism group. Since $\mathrm{Pic}\, X_{\mathrm{inv}}(3) \simeq \mathbb{Z}$, it follows from [LM04, thm. 11], that $X_{\mathrm{inv}}(3)$ could only be one of the subadjoint varieties. But none of them has Picard group $\mathbb{Z}$ and dimension 8.

Case m=4 — spinor variety $\mathbb{S}_6$

Proof. To prove (c) we only need to take 30 quadratic equations of Y as in (VI.8) and 36 quadratic equations from corollary VI.19 (a). By proposition VI.18(iii) the scheme X defined by those quadratic equations is G-invariant. As in the proofs of theorems VI.5 and VI.17, we only check that X is smooth at p_1 and p_2 and conclude it is smooth everywhere, hence those equations indeed define $X_{\mathrm{inv}}(4)$.

Therefore $X_{\mathrm{inv}}(4)$ is smooth, irreducible and its ideal is generated by quadrics, so it falls into the classification of [Buc06, thm. 5.11]. Hence we have two choices for $X_{\mathrm{inv}}(4)$ whose dimension is 15: the product of a line and a quadric $\mathbb{P}^1 \times Q_{14}$ or the spinor variety $\mathbb{S}_6$. The homogeneous ideal of polynomials vanishing on $\mathbb{P}^1 \times Q_{14} \subset \mathbb{P}^{31}$ is generated by $\dim(\mathbf{SL}_2 \times \mathbf{SO}_{16}) = 123$ linearly independent quadratic polynomials (see theorem III.5, alternatively, one can calculate the equations explicitly — see [Buc05, §7.2]). So $X_{\mathrm{inv}}(4)$, which by the above argument is generated by only 66 quadratic equations, must be isomorphic to $\mathbb{S}_6$.

□

Case m$\geq$ 5 — singular varieties

Proof. Finally we prove (d). We want to prove, that for $m \geq 5$ variety $X_{\mathrm{inv}}(m)$ is singular at p_1. To do that, we calculate the reduced tangent cone

$$T := \left(TC_{p_1} X_{\mathrm{inv}}(m)\right)_{red}.$$

From equations (VI.8) we easily get the following linear and quadratic equations of T (again we suggest to have a look at §VI.1):

$$b_{im} = b_{mi} = 0, \quad A_m B_m^T = B_m^T A_m = \lambda^2 \operatorname{Id}_{m-1}$$

for every $i \in \{1, \ldots m\}$ and some $\lambda \in \mathbb{C}^*$.

Next assume I and J are two sets of indices, both of cardinality $k = \left\lfloor \frac{1}{2} m \right\rfloor$ and such that neither I nor J contains m. Consider the equation of $X_{\mathrm{inv}}(m)$ as in corollary VI.19(b):

$$\left(\det A_{I,J}\right)^2 = \left(\det B_{I',J'}\right)^2 \cdot \left(a_{11} b_{11} + \ldots a_{1m} b_{1m}\right)^l.$$

To get an equation of T, we evaluate at $a_{mm} = 1$ and take the lowest degree part, which is simply $\left(\det\left((A_m)_{I,J}\right)\right)^2 = 0$. Since T is reduced, by varying I and J we get that:

$$\operatorname{rk} A_m \leq m - 1 - k - 1 = \left\lceil \frac{1}{2} m \right\rceil - 2$$

and therefore also:

$$A_m B_m^T = B_m^T A_m = 0.$$

Hence T is contained in the product of the linear space $W := \{A_m = 0, B = 0\}$ and the affine cone $\hat{U}$ over the union of $X_{\mathrm{deg}}(m-1, k)$ for $k \leq \left\lceil \frac{1}{2} m \right\rceil - 2$. We claim that $T = W \times \hat{U}$. By proposition VI.11(i), every component of $W \times \hat{U}$ has dimension $2m - 2 + (m-1)^2 = m^2 - 1 = \dim X_{\mathrm{inv}}(m)$, so by §I.3.8(1) the tangent cone must be a union of some of the components. Therefore to prove the claim it is enough to find for every $k \leq \left\lceil \frac{1}{2} m \right\rceil - 2$ a single element of $\mathcal{DEG}^{m-1}_{k, m-k-1}$ that is contained in the tangent cone.

So take α and β to be two strictly positive integers such that

$$\alpha = \left(\frac{1}{2} m - k - 1\right) \beta$$

and consider the curve in $\mathbb{P}(V)$ with the following parametrisation:

$$\left[\operatorname{diag}\{\underbrace{t^\alpha, \ldots, t^\alpha}_{k}, \underbrace{t^{\alpha+\beta}, \ldots, t^{\alpha+\beta}}_{m-k-1}, 1\}, \operatorname{diag}\{\underbrace{t^{\alpha+\beta}, \ldots, t^{\alpha+\beta}}_{k}, \underbrace{t^\alpha, \ldots, t^\alpha}_{m-k-1}, t^{2\alpha+\beta}\}\right].$$

It is easy to verify that this family is contained in $\mathcal{INV}^m$ for $t \neq 0$ and as t converges to 0, it gives rise to a tangent vector (i.e. an element of the reduced tangent cone - see point-wise definition in §I.3.8) that belongs to $\mathcal{DEG}^{m-1}_{k,m-k-1}$.

So indeed $T = W \times \hat{U}$, which for $m \geq 5$ contains more than 1 component, hence cannot be a linear space. Therefore by §I.3.8(3) variety $X_{\text{inv}}(m)$ is singular at p_1.

□

Chapter VII

Hyperplane sections of Legendrian subvarieties

The content of this chapter is partially published in [Buc07a].

The Legendrian variety $X_{\text{inv}}(3)$ constructed in chapter VI is isomorphic to a hyperplane section of another Legendrian variety $Gr(3,6)$. In this chapter we prove that general hyperplane sections of other Legendrian varieties also admit a Legendrian embedding. This gives numerous new examples of smooth Legendrian subvarieties.

Theorem VII.1. *Let $X \subset \mathbb{P}(V)$ be an irreducible Legendrian subvariety, which is smooth or has only isolated singularities. Then a general hyperplane section of X admits a Legendrian embedding into a projective space of appropriate dimension via a specific subsystem of the linear system $\mathcal{O}(1)$.*

More generally, assume $X \subset \mathbb{P}(V)$ is an irreducible Legendrian subvariety with singular locus of dimension k and $H \subset \mathbb{P}(V)$ is a general hyperplane. Then there exists a variety $\widetilde{X}_H$ whose singular locus has dimension at most $k-1$ and which has an open subset isomorphic to the smooth locus of $X \cap H$ such that $\widetilde{X}_H$ admits a Legendrian embedding.

The specific linear system and construction of $\widetilde{X}_H$ is described in §VII.1.1 and there we prove that the resulting variety is Legendrian. The proof that for a general section the result has the required smoothness property is presented in §VII.1.2.

This simple observation has quite strong consequences. Many researchers, including Landsberg, Manivel, Wiśniewski, Hwang and the author of this thesis, believed that the structure of smooth Legendrian subvarieties in projective space had to be somehow rigid at least in higher dimensions. So far the only non-rational examples known were in dimensions 1 and 2 (see §I.1.2) and these were also the only known to come in families. Already by a naive application of our theorem to the subadjoint varieties we get many more examples with various properties:

Example VII.2. *The following smooth varieties and families of smooth varieties admit Legendrian embedding:*

(a) *a family of $K3$ surfaces of genus 9;*

(b) *three different types of surfaces of general type;*

(c) *some Calabi-Yau 3-folds, some Calabi-Yau 5-folds and some Calabi-Yau 9-folds;*

(d) *some varieties of general type in dimensions 3, 4 (two families for every dimension), 5,6,7 and 8 (one family per dimension);*

(e) *some Fano varieties, like the blow up of a quadric Q^n in a codimension 2 hyperplane section Q^{n-2}, a family of Del Pezzo surfaces of degree 4 and others;*

(f) *infinitely many non-isomorphic, non-homogeneous Legendrian varieties in every dimension arising as a codimension k linear section of $\mathbb{P}^1 \times Q^{n+k}$.*

Example (a) agrees with the prediction of [LM04, §2.3]. Examples (b) and (d) give a partial answer to the question of a possible Kodaira dimension of a Legendrian variety (also see [LM04, §2.3]). Example (f) is a counterexample to the naive expectation that Legendrian variety in a sufficiently high dimension must be homogeneous.

We also note that our previous examples also arise in this way. Example (e) for $n = 2$ is described in example V.5. Hyperplane sections of $Gr(3, 6)$, $Gr_L(3, 6)$, $\mathbb{S}_6$ are studied in more details in chapter VI. Also non-homogeneous examples of other authors, Bryant [Bry82], Landsberg and Manivel [LM04] can be reconstructed by theorem VII.1 from some varieties with only isolated singularities (see §VII.3).

A more refined construction, using the decomposable Legendrian varieties (see §III.1.2), makes a much bigger list of examples, including smooth Legendrian varieties with maximal Kodaira dimension in every dimension or varieties with arbitrary rank of Picard group. This is described in detail in §VII.2.

All the varieties arising from theorem VII.1 and our construction in subsection VII.1.1 are embedded by a non-complete linear system. Therefore a natural question arises: what are the smooth Legendrian varieties whose Legendrian embedding is linearly normal. Another question is whether the construction can be inverted. So for a given Legendrian but not linearly normal embedding of some variety $\widetilde{X}$, can we find a bigger Legendrian variety X, such that $\widetilde{X}$ is a projection of a hyperplane section of X?

Building upon ideas of Bryant, Landsberg and Manivel we suggest a construction that provides some (but far from perfect) answer for the second question in §VII.3. In particular, we represent the example of Landsberg and Manivel as a hyperplane section of a 3-fold with only isolated singularities and the examples of Bryant as hyperplane sections of surfaces with at most isolated singularities.

VII.1 Hyperplane section

VII.1.1 Construction

The idea of the construction is built on the concept of symplectic reduction (see §II.1.3). Let $H \in \mathbb{P}(V^*)$ be a hyperplane in V. By

$$h := H^{\perp_\omega} \subset V$$

we denote the ω-perpendicular to H subspace of V, which in this case is a line contained in H. We think of h both as a point in the projective space $\mathbb{P}(V)$ and a line in V. We define

$$\pi : \mathbb{P}(H)\backslash\{h\} \longrightarrow \mathbb{P}(H/h)$$

to be the projection map and for a given Legendrian subvariety $X \subset \mathbb{P}(V)$ we let $\widetilde{X}_H := \pi(X \cap H)$.

We have the natural symplectic structure ω' on H/h determined by ω (see §II.1.3). Also $\widetilde{X}_H$ is always Legendrian by proposition II.1 and lemma I.4.

Note that so far we have not used any smoothness condition on X.

VII.1.2 Proof of smoothness

Hence to prove theorem VII.1 it is enough to prove that for a general $H \in \mathbb{P}(V^*)$, the map π gives an isomorphism of the smooth locus of $X \cap H$ onto its image, an open subset in $\widetilde{X}_H$.

