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Oriented coloring of 2-dimensional grids

PhD dissertation

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the dissertation is ready to be reviewed

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Abstract

Oriented coloring of 2-dimensional grids

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An oriented coloring of an oriented graph $\overrightarrow{G}$ is a homomorphism from $\overrightarrow{G}$ to an oriented graph $\overrightarrow{H}$ such that $\overrightarrow{H}$ has neither loops nor arcs in opposite directions. The oriented chromatic number $\overrightarrow{\chi}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the smallest number of vertices of $\overrightarrow{H}$ for which there exists a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$. The oriented chromatic number of an undirected graph is the maximal chromatic number over all possible orientations of G. In this thesis, we consider the oriented chromatic number of four families of graphs, namely: 2-dimensional grids, cylindrical grids, toroidal grids and strong-grids. A 2-dimensional grid $G(m, n)$ is the Cartesian product $P_m \Box P_n$ of two paths on m and n vertices, the Cartesian product of a cycle C_m and a path P_n is called a cylindrical grid $Cyl(m, n) = C_m \Box P_n$, whereas the Cartesian products of two cycles is called a toroidal grid $T(m, n) = C_m \Box C_n$. A strong-grid $G^{\boxtimes}(m, n)$ is the strong product $P_m \boxtimes P_n$ of two paths on m and n vertices. Closely related to oriented coloring is signed coloring. A signed graph is a pair (G, σ), where $G = (V(G), E(G))$ is an undirected graph and $\sigma : E(G) \to \{+, -\}$ is a function which marks each edge with "$+$" or "$-$". Two signed graphs are equivalent if one of them can be changed to the other by a sequence of resigning operations. The single resigning operation chooses a vertex $v \in V(G)$ and flips the signs of all edges incident to v. By $[G, \sigma]$ we shall denote the equivalence class of the signed graph (G, σ). Each element of class $[G, \sigma]$ is called a presentation. The coloring of signed graphs is defined through homomorphism. The signed graph $[G, \sigma]$ is colored by the signed graph (G_2, σ_2), if there exists a presentation (G, σ_1) of $[G, \sigma]$ and a vertex-mapping ϕ from G to G_2 which preserves signs of the edges. The signed chromatic number of the signed graph $[G, \sigma]$, denoted by $\chi_s([G, \sigma])$, is the size of the smallest graph which colors $[G, \sigma]$. The signed chromatic number $\chi_s(G)$ of the undirected graph G is the maximum of the signed chromatic numbers over all signed graphs with underlining graph G.

The main results of the thesis are listed below:

1. We establish the new lower bound of eight for oriented chromatic number of the family of all grids by showing that there exists an orientation of a grid that cannot be colored by seven colors. This also gives the new lower bound of eight for the family of all cylindrical grids and for the family of all toroidal grids. (Chapter 3.)

2. We present an oriented graph with ten vertices, namely, $\overrightarrow{H}_{10}$, which colors all orientations of all grids with eight rows. This gives the new upper bound of ten for the oriented chromatic number for the families of all grids with six, seven and eight rows. (Chapter 4.)

3. We show that every toroidal grid can be colored with twenty seven colors. (Chapter 6.)

4. We show that any orientation of any cylindrical grid with circuit at most seven can be colored by the graph $\overrightarrow{H}_{10}$. (Chapter 6.)

5. We give new lower and upper bounds for the oriented chromatic number of strong-grids. We show that there exists an orientation of a strong-grid $G^{\boxtimes}(2, 398)$ that cannot be colored with ten colors. This gives the new lower bound of eleven for the oriented chromatic number of the family of all strong-grids. Furthermore, we show that any orientation of any strong-grid can be colored with eighty eight colors. (Chapter 7.)

6. We show that any signed grid with at most seven rows can be colored with five colors. (Chapter 8.)

7. We show that the lower bound for the signed chromatic number of the family of all grids is five. (Chapter 8.)

The main results of the dissertation are published in ([11–13]) or are accepted for publication ([35]).

Keywords: grids, oriented coloring, signed coloring, combinatorial problem

AMS MS Classification 2010: 05C15, 05C20, 05C22, 05C76, 05C85

Streszczenie

Zorientowane kolorowanie dwuwymiarowych krat

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Zorientowane kolorownie skierowanego grafu $\overrightarrow{G}$ to homomorfizm grafu $\overrightarrow{G}$ w skierowany graf $\overrightarrow{H}$, który nie ma pętli ani łuków w przeciwnych kierunkach. Zorientowana liczba chromatyczna $\overrightarrow{\chi}(\overrightarrow{G})$ skierowanego grafu $\overrightarrow{G}$, to najmniejsza liczba wierzchołków w grafie $\overrightarrow{H}$, dla którego istnieje homomorfizm z grafu $\overrightarrow{G}$ w graf $\overrightarrow{H}$. Zorientowana liczba chromatyczna grafu nieskierowanego to maksymalna zorientowana liczba chromatyczna po wszystkich orientacjach grafu G. Tematem tej rozprawy jest zorientowana liczba chromatyczna czterech klas grafów: 2-wymiarowych krat, cylindrycznych krat, toroidalnych krat oraz silnych krat. 2-wymiarowa krata $G(m, n)$ to iloczyn kartezjański $P_m \square P_n$ dwóch ścieżek o m i n wierzchołkach, cylindryczna krata $Cyl(m, n) = C_m \square P_n$ to iloczyn kartezjański cyklu C_m i ścieżki P_n, natomiast iloczyn kartezjański dwóch cykli to krata toroidalna $T(m, n) = C_m \square C_n$. Silną kratą $G^{\boxtimes}(m, n) = P_m \boxtimes P_n$ nazywamy silny iloczyn dwóch ścieżek o m i n wierzchołkach. Ściśle powiązane z kolorowaniem zorientowanym jest kolorowanie oznakowane. Oznakowany graf (ang. signed graph) to para (G, σ), gdzie $G = (V(G), E(G))$ jest nieskierowanym grafem, a $\sigma : E(G) \to \{+, -\}$ to funkcja przyporządkowująca każdej krawędzi znak "+" lub "−". Dwa oznakowane grafy są równoważne, jeśli jeden z nich może być zamieniony w drugi za pomocą ciągu operacji zmiany znaków. Pojedyncza operacja zmiany znaków, dla wybranego wierzchołka $v \in V(G)$, zamienia znaki wszytkich krawędzi incydentnych z v. Poprzez $[G, \sigma]$ będziemy oznaczać klasę równoważności grafu oznakowanego (G, σ). Każdy element klasy $[G, \sigma]$ nazywamy reprezentantem tej klasy. Kolorowanie grafów oznakowanych definiujemy poprzez homomorfizm. Graf oznakowany $[G, \sigma]$ jest kolorowalny za pomocą (G_2, σ_2), jeżeli istnieje reprezentant (G, σ_1) klasy $[G, \sigma]$ i odwzorowanie ϕ z $V(G)$ w $V(G_2)$ zachowyjące znaki krawędzi. Oznakowana liczba chromatyczna oznakowanego grafu $[G, \sigma]$ to najmniejsza liczba wierzchołków grafu kolorującego graf $[G, \sigma]$. Oznakowana liczba chromatyczna $\chi_s(G)$ grafu G to maksymalna liczba spośród wszystkich oznakowanych liczb chromatycznych wszystkich oznakowanych grafów powstałych na bazie G.

Główne wyniki rozprawy są następujące:

1. Ustalamy nową dolną granicę osiem dla zorientowanej liczby chromatycznej krat, przedstawiając orientację kraty, która wymaga ośmiu kolorów do zorientowanego kolorowania. Daje to również nową dolną granicę osiem dla rodziny wszystkich cylindrycznych krat oraz dla rodziny wszystkich toroidalnych krat. (Rozdział 3.)

2. Przedstawiamy skierowany graf z dziesięcioma wierzchołkami, nazwany $\overrightarrow{H}_{10}$, taki, że dowolna orientacja kraty z ośmioma wierszami posiada homomrfizm w $\overrightarrow{H}_{10}$. Otrzymaliśmy w ten sposób nowe górne ograniczenie zorientowanej liczby chromatycznej dla rodzin wszystkich krat z sześcioma, siedmioma oraz ośmioma wierszami. (Rozdział 4.)

3. Pokazujemy, że dowolna orientacja toroidalnej kraty jest kolorowalna za pomocą dwudziestu siedmiu kolorów. (Rozdział 6.)

4. Pokazujemy, że dowolna orientacja cylindrycznej kraty o obwodzie cyklu co najwyżej siedem może być pokolorowana przy pomocy $\overrightarrow{H}_{10}$. (Rozdział 6.)

5. Ustanawiamy nową górną i dolną granicę zorientowanej liczby chromatycznej rodziny wszystkich silnych krat. Pokazujemy, że istnieje orientacja silnej kraty $G^{\boxtimes}(2, 398)$, która nie może być pokolorowana za pomocą 10 kolorów. Wyznaczamy w ten sposób nowe dolne ograniczenie zorientowanej liczby chromatycznej wszystkich silnych krat. Ponadto, pokazujemy, że dowolna orientacja dowolnej silnej kraty może być pokolorowana za pomocą osiemdziesięciu ośmiu kolorów. (Rozdział 7.)

6. Pokazujemy, że dowolna oznakowana krata z co najwyżej siedmioma wierszami może być pokolorowana za pomocą pięciu kolorów. (Rozdział 8.)

7. Pokazujemy, że dolna granica oznakowanej liczby chromatycznej rodziny wszystkich krat jest pięć. (Rozdział 8.)

Wyniki przedstawiowe w dysertacji zostały opublikowane w pracach ([11–13]) lub są przyjęte do publikacji w ([35]).

Słowa kluczowe: kraty, zorientowane kolorowanie, oznakowane kolorowanie, problem kombinatoryczny

Klasyfikacja AMS MSC 2010: 05C15, 05C20, 05C22, 05C76, 05C85

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Chapter 1

Introduction

Let $G = (V(G), E(G))$ be a simple undirected graph. *A proper vertex coloring* of graph G is a function that assigns colors to vertices of G such that any two adjacent vertices have different colors. *An orientation* of G is a directed graph $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$ that is obtained from G by replacing each edge $\{u, v\}$ by one of two arcs on vertices u and v. For example, Figure 1.1 presents all four orientations of path P_3. *A tournament* is

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FIGURE 1.1: Possible orientations of path P_3.

an orientation of a clique. We say that directed graph is *an oriented graph* if it has neither loops nor arcs in opposite directions. An *oriented coloring* c of an oriented graph $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$ is a coloring of vertices of $\overrightarrow{G}$ such that

(i) c is proper,

(ii) c respects the orientation: if the arc (y, z) goes from color $c(y)$ to $c(z)$, then no other arc can go in the opposite direction, i.e., from $c(z)$ to $c(y)$.

Figure 1.2 shows an orientation of cycle C_4 and an oriented coloring. Observe that this orientation cannot be colored with three colors. *The oriented chromatic number* $\overrightarrow{\chi}(\overrightarrow{G})$ of an oriented graph $\overrightarrow{G}$ is the smallest number of colors needed in its oriented coloring. The oriented coloring c of $\overrightarrow{G}$ can be viewed as a homomorphism from $\overrightarrow{G}$ to an oriented graph $\overrightarrow{H}$. In this case, the oriented chromatic number of an oriented graph $\overrightarrow{G}$ is the smallest number of vertices of $\overrightarrow{H}$ such that there exists a homomorphism $c : \overrightarrow{G} \to \overrightarrow{H}$. In

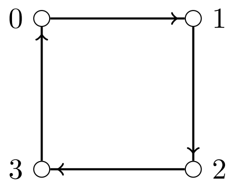

FIGURE 1.2: An orientation of cycle C_4 and its exemplary oriented coloring.

such a case, we shall call $\overrightarrow{H}$ *a coloring graph*. The oriented chromatic number $\overrightarrow{\chi}(G)$ of an undirected graph G is the maximal chromatic number over all possible orientations of G. For example, for C_4, we have $\overrightarrow{\chi}(C_4) = 4$. The oriented chromatic number of a family of graphs is the maximal chromatic number over all possible graphs of the family.

Oriented coloring was introduced by Courcelle in 1994. In his paper [8], oriented coloring was considered as a tool for encoding graph orientations with the help of vertex labels. Since then, several authors have studied links between the oriented chromatic number and other parameters of graphs. Raspaud and Sopena [42] considered relations between the oriented chromatic number and the acyclic chromatic number $\chi_a(G)$, i.e., the smallest number of colors needed for a proper vertex coloring of G with the additional condition that every cycle of G uses at least three colors. They proved the following:

Theorem 1.1 (Raspaud and Sopena [42]).
If $\chi_a(G) \leq k$, then $\overrightarrow{\chi}(G) \leq k \cdot 2^{k-1}$.

Ochem in [36] proved that this bound is tight for $k \geq 3$. Kostochka, Sopena and Zhu [26] proved the following:

Theorem 1.2 (Kostochka, Sopena and Zhu [26]).
If $\overrightarrow{\chi}(G) \leq k$, then $\chi_a(G) \leq k^{\lceil \log_2(\lceil \log_2 k \rceil + k/2) \rceil + 1}$.

Borodin, Kostochka, Nešetřil, Raspaud and Sopena [6] showed that there are relations between the oriented chromatic number of a graph and the maximum average degree $mad(G) = max\{\frac{2 \cdot |E(H)|}{|V(H)|} : H \subseteq G\}$, and girth, where $girth(G)$ is the length of the shortest cycle of G.

Theorem 1.3 (Borodin, Kostochka, Nešetřil, Raspaud and Sopena [6]).
For any graph G,

- *if $mad(G) < \frac{7}{3}$, then $\overrightarrow{\chi}(G) \leq 5$;*
- *if $mad(G) < \frac{11}{4}$ and $girth(G) \geq 5$, then $\overrightarrow{\chi}(G) \leq 7$;*
- *if $mad(G) < 3$, then $\overrightarrow{\chi}(G) \leq 11$;*

- *if* $mad(G) < \frac{10}{3}$, *then* $\overrightarrow{\chi}(G) \leq 19$.

Pinlou, Borodin, Ivanova, Kostochka and Marshall have improved some upper bounds on the oriented chromatic numbers of graphs with bounded maximum average degree by showing the following:

Theorem 1.4 (Borodin, Ivanova and Kostochka [7])**.**
For any graph G*, if* $mad(G) < \frac{12}{5}$ *and* $girth(G) \geq 5$*, then* $\overrightarrow{\chi}(G) \leq 5$.

Theorem 1.5 (Pinlou [38])**.**
For any graph G*, if* $mad(G) < \frac{10}{3}$ *and* $girth(G) \geq 5$*, then* $\overrightarrow{\chi}(G) \leq 16$.

Theorem 1.6 (Marshall [29])**.**
For any graph G *if* $mad(G) < \frac{28}{9}$*, then* $\overrightarrow{\chi}(G) \leq 11$. *This bound is sharp.*

In [25], Klostermeyer and MacGillivray proved that the oriented k-coloring problem (whether a given oriented graph $\overrightarrow{G}$ has $\overrightarrow{\chi}(\overrightarrow{G}) \leq k$) can be decided in polynomial time if $k \leq 3$, and is NP-complete if $k \geq 4$. Several authors bounded the oriented chromatic number for some classes of graphs, such as outerplanar graphs [16, 40, 48, 49], graphs with bounded degree [1, 14, 26, 29, 48, 51, 54], k-trees [48], Halin graphs [15, 23] and graphs with given excess [22].

For planar graphs, since every planar graph is acyclically 5-colorable (see [5]), using the result of Theorem 1.1, we obtain the following:

Theorem 1.7 (Raspaud and Sopena [42])**.**
For any planar graph G, $\overrightarrow{\chi}(G) \leq 80$.

This upper bound has not been improved yet. Sopena showed in [49] that there exists an oriented planar graph with an oriented chromatic number of at least 16. Marshall improved the lower bound to 17 in [27] and to 18 in [28].

In this dissertation, we discuss the oriented chromatic number of four families of graphs, namely, 2-dimensional grids, cylindrical grids, toroidal grids and strong-grids. *A 2-dimensional grid* $G(m,n)$ (or simply grid) is an undirected graph with vertices $V = \{(i,j) : 1 \leq i \leq m; 1 \leq j \leq n\}$ and edges $\{(i,j),(i+1,j)\}$ for $1 \leq i < m; 1 \leq j \leq n$, or $\{(i,j),(i,j+1)\}$ for $1 \leq i \leq m; 1 \leq j < n$. We shall say that the grid $G(m,n)$ has m rows and n columns. A grid $G(m,n)$ can be viewed as the Cartesian product of paths P_m and P_n. The Cartesian product of a cycle C_m and a path P_n is called a *cylindrical grid* $Cyl(m,n) = C_m \Box P_n$ or simply a cylinder and the Cartesian products of two cycles is called a *toroidal grid* $T(m,n) = C_m \Box C_n$ or simply a toroid.

Let $\mathcal{G}$ denote the set of all orientations of all grids and let $\mathcal{G}_m$ denote the set of all orientations of all grids with m rows. The problem of establishing the oriented chromatic number of grids was considered by several authors [4, 18, 41, 53]. The exact values of the oriented chromatic number of grids with two, three and four rows are summarized in Table 1.1. Fertin, Raspaud and Roychowdhury established the upper bound for the oriented chromatic number of grids.

Theorem 1.8 (Fertin, Raspaud and Roychowdhury [18]).

$$\overrightarrow{\chi}(\mathcal{G}) \leq 11.$$

Fertin et al. (see Lemma 2.24) showed that any orientation of any grid can be colored by the Paley tournament $\overrightarrow{T}_{11}$, see Definition 2.2. They also formulated two conjectures:

Conjecture 1.9 (Fertin, Raspaud and Roychowdhury [18]).
Every oriented grid can be colored with seven colors.

Conjecture 1.10 (Fertin, Raspaud and Roychowdhury [18]).
Every oriented grid can be colored by the Paley tournament $\overrightarrow{T}_7$.

Szepietowski and Targan [53] presented an orientation of $G(5, 35)$ that cannot be colored by $\overrightarrow{T}_7$, which contradicts Conjecture 1.10. However, their oriented grid can be colored by other coloring graphs with seven vertices. Hence, Conjecture 1.9 has been still an open problem.

Let $\mathcal{C}yl$ ($\mathcal{T}$, respectively) denote the set of all orientations of all cylinders (toroids, respectively) and let $\mathcal{C}yl_m$ ($\mathcal{T}_m$, respectively) denote the set of all orientations of all cylinders (toroids, respectively) with circuit m. Jamison and Matthews in [24] considered the acyclic chromatic numbers of cylinders and toroids. They proved that:

Theorem 1.11 (Jamison, Matthews [24]).

- $\chi_a(\mathcal{C}yl_m) = 3$, *for* $m \neq 4$,
- $\chi_a(\mathcal{C}yl_4) = 4$,
- $\chi_a(T(m, n)) = 4$, *for* $(m, n) \neq (3, 3)$,
- $\chi_a(T(3, 3)) = 5$.

Hence, by Theorem 1.1, we have:

Corollary 1.12.

- $\overrightarrow{\chi}(\mathcal{C}yl_m) \leq 12$, *for* $m \neq 4$,
- $\overrightarrow{\chi}(\mathcal{C}yl_4) \leq 32$,
- $\overrightarrow{\chi}(T(m,n)) \leq 32$, *for* $(m,n) \neq (3,3)$,
- $\overrightarrow{\chi}(T(3,3)) \leq 80$.

Marshall [29] has proved that:

Theorem 1.13 (Marshall [29])**.**
Every orientation of every cylindrical grid can be colored by the Paley tournament $\overrightarrow{T}_{11}$.

Aravind, Narayanan and Subramanian [2] discussed the oriented chromatic number of strong products of paths $G^{\boxtimes}(m,n)$, called strong-grids. *A strong-grid* $G^{\boxtimes}(m,n)$ is a graph with the vertex set $V(P_m) \times V(P_n)$ and where two vertices are adjacent if and only if they are adjacent in one coordinate and adjacent or equal in the other; see Figure 1.3. Let $\mathcal{G}^{\boxtimes}$ denote the set of all orientations of all strong-grids and let $\mathcal{G}^{\boxtimes}_m$ denote the

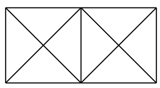

FIGURE 1.3: Strong-grid $G^{\boxtimes}(2,3)$.

set of all orientations of all strong-grids with m rows. They showed the following:

Theorem 1.14 (Aravind, Narayanan and Subramanian [2])**.**

- $8 \leq \overrightarrow{\chi}(G^{\boxtimes}(2,n)) \leq 11$, *for every* $n \geq 5$.
- $10 \leq \overrightarrow{\chi}(G^{\boxtimes}(3,n)) \leq 67$, *for every* $n \geq 5$.

Sopena proved the following:

Theorem 1.15 (Sopena [50])**.**

$$\overrightarrow{\chi}(\mathcal{G}^{\boxtimes}) \leq 126.$$

Closely related to oriented coloring is signed coloring. The graphs with edges marked by "+" and "−" signs were introduced by F. Harary in 1954 [21] to model social interaction within a group. Afterwards they were used as a way of extending classical problems in graph theory such as Hadwiger's conjecture. *A signed graph* is a pair (G,σ), where $G = (V(G), E(G))$ is an undirected graph and $\sigma : E(G) \to \{+,-\}$ is a function which marks each edge with ”+” or ”−”. Two signed graphs are *equivalent* if one of them can be changed to the other by a sequence of resigning operations. The single resigning operation chooses a vertex $v \in V(G)$ and flips the signs of all edges incident to v. By $[G, \sigma]$ we shall denote the equivalence class of the signed graph (G, σ). Each element of $[G, \sigma]$ is called *a presentation* of $[G, \sigma]$.

Some authors use different names for signed graph. They call a presentation (G, σ) a signified graph or a 2-edge-colored graph. In this thesis I shall follow [52] and call both (G, σ) and $[G, \sigma]$ signed graphs. The difference can be recognized by the braces. I hope that this will not lead to confusion. The coloring of signed graphs is defined through homomorphism. The signed graph $[G, \sigma]$ is colored by the signed graph (G_2, σ_2), if there exists a presentation (G, σ_1) of $[G, \sigma]$ and a vertex-mapping ϕ from G to G_2 which preserves signs of the edges. Observe that if $[G, \sigma]$ is colored by (G_2, σ_2), then $[G, \sigma]$ is also colored by any signed graph (G_2, σ_3) which is equivalent to (G_2, σ_2). Hence, we can say that $[G, \sigma]$ is colored by $[G_2, \sigma_2]$. The chromatic numbers are defined as usual. *The signed chromatic number* of the signed graph $[G, \sigma]$, denoted by $\chi_s([G, \sigma])$, is the size of the smallest graph which colors $[G, \sigma]$. The signed chromatic number $\chi_s(G)$ of the undirected graph G is the maximum of the signed chromatic numbers over all signed graphs with underlining graph G. The signed chromatic number $\chi_s(\mathcal{F})$ of the family of graphs $\mathcal{F}$ is the maximum over all graphs in $\mathcal{F}$.

Some authors consider coloring without resigning. Such coloring is called 2-edge-coloring. The signed graph (G, σ) is *2-edge-colored* by the signed graph (G_2, σ_2), if there exists a vertex-mapping ϕ from G to G_2 which preserves signs of the edges. *The 2-edge-colored chromatic number of the signed graph* (G, σ), denoted by $\chi_2((G, \sigma))$, is the size of the smallest graph which colors (G, σ).

The connection of signed graphs and their coloring with classical problems of graph coloring is described by Naserasr, Rollovâ, and Sopena in [32]. Colorings of signed graphs and homomorphisms of signed graphs have been widely studied in recent years [9, 31–34, 44, 52, 55]. The authors pay particular attention to study planar graphs and their subclass such as planar graphs with given girth [39] or grids [3, 10]. Some authors (see Sen [44]) observed that most results in oriented coloring had a similar ”signed version” and could be proved using the same proof techniques, with little adoptions.

Ochem, Pinlou and Sen proved that:

Theorem 1.16 (Ochem, Pinlou and Sen [39]).
If G admits an acyclic k-coloring, then $\chi_s(G) \leq k \cdot 2^{k-2}$.

Bensmail [3] studies 2-edge-colored chromatic number of grids. He showed that:

Theorem 1.17 (Bensmail [3]).

- $8 \leq \chi_2(\mathcal{G}) \leq 11$.
- $\chi_2(G(2,2)) = 4$ *and* $\chi_2(G(2,n)) = 5$, *for* $n \geq 3$,
- $\chi_2(G(3,3)) \geq 6$ *and* $\chi_2(G(3,n)) \geq 7$, *for large enough* n,
- $\chi_2(G(4,n)) \leq 9$, *for* $n \geq 1$,
- $\chi_2(G(5,n)) \geq 8$, *for large enough* n

Recently Dybizbański [10] showed that:

Theorem 1.18 (Dybizbański [10]).

$$\chi_2(\mathcal{G}) \leq 9.$$

In Chapter 3, we show that there exists an orientation of a grid that cannot be colored by seven colors. We shall call an orientation of a grid that cannot be colored by a given coloring graph $\overrightarrow{H}$ *a trap*. We use a greedy algorithm that finds a trap for a given oriented graph $\overrightarrow{H}$. We have found traps for all of 456 non-isomorphic tournaments on seven vertices. Any oriented grid, that contains all the abovementioned traps as subgraphs cannot be colored by any coloring graph with seven vertices. This constitutes a counterexample to Conjecture 1.9 and provides a new, best known so far, lower bound for the oriented chromatic number of grids:

$$8 \leq \overrightarrow{\chi}(\mathcal{G}).$$

The result was published in [11].

