University of Warsaw
Faculty of Mathematics, Informatics and Mechanics

Ali Rezaei Divroodi

Bisimulation Equivalence in Description Logics and Its Applications

PhD Dissertation

Supervisor:
dr hab. Linh Anh Nguyen
Institute of Informatics
University of Warsaw

Author's Declaration

Aware of legal responsibility I hereby declare that I have written this dissertation myself and all its contents have been obtained by legal means.

Date 26/02/2015 Author's signature

Supervisor's Declaration

The dissertation is ready to be reviewed.

Date 26/02/2015 Supervisor's signature

Abstract

In this dissertation, we study bisimulations and bisimulation-based comparisons in a uniform way for a large class of description logics that extend $\mathcal{ALC}_{reg}$ (a variant of propositional dynamic logic) with an arbitrary set of features among I (inverse roles), O (nominals), Q (qualified number restrictions), U (the universal role), `Self` (local reflexivity of a role) as well as role axioms.

We give results on invariance of concepts, TBoxes and ABoxes, preservation of RBoxes and knowledge bases, the Hennessy-Milner property w.r.t. bisimulations, the largest auto-bisimulations and quotient interpretations w.r.t. such equivalence relations. By adapting Hopcroft's automaton minimization algorithm and the Paige-Tarjan algorithm, we give efficient algorithms for computing the partition corresponding to the largest auto-bisimulation of a finite interpretation.

We provide results on preservation of semi-positive concepts, the Hennessy-Milner property w.r.t. bisimulation-based comparisons, characterizing bisimulation for tidy interpretations by semi-positive concepts, and minimization of interpretations that preserves semi-positive concepts.

We separate the expressiveness of the description logics that extend $\mathcal{L}$, where $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$, with any combination of the features I, O, Q, U, `Self`. Our separation results are w.r.t. concepts, positive concepts, TBoxes and ABoxes.

We prove that any concept in any description logic that extends $\mathcal{ALC}$ with some features amongst I, `Self`, Q_k (qualified number restrictions with numbers bounded by a constant k) can be learned if the training information system (specified as an interpretation) is good enough.

Keywords: description logics, bisimulation, minimization, expressiveness, concept learning, learnability.

ACM Classification: description logics, modeling and simulation, machine learning.

Contents

Chapter 1

Introduction

Description logics (DLs) are variants of modal logic. They are of particular importance in providing a logical formalism for ontologies and the Semantic Web. DLs represent the domain of interest in terms of concepts, individuals, and roles. A concept is interpreted as a set of individuals, while a role is interpreted as a binary relation among individuals. A DL is characterized by a set of concept constructors, a set of role constructors, and a set of allowed forms of role axioms and individual assertions. A knowledge base in a DL usually has three parts: an RBox consisting of axioms about roles, a TBox consisting of terminology axioms, and an ABox consisting of assertions about individuals. The basic DL $\mathcal{ALC}$ allows basic concept constructors listed in Table 1.1, but does not allow role constructors nor role axioms. The most common additional features for extending $\mathcal{ALC}$ are also listed in Table 1.1.

Given two individuals in an interpretation, sometimes we are interested in the question whether they are "similar" or not, i.e., whether they are indiscernible w.r.t. the considered description language. Indiscernibility is used, for example, in machine learning. In DLs, it is formally characterized by bisimulation. Roughly speaking, two individuals are indiscernible iff they are bisimilar.

Bisimulations arose in modal logic [60, 61, 62] and state transition systems [50, 27]. They were introduced by van Benthem under the name *p-relation* in [60, 61] and the name *zigzag relation* in [62]. Bisimulations reflect, in a particularly simple and direct way, the locality of the modal satisfaction definition. The famous Van Benthem Characterization Theorem states that modal logic is the bisimulation invariant fragment of first-order logic. Bisimulations have been used to analyze the expressivity of a wide range of extended modal logics (see, e.g., [6] for details). In state transition systems, bisimulation is viewed as a binary relation associating systems which behave in the same way in the sense that one system simulates the other and vice versa. Kripke models in modal logic are a special case of labeled state transition systems. Hennessy and Milner [27] showed that weak modal languages could be used to classify various notions of process invariance. In general, bisimulations are a natural notion of equivalence for both mathematical and computational investigations.[1]

Bisimilarity between two states is usually defined by three conditions (the states

[1] This paragraph is based on [6].

Concept constructors of $\mathcal{ALC}$		
Constructor	Syntax	Example
complement	$\neg C$	$\neg Male$
intersection	$C \sqcap D$	$Human \sqcap Male$
union	$C \sqcup D$	$Doctor \sqcup Lawyer$
existential restriction	$\exists r.C$	$\exists hasChild.Male$
universal restriction	$\forall r.C$	$\forall hasChild.Female$

Some additional constructors/features of other DLs		
Constructor/Feature	Syntax	Example
inverse roles ($\mathcal{I}$)	r^-	$hasChild^-$ (i.e., $hasParent$)
qualified number restrictions ($\mathcal{Q}$)	$\geq n\, R.C$ $\leq n\, R.C$	$\geq 3\, hasChild.Male$ $\leq 2\, hasParent.\top$
nominals ($\mathcal{O}$)	$\{a\}$	$\{John\}$
hierarchies of roles ($\mathcal{H}$)	$R \sqsubseteq S$	$hasChild \sqsubseteq hasDesc.$
transitive roles ($\mathcal{S}$)	$R \circ R \sqsubseteq R$	$hasDesc. \circ hasDesc. \sqsubseteq hasDesc.$

Table 1.1: Concept constructors for $\mathcal{ALC}$ and some additional constructors/features of other DLs.

have the same label, each transition from one of the states can be simulated by a similar transition from the other, and vice versa). As shown in [6], the four program constructors of PDL (propositional dynamic logic) are “safe” for these three conditions. That is, we need to specify the mentioned conditions only for atomic programs, and as a consequence, they hold also for complex programs. For bisimulation between two pointed-models, the initial states of the models are also required to be bisimilar. When converse is allowed (the case of CPDL), two additional conditions are required for bisimulation [6]. Bisimulation conditions for dealing with graded modalities were studied in [13, 12, 34]. In the field of hybrid logic, the bisimulation condition for dealing with nominals is well known (see, e.g., [3]).

In modal logic, bisimulation invariance has the form: if two states are bisimilar then they satisfy the same set of formulas (i.e., all modal formulas are invariant w.r.t. bisimulation). For the converse, the Hennessy-Milner property states that, in modally-saturated Kripke models, two states are bisimilar iff they satisfy the same set of formulas.

Simulation is a notion with weaker conditions than bisimulation. It is only “one way”, while bisimulation is “two way”. In the most common understanding, the “ways” are related with the “transitions” but not w.r.t. comparison between the sets of atomic formulas satisfied at the considered states. Such simulation preserves positive existential formulas (see, e.g., [6]).

In [36], Kurtonina and de Rijke introduced *directed (modal) simulation*, which preserves negation-free formulas. Such directed simulation uses the conditions of bisimulation for “transitions” and compares the sets of atomic formulas satisfied at the considered states. They first formulated directed simulation for a monomodal logic (denoted by $\mathcal{L}^{\neg}_{\Diamond,\Box}$, which is the monomodal logic K without negation) and then, as examples, they extended it to the DL $\mathcal{FLEUC}^{-}$, temporal logic, feature logics and languages with non-Boolean negation. They used directed simulation to obtain preservation, safety and definability results. They also proved the corresponding Hennessy-Milner property for the considered monomodal logic.

Being not aware of the work [36] by Kurtonina and de Rijke, in [18] we used the term "bisimulation-based comparison" instead of "directed simulation". We still prefer the term "bisimulation-based comparison". The reader can treat it as a synonym of "directed simulation" [36].

Bisimulation-based comparison between Kripke models is worth studying, because it can be used for minimizing a Kripke model w.r.t. the set of logical consequences being positive formulas. For example, after constructing a least Kripke model of a positive modal logic program in a serial modal logic [42, 44, 20], one can minimize it w.r.t. positive formulas to obtain a minimal Kripke model that characterizes the program w.r.t. positive consequences. Such minimization is also applicable to (non-serial) DLs [43, 46].

In this dissertation we study bisimulations and bisimulation-based comparisons between interpretations in DLs. The simplest among the considered logics is $\mathcal{ALC}_{reg}$, a variant of PDL (propositional dynamic logic). The others extend that logic with inverse roles, nominals, qualified number restrictions, the universal role, and/or the concept constructor $\exists r.\mathtt{Self}$ for expressing the local reflexivity of a role. Inverse roles are like converse modal operators, qualified number restrictions are like graded modalities, and nominals are as in hybrid logic. The considered logics also allow role axioms.

The topic is worth studying due to the following reasons:

1. Despite that bisimulation conditions are known for PDL and for some features like converse modal operators, graded modal operators and nominals, we are not aware of previous work on bisimulation conditions for the universal role and the concept constructor $\exists r.\mathtt{Self}$. More importantly, without proofs one cannot be sure that all the conditions can be combined together to guarantee standard properties like invariance and the Hennessy-Milner property.

   There are many papers on bisimulations, but just a few on bisimulations in DLs:

   - In [37] Kurtonina and de Rijke studied expressiveness of concept expressions in some DLs by using bisimulations. They considered a family of DLs that are sublogics of the DL $\mathcal{ALCNR}$, which extend $\mathcal{ALC}$ with (unqualified) number restrictions and role conjunction. They did not consider individuals, nominals, qualified number restrictions, the concept constructor $\exists r.\mathtt{Self}$, the universal role, and the role constructors like the program constructors of PDL.
   - In [40] Lutz et al. characterized the expressiveness of TBoxes in the DL $\mathcal{ALCQIO}$ and its sublogics, including the lightweight DLs such as DL-Lite and $\mathcal{EL}$. They also studied invariance of TBoxes and the problem of TBox rewritability. The logic $\mathcal{ALCQIO}$ lacks the role constructors of PDL, the concept constructor $\exists r.\mathtt{Self}$ and the universal role.

- The papers [16, 15, 18, 14, 19] are our works that form the contents of this dissertation.
- Bisimulation-based concept learning in DLs was studied in [48, 57, 26, 56, 58]. All of these works are based on the notion of bisimulation and its properties investigated in our papers [16, 19].

The family of DLs studied in this work is large and contains useful DLs. Not only concept constructors and role constructors are allowed, but role axioms are also allowed. In particular, the DL $\mathcal{SROIQ}$, which is the logical base of the Web Ontology Language OWL 2, belongs to this class.

2. DLs differ from other logics like modal logics and hybrid logics in the domain of applications and the settings. In DLs, there are special notions like named individual, RBox, TBox, ABox. Also, recall that a knowledge base in a DL usually consists of an RBox, a TBox and an ABox. Invariance of ABoxes and preservation of RBoxes and knowledge bases in DLs were not studied before. On the other hand, invariance of TBoxes was recently studied in the independent work [40] for the DL $\mathcal{ALCQIO}$ and its sublogics. Note that the first version [41] of [40] appeared to the public a few days later than our manuscript [17]. The works [40, 41] use the notion of global bisimulation to characterize invariance of TBoxes, whose condition is the same as the bisimulation conditions introduced in [17] for the universal role.

3. Bisimulation is a useful notion for DLs. The applications are analyzing expressiveness of DLs, minimizing interpretations and concept learning in DLs.

   - A DL $\mathcal{L}$ is more expressive than a DL $\mathcal{L}'$ w.r.t. concepts if every concept in $\mathcal{L}'$ has an equivalent concept in $\mathcal{L}$, but not vice versa. Similarly, DLs can be compared w.r.t. positive concepts, TBoxes and ABoxes.
   - Roughly speaking, two objects that are bisimilar to each other can be merged. This is the basis for minimizing interpretations. In automated reasoning in DLs, sometimes we want to return a model of a knowledge base (e.g., as a counter example for a subsumption problem or an instance checking problem). It is expected that the returned model is simple and as small as possible. One can just find some model and minimize it. As another example, given an information system specified by an acyclic knowledge base with a large ABox and a small TBox, one can compute that information system and minimize it to save space and increase efficiency of reasoning tasks.
   - Concept learning in DLs is similar to binary classification in traditional machine learning. The difference is that in DLs objects are described not only by attributes but also by relationship between the objects. As bisimulation is the notion for characterizing indiscernibility of objects in DLs, it is useful for concept learning in DLs [48, 57, 26, 15].

The Structure of the Rest of This Dissertation

Chapter 2: We present notation and semantics of DLs.

Chapter 3: We formulate bisimulations for the mentioned class of DLs. We give results about invariance of concepts, TBoxes and ABoxes, preservation of RBoxes and knowledge bases, and the Hennessy-Milner property w.r.t. bisimulations in the considered DLs. We also provide results on the largest auto-bisimulations and quotient interpretations w.r.t. such equivalence relations. Such results are useful for minimizing interpretations and concept learning in DLs. To deal with minimizing interpretations for the case when the considered logic allows qualified number restrictions and/or the constructor for the local reflexivity of a role, we introduce a new notion called QS-interpretation, which is needed for obtaining expected results. By adapting Hopcroft's automaton minimization algorithm [28] and the Paige-Tarjan algorithm [49], we give efficient algorithms for computing the partition corresponding to the largest auto-bisimulation of a finite interpretation.

Chapter 4: We study comparisons between interpretations in DLs with respect to logical consequences of the form of semi-positive concepts. Such comparisons are characterized by conditions similar to the ones of bisimulations. The studied problems are: preservation of semi-positive concepts with respect to comparisons, the Hennessy-Milner property for comparisons, characterizing bisimulation for tidy interpretations by semi-positive concepts, and minimization of interpretations that preserves semi-positive concepts.

Chapter 5: We compare the expressiveness of the considered DLs w.r.t. concepts, positive concepts, TBoxes and ABoxes. Our results about separating the expressiveness of description logics are based on bisimulations and bisimulation-based comparisons. They are naturally extended to the case when instead of $\mathcal{ALC}_{reg}$ we have any sublogic of $\mathcal{ALC}_{reg}$ that extends $\mathcal{ALC}$.

Chapter 6: This chapter concerns concept learning in DLs. In Section 6.1, we present a survey on bisimulation-based concept learning in DLs. In Section 6.2, we prove that any concept in any description logic that extends $\mathcal{ALC}$ with some features amongst I (inverse), Q_k (qualified number restrictions with numbers bounded by a constant k), `Self` (local reflexivity of a role) can be learned if the training information system (specified as an interpretation) is good enough. That is, there exists a learning algorithm such that, for every concept C of those logics, there exists a training information system consistent with C such that applying the learning algorithm to the system results in a concept equivalent to C. In Section 6.3, we generalize common types of queries for description logics, introduce interpretation queries and present some consequences.

Chapter 7: This chapter concludes our dissertation.

The bibliography, acknowledgements and an index of symbols and terms are provided at the end of this dissertation.

Chapter 2

Preliminaries

2.1 Notation of Description Logics

Our languages use a countable set Σ_C of *concept names* (atomic concepts), a countable set Σ_R of *role names* (atomic roles), and a countable set Σ_I of *individual names*. Let $\Sigma = \Sigma_C \cup \Sigma_R \cup \Sigma_I$. We denote concept names by letters like A and B, denote role names by letters like r and s, and denote individual names by letters like a and b.

We consider some (additional) *DL-features* denoted by I (*inverse*), O (*nominal*), Q (*qualified number restriction*), U (*universal role*), `Self`. A *set of DL-features* is a set consisting of some or zero of these names. We sometimes abbreviate sets of DL-features, writing e.g., IOQ instead of $\{I, O, Q\}$.

Definition 2.1 (**Syntax of Roles and Concepts).**
Let Φ be any set of DL-features and let $\mathcal{L}$ stand for $\mathcal{ALC}_{reg}$, which is the name of the DL corresponding to propositional dynamic logic (PDL). The DL language $\mathcal{L}_\Phi$ allows *roles* and *concepts* defined inductively as follows:

- if $r \in \Sigma_R$ then r is a role of $\mathcal{L}_\Phi$
- if $A \in \Sigma_C$ then A is a concept of $\mathcal{L}_\Phi$
- if R and S are roles of $\mathcal{L}_\Phi$ and C is a concept of $\mathcal{L}_\Phi$ then
  - ε, $R \circ S$, $R \sqcup S$, R^* and $C?$ are roles of $\mathcal{L}_\Phi$
  - $\top$, $\bot$, $\neg C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$ and $\exists R.C$ are concepts of $\mathcal{L}_\Phi$
  - if $I \in \Phi$ then R^- is a role of $\mathcal{L}_\Phi$
  - if $O \in \Phi$ and $a \in \Sigma_I$ then $\{a\}$ is a concept of $\mathcal{L}_\Phi$
  - if $Q \in \Phi$, $r \in \Sigma_R$ and n is a natural number
    then $\geq n\, r.C$ and $\leq n\, r.C$ are concepts of $\mathcal{L}_\Phi$
  - if $\{Q, I\} \subseteq \Phi$, $r \in \Sigma_R$ and n is a natural number
    then $\geq n\, r^-.C$ and $\leq n\, r^-.C$ are concepts of $\mathcal{L}_\Phi$
  - if $U \in \Phi$ then U is a role of $\mathcal{L}_\Phi$ (we assume $U \notin \Sigma_R$)
  - if $\mathtt{Self} \in \Phi$ and $r \in \Sigma_R$ then $\exists r.\mathtt{Self}$ is a concept of $\mathcal{L}_\Phi$. ■

We use letters like R and S to denote arbitrary roles, and use letters like C and D to denote arbitrary concepts. A role stands for a binary relation, while a concept stands for a unary relation.