For a variety $Y \subset \mathbb{P}^m$ we denote by $\sigma(Y) \subset \mathbb{P}^m$ its secant variety, i.e., closure of the union of all projective lines through y_1 and y_2, where (y_1, y_2) vary through all pairs of different points of Y.

Lemma VII.3. *Let $Y \subset \mathbb{P}^m$, choose such a point $y \in \mathbb{P}^m$ that $y \notin \sigma(Y)$ and let $\pi : \mathbb{P}^m\backslash\{y\} \longrightarrow \mathbb{P}^{m-1}$ be the projection map.*

(a) If Y is smooth, then π gives an isomorphism of Y and $\pi(Y)$.

(b) In general, π is 1 to 1 and π is an isomorphism of the smooth part of Y onto its image. In particular, the dimension of singular locus of Y is greater or equal to the dimension of singular locus of $\pi(Y)$.

Proof. See [Har77, prop. IV.3.4 and exercise IV.3.11(a)]. We only note that if Y is smooth, then the secant variety $\sigma(Y)$ contains all the embedded tangent spaces of Y. They arise when y_2 approaches y_1.

□

Now we can prove theorem VII.1:

Proof. By the lemma and the construction in §VII.1.1 it is enough to prove that there exists $h \in \mathbb{P}(V)$ s.t. $h \notin \sigma(X \cap h^{\perp_\omega})$.

Given two different points x_1 and x_2 in a projective space we denote by $\langle x_1, x_2\rangle$ the projective line through x_1 and x_2. Let

$$\begin{aligned}\tilde{\sigma}(X) \subset & X \times X \times \mathbb{P}(V),\\ \tilde{\sigma}(X) := & \overline{\{(x_1, x_2, p) |\ p \in \langle x_1, x_2\rangle\}},\end{aligned}$$

so that $\tilde{\sigma}(X)$ is the incidence variety for the secant variety of X. Obviously, $\dim(\tilde{\sigma}(X)) = 2\dim X + 1 = \dim(\mathbb{P}(V))$ and $\tilde{\sigma}(X)$ is irreducible. Also we let:

$$\begin{aligned}\kappa(X) \subset & \tilde{\sigma}(X),\\ \kappa(X) := & \overline{\{(x_1, x_2, h) |\ h \in \langle x_1, x_2\rangle \text{ and } x_1, x_2 \in h^{\perp_\omega}\}},\end{aligned}$$

so that the image of the projection of $\kappa(X)$ onto the last coordinate is the locus of 'bad' points. More precisely, for a point $h \in \mathbb{P}(V)$ there exist (x_1, x_2) such that $(x_1, x_2, h) \in \kappa(X)$ if and only if $h \in \sigma(X \cap h^{\perp_\omega})$.

We claim that the image of $\kappa(X)$ under the projection is not the whole $\mathbb{P}(V)$. To see this note that the condition defining $\kappa(X)$, i.e., $h \in \langle x_1, x_2\rangle,\ x_1, x_2 \in h^{\perp_\omega}$ is equivalent to $h \in \langle x_1, x_2\rangle$ and $\langle x_1, x_2\rangle$ is an isotropic subspace of V. Now either X is a linear subspace and then both the claim and the theorem are obvious or there exist two points $x_1, x_2 \in X$ such that $\omega(\hat{x_1}, \hat{x_2}) \neq 0$ where by $\hat{x_i}$ we mean some non-zero point in the line $x_i \subset V$. Therefore $\kappa(X)$ is strictly contained in $\tilde{\sigma}(X)$ and

$$\dim(\kappa(X)) < \dim(\tilde{\sigma}(X)) = \dim \mathbb{P}(V),$$

so the image of $\kappa(X)$ under the projection cannot be equal to $\mathbb{P}(V)$[1].

□

Corollary VII.4. *Let $X \subset \mathbb{P}(V)$ be an irreducible Legendrian subvariety whose singular locus has dimension at most $k-1$. Let F be the contact distribution on $\mathbb{P}(V)$ If $H \subset \mathbb{P}(V)$ is a general F-cointegrable linear subspace of codimension k, then $\widetilde{X}_H := X \cap H$ is smooth and admits a Legendrian embedding via an appropriate subsystem of linear system $\mathcal{O}_{\widetilde{X}_H}(1)$.*

□

We sketch some proofs of examples VII.2:

[1]The inequality on the dimensions, although simple, is essential for the proof. An analogous construction for Lagrangian subvarieties in symplectic manifolds is known as symplectic reduction (see §II.1.3 for linear algebra baby version of this), but does not produce smooth Lagrangian subvarieties.

Proof. $K3$ surfaces of (a) arise as codimension 4 linear sections of Lagrangian Grassmannian $Gr_L(3,6)$. Since the canonical divisor $K_{Gr_L(3,6)} = \mathcal{O}_{Gr_L(3,6)}(-4)$ (in other words $Gr_L(3,6)$ is Fano of index 4), by the adjunction formula, the canonical divisor of the section is indeed trivial. On the other hand, by [LM04, prop. 9] it must have genus 9. Although we take quite special (F-cointegrable) sections, they fall into the 19 dimensional family of Mukai's $K3$-surfaces of genus 9 [Muk88] and they form a 13 dimensional subfamily.

The other families of surfaces as in (b) arise as sections of the other exceptional subadjoint varieties: $Gr(3,6)$, $\mathbb{S}_6$ and E_7. They are all Fano of index 5, 10 and 18 respectively and their dimensions are 9, 15 and 27 hence taking successive linear sections we get to Calabi-Yau manifolds as stated in (c). Further the canonical divisor is very ample, so we have examples of general type as stated in (b) and (d).

The Fano varieties arise as intermediate steps, before coming down to the level of Calabi-Yau manifolds. Also $\mathbb{P}^1 \times Q^n$ is a subadjoint variety and its hyperplane section is the blow up of a quadric Q^n in a codimension 2 hyperplane section. The Del Pezzo surfaces are the hyperplane sections of the blow up of Q^3 in a conic curve.

□

VII.2 Linear sections of decomposable Legendrian varieties

Assume m_1 and m_2 are two positive integers, $m_1 \geq m_2$. Let $V_1 \simeq \mathbb{C}^{2m_1+2}$ and $V_2 \simeq \mathbb{C}^{2m_2+2}$ be two symplectic vector spaces, and let $X_1 \subset \mathbb{P}(V_1)$ and $X_2 \subset \mathbb{P}(V_2)$ be two smooth, irreducible, non-degenerate, Legendrian subvarieties. In this setup $\dim X_i = m_i$. Consider the decomposable variety $X_1 * X_2 \subset \mathbb{P}(V_1 \oplus V_2)$. Clearly $\operatorname{Sing}(X_1 * X_2) = X_1 \sqcup X_2$, hence $\dim\bigl(\operatorname{Sing}(X_1 * X_2)\bigr) = m_1$, while

$$\dim(X_1 * X_2) = m_1 + m_2 + 1.$$

Let L be the following line bundle on $X_1 \times X_2$:

$$L := \mathcal{O}_{X_1}(1) \boxtimes \mathcal{O}_{X_2}(-1).$$

Also let $(X_1 * X_2)_0$ be the smooth locus of $X_1 * X_2$.

Lemma VII.5. *$(X_1 * X_2)_0$ is isomorphic to $\mathbf{L}^\bullet$, the total space of the $\mathbb{C}^*$-bundle associated to L (see §I.3.7).*

Proof. Let $\mathbb{C}^*$ act on $V_1 \oplus V_2$ with weight -1 on V_1 and weight 1 on V_2. Then

$$\Bigl(\mathbb{P}(V_1 \oplus V_2) \setminus \bigl(\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2)\bigr)\Bigr)\Big/\mathbb{C}^* = \mathbb{P}(V_1) \times \mathbb{P}(V_2)$$

and the quotient map:

$$\Big(\mathbb{P}(V_1 \oplus V_2) \setminus \big(\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2)\big)\Big) \xrightarrow{/\mathbb{C}^*} \mathbb{P}(V_1) \times \mathbb{P}(V_2)$$

is a principal $\mathbb{C}^*$-bundle obtained by removing the zero section from the total space of the line bundle $\mathcal{O}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(d_1, d_2)$ for some integers d_1 and d_2. We have,

$$\operatorname{Pic}\Big(\mathbb{P}(V_1 \oplus V_2) \setminus \big(\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2)\big)\Big) = \operatorname{Pic}\mathbb{P}(V_1 \oplus V_2) = \mathbb{Z}[\mathcal{O}_{\mathbb{P}(V_1\oplus V_2)}(1)]$$
(by [Har77, prop. II.6.5(c)]).

On the other hand,

$$\operatorname{Pic}\Big(\mathbb{P}(V_1 \oplus V_2) \setminus \big(\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2)\big)\Big) = \operatorname{Pic}\big(\mathbb{P}(V_1) \times \mathbb{P}(V_2)\big) / \big\langle \mathcal{O}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(d_1, d_2)\big\rangle$$
(by lemma I.6).

Moreover via the isomorphism

$$\operatorname{Pic}\big(\mathbb{P}(V_1) \times \mathbb{P}(V_2)\big) / \big\langle \mathcal{O}_{\mathbb{P}(V_1)\times\mathbb{P}(V_2)}(d_1, d_2)\big\rangle \simeq \mathbb{Z}[\mathcal{O}_{\mathbb{P}(V_1\oplus V_2)}(1)]$$

the class of line bundle $\mathcal{O}_{\mathbb{P}(V_1\oplus V_2)}(e_1, e_2)$ is mapped to $\mathcal{O}_{\mathbb{P}(V_1\oplus V_2)}(e_1 + e_2)$. Hence $(d_1, d_2) = (1, -1)$ or $(-1, 1)$. In both cases the total spaces of the line bundles are the same after removing the zero sections (the difference is only in the sign of the weights of the $\mathbb{C}^*$-action, which we ignore at this point).

To finish the proof just note that:

$$(X_1 * X_2)_0 = (X_1 * X_2) \cap \Big(\mathbb{P}(V_1 \oplus V_2) \setminus \big(\mathbb{P}(V_1) \sqcup \mathbb{P}(V_2)\big)\Big)$$

and the image of $(X_1 * X_2)_0$ under the quotient map is equal to $X_1 \times X_2$.

□

Hence by lemma I.6 we have:

$$\operatorname{Pic}(X_1 \times X_2) \twoheadrightarrow \operatorname{Pic}(X_1 * X_2)_0 = \operatorname{Cl} X_1 * X_2$$

and the kernel of the epimorphic map is generated by L. If $L_1 \in \operatorname{Pic} X_1$ and $L_2 \in \operatorname{Pic} X_2$, by $[L_1 \boxtimes L_2]$ we will denote a line bundle on $(X_1 * X_2)_0$ which represents the image of $L_1 \boxtimes L_2$ under the epimorphic map.