In Chapter 4, we improve the upper bound for the oriented chromatic number of grids with eight rows. We present a coloring graph with ten vertices, namely, $\overrightarrow{H}_{10}$, which is the graph obtained from the Paley tournament $\overrightarrow{T}_{11}$ by removing one vertex. We show that any orientation of any grid with eight rows can be colored by $\overrightarrow{H}_{10}$. This improves upon the result of Theorem 1.8 published in 2003 in [18]. The results of this chapter were accepted for publication in [35].

In Chapter 5, we consider the family $\mathcal{G}_5$ of oriented grids with five rows. We improve the upper bound for the oriented chromatic number of $\mathcal{G}_5$ by presenting a coloring graph, with nine colors, which can be used to color any orientation of any grid in $\mathcal{G}_5$. Our proof uses similar algorithm to the one designed by Szepietowski and Targan in [53]. The result was published in [12].

In Chapter 6, we consider cylinders $Cyl(m,n)$ and toroids $T(m,n)$. Grids are subgraphs of cylinders and toroids. Hence, the orientation of a grid with seven rows found in

Chapter 3 gives the lower bound 8 for cylinders and toroids with $m \geq 7$. We also give another proof of Theorem 1.13 of Marshall. We think, that our proof is simpler and more direct. Furthermore, we improve the upper bound for the oriented chromatic number of toroids by showing that

$$\overrightarrow{\chi}(\mathcal{T}) \leq 27.$$

Moreover, using the method from Chapter 4 and the coloring graph $\overrightarrow{H}_{10}$ we show that $\overrightarrow{\chi}(\mathcal{C}yl_m) \leq 10$, for $m = 3, 4, 5, 6, 7$. The results are summarized in Table 1.2. Some of them were accepted for publication in [35].

In Chapter 7, we consider strong-grids $G^{\boxtimes}(m, n)$. We give new lower and upper bounds for the oriented chromatic number of the family $\mathcal{G}^{\boxtimes}$, by showing that

$$11 \leq \overrightarrow{\chi}(\mathcal{G}^{\boxtimes}) \leq 88.$$

The results for strong-grids are summarized in Table 1.3. Some of them were published in [12].

In Chapter 8, we consider signed grids $[G(m, n), \sigma]$. Using an algorithm similar to an algorithm described in Chapter 5, we show that every signed grid with at most seven rows can be colored with five colors, by the signed Paley graph SP_5. This gives the exact value $\chi_s(G(m, n)) = 5$, for $3 \leq m \leq 7$ and $n \geq 4$. Moreover, we show that the signed chromatic number for the class of all 2-dimensional grids lies between 5 and 6. It is worth noticing that the upper bound 6 follows from Theorem 1.16 of Ochem, Pinlou and Sen [39] or from Lemma 30(3) in [31] of Montejano et al. (note that grids admit an acyclic 3-coloring [17]). However, we think that our proof of the upper bound is much more direct and constructive. The results of Chapter 8 were published in [13] and are summarized in Table 1.4.

$\overrightarrow{\chi}(G(2,2)) = 4$ $\overrightarrow{\chi}(G(2,3)) = 5$ $\overrightarrow{\chi}(G(2,n)) = 6, n \geq 4$ $\overrightarrow{\chi}(G(3,3)) = \overrightarrow{\chi}(G(3,4)) =$ $= \overrightarrow{\chi}(G(3,5)) = 6$	Fertin, Raspaud and Roychowdhury (2003) [18]
$\overrightarrow{\chi}(G(3,6)) = 6$ $\overrightarrow{\chi}(G(3,n)) = 7, n \geq 7$	Szepietowski and Targan (2004) [53]
$\overrightarrow{\chi}(G(4,4)) = 6.$	Fertin, Raspaud and Roychowdhury (2003) [18]
$\overrightarrow{\chi}(G(4,n)) = 7, n \geq 5$	Szepietowski and Targan (2004) [53]
$\overrightarrow{\chi}(\mathcal{G}_5) \leq 9$	Dybizbański and Nenca (2019) [12], [this thesis]
$\overrightarrow{\chi}(\mathcal{G}_6) \leq 10$ $\overrightarrow{\chi}(\mathcal{G}_7) \leq 10$ $\overrightarrow{\chi}(\mathcal{G}_8) \leq 10$	Nenca (In Press) [35], [this thesis]
$8 \leq \overrightarrow{\chi}(G(7,n)), n \geq 162$	Dybizbański and Nenca (2012) [11], [this thesis]
$\overrightarrow{\chi}(\mathcal{G}) \leq 11.$	Fertin, Raspaud and Roychowdhury (2003) [18]

TABLE 1.1: Best known results for oriented chromatic numbers of grids with m rows.

$\overrightarrow{\chi}(\mathcal{C}yl_3) \leq 10$ $\overrightarrow{\chi}(\mathcal{C}yl_4) \leq 10$ $\overrightarrow{\chi}(\mathcal{C}yl_5) \leq 10$ $\overrightarrow{\chi}(\mathcal{C}yl_6) \leq 10$ $\overrightarrow{\chi}(\mathcal{C}yl_7) \leq 10$	Nenca (In Press) [35], [this thesis]
$\overrightarrow{\chi}(\mathcal{C}yl_m) \geq 8$, for $m \geq 7$	[this thesis]
$\overrightarrow{\chi}(\mathcal{C}yl) \leq 11$	Marshall [29], [this thesis]
$\chi(T(3,3)) = 9$ $\overrightarrow{\chi}(\mathcal{T}_3) \leq 16$ $\overrightarrow{\chi}(\mathcal{T}_4) \leq 18$ $\overrightarrow{\chi}(\mathcal{T}_5) \leq 20$ $\overrightarrow{\chi}(\mathcal{T}_6) \leq 22$ $\overrightarrow{\chi}(\mathcal{T}_7) \leq 24$ $\overrightarrow{\chi}(\mathcal{T}) \leq 27$	[this thesis]

TABLE 1.2: Best known results for oriented chromatic numbers of cylinders and toroids.

$\overrightarrow{\chi}(\mathcal{G}_2^{\boxtimes}) \leq 11,$	Aravind, Narayanan and Subramanian [2]
$\overrightarrow{\chi}(G^{\boxtimes}(2,n)) \geq 11,\ n \geq 398$ $\overrightarrow{\chi}(\mathcal{G}^{\boxtimes}) \geq 11,$ $\overrightarrow{\chi}(\mathcal{G}_3^{\boxtimes}) \leq 19$	Dybizbański and Nenca (2019) [12], [this thesis]
$\overrightarrow{\chi}(\mathcal{G}_4^{\boxtimes}) \leq 38$ $\overrightarrow{\chi}(\mathcal{G}^{\boxtimes}) \leq 88$	[this thesis]

TABLE 1.3: Best known results for oriented chromatic numbers of strong-grids.

$\chi_s(G(2,n)) = 4, n \geq 2$ $\chi_s(G(3,3)) = 4$, $\chi_s(G(3,n)) = 5, n \geq 4$, $\chi_s(G(4,n)) = 5, n \geq 4$, $\chi_s(G(5,n)) = 5, n \geq 4$, $\chi_s(G(6,n)) = 5, n \geq 4$, $\chi_s(G(7,n)) = 5, n \geq 4$,	Dybizbański, Nenca, Szepietowski (2020) [13], [this thesis]
$\chi_s(\mathcal{G}) \leq 6$	[31], [39], Dybizbański, Nenca, Szepietowski (2020) [13], [this thesis]

TABLE 1.4: Best known results for signed chromatic numbers of grids.

Chapter 2

Preliminaries

An undirected graph G consists of a non-empty finite set $V(G)$ of elements, which are called vertices, and a set $E(G)$ of unordered pairs of $V(G)$, which are called edges, i.e., $E(G) \subseteq \binom{V(G)}{2}$. *A directed graph* $\overrightarrow{G}$ consists of a non-empty finite set $V(\overrightarrow{G})$ of vertices and a set $A(\overrightarrow{G})$ of ordered pairs of elements of $V(\overrightarrow{G})$, which are called arcs, with $A(\overrightarrow{G}) \subseteq (V(\overrightarrow{G}))^2$. An arc $(u, v) \in A(\overrightarrow{G})$ is said to be directed from u to v. The vertices u, v are *incident* to the arc (u, v), and we shall say that u and v are adjacent to each other. For a vertex $u \in V(\overrightarrow{G})$, the *indegree* of u is the number of arcs that are directed to u, and *the outdegree* of u is the number of arcs that are directed from u. The vertex u is called *a source* if it is of indegree zero and *a sink* if it is of outdegree zero. If u is neither a source nor a sink, it is called *an internal vertex.* If $V(\overrightarrow{G}) = \{v_1, ..., v_n\}$, *the adjacency matrix of* $\overrightarrow{G}$ is the $n \times n$ matrix $A = (a_{ij})$ where a_{ij} is the number of arcs from v_i to v_j.

In this dissertation, we shall only deal with directed graphs without loops (i.e., no arc of the form (u, u)) and without arcs in opposite directions (i.e., if $(u, v) \in A(\overrightarrow{G})$, then $(v, u) \notin A(\overrightarrow{G})$). We shall call such graphs *oriented graphs. An orientation* of an undirected graph G is a directed graph $\overrightarrow{G}$ obtained from G by replacing each edge by one of the two possible arcs with the same ends. *A tournament* is an oriented graph in which every two vertices are adjacent. In other words, a tournament is an orientation of a clique K_n.

A homomorphism from a graph $\overrightarrow{G} = (V(\overrightarrow{G}), A(\overrightarrow{G}))$ to a graph $\overrightarrow{H} = (V(\overrightarrow{H}), A(\overrightarrow{H}))$ is a mapping $\phi : V(\overrightarrow{G}) \to V(\overrightarrow{H})$ such that $(\phi(u), \phi(v)) \in A(\overrightarrow{H})$ for all $(u, v) \in A(\overrightarrow{G})$. If there is a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$, we shall say that $\overrightarrow{H}$ *colors* $\overrightarrow{G}$ or that $\overrightarrow{G}$ is *colored by* $\overrightarrow{H}$. We shall call any oriented graph *a coloring graph*, especially when it is used for coloring.

Two oriented graphs $\overrightarrow{H}_1 = (V(\overrightarrow{H}_1), A(\overrightarrow{H}_1))$ and $\overrightarrow{H}_2 = (V(\overrightarrow{H}_2), A(\overrightarrow{H}_2))$ are *isomorphic* if there is a bijection $\pi\colon V(\overrightarrow{H}_1) \to V(\overrightarrow{H}_2)$ such that $(u,v) \in A(\overrightarrow{H}_1)$ if and only if $(\pi(u), \pi(v)) \in A(\overrightarrow{H}_2)$.

An automorphism of $\overrightarrow{H}$ is an isomorphism from $\overrightarrow{H}$ to $\overrightarrow{H}$. Let $Aut(\overrightarrow{H})$ denote the set of all automorphisms of $\overrightarrow{H}$. If $\alpha : \overrightarrow{H} \to \overrightarrow{H}$ is an automorphism of $\overrightarrow{H}$ and c is a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$, then $\alpha \circ c : \overrightarrow{G} \to \overrightarrow{H}$ is also a homomorphism.

We shall denote the *converse* of $\overrightarrow{G}$, i.e., the oriented graph obtained from $\overrightarrow{G}$ by reversing all arcs, i.e., $(u,v) \in A(\overrightarrow{G}^R) \Leftrightarrow (v,u) \in A(\overrightarrow{G})$, by $\overrightarrow{G}^R$.

An oriented graph $\overrightarrow{H}$ is *self-converse* if there is an isomorphism from $\overrightarrow{H}$ to its converse $\overrightarrow{H}^R$, and $\overrightarrow{H}$ is *arc-transitive* if for any two pairs of arcs $(u,v) \in A(\overrightarrow{H})$ and $(x,y) \in A(\overrightarrow{H})$, there is an automorphism $f : \overrightarrow{H} \to \overrightarrow{H}$ such that $f(u) = x$ and $f(v) = y$.

Lemma 2.1. *Let $\overrightarrow{H}$ be arc-transitive, $\overrightarrow{G}$ be an oriented graph, $(u,v) \in A(\overrightarrow{G})$ and $(x,y) \in A(\overrightarrow{H})$. Then the following two statements are equivalent:*

- *There is a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$.*
- *There is a homomorphism h from $\overrightarrow{G}$ to $\overrightarrow{H}$, that satisfies $h(u) = x$ and $h(v) = y$.*

□

Lemma 2.2. *Let h be a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}$. Then, h is a homomorphism from $\overrightarrow{G}^R$ to $\overrightarrow{H}^R$. Moreover, if $\overrightarrow{H}$ is self-converse, then there exists a homomorphism h' from $\overrightarrow{G}^R$ to $\overrightarrow{H}$.*

□

Corollary 2.3. *If an oriented graph $\overrightarrow{G}$ can be colored by $\overrightarrow{H}$, then its converse $\overrightarrow{G}^R$ can be colored by $\overrightarrow{H}^R$. Moreover, if $\overrightarrow{H}$ is self-converse, then $\overrightarrow{G}^R$ can be colored by $\overrightarrow{H}$.*

Lemma 2.4. *If an oriented graph $\overrightarrow{G}$ is colored by an oriented graph $\overrightarrow{H}$, then $\overrightarrow{G}$ can be colored by a tournament $\overrightarrow{T}$.*

Proof. The graph $\overrightarrow{G}$ can be colored by any tournament $\overrightarrow{T}$ that contains $\overrightarrow{H}$ as a subgraph. □

Definition 2.5. A 2-dimensional grid $G(m,n)$ is an undirected graph with vertices $V = \{(i,j) : 1 \leq i \leq m; 1 \leq j \leq n\}$ and edges of the form $\{(i,j),(i+1,j)\}$ for $1 \leq i < m; 1 \leq j \leq n$, or $\{(i,j),(i,j+1)\}$ for $1 \leq i \leq m; 1 \leq j < n$.

Definition 2.6. A cylindrical grid $Cyl(m,n)$, or simply a cylinder, is an undirected graph with vertices $V = \{(i,j) : 1 \leq i \leq m; 1 \leq j \leq n\}$ and edges of the form $\{(i,j),(i+1,j)\}$ for $1 \leq i < m; 1 \leq j \leq n$, or $\{(i,j),(i,j+1)\}$ for $1 \leq i \leq m; 1 \leq j < n$ or edges of the form $\{(m,j),(1,j)\}$ for $1 \leq j \leq n$.

Definition 2.7. A toroidal grid $T(m,n)$, or simply a toroid, is an undirected graph with vertices $V = \{(i,j) : 1 \le i \le m; 1 \le j \le n\}$ and edges of the form $\{(i,j),(i+1,j)\}$ for $1 \le i < m; 1 \le j \le n$, or $\{(i,j),(i,j+1)\}$ for $1 \le i \le m; 1 \le j < n$, or $\{(m,j),(1,j)\}$ for $1 \le j \le n$, or $\{(i,1),(i,n)\}$ for $1 \le i \le m$.

We denote the family of all orientations of all grids (cylinders, toroids, respectively) by $\mathcal{G}$ ($\mathcal{C}yl$, $\mathcal{T}$, respectively) and the family of all orientations of all grids with m rows by $\mathcal{G}_m$. By $\mathcal{C}yl_m$ ($\mathcal{T}_m$, respectively) we denote the set of all orientations of all cylinders (toroids, respectively) with circuit m.

Definition 2.8. The comb R_m is an undirected graph with the set of vertices $V(R_m) = \{(1,1),\dots,(m,1),\ (1,2),\dots,(m,2)\}$ and edges of the form $\{(i,1),(i,2)\}$ for $1 \le i \le m$ or $\{(i,2),(i+1,2)\}$ for $1 \le i < m$; see Figure 2.1. The vertices $(1,1),\dots,(m,1)$ form the first column of the comb R_m, while $(1,2),\dots,(m,2)$ form the second column.

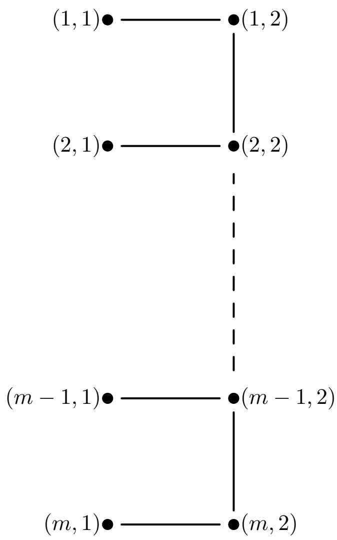

FIGURE 2.1: Comb R_m.

2.1 Reachable colorings

For an orientation $\overrightarrow{G}$ of $G(m,n)$ and $i \le n$, we denote the induced subgraph of $\overrightarrow{G}$ formed by the vertices of the first i columns by $\overrightarrow{G}(i)$. Note that, for $i \ge 1$, $\overrightarrow{G}(i+1)$ consists of $\overrightarrow{G}(i)$ and an oriented comb, which we denote by $\overrightarrow{R}(i+1)$, where vertices $(1,i),\dots,(m,i)$ form the first column of $\overrightarrow{R}(i+1)$ and vertices $(1,i+1),\dots,(m,i+1)$ form the second column of $\overrightarrow{R}(i+1)$; see Figure 2.2.

Definition 2.9. For an orientation $\overrightarrow{R}$ of the comb R_m, a coloring graph $\overrightarrow{H}$ and a sequence $s = (s_1,\dots,s_m) \in V(\overrightarrow{H})^m$, let

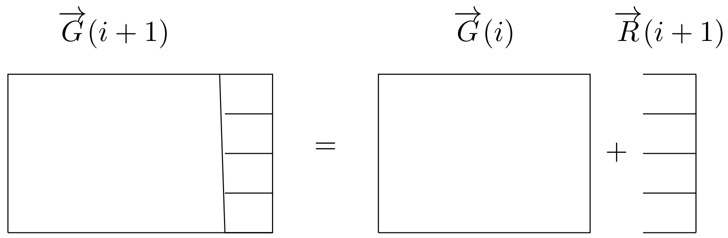

FIGURE 2.2: Formation of $\overrightarrow{G}(i+1)$.

$NEXT(s,\overrightarrow{R},\overrightarrow{H}) = \{(d_1,\ldots,d_m)$: there exists a coloring $c : \overrightarrow{R} \to \overrightarrow{H}$, such that $c(i,1) = s_i$ and $c(i,2) = d_i$, for $1 \leq i \leq m\}$.

The set $NEXT(s,\overrightarrow{R},\overrightarrow{H})$ is the set of reachable colorings of the second column of the orientation $\overrightarrow{R}$ of the comb R_m when the vertices of the first column of $\overrightarrow{R}$ are colored by s.

Definition 2.10. For an orientation $\overrightarrow{R}$ of the comb R_m, a coloring graph $\overrightarrow{H}$ and a set S of sequences $S \subseteq V(\overrightarrow{H})^m$, $NEXT(S,\overrightarrow{R},\overrightarrow{H}) = \bigcup_{s\in S} NEXT(s,\overrightarrow{R},\overrightarrow{H})$.

The set $NEXT(S,\overrightarrow{R},\overrightarrow{H})$ can be computed by a simple procedure.

Procedure $\mathsf{Next_m}(\mathsf{S},\overrightarrow{R},\overrightarrow{H})$
Input: a set $S \subset V(\overrightarrow{H})^m$, an orientation $\overrightarrow{R}$ of the comb R_m and a coloring graph $\overrightarrow{H}$.
Output: the set $D = NEXT(S,\overrightarrow{R},\overrightarrow{H})$.

D := $\emptyset$
for every $(s_1,\ldots,s_m) \in S$
    color the first column of $\overrightarrow{R}$ with $(s_1,\ldots,s_m)$
    for every coloring $(d_1,\ldots,d_m)$ of the second column of $\overrightarrow{R}$
        that is consistent with $\overrightarrow{H}$
        D := $D \cup \{(d_1,\ldots,d_m)\}$
return D

Definition 2.11. For an orientation $\overrightarrow{G}$ of the grid $G(m,n)$ and a coloring graph $\overrightarrow{H}$, we shall denote by $S(\overrightarrow{G},\overrightarrow{H})$ the set of reachable colorings of the last column of $\overrightarrow{G}$ by $\overrightarrow{H}$, i.e., $S(\overrightarrow{G},\overrightarrow{H}) = \{(c_1,\ldots,c_m)$: there exists a coloring $c : \overrightarrow{G} \to \overrightarrow{H}$, such that $c(1,n) = c_1,\ldots,c(m,n) = c_m\}$.
We will use the notation $S(\overrightarrow{G})$ if the coloring graph is obvious from the context.

The set $S(\overrightarrow{G},\overrightarrow{H})$ can be computed by the following procedure.

Procedure Reachablecolorings($\overrightarrow{G}, \overrightarrow{H}$)
Input: m, n, an orientation $\overrightarrow{G}$ of the grid $G(m, n)$ and a coloring graph $\overrightarrow{H}$.
Output: the set $S(\overrightarrow{G}, \overrightarrow{H})$.

```
1.  SG := S(G(1), H)
2.  for j := 2 to n
3.      SG := NEXT(SG, R(j), H)
4.  return SG
```

We denote by $\mathcal{S}(\overrightarrow{H})$ the family $\mathcal{S}(\overrightarrow{H}) = \{S(\overrightarrow{G}, \overrightarrow{H}) : \overrightarrow{G} \in \mathcal{G}\}$ and by $\mathcal{S}_m(\overrightarrow{H})$ the family $\mathcal{S}_m(\overrightarrow{H}) = \{S(\overrightarrow{G}, \overrightarrow{H}) : \overrightarrow{G} \in \mathcal{G}_m\}$.

Lemma 2.12. *All orientations in $\mathcal{G}$ are colorable by a coloring graph $\overrightarrow{H}$ if and only if $\emptyset \notin \mathcal{S}(\overrightarrow{H})$. All orientations in $\mathcal{G}_m$ are colorable by $\overrightarrow{H}$ if and only if $\emptyset \notin \mathcal{S}_m(\overrightarrow{H})$.*

Proof. We prove only the first part of the lemma.
If $\emptyset \notin \mathcal{S}(\overrightarrow{H})$, then for each $\overrightarrow{G} \in \mathcal{G}$ we have $\emptyset \neq S(\overrightarrow{G}, \overrightarrow{H})$ and $\overrightarrow{G}$ is colorable by $\overrightarrow{H}$.
If every $\overrightarrow{G} \in \mathcal{G}$ is colorable by $\overrightarrow{H}$, then $\emptyset \neq S(\overrightarrow{G}, \overrightarrow{H})$. Hence, $\emptyset \notin \mathcal{S}(\overrightarrow{H})$. □

Lemma 2.13. *If $t \in S(\overrightarrow{G}, \overrightarrow{H})$ and $h : \overrightarrow{H} \to \overrightarrow{H}$ is an automorphism of $\overrightarrow{H}$, then $h(t) \in S(\overrightarrow{G}, \overrightarrow{H})$.*

□

We define the equivalence relation on $V(\overrightarrow{G})^m$, by $t \sim s$ iff there exists an automorphism $h : \overrightarrow{H} \to \overrightarrow{H}$ such that $s = h(t)$.

In several parts of this dissertation, we shall use algorithms that generate the families $\mathcal{S}_m(\overrightarrow{H})$ for some m and $\overrightarrow{H}$. Here, we describe a simple version of such an algorithm:

Algorithm SimpleGenerating(m, $\overrightarrow{H}$)
INPUT: (1) m - the number of rows, (2) $\overrightarrow{H}$ - the coloring graph.
OUTPUT: the set $\mathcal{S}_m(\overrightarrow{H})$ is printed.

First, for every orientation $\overrightarrow{P}$ of the path P_m, the algorithm computes the set $S_m(\overrightarrow{P}, \overrightarrow{H})$ and puts it in a queue Q. Next, it repeats the following steps:

(1) It takes a set S from the queue Q.

(2) For every orientation $\overrightarrow{R}$ of the comb R_m, it computes the set $S' = NEXT(S, \overrightarrow{R}, \overrightarrow{H})$, and puts S' in queue Q if it is new.

Lemma 2.14. *The algorithm Algorithm SimpleGenerating(m, $\overrightarrow{H}$) generates the set $\mathcal{S}_m(\overrightarrow{H})$. If $T \in \mathcal{S}_m(\overrightarrow{H})$, i.e., $T = S(\overrightarrow{G}, \overrightarrow{H})$ for some $\overrightarrow{G} \in \mathcal{G}_m$, then T is generated by Algorithm SimpleGenerating(m, $\overrightarrow{H}$). On the other hand, if Algorithm SimpleGenerating(m, $\overrightarrow{H}$) generates T, then $T \in \mathcal{S}_m(\overrightarrow{H})$.*

□

Algorithm SimpleGenerating is not practical in the general case. Our experiments show that it is difficult to calculate the family $\mathcal{S}_m(\overrightarrow{H})$ even for small values of m. However, if $\overrightarrow{H}$ has a large group of automorphisms, then there are far fewer reachable colorings to check.

2.2 Paley tournament $\overrightarrow{T}_p$

Let p be a prime number such that $p \equiv 3 \bmod 4$, and let $\mathbb{Z}_p = \{0, \ldots, p-1\}$ be the ring of integers modulo p. We denote by $QR = \{r : r \neq 0, r = s^2, \text{ for some } s \in \mathbb{Z}_p\}$ the set of nonzero quadratic residues of $\mathbb{Z}_p$ and by $NR = \{-x : x \in QR\}$ the set of quadratic nonresidues. The set QR is a subgroup of $\mathbb{Z}_p^\star = \{1, \ldots, p-1\}$ of order $\frac{p-1}{2}$, $QR \cap NR = \emptyset$, and $QR \cup NR = \mathbb{Z}_p^\star$.