The intended meaning of the role constructors is the following:

- ε stands for the empty binary relation,
- $R \circ S$ stands for the sequential composition of R and S,
- $R \sqcup S$ stands for the set-theoretical union of R and S,
- R^* stands for the reflexive and transitive closure of R,
- $C?$ stands for the test operator (as of PDL),
- R^- stands for the *inverse* of R,
- U stands for the full binary relation (on the domain).

The symbols $\top$ and $\bot$ stand for *truth* and *falsity*, respectively. The constructors $\neg$, $\sqcap$ and $\sqcup$ stand for *complement*, *intersection* and *union*, respectively. A concept $\{a\}$, called a *nominal*, stands for a singleton set. The constructors $\forall R.C$ and $\exists R.C$ are called *universal restriction* and *existential restriction*, respectively. They correspond to the modal operators $\langle R\rangle C$ and $[R]C$ of PDL, respectively. The concept constructors $\geq n\, R.C$ and $\leq n\, R.C$ are called *qualified number restrictions*. They correspond to graded modal operators. The constructor $\exists r.\mathtt{Self}$ stands for *local reflexivity* of r.

We refer to elements of Σ_R also as *atomic roles*. Let $\Sigma_R^{\pm} = \Sigma_R \cup \{r^- \mid r \in \Sigma_R\}$. From now on, by *basic roles* we refer to elements of $\Sigma_R^{\pm}$ if the considered language allows inverse roles, and refer to elements of Σ_R otherwise. In general, the language decides whether inverse roles are allowed in the considered context.

We say that a role R is in the *converse normal form* (CNF) if the inverse constructor is applied in R only to role names and the role U is not under the scope of any other role constructor. Since every role can be translated to an equivalent role in CNF,[1] in this dissertation we assume that roles are presented in the CNF.

Definition 2.2 (RBox – Box of Role Axioms).
A *role (inclusion) axiom* in $\mathcal{L}_\Phi$ is an expression of the form $\varepsilon \sqsubseteq r$ or $R_1 \circ \ldots \circ R_k \sqsubseteq r$, where $k \geq 1$ and $R_1, \ldots, R_k$ are basic roles of $\mathcal{L}_\Phi$.[2] An *RBox* in $\mathcal{L}_\Phi$ is a finite set of role axioms in $\mathcal{L}_\Phi$. ■

Definition 2.3 (TBox – Box of Terminological Axioms).
A *terminological axiom* in $\mathcal{L}_\Phi$, also called a *general concept inclusion* (GCI) in $\mathcal{L}_\Phi$, is an expression of the form $C \sqsubseteq D$, where C and D are concepts in $\mathcal{L}_\Phi$. A *TBox* in $\mathcal{L}_\Phi$ is a finite set of terminological axioms in $\mathcal{L}_\Phi$. ■

[1] For example, $((r \sqcup s^-) \circ r^*)^- = (r^-)^* \circ (r^- \sqcup s)$.

[2] This definition depends only on whether $\mathcal{L}_\Phi$ allows inverse roles, i.e., whether $I \in \Phi$.

Definition 2.4 (**ABox – Box of Individual Assertions**).
An *individual assertion* in $\mathcal{L}_\Phi$ is an expression of one of the forms $C(a)$ (*concept assertion*), $R(a, b)$ (*positive role assertion*), $\neg R(a, b)$ (*negative role assertion*), $a \doteq b$, and $a \neq b$, where C is a concept and R is a role in $\mathcal{L}_\Phi$. An *ABox* in $\mathcal{L}_\Phi$ is a finite set of individual assertions in $\mathcal{L}_\Phi$. ∎

Definition 2.5 (**Knowledge Base**).
A *knowledge base* in $\mathcal{L}_\Phi$ is a triple $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$, where $\mathcal{R}$ (resp. $\mathcal{T}$, $\mathcal{A}$) is an RBox (resp. a TBox, an ABox) in $\mathcal{L}_\Phi$. ∎

2.2 Semantics of Description Logics

As usual, the semantics of a logic is specified by interpretations and the satisfaction relation.

Definition 2.6 (**Interpretation**).
An *interpretation* $\mathcal{I} = \langle \Delta^\mathcal{I}, \cdot^\mathcal{I} \rangle$ consists of a non-empty set $\Delta^\mathcal{I}$, called the *domain* of $\mathcal{I}$, and a function $\cdot^\mathcal{I}$, called the *interpretation function* of $\mathcal{I}$, which maps every concept name A to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$, maps every role name r to a binary relation $r^\mathcal{I}$ on $\Delta^\mathcal{I}$, and maps every individual name a to an element $a^\mathcal{I}$ of $\Delta^\mathcal{I}$. We say that $\mathcal{I}$ is a *finite interpretation* if $\Delta^\mathcal{I}$ and Σ are finite. The interpretation function $\cdot^\mathcal{I}$ is extended to complex roles and complex concepts as shown in Figure 2.1, where $\#\Gamma$ stands for the cardinality of the set Γ, $C^\mathcal{I}(x)$ denotes $x \in C^\mathcal{I}$, and $R^\mathcal{I}(x, y)$ denotes $\langle x, y \rangle \in R^\mathcal{I}$. ∎

For a finite set $\Gamma = \{C_1, \ldots, C_n\}$ of concepts, by $\bigsqcap \Gamma$ we denote the concept $C_1 \sqcap \ldots \sqcap C_n$, which is $\top$ when $n = 0$. For a set Γ of concepts, by $\Gamma^\mathcal{I}$ we denote the set $\bigcap\{C^\mathcal{I} \mid C \in \Gamma\}$. If $x \in \Gamma^\mathcal{I}$ then we say that x *satisfies* Γ, $\mathcal{I}$ *satisfies* Γ (at x) and Γ is *satisfied* (at x) in $\mathcal{I}$.

If $R^\mathcal{I}(x, y)$ holds then we call y an *R-successor* of x.

Definition 2.7 (**The Satisfaction Relation**).
Given an interpretation $\mathcal{I}$, define that:

$$
\begin{array}{lll}
\mathcal{I} \models C \sqsubseteq D & \text{if} & C^\mathcal{I} \subseteq D^\mathcal{I} \\
\mathcal{I} \models R_1 \circ \ldots \circ R_k \sqsubseteq r & \text{if} & R_1^\mathcal{I} \circ \ldots \circ R_k^\mathcal{I} \subseteq r^\mathcal{I} \\
\mathcal{I} \models \varepsilon \sqsubseteq r & \text{if} & \varepsilon^\mathcal{I} \subseteq r^\mathcal{I} \\
\mathcal{I} \models a \doteq b & \text{if} & a^\mathcal{I} = b^\mathcal{I} \\
\mathcal{I} \models a \neq b & \text{if} & a^\mathcal{I} \neq b^\mathcal{I} \\
\mathcal{I} \models C(a) & \text{if} & C^\mathcal{I}(a^\mathcal{I}) \text{ holds} \\
\mathcal{I} \models R(a, b) & \text{if} & R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \text{ holds} \\
\mathcal{I} \models \neg R(a, b) & \text{if} & R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I}) \text{ does not hold,}
\end{array}
$$

where the operator $\circ$ stands for the composition of binary relations. We say that $\mathcal{I}$ *validates* an axiom (resp. *satisfies* an assertion) φ if $\mathcal{I} \models \varphi$. In that case, we also say that φ is *validated* by (resp. *satisfied* in) $\mathcal{I}$. ∎

$$
\begin{array}{rclrcl}
(R \circ S)^{\mathcal{I}} & = & R^{\mathcal{I}} \circ S^{\mathcal{I}} & \top^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \\
(R \sqcup S)^{\mathcal{I}} & = & R^{\mathcal{I}} \cup S^{\mathcal{I}} & \bot^{\mathcal{I}} & = & \emptyset \\
(R^*)^{\mathcal{I}} & = & (R^{\mathcal{I}})^* & (\neg C)^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \\
(C?)^{\mathcal{I}} & = & \{\langle x, x\rangle \mid C^{\mathcal{I}}(x)\} & (C \sqcap D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
\varepsilon^{\mathcal{I}} & = & \{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\} & (C \sqcup D)^{\mathcal{I}} & = & C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
U^{\mathcal{I}} & = & \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} & \{a\}^{\mathcal{I}} & = & \{a^{\mathcal{I}}\} \\
(R^-)^{\mathcal{I}} & = & (R^{\mathcal{I}})^{-1} & (\exists r.\mathtt{Self})^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x, x)\}
\end{array}
$$

$$
\begin{array}{rcl}
(\forall R.C)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid \forall y\,[R^{\mathcal{I}}(x, y) \text{ implies } C^{\mathcal{I}}(y)]\} \\
(\exists R.C)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid \exists y\,[R^{\mathcal{I}}(x, y) \text{ and } C^{\mathcal{I}}(y)] \\
(\geq n\, R.C)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid \#\{y \mid R^{\mathcal{I}}(x, y) \text{ and } C^{\mathcal{I}}(y)\} \geq n\} \\
(\leq n\, R.C)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid \#\{y \mid R^{\mathcal{I}}(x, y) \text{ and } C^{\mathcal{I}}(y)\} \leq n\}
\end{array}
$$

Figure 2.1: Interpretation of complex roles and complex concepts.

Note that reflexiveness and transitiveness of atomic roles are expressible by role axioms. When $I \in \Phi$, symmetry of an atomic role can also be expressed by a role axiom.

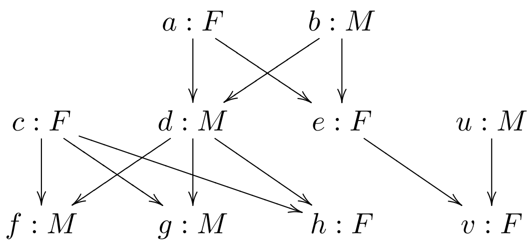
Definition 2.8 (Semantics).
An interpretation $\mathcal{I}$ is a *model* of a "box" (RBox, TBox or ABox) if it validates all the axioms/assertions of that "box". It is a *model* of a knowledge base $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ if it is a model of $\mathcal{R}$, $\mathcal{T}$ and $\mathcal{A}$. A knowledge base is *satisfiable* if it has a model. An individual a is said to be an *instance* of a concept C w.r.t. a knowledge base KB, denoted by $KB \models C(a)$, if, for every model $\mathcal{I}$ of KB, $a^{\mathcal{I}} \in C^{\mathcal{I}}$. ■

Example 2.9. Let

- $\Sigma_I = \{Alice, Bob, Claudia, Dave, Eva, Frank, George, Helen\}$,
- $\Sigma_C = \{Male, Female, Father, Mother\}$, and
- $\Sigma_R = \{hasChild, hasParent\}$.

Consider the interpretation $\mathcal{I}$ specified by:

- $\Delta^{\mathcal{I}} = \{a, b, c, d, e, f, g, h, u, v\}$,
- $Alice^{\mathcal{I}} = a$, $Bob^{\mathcal{I}} = b$, …, $Helen^{\mathcal{I}} = h$ (u and v are unnamed individuals),
- $hasChild^{\mathcal{I}}$ consists of elements illustrated by edges in the following graph:



(in this graph, the letter M denotes *Male*, and F denotes *Female*),

- $hasParent^{\mathcal{I}} = (hasChild^{-1})^{\mathcal{I}} = (hasChild^{\mathcal{I}})^{-1}$,
- $Male^{\mathcal{I}} = \{b, d, f, g, u\}$, $Female^{\mathcal{I}} = \Delta^{\mathcal{I}} \setminus Male^{\mathcal{I}} = \{a, c, e, h, v\}$‘,
- $Father^{\mathcal{I}} = (Male \sqcap \exists hasChild.\top)^{\mathcal{I}} = \{b, d, u\}$,
- $Mother^{\mathcal{I}} = (Female \sqcap \exists hasChild.\top)^{\mathcal{I}} = \{a, c, e\}$.

As examples, we have that:

- $(\exists hasChild.\mathtt{Self})^{\mathcal{I}} = \emptyset$,
- $(\geq 3\, hasChild.\top)^{\mathcal{I}} = \{c, d\}$,
- $(\geq 2\, hasChild.Male)^{\mathcal{I}} = \{c, d\}$,
- $(Female \sqcap {<}2\, hasChild.\top)^{\mathcal{I}} = \{e, h, v\}$. ■

Example 2.10. Let $\Sigma_I = \{a, b, c\}$, $\Sigma_C = \{F, M\}$ and $\Sigma_R = \{r\}$. One can think of these names as *Alice* (a), *Bob* (b), *Claudia* (c), *female* (F), *male* (M), and *has_child* (r). In Figure 2.2 we give three interpretations $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$.

The edges are instances of r. We have, for example, $\Delta^{\mathcal{I}_1} = \{a^{\mathcal{I}_1}, b^{\mathcal{I}_1}, c^{\mathcal{I}_1}, u_1, u_2, u_3\}$, where these six elements are pairwise different, $F^{\mathcal{I}_1} = \{a^{\mathcal{I}_1}, c^{\mathcal{I}_1}, u_2\}$, and $M^{\mathcal{I}_1} = \{b^{\mathcal{I}_1}, u_1, u_3\}$.[3] All of these interpretations are models of the following ABox in $\mathcal{L}_{IOQ}$, where r^- can be read as *has_parent*:

$$\{\ F(a),\ M(b),\ F(c),\ (\exists r.(\exists r^-.\{b\} \sqcap \geq 2\, r.\exists r^-.\{c\}))(a)\ \}$$

Assuming that r means *has_child*, then the last assertion of the above ABox means "a and b have a child which in turn has at least two children with c".

All the interpretations $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ validate the terminological axioms $\neg F \sqsubseteq M$ and $\{a\} \sqsubseteq \forall r^*.(\{a\} \sqcup \geq 2\, r^-.\top)$ of $\mathcal{L}_{IOQ}$. ■

[3]The elements u_i, v_j, w_k are unnamed objects. (The elements of Σ_I can be called *named individuals*, while the elements u_i, v_j, w_k can be called *unnamed individuals*.)

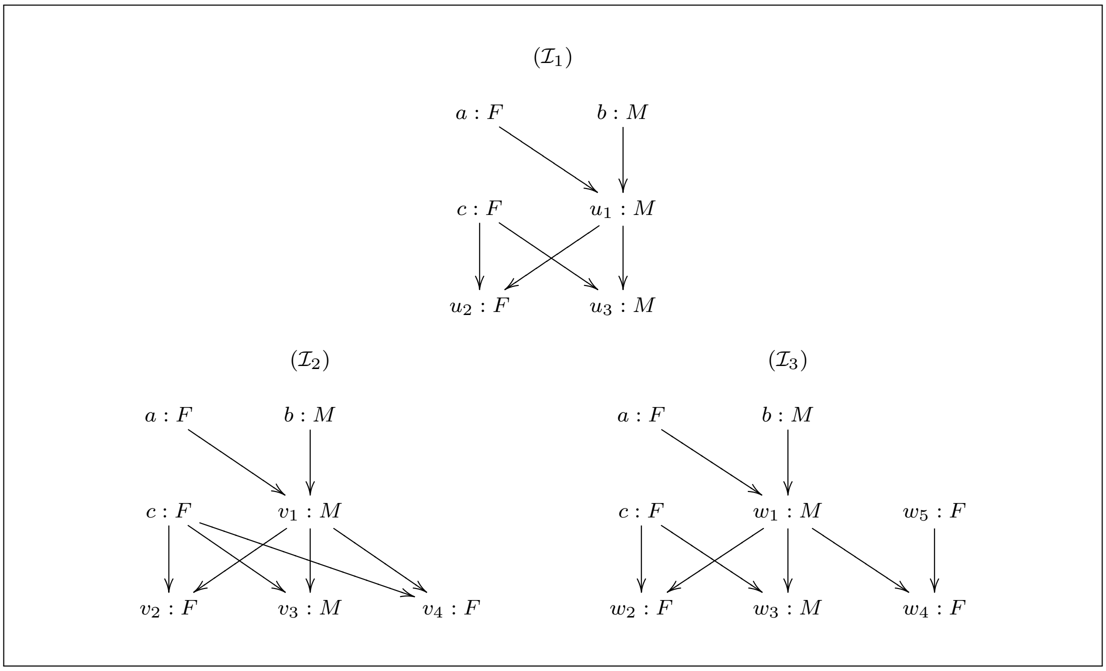


Figure 2.2: Interpretations used in Examples 2.10 and 3.4.

Chapter 3

Bisimulations for Description Logics

In this chapter we present conditions for bisimulation in a uniform way for the whole considered family of DLs. A special point of our approach is that named individuals are treated as initial states, which requires an appropriate condition for bisimulation. As far as we know, bisimulation conditions for the universal role and the concept constructor $\exists r.\mathtt{Self}$ are first given by us. We prove the standard invariance property (Theorem 3.4) and the Hennessy-Milner property (Theorem 3.12) and address the following problems:

- When is a TBox invariant for bisimulation? (Corollary 3.5 and Theorem 3.6)
- When is an ABox invariant for bisimulation? (Theorem 3.7)
- What can be said about preservation of RBoxes w.r.t. bisimulation? (Theorem 3.8)
- What can be said about invariance or preservation of knowledge bases w.r.t. bisimulation? (Theorems 3.9 and 3.10)

Furthermore, we give results (Theorems 3.15, 3.16, 3.17, 3.19 and 3.20) on the largest auto-bisimulation of an interpretation in a DL, the quotient interpretation w.r.t. that equivalence relation, and minimality of such a quotient interpretation. To deal with minimizing interpretations for the case when the considered logic allows qualified number restrictions and/or the concept constructor $\exists r.\mathtt{Self}$, we introduce a new notion called QS-interpretation, which is needed for obtaining expected results.