Theorem VII.6. *Let m_1, m_2, X_1, X_2 be as above. Let F be the contact distribution on $\mathbb{P}(V_1 \oplus V_2)$ and let $H \subset \mathbb{P}(V_1 \oplus V_2)$ be a general F-cointegrable linear subspace of codimension $m_1 + 1$. Then $X := (X_1 * X_2) \cap H$ is smooth, admits a Legendrian embedding and has the following properties:*

(a) $\deg X = \deg X_1 \cdot \deg X_2$;

(b) $K_X \simeq [K_{X_1} \boxtimes K_{X_2}]|_X \otimes \mathcal{O}_X(m_1 + 1)$;

(c) We have the restriction map on the Picard groups:

$$i^* : \operatorname{Pic}(X_1 \times X_2) / \langle L \rangle \longrightarrow \operatorname{Pic} X.$$

If $m_2 \geq 3$, *then* i^* *is an isomorphism. If* $m_2 = 2$, *then* i^* *is injective.*

In particular, we have:

(d) If $K_{X_1} \simeq \mathcal{O}_{X_1}(d_1)$ *and* $K_{X_2} \simeq \mathcal{O}_{X_2}(d_2)$, *then* $K_X \simeq \mathcal{O}_X(d_1 + d_2 + m_1 + 1)$;

(e) If $K_{X_1} \simeq \mathcal{O}_{X_1}(d_1) \otimes E_1$ *and* $K_{X_2} \simeq \mathcal{O}_{X_2}(d_2) \otimes E_2$, *where the* E_i*'s are line bundles corresponding to some effective divisors, then*

$$K_X \simeq \mathcal{O}_X(d_1 + d_2 + m_1 + 1) \otimes E$$

for some E *corresponding to an effective divisors;*

(f) If $m_2 \geq 3$, $\operatorname{Pic} X_1 = \mathbb{Z}[\mathcal{O}_{X_1}(1)]$, $\operatorname{Pic} X_2 = \mathbb{Z}[\mathcal{O}_{X_2}(1)]$ *and either* X_1 *or* X_2 *is simply connected (for example Fano), then* $\operatorname{Pic} X = \mathbb{Z}[\mathcal{O}_X(1)]$.

Proof. Part (a) is immediate, since $\deg(X_1 * X_2) = \deg X_1 \cdot \deg X_2$.

Part (b) follows from lemma VII.5, §I.3.7 and the adjunction formula (see [Har77, prop. II.8.20]).

Part (c) follows from [RS06].

Parts (d) and (e) are immediate consequences of (b) and (c).

Finally, part (f) follows from (c) and from [Har77, ex. III.12.6].

□

To conclude we give a further series of examples:

Example VII.7. *Apply the theorem to both* X_1 *and* X_2 *equal to the* E_7*-variety. As a result we get* X *which we denote by* $(E_7)^{*2}$, *a smooth Legendrian Fano variety of dimension 27, Picard group generated by a hyperplane section and of index 8. Now apply the theorem to* X_1 *being the* E_7*-variety again and* $X_2 = (E_7)^{*2}$. *The result,* $(E_7)^{*3}$ *again has the Picard group generated by a hyperplane section and* $K_{(E_7)^{*3}} = \mathcal{O}_{(E_7)^{*3}}(2)$, *hence is very ample. Analogously we construct* $(E_7)^{*k}$ *and combining this result with corollary VII.4, we get infinitely many families of smooth Legendrian varieties of general type with Picard group generated by a very ample class in every dimension* d, *where* $3 \leq d \leq 27$.

Example VII.8. *Let* $X_1 = \mathbb{P}^1 \times Q^{m_1 - 1}$ *and* X_2 *be arbitrary. If* $m_1 \geq 3$ *and* $\dim X_2 \geq 3$, *then* X *has Picard group isomorphic to* $\operatorname{Pic} X_2 \oplus \mathbb{Z}$. *Hence we can get a smooth Legendrian variety with arbitrarily big Picard rank.*

Example VII.9. *Let* $X_1 = X_2 = \mathbb{P}^1 \times Q^{m-1}$*. Let the resulting* X *be called* $(\mathbb{P}^1 \times Q^{m-1})^{*2}$*. Then* $K_{X_i} = \mathcal{O}_{X_i}(-m) \otimes E_i$*, where* E_i *is effective. Hence*

$$K_{(\mathbb{P}^1 \times Q^{m-1})^{*2}} = \mathcal{O}_{(\mathbb{P}^1 \times Q^{m-1})^{*2}}(-m+1) \otimes E$$

for an effective E*. Construct analogously* $(\mathbb{P}^1 \times Q^{m-1})^{*k}$ *by taking the section of*

$$\left(\left(\mathbb{P}^1 \times Q^{m-1}\right)^{*(k-1)}\right) * \left(\mathbb{P}^1 \times Q^{m-1}\right).$$

We get that

$$K_{(\mathbb{P}^1 \times Q^{m-1})^{*k}} = \mathcal{O}_{(\mathbb{P}^1 \times Q^{m-1})^{*k}}(-m-1+k) \otimes E$$

and for $k > m+1$ *we get that the canonical divisor can be written as an ample plus an effective, so it is big. Hence in every dimension, it is possible to construct many smooth Legendrian varieties with the maximal Kodaira dimension.*

VII.3 Extending Legendrian varieties

Our motivation is the example of Landsberg and Manivel [LM04, §4], a Legendrian embedding of a Kummer K3 surface blown up in 12 points. It can be seen, that this embedding is given by a codimension 1 linear system. We want to find a Legendrian 3-fold in $\mathbb{P}^7$ whose hyperplane section is this example. Unfortunately, we are not able to find a smooth 3-fold with these properties, but we get one with only isolated singularities.

We recall the setup for the construction of the example. Let W be a vector space of dimension $n+1$. Let Z be any subvariety in $\mathbb{P}^n = \mathbb{P}(W)$.

Definition. *We let* $Z^* \subset \check{\mathbb{P}}^n := \mathbb{P}(W^*)$ *be the closure of the set of hyperplanes tangent to* Z *at some point:*

$$Z^* := \left\{ H \in \check{\mathbb{P}}^n \mid \exists z \in Z \; T_z Z \subset H \right\}.$$

We say Z^* *is* ***the dual variety*** *to* Z*.*

Also let $Z^\sharp \subset \mathbb{P}(T^*\mathbb{P}^n) \subset \mathbb{P}^n \times \check{\mathbb{P}}^n$ be **the conormal variety**, i.e., the closure of the union of projectivised conormal spaces over smooth points of Z. Landsberg and Manivel study in details an explicit birational map $\varphi := \varphi_{H_0,p_0} : \mathbb{P}(T^*\mathbb{P}^n) \dashrightarrow \mathbb{P}^{2n-1}$ which depends on a hyperplane H_0 in $\mathbb{P}^n$ and on a point $p_0 \in H_0$. After Bryant [Bry82] they observe that $\overline{\varphi(Z)}$ (if only makes sense) is always a Legendrian subvariety, but usually singular. Next they study conditions under which $\overline{\varphi(Z)}$ is smooth. In particular, they prove that the conditions are satisfied when Z is a Kummer quartic surface in $\mathbb{P}^3$ in general position with respect to p_0 and H_0 and this gives rise to their example.

We want to modify the above construction just a little bit to obtain our 3-fold. Instead of considering $Z^\sharp$ as a subvariety in

$$\mathbb{P}(W) \times \mathbb{P}(W^*) = (W \setminus \{0\}) \times (W^* \setminus \{0\})/\mathbb{C}^* \times \mathbb{C}^*,$$

we consider a subvariety X in

$$\mathbb{P}^{2n+1} = \mathbb{P}(W \oplus W^*) = (W \times W^*) \setminus \{0\}/\mathbb{C}^*$$

such that the underlying affine cone of X in $W \times W^*$ is the same as the underlying affine pencil of $Z^\sharp$. In other words, we take X to be the closure of preimage of $Z^\sharp$ under the natural projection map:

$$p : \mathbb{P}(W \oplus W^*) \dashrightarrow \mathbb{P}(W) \times \mathbb{P}(W^*).$$

Both $\mathbb{P}(W)$ and $\mathbb{P}(W^*)$ are naturally embedded into $\mathbb{P}(W \oplus W^*)$. Let H be a hyperplane in $\mathbb{P}(W \oplus W^*)$ which does not contain $\mathbb{P}(W)$ nor $\mathbb{P}(W^*)$. Set $H_0 := \mathbb{P}(W) \cap H$ and p_0 to be the point in $\mathbb{P}(W)$ dual to $\mathbb{P}(W^*) \cap H$. Assume H is chosen in such a way that $p_0 \in H_0$.

Theorem VII.10. *Let $X \subset \mathbb{P}(W \oplus W^*) \simeq \mathbb{P}^{2n+1}$ be a subvariety constructed as above from any irreducible subvariety $Z \subset \mathbb{P}(W)$. On $W \oplus W^*$ consider the standard symplectic structure (see §II.1.5) and on $\mathbb{P}(W \oplus W^*)$ consider the associated contact structure. Also assume H, H_0 and p_0 are chosen as above. Then:*

(i) X is a Legendrian subvariety contained in the quadric $\overline{p^{-1}\Big(\mathbb{P}\big(T^*\mathbb{P}(W)\big)\Big)}$.

(ii) Let $\widetilde{X}_H$ be the Legendrian variety in $\mathbb{P}^{2n-1}$ constructed from X and H as in §VII.1.1. Also consider the closure of $\varphi_{H_0,p_0}(Z^\sharp)$ as in the construction of [LM04, §4]. Then the two constructions agree, i.e., the closure $\varphi_{H_0,p_0}(Z^\sharp)$ is a component of $\widetilde{X}_H$.

(iii) The singular locus of X equal to the union of following:

on $\mathbb{P}(W)$ the singular points of Z,

on $\mathbb{P}(W^*)$ the singular points of Z^* and

outside $\mathbb{P}(W) \cup \mathbb{P}(W^*)$ the preimage under p of the singular locus of the conormal variety $Z^\sharp$.

Proof. For part (i) consider $\widehat{Z} \subset W$, the affine cone over $Z \subset \mathbb{P}(W)$. The cotangent bundle to W is equal to $W \oplus W^*$. Furthermore, by our definition $\widehat{X} \subset V$, the affine cone over $X \subset \mathbb{P}(W \oplus W^*)$ is the conormal variety of $\widehat{X}$, so a Lagrangian subvariety (see example II.6).

For part (ii), we choose coordinates $x_0, x_1, \dots, x_n$ on W and dual coordinates $y^0, y^1, \dots, y^n$ on W^* such that in the induced coordinates on V the hyperplane H has the equation $x_0 - y^n = 0$. Now restrict to the affine piece $x_0 = y^n = 1$ on both H and $\mathbb{P}(W) \times \mathbb{P}(W^*)$. We see explicitly, that the projection map $H \to \mathbb{P}^{2n-1}$

$$[1, x_1, \dots, x_n, y^0, \dots, y^{n-1}, 1] \mapsto [y^1, \dots, y^{n-1}, y^0 - x_n, x_1, \dots, x_{n-1}, 1]$$

agrees with the map φ from [LM04, §4].