Lemma 2.15 ([43])**.** *The product of two quadratic residues or of two quadratic nonresidues is a quadratic residue, whereas the product of a quadratic residue and a quadratic nonresidue is a quadratic nonresidue.*

Definition 2.16. The directed graph $\overrightarrow{T}_p$ with set of vertices $V(\overrightarrow{T}_p) = \mathbb{Z}_p$ and set of arcs $A(\overrightarrow{T}_p) = \{(x, y) : x, y \in \mathbb{Z}_p \text{ and } y - x \in QR\}$ is called the *Paley tournament* of order p.

Observe that $\overrightarrow{T}_p$ is a tournament; i.e., for every pair u, v of distinct vertices in $V(\overrightarrow{T}_p)$, either $(u, v) \in A(\overrightarrow{T}_p)$ and $(v, u) \notin A(\overrightarrow{T}_p)$, or $(u, v) \notin A(\overrightarrow{T}_p)$ and $(v, u) \in A(\overrightarrow{T}_p)$.

Lemma 2.17. *If $a \in QR$ and $b \in \mathbb{Z}_p$, then the mapping $f : \overrightarrow{T}_p \to \overrightarrow{T}_p$ defined by $f(x) = a \cdot x + b$ is an automorphism.*

Proof. By Lemma 2.15, we have $f(v) - f(u) = a \cdot (v - u) \in QR$ if and only if $v - u \in QR$. □

Lemma 2.18 (Fried [19])**.** *The Paley tournament $\overrightarrow{T}_p$ is arc-transitive; i.e., for any two pairs of arcs (u, v), $(x, y) \in A(\overrightarrow{T}_p)$, there exists an automorphism h such that $h(u) = x$ and $h(v) = y$.*

Proof. Let h be the mapping $h : \overrightarrow{T}_p \to \overrightarrow{T}_p$ defined by

$$h(z) = (y - x) \cdot (v - u)^{-1} \cdot z + x - u \cdot (y - x) \cdot (v - u)^{-1}.$$

By Lemma 2.17, h is an automorphism, $h(u) = x$, and $h(v) = y$. □

Lemma 2.19. *The Paley tournament $\overrightarrow{T}_p$ is self-converse; i.e., $\overrightarrow{T}_p$ and its converse $\overrightarrow{T}_p^R$ are isomorphic.*

Proof. Consider the function $f : \overrightarrow{T}_p^R \to \overrightarrow{T}_p$ defined by $f(x) = -x$. An arc $(x, y) \in A(\overrightarrow{T}_p^R)$ if and only if $(-x, -y) \in A(\overrightarrow{T}_p)$.

□

Suppose that i and j are positive integers. Consider the star $K_{1,i}$ with the set of vertices $V(K_{1,i}) = \{x, v_1, v_2, \ldots, v_i\}$ and edges of the form $\{x, v_k\}$ for $1 \leq k \leq i$. Let $\overrightarrow{K}$ be an orientation of the star $K_{1,i}$.

Definition 2.20. We say that the coloring graph $\overrightarrow{H}$ has the property $P(i, j)$ if: $|V(\overrightarrow{H})| \geq i$ and for every orientation of the star $K_{1,i}$ and every sequence of different colors given for $v_1, \ldots, v_i$, we can choose j different ways to color x, the universal vertex of the star.

Lemma 2.21 ([14])**.** *If p is a prime with $p \equiv 3 \pmod 4$, then the Paley tournament $\overrightarrow{T}_p$ has properties $P(1, \frac{p-1}{2})$ and $P(2, \frac{p-3}{4})$.*

If $p = 7$, we have the Paley tournament $\overrightarrow{T}_7 = (V(\overrightarrow{T}_7), A(\overrightarrow{T}_7))$, where $V(\overrightarrow{T}_7) = \mathbb{Z}_7$ and $A(\overrightarrow{T}_7) = \{(x, x+b \bmod 7) : x \in V(\overrightarrow{T}_7),\ \ b = 1, 2,\ \text{or}\ \ 4\}$; see Figure 2.3. If $p = 11$, then $\overrightarrow{T}_{11} = (V(\overrightarrow{T}_{11}), A(\overrightarrow{T}_{11}))$, where $V(\overrightarrow{T}_{11}) = \mathbb{Z}_{11}$ and $A(\overrightarrow{T}_{11}) = \{(x, x + b \bmod 11) : x \in V(\overrightarrow{T}_{11}),\ \ b = 1, 3, 4, 5 \text{ or }\ 9\}$ is the Paley tournament of order eleven; see Figure 2.4.

Lemma 2.22 (Borodin et al. [6])**.** *For any two distinct vertices $u, v \in \overrightarrow{T}_{11}$ and any orientation $\overrightarrow{P}$ of P_3 (see Figure 1.1), there exist at least two distinct paths of length 2 with orientation $\overrightarrow{P}$ that join the vertices u and v.*

Lemma 2.23. *No graph on ten vertices satisfies Lemma 2.22.*

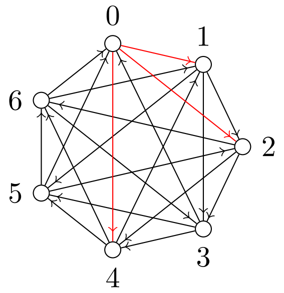

FIGURE 2.3: Paley tournament of order 7.

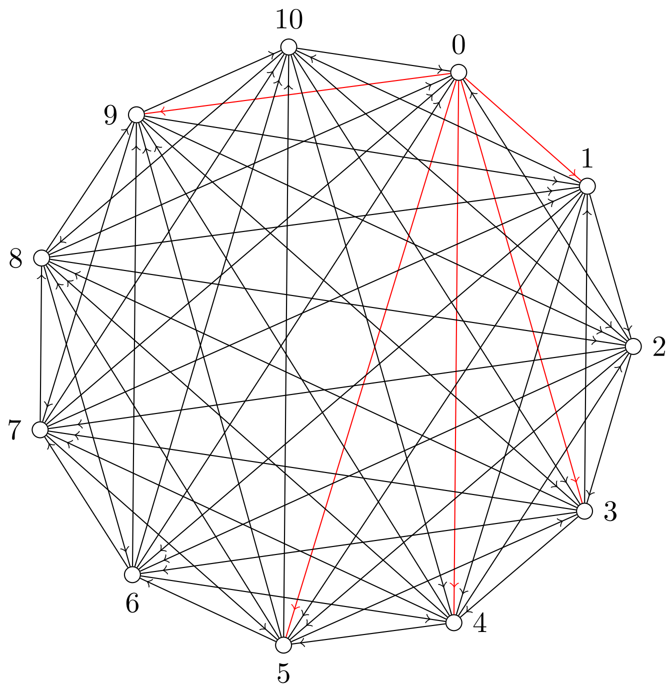

FIGURE 2.4: Paley tournament of order 11.

Proof. Any vertex $u \in \overrightarrow{H}$ has at least four outgoing arcs and at least four ingoing arcs. Since

$$\sum_{v \in V(\overrightarrow{H})} indegree(v) = \sum_{v \in V(\overrightarrow{H})} outdegree(v),$$

there is a vertex u with indegree 4. Let v be a vertex with $(v, u) \in A(\overrightarrow{H})$. There do not exist four distinct vertices x_1, x_2, x_3, x_4 such that

- $u \leftarrow x_1 \leftarrow v$,
- $u \leftarrow x_2 \leftarrow v$,
- $u \leftarrow x_3 \rightarrow v$,
- $u \leftarrow x_4 \rightarrow v$.

□

Lemma 2.24 (Fertin, Raspaud and Roychowdhury [18]).
Any orientation of any grid can be colored by $\overrightarrow{T}_{11}$.

2.3 Tromp graph $\overrightarrow{Tr}(\overrightarrow{G})$

Definition 2.25. Let $\overrightarrow{G}$ be an oriented graph and $\overrightarrow{G}'$ be an isomorphic copy of $\overrightarrow{G}$. For every $u \in V(\overrightarrow{G})$, by $u' \in V(\overrightarrow{G}')$ we denote the isomorphic copy of u. The Tromp graph $\overrightarrow{Tr}(\overrightarrow{G})$ has $2|V(\overrightarrow{G})| + 2$ vertices and is defined as follows, see [38]:

- $V(\overrightarrow{Tr}(\overrightarrow{G})) = V(\overrightarrow{G}) \cup V(\overrightarrow{G}') \cup \{\infty, \infty'\}$,
- $\forall_{u \in V(\overrightarrow{G})} (u, \infty), (\infty, u'), (u', \infty'), (\infty', u) \in A(\overrightarrow{Tr}(\overrightarrow{G}))$,
- $\forall_{u,v \in V(\overrightarrow{G}),\ (u,v) \in A(\overrightarrow{G})} (u, v), (u', v'), (v, u'), (v', u) \in A(\overrightarrow{Tr}(\overrightarrow{G}))$,

see Figure 2.5. The vertices $u \in V(\overrightarrow{G})$ and $u' \in V(\overrightarrow{G}')$ are called *twin vertices*.

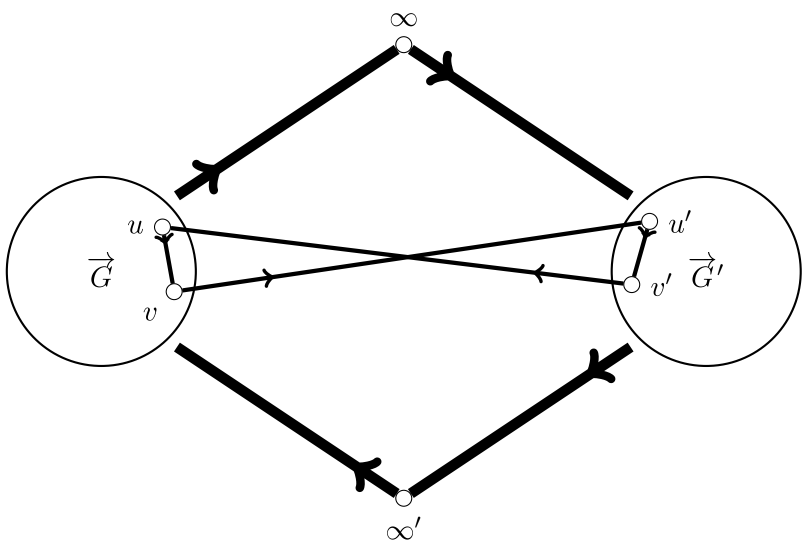

FIGURE 2.5: Tromp graph $\overrightarrow{Tr}(\overrightarrow{G})$.

Suppose that i and j are positive integers. Consider the star $K_{1,i}$ with the set of vertices $V(K_{1,i}) = \{x, v_1, v_2, \ldots, v_i\}$ and edges of the form $\{x, v_k\}$ for $1 \leq k \leq i$; and a Tromp graph $\overrightarrow{Tr}(\overrightarrow{G})$. Let $\overrightarrow{K}$ be an orientation of the star $K_{1,i}$ and $c : \overrightarrow{K} \to \overrightarrow{Tr}(\overrightarrow{G})$ be a homomorphism. We say that the sequence of colors $(c(v_1), c(v_2), \ldots, c(v_i))$ *is compatible* with the orientation $\overrightarrow{K}$ if for every pair of vertices v_k, v_l, $k \neq l$ colors $c(v_k)$ and $c(v_l)$ are compatible, which means:

(1) $c(v_k) \neq c(v_l)$ if (v_k, x) and $(x, v_l) \in \overrightarrow{K}$ or if (v_l, x) and $(x, v_k) \in \overrightarrow{K}$,

(2) $c(v_k) \neq c(v_l)'$ if (v_k, x) and $(v_l, x) \in \overrightarrow{K}$ or if (x, v_l) and $(x, v_k) \in \overrightarrow{K}$.

Definition 2.26. We say that the Tromp graph $\overrightarrow{Tr}(\overrightarrow{G})$ has the property $P_c(i, j)$ if: $|V(\overrightarrow{T})| \geq i$ and for every orientation $\overrightarrow{K}$ of the star $K_{1,i}$ and every sequence of colors $(c(v_1), c(v_2), \ldots, c(v_i))$ compatible with $\overrightarrow{K}$, we can choose j different ways to color x, the universal vertex of the star.

Lemma 2.27 (Ochem, Pinlou [37])**.** *If p is a prime with $p \equiv 3 \pmod 4$, and the Paley tournament $\overrightarrow{T}_p$ has property $P(i-1, j)$ for some $i, j \geq 2$, then the Tromp graph $\overrightarrow{Tr}(\overrightarrow{T}_p)$ has property $P_c(i, j)$.*

Chapter 3

A lower bound for oriented chromatic number of grids

In this chapter, we disprove Conjecture 1.9, which was put forward by Fertin, Raspaud, and Roychowdhury [18] in 2003, by showing the following:

Theorem 3.1 (Dybizbański, Nenca (2012) [11])**.**
There exists an orientation of a grid that cannot be colored by any coloring graph with seven vertices.

Corollary 3.2. $\overrightarrow{\chi}(\mathcal{G}) \geq 8$.

We shall say that an oriented grid $\overrightarrow{G}$ is *a trap* for $\overrightarrow{H}$ if $\overrightarrow{G}$ is not colorable by $\overrightarrow{H}$. After many trials, we have found an orientation $\overrightarrow{A}$, which is shown in Figure 3.1, with the following property.

Lemma 3.3. *The orientation $\overrightarrow{A}$ of the grid $G(5,33)$, which is shown in Figure 3.1, can be colored by only nine (non-isomorphic) tournaments on seven vertices, namely, by $\overrightarrow{H}_0$, $\overrightarrow{H}_1$, $\overrightarrow{H}_2$, $\overrightarrow{H}_3$, $\overrightarrow{H}_4$, $\overrightarrow{H}_5$, $\overrightarrow{H}_6$, $\overrightarrow{H}_7$, and $\overrightarrow{H}_8$, which are defined by the adjacency matrices shown in Figures 3.2, 3.3, 3.4, and 3.5.*

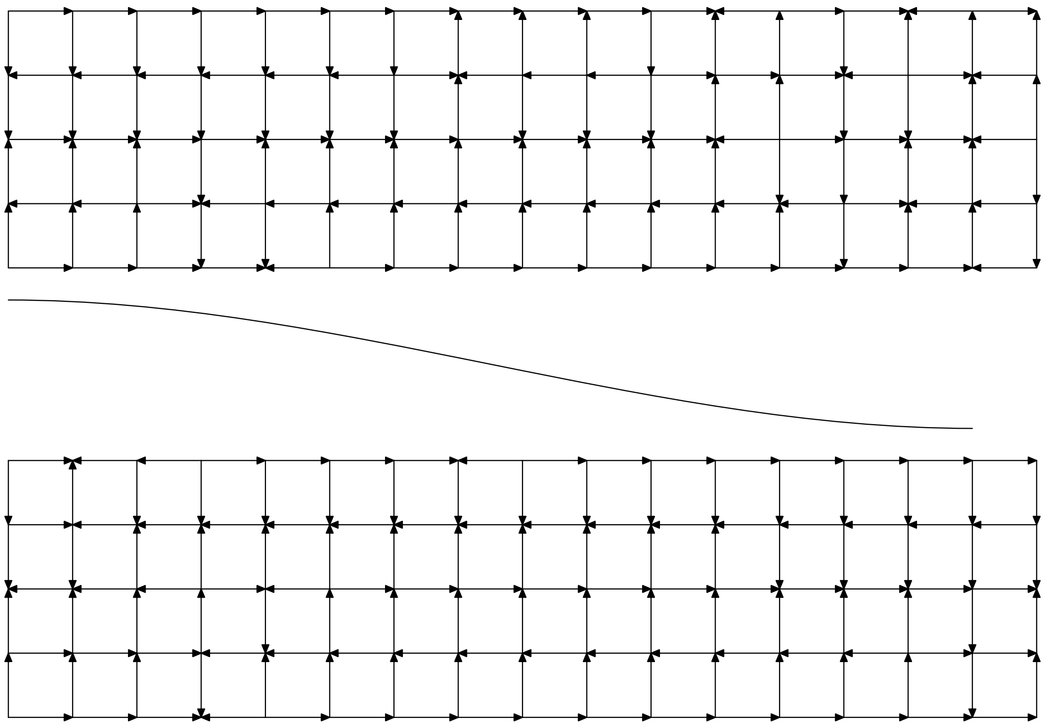

FIGURE 3.1: Orientation $\overrightarrow{A}$ of $G(5, 33)$.

Proof. There are 456 non-isomorphic tournaments on seven vertices. We use `nauty` [30] to generate all of them. We have found that the set $S(\overrightarrow{A}, \overrightarrow{H})$ of reachable colorings of the last column of $\overrightarrow{A}$ (see Definition 2.11) is empty for each of these 456 tournaments except the following: $\overrightarrow{H}_0$, $\overrightarrow{H}_1$, $\overrightarrow{H}_2$, $\overrightarrow{H}_3$, $\overrightarrow{H}_4$, $\overrightarrow{H}_5$, $\overrightarrow{H}_6$, $\overrightarrow{H}_7$, and $\overrightarrow{H}_8$, for which the adjacency matrices are presented in Figures 3.2, 3.3, 3.4, and 3.5.

$$\begin{bmatrix} 0&0&0&0&0&1&1\\ 1&0&0&0&1&0&1\\ 1&1&0&0&0&1&0\\ 1&1&1&0&0&0&0\\ 1&0&1&1&0&0&0\\ 0&1&0&1&1&0&0\\ 0&0&1&1&1&1&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1&1&0&1\\ 1&0&0&0&0&1&0\\ 1&1&0&0&0&0&1\\ 0&1&1&0&0&1&0\\ 0&1&1&1&0&0&0\\ 1&0&1&0&1&0&0\\ 0&1&0&1&1&1&0 \end{bmatrix}$$

FIGURE 3.2: Adjacency matrices of coloring graphs $\overrightarrow{H}_0$ and $\overrightarrow{H}_1$

$$\begin{bmatrix} 0&0&1&0&1&0&0\\ 1&0&0&0&0&1&1\\ 0&1&0&0&0&1&0\\ 1&1&1&0&0&0&0\\ 0&1&1&1&0&0&1\\ 1&0&0&1&1&0&0\\ 1&0&1&1&0&1&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1&1&0&1\\ 1&0&0&0&0&1&1\\ 1&1&0&0&0&0&1\\ 0&1&1&0&0&1&0\\ 0&1&1&1&0&0&0\\ 1&0&1&0&1&0&0\\ 0&0&0&1&1&1&0 \end{bmatrix} \begin{bmatrix} 0&0&1&0&1&0&1\\ 1&0&0&0&0&1&0\\ 0&1&0&0&0&1&1\\ 1&1&1&0&0&0&0\\ 0&1&1&1&0&0&0\\ 1&0&0&1&1&0&0\\ 0&1&0&1&1&1&0 \end{bmatrix}$$

FIGURE 3.3: Adjacency matrices of coloring graphs $\overrightarrow{H}_2, \overrightarrow{H}_3$, and $\overrightarrow{H}_4$

$$\begin{bmatrix} 0&0&0&1&1&0&1\\ 1&0&0&0&0&1&1\\ 1&1&0&0&0&0&0\\ 0&1&1&0&0&1&0\\ 0&1&1&1&0&0&0\\ 1&0&1&0&1&0&0\\ 0&0&1&1&1&1&0 \end{bmatrix} \begin{bmatrix} 0&0&0&0&0&1&1\\ 1&0&0&0&1&0&0\\ 1&1&0&0&0&1&0\\ 1&1&1&0&0&0&1\\ 1&0&1&1&0&0&0\\ 0&1&0&1&1&0&0\\ 0&1&1&0&1&1&0 \end{bmatrix}$$

FIGURE 3.4: Adjacency matrices of coloring graphs $\overrightarrow{H}_5$ and $\overrightarrow{H}_6$.

$$\begin{bmatrix} 0&0&0&1&0&1&1\\ 1&0&0&0&1&0&1\\ 1&1&0&0&0&1&0\\ 0&1&1&0&0&0&1\\ 1&0&1&1&0&0&0\\ 0&1&0&1&1&0&0\\ 0&0&1&0&1&1&0 \end{bmatrix} \begin{bmatrix} 0&0&0&1&0&1&1\\ 1&0&0&0&1&0&1\\ 1&1&0&0&0&1&0\\ 0&1&1&0&0&0&0\\ 1&0&1&1&0&0&0\\ 0&1&0&1&1&0&0\\ 0&0&1&1&1&1&0 \end{bmatrix}$$

FIGURE 3.5: Adjacency matrices of coloring graphs $\overrightarrow{H}_7$ and $\overrightarrow{H}_8$

□

3.1 Traps

Let m be a specified number of rows. To find traps for tournaments $\overrightarrow{H}_0$, $\overrightarrow{H}_1$, $\overrightarrow{H}_2$, $\overrightarrow{H}_3$, $\overrightarrow{H}_4$, $\overrightarrow{H}_5$, $\overrightarrow{H}_6$ and $\overrightarrow{H}_8$, we use a greedy algorithm $\mathsf{F}indTrap_m$; see below. For a given tournament $\overrightarrow{H}$, if the algorithm stops, then it returns a trap, i.e., an orientation of a grid with m rows that cannot be colored by $\overrightarrow{H}$.

The algorithm starts with the grid $G(m,1)$, which is shown in Figure 3.6. There are 2^{m-1} orientations of $G(m,1)$. For every orientation $\overrightarrow{G}_1$ of $G(m,1)$, the algorithm computes the set $S(\overrightarrow{G}_1, \overrightarrow{H})$ of reachable colorings of $\overrightarrow{G}_1$. It chooses an orientation $\overrightarrow{G}$ that gives a smallest set $T = S(\overrightarrow{G}_1, \overrightarrow{H})$. If there is more than one orientation with the same cardinality of the set of reachable colorings, the algorithm chooses the one that occurs first.

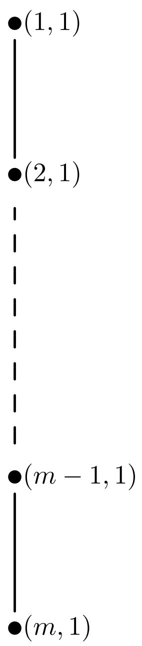

FIGURE 3.6: Grid G(m,1)

Next, for every orientation $\overrightarrow{R}$ of the comb R_m, the algorithm computes the set $NEXT(T, \overrightarrow{R}, \overrightarrow{H})$; see Definition 2.10. Once again, it chooses an orientation of R_m that gives a smallest set of reachable colorings of the second column of the comb. The grid $\overrightarrow{G}$ is extended by the chosen orientation of R_m. In the same way, the algorithm adds the next oriented combs one by one, each time choosing an orientation that gives a smallest set of reachable colorings on the second column of the comb. The algorithm stops when the empty set of colorings is reached. Now, we shall present a more formal description of the algorithm.

FindTrap$_{\mathbf{m}}$ $(\overrightarrow{H})$.
INPUT: a coloring graph $\overrightarrow{H}$.
OUTPUT: if the algorithm stops, it returns an oriented grid $\overrightarrow{G}$ with m rows that is not colorable by $\overrightarrow{H}$.

```
1.  min := $\infty$
2.  for every orientation $\overrightarrow{P}$ of $G(m,1)$
3.       $SG$ := $S(\overrightarrow{P},\overrightarrow{H})$
4.       if min > $\#SG$ then
5.            min := $\#SG$
6.            $T$ := $SG$
7.            $\overrightarrow{G}$ := $\overrightarrow{P}$
8.  while $T \neq \emptyset$ do
9.        min := $\infty$
10.       for every orientation $\overrightarrow{R}$ of the comb $R_m$
11.            $SG$ := $NEXT(T,\overrightarrow{R},\overrightarrow{H})$
12.            if min > $\#SG$ then
13.                 min := $\#SG$
14.                 $T'$ := $SG$
15.                 $extComb$ := $\overrightarrow{R}$
16.       $\overrightarrow{G}$ := $\overrightarrow{G}$ + $extComb$
17.       $T$ := $T'$
18.  return $\overrightarrow{G}$
```

First, in lines 1–7, the algorithm finds $\overrightarrow{G}$ — the orientation of the first column that gives a smallest set of reachable colorings. The set of reachable colorings is stored in T. The `while` loop (lines 8–17) builds the next columns of the grid one by one. At the beginning of the loop, in line 8, $\overrightarrow{G}$ stores the oriented grid that has been built so far, and T stores the set of colorings that are reachable on the last column of $\overrightarrow{G}$. If the set T is not empty, then the algorithm searches for an orientation of the comb, which will be added as the next column. The `for` loop, in lines 10–15, for every orientation $\overrightarrow{R}$ of the comb, computes the set $SG = NEXT(T,\overrightarrow{R},\overrightarrow{H})$ of reachable colorings on the second column of the comb $\overrightarrow{R}$. It stores in *extComb* the orientation that gives a smallest set and in T' the set of colorings that are reachable on *extComb*. After checking all orientations, in lines 12–15, the grid $\overrightarrow{G}$ is extended by the orientation *extComb*, and the set T stores the set of reachable colorings on the new last column.

Lemma 3.4. *For every coloring graph $\overrightarrow{H} \in \{\overrightarrow{H}_2,\overrightarrow{H}_3,\overrightarrow{H}_4,\overrightarrow{H}_5,\overrightarrow{H}_6\}$, there exists an orientation of a grid $\overrightarrow{G}$ with five rows such that $\overrightarrow{G}$ cannot be colored by $\overrightarrow{H}$.*

Proof. Using $\mathsf{F}indTrap_m$, we have found five orientations of grids: $\overrightarrow{B}$, $\overrightarrow{C}$, $\overrightarrow{D}$, $\overrightarrow{E}$ and $\overrightarrow{F}$ (see Figures 3.7, 3.8, 3.9, 3.10, 3.11) such that

- $S(\overrightarrow{B}, \overrightarrow{H}_2) = \emptyset$,
- $S(\overrightarrow{C}, \overrightarrow{H}_3) = \emptyset$,
- $S(\overrightarrow{D}, \overrightarrow{H}_4) = \emptyset$,
- $S(\overrightarrow{E}, \overrightarrow{H}_5) = \emptyset$,
- $S(\overrightarrow{F}, \overrightarrow{H}_6) = \emptyset$.