Computing the largest auto-bisimulations in modal logics and state transition systems is standard like Hopcroft's automaton minimization algorithm [28] and the Paige-Tarjan algorithm [49]. By adapting these algorithms, we give efficient algorithms for computing the partition corresponding to the largest auto-bisimulation of a finite interpretation in any DL of the considered family. The adaptation involves the allowed constructors of the considered DLs.

This chapter is structured as follows. In Section 3.1 we define bisimulations in those DLs and give our results on invariance and preservation w.r.t. such bisimulations. In

Section 3.2 we give our results on the Hennessy-Milner property of the considered DLs. Section 3.3 is devoted to auto-bisimulation and minimization. Section 3.4 is devoted to computing the partition corresponding to the largest auto-bisimulation of a finite interpretation. Section 3.5 discusses applications of minimizing interpretations.

3.1 Bisimulations and Invariance Results

Definition 3.1 (Bisimulation).
Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations. A non-empty binary relation $Z \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}$ is called an $\mathcal{L}_\Phi$-*bisimulation* between $\mathcal{I}$ and $\mathcal{I}'$ if the following conditions hold for every $a \in \Sigma_I$, $A \in \Sigma_C$, $r \in \Sigma_R$, $x, y \in \Delta^{\mathcal{I}}$, $x', y' \in \Delta^{\mathcal{I}'}$:

$$Z(a^{\mathcal{I}}, a^{\mathcal{I}'}) \tag{3.1}$$
$$Z(x, x') \Rightarrow [A^{\mathcal{I}}(x) \Leftrightarrow A^{\mathcal{I}'}(x')] \tag{3.2}$$
$$[Z(x, x') \wedge r^{\mathcal{I}}(x, y)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y, y') \wedge r^{\mathcal{I}'}(x', y')] \tag{3.3}$$
$$[Z(x, x') \wedge r^{\mathcal{I}'}(x', y')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z(y, y') \wedge r^{\mathcal{I}}(x, y)], \tag{3.4}$$

if $I \in \Phi$ then

$$[Z(x, x') \wedge r^{\mathcal{I}}(y, x)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y, y') \wedge r^{\mathcal{I}'}(y', x')] \tag{3.5}$$
$$[Z(x, x') \wedge r^{\mathcal{I}'}(y', x')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z(y, y') \wedge r^{\mathcal{I}}(y, x)], \tag{3.6}$$

if $O \in \Phi$ then

$$Z(x, x') \Rightarrow [x = a^{\mathcal{I}} \Leftrightarrow x' = a^{\mathcal{I}'}], \tag{3.7}$$

if $Q \in \Phi$ then

if $Z(x, x')$ holds and $y_1, \ldots, y_n$ ($n \geq 1$) are pairwise different elements of $\Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y_i)$ holds for every $1 \leq i \leq n$ then there exist pairwise different elements $y'_1, \ldots, y'_n$ of $\Delta^{\mathcal{I}'}$ such that $r^{\mathcal{I}'}(x', y'_i)$ and $Z(y_i, y'_i)$ hold for every $1 \leq i \leq n$ (3.8)

if $Z(x, x')$ holds and $y'_1, \ldots, y'_n$ ($n \geq 1$) are pairwise different elements of $\Delta^{\mathcal{I}'}$ such that $r^{\mathcal{I}'}(x', y'_i)$ holds for every $1 \leq i \leq n$ then there exist pairwise different elements $y_1, \ldots, y_n$ of $\Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y_i)$ and $Z(y_i, y'_i)$ hold for every $1 \leq i \leq n$, (3.9)

if $\{Q, I\} \subseteq \Phi$ then (additionally)

if $Z(x, x')$ holds and $y_1, \ldots, y_n$ ($n \geq 1$) are pairwise different elements of $\Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(y_i, x)$ holds for every $1 \leq i \leq n$ then there exist pairwise different elements $y'_1, \ldots, y'_n$ of $\Delta^{\mathcal{I}'}$ such that $r^{\mathcal{I}'}(y'_i, x')$ and $Z(y_i, y'_i)$ hold for every $1 \leq i \leq n$ (3.10)

if $Z(x, x')$ holds and $y'_1, \ldots, y'_n$ ($n \geq 1$) are pairwise different elements of $\Delta^{\mathcal{I}'}$ such that $r^{\mathcal{I}'}(y'_i, x')$ holds for every $1 \leq i \leq n$ then there exist pairwise different elements $y_1, \ldots, y_n$ of $\Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(y_i, x)$ and $Z(y_i, y'_i)$ hold for every $1 \leq i \leq n$, (3.11) if $U \in \Phi$ then

$$\forall x \in \Delta^{\mathcal{I}}\, \exists x' \in \Delta^{\mathcal{I}'}\, Z(x, x') \tag{3.12}$$

$$\forall x' \in \Delta^{\mathcal{I}'}\, \exists x \in \Delta^{\mathcal{I}}\, Z(x, x'), \tag{3.13}$$

if $\mathtt{Self} \in \Phi$ then

$$Z(x, x') \Rightarrow [r^{\mathcal{I}}(x, x) \Leftrightarrow r^{\mathcal{I}'}(x', x')]. \tag{3.14}$$

For example, if $\Phi = \{I, Q\}$ then only the conditions (3.1)-(3.6) and (3.8)-(3.11) (and all of them) are essential. ■

Notice that our bisimulation conditions (3.8)-(3.11) for qualified number restrictions are relatively simpler than the ones given for graded modalities in [13, 12].

Definition 3.2 (Finitely Branching Interpretation).
An interpretation $\mathcal{I}$ is *finitely branching* (or *image-finite*) w.r.t. $\mathcal{L}_\Phi$ if, for every $x \in \Delta^{\mathcal{I}}$ and every basic role R of $\mathcal{L}_\Phi$, the set $\{y \in \Delta^{\mathcal{I}} \mid R^{\mathcal{I}}(x, y)\}$ is finite. ■

Observe that, if $\mathcal{I}$ and $\mathcal{I}'$ are finitely branching interpretations, then:

- the combination of the conditions (3.8) and (3.9) is equivalent to:

  if $Z(x, x')$ holds then there exists a bijection $h : \{y \mid r^{\mathcal{I}}(x, y)\} \to \{y' \mid r^{\mathcal{I}'}(x', y')\}$ such that $h \subseteq Z$,

- the combination of the conditions (3.10) and (3.11) is equivalent to:

  if $Z(x, x')$ holds then there exists a bijection $h : \{y \mid r^{\mathcal{I}}(y, x)\} \to \{y' \mid r^{\mathcal{I}'}(y', x')\}$ such that $h \subseteq Z$.

Lemma 3.1.

1. *The relation $\{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}$.*
2. *If Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ then Z^{-1} is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}'$ and $\mathcal{I}$.*
3. *If Z_1 is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1$, and Z_2 is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$, then $Z_1 \circ Z_2$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_2$.*
4. *If $\mathcal{Z}$ is a set of $\mathcal{L}_\Phi$-bisimulations between $\mathcal{I}$ and $\mathcal{I}'$ then $\bigcup \mathcal{Z}$ is also an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.*

The proof of this lemma is straightforward.

Definition 3.3 ($\mathcal{L}_\Phi$-Bisimilarity).
An interpretation $\mathcal{I}$ is $\mathcal{L}_\Phi$-*bisimilar* to $\mathcal{I}'$ if there exists an $\mathcal{L}_\Phi$-bisimulation between them. We say that $x \in \Delta^{\mathcal{I}}$ is $\mathcal{L}_\Phi$-*bisimilar* to $x' \in \Delta^{\mathcal{I}'}$ if there exists an $\mathcal{L}_\Phi$-bisimulation Z between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z(x, x')$ holds. ■

Algorithm 1: checking $\mathcal{L}_\Phi$-bisimilarity of two finite interpretations

input : a set Φ of DL-features and finite interpretations $\mathcal{I}$, $\mathcal{I}'$
output: an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ if they are $\mathcal{L}_\Phi$-bisimilar, or *false* otherwise.

$Z := \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}$;
repeat
  foreach $x \in \Delta^{\mathcal{I}}$ *and* $x' \in \Delta^{\mathcal{I}'}$ **do**
    if *some condition among (3.2)-(3.11), (3.14) is related to* Φ *but not satisfied for some* A, r, y, y', a **then** delete the pair $\langle x, x' \rangle$ from Z
until Z *was not modified during the last iteration*;
if *the condition (3.1) is not satisfied for some* $a \in \Sigma_I$ **then return** *false*;
if $U \in \Phi$ *and the condition (3.12) or (3.13) is not satisfied* **then return** *false*;
return Z;

By Lemma 3.1, the former $\mathcal{L}_\Phi$-bisimilarity relation is an equivalence relation between interpretations. The latter $\mathcal{L}_\Phi$-bisimilarity relation is also an equivalence relation (between elements of interpretations' domains).

To check whether two finite interpretations $\mathcal{I}$ and $\mathcal{I}'$ are $\mathcal{L}_\Phi$-bisimilar to each other, one can use Algorithm 1 (on page 20). It is straightforward to prove the following proposition.

Proposition 3.2. *Algorithm 1 is correct. Furthermore, if it returns Z (but not "false") then Z is a maximal $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.*

Example 3.4. Consider the interpretations $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ given in Figure 2.2 (on page 16) and described in Example 2.10.

- By using Algorithm 1, it can be checked that all the interpretations $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ are $\mathcal{L}$-bisimilar. For example, running Algorithm 1 for $\mathcal{I}_1$ and $\mathcal{I}_2$ with $\Phi = \emptyset$ results in $Z = \{\langle a^{\mathcal{I}_1}, a^{\mathcal{I}_2}\rangle, \langle b^{\mathcal{I}_1}, b^{\mathcal{I}_2}\rangle, \langle c^{\mathcal{I}_1}, c^{\mathcal{I}_2}\rangle, \langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \langle u_2, v_4\rangle, \langle u_3, v_3\rangle\}$. By Proposition 3.2, this is a maximal $\mathcal{L}$-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$.

- Let us construct a minimal $\mathcal{L}$-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$. We try to construct a minimal relation $Z \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ that satisfies the conditions (3.1)-(3.4). Recall that $\Delta^{\mathcal{I}_1} = \{a^{\mathcal{I}_1}, b^{\mathcal{I}_1}, c^{\mathcal{I}_1}, u_1, u_2, u_3\}$ and $\Delta^{\mathcal{I}_2} = \{a^{\mathcal{I}_2}, b^{\mathcal{I}_2}, c^{\mathcal{I}_2}, v_1, v_2, v_3, v_4\}$. To satisfy the condition (3.1), $Z(x^{\mathcal{I}_1}, x^{\mathcal{I}_2})$ must hold for $x \in \{a, b, c\}$. To satisfy the condition (3.3) for the case when $x = a^{\mathcal{I}_1}$, $x' = a^{\mathcal{I}_2}$ and $y = u_1$, $Z(u_1, v_1)$ must hold. Observe that, due to the condition (3.2) with $A = M$, none of the pairs $\langle u_2, v_3\rangle$, $\langle u_3, v_2\rangle$, $\langle u_3, v_4\rangle$ belongs to Z. Since $Z(u_1, v_1)$ holds, to satisfy the condition (3.3) for the case when $x = u_1$, $x' = v_1$ and $y = u_3$, $Z(u_3, v_3)$ must hold. Similarly, to satisfy the condition (3.4) for the case when $x = u_1$, $x' = v_1$ and $y' = v_2$ (resp. $y' = v_4$), we must have that $\langle u_2, v_2\rangle \in Z$ (resp. $\langle u_2, v_4\rangle \in Z$). Summing up, we must have $\{\langle a^{\mathcal{I}_1}, a^{\mathcal{I}_2}\rangle, \langle b^{\mathcal{I}_1}, b^{\mathcal{I}_2}\rangle, \langle c^{\mathcal{I}_1}, c^{\mathcal{I}_2}\rangle, \langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \langle u_2, v_4\rangle, \langle u_3, v_3\rangle\} \subseteq Z$. Let Z be the set in the left hand side of this inclusion. It is easy to check that

Z satisfies all the conditions (3.1)-(3.4). Hence, Z is a minimal $\mathcal{L}$-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$. Together with the above item, it follows that this is the unique $\mathcal{L}$-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$.

- Running Algorithm 1 for $\mathcal{I}_1$ and $\mathcal{I}_2$ with $\Phi = \{I, O\}$ results in the same set $Z = \{\langle a^{\mathcal{I}_1}, a^{\mathcal{I}_2}\rangle, \langle b^{\mathcal{I}_1}, b^{\mathcal{I}_2}\rangle, \langle c^{\mathcal{I}_1}, c^{\mathcal{I}_2}\rangle, \langle u_1, v_1\rangle, \langle u_2, v_2\rangle, \langle u_2, v_4\rangle, \langle u_3, v_3\rangle\}$ as in the case $\Phi = \emptyset$. By Proposition 3.2, Z is a maximal $\mathcal{L}_{IO}$-bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$. It follows that the elements u_2 (of $\mathcal{I}_1$) and v_2, v_4 (of $\mathcal{I}_2$) are $\mathcal{L}_{IO}$-bisimilar.

- Running Algorithm 1 for $\mathcal{I}_1$ and $\mathcal{I}_2$ with $\Phi = \{Q\}$ results in *false*. Hence, $\mathcal{I}_1$ and $\mathcal{I}_2$ are not $\mathcal{L}_Q$-bisimilar. Similarly, the interpretation $\mathcal{I}_3$ is not $\mathcal{L}_I$-bisimilar to $\mathcal{I}_1$ nor $\mathcal{I}_2$. ■

Lemma 3.3. *Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations and Z be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Then the following properties hold for every concept C in $\mathcal{L}_\Phi$, every role R in $\mathcal{L}_\Phi$, every $x, y \in \Delta^\mathcal{I}$, every $x', y' \in \Delta^{\mathcal{I}'}$, and every $a \in \Sigma_I$:*

$$Z(x, x') \Rightarrow [C^\mathcal{I}(x) \Leftrightarrow C^{\mathcal{I}'}(x')] \tag{3.15}$$

$$[Z(x, x') \wedge R^\mathcal{I}(x, y)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y, y') \wedge R^{\mathcal{I}'}(x', y')] \tag{3.16}$$

$$[Z(x, x') \wedge R^{\mathcal{I}'}(x', y')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z(y, y') \wedge R^{\mathcal{I}}(x, y)] \tag{3.17}$$

if $O \in \Phi$ then:

$$Z(x, x') \Rightarrow [R^\mathcal{I}(x, a^\mathcal{I}) \Leftrightarrow R^{\mathcal{I}'}(x', a^{\mathcal{I}'})]. \tag{3.18}$$

Proof. We prove this lemma by induction on the structures of C and R.

Consider the assertion (3.16). Suppose $Z(x, x')$ and $R^\mathcal{I}(x, y)$ hold. By induction on the structure of R we prove that there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y, y')$ and $R^{\mathcal{I}'}(x', y')$ hold. The base case occurs when R is a role name and the assertion for it follows from (3.3). The induction steps are given below.

- Case $R = S_1 \circ S_2$: We have that $(S_1 \circ S_2)^\mathcal{I}(x, y)$ holds. Hence, there exists $z \in \Delta^\mathcal{I}$ such that $S_1^\mathcal{I}(x, z)$ and $S_2^\mathcal{I}(z, y)$ hold. Since $Z(x, x')$ and $S_1^\mathcal{I}(x, z)$ hold, by the inductive assumption of (3.16), there exists $z' \in \Delta^{\mathcal{I}'}$ such that $Z(z, z')$ and $S_1^{\mathcal{I}'}(x', z')$ hold. Since $Z(z, z')$ and $S_2^\mathcal{I}(z, y)$ hold, by the inductive assumption of (3.16), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y, y')$ and $S_2^{\mathcal{I}'}(z', y')$ hold. Since $S_1^{\mathcal{I}'}(x', z')$ and $S_2^{\mathcal{I}'}(z', y')$ hold, we have that $(S_1 \circ S_2)^{\mathcal{I}'}(x', y')$ holds, i.e. $R^{\mathcal{I}'}(x', y')$ holds.

- Case $R = S_1 \sqcup S_2$ is trivial.

- Case $R = S^*$: Since $R^\mathcal{I}(x, y)$ holds, there exist $x_0, \ldots, x_k \in \Delta^\mathcal{I}$ with $k \geq 0$ such that $x_0 = x$, $x_k = y$ and, for $1 \leq i \leq k$, $S^\mathcal{I}(x_{i-1}, x_i)$ holds. Let $x'_0 = x'$. For each $1 \leq i \leq k$, since $Z(x_{i-1}, x'_{i-1})$ and $S^\mathcal{I}(x_{i-1}, x_i)$ hold, by the inductive assumption of (3.16), there exists $x'_i \in \Delta^{\mathcal{I}'}$ such that $Z(x_i, x'_i)$ and $S^{\mathcal{I}'}(x'_{i-1}, x'_i)$ hold. Hence, $Z(x_k, x'_k)$ and $(S^*)^{\mathcal{I}'}(x'_0, x'_k)$ hold. Let $y' = x'_k$. Thus, $Z(y, y')$ and $R^{\mathcal{I}'}(x', y')$ hold.

- Case $R = (D?)$: Since $R^{\mathcal{I}}(x, y)$ holds, we have that $D^{\mathcal{I}}(x)$ holds and $x = y$. Since $Z(x, x')$ and $D^{\mathcal{I}}(x)$ hold, by the inductive assumption of (3.15), $D^{\mathcal{I}'}(x')$ also holds, and hence $R^{\mathcal{I}'}(x', x')$ holds. By choosing $y' = x'$, both $Z(y, y')$ and $R^{\mathcal{I}'}(x', y')$ hold.