To find the singularities of X on $X \cap \mathbb{P}(W)$ as in part (iii) note that $X \subset \mathbb{P}(W \oplus W^*)$ is invariant under the following action of $\mathbb{C}^*$:

$$t \cdot [w, \alpha] := [tw, t^{-1}\alpha].$$

In particular, points of $X \cap \mathbb{P}(W)$ are fixed points of the action. So let $[w, 0] \in X$ and then $T_{[w,0]}X$ decomposes into the eigenspaces of the action:

$$T_{[w,0]}X = T_{[w,0]}(X \cap \mathbb{P}(W)) \oplus T_{[w,0]}(X \cap F_w) \tag{VII.11}$$

where F_w is the fibre of the projection $\rho : (\mathbb{P}(W \oplus W^*) \backslash \mathbb{P}(W^*)) \to \mathbb{P}(W)$, $F_w := \rho^{-1}([w])$. Clearly the image of X under the projection ρ is Z, so the dimension of a general fibre of $\rho|_X : X \to Z$ is equal to $\dim X - \dim Z = \dim \mathbb{P}(W) - \dim Z = \operatorname{codim}_{\mathbb{P}(W)} Z$. Therefore, since the dimension of the fibre can only grow at special points, we have:

$$\dim T_{[w,0]}(X \cap F_w) \geq \dim(X \cap F_w) \geq \operatorname{codim}_{\mathbb{P}(W)} Z. \tag{VII.12}$$

Also $\mathrm{d}_{[w,0]}(\rho|_X) : T_{[w,0]}X \to T_{[w]}Z$ maps $T_{[w,0]}(X \cap F_w)$ to 0 and $T_{[w,0]}(X \cap \mathbb{P}(W))$ onto $T_{[w]}Z$. Therefore:

$$\dim T_{[w,0]}(X \cap \mathbb{P}(W)) \geq \dim T_{[w]}Z \geq \dim Z. \tag{VII.13}$$

Now assume $[w, 0]$ is a smooth point of X. Then adding (VII.12) and (VII.13) we get:

$$\begin{aligned}\dim X = \dim T_{[w,0]}X &\\ \overset{\text{by (VII.11)}}{=} \dim T_{[w,0]}(X \cap F_w) + \dim T_{[w,0]}(X \cap \mathbb{P}(W)) &\\ \geq \operatorname{codim}_{\mathbb{P}(W)} Z + \dim Z = \dim \mathbb{P}(W).\end{aligned}$$

By (i) the $\dim X$ is equal to the $\dim \mathbb{P}(W)$, so in (VII.12) and (VII.13) all the inequalities are in fact equalities. In particular $\dim T_{[w]}Z = \dim Z$, so $[w]$ is a smooth point of Z.

Conversely, assume $[w]$ is a smooth point of Z, then the tangent space

$$T_{[w,0]}X = T_{[w]}Z \oplus N^*_{[w]}(Z \subset \mathbb{P}(W)),$$

therefore clearly $[w, 0]$ is a smooth point of X.

Exactly the same argument shows that X is singular at a point $[0, \alpha] \in X \cap \mathbb{P}(W^*)$ if and only if Z^* is singular at $[\alpha]$.

For the last part of (iii) it is enough to note that p is a locally trivial $\mathbb{C}^*$-bundle when restricted to $\mathbb{P}(W \oplus W^*) \setminus (\mathbb{P}(W) \cup \mathbb{P}(W^*))$.

□

Corollary VII.14. *Given a Legendrian subvariety $\widetilde{Z} \subset \mathbb{P}^{2n-1}$ we can take $Z^{\#} := \phi^{-1}_{H_0,p_0}(\widetilde{Z})$ to construct a Legendrian subvariety in $\mathbb{P}(T^*\mathbb{P}^n)$. Such a variety must be the conormal variety to some variety $Z \subset \mathbb{P}^n$ (see corollary III.19). Let $X \subset \mathbb{P}^{2n+1}$ be the Legendrian variety constructed from Z as above. By theorem VII.10 (ii), a component of a hyperplane section of X can be projected onto $\widetilde{Z}$.*

Unfortunately, in the setup of the theorem X is almost always singular (see §VII.4).

Example VII.15. *If Z is a Kummer quartic surface in $\mathbb{P}^3$, then X is a 3-fold with 32 isolated singular points (it follows from theorem VII.10(iii) because the Kummer quartic surface has 16 singular points, it is isomorphic to its dual and it has smooth conormal variety in $\mathbb{P}(T^*\mathbb{P}^3)$). Therefore by theorem VII.1 a general hyperplane section of X is smooth and admits a Legendrian embedding. By theorem VII.10 the example of Landsberg and Manivel is a special case of this hyperplane section. Even though the condition $p_0 \in H_0$ is a closed condition, it satisfies the generality conditions of theorem VII.1 and therefore this hyperplane section consists of a unique smooth component that is projected isomorphically onto $\widetilde{Z}$.*

Example VII.16. *Similarly, if Z is a curve in $\mathbb{P}^2$ satisfying the generality conditions of Bryant [Bry82, thm G], then X is a surface with only isolated singularities and its hyperplane section projects isomorphically onto a Bryant's Legendrian curve.*

VII.4 Smooth varieties with smooth dual

Furthermore we observe that a classical problem of classifying smooth varieties with smooth dual variety can be expressed in terms of Legendrian varieties:

Corollary VII.17. *Using the notation of the previous section, let $Q_W \subset \mathbb{P}(W \oplus W^*)$ be the quadric $\overline{p^{-1}(\mathbb{P}(T^*\mathbb{P}(W)))}$ — see VII.10(i). On $W \oplus W^*$ consider the standard symplectic structure (see §II.1.5) and on $\mathbb{P}(W \oplus W^*)$ consider the associated contact structure (see §III.1).*

(i) Let $Z \subset \mathbb{P}(W)$ be a smooth subvariety with $Z^* \subset \mathbb{P}(W^*)$ smooth. Let $X \subset \mathbb{P}(W \oplus W^*)$ be as in the above construction. Then X is a smooth Legendrian variety contained in Q_W.

(ii) *Conversely, assume $X \subset \mathbb{P}(W \oplus W^*)$ is irreducible, Legendrian and contained in Q_W. Let $Z = X \cap \mathbb{P}(W)$. Then $Z^* = X \cap \mathbb{P}(W^*)$ and the variety arising from Z in the above construction is exactly X. Moreover, if X is smooth, then Z and Z^* are smooth.*

We underline that although all the smooth quadrics of a given dimension are projectively isomorphic, the classification of quadrics relatively to the contact structure is more complicated. The quadric Q_W can therefore be written as $x_0y_0 + \ldots + x_ny_n = 0$ in some symplectic coordinates $x_0, \ldots, x_n, y_0, \ldots, y_n$ on $W \oplus W^*$. We note (without proof), that such quadric Q_W determines uniquely the pair of Lagrangian subspaces W and W^*.

Proof. Part (i) follows immediately from theorem VII.10(i) and (iii).

To prove part (ii), consider $p(X) \subset \mathbb{P}(T^*\mathbb{P}(W))$. By lemma I.4 and proposition II.1 $p(X)$ is Legendrian. By corollary III.19, $p(X)$ is a conormal variety to some subvariety $Z \subset \mathbb{P}(W)$. The next thing to prove is that X coincides with the variety constructed above from Z, i.e. that

$$X = \overline{p^{-1}\big(p(X)\big)}.$$

Equivalently, it is enough to prove that X is $\mathbb{C}^*$-invariant. This is provided by theorem III.5 since the quadric Q_W produces exactly the required action. Finally, it follows that $Z = X \cap \mathbb{P}(W)$. Moreover, $p(X)$ is also the conormal variety to $Z^* \subset \mathbb{P}(W^*)$ and hence $Z^* = X \cap \mathbb{P}(W^*)$. If X is in addition smooth, then Z and Z^* are smooth by theorem VII.10(iii).

□

smooth self-dual variety $Z \subset \mathbb{P}^n$	the corresponding Legendrian variety $X \subset \mathbb{P}^{2n+1}$
Q^m	$\mathbb{P}^1 \times Q^m$
$\mathbb{P}^1 \times \mathbb{P}^m$	$\mathbb{P}^1 \times Q^{2m}$
$Gr(2,5)$	$Gr(3,6)$
$\mathbb{S}_5$	$\mathbb{S}_6$

Table VII.1: The known self-dual varieties and their corresponding Legendrian varieties. Note that Q^{2m} and $\mathbb{P}^1 \times \mathbb{P}^m$ lead to isomorphic Legendrian varieties. Yet their embeddings in the distinguished quadrics are not isomorphic.

Therefore the classification of smooth varieties with smooth dual is equivalent to the classification of pairs (X, Q), where $Q \subset \mathbb{P}^{2n+1}$ is a quadric which can be written as $x_0y_0+\ldots+x_ny_n = 0$ in some symplectic coordinates $x_0, \ldots, x_n, y_0, \ldots, y_n$ on $\mathbb{C}^{2n+2}$ and $X \subset \mathbb{P}^{2n+1}$ is a smooth Legendrian variety, which is contained in Q. So far the only known examples of smooth varieties with smooth dual are the smooth self-dual varieties (see [Ein86]). From these we get some of the homogeneous Legendrian varieties (see table VII.1). Therefore we cannot hope to produce new examples of smooth Legendrian varieties in this way. What we hope for is to classify the pairs (X, Q) as above and hence finish the classification of smooth varieties with smooth dual.

Appendix A

Vector fields, forms and automorphisms

In the course of the main part of this dissertation, particularly in chapter III we used some differential geometric facts, which we summarise in this appendix. Although all these facts are standard or follow easily from the standard material, we reproduce or at least sketch most of the proofs. We do this for the sake of completeness of the material presented in the thesis and also because various authors of textbooks use various notations and combining these one can get very confused (at least this has happened to the author of this thesis).

A.1 Homogeneous differential forms and vector fields

Let Y, Y' be two complex manifolds and let $\phi : Y' \longrightarrow Y$ be a holomorphic map. For a k-form $\omega \in H^0(Y, \Omega^k Y)$, by $\phi^*\omega \in H^0(Y', \Omega^k Y')$ we denote the pull-back of ω:

$$(\phi^*\omega)_y(v_1, \ldots, v_k) := \omega_{\phi(y)}\big(\mathrm{D}_y\phi(v_1), \ldots, \mathrm{D}_y\phi(v_k)\big).$$

Now assume we have a $\mathbb{C}^*$-action on Y:

$$(t, y) \longmapsto \lambda_t(y).$$

We say that $\omega \in H^0(Y, \Omega^k Y)$ is **homogeneous of weight** $\mathrm{wt}(\omega)$ if

$$\forall t \in \mathbb{C}^* \quad \lambda_t^*\omega = t^{\mathrm{wt}(\omega)}\omega.$$

For example, assume $Y = \mathbb{A}^n = \mathrm{Spec}(\mathbb{C}[y_1, \ldots, y_n])$ and $\mathbb{C}^*$ acts via homotheties. We say $\omega \in \Omega^k\mathbb{A}^n$ is **constant**, if it is a $\mathbb{C}$-linear combination of $\mathrm{d}y_{i_1} \wedge \ldots \wedge \mathrm{d}y_{i_k}$. Constant k forms are homogeneous of weight k (not of weight 0 as one could possibly expect). Conversely, if $\omega \in H^0(\mathbb{A}^n, \Omega^k\mathbb{A}^n)$ is homogeneous of weight k, then it is constant, because every global form can be written as $\sum f_{i_1,\dots,i_k} \mathrm{d}y_{i_1} \wedge \ldots \wedge \mathrm{d}y_{i_k}$. Since $\mathrm{d}y_{i_1} \wedge \ldots \wedge \mathrm{d}y_{i_k}$ are already of weight k, it follows that $f_{i_1,\dots,i_k}$ are constant functions.