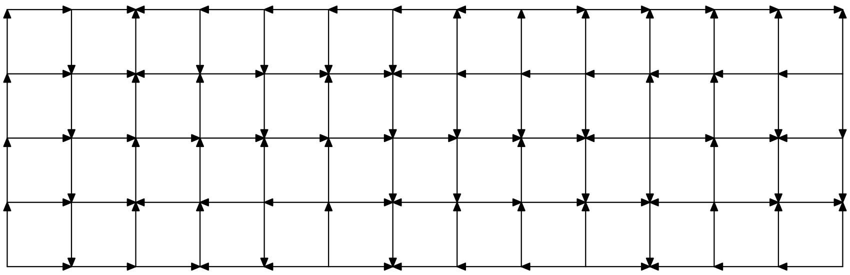

FIGURE 3.7: Orientation $\overrightarrow{B}$ of $G(5, 14)$ that is not colorable by $\overrightarrow{H}_2$.

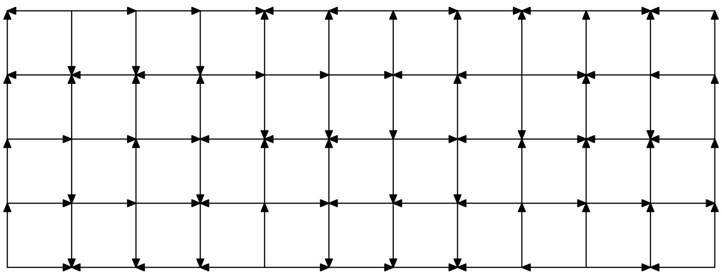

FIGURE 3.8: Orientation $\overrightarrow{C}$ of $G(5, 12)$ that is not colorable by $\overrightarrow{H}_3$.

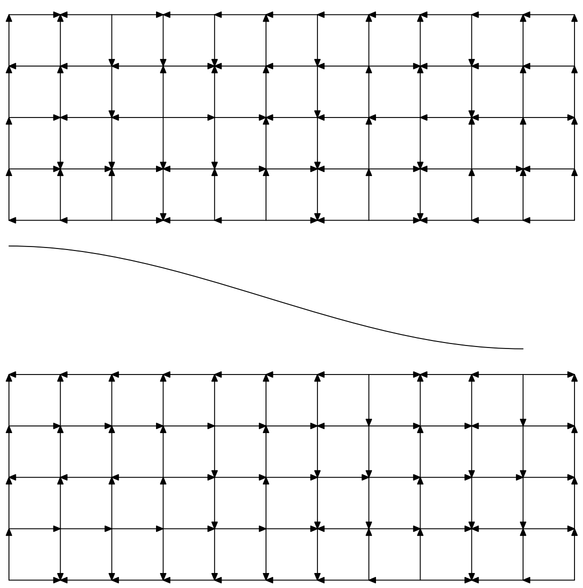

FIGURE 3.9: Orientation $\overrightarrow{D}$ of $G(5,23)$ that is not colorable by $\overrightarrow{H}_4$.

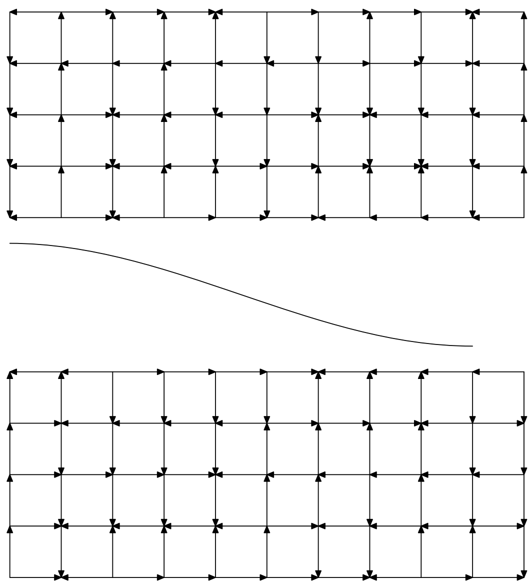

FIGURE 3.10: Orientation $\overrightarrow{E}$ of $G(5,21)$ that is not colorable by $\overrightarrow{H}_5$.

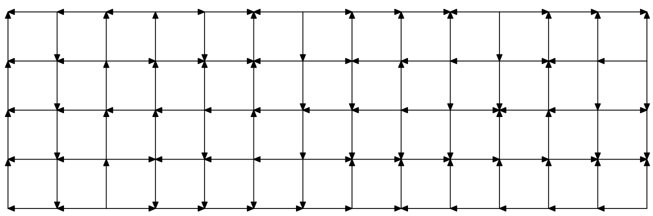

FIGURE 3.11: Orientation $\overrightarrow{F}$ of $G(5,14)$ that is not colorable by $\overrightarrow{H}_6$.

$\square$

Hence, we have found traps (non-colorable grids) with five rows for all coloring graphs with seven vertices except four: $\overrightarrow{H}_0$, $\overrightarrow{H}_1$, $\overrightarrow{H}_7$ and $\overrightarrow{H}_8$. The graph $\overrightarrow{H}_7$ is isomorphic to the Paley tournament $\overrightarrow{T}_7$ (see Section 2.2) and the graph $\overrightarrow{H}_8$, which we call $\overleftarrow{T}_7$, is obtained from $\overrightarrow{T}_7$ by reversing one arc. When we run F$indTrap_5$ on these graphs for grids with 5 rows, it loops and does not stop. This does not mean that every grid with five rows can be colored by them because the algorithm omits many orientations of grids.

Lemma 3.5. *There exists an orientation $\overrightarrow{I}$ with 5 rows (see Figure 3.12) such that $\overrightarrow{I}$ cannot be colored by $\overrightarrow{T}_7$ and there exist orientations $\overrightarrow{J}$, $\overrightarrow{K}$ and $\overrightarrow{L}$ with 7 rows (see Figures 3.13, 3.14, 3.15) such that*

- *$\overrightarrow{J}$ cannot be colored by $\overrightarrow{H}_0$.*
- *$\overrightarrow{K}$ cannot be colored by $\overrightarrow{H}_1$.*
- *$\overrightarrow{L}$ cannot be colored by $\overleftarrow{T}_7$.*

Proof. For $\overrightarrow{T}_7$, Szepietowski and Targan [53] found a trap with 35 columns. We have used a similar method and found an orientation $\overrightarrow{I}$ of $G(5, 28)$ that cannot be colored by $\overrightarrow{T}_7$; see Figure 3.12.

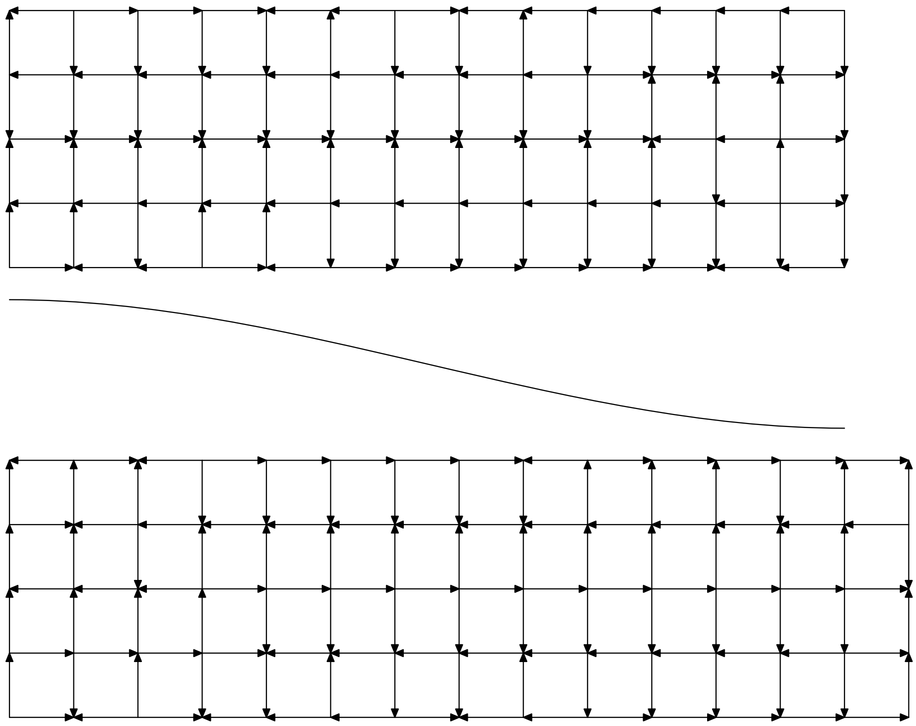

FIGURE 3.12: Orientation $\overrightarrow{I}$ of $G(5, 28)$ that is not colorable by $\overrightarrow{T}_7$.

We have found the orientation $\overrightarrow{J}$ of $G(7, 10)$, the orientation $\overrightarrow{K}$ of $G(7, 13)$ and the orientation $\overrightarrow{L}$ of $G(7, 22)$ by running F$indTrap_7$ for grids with 7 rows.

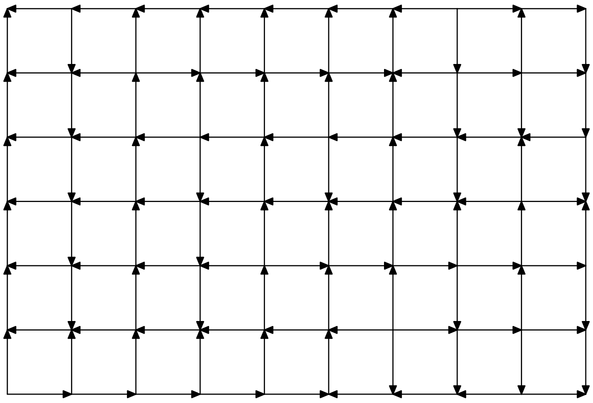

FIGURE 3.13: Orientation $\overrightarrow{J}$ of $G(7,10)$ that is not colorable by $\overrightarrow{H}_0$.

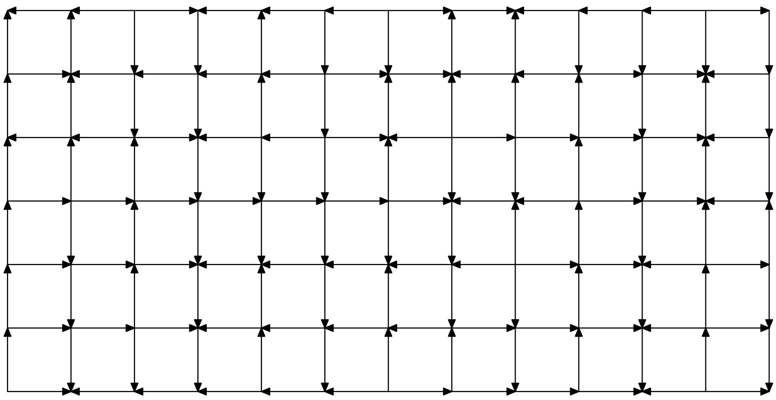

FIGURE 3.14: Orientation $\overrightarrow{K}$ of $G(7,13)$ that is not colorable by $\overrightarrow{H}_1$.

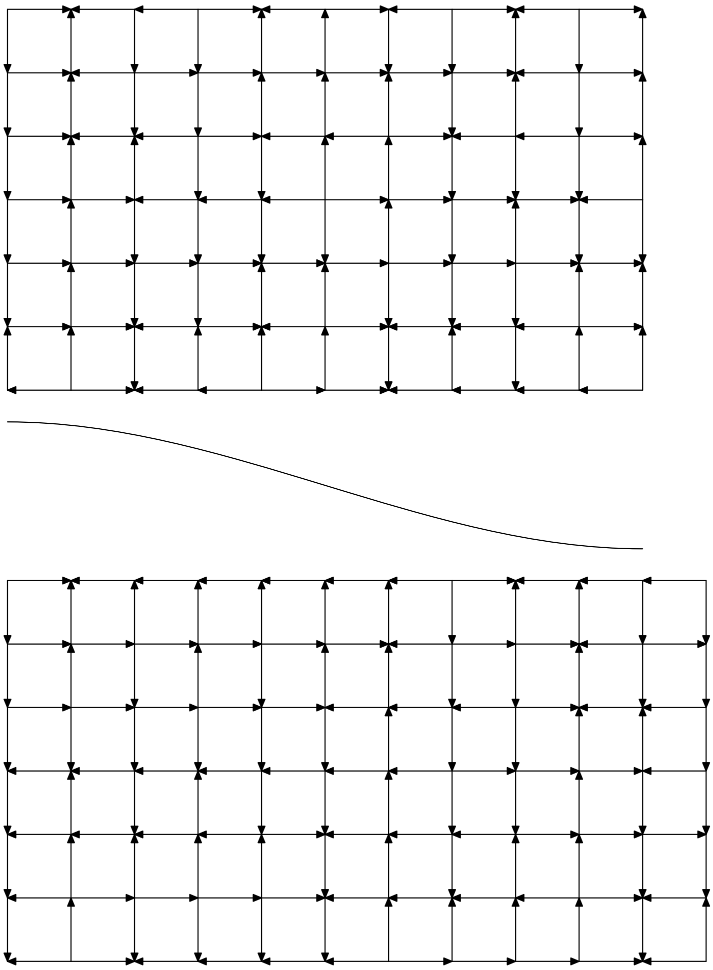

FIGURE 3.15: Orientation $\overrightarrow{L}$ of $G(7,22)$ that is not colorable by $\overleftarrow{T}_7$.

□

3.2 Proof of Theorem 3.1

If an orientation $\overrightarrow{G}$ of a grid cannot be colored by a given tournament $\overrightarrow{H}$, then any orientation that contains $\overrightarrow{G}$ as a subgraph cannot be colored by $\overrightarrow{H}$. Hence, if we join the orientations $\overrightarrow{A}$, $\overrightarrow{B}$, $\overrightarrow{C}$, $\overrightarrow{D}$, $\overrightarrow{E}$, $\overrightarrow{F}$, $\overrightarrow{I}$, $\overrightarrow{J}$, $\overrightarrow{K}$ and $\overrightarrow{L}$ in a single large grid $\overrightarrow{G}_{7,162}$ with 7 rows and 162 columns, we obtain an oriented grid that cannot be colored by any coloring graph with seven vertices.

□

After many experiments with several heuristic algorithms we have found traps $\overrightarrow{J}'$ and $\overrightarrow{K}'$ with five rows such that $\overrightarrow{J}'$ cannot be colored by $\overrightarrow{H}_0$ and $\overrightarrow{K}'$ cannot be colored by $\overrightarrow{H}_1$ (see Figures 3.16 and 3.17). Hence we have found traps with five rows for all coloring graphs with seven vertices except $\overleftarrow{T}_7$.

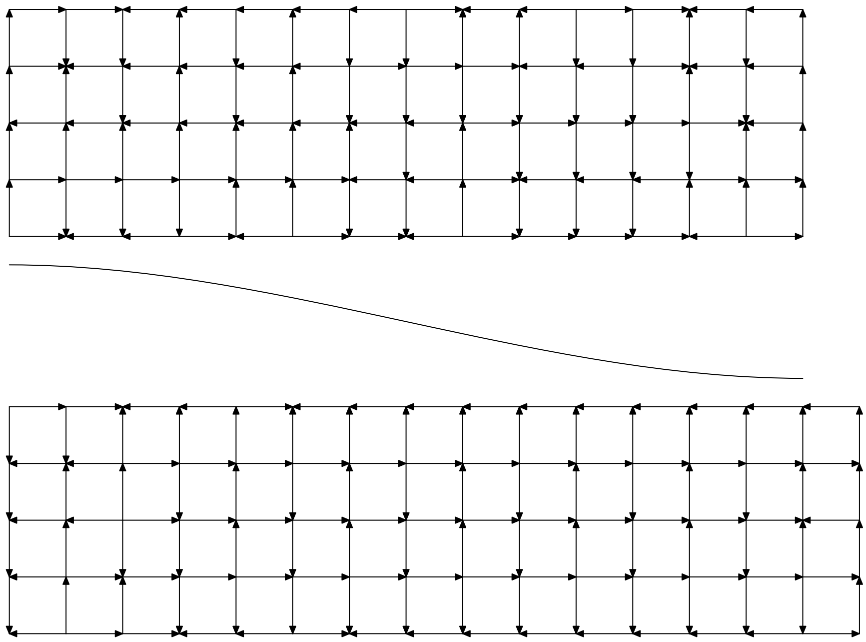

FIGURE 3.16: Orientation $\overrightarrow{J}'$ of $G(5, 30)$ that is not colorable by $\overrightarrow{H}_0$.

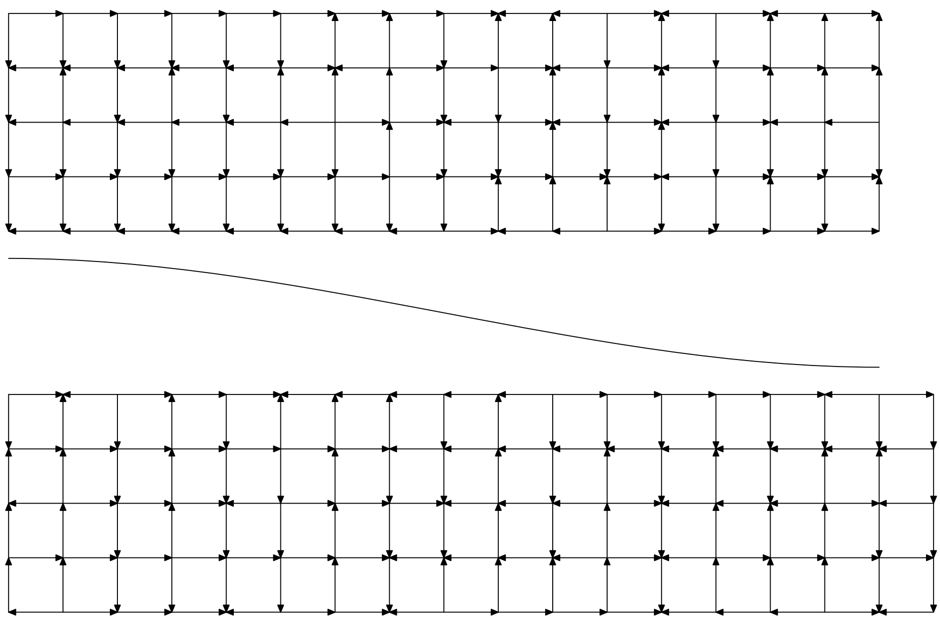

FIGURE 3.17: Orientation $\overrightarrow{K}'$ of $G(5,34)$ that is not colorable by $\overrightarrow{H}_1$.

Chapter 4

An upper bound for oriented chromatic number of grids with 6, 7, or 8 rows

In this chapter, we present a new upper bound for the oriented chromatic number of grids with eight rows by showing the following:

Theorem 4.1. *[Nenca [35]]*
Every orientation of every grid with eight rows can be colored by ten colors. Moreover, there exists a coloring graph $\overrightarrow{H}_{10}$ with ten vertices that colors every grid with eight rows.

Proof. The theorem follows immediately from Lemma 4.7. □

Corollary 4.2. *Every orientation of every grid with less than eight rows can be colored by ten colors.*

To prove Theorem 4.1, we present a coloring graph with ten vertices, which is denoted by $\overrightarrow{H}_{10}$, that can be used to color every orientation of grids with eight rows. This improves by one the bound presented in 2003 in Theorem 1.8 of Fertin, Raspaud and Roychowdhury.

4.1 The coloring graph $\overrightarrow{H}_{10}$.

Consider the coloring graph $\overrightarrow{H}_{10}$ obtained from the Paley tournament $\overrightarrow{T}_{11}$ by removing one vertex, say 0, i.e., $V(\overrightarrow{H}_{10}) = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} = \mathbb{Z}_{11} - \{0\}$ and $(u, v) \in A(\overrightarrow{H}_{10})$ if $(v - u) \in \{1, 3, 4, 5, 9\} = QR$; see Figure 4.1.

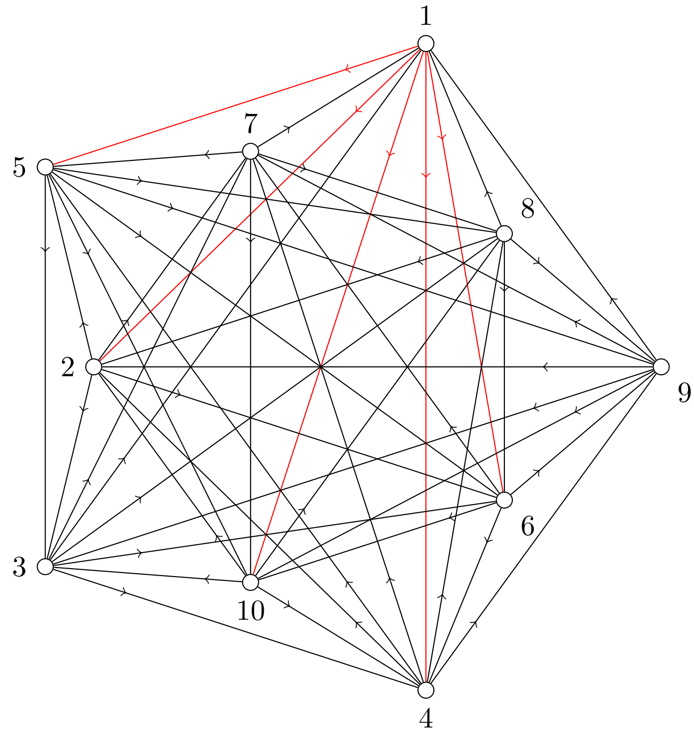

FIGURE 4.1: Coloring graph $\overrightarrow{H}_{10}$.

Lemma 4.3. *For every $a \in \{1,3,4,5,9\}$, the function $h_a(x) = ax \pmod{11}$ is an automorphism of $\overrightarrow{H}_{10}$.*

Proof. By Lemma 2.17, for each $a \in \{1,3,4,5,9\} = QR$, the function $h_a(x)$ is an automorphism of $\overrightarrow{T}_{11}$ such that $h_a(0) = 0$. Therefore, h_a is an automorphism of $\overrightarrow{H}_{10}$. □

Lemma 4.4. *Let $\overrightarrow{G}$ be an orientation of a grid, and let v be one of its vertices. Then, the following two statements are equivalent:*

(a) There exists an oriented coloring $c : \overrightarrow{G} \to \overrightarrow{H}_{10}$.

(b) There exists an oriented coloring $c' : \overrightarrow{G} \to \overrightarrow{H}_{10}$ such that $c'(v) \in \{1, 10\}$.

Proof. It suffices to show that (a) implies (b). If there exists an oriented coloring $c : \overrightarrow{G} \to \overrightarrow{H}_{10}$, then the composition $h_a \circ c$ is an oriented coloring for every $a \in QR$. Table 4.1 shows automorphisms of $\overrightarrow{H}_{10}$. If $c(v) \in QR$, then there exists $a \in QR$ such that $a \cdot c(v) = 1$. If $c(v) \in \{2,6,7,8,10\}$, then there exists $a \in QR$ such that $a \cdot c(v) = 10$. □

x	1	2	3	4	5	6	7	8	9	10
$h_1(x) = x \pmod{11}$	1	2	3	4	5	6	7	8	9	10
$h_3(x) = 3x \pmod{11}$	3	6	9	1	4	7	10	2	5	8
$h_4(x) = 4x \pmod{11}$	4	8	1	5	9	2	6	10	3	7
$h_5(x) = 5x \pmod{11}$	5	10	4	9	3	8	2	7	1	6
$h_9(x) = 9x \pmod{11}$	9	7	5	3	1	10	8	6	4	2

TABLE 4.1: Automorphisms of $\overrightarrow{H}_{10}$.

4.2 Set closed under extension

Let R_8 be the comb with eight rows; see Definition 2.8.

Definition 4.5. A set $S \subseteq (V(\overrightarrow{H}_{10}))^8$ is closed under extension if

(a) for every orientation $\overrightarrow{P}$ of the path $P_8 = (v_1, \ldots, v_8)$, there exists an oriented coloring $c : \overrightarrow{P} \to \overrightarrow{H}_{10}$ such that $(c(v_1), \ldots, c(v_8)) \in S$;

(b) for every orientation $\overrightarrow{R}$ of the comb R_8 and every sequence $(c_1, \ldots, c_8) \in S$, there exists an oriented coloring $c : \overrightarrow{R} \to \overrightarrow{H}_{10}$ and a automorphism h_a of $\overrightarrow{H}_{10}$ such that

 (1) $(c(1,1), \ldots, c(8,1)) = (c_1, \ldots, c_8)$ and

 (2) $(h_a \circ c(1,2), \ldots, h_a \circ c(8,2)) \in S$.

Lemma 4.6. *There exists a nonempty set $S \subseteq (V(\overrightarrow{H}_{10}))^8$ that is closed under extension.*

Proof. To prove the lemma, we use a computer. We have designed an algorithm that finds a proper set S. Let

$$S^*_{max}(P_8) = \{(c_1, \ldots, c_8) : c_1 \in \{1, 10\}, \text{ and } \forall_{2 \le i \le n}\ c_i \in V(\overrightarrow{H}_{10}), \text{ and } c_{i-1} \neq c_i\}.$$

For every sequence $t = (t_1, \ldots, t_8) \in S^*_{max}(P_8)$, there exists an orientation $\overrightarrow{P}$ of the path $P_8 = (v_1, \ldots, v_8)$ and a coloring $c : \overrightarrow{P} \to \overrightarrow{H}_{10}$ such that $(c(v_1), \ldots, c(v_8)) = t$. For a set T, a sequence $t = (t_1, \ldots, t_8) \in T$, and an orientation $\overrightarrow{R}$ of the comb R_8, we say that t *can be extended* in T on $\overrightarrow{R}$ if there exists a coloring $c : \overrightarrow{R} \to \overrightarrow{H}_{10}$ and a automorphism h_a such that

- $(c(1,1), \ldots, c(8,1)) = t$
- $(h_a \circ c(1,2), \ldots, h_a \circ c(8,2)) \in T$.