- Case $I \in \Phi$ and $R = r^-$: The assertion for this case follows from (3.5).

By Lemma 3.1(2), the assertion (3.17) follows from the assertion (3.16).

Consider the assertion (3.18) and suppose $O \in \Phi$. By Lemma 3.1(2), it suffices to show that if $Z(x, x')$ and $R^{\mathcal{I}}(x, a^{\mathcal{I}})$ hold then $R^{\mathcal{I}'}(x', a^{\mathcal{I}'})$ also holds. We prove this by using similar argumentation as for (3.16). Suppose $Z(x, x')$ and $R^{\mathcal{I}}(x, a^{\mathcal{I}})$ hold. We prove that $R^{\mathcal{I}'}(x', a^{\mathcal{I}'})$ also holds by induction on the structure of R. The base case occurs when R is a role name and the assertion for it follows from (3.3) and (3.7). The induction steps are given below.

- Case $R = S_1 \circ S_2$: We have that $(S_1 \circ S_2)^{\mathcal{I}}(x, a^{\mathcal{I}})$ holds. Hence, there exists $y \in \Delta^{\mathcal{I}}$ such that $S_1^{\mathcal{I}}(x, y)$ and $S_2^{\mathcal{I}}(y, a^{\mathcal{I}})$ hold. Since $Z(x, x')$ and $S_1^{\mathcal{I}}(x, y)$ hold, by the inductive assumption of (3.16), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y, y')$ and $S_1^{\mathcal{I}'}(x', y')$ hold. Since $Z(y, y')$ and $S_2^{\mathcal{I}}(y, a^{\mathcal{I}})$ hold, by the inductive assumption of (3.18), $S_2^{\mathcal{I}'}(y', a^{\mathcal{I}'})$ holds. Since $S_1^{\mathcal{I}'}(x', y')$ and $S_2^{\mathcal{I}'}(y', a^{\mathcal{I}'})$ hold, we have that $(S_1 \circ S_2)^{\mathcal{I}'}(x', a^{\mathcal{I}'})$ holds, i.e. $R^{\mathcal{I}'}(x', a^{\mathcal{I}'})$ holds.

- Case $R = S_1 \sqcup S_2$ is trivial.

- Case $R = S^*$: Since $R^{\mathcal{I}}(x, a^{\mathcal{I}})$ holds, there exist $x_0, \ldots, x_k \in \Delta^{\mathcal{I}}$ with $k \geq 0$ such that $x_0 = x$, $x_k = a^{\mathcal{I}}$ and, for $1 \leq i \leq k$, $S^{\mathcal{I}}(x_{i-1}, x_i)$ holds.
  - Case $k = 0$: We have that $x = a^{\mathcal{I}}$. Since $Z(x, x')$ holds, by (3.7), it follows that $x' = a^{\mathcal{I}'}$. Hence $R^{\mathcal{I}'}(x', a^{\mathcal{I}'})$ holds.
  - Case $k > 0$: Let $x'_0 = x'$. For each $1 \leq i < k$, since $Z(x_{i-1}, x'_{i-1})$ and $S^{\mathcal{I}}(x_{i-1}, x_i)$ hold, by the inductive assumption of (3.16), there exists $x'_i \in \Delta^{\mathcal{I}'}$ such that $Z(x_i, x'_i)$ and $S^{\mathcal{I}'}(x'_{i-1}, x'_i)$ hold. Hence, $Z(x_{k-1}, x'_{k-1})$ and $(S^*)^{\mathcal{I}'}(x'_0, x'_{k-1})$ hold. Since $Z(x_{k-1}, x'_{k-1})$ and $S^{\mathcal{I}}(x_{k-1}, a^{\mathcal{I}})$ hold, by the inductive assumption of (3.18), we have that $S^{\mathcal{I}'}(x'_{k-1}, a^{\mathcal{I}'})$ holds. Since $(S^*)^{\mathcal{I}'}(x'_0, x'_{k-1})$ holds, it follows that $R^{\mathcal{I}'}(x', a^{\mathcal{I}'})$ holds.

- Case $R = (D?)$: Since $R^{\mathcal{I}}(x, a^{\mathcal{I}})$ holds, we have that $x = a^{\mathcal{I}}$ and $D^{\mathcal{I}}(a^{\mathcal{I}})$ holds. Since $Z(x, x')$ holds, by (3.7), it follows that $x' = a^{\mathcal{I}'}$. Since $Z(a^{\mathcal{I}}, a^{\mathcal{I}'})$ and $D^{\mathcal{I}}(a^{\mathcal{I}})$ hold, by the inductive assumption of (3.15), $D^{\mathcal{I}'}(a^{\mathcal{I}'})$ also holds. Since $x' = a^{\mathcal{I}'}$, it follows that $R^{\mathcal{I}'}(x', a^{\mathcal{I}'})$ holds.

- Case $I \in \Phi$ and $R = r^-$: The assertion for this case follows from (3.5) and (3.7).

Consider the assertion (3.15). By Lemma 3.1(2), it suffices to show that if $Z(x, x')$ and $C^{\mathcal{I}}(x)$ hold then $C^{\mathcal{I}'}(x')$ also holds. Suppose $Z(x, x')$ and $C^{\mathcal{I}}(x)$ hold. The cases when C is of the form $\top$, $\bot$, A, $\neg D$, $D \sqcup D'$ or $D \sqcap D'$ are trivial.

- Case $C = \exists R.D$: Since $C^{\mathcal{I}}(x)$ holds, there exists $y \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y)$ and $D^{\mathcal{I}}(y)$ hold. Since $Z(x, x')$ and $R^{\mathcal{I}}(x, y)$ hold, by the assertion (3.16) (proved earlier), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y, y')$ and $R^{\mathcal{I}'}(x', y')$ hold. Since $Z(y, y')$ and $D^{\mathcal{I}}(y)$ hold, by the inductive assumption of (3.15), it follows that $D^{\mathcal{I}'}(y')$ holds. Therefore, $C^{\mathcal{I}'}(x')$ holds.

- Case $C = \forall R.D$ is reduced to the above case, treating $\forall R.D$ as $\neg\exists R.\neg D$.
- Case $O \in \Phi$ and $C = \{a\}$: Since $C^{\mathcal{I}}(x)$ holds, we have that $x = a^{\mathcal{I}}$. Since $Z(x, x')$ holds, by (3.7), it follows that $x' = a^{\mathcal{I}'}$. Hence $C^{\mathcal{I}'}(x')$ holds.
- Case $Q \in \Phi$ and $C = (\geq n\, R.D)$, where R is a basic role: Since $C^{\mathcal{I}}(x)$ holds, there exist pairwise different $y_1, \ldots, y_n \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y_i)$ and $D^{\mathcal{I}}(y_i)$ hold for all $1 \leq i \leq n$. Since $Z(x, x')$ holds, by the conditions (3.8) and (3.10), there exist pairwise different $y'_1, \ldots, y'_n \in \Delta^{\mathcal{I}'}$ such that $R^{\mathcal{I}'}(x', y'_i)$ and $Z(y_i, y'_i)$ hold for all $1 \leq i \leq n$. Since $Z(y_i, y'_i)$ and $D^{\mathcal{I}}(y_i)$ hold, by the inductive assumption of (3.15), it follows that $D^{\mathcal{I}'}(y'_i)$ holds. Since $R^{\mathcal{I}'}(x', y'_i)$ and $D^{\mathcal{I}'}(y'_i)$ hold for all $1 \leq i \leq n$, it follows that $C^{\mathcal{I}'}(x')$ holds.
- Case $Q \in \Phi$ and $C = (\leq n\, R.D)$, where R is a basic role: This case is reduced to the above case, treating $\leq n\, R.D$ as $\neg(\geq (n+1)\, R.D)$.
- Case $\mathtt{Self} \in \Phi$ and $C = \exists r.\mathtt{Self}$: Since $C^{\mathcal{I}}(x)$ holds, we have that $r^{\mathcal{I}}(x, x)$ holds. By (3.14), it follows that $r^{\mathcal{I}'}(x', x')$ holds. Hence $C^{\mathcal{I}'}(x')$ holds. ∎

Definition 3.5 (**Invariance of a Concept).**
A concept C in $\mathcal{L}_\Phi$ is said to be *invariant for $\mathcal{L}_\Phi$-bisimulation* if, for any interpretations $\mathcal{I}$, $\mathcal{I}'$ and any $\mathcal{L}_\Phi$-bisimulation Z between $\mathcal{I}$ and $\mathcal{I}'$, if $Z(x, x')$ holds then $x \in C^{\mathcal{I}}$ iff $x' \in C^{\mathcal{I}'}$. ∎

Theorem 3.4. *All concepts in $\mathcal{L}_\Phi$ are invariant for $\mathcal{L}_\Phi$-bisimulation.*

This theorem follows immediately from the assertion (3.15) of Lemma 3.3.

Definition 3.6 (**Invariance of a TBox, an ABox or a Knowledge Base).**
A TBox $\mathcal{T}$ in $\mathcal{L}_\Phi$ is said to be *invariant for $\mathcal{L}_\Phi$-bisimulation* if, for every interpretations $\mathcal{I}$ and $\mathcal{I}'$, if there exists an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ then $\mathcal{I}$ is a model of $\mathcal{T}$ iff $\mathcal{I}'$ is a model of $\mathcal{T}$. The notions of whether an ABox or a knowledge base in $\mathcal{L}_\Phi$ is invariant for $\mathcal{L}_\Phi$-bisimulation are defined similarly. ∎

Corollary 3.5. *If $U \in \Phi$ then all TBoxes in $\mathcal{L}_\Phi$ are invariant for $\mathcal{L}_\Phi$-bisimulation.*

Proof. Suppose $U \in \Phi$ and let $\mathcal{T}$ be a TBox in $\mathcal{L}_\Phi$ and $\mathcal{I}, \mathcal{I}'$ be interpretations. Suppose that $\mathcal{I}$ is a model of $\mathcal{T}$, and Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. We show that $\mathcal{I}'$ is a model of $\mathcal{T}$. Let $C \sqsubseteq D$ be an axiom from $\mathcal{T}$ and let $x' \in \Delta^{\mathcal{I}'}$. We need to show that $x' \in (\neg C \sqcup D)^{\mathcal{I}'}$. By (3.13), there exists $x \in \Delta^{\mathcal{I}}$ such that $Z(x, x')$ holds. Since $\mathcal{I}$ is a model of $\mathcal{T}$, we have that $x \in (\neg C \sqcup D)^{\mathcal{I}}$, which, by Theorem 3.4, implies that $x' \in (\neg C \sqcup D)^{\mathcal{I}'}$. ∎

Definition 3.7 (**Unreachable-Objects-Free Interpretation).**
An interpretation $\mathcal{I}$ is said to be *unreachable-objects-free* (w.r.t. the considered language) if every element of $\Delta^{\mathcal{I}}$ is reachable from some $a^{\mathcal{I}}$, where $a \in \Sigma_I$, via a path consisting of edges being instances of basic roles. ∎

It is clear that, if $U \notin \Phi$, C is a concept of $\mathcal{L}_\Phi$ and $a \in \Sigma_I$, then $\mathcal{I} \models C(a)$ iff $\mathcal{I}' \models C(a)$, where $\mathcal{I}'$ is the unreachable-objects-free interpretation obtained from $\mathcal{I}$ by deleting from the domain unreachable objects. That is, when $U \notin \Phi$, unreachable objects are redundant for the instance checking problem. Therefore, it is worth considering the class of unreachable-objects-free interpretations.

Like Corollary 3.5, the following theorem concerns invariance of TBoxes w.r.t. $\mathcal{L}_\Phi$-bisimulation.

Theorem 3.6. *Let $\mathcal{T}$ be a TBox in $\mathcal{L}_\Phi$ and $\mathcal{I}$, $\mathcal{I}'$ be unreachable-objects-free interpretations (w.r.t. $\mathcal{L}_\Phi$) such that there exists an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Then $\mathcal{I}$ is a model of $\mathcal{T}$ iff $\mathcal{I}'$ is a model of $\mathcal{T}$.*

Proof. Let Z be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. By Lemma 3.1(2), it suffices to show that if $\mathcal{I}$ is a model of $\mathcal{T}$ then $\mathcal{I}'$ is also a model of $\mathcal{T}$. Suppose $\mathcal{I}$ is a model of $\mathcal{T}$. Let $C \sqsubseteq D$ be an axiom from $\mathcal{T}$. We need to show that $C^{\mathcal{I}'} \subseteq D^{\mathcal{I}'}$. Let $x' \in C^{\mathcal{I}'}$. We show that $x' \in D^{\mathcal{I}'}$.

Since $\mathcal{I}'$ is an unreachable-objects-free interpretation, there exist elements $x'_0, \ldots, x'_k$ of $\Delta^{\mathcal{I}'}$ and basic roles $R_1, \ldots, R_k$ with $k \geq 0$ such that $x'_0 = a^{\mathcal{I}'}$ for some $a \in \Sigma_I$, $x'_k = x'$ and, for $1 \leq i \leq k$, $R_i^{\mathcal{I}'}(x'_{i-1}, x'_i)$ holds.

By (3.1), $Z(a^{\mathcal{I}}, a^{\mathcal{I}'})$ holds. Let $x_0 = a^{\mathcal{I}}$. For each $1 \leq i \leq k$, since $Z(x_{i-1}, x'_{i-1})$ and $R_i^{\mathcal{I}'}(x'_{i-1}, x'_i)$ hold, by (3.17), there exists $x_i \in \Delta^{\mathcal{I}}$ such that $Z(x_i, x'_i)$ and $R_i^{\mathcal{I}}(x_{i-1}, x_i)$ hold. Let $x = x_k$. Thus, $Z(x, x')$ holds. Since $x' \in C^{\mathcal{I}'}$, by Theorem 3.4, we have that $x \in C^{\mathcal{I}}$. Since $\mathcal{I}$ is a model of $\mathcal{T}$, it follows that $x \in D^{\mathcal{I}}$. By Theorem 3.4, we derive that $x' \in D^{\mathcal{I}'}$, which completes the proof. ■

To justify that Corollary 3.5 and Theorem 3.6 are as strong as possible, we present here a simple example with $U \notin \Phi$ and one of $\mathcal{I}$, $\mathcal{I}'$ being not unreachable-objects-free such that $\mathcal{I}$ and $\mathcal{I}'$ are $\mathcal{L}_\Phi$-bisimilar but there exists a TBox $\mathcal{T}$ such that $\mathcal{I} \models \mathcal{T}$ and $\mathcal{I}' \not\models \mathcal{T}$:

Example 3.8. Assume that $U \notin \Phi$ and let $\Sigma_C = \{A\}$, $\Sigma_R = \emptyset$, $\Sigma_I = \{a\}$ (i.e., the signature consists of only concept name A and individual name a). Let $\mathcal{I}$ and $\mathcal{I}'$ be the interpretations specified by: $\Delta^{\mathcal{I}} = \{a\}$, $\Delta^{\mathcal{I}'} = \{a, u\}$, $a^{\mathcal{I}} = a^{\mathcal{I}'} = a$, $A^{\mathcal{I}} = A^{\mathcal{I}'} = \{a\}$. It is easy to see that $Z = \{\langle a, a\rangle\}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ (it satisfies all of the conditions (3.1)-(3.11) and (3.14)). However, $\mathcal{I}$ is a model of the TBox $\{\top \sqsubseteq A\}$, while $\mathcal{I}'$ is not. ■

As mentioned in the introduction, in the independent work [40] Lutz et al. use the notion of global bisimulation to characterize invariance of TBoxes, whose condition is the same as our bisimulation conditions (3.12) and (3.13) for the universal role. Their result on invariance of TBoxes is not stronger than our Corollary 3.5: one can just add U to Φ, and the considered TBox, which may not use U, is invariant w.r.t. the corresponding bisimulation satisfying the conditions (3.12) and (3.13). Furthermore, the family of DLs considered in this dissertation contains other logics than the DL $\mathcal{ALCQIO}$ considered in [40]. On the matter of originality of our Corollary 3.5 and Theorem 3.6, note that they appeared to the public early in [17].

The following theorem concerns invariance of ABoxes w.r.t. $\mathcal{L}_\Phi$-bisimulation.

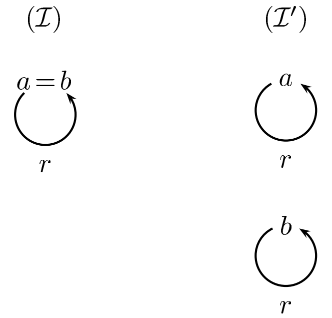
Theorem 3.7. *Let $\mathcal{A}$ be an ABox in $\mathcal{L}_\Phi$. If $O \in \Phi$ or $\mathcal{A}$ contains only assertions of the form $C(a)$ then $\mathcal{A}$ is invariant for $\mathcal{L}_\Phi$-bisimulation.*

Proof. Suppose that $O \in \Phi$ or $\mathcal{A}$ contains only assertions of the form $C(a)$. Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations and let Z be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. By Lemma 3.1(2), it suffices to show that if $\mathcal{I}$ is a model of $\mathcal{A}$ then $\mathcal{I}'$ is also a model of $\mathcal{A}$. Suppose $\mathcal{I}$ is an model of $\mathcal{A}$. Let φ be an assertion from $\mathcal{A}$. We need to show that $\mathcal{I}' \models \varphi$.