Let $\mu \in H^0(Y, TY)$ be a vector field. We say μ is **homogeneous of weight** $\mathrm{wt}(\mu)$ if

$$\mathrm{D}\lambda_{t^{-1}}\mu = t^{\mathrm{wt}(\mu)}\mu.$$

Lemma A.1. *Let Y, Y' be complex manifolds, both with a $\mathbb{C}^*$-action. Moreover assume $\phi : Y' \longrightarrow Y$ is a $\mathbb{C}^*$-equivariant map, $\omega \in H^0(Y, \Omega^k Y)$ is a homogeneous k-form for some $k \in \{0, 1, \dots, \dim Y\}$ and $\mu \in H^0(Y, TY)$, $\nu \in H^0(Y', TY')$ are two homogeneous vector fields.*

(i) $\omega(\mu)$ *is homogeneous and* $\mathrm{wt}\big(\omega(\mu)\big) = \mathrm{wt}(\omega) + \mathrm{wt}(\mu)$*;*

(ii) $\phi^*\omega$ *is homogeneous of weight* $\mathrm{wt}(\omega)$ *and* $\mathrm{D}\phi(\nu)$ *is homogeneous of weight* $\mathrm{wt}(\nu)$*;*

(iii) $\mathrm{d}\omega$ *is homogeneous of weight* $\mathrm{wt}(\omega)$*.*

Proof. This is an immediate calculation. For instance (i):

$$\lambda_t^*(\omega(\mu))_x(v_1, \dots, v_{k-1}) = \omega_{\lambda_t(x)}(\mu, \mathrm{D}\lambda_t(v_1), \dots, \mathrm{D}\lambda_t(v_{k-1})) =$$

$$= (\lambda_t^*\omega)_x(\mathrm{D}\lambda_{t^{-1}}(\mu), v_1, \dots, v_{k-1}) = t^{\mathrm{wt}(\omega)} t^{\mathrm{wt}(\mu)} (\omega(\mu))_x(v_1, \dots, v_{k-1}).$$

□

A.2 Vector fields and automorphisms

A.2.1 Vector fields, Lie bracket and distributions

Let Y be a complex manifold or a smooth algebraic variety, let $F \subset TY$ be a corank 1 subbundle[1] and let $\theta : TY \to TY/F =: L$ be the quotient map, so that the following sequence is exact:

$$0 \longrightarrow F \longrightarrow TY \xrightarrow{\theta} L \longrightarrow 0.$$

Also assume U is an open subset. We say that a (possibly singular) subvariety $X \subset U$ with its smooth locus X_0 is **F-integrable** if TX_0 is contained in F.

[1] One could also consider F to be a corank r subbundle for any $r \in \{1 \dots \dim Y\}$. Some of the statements below can be generalised to any r (not necessary $r = 1$), but the proofs get more complicated, especially in notation. We restrict our considerations to the $r = 1$ case, as this is the only one used in the thesis.

Proposition A.2. *With the assumptions as above:*

(i) $\mathrm{d}\theta$ *gives a well defined map of* $\mathcal{O}_Y$*-modules:*

$$\mathrm{d}\theta : \bigwedge^2 F \longrightarrow L.$$

We refer to this map as ***the twisted 2-form*** $\mathrm{d}\theta$.

(ii) Assume μ *and* ν *are two vector fields on* U*, both contained in* F*. Then* $\theta([\mu,\nu])(y) = \mathrm{d}\theta_y(\mu(y),\nu(y))$*. In particular* $\theta([\mu,\nu])(y)$ *does not depend on the vector fields, but only on their values at* y*.*

(iii) Again assume μ *and* ν *are two vector fields on* U*, but now only* ν *is contained in* F*. Then again* $\theta([\mu,\nu])(y)$ *depends only on the value of* ν *at* y*, but not on the whole vector field. In other words the map of sheaves* $F \longrightarrow L$ *given by* $\theta\big([\mu,\cdot]\big)$ *is* $\mathcal{O}_Y$*-linear and hence it determines a map of vector bundles* $F \longrightarrow L$*.*

(iv) If X *is* F*-integrable, then* $\mathrm{d}\theta|_{X_0} \equiv 0$*. In particular if* $r=1$*, then*

$$\dim X \le \operatorname{rk} F - \frac{1}{2}\min_{x\in X}(\operatorname{rk} \mathrm{d}\theta_x)$$

Proof. All the statements are analytically local, so it is enough to assume that Y is a disc $D^{2n} \subset \mathbb{C}^n$ with coordinates $y_1,\dots,y_m$, $U=Y$, $y=0$ and that θ is a nowhere vanishing section of $\Omega^1 Y \otimes L \simeq \Omega^1 Y$ (the choice of the trivialisation of L is of course not unique):

$$\theta = \sum_i A_i \mathrm{d}y_i = \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{y},$$

where the collection $(A_1,\dots,A_m)$ (respectively $(\mathrm{d}y_1,\dots,\mathrm{d}y_m)^T$) we denote by $\boldsymbol{A}$ (respectively $\mathrm{d}\boldsymbol{y}$). Then:

$$F := \left\{ v \in TD^{2n} \mid \sum_i A_i \mathrm{d}y_i(v) = 0 \right\}.$$

To prove (i) note that:

$$\mathrm{d}\theta = \sum_i \mathrm{d}A_i \wedge \mathrm{d}y_i = \mathrm{d}\boldsymbol{A}\wedge \mathrm{d}\boldsymbol{y}.$$

We must check that this does not depend on the choice of the trivialisation $\boldsymbol{A}$ of L. So assume $\boldsymbol{B}$ is a different trivialisation, so there exists $g : Y \longrightarrow \mathbf{GL}(1) \simeq \mathbb{C}^*$ such that:

$$\boldsymbol{B} = g\cdot \boldsymbol{A}.$$

We must prove that $\mathrm{d}\boldsymbol{B} \wedge \mathrm{d}\boldsymbol{y}$ restricted to F transforms in the same manner:

$$\mathrm{d}\boldsymbol{B} \wedge \mathrm{d}\boldsymbol{y} = \mathrm{d}(g \cdot \boldsymbol{A}) \wedge \mathrm{d}\boldsymbol{y} = (\mathrm{d}g \cdot \boldsymbol{A} + g \cdot \mathrm{d}\boldsymbol{A}) \wedge \mathrm{d}\boldsymbol{y} =$$

$$\overset{\text{since } \boldsymbol{A} \text{ vanish on } F}{=} (g \cdot \mathrm{d}\boldsymbol{A}) \wedge \mathrm{d}\boldsymbol{y}.$$

To prove (ii) let

$$\mu = \sum_k \mu_k \frac{\partial}{\partial y_k}, \tag{A.3}$$

$$\nu = \sum_k \nu_k \frac{\partial}{\partial y_k} \tag{A.4}$$

for some holomorphic functions μ_k and ν_k. Since μ and ν are contained in F we have:

$$\sum_k A_k \mu_k = 0 \quad \text{ and } \quad \sum_l A_l \nu_l = 0.$$

Therefore for every k or l we have:

$$\sum_k \frac{\partial A_k}{\partial y_l} \mu_k = -\sum_k A_k \frac{\partial \mu_k}{\partial y_l}; \tag{A.5a}$$

$$\sum_l \frac{\partial A_l}{\partial y_k} \nu_l = -\sum_l A_l \frac{\partial \nu_l}{\partial y_k}. \tag{A.5b}$$

Since

$$[\mu, \nu] = \sum_{k,l} \left(\nu_k \frac{\partial \mu_l}{\partial y_k} \frac{\partial}{\partial y_l} - \mu_l \frac{\partial \nu_k}{\partial y_l} \frac{\partial}{\partial y_k} \right),$$

hence:

$$\begin{aligned}
\theta([\mu,\nu]) &= \sum_{k,l} \left(A_l \nu_k \frac{\partial \mu_l}{\partial y_k} - A_k \mu_l \frac{\partial \nu_k}{\partial y_l} \right) = \\
&\overset{\text{by (A.5)}}{=} \sum_{k,l} \left(-\frac{\partial A_l}{\partial y_k} \mu_l \nu_k + \frac{\partial A_k}{\partial y_l} \mu_l \nu_k \right) = \\
&= \sum_{k,l} \left(\frac{\partial A_l}{\partial y_k} (\mu_k \nu_l - \mu_l \nu_k) \right) = \\
&= \sum_{k,l} \left(\frac{\partial A_l}{\partial y_k} (\mathrm{d}y_k \wedge \mathrm{d}y_l)(\mu, \nu) \right) = \\
&= \mathrm{d}\theta (\mu, \nu).
\end{aligned}$$

We note that the above calculation is a special case of [KN96, prop. I.3.11], though the reader should be careful, as the notation in [KN96] is different than ours and as a consequence a constant factor -2 is “missing” in our formula.

The proof of (iii) is identical as the beginning of the proof of (ii).

Finally to prove (iv) just use (ii) and the fact that the Lie bracket of two vector fields tangent to X must be tangent to X.

□

A.2.2 Automorphisms

Here we introduce the notation about several types of automorphisms of a manifold Y and its subvariety X. Also we recall some standard properties and relations between them.

Let Y be a complex manifold (or respectively, smooth algebraic variety) and let $U \subset Y$ be an open subset in analytic (or respectively, Zariski) topology. By $\mathrm{Aut}^{hol}(U)$ (respectively, $\mathrm{Aut}^{alg}(U)$) we denote the group of holomorphic (respectively, algebraic) automorphisms of U. By $\mathrm{Aut}^{\bullet}(U)$ we mean either $\mathrm{Aut}^{hol}(U)$ or $\mathrm{Aut}^{alg}(U)$, whenever specifying is not necessary.

Assume that a complex Lie group (respectively, an algebraic group) G acts on U, i.e. we have a group homomorphism $G \longrightarrow \mathrm{Aut}^{\bullet}(U)$. Also let $\mathfrak{g}$ be the Lie algebra of G. By G^0 we denote the the connected component of identity in G.

An infinitesimal automorphism of U is a vector field $\mu \in H^0(U, TY)$. Differentiating the action map $G \times U \longrightarrow U$ by the first coordinate we get the induced map $\mathfrak{g} \times Y \longrightarrow TY$ or more precisely $\mathfrak{g} \longrightarrow H^0(U, TY)$. This map preserves the Lie bracket (see [Akh95, thm in §1.7]) and if the action is faithful, then it is injective (see [Akh95, thm in §1.5]).