The algorithm starts with $T = S^*_{max}(P_8)$. In the while loop, for each sequence $t \in T$ and for each orientation $\overrightarrow{R}$ of the comb R_8, the algorithm checks if t can be extended in T on $\overrightarrow{R}$. If the sequence t cannot be extended, then t is removed from T. After the while loop, the set T satisfies condition b) of Definition 4.5. If T is not empty, then it also satisfies condition a). In this case, $S = T$ is returned. If T is empty, then the algorithm returns NO.

Algorithm ComputeSet$S$

OUTPUT: a nonempty set $S \subset (V(\overrightarrow{H}_{10}))^8$ that is closed under extension or NO if such a set does not exist.

```
compute the set $S^*_{max}(P_8)$
$T$ := $S^*_{max}(P_8)$
SetIsReady := false
while not SetIsReady
      SetIsReady := true
      for every sequence $t = (t_1, \ldots, t_8) \in T$
           color the first column of the comb $R_8$
                   by setting $c(i,1) = t_i$ for $1 \le i \le 8$
           SeqCanBeExtended := true
           for every orientation $\overrightarrow{R}$ of the comb $R_8$
                if $t$ cannot be extended on $\overrightarrow{R}$
                      SeqCanBeExtended := false
           if not SeqCanBeExtended
                $T$ := $T - t$
                SetIsReady := false
if $T = \emptyset$
      return NO
else
      $S$ := $T$
      return the set $S$
```

Using the Algorithm ComputeSet, we have found a nonempty set S that is closed under extension. The set S is posted on the website https://inf.ug.edu.pl/grids/. $\square$

Lemma 4.7. *Every orientation $\overrightarrow{G}$ of the grid $G(8, n)$ can be colored by $\overrightarrow{H}_{10}$.*

Proof. For a specified orientation $\overrightarrow{G}$ of $G(8, n)$ and $i \leq n$, we denote the induced subgraph of $\overrightarrow{G}$ that is formed by the first i columns by $\overrightarrow{G}(i)$. It is easy to show by induction that for every i, there is a coloring $c : \overrightarrow{G}(i) \to \overrightarrow{H}_{10}$ and an automorphism h_a of $\overrightarrow{H}_{10}$ such that $(h_a \circ c(1, i), \ldots, h_a \circ c(8, i)) = s$ for some $s \in S$. $\square$

Theorem 4.1 provides the new upper bound of ten for the oriented chromatic number of grids with eight rows, which also holds for grids with six or seven rows. In the next chapter, we prove that nine colors are sufficient for the oriented coloring of grids with five rows.

Chapter 5

Grids with five rows

In this chapter, we focus on grids with five rows. In Chapter 4, we showed that every orientation of a grid with at most eight rows can be colored with 10 colors. Fertin et al. in [18] presented an orientation of the grid $G(4,5)$ that cannot be colored with six colors. Thus, the bounds for grids with five rows, were $7 \leq \overrightarrow{\chi}(\mathcal{G}_5) \leq 10$. In this chapter, we show that there exists a coloring graph with nine vertices, which is denoted by $\overrightarrow{H}_9$, that can be used for the oriented coloring of any orientation of grids with five rows. The coloring graph $\overrightarrow{H}_9$ is obtained from the Paley tournament $\overrightarrow{T}_7$ by adding two vertices, namely, one sink and one source; see Figure 5.1. More precisely, $V(\overrightarrow{H}_9) = \{0,1,2,3,4,5,6,7,8\}$ and $(u,v) \in A(\overrightarrow{H}_9)$ if

- $u, v < 7$ and $v - u \equiv 1, 2, \text{ or } 4 \pmod 7$,
- $u = 7$, or $v = 8$.

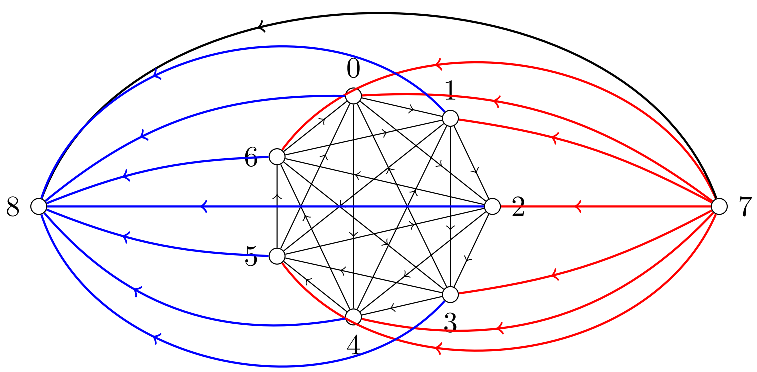

FIGURE 5.1: Coloring graph $\overrightarrow{H}_9$.

Theorem 5.1 (Dybizbański, Nenca [12]).
Every oriented grid with five rows can be colored with nine colors. Moreover, there is an oriented graph $\overrightarrow{H}_9$ with nine vertices that colors every oriented grid with five rows.

First, we consider the family $\mathcal{G}_5^*$ of all orientations of all grids with five rows such that all vertices in the first, third and fifth rows, except possibly those in the last column, are internal. In Section 5.1, we show that every $\overrightarrow{G} \in \mathcal{G}_5^*$ can be colored by the Paley tournament $\overrightarrow{T}_7$ by showing that the empty set is not reachable on the last column of any grid $\overrightarrow{G} \in \mathcal{G}_5^*$. Since the Paley tournament $\overrightarrow{T}_7$ is arc-transitive and self-converse, we do not need to generate the whole family $\mathcal{S}_5(\overrightarrow{T}_7)$. In Section 5.2, we prove that every grid $\overrightarrow{G} \in \mathcal{G}_5$ can be colored by $\overrightarrow{H}_9$.

5.1 Coloring $\mathcal{G}_5^*$

In this section, we prove that every grid $\overrightarrow{G}$ in $\mathcal{G}_5^*$ can be colored by the Paley tournament $\overrightarrow{T}_7$; see Section 2.2. The tournament $\overrightarrow{T}_7$ is self-converse, and the function $f(x) = -x$ is an isomorphism from $\overrightarrow{T}_7$ to $\overrightarrow{T}_7^R$; see Lemma 2.19. Therefore, we can only consider grids $\overrightarrow{G}$ that have the arc $((1,n),(2,n))$ in the last column. Let $\mathcal{G}_5^{\downarrow}$ denote the family

$$\mathcal{G}_5^{\downarrow} = \{\overrightarrow{G} \in \mathcal{G}_5 : ((1,n),(2,n)) \in A(\overrightarrow{G}),\ \text{where } n \text{ is the number of columns}\}.$$

Similarly, let $\mathcal{G}_5^{\uparrow}$ denote the family of all orientations of grids with five rows with the arc $((2,n),(1,n))$ in the last column. Let $\mathcal{G}_5^{*\downarrow} = \mathcal{G}_5^* \cap \mathcal{G}_5^{\downarrow}$ and $\mathcal{G}_5^{*\uparrow} = \mathcal{G}_5^* \cap \mathcal{G}_5^{\uparrow}$. Clearly, $\overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}$ if and only if $\overrightarrow{G}^R \in \mathcal{G}_5^{*\uparrow}$.
Let $t = (t_1, \ldots, t_5)$ be a sequence of elements of $\mathbb{Z}_7$. We denote the sequence $(-t_1, \ldots, -t_5)$ by $-t$. For a set of sequences T, we denote by $-T$ the set $\{-t : t \in T\}$.

Lemma 5.2. *If $\overrightarrow{G}$ is colorable by $\overrightarrow{T}_7$, then $\overrightarrow{G}^R$ is colorable by $\overrightarrow{T}_7$. If a set $T \subseteq (\mathbb{Z}_7)^5$ is reachable on the last column of $\overrightarrow{G}$, then $-T$ is reachable on the last column of $\overrightarrow{G}^R$.*

□

By Lemma 2.18, the tournament $\overrightarrow{T}_7$ is arc-transitive. Arc $(0,1) \in A(\overrightarrow{T}_7)$. Hence, by Lemma 2.1, if a grid $\overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}$ can be colored by $\overrightarrow{T}_7$, then there is a coloring $c : \overrightarrow{G} \to \overrightarrow{T}_7$ such that $c(1,n) = 0$ and $c(2,n) = 1$. Hence, to determine whether all grids in $\mathcal{G}_5^*$ are colorable by $\overrightarrow{T}_7$, it suffices to check whether all grids $\overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}$ are colorable by $\overrightarrow{T}_7$. Moreover, we can only check if there exists a coloring $c : \overrightarrow{G} \to \overrightarrow{T}_7$ such that $c(1,n) = 0$ and $c(2,n) = 1$.
For $\overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}$, we denote by $S_5^{0,1}(\overrightarrow{G})$ the set

$$S_5^{0,1}(\overrightarrow{G}) = \{(c_3, c_4, c_5) : (0, 1, c_3, c_4, c_5) \in S(\overrightarrow{G}, \overrightarrow{T}_7)\},$$

where $S(\overrightarrow{G}, \overrightarrow{T}_7)$ is the set of reachable colorings of the last column of $\overrightarrow{G}$ by $\overrightarrow{T}_7$ (see Definition 2.11), and we denote by $\mathcal{S}_5^{0,1}$the family $\{S_5^{0,1}(\overrightarrow{G}) : \overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}\}$.

Lemma 5.3. *If* $(0, 1, s_3, s_4, s_5) \sim t$ *for some* $t \in S(\overrightarrow{G}, \overrightarrow{T}_7)$*, then* $(s_3, s_4, s_5) \in S_5^{0,1}(\overrightarrow{G})$*.*

Proof. Suppose that $c : \overrightarrow{G} \to \overrightarrow{T}_7$ is a coloring of $\overrightarrow{G}$ such that $c(\text{last column}) = t$. Since $(0, 1, s_3, s_4, s_5) \sim t$, there is an automorphism $h : \overrightarrow{T}_7 \to \overrightarrow{T}_7$ such that $h(t) = (0, 1, s_3, s_4, s_5)$. Hence, the coloring $h \circ c : \overrightarrow{G} \to \overrightarrow{T}_7$ is a coloring of $\overrightarrow{G}$ with $(0, 1, s_3, s_4, s_5) = h \circ c(\text{last column})$. Hence, $(0, 1, s_3, s_4, s_5) \in S(\overrightarrow{G}, \overrightarrow{T}_7)$, and $(s_3, s_4, s_5) \in S_5^{0,1}(\overrightarrow{G})$. □

Lemma 5.4. *The following three conditions are equivalent.*

(1) Every grid $\overrightarrow{G} \in \mathcal{G}_5^*$ *is colored by* $\overrightarrow{T}_7$*.*

(2) Every grid $\overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}$ *is colored by* $\overrightarrow{T}_7$*.*

(3) $\emptyset \notin \mathcal{S}_5^{0,1}(\overrightarrow{T}_7)$*.*

Proof. (1) ⇔ (2) follows from Lemma 5.2.
(2) ⇔ (3) $\emptyset \notin \mathcal{S}_5^{0,1}(\overrightarrow{T}_7)$ if and only if $\forall_{\overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}} \ \exists_c \ \ c : \overrightarrow{G} \to \overrightarrow{T}_7$.

□

For a given set $S \in \mathcal{S}_5^{0,1}$ and an orientation $\overrightarrow{R}$ of the comb R_5, we define the set $NEXT_5^{0,1}(S, \overrightarrow{R})$:

- If $\overrightarrow{R} \in \mathcal{G}_5^{\downarrow}$, then

$$NEXT_5^{0,1}(S, \overrightarrow{R}) = \{(c_3, c_4, c_5) : \exists_t \quad t \in NEXT(S, \overrightarrow{R}, \overrightarrow{T}_7) \text{ and } t \sim (0, 1, c_3, c_4, c_5)\};$$

- if $\overrightarrow{R} \in \mathcal{G}_5^{\uparrow}$, then

$$NEXT_5^{0,1}(S, \overrightarrow{R}) = \{(c_3, c_4, c_5) : \exists_t \quad t \in NEXT(S, \overrightarrow{R}, \overrightarrow{T}_7) \text{ and } -t \sim (0, 1, c_3, c_4, c_5)\},$$

where $NEXT(S, \overrightarrow{R}, \overrightarrow{T}_7)$ is the set of colorings of the second column of $\overrightarrow{R}$ when the vertices of the first column of $\overrightarrow{R}$ are colored by sequences $s \in S$; see Definition 2.10. Let us denote by $\Phi(\overrightarrow{G})$ the sequence $(\delta(1,n), \delta(3,n), \delta(5,n))$, where

$$\delta(u) = \begin{cases} 0 & \text{if } u \text{ is a sink or a source,} \\ 1 & \text{otherwise.} \end{cases}$$

Consider the comb R_5, where $(x_1, x_2, x_3, x_4, x_5)$ is the first column of R_5. Let $T = S_5^{0,1}(\overrightarrow{G})$ for some orientation $\overrightarrow{G}$ of $G(5,n)$. Suppose that we glue together the grid $G(5,n)$ and the comb R_5 in such a way that the last column of $G(5,n)$ becomes the first column of R_5. We shall say that an orientation $\overrightarrow{R}$ of the comb R_5 *is good* for T and $\Phi(\overrightarrow{G}) = \Phi = (\delta_1, \delta_3, \delta_5) \in \{0,1\}^3$ if for each $i \in \{1,3,5\}$

- $\delta_i = \delta(x_i) = 1$ or
- $\delta_i = \delta(x_i) = 0$ and the arcs of the last column of $\overrightarrow{G}$ and the arcs from $\overrightarrow{R}$ do not form a sink or a source in y.

Now, we describe an algorithm that generates the set $\mathcal{S}_5^{0,1}$. The algorithm uses a queue Q.

The algorithm starts with the grid $G(5,1)$. For every orientation $\overrightarrow{P}$ of $G(5,1)$ with the arc $((1,1),(2,1)) \in \overrightarrow{P}$, it computes the pair $(S_5^{0,1}(\overrightarrow{P}), \Phi(\overrightarrow{P}))$ and puts it into the queue Q. Next, the algorithm takes from the queue one by one a pair (T, Φ) and, for every orientation $\overrightarrow{R}$ of the comb R_5, checks whether the orientation $\overrightarrow{R}$ is good for T and Φ. If so, the algorithm computes $\Phi' = (\delta(1,2), \delta(3,2), \delta(5,2))$ and the set $T' = NEXT_5^{0,1}(T, \overrightarrow{R})$ and adds the pair (T', Φ') to the queue Q if it is new.

Algorithm GenerateSets

OUTPUT: the set $\mathcal{S}_5^{0,1}$ is printed on the screen.

`for every orientation` $\overrightarrow{P} \in \mathcal{G}_5^{*\downarrow}$
    `compute` $(S_5^{0,1}(\overrightarrow{P}), \Phi(\overrightarrow{P}))$
    $Q \leftarrow (S_5^{0,1}(\overrightarrow{P}), \Phi(\overrightarrow{P}))$
    `print` $S_5^{0,1}(\overrightarrow{P})$
`while` $Q \neq \emptyset$ `do`
    $(T, \Phi) \leftarrow Q$
    `for each orientation` $\overrightarrow{R}$ `of the comb` R_5
        `if the orientation` $\overrightarrow{R}$ `is good for` T `and` Φ

T' := $NEXT_5^{0,1}(T, \overrightarrow{R})$

$\Phi' := (\delta(1,2), \delta(3,2), \delta(5,2))$

`if` (T', Φ') `was never in the queue`

  $Q \leftarrow (T', \Phi')$

  `print` T'

Lemma 5.5. *(1) For every $\overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}$, the pair $(S_5^{0,1}(\overrightarrow{G}), \Phi(\overrightarrow{G}))$ appears in the queue.*

(2) For every pair (T, Φ) that appears in the queue, there exists a grid $\overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}$ such that $(T, \Phi) = (S_5^{0,1}(\overrightarrow{G}), \Phi(\overrightarrow{G}))$.

Proof. (1) Proceed by induction on the number n of columns of $\overrightarrow{G}$.
For $n = 1$, the lemma clearly holds. Suppose that the lemma holds for grids with n columns and $\overrightarrow{G}$ is the orientation of the grid $G(5, n+1)$. The orientation $\overrightarrow{G} = \overrightarrow{G}(n+1)$ consists of $\overrightarrow{G}(n)$ and the orientation $\overrightarrow{R} = \overrightarrow{R}(n+1)$ of the comb R_5; see Figure 5.2. Since $\overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}$, $\overrightarrow{G}$ has only internal vertices in the first, third and fifth rows, except possibly in the last column. Hence, the orientation $\overrightarrow{R}$ is good for $S_5^{0,1}(\overrightarrow{G}(n))$ and $(\delta(1,n), \delta(3,n), \delta(5,n))$. First, suppose that $\overrightarrow{G}(n) \in \mathcal{G}_5^{*\downarrow}$; see Figure 5.2.

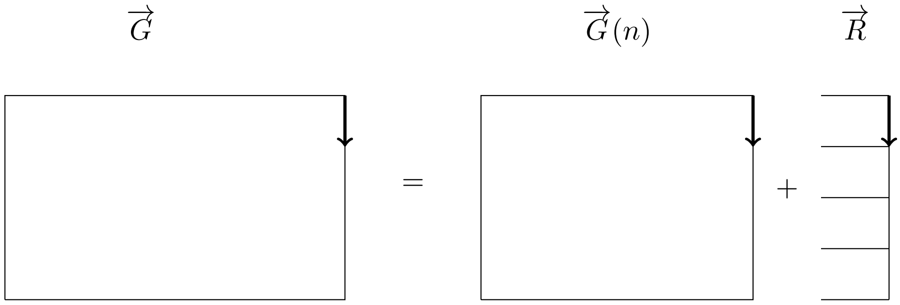

FIGURE 5.2: Formation of $\overrightarrow{G}$, when $\overrightarrow{G}(n) \in \mathcal{G}_5^{*\downarrow}$.

By the induction hypothesis, $(S_5^{0,1}(\overrightarrow{G}(n)), \Phi(\overrightarrow{G}(n)))$ appears in the queue. Now, we shall show that

$$S_5^{0,1}(\overrightarrow{G}) = NEXT_5^{0,1}(S_5^{0,1}(\overrightarrow{G}(n)), \overrightarrow{R}),$$

and the pair $(S_5^{0,1}(\overrightarrow{G}), \Phi(\overrightarrow{G}))$ also appears in the queue. Note that $\Phi(\overrightarrow{G}) = \Phi(\overrightarrow{R})$.
First inclusion $\subset$:
Suppose that $t = (t_3, t_4, t_5) \in S_5^{0,1}(\overrightarrow{G})$; see Figure 5.3. There exists a coloring $c : \overrightarrow{G} \to \overrightarrow{T}_7$ such that c(n+1 th column) $= (0, 1, t_3, t_4, t_5)$. Consider c(n th column) $= t' = (t'_1, \ldots, t'_5) \in S(\overrightarrow{G}(n), \overrightarrow{T}_7)$. There exists an automorphism $\alpha \in Aut(\overrightarrow{T}_7)$ such that $\alpha(t') = (0, 1, t''_3, t''_4, t''_5) \in S_5^{0,1}(\overrightarrow{G}(n))$.

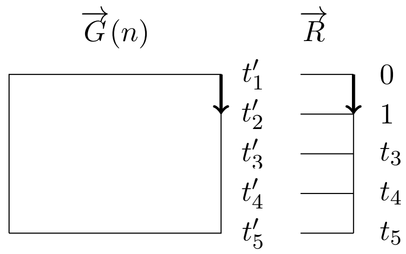

FIGURE 5.3: Coloring c.

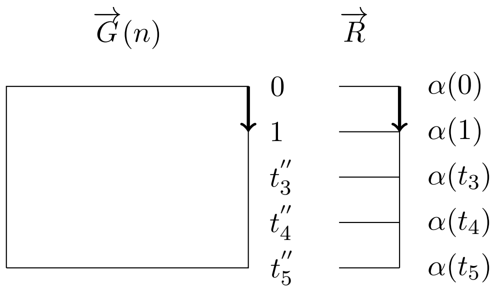

FIGURE 5.4: Coloring $\alpha \circ c$.

$$\alpha \circ c(\text{n+1 th column}) \in NEXT(\alpha(t'), \overrightarrow{R}, \overrightarrow{T}_7),$$

and

$$\alpha \circ c(\text{n+1 th column}) = \alpha(0, 1, t_3, t_4, t_5).$$

Hence,

$$\alpha \circ c(\text{n+1 th column}) \sim (0, 1, t_3, t_4, t_5)$$

and

$$(t_3, t_4, t_5) \in NEXT_5^{0,1}(S_5^{0,1}(\overrightarrow{G}(n)), \overrightarrow{R}).$$

Now, inclusion $\supset$:
Let $t = (t_3, t_4, t_5) \in NEXT_5^{0,1}(S_5^{0,1}(\overrightarrow{G}(n)), \overrightarrow{R})$. Then, there exists $t' = (t'_1, \ldots, t'_5)$ such that $t' \sim (0, 1, t_3, t_4, t_5)$ and $t' \in NEXT(S_5^{0,1}(\overrightarrow{G}(n)), \overrightarrow{R}, \overrightarrow{T}_7)$. Hence, there exists a coloring $c' : \overrightarrow{R} \to \overrightarrow{T}_7$ such that

$$c'(\text{first column of } \overrightarrow{R}) = (0, 1, s_3, s_4, s_5) \in S_5^{0,1}(\overrightarrow{G}(n)),$$

and

$$c'(\text{second column of } \overrightarrow{R}) = (t'_1, t'_2, t'_3, t'_4, t'_5),$$

and there exists a coloring $c'' : \overrightarrow{G}(n) \to \overrightarrow{T}_7$ such that

$$c''(\text{n th column of } \overrightarrow{G}(n)) = (0, 1, s_3, s_4, s_5).$$

By combining colorings c' and c'', we obtain a coloring c of $\overrightarrow{G}$ with $t' = c(\text{n+1 column})$. Hence, by Lemma 5.3, $(t_3, t_4, t_5) \in S_5^{0,1}(\overrightarrow{G})$.

Now, suppose that $\overrightarrow{G}(n) \in \mathcal{G}_5^{*\uparrow}$; see Figure 5.5. Consider $\overrightarrow{G}^R$, which is composed of

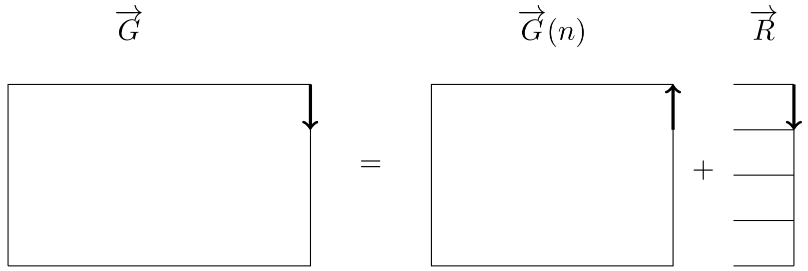

FIGURE 5.5: Case 2, $\overrightarrow{G}(n) \in \mathcal{G}_5^{*\uparrow}$.

$\overrightarrow{G}^R(n)$ and $\overrightarrow{R}^R$; see Figure 5.6.

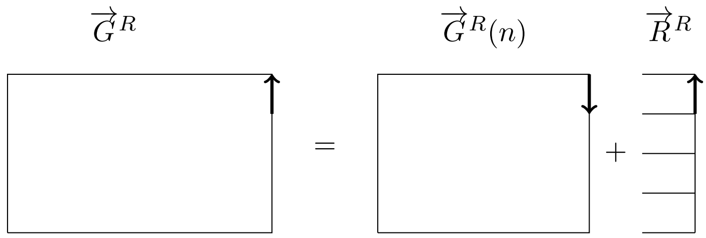

FIGURE 5.6: Case 2, $\overrightarrow{G}^R(n) \in \mathcal{G}_5^{*\downarrow}$.

By the induction hypothesis, $S_5^{0,1}(\overrightarrow{G}^R(n))$ appears in the queue. Now, we shall show that

$$S_5^{0,1}(\overrightarrow{G}) = NEXT_5^{0,1}(S_5^{0,1}(\overrightarrow{G}^R(n)), \overrightarrow{R}^R).$$

Thus, the pair $(S_5^{0,1}(\overrightarrow{G}), \Phi(\overrightarrow{R}))$ appears in the queue.

First inclusion $\subset$:

Suppose $t = (t_3, t_4, t_5) \in S_5^{0,1}(\overrightarrow{G})$. Then there exists a coloring $c : \overrightarrow{G} \to \overrightarrow{T}_7$ such that $c(\text{n+1 th column}) = t = (0, 1, t_3, t_4, t_5)$. Consider coloring $c(\text{n th column}) = t' = (t'_1, \ldots, t'_5)$.

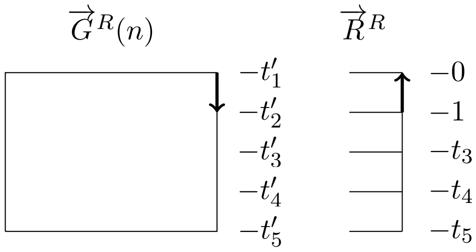

FIGURE 5.7: Coloring c.