- Case $\varphi = (a \doteq b)$: Since $\mathcal{I} \models \varphi$, we have that $a^\mathcal{I} = b^\mathcal{I}$. By (3.1), $Z(a^\mathcal{I}, a^{\mathcal{I}'})$ and $Z(b^\mathcal{I}, b^{\mathcal{I}'})$ hold. Since $a^\mathcal{I} = b^\mathcal{I}$, by (3.7), it follows that $a^{\mathcal{I}'} = b^{\mathcal{I}'}$. Hence $\mathcal{I}' \models \varphi$.
- Case $\varphi = (a \not\doteq b)$ is reduced to the above case, by using Lemma 3.1(2).
- Case $\varphi = C(a)$: By (3.1), $Z(a^\mathcal{I}, a^{\mathcal{I}'})$ holds. Since $\mathcal{I} \models \varphi$, $C^\mathcal{I}(a^\mathcal{I})$ holds. By (3.15), it follows that $C^{\mathcal{I}'}(a^{\mathcal{I}'})$ holds. Thus $\mathcal{I}' \models \varphi$.
- Case $\varphi = R(a, b)$: By (3.1), $Z(a^\mathcal{I}, a^{\mathcal{I}'})$ holds. Since $\mathcal{I} \models \varphi$, $R^\mathcal{I}(a^\mathcal{I}, b^\mathcal{I})$ holds. By (3.16), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(b^\mathcal{I}, y')$ and $R^{\mathcal{I}'}(a^{\mathcal{I}'}, y')$ hold. Consider $C = \{b\}$ (the assumption $O \in \Phi$ is used here). Since $Z(b^\mathcal{I}, y')$ and $C^\mathcal{I}(b^\mathcal{I})$ hold, by (3.15), $C^{\mathcal{I}'}(y')$ holds, which means $y' = b^{\mathcal{I}'}$. Thus $R^{\mathcal{I}'}(a^{\mathcal{I}'}, b^{\mathcal{I}'})$ holds, i.e., $\mathcal{I}' \models \varphi$.
- Case $\varphi = \neg R(a, b)$ is reduced to the above case, by using Lemma 3.1(2). ■

Clearly, the condition "$O \in \Phi$ or $\mathcal{A}$ contains only assertions of the form $C(a)$" of the above theorem covers many useful cases. The following example justifies that this theorem is as strong as possible.

Example 3.9. We show that if $O \notin \Phi$ then none of the ABoxes $\mathcal{A}_1 = \{a \doteq b\}$, $\mathcal{A}_2 = \{a \not\doteq b\}$, $\mathcal{A}_3 = \{r(a, b)\}$, $\mathcal{A}_4 = \{\neg r(a, b)\}$ is invariant for $\mathcal{L}_\Phi$-bisimulation. Assume that $O \notin \Phi$ and let $\Sigma_C = \emptyset$, $\Sigma_I = \{a, b\}$, $\Sigma_R = \{r\}$. Let $\mathcal{I}$ and $\mathcal{I}'$ be the interpretations specified by:



$\Delta^\mathcal{I} = \Delta^{\mathcal{I}'} = \{u, v\}$ with $u \neq v$, $a^\mathcal{I} = b^\mathcal{I} = a^{\mathcal{I}'} = u$, $b^{\mathcal{I}'} = v$, and $r^\mathcal{I} = r^{\mathcal{I}'} = \{\langle u, u\rangle, \langle v, v\rangle\}$. It can be checked that $Z = \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. However:

- $\mathcal{I}$ is a model of $\mathcal{A}_1$, while $\mathcal{I}'$ is not

- $\mathcal{I}'$ is a model of $\mathcal{A}_2$, while $\mathcal{I}$ is not
- $\mathcal{I}$ is a model of $\mathcal{A}_3$, while $\mathcal{I}'$ is not
- $\mathcal{I}'$ is a model of $\mathcal{A}_4$, while $\mathcal{I}$ is not. ■

In general, RBoxes are not invariant for $\mathcal{L}_\Phi$-bisimulations. (The Van Benthem Characterization Theorem states that a first-order formula is invariant for bisimulations iff it is equivalent to the translation of a modal formula (see, e.g., [6]).) We give below a simple example about this:

Example 3.10. Let $\Sigma_C = \emptyset$, $\Sigma_R = \{r\}$, $\Sigma_I = \{a\}$ (i.e., the signature consists of only role name r and individual name a) and $\Phi = \emptyset$. Let $\mathcal{I}$ and $\mathcal{I}'$ be the interpretations specified by:

$$(\mathcal{I}) \quad \begin{tikzcd} a \arrow[r, "r"] & u \arrow[r, "r"] & v \arrow[loop right, "r"] \end{tikzcd}$$

$$(\mathcal{I}') \quad \begin{tikzcd} a \arrow[r, "r"] \arrow[rr, bend right, "r"'] & u \arrow[r, "r"] & v \arrow[loop right, "r"] \end{tikzcd}$$

$\Delta^\mathcal{I} = \Delta^{\mathcal{I}'} = \{a, u, v\}$, $a^\mathcal{I} = a^{\mathcal{I}'} = a$, $r^\mathcal{I} = \{\langle a, u\rangle, \langle u, v\rangle, \langle v, v\rangle\}$ and $r^{\mathcal{I}'} = r^\mathcal{I} \cup \{\langle a, v\rangle\}$. It can be checked that $Z = \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. However, $\mathcal{I}'$ is a model of the RBox $\{r \circ r \sqsubseteq r\}$, while $\mathcal{I}$ is not. ■

Definition 3.11 (Least R-Extension of an Interpretation).
An interpretation $\mathcal{I}'$ is an r-extension of an interpretation $\mathcal{I}$ if $\Delta^{\mathcal{I}'} = \Delta^\mathcal{I}$, $\cdot^{\mathcal{I}'}$ differs from $\cdot^\mathcal{I}$ only in interpreting role names, and for all $r \in \Sigma_R$, $r^{\mathcal{I}'} \supseteq r^\mathcal{I}$.

Given an interpretation $\mathcal{I}$ and an RBox $\mathcal{R}$, the *least r-extension of $\mathcal{I}$ validating $\mathcal{R}$* is the r-extension $\mathcal{I}'$ of $\mathcal{I}$ such that $\mathcal{I}'$ is a model of $\mathcal{R}$ and, for every r-extension $\mathcal{I}''$ of $\mathcal{I}$, if $\mathcal{I}''$ is a model of $\mathcal{R}$ then $r^{\mathcal{I}'} \subseteq r^{\mathcal{I}''}$ for all $r \in \Sigma_R$. ■

The least r-extension exists and is unique because the axioms of $\mathcal{R}$ correspond to non-negative Horn clauses of first-order logic. Namely, a role axiom $\varepsilon \sqsubseteq r$ corresponds to the following Horn clause

$$\forall x\ r(x, x),$$

and a role axiom $R_1 \circ \ldots \circ R_k \sqsubseteq r$ corresponds to the following Horn clause

$$\forall x_0 \ldots \forall x_k\ [R_1(x_0, x_1) \land \ldots \land R_k(x_{k-1}, x_k) \to r(x_0, x_k)],$$

where $s^-(x, y)$ stands for $s(y, x)$. It is clear that one can extend the relations standing for the roles in a minimal way to satisfy all of the Horn clauses corresponding to the axioms of $\mathcal{R}$.

Theorem 3.8. *Suppose $\Phi \subseteq \{I, O, U\}$ and let $\mathcal{R}$ be an RBox in $\mathcal{L}_\Phi$. Let $\mathcal{I}_0$ be a model of $\mathcal{R}$, Z be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and an interpretation $\mathcal{I}_1$, and $\mathcal{I}_1'$ be the least r-extension of $\mathcal{I}_1$ validating $\mathcal{R}$. Then Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1'$.*

This theorem states that, even in the case when interpretations $\mathcal{I}_0$ and $\mathcal{I}_1$ are $\mathcal{L}_\Phi$-bisimilar but $\mathcal{I}_0 \models \mathcal{R}$ while $\mathcal{I}_1 \not\models \mathcal{R}$, we can modify $\mathcal{I}_1$ slightly by adding some edges (i.e. instances of roles) to obtain a model $\mathcal{I}_1'$ of $\mathcal{R}$ that is $\mathcal{L}_\Phi$-bisimilar with $\mathcal{I}_0$ (and hence also with $\mathcal{I}_1$). This theorem is thus natural.

Proof. We only need to prove that, for every $r \in \Sigma_R$, $x \in \Delta_0^{\mathcal{I}}$, $x', y' \in \Delta^{\mathcal{I}_1'}$:

1. $[Z(x,x') \land r^{\mathcal{I}_1'}(x',y')] \Rightarrow \exists y \in \Delta^{\mathcal{I}_0}[Z(y,y') \land r^{\mathcal{I}_0}(x,y)]$

2. if $I \in \Phi$ then $[Z(x,x') \land r^{\mathcal{I}_1'}(y',x')] \Rightarrow \exists y \in \Delta^{\mathcal{I}_0}[Z(y,y') \land r^{\mathcal{I}_0}(y,x)]$.

We prove these assertions by induction on the timestamps of the steps that extend relations $r^{\mathcal{I}_1}$ to $r^{\mathcal{I}_1'}$, for $r \in \Sigma_R$.

Consider the first assertion. Suppose $Z(x,x')$ and $r^{\mathcal{I}_1'}(x',y')$ hold. We need to show there exists $y \in \Delta^{\mathcal{I}_0}$ such that $Z(y,y')$ and $r^{\mathcal{I}_0}(x,y)$ hold. There are the following three cases:

- Case $r^{\mathcal{I}_1'}(x',y')$ holds because $r^{\mathcal{I}_1}(x',y')$ holds: The assertion holds because Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1$.

- Case $r^{\mathcal{I}_1'}(x',y')$ holds because $(\varepsilon \sqsubseteq r) \in \mathcal{R}$ and $y' = x'$: Take $y = x$. Thus, $Z(y,y')$ holds. Since $\mathcal{I}_0$ is a model of $\mathcal{R}$, it validates $\varepsilon \sqsubseteq r$, and hence $r^{\mathcal{I}_0}(x,y)$ also holds.

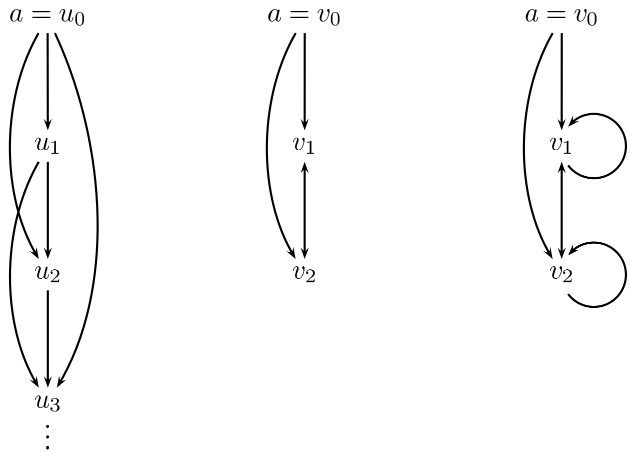
- Case $r^{\mathcal{I}_1'}(x',y')$ holds because $R_1 \circ \ldots \circ R_k \sqsubseteq r$ is an axiom of $\mathcal{R}$ and there exist $x_0' = x, x_1', \ldots, x_{k-1}', x_k' = y'$ such that $R_i^{\mathcal{I}_1'}(x_{i-1}', x_i')$ holds for all $1 \leq i \leq k$: Let $x_0 = x$. For each $1 \leq i \leq k$, since $Z(x_{i-1}, x_{i-1}')$ and $R_i^{\mathcal{I}_1'}(x_{i-1}', x_i')$ hold, by the inductive assumptions of both the assertions, there exists $x_i \in \Delta^{\mathcal{I}_0}$ such that $Z(x_i, x_i')$ and $R_i^{\mathcal{I}_0}(x_{i-1}, x_i)$ hold. Thus, $Z(x_k, x_k')$ holds. Since $\mathcal{I}_0$ validates the axiom $R_1 \circ \ldots \circ R_k \sqsubseteq r$ of $\mathcal{R}$, we also have that $r^{\mathcal{I}_0}(x_0, x_k)$ holds. We choose $y = x_k$ and finish with the proof of the first assertion.

The proof of the second assertion is similar to the proof of the first one. ■

Example 3.12. To justify that the form of the above theorem is as strong as possible, we show that allowing either Q or `Self` in Φ can make the theorem wrong. In the following: $u_i \neq u_j$ if $i \neq j$; $v_i \neq v_j$ if $i \neq j$; and $u_i \neq v_j$ for all i, j. Here are examples:

1. Assume that `Self` $\in \Phi$ and $\Phi \subseteq \{\mathtt{Self}, O, U\}$. Let $\Sigma_C = \emptyset$, $\Sigma_I = \{a\}$ and $\Sigma_R = \{r\}$. Let $\mathcal{I}_0$ and $\mathcal{I}_1$ be the interpretations specified by:

$(\mathcal{I}_0)$ $(\mathcal{I}_1)$ $(\mathcal{I}'_1)$

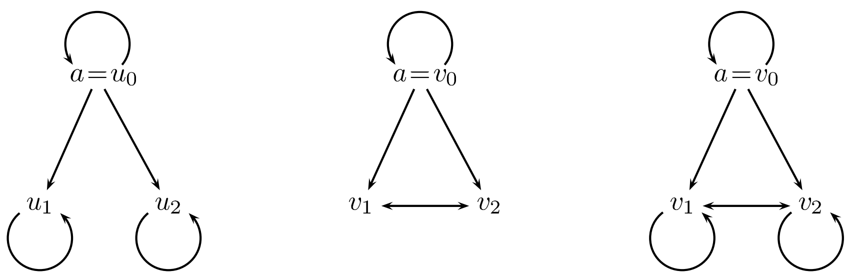


- $\Delta^{\mathcal{I}_0} = \{u_i \mid i \geq 0\}$, $a^{\mathcal{I}_0} = u_0$, $r^{\mathcal{I}_0} = \{\langle u_i, u_j\rangle \mid i < j\}$
- $\Delta^{\mathcal{I}_1} = \{v_0, v_1, v_2\}$, $a^{\mathcal{I}_1} = v_0$, $r^{\mathcal{I}_1} = \{\langle v_0, v_1\rangle, \langle v_1, v_2\rangle, \langle v_2, v_1\rangle, \langle v_0, v_2\rangle\}$.

Let $Z = \{\langle u_0, v_0\rangle\} \cup \{\langle u_i, v_j\rangle \mid i \geq 1 \text{ and } (j = 1 \text{ or } j = 2)\}$. It is easy to check that Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1$, $\mathcal{I}_0$ is a model of the RBox $\mathcal{R} = \{r \circ r \sqsubseteq r\}$, but $\mathcal{I}_1$ is not. Let $\mathcal{I}'_1$ be the least r-extension of $\mathcal{I}_1$ validating $\mathcal{R}$. We have that $\{\langle v_1, v_1\rangle, \langle v_2, v_2\rangle\} \subseteq r^{\mathcal{I}'_1}$, while $\langle u_i, u_i\rangle \notin r^{\mathcal{I}_0}$ for all $i \geq 0$. Hence $\{v_1, v_2\} \subseteq (\exists \mathtt{Self}.r)^{\mathcal{I}'_1}$, while $u_i \notin (\exists \mathtt{Self}.r)^{\mathcal{I}_0}$ for all $i \geq 0$, Thus, it is easy to check that Z is not an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}'_1$.

2. Assume that $Q \in \Phi$ and $\mathtt{Self} \notin \Phi$. Let $\Sigma_C = \emptyset$, $\Sigma_I = \{a\}$ and $\Sigma_R = \{r\}$. Let $\mathcal{I}_0$ and $\mathcal{I}_1$ be the interpretations specified by:

$(\mathcal{I}_0)$ $(\mathcal{I}_1)$ $(\mathcal{I}'_1)$

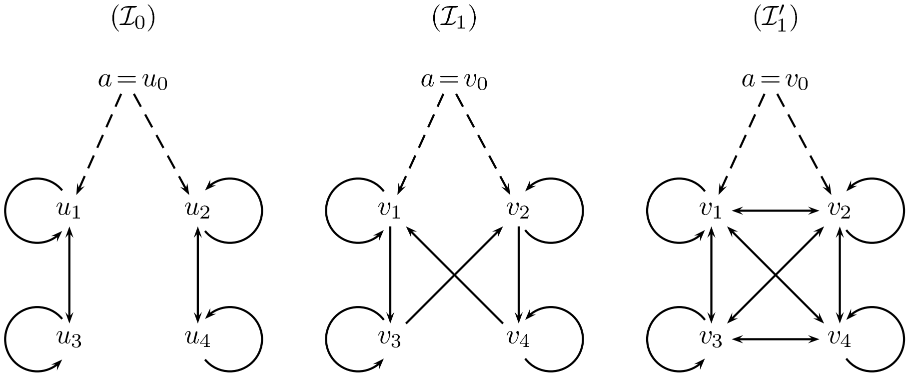


- $\Delta^{\mathcal{I}_0} = \{u_0, u_1, u_2\}$, $a^{\mathcal{I}_0} = u_0$,
  $r^{\mathcal{I}_0} = \{\langle u_0, u_0\rangle, \langle u_0, u_1\rangle, \langle u_0, u_2\rangle, \langle u_1, u_1\rangle, \langle u_2, u_2\rangle\}$
- $\Delta^{\mathcal{I}_1} = \{v_0, v_1, v_2\}$, $a^{\mathcal{I}_1} = v_0$,
  $r^{\mathcal{I}_1} = \{\langle v_0, v_0\rangle, \langle v_0, v_1\rangle, \langle v_0, v_2\rangle, \langle v_1, v_2\rangle, \langle v_2, v_1\rangle\}$.