The particular case is when $G = \mathbb{C}^*$. Then we get a map $\mathbb{C} \longrightarrow H^0(U, TY)$ and we set $\mu_{\mathbb{C}^*}$ to be the image of $1 \in \mathbb{C}$ under this map. We say $\mu_{\mathbb{C}^*}$ is **the vector field related to the $\mathbb{C}^*$-action.** Note that $\mu_{\mathbb{C}^*}$ is homogeneous of weight 0.

The infinitesimal automorphisms make a sheaf TY of Lie algebras, which at the same time is an $\mathcal{O}_Y$-module. The two structures are related by the following Leibniz rule:

$$\forall f \in H^0(U, \mathcal{O}_Y),\ \forall \mu, \nu \in H^0(U, TY) \quad [f\mu, \nu] = f[\mu, \nu] + \mathrm{d}f(\nu)\mu. \tag{A.6}$$

The following theorem comparing infinitesimal, algebraic and holomorphic automorphisms for a projective variety is well known and standard:

Theorem A.7. *Let Y be a projective variety. Then:*

(i) $\mathrm{Aut}^{hol}(Y)$ *is a complex Lie group.*

(ii) Every holomorphic automorphism of Y is algebraic and hence

$$\mathrm{Aut}(Y) := \mathrm{Aut}^{hol}(Y) = \mathrm{Aut}^{alg}(Y).$$

(iii) By $\mathfrak{aut}(Y)$ *we denote the tangent Lie algebra to* $\mathrm{Aut}(Y)$*. Every infinitesimal automorphism is tangent to some 1-parameter subgroup of* $\mathrm{Aut}^{hol}(Y)$*, so that* $\mathfrak{aut}(Y) = H^0(Y, TY)$*.*

Proof. Part (i) is proved in [Akh95, §2.3]. Part (ii) is a consequence of [Gri74, thm IV.A]. Part (iii) is explained in [Akh95, prop. in §1.5 & cor. 1 in §1.8].

□

Clearly $H^0(U, \mathcal{O}_Y)$ is a representation of G and hence also of $\mathfrak{g}$. We also have the following Lie algebra action of the sheaf of infinitesimal automorphisms:

$$\begin{array}{rcl} TY \times \mathcal{O}_Y & \longrightarrow & \mathcal{O}_Y \\ (\mu, f) & \mapsto & \mathrm{d}f(\mu), \end{array}$$

which is given by the derivation in the direction of the vector field.

The action of $\mathfrak{g}$ on $H^0(U, \mathcal{O}_Y)$ is the composition

$$\mathfrak{g} \longrightarrow H^0(U, TY) \longrightarrow \mathfrak{gl}\left(H^0(U, \mathcal{O}_Y)\right).$$

Let $X \subset Y$ be a subvariety. By $\mathrm{Aut}^{\bullet}(U, X)$ we denote the respective subgroup of $\mathrm{Aut}^{\bullet}(U)$ preserving the intersection $U \cap X$. If Y is projective, then by $\mathfrak{aut}(Y, X)$ we mean the Lie algebra tangent to $\mathrm{Aut}^{\bullet}(Y, X)$. By $\mathfrak{aut}^{\mathrm{inf}}(U, X)$ we denote the Lie algebra of infinitesimal automorphisms of U preserving X, i.e.:

$$\mathfrak{aut}^{\mathrm{inf}}(U, X) := \left\{\mu \in H^0(U, TY) \mid \forall f \in \mathcal{I}(X)|_U \quad (\mathrm{d}f)(\mu) \in \mathcal{I}(X)|_U\right\},$$

where $\mathcal{I}(X) \lhd \mathcal{O}_Y$ is the sheaf of ideals of X.

Clearly, if G preserves X, then the image of $\mathfrak{g} \longrightarrow H^0(U, TY)$ is contained in $\mathfrak{aut}^{\mathrm{inf}}(U, X)$. Conversely, if the image is contained in $\mathfrak{aut}^{\mathrm{inf}}(U, X)$, then the action of the connected component G^0 preserves X.

Corollary A.8. *If* Y *is projective, then* $\mathfrak{aut}^{\mathrm{inf}}(Y, X) = \mathfrak{aut}(Y, X)$*.*

□

Moreover $\mathfrak{aut}^{\mathrm{inf}}(\cdot, X)$ makes in TY a subsheaf of Lie algebras and $\mathcal{O}_Y$-modules.

A.2.3 Distributions and automorphisms preserving them

If $F \subset TY$ is a corank 1 vector subbundle (particularly a contact distribution - see §III.2 for the definition), then by $\mathrm{Aut}^{\bullet}_F(U)$, $\mathfrak{aut}_F(Y)$, $\mathfrak{aut}^{\mathrm{inf}}_F(U)$, $\mathrm{Aut}^{\bullet}_F(U, X)$, $\mathfrak{aut}_F(Y, X)$ and $\mathfrak{aut}^{\mathrm{inf}}_F(U, X)$ we denote the appropriate automorphisms or infinitesimal automorphisms preserving F and possibly the subvariety X.

For instance,

$$\mathfrak{aut}^{\mathrm{inf}}_F(U) = \left\{\mu \in H^0(U, TY) \mid [\mu, F] \subset F\right\}. \tag{A.9}$$

Also $\mathfrak{aut}_F^{\inf}$ makes a sheaf of Lie algebras, but usually it is not an $\mathcal{O}_Y$-submodule of TY. To see that take any $\mu \in \mathfrak{aut}_F^{\inf}(U)$ for U small enough. Assume for all $f \in \mathcal{O}_Y(U)$ we have $f\mu \in \mathfrak{aut}_F^{\inf}(U)$. Then by Leibniz rule (see equation A.6):

$$\forall \nu \in H^0(U,F) \quad \mathrm{d}f(\nu)\cdot\mu \in H^0(U,F).$$

This can only happen if either:

- $\mu \in H^0(U,F)$ or
- $F = 0$, i.e. F is the rank 0 bundle.

We have seen that the first case does not happen if F is a contact distribution (unless $\mu = 0$, see theorem III.151)). In fact, one can prove that it never happens for all $\mu \in H^0(U,F)$ (remember that U is small enough), unless $F = 0$.

If G acts on U and preserves the distribution F, then the map $\mathfrak{g} \to H^0(U,TY)$ factors through $\mathfrak{aut}_F^{\inf}(U)$. Conversely, if G is connected, it acts on U and the map $\mathfrak{g} \to H^0(U,TY)$ factors through $\mathfrak{aut}_F^{\inf}(U)$, then the action of G preserves F. As a consequence we get:

Corollary A.10. *If Y is projective and $X \subset Y$ is a subvariety, then:*

(i) $\mathfrak{aut}_F(Y) = \mathfrak{aut}_F^{\inf}(Y)$

(ii) $\mathfrak{aut}_F(Y,X) = \mathfrak{aut}_F^{\inf}(Y,X)$

Proof. This follows from the above considerations and from theorem A.7. □

Further, let L be the quotient bundle and θ be the quotient map:

$$0 \longrightarrow F \longrightarrow TY \xrightarrow{\theta} L \longrightarrow 0.$$

If the action of G on U extended to $TY|_U$ preserves F, then in the obvious way we get **the induced action of G on the total spaces of $L|_U$ and $L^*|_U$**. These actions preserve the zero sections.

Let $\mathbf{L}^\bullet$ and $\mathcal{R}_L$ be as in §I.3.7.

By analogy with above we want to define the action of $\mathfrak{aut}_F^{\inf}$ on $\mathbf{L}^\bullet$. In other words, we define a special lifting of the vector fields from $\mathfrak{aut}_F^{\inf} \subset TY$ to vector fields on $\mathbf{L}^\bullet$.

First observe that the sheaf of Lie algebras $\mathfrak{aut}_F^{\inf}$ acts on the sheaf L: if $s \in H^0(U,L)$, then choose an open subset $V \subset U$ small enough and any lifting $s_{TY} \in H^0(V,TY)$, $\theta(s_{TY}) = s|_V$ and let $\mu \in \mathfrak{aut}_F^{\inf}(U)$ act on $H^0(U,L)$ locally by

$$s|_V \mapsto (\mu.s)|_V := \theta\left([s_{TY}, \mu|_V]\right). \tag{A.11}$$

By equation (A.9), this does not depend on the choice of s_{TY} and hence, by elementary properties of sheaves, it glues uniquely to an element of $H^0(U,L)$. Hence we get a Lie algebra representation $\mathfrak{aut}_F^{\inf}(U) \longrightarrow \mathfrak{gl}\left(H^0(U,L)\right)$.

Secondly, we can extend the action of $\mathfrak{aut}_F^{\inf}$ on the locally free sheaf L defined in equation (A.11) to an action on $\mathcal{R}_L$, by requesting that the action must satisfy the Leibniz rule:

$$t, s \in \mathcal{R}_L,\ \mu \in \mathfrak{aut}_F^{\inf} \implies \mu.(ts) = (\mu.t)s + t(\mu.s) \tag{A.12}$$

— locally every section of L^m can be written as a sum of products of sections of L (or their inverses, if $m < 0$).

Finally, we can extend this action to $\mathcal{O}_{\mathbf{L}^\bullet}$, again requesting the Leibniz rule. Eventually, we get the action, which we will call **the induced action of $\mathfrak{aut}_F^{\inf}$ on $\mathbf{L}^\bullet$**. The following property justifies the name:

Proposition A.13. *If the action of G preserves F, then the tangent action to the induced action of G on $\mathbf{L}^\bullet|_U := \pi^{-1}(U)$ is the composition of $\mathfrak{g} \longrightarrow \mathfrak{aut}_F^{\inf}(U)$ and the induced action of $\mathfrak{aut}_F^{\inf}$ on $\mathbf{L}^\bullet$.*

□

For a fixed $\mu \in \mathfrak{aut}_F^{\inf}(U)$, the induced map $\mathcal{O}_{\mathbf{L}^\bullet}|_{\pi^{-1}(U)} \longrightarrow \mathcal{O}_{\mathbf{L}^\bullet}|_{\pi^{-1}(U)}$ is a derivation, so it corresponds to a vector field $\breve{\mu} \in H^0(\pi^{-1}(U), T\mathbf{L}^\bullet)$, such that

$$\forall f \in \mathcal{O}_{\mathbf{L}^\bullet} \quad \mu.f = \mathrm{d}f(\breve{\mu}). \tag{A.14}$$

By construction we also have $\mathrm{D}\pi(\breve{\mu}) = \mu$.