Sequence $t' \in S(\overrightarrow{G}(n), \overrightarrow{T}_7)$, and sequence $-t' \in S(\overrightarrow{G}^R(n), \overrightarrow{T}_7)$; see Figure 5.7. There exists an automorphism $\alpha \in Aut(\overrightarrow{T}_7)$ such that

$$\alpha(-t') = t'' = (0, 1, t_3^{''}, t_4^{''}, t_5^{''})$$

and

$$(t_3^{''}, t_4^{''}, t_5^{''}) \in S_5^{0,1}(\overrightarrow{G}^R(n)).$$

Since $\overrightarrow{R}^R \in \mathcal{G}_5^{*\uparrow}$, the function $NEXT_5^{0,1}(S_5^{0,1}(\overrightarrow{G}^R(n)), \overrightarrow{R}^R)$ returns, among other se-

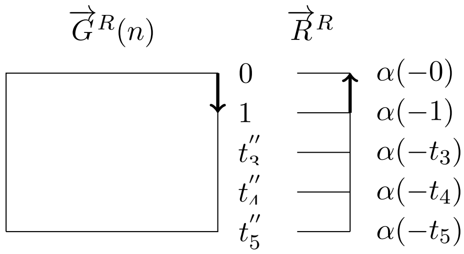

FIGURE 5.8: Coloring $\alpha \circ c$.

quences, the sequence equivalent to $-\alpha(-(0, 1, t_3, t_4, t_5))$, which is equivalent to the sequence $(0, 1, t_3, t_4, t_5)$. Hence,

$$(t_3, t_4, t_5) \in NEXT_5^{0,1}(S_5^{0,1}(\overrightarrow{G}^R(n)), \overrightarrow{R}^R).$$

Now, inclusion $\supset$:
Let $t = (t_3, t_4, t_5) \in NEXT_5^{0,1}(S(\overrightarrow{G}^R(n), \overrightarrow{T}_7), \overrightarrow{R}^R)$. Since $\overrightarrow{R}^R \in \mathcal{G}_5^{*\uparrow}$, there exists $t' = (t_1', \dots, t_5')$ such that

$$-t' \sim (0, 1, t_3, t_4, t_5)$$

and

$$t' \in NEXT(S_5^{0,1}(\overrightarrow{G}^R(n)), \overrightarrow{R}^R, \overrightarrow{T}_7).$$

Hence, there exists a coloring $c' : \overrightarrow{R}^R \to \overrightarrow{T}_7$ such that

$$c'(\text{first column of } \overrightarrow{R}^R) = (0, 1, s_3, s_4, s_5) \in S_5^{0,1}(\overrightarrow{G}^R(n))$$

and

$$c'(\text{second column of } \overrightarrow{R}^R) = (t'_1, \ldots, t'_5),$$

and there exists a coloring $c'' : \overrightarrow{G}^R(n) \to \overrightarrow{T}_7$ such that

$$c''(\text{n th column}) = (0, 1, s_3, s_4, s_5).$$

By combining colorings c' and c'', we obtain a coloring c of $\overrightarrow{G}$ with

$$-t' = c(\text{n+1 column}).$$

Hence, $t \in S_5^{0,1}(\overrightarrow{G})$.

(2) Now we prove that if a pair (T, Φ) appears in the queue, then $T \in \mathcal{S}_5^{0,1}$. The proof is by induction on the iteration number of the while loop at which (T, Φ) appears in the queue. If T is generated before the while loop, then $T \in \mathcal{S}_5^{0,1}$. Assume that we are at the i th iteration and a pair (T, Φ) is in the queue Q. Then, $T \in \mathcal{S}_5^{0,1}$. We need to show that $NEXT_5^{0,1}(T, \overrightarrow{R}) \in \mathcal{S}_5^{0,1}$. Since $T \in \mathcal{S}_5^{0,1}$, there is an orientation $\overrightarrow{G}$ of $G(5, n)$ for some n such that $S_5^{0,1}(\overrightarrow{G}) = T$. Consider the orientation $\overrightarrow{G}'$ of the grid $G(5, n+1)$ such that $\overrightarrow{G}'(n) = \overrightarrow{G}$ and $\overrightarrow{R}(n+1) = \overrightarrow{R}$. Then, the set $S = NEXT_5^{0,1}(T, \overrightarrow{R})$ is the set of reachable colorings on the last column of $\overrightarrow{G}'$, i.e., $S = S_5^{0,1}(\overrightarrow{G}')$. Hence, $S \in \mathcal{S}_5^{0,1}$.

□

After running Algorithm GenerateSets, we found that the algorithm ends with empty queue Q and that no pair of the form $(\emptyset, \Phi)$ is reachable. Thus, we have the following lemma:

Lemma 5.6. *Every $\overrightarrow{G} \in \mathcal{G}_5^{*\downarrow}$ can be colored with $\overrightarrow{T}_7$.*

5.2 Proof of Theorem 5.1

Proof. Let $\overrightarrow{G}$ be an orientation of $G(5, n)$, where $n \geq 5$. We show that there exists a homomorphism γ from $\overrightarrow{G}$ to $\overrightarrow{H}_9$. To show this, we construct a new orientation $\overrightarrow{G}'$ of $G(5, n)$ by reversing some arcs in $\overrightarrow{G}$. For every column i,

- if $(1, i)$ is a sink or a source, then we reverse the arc between $(1, i)$ and $(2, i)$;
- if $(3, i)$ is a sink or a source, then we reverse the arc between $(3, i)$ and $(2, i)$;
- if $(5, i)$ is a sink or a source, then we reverse the arc between $(5, i)$ and $(4, i)$.

We leave every horizontal arc unchanged, and we reverse at most one arc incident to any vertex in the first, third or fifth row. Thus, there is no sink or source vertex in the first, third or fifth row of $\overrightarrow{G}'$. Hence, by Lemma 5.6, there is a coloring $\gamma' : \overrightarrow{G}' \to \overrightarrow{T}_7$. We can color $\overrightarrow{G}$ using a homomorphism $\gamma : \overrightarrow{G} \to \overrightarrow{H}_9$ in the following way:

- if vertex a is in the second or fourth row, we set $\gamma(a) := \gamma'(a)$;
- if vertex a is in the first, third or fifth row and it is not a sink or a source in $\overrightarrow{G}$, we set $\gamma(a) := \gamma'(a)$;
- if vertex a is in the first, third or fifth row and it is a sink or a source in $\overrightarrow{G}$, then we set
  * $\gamma(a) := 7$ if a is a source in $\overrightarrow{G}$;
  * $\gamma(a) := 8$ if a is a sink in $\overrightarrow{G}$.

To show that γ is a homomorphism from $\overrightarrow{G}$ to $\overrightarrow{H}_9$, consider an arc $(a, b) \in \overrightarrow{G}$. There are four possible cases:

- If the arc (a, b) has not been reversed and the colors of a and b have not been changed, then these colors fit in both $\overrightarrow{T}_7$ and $\overrightarrow{H}_9$.
- If the arc (a, b) has not been reversed and the color of one of its ends, say a, has been changed, then vertex a is a sink or a source and its color matches the color of vertex b because $\gamma(b) \in V(\overrightarrow{T}_7)$.
- If the arc (a, b) has not been reversed but the colors of both its ends has been changed, then a and b are in the same row, namely, the first, third or fifth row. This is possible only if one of them is a sink and the other is a source. Then, their colors are 7 and 8 and fit in $\overrightarrow{H}_9$.
- If the arc (a, b) has been reversed, then a is a source (or b is a sink, respectively) in the first, third, or fifth row and receives color 7 (or 8, respectively). The other end of the arc is in the second or the fourth column. Thus, it has a color from $\overrightarrow{T}_7$ and $(7, \gamma(b)) \in A(\overrightarrow{H}_9)$ (or $(\gamma(a), 8) \in A(\overrightarrow{H}_9)$, respectively).

□

Lemma 5.7. *There exists an orientation of $G(7,28)$ that cannot be colored by $\overrightarrow{H}_9$.*

Proof. We use a similar algorithm to the algorithm described in the proof of Lemma 5.6 and construct an orientation $\overrightarrow{G}_{5,28}$ of the grid $G(5,28)$ (see Figure 5.9) with only internal vertices in the second, third and fourth rows that cannot be colored by the Paley tournament $\overrightarrow{T}_7$. An orientation of $G(7,28)$ is created from $\overrightarrow{G}_{5,28}$ by adding two extra rows: one above the first row – row 0 – and one below the fifth row – row 6 – in such a way that there are only internal vertices in the first and fifth rows. Hence, we cannot use color 7 or 8 to color any vertex in rows 1–5, and the obtained orientation cannot be colored by $\overrightarrow{H}_9$. □

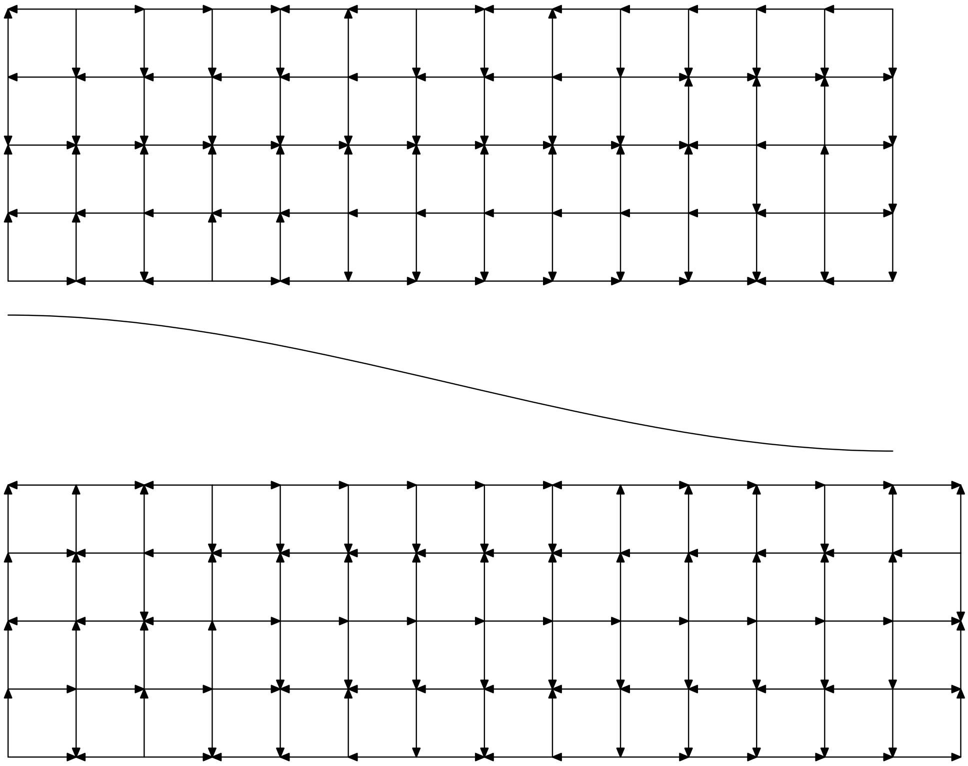

FIGURE 5.9: Oriented grid $\overrightarrow{G}_{5,28}$.

Chapter 6

Cylindrical grid and toroids

In this chapter, we consider the oriented chromatic number of cylinders $Cyl(m,n) = C_m \Box P_n$ and toroids $T(m,n) = C_m \Box C_n$, see Figure 6.1 and Figure fig:t.

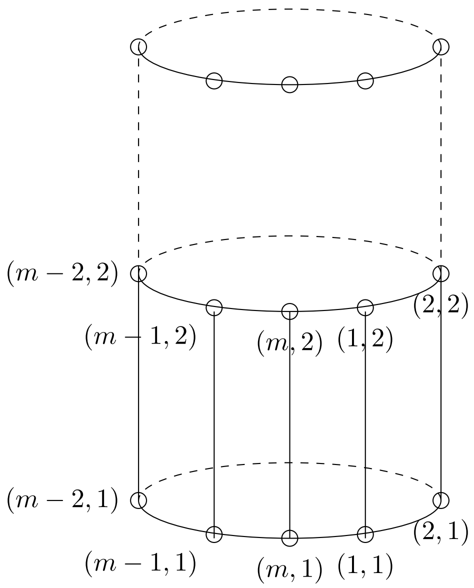

FIGURE 6.1: Cylindrical grid $Cyl(m,n)$.

We know that:

Theorem 6.1 (Marshall [29]).
Every orientation of every cylindrical grid can be colored by the Paley tournament $\overrightarrow{T}_{11}$*, see Figure 2.4.*

This theorem was proved by Marshall in [29] in a more general form. We give another proof of the theorem. We think that our proof is simpler and more direct.

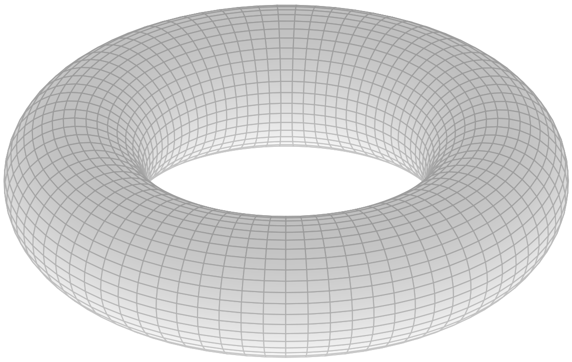

FIGURE 6.2: Toroid $T(m,n)$.

Theorem 6.2.
There exists an orientation of a cylindrical grid that cannot be colored by any coloring graph with seven vertices.

Moreover, we show that:

Theorem 6.3 (Nenca [35])**.**
Every orientation of every cylindrical grid with circuit $m = 3, 4, 5, 6, 7$ *can be colored by the coloring graph* $\overrightarrow{H}_{10}$*, see Section 4.1 and Figure 4.1.*

Furthermore, we improve the upper bound for oriented chromatic number of toroids, by showing that:

Theorem 6.4.
Every orientation of a toroid $C_m \Box C_n$*, with* $m, n \geq 3$*, can be colored with 27 colors.*

We also show, that:

Theorem 6.5. *a)* $\overrightarrow{\chi}(C_3 \Box C_3) = 9$,

b) $\overrightarrow{\chi}(C_3 \Box C_n) \leq 16$*, for* $n > 3$*,*

c) $\overrightarrow{\chi}(C_4 \Box C_n) \leq 18$*, for* $n > 3$*,*

d) $\overrightarrow{\chi}(C_5 \Box C_n) \leq 20$*, for* $n > 4$*,*

e) $\overrightarrow{\chi}(C_6 \Box C_n) \leq 22$*, for* $n > 5$*,*

f) $\overrightarrow{\chi}(C_7 \Box C_n) \leq 24$*, for* $n > 6$*.*

6.1 Upper and lower bounds for $\overrightarrow{\chi}(Cyl)$

In Chapter 3 we showed that there exists an oriented grid $G(7, 162)$ which cannot be colored with 7 colors. This implies Theorem 6.2 and the lower bound $8 \leq \overrightarrow{\chi}(Cyl)$, since grids are subgraphs of cylindrical grids.

Before presenting the proof of Theorem 6.1 we first give some properties of the coloring graph $\overrightarrow{T}_{11}$.

Lemma 6.6 (Marshall [29]). *For every set $S \subset V(T_{11})$ with $|S| = 4$, we have $|outdegree(S)| \geq 10$ and $|indegree(S)| \geq 10$. In other words both oriented neighborhoods of S contain all colors with possible one exception.*

Lemma 6.7. *Consider the comb R_4, see Figure 6.3, with arbitrary orientation $\overrightarrow{R}$ and suppose that the first column of $\overrightarrow{R}$, i.e. vertices u_1, u_2, u_3, u_4, is already colored, by colors (c_1, c_2, c_3, c_4) in such a way that $c_i \neq c_{i+1}$, for $1 \leq i \leq 3$. Then there are at least two colors available for v_1, such that for each of them we can obtain four different colors for v_4 in an oriented coloring.*

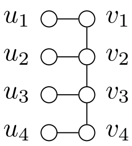

FIGURE 6.3: Comb R_4.

Proof. By computer. □

Suppose that i and j are positive integers. Consider the star $K_{1,i}$ with the set of vertices $V(K_{1,i}) = \{x, v_1, v_2, \ldots, v_i\}$ and edges of the form $\{x, v_k\}$ for $1 \leq k \leq i$.

Lemma 6.8. *Suppose that:*

- *$\overrightarrow{K}$ is an orientation of the star $K_{1,2}$,*
- *S is any subset of the vertex set of $\overrightarrow{T}_{11}$ of cardinality 4 whose elements correspond to available colors for the vertex v_1 of $\overrightarrow{K}$,*
- *and $c(v_2)$ is a color chosen for the vertex v_2.*

Then there are at least four different ways to color x - the universal vertex of the star.

Proof. It follows from Lemma 6.6 and the fact that for any vertex $u \in \overrightarrow{T}_{11}$ we have $|outdegree(u)| = |indegree(u)| = 5$. □

Lemma 6.9. *Suppose that:*

- $\overrightarrow{K}$ *is an orientation of the star* $K_{1,3}$
- S *is any subset of the vertex set of* $\overrightarrow{T}_{11}$ *of cardinality* 4 *whose elements correspond to available colors for the vertex* v_1 *of* $\overrightarrow{K}$*,*
- *and* $c(v_2)$ *and* $c(v_3)$ *are any two different colors for vertices* v_2 *and* v_3*.*

Then we can color x *- the universal vertex of the star.*

Proof. It follows from Lemma 6.6. □

Now we shall prove Theorem 6.1.

Proof of Theorem 6.1. Since $\overrightarrow{H}_{10}$ is a subgraph of $\overrightarrow{T}_{11}$, Theorem 6.3 implies that cylindrical grids with circuit $m = 3, 4$ can be colored by $\overrightarrow{T}_{11}$. Let $m \geq 5$ and let $\overrightarrow{Y}$ be any orientation of cylindrical grid $Cyl(m, n) = C_m \square P_n$. We identify each vertex $u \in \overrightarrow{Y}$ with the pair of its coordinates (i, j), $1 \leq i \leq m$, $1 \leq j \leq n$. We shall show that $\overrightarrow{Y}$ can be colored by $\overrightarrow{T}_{11}$. We color the vertices of $\overrightarrow{Y}$ row by row. For the first row, clearly, it is always possible to color any oriented cycle by homomorphism to $\overrightarrow{T}_{11}$ using Lemma 2.22. Now, suppose that $i > 1$ and the rows from 1 to $i - 1$ are already colored. By Lemma 6.7, there are two possible colors for vertex $(1, i)$, such that each of them ensures four possibilities of colors for vertex $(4, i)$. We choose the one which is different from the color of vertex $(m, i - 1)$. Let us denote by $S(j)$ the set of possible colors for vertex (j, i), for $2 \leq j \leq 4$. Now for each $j = 5, 6, \ldots, m - 1$ we define set $S(j)$ as a result of applying Lemma 6.8 for the set $S(j - 1)$ and the color of $(j, i - 1)$. By Lemma 6.9, we can color the vertex (m, i). Now we can color vertices $(2, i)$, $(3, i)$, ..., $(m - 1, i)$ in reverse order. First, for the vertex $(m - 1, i)$ we choose the color $c(m - 1, i) \in S(m - 1)$ such that the orientation of the edge $\{c((m - 1, i)), c((m, i))\}$ in $\overrightarrow{T}_{11}$ equals to the orientation of the edge $\{(m - 1, i), (m, i)\}$ in $\overrightarrow{Y}$. Consecutive vertices we color in the same way. Similarly, we color the following rows. □

6.2 Cylindrical grids $Cyl(m, n)$ with $m = 3, 4, 5, 6, 7$

In order to proof Theorem 6.3 we use a computer. We have designed an algorithm, similar to the one described in Chapter 4, that finds a set S that is cycle-closed under extension.

Definition 6.10. For $m \geq 3$, the m-sunlet graph Sun_m is an undirected graph with the set of vertices $V(Sun_m) = \{(1,1), \ldots, (m,1), (1,2), \ldots, (m,2)\}$ and edges of the form $\{(i,1),(i,2)\}$ for $1 \leq i \leq m$, or $\{(i,2),(i+1,2)\}$ for $1 \leq i < m$, or $\{(m,2),(1,2)\}$; see Figure 6.4.

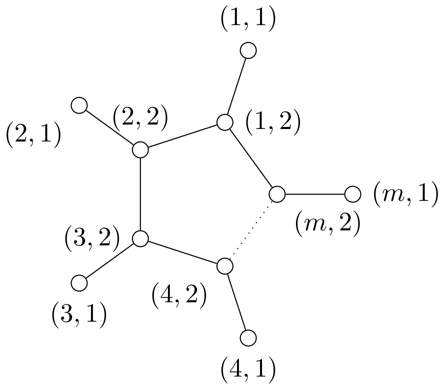

FIGURE 6.4: m-sunlet graph.

Definition 6.11. Let $\overrightarrow{H}$ be a coloring graph. A set $S_m(\overrightarrow{H}) \subseteq (V(\overrightarrow{H}))^m$ is cycle-closed under extension if:

(a) for every orientation $\overrightarrow{C}$ of the cycle $C_m = (v_1, \ldots, v_m)$, there exists a coloring $c : \overrightarrow{C} \to \overrightarrow{H}$ such that $(c(v_1), \ldots, c(v_m)) \in S_m(\overrightarrow{H})$,

(b) for every orientation $\overrightarrow{Sun}$ of the m-sunlet graph Sun_m and for every sequence $(c_1, \ldots, c_m) \in S_m(\overrightarrow{H})$, there exists a coloring $c : \overrightarrow{Sun} \to \overrightarrow{H}$ such that:

(1) $(c(1,1), \ldots, c(m,1)) = (c_1, \ldots, c_m)$ and

(2) $(c(1,2), \ldots, c(m,2)) \in S_m(\overrightarrow{H})$

Lemma 6.12 (Nenca [35])**.** *For each $m = 3, 4, 5, 6{,}7$, there exists a nonempty set $S_m(\overrightarrow{H}_{10}) \subseteq (V(\overrightarrow{H}_{10}))^m$, which is cycle-closed under extension.*

Proof. In order to prove the lemma we use a computer. We have designed an algorithm, similar to the Algorithm ComputeSet$S$ described in Chapter 4, which finds a set cycle-closed under extension. The algorithm, for a given m, uses the m-sunlet Sun_m instead of a comb R_8. Using the algorithm we have found that for each $m = 3, \ldots, 7$, there exists a nonempty set cycle-closed under extension. □

Lemma 6.13 (Nenca [35])**.** *For each $m = 3, \ldots, 7$ every orientation $\overrightarrow{Y}$ of the cylindrical grid $Cyl(m,n)$ can be colored by $\overrightarrow{H}_{10}$.*

Proof. For a given m, let $\overrightarrow{Y}$ be an arbitrary given orientation of $Cyl(m,n)$. For $i \leq n$, we denote the induced subgraph of $\overrightarrow{Y}$ that is formed by the first i cycles by $\overrightarrow{Y}(i)$. It is easy to show by induction that for every i, there is a coloring $c : \overrightarrow{Y}(i) \to \overrightarrow{H}_{10}$ such that $c(1,i),\ldots,c(m,i)) = s$ for some $s \in S_m$. $\square$

6.3 Toroid

In Section 2.3 we describe the Tromp graphs, see Definition 2.25. Consider the Tromp graph $\overrightarrow{T}_{24} = \overrightarrow{Tr}(\overrightarrow{T}_{11})$. Applying Lemma 2.21 and Lemma 2.27 we have:

Lemma 6.14.

(a) The Paley tournament $\overrightarrow{T}_{11}$ has properties $P(1,5)$, $P(2,2)$.

(b) The Tromp graph $\overrightarrow{Tr}(\overrightarrow{T}_{11})$ has properties $P_c(2,5)$, $P_c(3,2)$.

Proof of Theorem 6.4. Let $\overrightarrow{T}$ be any orientation of a toroid $T(m,n)$. We identify each vertex $u \in \overrightarrow{T}$ with the pair of its coordinates (i,j), $1 \le i \le m$, $1 \le j \le n$.

Consider the cylinder $\overrightarrow{C}$ obtained from $\overrightarrow{T}$ by removing the vertices $(i,n-1)$ and (i,n), for $1 \le i \le m$. By Theorem 6.1 there exists a coloring $c : \overrightarrow{C} \to \overrightarrow{T}_{11}$. We construct the coloring of $\overrightarrow{T}$ in following way:

Step 1 color vertices (i,j), $1 \le i \le m$, $1 \le j \le n-2$ by $\overrightarrow{T}_{11}$,

Step 2 color vertices $(1,n-1)$ and $(1,n)$ by $\overrightarrow{T}_{24}$, see Figure 6.5, in such a way that:

- colors of vertices $(1,n-1)$ and $(2,n-2)$ are compatible in the star $\{(2,n-1),(2,n-2),(1,n-1)\}$
- colors of vertices $(1,n-1)$ and $(m,n-2)$ are compatible in the star $\{(m,n-1),(m,n-2),(1,n-1)\}$
- colors of vertices $(1,n-1)$ and $(1,1)$ are compatible in the star $\{(1,n),(1,n-1),(1,1)\}$
- colors of vertices $(1,n)$ and $(2,1)$ are compatible in the star $\{(2,n),(2,1),(1,n)\}$
- colors of vertices $(1,n)$ and $(m,1)$ are compatible in the star $\{(m,n),(m,1),(1,n)\}$

Step 3 color vertices $(2,n-1)$ and $(2,n)$ by $\overrightarrow{T}_{24}$, see Figure 6.5.