Let $Z = \{\langle u_0, v_0\rangle\} \cup (\{u_1, u_2\} \times \{v_1, v_2\})$. It is easy to check that Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1$, $\mathcal{I}_0$ is a model of the RBox $\mathcal{R} = \{\varepsilon \sqsubseteq r\}$, but

$\mathcal{I}_1$ is not. Let $\mathcal{I}'_1$ be the least r-extension of $\mathcal{I}_1$ validating $\mathcal{R}$. We have that $\{\langle v_1, v_1\rangle, \langle v_2, v_2\rangle\} \subseteq r^{\mathcal{I}'_1}$. Hence $\{v_1, v_2\} \subseteq (\geq 2\, r.\top)^{\mathcal{I}'_1}$, while $u_i \notin (\geq 2\, r.\top)^{\mathcal{I}_0}$ for both $i \in \{1, 2\}$. Thus, it is easy to check that Z is not an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}'_1$.

3. Assume that $Q \in \Phi$. Let $\Sigma_C = \emptyset$, $\Sigma_I = \{a\}$, $\Sigma_R = \{r, s\}$ and let $\mathcal{I}_0$, $\mathcal{I}_1$ be the interpretations specified by:



- $\Delta^{\mathcal{I}_0} = \{u_0, \ldots, u_4\}$, $a^{\mathcal{I}_0} = u_0$, $r^{\mathcal{I}_0} = \{\langle u_0, u_1\rangle, \langle u_0, u_2\rangle\}$,
  $s^{\mathcal{I}_0} = \{\langle u_i, u_j\rangle \mid \{i, j\} \subseteq \{1, 3\} \text{ or } \{i, j\} \subseteq \{2, 4\}\}$
- $\Delta^{\mathcal{I}_1} = \{v_0, \ldots, v_4\}$, $a^{\mathcal{I}_1} = v_0$, $r^{\mathcal{I}_1} = \{\langle v_0, v_1\rangle, \langle v_0, v_2\rangle\}$,
  $s^{\mathcal{I}_1} = \{\langle v_i, v_i\rangle \mid 1 \leq i \leq 4\} \cup \{\langle v_1, v_3\rangle, \langle v_3, v_2\rangle, \langle v_2, v_4\rangle, \langle v_4, v_1\rangle\}$.

Let $Z = \{\langle u_0, v_0\rangle\} \cup (\{u_1, u_2\} \times \{v_1, v_2\}) \cup (\{u_3, u_4\} \times \{v_3, v_4\})$. It is easy to check that Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}_1$, $\mathcal{I}_0$ is a model of the RBox $\mathcal{R} = \{s \circ s \sqsubseteq s\}$, but $\mathcal{I}_1$ is not. Let $\mathcal{I}'_1$ be the least r-extension of $\mathcal{I}_1$ validating $\mathcal{R}$. We have that $\{\langle v_3, v_4\rangle, \langle v_3, v_1\rangle\} \subseteq s^{\mathcal{I}'_1}$. Hence $v_3 \in (\geq 4\, s.\top)^{\mathcal{I}'_1}$, while $u_i \notin (\geq 4\, s.\top)^{\mathcal{I}_0}$ for all $0 \leq i \leq 4$. Thus, it is easy to check that Z is not an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}'_1$. ■

The following theorem concerns invariance of knowledge bases w.r.t. $\mathcal{L}_\Phi$-bisimulation. As stated before, in general, RBoxes are not invariant for $\mathcal{L}_\Phi$-bisimulations. Thus, it is natural to consider the case when the considered RBox is empty. Restricting to this case, generality of the below theorem follows from the generality of Theorems 3.6 and 3.7. The case when the considered RBox is not empty is addressed in Theorem 3.10.

Theorem 3.9. *Let $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ be a knowledge base in $\mathcal{L}_\Phi$ such that $\mathcal{R} = \emptyset$ and either $O \in \Phi$ or $\mathcal{A}$ contains only assertions of the form $C(a)$. Let $\mathcal{I}$ and $\mathcal{I}'$ be unreachable-objects-free interpretations (w.r.t. $\mathcal{L}_\Phi$) such that there exists an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Then $\mathcal{I}$ is a model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$ iff $\mathcal{I}'$ is a model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A}\rangle$.*

This theorem follows immediately from Theorems 3.6 and 3.7.

The following theorem concerns preservation of knowledge bases under $\mathcal{L}_\Phi$-bisimulation. Its generality follows from the generality of Theorems 3.6, 3.7 and 3.8. Clearly, it covers many useful cases.

Theorem 3.10. *Suppose* $\Phi \subseteq \{I, O, U\}$ *and let* $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ *be a knowledge base in* $\mathcal{L}_\Phi$ *such that if* $O \notin \Phi$ *then* $\mathcal{A}$ *contains only assertions of the form* $C(a)$*. Let* $\mathcal{I}_0$ *and* $\mathcal{I}_1$ *be interpretations such that:* $\mathcal{I}_0$ *is a model of* $\mathcal{R}$*, there is an* $\mathcal{L}_\Phi$*-bisimulation* Z *between* $\mathcal{I}_0$ *and* $\mathcal{I}_1$*, and if* $U \notin \Phi$ *then* $\mathcal{I}_0$ *and* $\mathcal{I}_1$ *are unreachable-objects-free (w.r.t.* $\mathcal{L}_\Phi$*). Let* $\mathcal{I}'_1$ *be the least r-extension of* $\mathcal{I}_1$ *validating* $\mathcal{R}$*. Then:*

1. *Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}_0$ and $\mathcal{I}'_1$,*

2. *$\mathcal{I}'_1$ is a model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$ iff $\mathcal{I}_0$ is a model of $\langle \mathcal{R}, \mathcal{T}, \mathcal{A} \rangle$.*

This theorem follows immediately from Corollary 3.5 and Theorems 3.6, 3.7, 3.8.

3.2 The Hennessy-Milner Property

Definition 3.13 (**Modally Saturated Interpretation).**
An interpretation $\mathcal{I}$ is said to be *modally saturated* w.r.t. $\mathcal{L}_\Phi$ if the following conditions hold:

- for every $x \in \Delta^\mathcal{I}$, every basic role R of $\mathcal{L}_\Phi$ and every infinite set Γ of concepts in $\mathcal{L}_\Phi$, if for every finite subset Λ of Γ there exists an R-successor of x that satisfies Λ, then there exists an R-successor of x that satisfies Γ;

- if $Q \in \Phi$ then, for every $x \in \Delta^\mathcal{I}$, every basic role R of $\mathcal{L}_\Phi$, every infinite set Γ of concepts in $\mathcal{L}_\Phi$ and every natural number n, if for every finite subset Λ of Γ there exist n pairwise different R-successors of x that satisfy Λ, then there exist n pairwise different R-successors of x that satisfy Γ;

- if $U \in \Phi$ and $\mathcal{I}$ is not unreachable-objects-free then, for every infinite set Γ of concepts in $\mathcal{L}_\Phi$, if every finite subset Λ of Γ is satisfied in $\mathcal{I}$ (i.e. $\Lambda^\mathcal{I} \neq \emptyset$) then Γ is also satisfied in $\mathcal{I}$ (i.e. $\Gamma^\mathcal{I} \neq \emptyset$). ■

Observe that *ω-saturated* interpretations (defined, e.g., as in [13]) are modally saturated.

Proposition 3.11. *Every finite interpretation is modally saturated. Every finitely branching and unreachable-objects-free interpretation is modally saturated. If* $U \notin \Phi$ *then every finitely branching interpretation is modally saturated.*

The proof of this proposition is straightforward.

Definition 3.14 (**$\mathcal{L}_\Phi$-Equivalence).**
Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations, and let $x \in \Delta^\mathcal{I}$ and $x' \in \Delta^{\mathcal{I}'}$. We say that x is *$\mathcal{L}_\Phi$-equivalent* to x' if, for every concept C in $\mathcal{L}_\Phi$, $x \in C^\mathcal{I}$ iff $x' \in C^{\mathcal{I}'}$. ■

Theorem 3.12 (The Hennessy-Milner Property)**.** *Let* $\mathcal{I}$ *and* $\mathcal{I}'$ *be modally saturated interpretations (w.r.t.* $\mathcal{L}_\Phi$*) such that, for every* $a \in \Sigma_I$*,* $a^\mathcal{I}$ *is* $\mathcal{L}_\Phi$*-equivalent to* $a^{\mathcal{I}'}$*. Suppose that if* $U \in \Phi$ *then either both* $\mathcal{I}$ *and* $\mathcal{I}'$ *are unreachable-objects-free or both of them are not unreachable-objects-free. Then* $x \in \Delta^\mathcal{I}$ *is* $\mathcal{L}_\Phi$*-equivalent to* $x' \in \Delta^{\mathcal{I}'}$ *iff there exists an* $\mathcal{L}_\Phi$*-bisimulation* Z *between* $\mathcal{I}$ *and* $\mathcal{I}'$ *such that* $Z(x, x')$ *holds. In particular, the relation* $\{\langle x, x' \rangle \in \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'} \mid x$ *is* $\mathcal{L}_\Phi$*-equivalent to* $x'\}$ *is an* $\mathcal{L}_\Phi$*-bisimulation between* $\mathcal{I}$ *and* $\mathcal{I}'$ *when it is not empty.*

Proof. Consider the "$\Leftarrow$" direction. Suppose Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z(x, x')$ holds. By (3.15), for every concept C in $\mathcal{L}_\Phi$, $C^{\mathcal{I}}(x)$ holds iff $C^{\mathcal{I}'}(x')$ holds. Therefore, x is $\mathcal{L}_\Phi$-equivalent to x'.

Now consider the "$\Rightarrow$" direction. Define $Z = \{\langle x, x'\rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'} \mid x$ is $\mathcal{L}_\Phi$-equivalent to $x'\}$ and assume that Z is not empty. We show that Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.

- The assertion (3.1) follows from the assumption of the theorem.
- Consider the assertion (3.2) and suppose $Z(x, x')$ holds. By the definitions of Z and $\mathcal{L}_\Phi$-equivalence, it follows that, for every concept name A, $A^{\mathcal{I}}(x)$ holds iff $A^{\mathcal{I}'}(x')$ holds.
- Consider the assertion (3.3) and suppose that $Z(x, x')$ and $r^{\mathcal{I}}(x, y)$ hold. Let $\mathbf{S} = \{y' \mid r^{\mathcal{I}'}(x', y')\}$. We want to show there exists $y' \in \mathbf{S}$ such that $Z(y, y')$ holds. For the sake of contradiction, suppose that, for every $y' \in \mathbf{S}$, $Z(y, y')$ does not hold, which means y is not $\mathcal{L}_\Phi$-equivalent to y'. Thus, for every $y' \in \mathbf{S}$, there exists a concept $C_{y'}$ such that $y \in C_{y'}^{\mathcal{I}}$ but $y' \notin C_{y'}^{\mathcal{I}'}$. Let $\Gamma = \{C_{y'} \mid y' \in \mathbf{S}\}$. Thus, no $y' \in \mathbf{S}$ satisfies Γ (i.e. $\mathbf{S} \cap \Gamma^{\mathcal{I}'} = \emptyset$). Since $\mathcal{I}'$ is modally saturated, it follows that there exists a finite set Λ of Γ such that, for every $y' \in \mathbf{S}$, $y' \notin \Lambda^{\mathcal{I}'}$. Let $C = \exists r.\bigsqcap\Lambda$, where $\bigsqcap\{C_1, \ldots, C_n\} = C_1 \sqcap \ldots \sqcap C_n$ and $\bigsqcap\emptyset = \top$. Thus, $x \in C^{\mathcal{I}}$ but $x' \notin C^{\mathcal{I}'}$, which contradicts the fact that x is $\mathcal{L}_\Phi$-equivalent to x'. Therefore, there exists $y' \in \mathbf{S}$ such that $Z(y, y')$ holds.
- The assertion (3.4) can be proved analogously as for (3.3).
- Consider the assertions (3.5) and (3.6) and the case $I \in \Phi$. Observe that the argumentation used for proving (3.3) are still applicable when replacing r by r^-. Hence the assertion (3.5) holds. Similarly, the assertion (3.6) also holds.
- Consider the assertion (3.7) and the case $O \in \Phi$. Suppose $Z(x, x')$ holds. Take $C = \{a\}$. Since x is $\mathcal{L}_\Phi$-equivalent to x', $x \in C^{\mathcal{I}}$ iff $x' \in C^{\mathcal{I}'}$. Hence, $x = a^{\mathcal{I}}$ iff $x' = a^{\mathcal{I}'}$.
- Consider the assertion (3.8) and the case $Q \in \Phi$. Suppose $Z(x, x')$ holds. Let $\mathbf{S} = \{y \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x, y)\}$ and $\mathbf{S}' = \{y' \in \Delta^{\mathcal{I}'} \mid r^{\mathcal{I}'}(x', y')\}$. Let $y_1, \ldots, y_n$ be pairwise different elements of $\mathbf{S}$. We need to show that there exist pairwise different $y'_1, \ldots, y'_n \in \mathbf{S}'$ such that y'_i is $\mathcal{L}_\Phi$-equivalent to y_i for every $1 \leq i \leq n$. Without loss of generality, assume that $y_1, \ldots, y_n$ are $\mathcal{L}_\Phi$-equivalent to each other. Let $\mathbf{S}'' = \{y' \in \mathbf{S}' \mid y'$ is not $\mathcal{L}_\Phi$-equivalent to $y_1\}$. Thus, for every $y' \in \mathbf{S}''$, there exists a concept $C_{y'}$ such that $y_1 \in C_{y'}^{\mathcal{I}}$ but $y' \notin C_{y'}^{\mathcal{I}'}$. Let $\Gamma = \{C_{y'} \mid y' \in \mathbf{S}''\}$. Note that every element of $\Gamma^{\mathcal{I}'}$ is $\mathcal{L}_\Phi$-equivalent to y_1. For every finite subset Λ of Γ, since $y_1, \ldots, y_n \in \Lambda^{\mathcal{I}}$, we have $x \in ({\geq}\, n\, r.\bigsqcap\Lambda)^{\mathcal{I}}$, and since $Z(x, x')$ holds, we also have that $x' \in ({\geq}\, n\, r.\bigsqcap\Lambda)^{\mathcal{I}'}$, which means there are at least n pairwise different $y'_1, \ldots, y'_n \in \mathbf{S}'$ that belong to $\Lambda^{\mathcal{I}'}$. Since $\mathcal{I}'$ is modally saturated, it follows that there are at least n pairwise different $y'_1, \ldots, y'_n \in \mathbf{S}'$ that belong to $\Gamma^{\mathcal{I}'}$ and are thus $\mathcal{L}_\Phi$-equivalent to y_1 and any y_i with $2 \leq i \leq n$.

- The assertion (3.9) for the case $Q \in \Phi$ and the assertions (3.10) and (3.11) for the case $\{Q, I\} \subseteq \Phi$ can be proved analogously as for (3.8).

- Consider the assertion (3.12) and the case $U \in \Phi$. If $\mathcal{I}$ is unreachable-objects-free then the assertion (3.12) follows from the assertions (3.1), (3.3) and (3.5). So, assume that $\mathcal{I}$ is not unreachable-objects-free. Thus, $\mathcal{I}'$ is also not unreachable-objects-free. Since Z is not empty, there exists $\langle y, y' \rangle \in Z$. Let $x \in \Delta^{\mathcal{I}}$. For the sake of contradiction suppose that no $x' \in \Delta^{\mathcal{I}'}$ is $\mathcal{L}_\Phi$-equivalent to x. Thus, for every $x' \in \Delta^{\mathcal{I}'}$, there exists a concept $C_{x'}$ such that $x \in C_{x'}^{\mathcal{I}}$ but $x' \notin C_{x'}^{\mathcal{I}'}$. Let $\Gamma = \{C_{x'} \mid x' \in \Delta^{\mathcal{I}'}\}$. For any finite subset Λ of Γ, since $x \in \Lambda^{\mathcal{I}}$, we have that $y \in (\exists U.\bigsqcap \Lambda)^{\mathcal{I}}$, which implies that $y' \in (\exists U.\bigsqcap \Lambda)^{\mathcal{I}'}$, which means Λ is satisfied in $\mathcal{I}'$. Since $\mathcal{I}'$ is modally saturated and not unreachable-objects-free, it follows that Γ is satisfied in $\mathcal{I}'$, which is a contradiction.

- The assertion (3.13) can be proved analogously as for (3.12).

- Consider the assertion (3.14) and the case $\mathtt{Self} \in \Phi$. Suppose $Z(x, x')$ holds. Thus, $x \in (\exists r.\mathtt{Self})^{\mathcal{I}}$ iff $x' \in (\exists r.\mathtt{Self})^{\mathcal{I}'}$. Hence, $r^{\mathcal{I}}(x, x)$ holds iff $r^{\mathcal{I}'}(x', x')$ holds. ■

3.3 Auto-Bisimulation and Minimization

Definition 3.15 ($\mathcal{L}_\Phi$-Auto-Bisimulation).
An $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and itself is called an *$\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$*. An $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$ is said to be the *largest* if it is larger than or equal to ($\supseteq$) any other $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$. ■

Proposition 3.13. *For every interpretation $\mathcal{I}$, the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$ exists and is an equivalence relation.*

This proposition follows from Lemma 3.1.