A.2.4 1-form $\theta^\bullet$

With the notation and assumptions as in the previous sections, we have a canonical isomorphism of line bundles $\tau : \pi^* L \xrightarrow{\simeq} \mathcal{O}_{\mathbf{L}^\bullet}$: if $y \in Y$, $\lambda \in \mathbf{L}^\bullet_y = \pi^{-1}(y)$, $l \in L_y$, then we set

$$\tau(y, \lambda, l) := (y, \lambda, \lambda(l)).$$

We let $\theta^\bullet := \tau \circ \pi^*\theta \circ \mathrm{D}\pi$:[2]

$$T\mathbf{L}^\bullet \xrightarrow{\mathrm{D}\pi} \pi^*TY \xrightarrow{\pi^*\theta} \pi^*L \xrightarrow{\tau} \mathcal{O}_{\mathbf{L}^\bullet}.$$

Lemma A.15. *For every $\mu \in \mathfrak{aut}_F^{\inf}(U)$ the induced infinitesimal automorphism $\breve{\mu}$ preserves $\theta^\bullet$, i.e.:*

$$L_{\breve{\mu}}(\theta^\bullet) := \lim_{t \to 0} \frac{\gamma_{\breve{\mu}}(t)^*\theta^\bullet - \theta^\bullet}{t} = 0,$$

where $L_{\breve{\mu}}$ is the Lie derivative operator and $\gamma_{\breve{\mu}}(t)$ is the local 1-parameter group of transformations of $\mathbf{L}^\bullet$ determined by $\breve{\mu}$.

[2]In [Bea98], [LeB95] the authors denote $\theta^\bullet$ simply as $\pi^*\theta$, since the other maps are natural. This is a bit confusing to some people (including the author of this thesis, but see also a comment in [SCW04] about a small mistake in [KPSW00]) and therefore we underline that $\theta^\bullet$ is the composition of three maps.

Proof. For the simplicity of notation assume $\gamma_{\breve{\mu}}(t)$ is a global transformation. The following diagram of vector bundles is commutative:

$$\begin{array}{ccccccc}
T\mathbf{L}^{\bullet} & \xrightarrow{\mathrm{D}\pi} & \pi^*TY & \xrightarrow{\pi^*\theta} & \pi^*L & \xrightarrow{\tau} & \mathcal{O}_{\mathbf{L}^{\bullet}} \\
\big\downarrow{\scriptstyle \mathrm{D}\gamma_{\breve{\mu}}(t)} & & \big\downarrow{\scriptstyle \mathrm{D}^{\pi}\gamma_{\mu}(t)} & & \big\downarrow{\scriptstyle \gamma_{\mu}^{\pi^*L}(t)} & & \big\downarrow{\scriptstyle \gamma_{\breve{\mu}}(t)\times \mathrm{id}_{\mathbb{C}}} \\
T\mathbf{L}^{\bullet} & \xrightarrow{\mathrm{D}\pi} & \pi^*TY & \xrightarrow{\pi^*\theta} & \pi^*L & \xrightarrow{\tau} & \mathcal{O}_{\mathbf{L}^{\bullet}}
\end{array}$$

where by $\mathrm{D}^{\pi}\gamma_{\mu}(t)$ we mean the automorphism of π^*TY, which is determined by $\mathrm{D}\gamma_{\mu}(t) : TY \to TY$ and $\gamma_{\breve{\mu}}(t) : \mathbf{L}^{\bullet} \to \mathbf{L}^{\bullet}$; similarly $\gamma_{\mu}^{\pi^*L}(t)$ is determined by $\mathrm{D}\gamma_{\mu}(t) : TY/F \to TY/F$ and $\gamma_{\breve{\mu}}(t) : \mathbf{L}^{\bullet} \to \mathbf{L}^{\bullet}$. The composition of the whole upper row is equal to $\theta^{\bullet}$. The composition of the left most vertical arrow and the whole lower row is equal to $\gamma_{\breve{\mu}}(t)^*\theta^{\bullet}$. Since the right arrow is the identity on the second component $\mathcal{O}_{\mathbf{L}^{\bullet}} = \mathbf{L}^{\bullet} \times \mathbb{C}$ and since the diagram is commutative, both forms take the same values on every vector $v \in T\mathbf{L}^{\bullet}$, hence are equal and the claim follows.

□

We also give a local description of $\theta^{\bullet}$ and $\mathrm{d}\theta^{\bullet}$. So now assume $Y \simeq D^{2m}$ and let $y_1, \dots, y_m$ be some coordinates on Y. Let z be a linear coordinate on the fibre of $\mathbf{L}^{\bullet} \simeq Y \times \mathbb{C}^*$. This means that z determines a section of L which trivialises L over D^{2m}. So we can think of θ as of a holomorphic 1-form on $\mathbf{L}^{\bullet}$ depending only on y_i's and $\mathrm{d}y_i$'s. Let (y, z_0) be any point of $\mathbf{L}^{\bullet}$ and let $\bar{v}$ be any vector tangent to $\mathbf{L}^{\bullet}$ at (y, z_0). We write $\bar{v} = v + w$, where v is the component tangent to Y, while w is tangent to $\mathbb{C}^*$. Then:

$$\theta^{\bullet}_{(y,z_0)}(\bar{v}) = (\tau \circ \pi^*\theta \circ \mathrm{D}\pi)_{y,z_0}(\bar{v}) = (\tau \circ \pi^*\theta)_{y,z_0}(v) = z_0(\theta_y(v)) = z_0 \cdot \theta_y(v).$$

Or more concisely (in local coordinates)

$$\theta^{\bullet} = z\theta, \qquad \text{(A.16)}$$

and therefore

$$\mathrm{d}\theta^{\bullet} = \mathrm{d}\,(z\theta) = z\mathrm{d}\theta + \mathrm{d}z \wedge \theta. \qquad \text{(A.17)}$$

Since in this notation θ is a homogeneous 1-form of weight 0 and $\mathrm{wt}(z) = 1$, $\theta^{\bullet}$ and $\mathrm{d}\theta^{\bullet}$ are homogeneous forms of weight 1 (see §A.1).

In the above coordinates, the vector field $\mu_{\mathbb{C}^*}$ related to the $\mathbb{C}^*$-action can be expressed as follows:

$$\mu_{\mathbb{C}^*} = z\frac{\partial}{\partial z}.$$

Proposition A.18. *Let Y be a complex manifold or smooth algebraic variety and let L be a line bundle on Y. Also let $\mathbf{L}^{\bullet}$ be the principal $\mathbb{C}^*$-bundle over Y as in §I.3.7 and let $\mu_{\mathbb{C}^*}$ be the vector field on $\mathbf{L}^{\bullet}$ associated to the action of $\mathbb{C}^*$. Finally, let ω be a homogeneous closed 2-form on $\mathbf{L}^{\bullet}$ of weight* 1. *Then:*

(i) $\omega = \mathrm{d}\big(\omega(\mu_{\mathbb{C}^*})\big)$*;*

(ii) There exists a unique twisted 1-form $\theta : TY \longrightarrow L$*, such that* $\omega(\mu_{\mathbb{C}^*}) = \theta^\bullet$*, where* $\theta^\bullet$ *is defined from* θ *as above;*

(iii) Moreover, $\omega(\mu_{\mathbb{C}^*})$ *is nowhere vanishing if and only if* θ *is nowhere vanishing. If this is the case, then* ω *is non-degenerate if and only if* $\mathrm{d}\theta|_F$ *is non-degenerate.*

Proof. To prove (i) let z be a local coordinate linear on the fibres of $\pi : \mathbf{L}^\bullet \to Y$. Since ω is closed, locally it is exact, so

$$\omega = \mathrm{d}(z\phi' + g\mathrm{d}z)$$

for some function g and 1-form ϕ', both homogeneous of weight 0. However,

$$\mathrm{d}(z\phi' + g\mathrm{d}z) = \mathrm{d}\big(z(\phi' - \mathrm{d}g)\big).$$

Set $\phi := \phi' - \mathrm{d}g$, so that $\omega = \mathrm{d}(z\phi)$. Note that although ϕ' and g are not uniquely determined, ϕ is the unique homogeneous 1-form of weight 0 such that $\omega = \mathrm{d}(z\phi)$. Then,

$$\omega\left(\mu_{\mathbb{C}^*}\right) = (\mathrm{d}z \wedge \phi)\left(z\frac{\partial}{\partial z}\right) + z\mathrm{d}\phi\left(z\frac{\partial}{\partial z}\right) = \mathrm{d}z\left(z\frac{\partial}{\partial z}\right) \cdot \phi = z\phi.$$

Hence $\mathrm{d}\big(\omega\left(\mu_{\mathbb{C}^*}\right)\big) = \omega$, as claimed in (i).

To prove (ii), define θ to be locally the form ϕ from the above argument. One must verify that ϕ glues uniquely to a twisted 1-form $\theta : TY \longrightarrow L$.

Part (iii) follows from the local descriptions of $\theta^\bullet$ and $\mathrm{d}\theta^\bullet$, see (A.16) and (A.17). For instance, if $n = \frac{1}{2}(\dim Y - 1)$, then:

$$(\mathrm{d}\theta^\bullet)^{\wedge^{n+1}} = (n+1)\mathrm{d}z \wedge \theta \wedge (\mathrm{d}\theta)^{\wedge^n}.$$

Therefore $\mathrm{d}\theta^\bullet$ is non-degenerate at a given point if and only if θ does not vanish at that point and $\mathrm{d}\theta$ is non-degenerate on the kernel of θ.

□

Lemma A.19. *Let* $X \subset Y$ *be any subvariety and* X_0 *its smooth locus. Then* X *is* F*-integrable if and only if* $\mathrm{d}\theta^\bullet$ *vanishes identically on the tangent space to* $\pi^{-1}(X_0)$*.*

Proof. First assume X is F-integrable. Then $\mathrm{d}\theta$ vanishes on $T\left(\pi^{-1}(X_0)\right)$ by proposition A.2(iv) and θ vanishes by definition. Hence from the local description of $\mathrm{d}\theta^\bullet$ (see equation (A.17)) we get the result.

On the other hand if $\mathrm{d}\theta^\bullet|_{T(\pi^{-1}(X_0))} \equiv 0$, since

$$\mu_{\mathbb{C}^*}|_{\pi^{-1}(X_0)} \in H^0\left(\pi^{-1}(X_0), T\left(\pi^{-1}(X_0)\right)\right),$$

then in particular

$$\mathrm{d}\theta^\bullet\left(\mu_{\mathbb{C}^*}, T\left(\pi^{-1}(X_0)\right)\right) \equiv 0.$$

But $\mathrm{d}\theta^\bullet(\mu_{\mathbb{C}^*}) = \theta^\bullet$ (see proposition A.18(ii)), hence $\pi^{-1}X$ is (π^*F)-integrable and therefore X is F-integrable.

□

For $s \in \mathcal{R}_L = \pi_*\mathcal{O}_{\mathbf{L}^\bullet}$, by $\tilde{s} \in \mathcal{O}_{\mathbf{L}^\bullet}$ we denote the lifting of s, i.e. $\tilde{s} := \tau \circ \pi^* s$.

Hence we have two possibilities of lifting an infinitesimal automorphism $\mu \in \mathfrak{aut}_F^{\inf}$ to an object on $\mathbf{L}^\bullet$: either we lift it to a vector field $\breve{\mu}$ (see (A.14)) or we lift $\theta(\mu)$ to function $\widetilde{\theta(\mu)}$. We will compare these two liftings and how they behave with respect to the Lie bracket of vector fields in the following statements.