In order to color $(2,n-1)$, consider the star $\{(2,n-1),(1,n-1),(2,n-2)\}$, see Figure 6.5. Colors in the leaves of the star are compatible. Indeed: the color in $(1,n-1)$ has been chosen to be compatible with the color of $(2,n-2)$. Hence, by Property $P_c(2,5)$, we have at least five colors to color $(2,n-1)$. We choose one which is compatible with

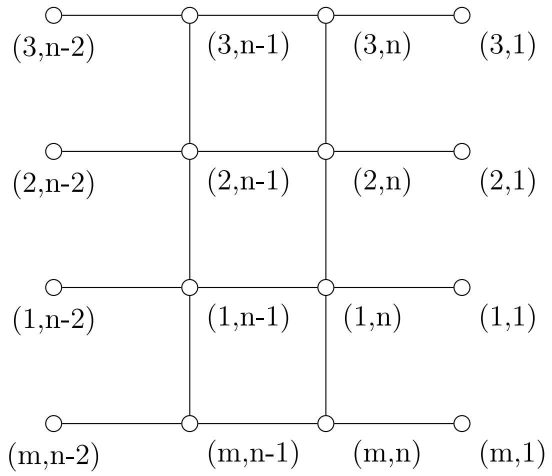

FIGURE 6.5: Vertices of toroid $\overrightarrow{T}$

- the color of $(3, n-2)$ in the star $\{(3, n-1), (3, n-2), (2, n-1)\}$,
- the colors of $(2, 1)$ and $(1, n)$ in the star $\{(2, n), (2, 1), (2, n-1), (1, n)\}$,

To color vertex $(2, n)$ consider the star $\{(2, n), (2, 1), (2, n-1), (1, n)\}$, see Figure 6.5. Colors in the leaves are compatible, so by Property $P_c(3, 2)$ we have at least two ways to color vertex $(2, n)$. We choose the one which is compatible with the color of vertex $(3, 1)$ in the star $\{(3, n), (3, 1), (2, n)\}$.

Step 4 In the same way we color vertices $(j, n-1)$ and (j, n), for $3 \leq j \leq m-2$.

Step 5 Consider last four vertices of $\overrightarrow{T}$, namely $a = (m-1, n-1)$, $b = (m-1, n)$, $c = (m, n-1)$ and $d = (m, n)$, see Figure 6.6. Consider the star $\{a, (m-1, n-2), (m-2, n-1)\}$. The colors in leaves are compatible, so using Property $P_c(2, 5)$ there are at least five colors to color vertex a. We choose the one which is compatible:

- with colors of vertices $(m, n-2)$ and $(1, n-1)$ in the star $\{c, (1, n-1), (m, n-2), a\}$ and
- with colors of vertices $(m-1, 1)$ and $(m-2, n)$ in the star $\{b, (m-1, 1), (m-2, n), a\}$

We have used 24 colors so far. For vertices b, c, d we use three additional colors.

□

In order to prove Lemma 6.5 first we prove the following lemma:

Lemma 6.15. $\overrightarrow{\chi}(C_m \square C_n) \leq \overrightarrow{\chi}(C_m \square P_{n-2}) + 2m$, *for* $m, n \geq 3$.

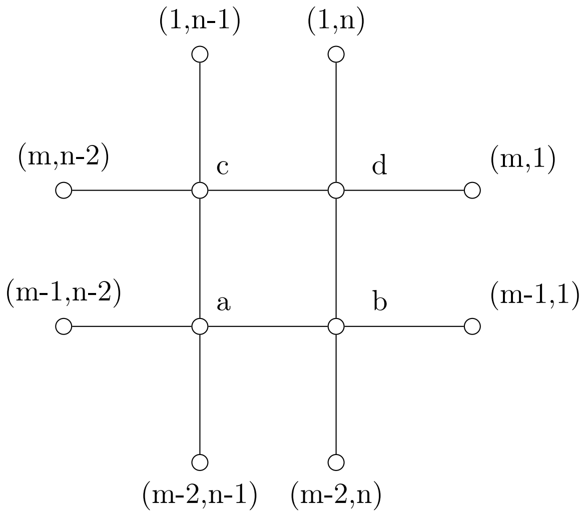

FIGURE 6.6: Last four vertices of $\overrightarrow{T}$

Proof. The proof resembles the Proof of Theorem 6.4. Let $\overrightarrow{T}$ be any orientation of a toroid $T(m,n)$. Once again, we identify each vertex $u \in \overrightarrow{T}$ with the pair of its coordinates (i,j), $1 \leq i \leq m$, $1 \leq j \leq n$. Consider the cylinder $\overrightarrow{C}$ obtained from $\overrightarrow{T}$ by removing the vertices $(i, n-1)$ and (i,n), for $1 \leq i \leq m$. Let $\overrightarrow{H}$ be a coloring graph, such that there exists a coloring $c : \overrightarrow{C} \to \overrightarrow{H}$. We construct the coloring of $\overrightarrow{T}$ in following way:

Step 1 color vertices (i,j), $1 \leq i \leq m$, $1 \leq j \leq n-2$ by $\overrightarrow{H}$,

Step 2 color vertices $(j, n-1)$ and (j,n), for $1 \leq j \leq m$ with additional $2m$ colors.

□

Proof of Theorem 6.5.
It is easy to see that nine colors are sufficient to color any orientation of $T(3,3)$. On the other hand, it is tedious but easy task to check that there exists an orientation of $T(3,3)$ such that any two distinct vertices u, v of $T(3,3)$ are connected by a direct path of length at most 2. The remaining cases follow from Lemma 6.15 and Lemma 6.3. □

Chapter 7

The oriented chromatic number of strong-grids

In this chapter, we improve the bounds for the oriented chromatic number of the family $\mathcal{G}^{\boxtimes}$ of all strong-grids by showing that:

Theorem 7.1. $11 \leq \overrightarrow{\chi}(\mathcal{G}^{\boxtimes}) \leq 88$.

Moreover, we show that:

Theorem 7.2 (Dybizbański and Nenca (2017))**.** *There exists an orientation of the strong-grid $G^{\boxtimes}(2, 398)$ that cannot be colored by any coloring graph with ten vertices.*

This gives the exact value for the oriented chromatic number of the strong-grids with two rows.

Corollary 7.3. $\overrightarrow{\chi}(\mathcal{G}_2^{\boxtimes}) = 11$.

Furthermore, we give new upper bounds for the oriented chromatic numbers of the family of strong-grids with three or four rows by showing that:

Theorem 7.4 (Dybizbański and Nenca (2017))**.** *Every orientation of every strong-grid with 3 rows can be colored by the Paley tournament $\overrightarrow{T}_{19}$.*

Corollary 7.5. $11 \leq \overrightarrow{\chi}(\mathcal{G}_3^{\boxtimes}) \leq 19$.

Theorem 7.6. *Every orientation of every strong-grid with 4 rows can be colored by a coloring graph with 38 vertices.*

Corollary 7.7. $11 \leq \overrightarrow{\chi}(\mathcal{G}_4^{\boxtimes}) \leq 38$.

Similarly as in Chapter 2 we define a strong-comb.

Definition 7.8. The strong-comb $R_m^{\boxtimes}$ is an undirected graph with the set of vertices $V(R_m^{\boxtimes}) = \{(1,1),\ldots,(m,1),(1,2),\ldots,(m,2)\}$ and edges of the form $\{(i,1),(i,2)\}$, for $1 \le i \le m$, and edges of the form $\{(i,2),(i+1,2)\}$, $\{(i,1),(i+1,2)\}$, or $\{(i+1,1),(i,2)\}$, for $1 \le i < m$; see Figure 7.1. The vertices $(1,1),\ldots,(m,1)$ form the first column of the comb while $(1,2),\ldots,(m,2)$ form the second column.

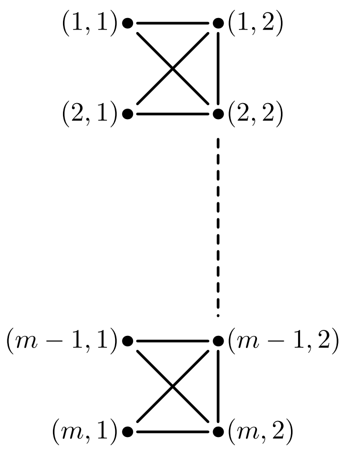

FIGURE 7.1: The strong-comb $R_m^{\boxtimes}$.

Let $\overrightarrow{H}$ be a coloring graph, s be a sequence $s = (s_1,\ldots,s_m) \in V(\overrightarrow{H})^m$ and $\overrightarrow{R}^{\boxtimes}$ be an orientation of the strong-comb $R_m^{\boxtimes}$. By $NEXT(s,\overrightarrow{R}^{\boxtimes},\overrightarrow{H})$ we define the set of reachable colorings of the second column of the orientation $\overrightarrow{R}^{\boxtimes}$, when the vertices of the first column of $\overrightarrow{R}^{\boxtimes}$ are colored by s. Furthermore, $NEXT(S,\overrightarrow{R}^{\boxtimes},\overrightarrow{H}) = \bigcup_{s\in S} NEXT(s,\overrightarrow{R}^{\boxtimes},\overrightarrow{H})$. For an orientation $\overrightarrow{G}^{\boxtimes}$ of a strong-grid $G^{\boxtimes}$, and a coloring graph $\overrightarrow{H}$, we define:

- $S(\overrightarrow{G}^{\boxtimes},\overrightarrow{H})$ the set of reachable colorings of the last column of $\overrightarrow{G}^{\boxtimes}$,
- $S^{\boxtimes}(\overrightarrow{H}) = \{S(\overrightarrow{G}^{\boxtimes},\overrightarrow{H}) : \overrightarrow{G}^{\boxtimes} \in \mathcal{G}^{\boxtimes}\}$,
- $\mathcal{S}_m^{\boxtimes}(\overrightarrow{H})$ the family $\mathcal{S}_m^{\boxtimes}(\overrightarrow{H}) = \{S(\overrightarrow{G}^{\boxtimes},\overrightarrow{H}) : \overrightarrow{G}^{\boxtimes} \in \mathcal{G}_m^{\boxtimes}\}$.

7.1 Upper bound for $\overrightarrow{\chi}(G^{\boxtimes}(m,n))$

Consider the Paley tournament $\overrightarrow{T}_{43}$. Applying Lemma 2.21 we have:

Corollary 7.9. *The Paley tournament $\overrightarrow{T}_{43}$ has properties $P(1,21)$, $P(2,10)$.*

Lemma 7.10 ([14])**.** *The smallest Paley tournament with the property $P(3,3)$ is $\overrightarrow{T}_{43}$.*

Consider the Tromp graph $\overrightarrow{T}_{88} = \overrightarrow{Tr}(\overrightarrow{T}_{43})$, see Definition 2.25. Applying Lemma 2.21 and Lemma 2.27 we have:

Corollary 7.11. *The Tromp graph $\overrightarrow{Tr}(\overrightarrow{T}_{43})$ has properties $P_c(2,21)$, $P_c(3,10)$, $P_c(4,3)$.*

Lemma 7.12. *Let $(u,v) \in A(\overrightarrow{Tr}(\overrightarrow{G}))$ be an arc. Then neither $u = v$ nor u and v are twin vertices.*

Corollary 7.13. *Suppose that x and y form an arc in a grid, they are leaves in an oriented star $\overrightarrow{K}$, and they are already colored by a Tromp graph $\overrightarrow{Tr}(\overrightarrow{G})$, then the colors $c(x)$ and $c(y)$ are compatible.*

The upper bound from Theorem 7.1 follows from the following theorem:

Theorem 7.14. *Every orientation of every strong-grid $G^{\boxtimes}(m,n)$ can be colored by a coloring graph with 88 vertices, namely, by the Tromp graph $\overrightarrow{Tr}(\overrightarrow{T}_{43})$.*

Proof. Let $\overrightarrow{G}^{\boxtimes}$ be any orientation of strong-grid $G^{\boxtimes}(m,n)$. We identify each vertex $u \in \overrightarrow{G}^{\boxtimes}$ with the pair of its coordinates (i,j), $1 \le i \le m$, $1 \le j \le n$.

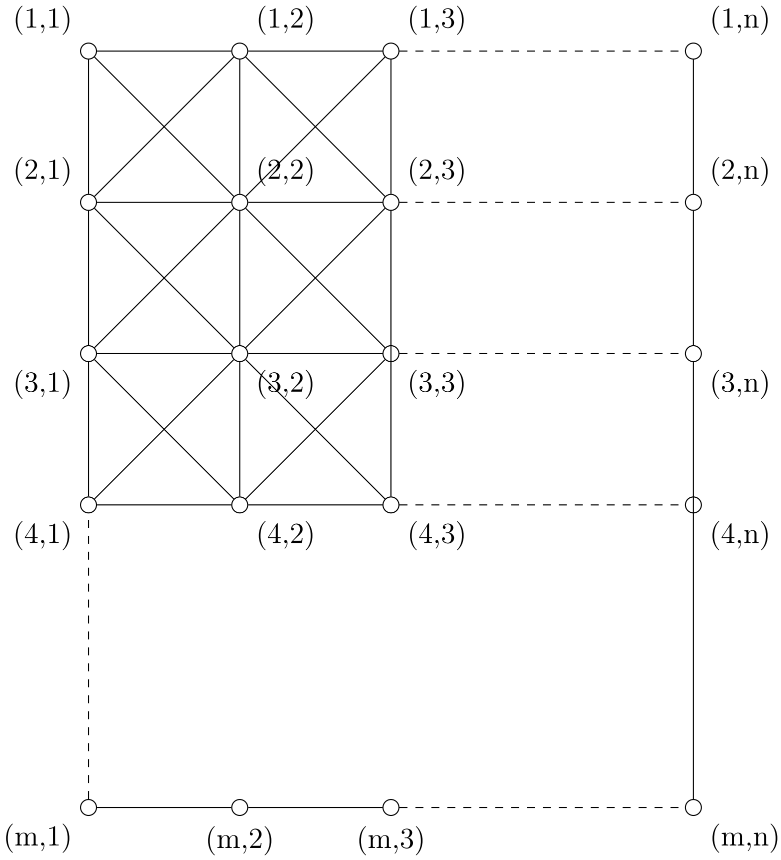

FIGURE 7.2: Strong-grid $G^{\boxtimes}(m,n)$.

We shall show that $\overrightarrow{G}^{\boxtimes}$ can be colored by $\overrightarrow{T}_{88} = \overrightarrow{Tr}(\overrightarrow{T}_{43})$. We color the vertices of $\overrightarrow{G}^{\boxtimes}$ column by column in such a way that colors $c(i,j)$ and $c(i+2,j)$ are compatible in the star $(i+1, j+1), (i, j), (i+2, j)$. For the first column, clearly, it is always possible to color any oriented path by homomorphism to $\overrightarrow{T}_{88}$ in such a way, that the vertices $(i, 1)$ and $(i+2, 1)$ has colors compatible in the star $\{(i+1, 2), (i, 1), (i+2, 1)\}$. Now, suppose that $j > 1$ and the columns from 1 to $j-1$ are already colored in this way. In order to color the vertex $(1, j)$ consider the star $\{(1, j), (1, j-1), (2, j-1)\}$, see Figure 7.3. Vertices $(1, j-1)$ and $(2, j-1)$ are neighbors, hence by Corollary 7.13, their colors are

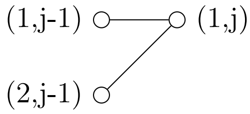

FIGURE 7.3: Star $\{(1, j), (1, j-1), (2, j-1)\}$.

compatible. Thus, by Property $P_c(2, 21)$, there are 21 colors available for $(1, j)$, and we can choose a color which is compatible with the colors of vertices $(2, j-1)$ and $(3, j-1)$ in the star $\{(2, j), (1, j), (1, j-1), (2, j-1), (3, j-1)\}$, see Figure 7.4.

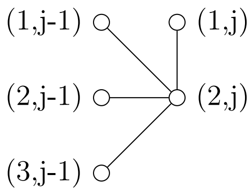

FIGURE 7.4: Star $\{(2, j), (1, j), (1, j-1), (2, j-1), (3, j-1)\}$.

In order to color $(2, j)$, consider the star $\{(2, j), (1, j), (1, j-1), (2, j-1), (3, j-1)\}$, see Figure 7.4. Colors in the leaves of the star are compatible. Indeed: colors in the pair $(i, j-1), (i+2, j-1)$ are compatible, by induction hypothesis. The color in $(1, j)$ has been chosen to be compatible with the color of $(3, j-1)$, and colors in every other pair are compatible, because it (the pair) forms an arc. Hence, by Property $P_c(4, 3)$, we have at least three colors to color $(2, j)$. We choose one which is compatible with the color of $(4, j-1)$ in the star $\{(3, j), (2, j), (2, j-1), (3, j-1), (4, j-1)\}$, see Figure 7.5.

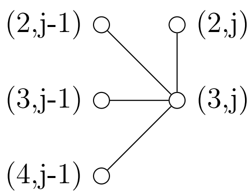

FIGURE 7.5: Star $\{(3, j), (2, j), (2, j-1), (3, j-1), (4, j-1)\}$.

To color $(3, j)$, consider the star $\{(3, j), (2, j), (2, j-1), (3, j-1), (4, j-1)\}$, see Figure 7.5. Colors in the leaves are compatible, so by Property $P_c(4, 3)$, we have at least three colors to color $(3, j)$. We choose the one which is compatible

- with the colors of vertex $(5, j-1)$ in the star $\{(4, j), (3, j), (3, j-1), (4, j-1), (5, j-1)\}$, see Figure 7.6, and

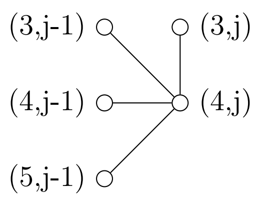

FIGURE 7.6: Star $\{(4, j), (3, j), (3, j-1), (4, j-1), (5, j-1)\}$.

- with the color of vertex $(1, j)$ in the star $\{(2, j+1), (1, j), (3, j)\}$, see Figure 7.7.

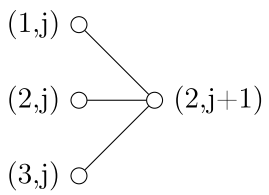

FIGURE 7.7: Star $\{(2, j+1), (1, j), (3, j)\}$.

We continue this way to color vertices $(4, j), \ldots, (m, j)$ and similarly we color the next columns.

$\square$

7.2 Strong-grids with two rows

First we show that every orientation $\overrightarrow{G}_1$ of $G^{\boxtimes}(2, n)$ is isomorphic to a grid $\overrightarrow{G}_2$ which has all vertical arc oriented downwards. In order to show this, suppose in $\overrightarrow{G}_1$ we have an arc $((2, i), (1, i))$ for some $1 \leq i \leq n$, and consider the orientation $\overrightarrow{G}_2$ such that $((1, i), (2, i)) \in A(\overrightarrow{G}_2)$ and for every common neighbor x of $(1, i)$ and $(2, i)$

$$(x, (1, i)) \in A(\overrightarrow{G}_2) \Longleftrightarrow (x, (2, i)) \in A(\overrightarrow{G}_1)$$

and

$$(x, (2, i)) \in A(\overrightarrow{G}_2) \Longleftrightarrow (x, (1, i)) \in A(\overrightarrow{G}_1).$$

All other arcs in $\overrightarrow{G}_2$ are the same as in $\overrightarrow{G}_1$. It is easy to see that the function $f : \overrightarrow{G}_1 \to \overrightarrow{G}_2$, where

$$f(u) = \begin{cases} (2,i) & \text{if } u = (1,i), \\ (1,i) & \text{if } u = (2,i), \\ u & \text{otherwise} \end{cases}$$

is an isomorphism. Using this observations we can reverse other arcs in $\overrightarrow{G}_1$ one by one.

Proof of Theorem 7.2.
Using nauty [30] we generate a sequence $\overrightarrow{H}_1, \ldots, \overrightarrow{H}_M$, where $M = 9733056$, of all non-isomorphic tournaments with ten vertices. By the remarks above, we can consider only those orientation of $G^{\boxtimes}(2,n)$ with all vertical arcs of the form $((1,i),(2,i))$.
To construct an orientation of a strong-grid that cannot be colored by ten colors we use the following algorithm:

Algorithm
Output: Strong-grid $\overrightarrow{G}^{\boxtimes}_M$ not colorable by any tournament with ten colors.

1. $\overrightarrow{G}_0$ `is the grid with` $V(\overrightarrow{G}_0) = \{(1,1),(2,1)\}$ `and` $A(\overrightarrow{G}_0) = ((1,1),(2,1))$
2. `for i := 1 to M`
3. $\quad\overrightarrow{G}_i := \mathsf{E}xtend(\overrightarrow{G}_{i-1}, \overrightarrow{H}_i)$
4. `return` $\overrightarrow{G}_M$

The function Extend uses a queue Q. It starts with an empty Q and computes the set $S = S(\overrightarrow{G}_{i-1}, \overrightarrow{H}_i)$. If the set S is not empty then it is put to the queue Q. Next, the algorithm takes one by one a set S from the queue Q and for every orientation $\overrightarrow{R}^{\boxtimes}$ of the strong-comb $R^{\boxtimes}_2$, computes the set $S' = NEXT(S, \overrightarrow{R}^{\boxtimes}.\overrightarrow{H})$ and adds S' to the queue, provided it is new. Moreover, the function puts to an additional memory the triple: the set S, the orientation $\overrightarrow{R}^{\boxtimes}$ and the set S'. The function stops when the empty set of colorings is reached. After this the function reconstructs a grid $\overrightarrow{G}_i$, such that $S(\overrightarrow{G}_i, \overrightarrow{H}_i) = \emptyset$ and $\overrightarrow{G}_i$ is an extension of $\overrightarrow{G}_{i-1}$. To do this the function uses the information kept in the additional memory.

Using the algorithm we have found the orientation $\overrightarrow{G}^{\boxtimes}_*$ of $G^{\boxtimes}(2,398)$ that cannot be colored with ten colors. The orientation $\overrightarrow{G}^{\boxtimes}_*$ is posted on the website `https://inf.ug.edu.pl/grids/`. We can obtain a grid of different size, when we change the order of graphs $\overrightarrow{H}_1, \ldots, \overrightarrow{H}_M$.

□

7.3 Strong-grids with three rows

Proof of Theorem 7.4.

We use an algorithm, similar to the algorithm described in Section 5.1 to show that any orientation $\overrightarrow{G}^{\boxtimes} \in \mathcal{G}_3^{\boxtimes}$ can be colored by the Paley tournament $\overrightarrow{T}_{19}$. We show that the empty set is not reachable on the last column of any $\overrightarrow{G}^{\boxtimes} \in \mathcal{G}_3^{\boxtimes}$.

Since the Paley tournament $\overrightarrow{T}_{19}$ is arc-transitive and self-converse, we can store in memory only those orientations $\overrightarrow{G}^{\boxtimes}$ of $G^{\boxtimes}(3,n)$ where $((1,n),(2,n)) \in A(\overrightarrow{G}^{\boxtimes})$ and only those colorings where $c(1,n) = 0$ and $c(2,n) = 1$. Thus, there are only two orientations of the last column and only nine colorings of the last column.

The algorithm starts with the path $P_3 = G^{\boxtimes}(3,1)$. For every orientation $\overrightarrow{P}$ of P_3, such that $((1,1),(2,1)) \in \overrightarrow{P}$, it computes the set of reachable colorings $S(\overrightarrow{P}, \overrightarrow{T}_{19})$. The algorithm uses queue to store such sets. Next the algorithm takes from the queue one by one a set T and, for every orientation $\overrightarrow{R}^{\boxtimes}$ of the strong-comb $R_3^{\boxtimes}$, computes the set $T' = NEXT(T, \overrightarrow{R}^{\boxtimes}, \overrightarrow{T}_{19})$. The set T' is added to the queue if it is new. After running the algorithm, we found that the empty set of colorings of the last column is not reachable. $\square$

7.4 Strong-grids with four rows

We use an algorithm similar to algorithm described in Chapter 4 to show that every orientation of every strong-grid with four rows can be colored by 38 colors.

Definition 7.15. Let $\overrightarrow{G}$ be an oriented graph and $\overrightarrow{G}'$ be an isomorphic copy of $\overrightarrow{G}$. For every $u \in V(\overrightarrow{G})$ by $u' \in V(\overrightarrow{G}')$ we denote the isomorphic copy of u. The *push-graph* $\overrightarrow{P}(\overrightarrow{G})$ has $2|V(\overrightarrow{G})|$ vertices and is defined as follows (see [25]):

- $V(\overrightarrow{P}(\overrightarrow{G})) = V(\overrightarrow{G}) \cup V(\overrightarrow{G}')$,
- $\forall_{u,v \in V(\overrightarrow{G}),\, (u,v) \in A(\overrightarrow{G})} (u,v), (u',v'), (v,u'), (v',u) \in A(\overrightarrow{P}(\overrightarrow{G}))$,

see Figure 7.8. The vertices $u \in V(\overrightarrow{G})$ and $u' \in V(\overrightarrow{G}')$ are called *twin vertices.*

Consider an oriented graph $\overrightarrow{H}_{38} = \overrightarrow{P}(\overrightarrow{T}_{19})$ and the set

$$S = \{(c_1, c_2, c_3, c_4) : c_1, \ldots, c_4 \in V(\overrightarrow{H}_{38}),\ c_i \neq c_{i+1} \text{ and } c_i \neq c'_{i+1} \text{ for } 1 \leq i < 4 \text{ and } c_i \neq c_{i+2} \text{ and } c_i \neq c'_{i+2} \text{ for } 1 \leq i \leq 2\}.$$

Using a computer, we have checked that the set S has the following properties

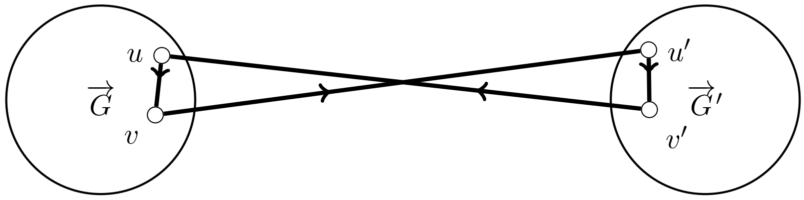

FIGURE 7.8: Push-graph $\overrightarrow{P}(\overrightarrow{G})$.