Definition 3.16. Given an interpretation $\mathcal{I}$, by $\sim_{\Phi,\mathcal{I}}$ we denote the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$, and by $\equiv_{\Phi,\mathcal{I}}$ we denote the binary relation on $\Delta^{\mathcal{I}}$ with the property that $x \equiv_{\Phi,\mathcal{I}} x'$ iff x is $\mathcal{L}_\Phi$-equivalent to x'. ■

Proposition 3.14. *For every modally saturated interpretation $\mathcal{I}$, $\equiv_{\Phi,\mathcal{I}}$ is the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$ (i.e. the relations $\equiv_{\Phi,\mathcal{I}}$ and $\sim_{\Phi,\mathcal{I}}$ coincide).*

Proof. By Theorem 3.12, $\equiv_{\Phi,\mathcal{I}}$ is an $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$. We now show that it is the largest one. Suppose Z is another $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$. If $Z(x, x')$ holds then, by (3.15), for every concept C of $\mathcal{L}_\Phi$, $C^{\mathcal{I}}(x)$ holds iff $C^{\mathcal{I}}(x')$ holds, and hence $x \equiv_{\Phi,\mathcal{I}} x'$. Therefore, $Z \subseteq \equiv_{\Phi,\mathcal{I}}$. ■

3.3.1 The Case without Q and $\mathtt{Self}$

Definition 3.17 (**Quotient Interpretation**).
Given an interpretation $\mathcal{I}$, the *quotient interpretation* $\mathcal{I}/_\sim$ of $\mathcal{I}$ w.r.t. an equivalence relation $\sim \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$ is defined as usual:

- $\Delta^{\mathcal{I}/\sim} = \{[x]_\sim \mid x \in \Delta^\mathcal{I}\}$, where $[x]_\sim$ is the equivalence class of x w.r.t. $\sim$,
- $a^{\mathcal{I}/\sim} = [a^\mathcal{I}]_\sim$, for $a \in \Sigma_I$,
- $A^{\mathcal{I}/\sim} = \{[x]_\sim \mid x \in A^\mathcal{I}\}$, for $A \in \Sigma_C$,
- $r^{\mathcal{I}/\sim} = \{\langle [x]_\sim, [y]_\sim \rangle \mid \langle x, y\rangle \in r^\mathcal{I}\}$, for $r \in \Sigma_R$. ∎

Theorem 3.15. *If $\Phi \subseteq \{I, O, U\}$ then, for every interpretation $\mathcal{I}$, the relation $Z = \{\langle x, [x]_{\sim_{\Phi,\mathcal{I}}}\rangle \mid x \in \Delta^\mathcal{I}\}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$.*

Proof. Suppose $\Phi \subseteq \{I, O, U\}$. We have to consider the assertions (3.1)-(3.7), (3.12), (3.13) for $\mathcal{I}' = \mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$. By the definition of $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$, the assertions (3.1) and (3.2) clearly hold. Similarly, the assertion (3.7) for the case $O \in \Phi$ and the assertions (3.12), (3.13) for the case $U \in \Phi$ also hold.

Consider the assertion (3.3). Suppose $Z(x, x')$ and $r^\mathcal{I}(x, y)$ hold. We need to show there exists $y' \in \Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ such that $Z(y, y')$ and $r^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}(x', y')$ hold. We must have that $x' = [x]_{\sim_{\Phi,\mathcal{I}}}$. Take $y' = [y]_{\sim_{\Phi,\mathcal{I}}}$. Clearly, the goals are satisfied.

For a similar reason, the assertion (3.5) for the case $I \in \Phi$ holds.

Consider the assertion (3.4). Suppose $Z(x, x')$ and $r^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}(x', y')$ hold. We need to show there exists $y \in \Delta^\mathcal{I}$ such that $Z(y, y')$ and $r^\mathcal{I}(x, y)$ hold. We must have that $x' = [x]_{\sim_{\Phi,\mathcal{I}}}$. Since $r^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}(x', y')$ holds, there exists $y \in y'$ such that $r^\mathcal{I}(x, y)$ holds. Clearly, $y' = [y]_{\sim_{\Phi,\mathcal{I}}}$ and $Z(y, y')$ holds.

For a similar reason, the assertion (3.6) for the case $I \in \Phi$ holds. ∎

The following theorem concerns invariance of terminological axioms and concept assertions, as well as preservation of role axioms and other individual assertion under the transformation of an interpretation to its quotient using the largest $\mathcal{L}_\Phi$-auto-bisimulation.

Theorem 3.16. *Suppose $\Phi \subseteq \{I, O, U\}$ and let $\mathcal{I}$ be an interpretation. Then:*

1. *For every expression φ which is either a terminological axiom in $\mathcal{L}_\Phi$ or a concept assertion (of the form $C(a)$) in $\mathcal{L}_\Phi$, $\mathcal{I} \models \varphi$ iff $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*
2. *For every expression φ which is either a role inclusion axiom or an individual assertion of the form $R(a, b)$ or $a \doteq b$, if $\mathcal{I} \models \varphi$ then $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*

Proof. The first assertion follows from Theorems 3.15, 3.4 and the definition of $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$. Consider the second assertion. This assertion for the cases when φ is of the form $\varepsilon \sqsubseteq r$, $R(a, b)$ or $a \doteq b$ follows immediately from the definition of $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$. Let $\varphi = (R_1 \circ \ldots \circ R_k \sqsubseteq r)$ and suppose $\mathcal{I} \models \varphi$. We show that $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \models \varphi$. Let $v_0, \ldots, v_k$ be elements of $\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ such that, for $1 \leq i \leq k$, $R_i^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}(v_{i-1}, v_i)$ holds. We need to show that $r^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}(v_0, v_k)$ holds.

For $1 \leq i \leq k$, since $R_i^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}(v_{i-1}, v_i)$ holds, there exist $y_{i-1} \in v_{i-1}$ and $z_i \in v_i$ such that $R_i^{\mathcal{I}}(y_{i-1}, z_i)$ holds. Let $x_0 = y_0$. For $1 \leq i \leq k$, since $x_{i-1} \sim_{\Phi,\mathcal{I}} y_{i-1}$ and $R_i^{\mathcal{I}}(y_{i-1}, z_i)$ hold, by (3.16), there exists x_i such that $x_i \sim_{\Phi,\mathcal{I}} z_i$ and $R_i^{\mathcal{I}}(x_{i-1}, x_i)$ hold, which implies $x_i \in v_i$ and $x_i \sim_{\Phi,\mathcal{I}} y_i$ (when $i < k$). Since $\mathcal{I} \models (R_1 \circ \ldots \circ R_k \sqsubseteq r)$, it follows that $r^{\mathcal{I}}(x_0, x_k)$ holds. Therefore, by definition, $r^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}(v_0, v_k)$ holds. ∎

An interpretation $\mathcal{I}$ is said to be *minimal* among a class of interpretations if $\mathcal{I}$ belongs to that class and, for every other interpretation $\mathcal{I}'$ of that class, $\#\Delta^{\mathcal{I}} \leq \#\Delta^{\mathcal{I}'}$ (the cardinality of $\Delta^{\mathcal{I}}$ is less than or equal to the cardinality of $\Delta^{\mathcal{I}'}$). The following theorem concerns minimality of quotient interpretations generated by using the largest $\mathcal{L}_\Phi$-auto-bisimulations.

Theorem 3.17. *Suppose $\Phi \subseteq \{I, O, U\}$ and let $\mathcal{I}$ be an interpretation.*

1. *If $\mathcal{I}$ is unreachable-objects-free or $U \in \Phi$ then $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is a minimal interpretation $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$.*
2. *If $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is finite then it is a minimal interpretation that validates the same set of terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*
3. *If $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is unreachable-objects-free and finitely branching then it is a minimal interpretation that satisfies the same set of concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*

Proof. By Theorems 3.15 and 3.16, $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$, validates the same set of terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$, and satisfies the same set of concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$.

Since $\sim_{\Phi,\mathcal{I}}$ is the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$, by Lemma 3.1(4), for $u, v \in \Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$, if $u \neq v$ then u is not $\mathcal{L}_\Phi$-bisimilar to v. Let $Z = \{\langle [x]_{\sim_{\Phi,\mathcal{I}}}, x\rangle \mid x \in \Delta^{\mathcal{I}}\}$. By Theorem 3.15 and Lemma 3.1(2), Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ and $\mathcal{I}$.

Consider the first assertion and suppose that either $\mathcal{I}$ is unreachable-objects-free or $U \in \Phi$. Let $\mathcal{I}'$ be any interpretation $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$. We show that $\#\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} \leq \#\Delta^{\mathcal{I}'}$. Let Z' be an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$, and let $Z'' = Z \circ Z'$. By Lemma 3.1(3), Z'' is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ and $\mathcal{I}'$. If $\mathcal{I}$ is unreachable-objects-free, then $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is also unreachable-objects-free, and by (3.1), (3.3) and (3.5), for every $u \in \Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$, there exists $x_u \in \Delta^{\mathcal{I}'}$ such that $Z''(u, x_u)$ holds. If $U \in \Phi$ then, by (3.12), we also have that, for every $u \in \Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$, there exists $x_u \in \Delta^{\mathcal{I}'}$ such that $Z''(u, x_u)$ holds. Let $u, v \in \Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ and $u \neq v$. If $x_u = x_v$ then, since u is $\mathcal{L}_\Phi$-bisimilar to x_u and x_v is $\mathcal{L}_\Phi$-bisimilar to v, we would have that u is $\mathcal{L}_\Phi$-bisimilar to v, which is a contradiction. Therefore $x_u \neq x_v$ and we conclude that $\#\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} \leq \#\Delta^{\mathcal{I}'}$.

Consider the second assertion and suppose $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is finite. Let $\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} = \{v_1, \ldots, v_n\}$. Since $\sim_{\Phi,\mathcal{I}}$ is the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$, by Theorem 3.12 and Lemma 3.1, if $1 \leq i < j \leq n$ then v_i is not $\mathcal{L}_\Phi$-equivalent to v_j. For $1 \leq i, j \leq n$ with $i \neq j$, let $C_{i,j}$ be a concept in $\mathcal{L}_\Phi$ such that $v_i \in C_{i,j}^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ and $v_j \notin C_{i,j}^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$. For $1 \leq i \leq n$, let $C_i = (C_{i,1} \sqcap \ldots \sqcap C_{i,i-1} \sqcap C_{i,i+1} \sqcap \ldots \sqcap C_{i,n})$. We have that

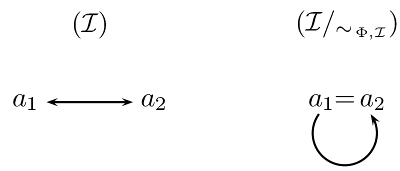
$v_i \in C_i^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ and $v_j \notin C_i^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ if $j \neq i$. Let $C = (C_1 \sqcup \ldots \sqcup C_n)$ and, for $1 \leq i \leq n$, let $D_i = (C_1 \sqcup \ldots \sqcup C_{i-1} \sqcup C_{i+1} \sqcup \ldots \sqcup C_n)$. Thus, $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ validates $\top \sqsubseteq C$ but does not validate any $\top \sqsubseteq D_i$ for $1 \leq i \leq n$. Any other interpretation with such properties must have at least n elements in the domain. That is, $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is a minimal interpretation that validates the same set of terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$.

Consider the third assertion and suppose $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is unreachable-objects-free and finitely branching. Let $\mathcal{I}'$ be any interpretation that satisfies the same set of concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$. We show that $\#\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} \leq \#\Delta^{\mathcal{I}'}$. By Theorem 3.16, $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ satisfies the same set of concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$ and $\mathcal{I}'$. Thus, for every individual name a, $a^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ is $\mathcal{L}_\Phi$-equivalent to $a^{\mathcal{I}'}$. Since Σ_I is countable and $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is unreachable-objects-free and finitely branching, $\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}}$ is countable. If $\mathcal{I}'$ is not finitely branching then it is infinite and the assertion clearly holds. So, assume that $\mathcal{I}'$ is finitely branching. Let $Z = \{\langle x, x' \rangle \in \Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} \times \Delta^{\mathcal{I}'} \mid x \text{ is } \mathcal{L}_\Phi\text{-equivalent to } x'\}$. Like the proof of Theorem 3.12, the conditions (3.1), (3.3) and (3.5) hold, and since $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is unreachable-objects-free, the condition (3.12) also holds. Analogously to the proof of the first assertion, it follows that $\#\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} \leq \#\Delta^{\mathcal{I}'}$. ∎

3.3.2 The Case with Q and/or `Self`

The following two examples show that we cannot make Theorems 3.15 and 3.16 stronger by allowing $\texttt{Self} \in \Phi$ or $Q \in \Phi$.

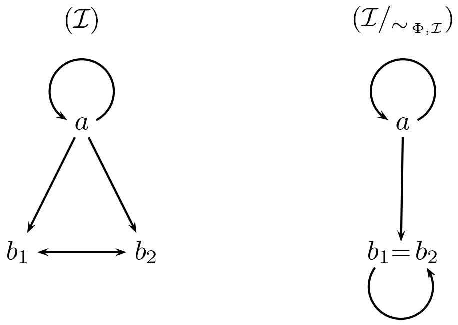
Example 3.18. Let $\Sigma_C = \emptyset$, $\Sigma_I = \{a_1, a_2\}$ and $\Sigma_R = \{r\}$, where $a_1 \neq a_2$. Consider the interpretation $\mathcal{I}$ specified by:



$\Delta^{\mathcal{I}} = \{a_1, a_2\}$, $a_1^{\mathcal{I}} = a_1$, $a_2^{\mathcal{I}} = a_2$ and $r^{\mathcal{I}} = \{\langle a_1, a_2 \rangle, \langle a_2, a_1 \rangle\}$. For any Φ, we have that $a_1 \sim_{\Phi,\mathcal{I}} a_2$. Denote $a = [a_1]_{\sim_{\Phi,\mathcal{I}}}$ $(= \{a_1, a_2\})$. The quotient interpretation $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is thus specified by: $\Delta^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} = \{a\}$, $a_1^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} = a_2^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} = a$ and $r^{\mathcal{I}/\sim_{\Phi,\mathcal{I}}} = \{\langle a, a \rangle\}$. Observe that if $\texttt{Self} \in \Phi$ then:

- $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is not $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$,
- for φ being any of the axioms/assertions $\top \sqsubseteq \exists r.\texttt{Self}$, $\varepsilon \sqsubseteq r$, $(\exists r.\texttt{Self})(a_1)$, $a_1 = a_2$, $r(a_1, a_1)$, we have that $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \models \varphi$, but $\mathcal{I} \not\models \varphi$. ∎

Example 3.19. Let $\Sigma_C = \emptyset$, $\Sigma_I = \{a, b_1, b_2\}$ and $\Sigma_R = \{r\}$, where a, b_1, b_2 are pairwise disjoint. Assume that $Q \in \Phi$ and consider the interpretation $\mathcal{I}$ specified by:



$\Delta^{\mathcal{I}} = \{a, b_1, b_2\}$, $a^{\mathcal{I}} = a$, $b_1^{\mathcal{I}} = b_1$, $b_2^{\mathcal{I}} = b_2$ and $r^{\mathcal{I}} = \{\langle a, a\rangle, \langle a, b_1\rangle, \langle a, b_2\rangle, \langle b_1, b_2\rangle, \langle b_2, b_1\rangle\}$. Note that b_1 is $\mathcal{L}_\Phi$-bisimilar to b_2 and is not $\mathcal{L}_\Phi$-bisimilar to a. Denote $a' = [a]_{\sim_{\Phi,\mathcal{I}}}$ and $b' = [b_1]_{\sim_{\Phi,\mathcal{I}}}$ $(= \{b_1, b_2\})$. The quotient interpretation $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is thus specified by: $\Delta^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = \{a', b'\}$, $a^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = a'$, $b_1^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = b_2^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = b'$ and $r^{\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}} = \{\langle a', a'\rangle, \langle a', b'\rangle, \langle b', b'\rangle\}$. Observe that:

- $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ is not $\mathcal{L}_\Phi$-bisimilar to $\mathcal{I}$,
- for φ being any of the axioms/assertions $\geq 2\, r.\top \sqsubseteq\ \geq 3\, r.\top$, $\varepsilon \sqsubseteq r$, $(\geq 3\, r.\top)(a)$, $b_1 = b_2$, $r(b_1, b_1)$, we have that $\mathcal{I} \models \varphi$ iff $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}} \not\models \varphi$. ■

For the case when $Q \in \Phi$ or $\mathtt{Self} \in \Phi$, in order to obtain results similar to Theorems 3.16 and 3.17, we introduce QS-interpretations as follows.

Definition 3.20 (QS-Interpretation).
A *QS-interpretation* is a tuple $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}, \mathtt{Q}^{\mathcal{I}}, \mathtt{S}^{\mathcal{I}}\rangle$, where

- $\langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}}\rangle$ is an interpretation,
- $\mathtt{Q}^{\mathcal{I}}$ is a function that maps every basic role to a function $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \to \mathbb{N}$ such that $\mathtt{Q}^{\mathcal{I}}(R)(x, y) > 0$ iff $\langle x, y\rangle \in R^{\mathcal{I}}$, where $\mathbb{N}$ is the set of natural numbers,
- $\mathtt{S}^{\mathcal{I}}$ is a function that maps every role name to a subset of $\Delta^{\mathcal{I}}$.