Lemma A.20. *We have:*

$$\forall \nu \in \mathfrak{aut}_F^{\inf}(U), \mu \in H^0(U, TY) \qquad \widetilde{\theta([\mu,\nu])} = \mathrm{d}\left(\widetilde{\theta(\mu)}\right)(\breve{\nu}).$$

Proof. By (A.11):

$$\theta([\mu,\nu]) = \nu.\theta(\mu)$$

and hence $\widetilde{\theta([\mu,\nu])} = \nu.\widetilde{\theta(\mu)}$. By (A.14), this is equal to $\mathrm{d}\left(\widetilde{\theta(\mu)}\right)(\breve{\nu})$.

□

Proposition A.21. *If $\mu \in \mathfrak{aut}_F^{\inf}(U)$, then:*

$$\mathrm{d}\left(\widetilde{\theta(\mu)}\right) = -(\mathrm{d}\theta^\bullet)(\breve{\mu}).$$

Proof. The following proof is quoted from [Bea98, prop. 1.6]. Since $L_{\breve{\mu}}(\theta^\bullet) = 0$ (see lemma A.15), by [KN96, prop. I.3.10(a)] we have:

$$(\mathrm{d}\theta^\bullet)(\breve{\mu}) = -\mathrm{d}\big(\theta^\bullet(\breve{\mu})\big).$$

On the other hand:

$$\theta^\bullet(\breve{\mu}) = \tau \circ \pi^*\theta \circ \mathrm{D}\pi(\breve{\mu}) = \tau \circ \pi^*\left(\theta(\mu)\right) = \widetilde{\theta(\mu)}.$$

Combining the two equalities, we get the result.

□

Corollary A.22. *If $\mu, \nu \in \mathfrak{aut}_F^{\inf}(U)$, then*

$$\widetilde{\theta([\mu,\nu])} = -(\mathrm{d}\theta^\bullet)(\breve{\mu}, \breve{\nu}).$$

Proof. This combines lemma A.20 and proposition A.21.

□

Bibliography

[Akh95] Dmitri N. Akhiezer. *Lie group actions in complex analysis*. Aspects of Mathematics, E27. Friedr. Vieweg & Sohn, Braunschweig, 1995.

[Arn74] V. I. Arnold. *Matematicheskie metody klassicheskoi mekhaniki*. Izdat. "Nauka", Moscow, 1974.

[Bea98] Arnaud Beauville. Fano contact manifolds and nilpotent orbits. *Comment. Math. Helv.*, 73(4):566–583, 1998.

[Bea99] Arnaud Beauville. Riemannian holonomy and algebraic geometry. arXiv:math.AG/9902110, 1999.

[Ber55] Marcel Berger. Sur les groupes d'holonomie homogène des variétés à connexion affine et des variétés riemanniennes. *Bull. Soc. Math. France*, 83:279–330, 1955.

[BK94] Ranee Brylinski and Bertram Kostant. Minimal representations of E_6, E_7, and E_8 and the generalized Capelli identity. *Proc. Nat. Acad. Sci. U.S.A.*, 91(7):2469–2472, 1994.

[Bry82] Robert L. Bryant. Conformal and minimal immersions of compact surfaces into the 4-sphere. *J. Differential Geom.*, 17(3):455–473, 1982.

[Buc03] Jarosław Buczyński. Podrozmaitości legendrowskie zespolonej przestrzeni rzutowej. Master's thesis, Institute of Mathematics, Warsaw University, 2003.

[Buc05] Jarosław Buczyński. Properties of Legendrian subvarieties of projective space. arXiv: math.AG/0503528v1, 2005.

[Buc06] Jarosław Buczyński. Legendrian subvarieties of projective space. *Geom. Dedicata*, 118:87–103, 2006.

[Buc07a] Jarosław Buczyński. Hyperplane sections of Legendrian varieties. arXiv: math.AG/0701590, 2007.

[Buc07b] Jarosław Buczyński. Some quasihomogeneous Legendrian varieties. arXiv: math.AG/0701750, 2007.

[Buc07c] Jarosław Buczyński. Toric Legendrian subvarieties. arXiv: math.AG/0609550, to appear in Transformation Groups, 2007.

[Cou95] S. C. Coutinho. *A primer of algebraic D-modules*, volume 33 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1995.

[Dem02] Jean-Pierre Demailly. On the Frobenius integrability of certain holomorphic p-forms. In *Complex geometry (Göttingen, 2000)*, pages 93–98. Springer, Berlin, 2002.

[Dru98] Stéphane Druel. Structures de contact sur les variétés algébriques de dimension 5. *C. R. Acad. Sci. Paris Sér. I Math.*, 327(4):365–368, 1998.

[Ein86] Lawrence Ein. Varieties with small dual varieties. I. *Invent. Math.*, 86(1):63–74, 1986.

[Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[Gri74] Phillip Griffiths. *Topics in algebraic and analytic geometry*. Princeton University Press, Princeton, N.J., 1974. Written and revised by John Adams, Mathematical Notes, No. 13.

[Gro61] A. Grothendieck. Éléments de géométrie algébrique. II. Étude globale élémentaire de quelques classes de morphismes. *Inst. Hautes Études Sci. Publ. Math.*, (8):222, 1961.

[Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[Har95] Joe Harris. *Algebraic geometry*, volume 133 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. A first course, Corrected reprint of the 1992 original.

[HH02] Haydeé Herrera and Rafael Herrera. $\hat{A}$-genus on non-spin manifolds with S^1 actions and the classification of positive quaternion-Kähler 12-manifolds. *J. Differential Geom.*, 61(3):341–364, 2002.

[Hit81] N. J. Hitchin. Kählerian twistor spaces. *Proc. London Math. Soc. (3)*, 43(1):133–150, 1981.

[Hon00] Jaehyun Hong. Fano manifolds with geometric structures modeled after homogeneous contact manifolds. *Internat. J. Math.*, 11(9):1203–1230, 2000.

[Hum75] James E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[Huy03] Daniel Huybrechts. Compact hyperkähler manifolds. In *Calabi-Yau manifolds and related geometries (Nordfjordeid, 2001)*, Universitext, pages 161–225. Springer, Berlin, 2003.

[JJ04] Stanisław Janeczko and Zbigniew Jelonek. Linear automorphisms that are symplectomorphisms. *J. London Math. Soc. (2)*, 69(2):503–517, 2004.

[Joy00] Dominic D. Joyce. *Compact manifolds with special holonomy*. Oxford Mathematical Monographs. Oxford University Press, Oxford, 2000.

[Keb01] Stefan Kebekus. Lines on contact manifolds. *J. Reine Angew. Math.*, 539:167–177, 2001.

[Keb05] Stefan Kebekus. Lines on complex contact manifolds. II. *Compos. Math.*, 141(1):227–252, 2005.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry. Vol. I*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.

[KO73] Shoshichi Kobayashi and Takushiro Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973.

[KPSW00] Stefan Kebekus, Thomas Peternell, Andrew J. Sommese, and Jarosław A. Wiśniewski. Projective contact manifolds. *Invent. Math.*, 142(1):1–15, 2000.

[Laz04] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[LeB95] Claude LeBrun. Fano manifolds, contact structures, and quaternionic geometry. *Internat. J. Math.*, 6(3):419–437, 1995.

[Leh04] Manfred Lehn. Symplectic moduli spaces. In *Intersection theory and moduli*, ICTP Lect. Notes, XIX, pages 139–184 (electronic). Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004.

[LM01] J. M. Landsberg and L. Manivel. The projective geometry of Freudenthal's magic square. *J. Algebra*, 239(2):477–512, 2001.

[LM02] Joseph M. Landsberg and Laurent Manivel. Construction and classification of complex simple Lie algebras via projective geometry. *Selecta Math. (N.S.)*, 8(1):137–159, 2002.

[LM04] Joseph M. Landsberg and Laurent Manivel. Legendrian varieties. arXiv: math.AG/0407279, to appear in Asian J. Math, 2004.

[LS94] Claude LeBrun and Simon Salamon. Strong rigidity of positive quaternion-Kähler manifolds. *Invent. Math.*, 118(1):109–132, 1994.

[MS98] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1998.

[MT99] V. B. Mehta and V. Trivedi. The variety of circular complexes and F-splitting. *Invent. Math.*, 137(2):449–460, 1999.

[Muk88] Shigeru Mukai. Curves, $K3$ surfaces and Fano 3-folds of genus ≤ 10. In *Algebraic geometry and commutative algebra, Vol. I*, pages 357–377. Kinokuniya, Tokyo, 1988.

[Muk98] Shigeru Mukai. Simple Lie algebra and Legendre variety. http://www.math.nagoya-u.ac.jp /˜mukai/, 1998.

[Mum99] David Mumford. *The red book of varieties and schemes*, volume 1358 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, expanded edition, 1999. Includes the Michigan lectures (1974) on curves and their Jacobians, With contributions by Enrico Arbarello.

[Oda88] Tadao Oda. *Convex bodies and algebraic geometry*, volume 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1988. An introduction to the theory of toric varieties, Translated from the Japanese.

[Ott95] Giorgio Ottaviani. Rational homogeneous varieties. http://web.math.unifi.it/users/ottavian/rathomo/rathomo.ps, 1995.

[PS91] Y. S. Poon and S. M. Salamon. Quaternionic Kähler 8-manifolds with positive scalar curvature. *J. Differential Geom.*, 33(2):363–378, 1991.

[RS06] G. V. Ravindra and V. Srinivas. The Grothendieck-Lefschetz theorem for normal projective varieties. *J. Algebraic Geom.*, 15(3):563–590, 2006.

[SCW04] Luis Eduardo Solá Conde and Jarosław A. Wiśniewski. On manifolds whose tangent bundle is big and 1-ample. *Proc. London Math. Soc. (3)*, 89(2):273–290, 2004.

[Sła91] Jan J. Sławianowski. *Geometry of phase spaces*. A Wiley-Interscience Publication. John Wiley & Sons Ltd., Chichester, 1991. Translated from the Polish.

[Str82] Elisabetta Strickland. On the conormal bundle of the determinantal variety. *J. Algebra*, 75(2):523–537, 1982.

[Stu97] Bernd Sturmfels. Equations defining toric varieties. In *Algebraic geometry—Santa Cruz 1995*, volume 62 of *Proc. Sympos. Pure Math.*, pages 437–449. Amer. Math. Soc., Providence, RI, 1997.

[Tim03] D. A. Timashëv. Equivariant compactifications of reductive groups. *Mat. Sb.*, 194(4):119–146, 2003.

[Wiś00] Jarosław Wiśniewski. Lines and conics on fano contact manifolds. http://www.mimuw.edu.pl/˜jarekw/, 2000.

[Wol65] Joseph A. Wolf. Complex homogeneous contact manifolds and quaternionic symmetric spaces. *J. Math. Mech.*, 14:1033–1047, 1965.

[Ye94] Yun-Gang Ye. A note on complex projective threefolds admitting holomorphic contact structures. *Invent. Math.*, 115(2):311–314, 1994.