(a) for every orientation $\overrightarrow{P}$ of path $P_4 = (v_1, \ldots, v_4)$, there exists a coloring $c : \overrightarrow{P} \to \overrightarrow{H}_{38}$ such that $(c(v_1), \ldots, c(v_4)) \in S$,

(b) for every orientation $\overrightarrow{R}^{\boxtimes}$ of the strong-comb $R_4^{\boxtimes}$ and every sequence $(c_1, \ldots, c_4) \in S$, there exists a coloring $c : \overrightarrow{R}^{\boxtimes} \to \overrightarrow{H}_{38}$ such that:

(1) $(c(1,1), \ldots, c(4,1)) = (c_1, \ldots, c_4)$ and

(2) $(c(1,2), \ldots, c(4,2)) \in S$.

Theorem 7.6 follows from the following lemma:

Lemma 7.16. *For every orientation $\overrightarrow{G}^{\boxtimes}$ of $G^{\boxtimes}(4, n)$ there exists a coloring $c : \overrightarrow{G}^{\boxtimes} \to \overrightarrow{H}_{38}$.*

Chapter 8

Signed coloring

In this chapter we consider signed chromatic number of grids. We show that:

Theorem 8.1 (Dybizbański, Nenca, Szepietowski (2020) [13])**.**
$5 \leq \chi_s(\mathcal{G}) \leq 6$.

The upper bound 6 follows from Theorem 1.16 and the fact that grids are acyclic 3-colorable. However, we think that our proof of the upper bound is much more direct and constructive. Moreover, we show that:

Theorem 8.2 (Dybizbański, Nenca, Szepietowski (2020) [13])**.**
Every signed grid with at most seven rows can be colored by the Paley graph SP_5.

Corollary 8.3. *$\chi_s(G(m,n)) = 5$, for $3 \leq m \leq 7$ and $n \geq 4$.*

Furthermore, we show that:

Theorem 8.4 (Dybizbański, Nenca, Szepietowski (2020) [13])**.**

(a) $\chi_s(G(2,n)) = 4$ *for* $n \geq 2$ *and*

(b) $\chi_s(G(3,3)) = 4$.

8.1 Definitions

Consider a grid $G(m,n)$. For each pair x, y, with $0 \leq x \leq m-2$ and $0 \leq y \leq n-2$, let $C_{x,y}$ denote the cycle which consists of the vertices: (x,y), $(x,y+1)$, $(x+1,y+1)$, $(x+1,y)$. We shall call $C_{x,y}$ a square. Suppose that σ is a signature of the grid $G(m,n)$. We say that the square $C_{x,y}$ *is balanced* if it has an even number of "$-$" edges, and unbalanced otherwise. The type of the signed grid $(G(m,n),\sigma)$ is the array $[g_{x,y}]$ with $g_{x,y}=b$ if the square $C_{x,y}$ is balanced, and $g_{x,y}=u$ otherwise, where $0\le x\le m-2$, $0\le y\le n-2$.

Lemma 8.5 ([32])**.** *If a graph G has M edges, N vertices, and C components, then there are 2^{M-N+C} distinct equivalent classes of signed graphs on G.*

Lemma 8.6. *Two signed grids are equivalent if and only if their types are equal.*

Proof. The resigning does not change the type of the grid. Hence, two equivalent grids are of the same type. On the other hand, there are $2^{(m-1)(n-1)}$ types and $2^{(m-1)(n-1)}$ equivalent classes, hence, grids with the same type are equivalent. □

Corollary 8.7. *Two signed grids (G,σ) and $(G,-\sigma)$ are equivalent.*

8.2 Signed coloring graphs

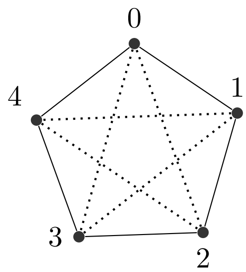

FIGURE 8.1: Coloring graph SP_5. Dotted (solid, respectively) edges have sign "−" ("+", respectively).

Definition 8.8. The signed graph SP_5 with the set of vertices $\{0,1,2,3,4\}$ with

$$\sigma(uv)=\begin{cases} - & \text{if} \quad v-u=2 \text{ or } 3 \pmod 5,\\ + & \text{otherwise.}\end{cases}$$

is called the *signed Paley graph SP_5*.

Lemma 8.9. *For each $a\in\{1,4\}$ and $b\in\{0,1,2,3,4\}$, the function $h(x)=a\cdot x+b \pmod 5$ is an automorphism in SP_5.*

Lemma 8.10. *For every edge $(u,v)\in SP_5$ signed by "+", there exists an automorphism h such that $h(u,v)=(0,1)$. For every edge (u,v) signed by "−", there exists an automorphism h such that $h(u,v)=(0,2)$.*

Lemma 8.11. *Every two different vertices $u,v\in V(SP_5)$ can be connected by one balanced and two unbalanced paths of length two. A path of length two is unbalanced if its edges have different signs, and is balanced otherwise.*

8.3 Signed grids with at most seven rows

In this section, we prove the Theorem 8.2. We show that every signed grid $[G, \sigma]$ with at most seven rows can be colored by the signed Paley graph SP_5; see Figure 8.1. The proof is computer aided. Consider a signed grid $[G, \sigma]$ with m rows. We say that a vector $t = (t_1, \ldots, t_m) \in \{0, 1, 2, 3, 4\}^m$ is reachable on the signed grid $[G, \sigma]$ if there is a presentation (G, σ) of $[G, \sigma]$ and a coloring of (G, σ) by SP_5, such that t is the coloring of the last column of (G, σ). The set

$$S_m([G, \sigma]) = \{t \in \{0, 1, 2, 3, 4\}^m : t \text{ is reachable on } [G, \sigma]\}$$

is called the set (of colorings) reachable on $[G, \sigma]$. We have designed an algorithm, similar to the algorithm described in Chapter 5 which for given natural m, $5 \leq m \leq 7$, generates all sets reachable on signed grids with m rows. We have found that the empty set of colorings is not reachable, for $m = 5, 6, 7$.
Let t be a sequence $t = (t_1, \ldots, t_m) \in \{0, \ldots, 4\}^m$ and (R, σ) be a presentation of a signed comb with $R = R_m$.

By $NEXT(t, (R, \sigma))$ we donote the set of reachable colorings of the second column of the comb (R, σ), when the vertices of the first column of (R, σ) are colored by t.

By $Type(t, (R, \sigma)) \in \{b, u\}^{m-1}$ we define the type of signed grid $(G(m, 2), \lambda)$, which is obtained from (R, σ) by adding signs to edges $\{(i, 1), (i + 1, 1)\}$, in such a way that the sign of each edge $\{(i, 1), (i + 1, 1)\}$ in the grid $(G(m, 2), \lambda)$ equals to the sign of the edge $\{t_i, t_{i+1}\}$ in SP_5.

We give a simple version of an algorithm for $m = 5$, namely $\mathsf{SimpleGenerateSets}_5$.

Algorithm $\mathsf{SimpleGenerateSets}_5$
OUTPUT: an information whether there exists a signed grid $[G(5, n), \sigma]$ which is not colorable by SP_5 or, in other words, whether the set of colorings reachable on the last column of $[G(5, n), \sigma]$) is empty

1. `compute` $S_5([SPath_5, \sigma])$
2. $Q \leftarrow S_5([SPath_5, \sigma])$
3. `while` $Q \neq \emptyset$ `do`
4. $\quad T \leftarrow Q$
5. $\quad$ `for each type` $\tau \in \{b, u\}^4$
6. $\quad\quad T'$ `:=` $\emptyset$

7. for each signature (R, σ) of the comb R_5
8. for each sequence $t \in T$
9. if the $Type(t, (R, \sigma)) = \tau$
10. add $NEXT(t, (R, \sigma))$ to T'
11. if T' is empty
12. return "found empty set"
13. if T' was never in the queue
14. $Q \leftarrow T'$
15. return "not found empty set"

First, in line 2, the algorithm computes the set $S_5([SPath_5, \sigma])$ (of colorings reachable on the first column) and puts it in a queue Q. Observe that all signatures of the path are equivalent, so we have one set of colorings.

Next, it repeats the following steps (lines 3–14):

(1) It takes a set T from the queue Q,

(2) It computes the sets of colorings that are reachable from T by adding comb on the last column of grid (on which T is reachable).

Note, that a signature (R, σ) of R_5 can create different types of the squares formed by two last columns of glued grid, depending on the sequence t. Hence, the algorithm compute sets of reachable colorings separately for every type τ (lines 5–14). For every type τ the algorithm create a new set T', and then, for every signature (R, σ) of the comb R_5 and for every sequence $t \in T$, it checks if the $Type(t, (R, \sigma))$ is the same as τ. If so, the algorithm computes the set $NEXT(t, (R, \sigma))$ and add it to T'. After checking all signatures and all colorings t, the set T' stores the set of reachable colorings on the new last column. If the set is empty, then the algorithm returns an information that an empty set has been found. Otherwise, it puts the set T' in queue Q if it is new.

We have also designed another more efficient version of the algorithm, by changing the order of the loops and by using symmetries in SP_5.

8.4 Lower bound

Theorem 8.12 (Dybizbański, Nenca, Szepietowski (2020) [13]). *There exists a signed grid $[G(3, 4), \sigma]$, with three rows and four columns which cannot be colored with four colors.*

Proof. Consider the signed grid $[G(3,4),\sigma]$ presented on Figure 8.2. The middle square

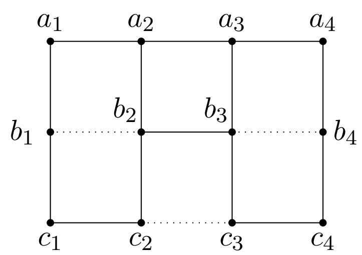

FIGURE 8.2: Signed grid $[G(3,4),\sigma]$

in the first row is balanced and all other squares are unbalanced. We shall show that the signed grid $[G(3,4),\sigma]$ cannot be colored by any graph $H = (K_4,\delta)$, where K_4 is the clique with four vertices. Without loss of generality we may assume that H has at least three edges signed by "+". Indeed, if a signed grid (G,σ) is colored by a graph H with less than three "+" edges, then we can flip all signs both in (G,σ) and H, and obtain a coloring of the grid which is equivalent with (G,σ), by the graph with more than three "+" edges. Observe that since $[G(3,4),\sigma]$ has unbalanced squares, H should have an unbalanced cycle of length four. There are three nonisomorphic such graphs, namely SH_1, SH_2, and (C_5^{-0}), presented on Figure 8.3, Figure 8.5 and Figure 8.6.

Case 1. Coloring graph SH_1 – the clique K_4 with one edge signed by "$-$". Note that in

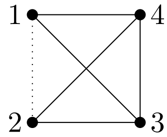

FIGURE 8.3: Coloring graph SH_1

each unbalanced square colored by SH_1: vertices have different colors, exactly one edge has sign "$-$", and this edge is colored with $1,2$.

First, consider the signed grid $[G(2,3),\lambda]$ with two unbalanced squares presented on Figure 8.4. Suppose that there is a presentation $(G(2,3),\lambda_1)$ of $[G(2,3),\lambda]$ which is

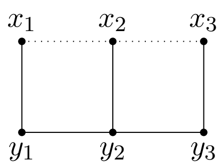

FIGURE 8.4: $[G(2,3),\lambda]$.

colored by SH_1. It is easy to observe that if an edge in the first row, say x_1x_2, has the sign "$-$", then: both edges in the first row have "$-$", the vertices x_1,x_2,x_3 have colors in $\{1,2\}$, and the vertices y_1,y_2,y_3 have colors in $\{3,4\}$. Similarly, if an edge in the second row has the sign "−", then the vertices y_1, y_2, y_3 have colors in $\{1, 2\}$ and x_1, x_2, x_3 have colors in $\{3, 4\}$.

Now, suppose, for a contradiction, that there is a presentation $(G(3, 4), \sigma_1)$ of $[G(3, 4), \sigma]$ which is colored by SH_1. The square b_2, b_3, c_3, c_2 is unbalanced, so one of its edges is signed with "−" and has colors $1, 2$. We have four subcases:

(1.1) The edge (b_2, b_3) has "−". All three squares in the second row are unbalanced. Hence, by the above observation, all vertices in the second row (b_1, b_2, b_3, b_4) have colors in $\{1, 2\}$. Furthermore, all vertices in the first row have colors in $\{3, 4\}$, because the squares a_1, a_2, b_2, b_1 and a_3, a_4, b_4, b_3 are unbalanced. Hence, we have a wrong coloring of the balanced square a_2, a_3, b_3, b_2.

(1.2) The edge (c_2, c_3) has "−". Similarly as in subcase (1.1), all vertices in the second row have colors in $\{3, 4\}$, and all vertices in the first row have colors in $\{1, 2\}$. Again, we have a wrong coloring of the square a_2, a_3, b_3, b_2.

(1.3) The edge (b_2, c_2) has "−". Then: vertices in column a_2, b_2, c_2 have colors in $\{1, 2\}$, vertices in column a_4, b_4, c_4 have colors in $\{1, 2\}$, and vertices in column a_3, b_3, c_3 have colors in $\{3, 4\}$. Hence, we have a wrong coloring of the square a_2, a_3, b_3, b_2.

(1.4) The case where the edge (b_3, c_3) has "−", is similar to the subcase (1.3).

Case 2. Coloring graph SH_2 – the clique K_4 with two incident edges signed by "−", see Figure 8.5. In SH_2 edges $(1, 3)$ and $(1, 4)$ have sign "−". Note that in each unbalanced

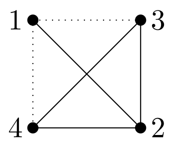

FIGURE 8.5: Coloring graph SH_2.

square colored by SH_2: vertices have different colors and exactly one edge has sign "−". Moreover one of the neighbors of the vertex colored by 1 is colored by 2. Hence, each unbalanced square has an edge colored by $\{1, 2\}$. The rest of the proof goes exactly like in Case 1.

Case 3. Coloring graph (C_5^{-0})– the signed graph obtained from SP_5 by removing the vertex 0, see Figure 8.6.

Lemma 8.13. *The function* $f(x) = 4 \cdot x \pmod 5$ *is an automorphism in* (C_5^{-0}).

Lemma 8.14. *Let* (G, σ) *be a signed grid,* ψ *be a homomorphism from* (G, σ) *to* (C_5^{-0}), *and* (u, v) *be an edge in* G. *Then there is a signed grid* (G, σ') *equivalent to* (G, σ) *and*

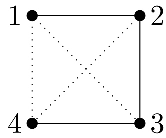

FIGURE 8.6: Coloring graph (C_5^{-0})

a homomorphism $\phi : (G, \sigma') \to (C_5^{-0})$ *such that* $\{\phi(u), \phi(v)\} = \{1, 2\}$ *or* $\{\phi(u), \phi(v)\} = \{2, 3\}$.

First, consider the signed grid $[G(2, 3), \lambda]$ presented on Figure 8.4. Suppose that there is a signature λ_3 and a homomorphism $\phi_2 : (G(2, 3), \lambda_3) \to H_3$. It is easy to observe that: if an edge in the first row has the both colors in $\{2, 3\}$ or in $\{1, 4\}$, then $\phi_3(x_1) = \phi_3(x_3)$. Similarly, if an edge in the second row has the colors in $\{2, 3\}$ or in $\{1, 4\}$, then $\phi_3(y_1) = \phi_3(y_3)$.

Now, suppose, for a contradiction, that there is a presentation $(G(3, 4), \sigma_2)$ of $[G(3, 4), \sigma]$ which is colored by SP_5. By Lemma 8.14, we may assume that the edge (b_2, b_3) has the sign "+" and is colored with $1, 2$ or $2, 3$.

(3.1) If the edge (b_2, b_3) has colors $(1, 2)$, then the edge (c_2, c_3) has colors $(4, 3)$. By the above observation, a_2 is colored by 4 and a_3 is colored by 3, hence, we have a wrong coloring of the balanced square a_2, a_3, b_3, b_2.

(3.2) If the edge (b_2, b_3) has colors in $(2, 3)$, then, all vertices in the second row have colors in $\{2, 3\}$. Hence, all vertices in the first row have colors 1 or 4, because the squares a_1, a_2, b_2, b_1 and a_3, a_4, b_4, b_3 are unbalanced. Hence, we have a wrong coloring of the balanced square a_2, a_3, b_3, b_2.

□

8.5 Upper bound

Let (C_5^{+5}) denote the signed graph obtained from SP_5 by adding the vertex 5 and connecting it with all other vertices by edges signed with "−", see Figure 8.7.

Lemma 8.15. *For each* $a \in \{1, 4\}$ *and* $b \in \{0, 1, 2, 3, 4\}$*, let* h *be the function defined by:*

$$h(x) = \begin{cases} a \cdot x + b \pmod 5 & \text{if } x \neq 5, \\ 5 & \text{otherwise.} \end{cases}$$

The function h *is an automorphism in* (C_5^{+5}).

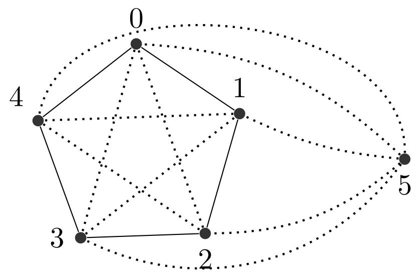

FIGURE 8.7: Signed graph (C_5^{+5})

Lemma 8.16. *Each edge* $(u, v) \in E(C_5^{+5})$ *can be mapped by an automorphism on one of the following three edges:* $(0, 1)$*,* $(0, 2)$*,* $(0, 5)$*. More precisely:*

- *if* $u, v \neq 5$ *and* (u, v) *is signed by "+", there is an automorphism* ϕ *such that* $\phi(u, v) = (0, 1)$*;*
- *if* $u, v \neq 5$ *and* (u, v) *is signed by "−", there is an automorphism* ϕ *such that* $\phi(u, v) = (0, 2)$*;*
- *if* $v = 5$*, there is an automorphism* ϕ *such that* $\phi(u, v) = (0, 5)$*.*

Lemma 8.17. *Every two different vertices* $u, v \in V(C_5^{+5})$ *can be connected, by two unbalanced paths of length two and by two balanced paths of length two.*

Lemma 8.18. *Consider a signed path* $[P_3, \sigma]$ *with vertices* x*,* y*, and* z*, and a coloring* $\phi : P_3 \to (C_5^{+5})$*. Suppose that the two end-points* x *and* z *are already colored by* $\phi(x)$ *and* $\phi(z)$*, with* $\phi(x) \neq \phi(z)$*. Then we can color* y *by two colors. When coloring some resigning in* y *may be necessary.*

Theorem 8.19 (Dybizbański, Nenca, Szepietowski (2020) [13])**.** *Every signed grid* $[G(m, n), \sigma]$ *can be colored by the graph* (C_5^{+5})*.*

Proof. We color the first row vertex by vertex without any resigning. We always have at least two possibilities to color the next vertex. Next, assume that $k - 1$ rows are colored and we want to color the kth row. Let us denote by $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ the consecutive vertices of the $(k - 1)$th and kth rows. First, note that the vertex b_1 can be colored in at least two ways (without resigning). We choose for b_1 the color which is different from the color in a_2. By Lemma 8.18, we can color the vertex b_2 by two colors col_1 and col_2. It is possible that the resigning in b_2 is necessary. Now, we choose for b_2 the color which is different from the color in a_3. In the same way we color the rest of the kth row. □

8.6 Grids with two, three or four rows

Theorem 8.20 (Dybizbański, Nenca, Szepietowski (2020) [13]). *Every signed grid $[G, \sigma]$ with two rows can be colored by the graph (C_5^{-0}).*

Proof. Let us denote by $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ the consecutive vertices of the first and second rows of $[G, \sigma]$. We shall show that G can be colored in such a way that every column has colors in $\{1, 4\}$ or in $\{2, 3\}$.

First, we color a_1 and b_1 with colors in $\{1, 4\}$. Assume now, that the vertices $a_1, \ldots, a_{i-1}$, $b_1, \ldots, b_{i-1}$ have been colored in the previous steps.

- If the square $a_{i-1}, a_i, b_i, b_{i-1}$ is balanced, then we color the edge (a_i, b_i) with the same set of colors as we use in the column (a_{i-1}, b_{i-1}).
- If the square $a_{i-1}, a_i, b_i, b_{i-1}$ is unbalanced, then we color the edge (a_i, b_i) with the set of colors which is disjoint with colors in the column (a_{i-1}, b_{i-1}).

It is easy to see that such coloring is always possible, because we can make resigning in the vertices a_i and b_i. □

Theorem 8.21 (Dybizbański, Nenca, Szepietowski (2020) [13]). *Every signed grid $[G(3, 3), \sigma]$ with three rows and three columns can be colored by the graph (C_5^{-0}).*

Proof. Let us denote by a_1, a_2, a_3, b_1, b_2, b_3 and c_1, c_2, c_3 the consecutive vertices of the first, second and third rows of $[G(3, 3), \sigma]$. We color the grid in the following way: the vertex b_2 is colored with 2. The vertices a_2 and c_2 are colored with 1, and the vertices b_1 and b_3 are colored with 3. Now, for every vertex $x \in \{a_1, a_3, c_1, c_3\}$, the vertex x is colored with 2, if the square containing x is balanced, and is colored with 4, if the square is unbalanced.

It is easy to see that the signature induced by the coloring is equivalent with the original signature σ. □

Theorem 8.22 (Dybizbański, Nenca, Szepietowski (2020) [13]). *Every signed grid $[G, \sigma]$ with four rows can be colored by the graph SP_5.*

Proof. The theorem follows from the Theorem 8.2. Here we present a proof which does not use a computer. Let us denote by $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, $c_1, \ldots, c_n$, and $d_1, \ldots, d_n$ the consecutive vertices of the first, second, third, and fourth row of $[G, \sigma]$. We shall show that $[G, \sigma]$ can be colored column by column. The first column is colored in an arbitrary proper way. Now, assume that the columns from 1 to $i-1$ have been colored in the previous steps. Without loos of generality we may assume that $\phi(b_{i-1}) = 0$ and $\phi(c_{i-1}) = 1$. In order to color the ith column: first, we color vertices b_i and c_i in such a way that $\phi(b_i) \neq \phi(a_{i-1})$ and $\phi(c_i) \neq \phi(d_{i-1})$. If necessary we make some resigning in b_i or c_i. Next, we color vertices a_i and d_i. Again we may make some resigning in a_i or d_i. It is easy to see that such coloring is possible. $\square$

Chapter 9

Further Work

In this chapter, we present several problems left for future work.

9.1 Is $\overrightarrow{\chi}(\mathcal{G}) > 8$?

In Chapter 3, we have shown that there exists an orientation of a grid with seven rows that cannot be colored with seven colors. We have tried to find traps for all coloring graphs with 8 vertices. There are 6880 nonisomorphic tournaments on 8 vertices. When Algorithm $\mathsf{F}indTrap_m$ (see Section 3.1) is applied to grids with seven or eight rows, it often loops and does not obtain the answer. We have found traps with seven rows for all nonisomorphic tournaments on 8 vertices except 1127. We pose the following conjecture:

Conjecture 9.1. $\overrightarrow{\chi}(\mathcal{G}) > 8$.

9.2 Is $\overrightarrow{\chi}(\mathcal{G}) \leq 10$?

In Chapter 4, we have shown that any orientation of any grid with six, seven or eight rows can be colored by the coloring graph with ten vertices $\overrightarrow{H}_{10}$; see Theorem 4.1. For grids with eight rows, we have found that the set S is closed under extension; see Section 4.2. The set S consists of over 90% of all possible sequences. Thus, we pose the following conjecture:

Conjecture 9.2. $\overrightarrow{\chi}(\mathcal{G}) \leq 10$.

9.3 Problems related to the coloring graph $\overleftarrow{T}_7$

The graph $\overleftarrow{T}_7$ is obtained from the Paley tournament $\overrightarrow{T}_7$ by reversing one arc. In Chapter 3, we have shown that there exists the orientation of a grid $G(7, 22)$ (see Figure 3.15) that cannot be colored by $\overleftarrow{T}_7$.

The graph $\overleftarrow{T}_7$ is neither self-converse nor arc-transitive. Hence, the families $\mathcal{S}_5(\overleftarrow{T}_7)$, $\mathcal{S}_6(\overleftarrow{T}_7)$ and even $\mathcal{S}_4(\overleftarrow{T}_7)$ are very big and cannot be handled with simple algorithms. We have only generated the family $\mathcal{S}_3(\overleftarrow{T}_7)$ that has 980 elements of $4 \cdot 2^{112}$ possible. We know that $\emptyset \notin \mathcal{S}_3(\overleftarrow{T}_7)$. Thus, all graphs in $\mathcal{G}_3$ can be colored by $\overleftarrow{T}_7$. Although we do not know whether $\overleftarrow{T}_7$ can color all graphs in $\mathcal{G}_m$, for $m = 5$ or 6, we know that no other coloring graph with seven vertices can color the whole $\mathcal{G}_5$ or $\mathcal{G}_6$. For grids with four rows, we do not know whether $\mathcal{G}_4$ can be colored by $\overleftarrow{T}_7$, although we know that it can be colored by $\overrightarrow{T}_7$; see [53].

Problem 9.3. *Is* $\overrightarrow{\chi}(\mathcal{G}_5) = 7$*?*

Problem 9.4. *Is* $\overrightarrow{\chi}(\mathcal{G}_6) = 7$*?*

9.4 Signed coloring

In Chapter 8 we have shown that any orientation of any grid with three, four, five, six or seven rows can be colored by the signed Paley graph SP_5; see Theorem 8.2. We have tried several algorithms to generate a grid which is not colored by SP_5 and so far we have not succeeded. Therefore, we pose the following conjecture:

Conjecture 9.5. *Every signed grid* $[G, \sigma]$ *can be colored by the graph* SP_5*.*

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