If $\mathcal{I}$ is a QS-interpretation then we redefine

$$\begin{array}{rcl}
(\exists r.\mathtt{Self})^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid x \in \mathtt{S}^{\mathcal{I}}(r)\} \\
(\geq n\, R.C)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid \Sigma\{\mathtt{Q}^{\mathcal{I}}(R)(x, y) \mid C^{\mathcal{I}}(y)\} \geq n\} \\
(\leq n\, R.C)^{\mathcal{I}} & = & \{x \in \Delta^{\mathcal{I}} \mid \Sigma\{\mathtt{Q}^{\mathcal{I}}(R)(x, y) \mid C^{\mathcal{I}}(y)\} \leq n\},
\end{array}$$

where the sum of a set of natural numbers is assumed to be $+\infty$ if it is not finitely bounded, and $+\infty$ is greater than any natural number . Other notions for interpretations remain unchanged for QS-interpretations. ■

Definition 3.21 (Quotient QS-Interpretation).
Given a finitely branching interpretation $\mathcal{I}$, the *quotient QS-interpretation* of $\mathcal{I}$ w.r.t. an equivalence relation $\sim\ \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, denoted by $\mathcal{I}/^{QS}_{\sim}$, is the QS-interpretation $\mathcal{I}' = \langle \Delta^{\mathcal{I}'}, \cdot^{\mathcal{I}'}, \mathtt{Q}^{\mathcal{I}'}, \mathtt{S}^{\mathcal{I}'}\rangle$ such that:

- $\langle \Delta^{\mathcal{I}'}, \cdot^{\mathcal{I}'}\rangle$ is the quotient interpretation of $\mathcal{I}$ w.r.t. $\sim$,

- for every basic role R and every $x, y \in \Delta^{\mathcal{I}}$,

$$\mathtt{Q}^{\mathcal{I}'}(R)([x]_\sim, [y]_\sim) = \min_{x' \in [x]_\sim} \#\{y' \in [y]_\sim \mid \langle x', y' \rangle \in R^{\mathcal{I}}\},$$

- for every role name r,

$$\mathtt{S}^{\mathcal{I}'}(r) = \{[x]_\sim \mid \langle x, x \rangle \in r^{\mathcal{I}}\}.$$

■

Note that, in the case when $Q \in \Phi$, we have

$$\mathtt{Q}^{\mathcal{I}'}(R)([x]_{\sim_{\Phi,\mathcal{I}}}, [y]_{\sim_{\Phi,\mathcal{I}}}) = \#\{y' \in [y]_{\sim_{\Phi,\mathcal{I}}} \mid \langle x, y' \rangle \in R^{\mathcal{I}}\}.$$

Lemma 3.18. *Let $\mathcal{I}$ be a finitely branching interpretation and let $\mathcal{I}' = \mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$. Then $Z = \{\langle x, [x]_{\sim_{\Phi,\mathcal{I}}} \rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}\}$ satisfies all the properties (3.1)-(3.7), (3.12), (3.13), (3.15)-(3.18). In particular, the assertion (3.15) states that, for every concept C in $\mathcal{L}_\Phi$ and every $x \in \Delta^{\mathcal{I}}$, $x \in C^{\mathcal{I}}$ iff $[x]_{\sim_{\Phi,\mathcal{I}}} \in C^{\mathcal{I}'}$.*

Proof. The properties (3.1)-(3.7), (3.12) and (3.13) can be shown as in the proof of Theorem 3.15. The properties (3.15)-(3.18) can be shown as in Lemma 3.3 except that the case when $Q \in \Phi$ and $C = (\geq n\, R.D)$ and the case when $\mathtt{Self} \in \Phi$ and $C = \exists r.\mathtt{Self}$ in the proof of the assertion (3.15) are changed to the following:

- Case $Q \in \Phi$ and $C = (\geq n\, R.D)$, where R is a basic role: Since $Z(x, x')$ holds, we have that $x' = [x]_{\sim_{\Phi,\mathcal{I}}}$. Since $C^{\mathcal{I}}(x)$ holds, there exist pairwise different $y_1, \ldots, y_n \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y_i)$ and $D^{\mathcal{I}}(y_i)$ hold for all $1 \leq i \leq n$. Let the partition of $\{y_1, \ldots, y_n\}$ that corresponds to the equivalence relation $\sim_{\Phi,\mathcal{I}}$ consist of pairwise different blocks $Y_{i_1}, \ldots, Y_{i_k}$, where $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ and $y_{i_j} \in Y_{i_j}$ for all $1 \leq j \leq k$. By the inductive assumption, $D^{\mathcal{I}'}([y_{i_j}]_{\sim_{\Phi,\mathcal{I}}})$ holds for all $1 \leq j \leq k$. By the definition of $\mathcal{I}'$, $\mathtt{Q}^{\mathcal{I}'}(R)([x]_{\sim_{\Phi,\mathcal{I}}}, [y_{i_j}]_{\sim_{\Phi,\mathcal{I}}}) \geq \#Y_{i_j}$ for all $1 \leq j \leq k$. Hence $C^{\mathcal{I}'}([x]_{\sim_{\Phi,\mathcal{I}}})$ holds, which means $C^{\mathcal{I}'}(x')$ holds.
- Case $\mathtt{Self} \in \Phi$ and $C = \exists r.\mathtt{Self}$: Since $Z(x, x')$ holds, we have that $x' = [x]_{\sim_{\Phi,\mathcal{I}}}$. Since $C^{\mathcal{I}}(x)$ holds, we have that $r^{\mathcal{I}}(x, x)$ holds. Hence $[x]_{\sim_{\Phi,\mathcal{I}}} \in \mathtt{S}^{\mathcal{I}'}(r)$ and consequently $[x]_{\sim_{\Phi,\mathcal{I}}} \in (\exists r.\mathtt{Self})^{\mathcal{I}'}$, which means $C^{\mathcal{I}'}(x')$ holds. ■

The following theorem is a counterpart of Theorem 3.16, with no restrictions on Φ.

Theorem 3.19. *Let $\mathcal{I}$ be a finitely branching interpretation. Then:*

1. *For every expression φ which is either a terminological axiom in $\mathcal{L}_\Phi$ or a concept assertion (of the form $C(a)$) in $\mathcal{L}_\Phi$, $\mathcal{I} \models \varphi$ iff $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*
2. *For every expression φ which is either a role inclusion axiom or an individual assertion of the form $R(a, b)$ or $a \doteq b$, if $\mathcal{I} \models \varphi$ then $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}} \models \varphi$.*

Proof. Denote $\mathcal{I}' = \mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ and let $Z = \{\langle x, [x]_{\sim_{\Phi,\mathcal{I}}}\rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}\}$. By Lemma 3.18, for every concept C in $\mathcal{L}_\Phi$, $x \in C^{\mathcal{I}}$ iff $[x]_{\sim_{\Phi,\mathcal{I}}} \in C^{\mathcal{I}'}$. The first assertion follows immediately from this property. The second assertion can be proved as for Theorem 3.16. ■

The following theorem is a counterpart of Theorem 3.17, with no restrictions on Φ.

Theorem 3.20. *Let $\mathcal{I}$ be a finitely branching interpretation.*

1. *If $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ is finite then it is a minimal QS-interpretation that validates the same set of terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*
2. *If $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ is unreachable-objects-free then it is a minimal QS-interpretation that satisfies the same set of concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$.*

Proof. By Lemma 3.18, every $x \in \Delta^{\mathcal{I}}$ is $\mathcal{L}_\Phi$-equivalent to $[x]_{\sim_{\Phi,\mathcal{I}}}$. Since $\equiv_{\Phi,\mathcal{I}}$ and $\sim_{\Phi,\mathcal{I}}$ coincide, if $[x]_{\sim_{\Phi,\mathcal{I}}} \neq [x']_{\sim_{\Phi,\mathcal{I}}}$ then $[x]_{\sim_{\Phi,\mathcal{I}}}$ and $[x']_{\sim_{\Phi,\mathcal{I}}}$ are not $\mathcal{L}_\Phi$-equivalent to each other. Denote $\mathcal{I}' = \mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$.

Consider the first assertion and suppose $\mathcal{I}'$ is finite. Let $\Delta^{\mathcal{I}'} = \{v_1, \ldots, v_n\}$, where $v_1, \ldots, v_n$ are pairwise different and each v_i is some $[x_i]_{\sim_{\Phi,\mathcal{I}}}$. For $1 \leq i, j \leq n$ with $i \neq j$, let $C_{i,j}$ be a concept in $\mathcal{L}_\Phi$ such that $v_i \in C_{i,j}^{\mathcal{I}'}$ and $v_j \notin C_{i,j}^{\mathcal{I}'}$. For $1 \leq i \leq n$, let $C_i = (C_{i,1} \sqcap \ldots \sqcap C_{i,i-1} \sqcap C_{i,i+1} \sqcap \ldots \sqcap C_{i,n})$. We have that $v_i \in C_i^{\mathcal{I}'}$ and $v_j \notin C_i^{\mathcal{I}'}$ if $j \neq i$. Let $C = (C_1 \sqcup \ldots \sqcup C_n)$ and, for $1 \leq i \leq n$, let $D_i = (C_1 \sqcup \ldots \sqcup C_{i-1} \sqcup C_{i+1} \sqcup \ldots \sqcup C_n)$. Thus, $\mathcal{I}'$ validates $\top \sqsubseteq C$ but does not validate any $\top \sqsubseteq D_i$ for $1 \leq i \leq n$. Any other QS-interpretation with such properties must have at least n elements in the domain. That is, $\mathcal{I}'$ is a minimal QS-interpretation that validates the same set of terminological axioms in $\mathcal{L}_\Phi$ as $\mathcal{I}$.

Consider the second assertion and suppose $\mathcal{I}'$ is unreachable-objects-free. Since $\mathcal{I}$ is finitely branching, $\mathcal{I}'$ is also finitely branching. Let $\mathcal{I}''$ be any QS-interpretation that satisfies the same set of concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$. We show that $\#\Delta^{\mathcal{I}'} \leq \#\Delta^{\mathcal{I}''}$. Since Σ_I is countable and $\mathcal{I}'$ is unreachable-objects-free and finitely branching, $\Delta^{\mathcal{I}'}$ is countable. If $\mathcal{I}''$ is not finitely branching then it is infinite and the assertion clearly holds. So, assume that $\mathcal{I}''$ is finitely branching. Since $\mathcal{I}'$ is unreachable-objects-free and $\mathcal{I}''$ is a finitely branching QS-interpretation that satisfies the same set of concept assertions in $\mathcal{L}_\Phi$ as $\mathcal{I}$ and $\mathcal{I}'$, it can be shown that, for every $x' \in \Delta^{\mathcal{I}'}$, there exists $x'' \in \Delta^{\mathcal{I}''}$ that is $\mathcal{L}_\Phi$-equivalent to x'. Recall that if x'_1 and x'_2 are different elements of $\Delta^{\mathcal{I}'}$ then they are not $\mathcal{L}_\Phi$-equivalent to each other. This implies that $\#\Delta^{\mathcal{I}'} \leq \#\Delta^{\mathcal{I}''}$. ■

3.4 Minimizing Interpretations

In this section, we adapt Hopcroft's automaton minimization algorithm [28] and the Paige-Tarjan algorithm [49] to obtain efficient algorithms for computing the partition corresponding to the equivalence relation $\sim_{\Phi,\mathcal{I}}$ for the case when $\mathcal{I}$ is finite. The partition is used to minimize $\mathcal{I}$ to obtain $\mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ for the case $\{Q, \texttt{Self}\} \cap \Phi = \emptyset$, or $\mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ for the other case. We do not require any restrictions on Φ.

For $x \in \Delta^{\mathcal{I}}$, $Y \subseteq \Delta^{\mathcal{I}}$ and a basic role R of $\mathcal{L}_\Phi$, define

$$deg_R(x,Y) = \#\{y \in Y \mid \langle x,y \rangle \in R^{\mathcal{I}}\} \text{ and } deg_R(x) = deg_R(x, \Delta^{\mathcal{I}}).$$

The similarity between minimizing automata and minimizing interpretations relies on that equivalence between two states in a finite deterministic automaton is similar to $\mathcal{L}_\Phi$-equivalence between two objects (i.e. elements of the domain) of an interpretation. The alphabet Σ of an automaton corresponds to Σ_R in the case $I \notin \Phi$, and $\Sigma_R^{\pm}$ in the other case. Note that the end conditions for equivalence are different; in the case of automata, it is required that the two considered states are either both accepting states or both unaccepting states; in the case of interpretations, it is required that the two considered objects x and x' satisfy the conjunction of the following conditions:

- for every $A \in \Sigma_C$, $x \in A^{\mathcal{I}}$ iff $x' \in A^{\mathcal{I}}$,
- if $Q \notin \Phi$ then, for every basic role R of $\mathcal{L}_\Phi$, $deg_R(x) = 0$ iff $deg_R(x') = 0$,
- if $Q \in \Phi$ then, for every basic role R of $\mathcal{L}_\Phi$, $deg_R(x) = deg_R(x')$,
- if $O \in \Phi$ then, for every $a \in \Sigma_I$, $x = a^{\mathcal{I}}$ iff $x' = a^{\mathcal{I}}$,
- if $\texttt{Self} \in \Phi$ then, for every $r \in \Sigma_R$, $\langle x,x \rangle \in r^{\mathcal{I}}$ iff $\langle x',x' \rangle \in r^{\mathcal{I}}$.

Denote the conjunction of the above conditions by $ECond_\Phi(x,x')$.

Interpretations are like nondeterministic automata, while Hopcroft's algorithm [28] works only for deterministic automata. The Paige-Tarjan algorithm [49] for the relational coarsest partition problem works on a graph and exploits the idea "process the smaller half" of Hopcroft's algorithm. We adapt it for computing the partition corresponding to $\sim_{\Phi,\mathcal{I}}$ for the case $Q \notin \Phi$. We directly adapt Hopcroft's algorithm for the case $Q \in \Phi$. The idea for a similar problem related with number restrictions was formulated for graphs in [49] and efficient algorithms for other similar problems were proposed even earlier (see [49]).

3.4.1 The Case $Q \in \Phi$

Algorithm 2 (given on page 40) computes the partition corresponding to $\sim_{\Phi,\mathcal{I}}$ for the case when $Q \in \Phi$ and $\mathcal{I}$ is finite. It starts by splitting $\Delta^{\mathcal{I}}$ into blocks using the equivalence relation $ECond_\Phi$ and after that follows the idea of Hopcroft's algorithm [28] to refine that partition. Like Hopcroft's algorithm, Algorithm 2 keeps the current partition $\mathbb{P}$ and a collection $\mathbb{L}$ of pairs $\langle Y,R \rangle$ for refining the partition, where $Y \in \mathbb{P}$ and R is a basic role. Splitting a block $X \in \mathbb{P}$ by a pair $\langle Y,R \rangle$ is done so that $x,x' \in X$ are separated when $deg_R(x,Y) \neq deg_R(x',Y)$, and may result in more than two blocks. Technically, it is done as follows: for each $y \in Y$ and for each edge coming to y via R from some x (i.e. for each $\langle x,y \rangle \in R^{\mathcal{I}}$), do

1. if it is the first time x is considered for this task then set $count(x) := 1$, remove x from its current block X and put x into the block $X_{count=1}$,

Algorithm 2: computing the partition corresponding to $\sim_{\Phi,\mathcal{I}}$ for the case $Q \in \Phi$

input : a set Φ of DL-features with $Q \in \Phi$ and a finite interpretation $\mathcal{I}$
output: the partition $\mathbb{P}$ corresponding to the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$

if $I \notin \Phi$ **then** let $\Sigma_R^\dagger = \Sigma_R$ **else** let $\Sigma_R^\dagger = \Sigma_R^\pm$;
set $\mathbb{P}$ to the partition corresponding to the equivalence relation $ECond_\Phi$;
set Z to a maximal block of $\mathbb{P}$;
set $\mathbb{L}$ to the empty collection;
foreach $X \in \mathbb{P} \setminus \{Z\}$ *and* $R \in \Sigma_R^\dagger$ **do**
  add $\langle X, R\rangle$ to $\mathbb{L}$;
while $\mathbb{L} \neq \emptyset$ **do**
  extract a pair $\langle Y, R\rangle$ from $\mathbb{L}$;
  foreach $X \in \mathbb{P}$ *split by* $\langle Y, R\rangle$ **do**
    split X by $\langle Y, R\rangle$ into a set $\mathbb{X}$ of blocks;
    replace X in $\mathbb{P}$ by all the blocks of $\mathbb{X}$;
    set Z to a maximal block of $\mathbb{X}$;
    foreach $S \in \Sigma_R^\dagger$ **do**
      if $\langle X, S\rangle \in \mathbb{L}$ **then**
        replace $\langle X, S\rangle$ in $\mathbb{L}$ by all the pairs $\langle X', S\rangle$ with $X' \in \mathbb{X}$;
      else
        add all the pairs $\langle X', S\rangle$ with $X' \in \mathbb{X} \setminus \{Z\}$ to $\mathbb{L}$;

2. else if $count(x) = k$ then increase $count(x)$ by 1, remove x from its current block $X_{count=k}$ and put it into the block $X_{count=k+1}$.

The non-empty blocks created from X together with the modified block X, if not empty, form the set $\mathbb{X}$ mentioned in the algorithm.

Lemma 3.21. *Consider an execution of Algorithm 2. The resulting partition $\mathbb{P}$ corresponds to an $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$.*

Proof. Let Z be the equivalence relation corresponding to the partition $\mathbb{P}$. Consider the conditions (3.1)–(3.14) with $\mathcal{I}' = \mathcal{I}$. Clearly, (3.1), (3.2), (3.7), (3.12), (3.13), (3.14) hold. As (3.3)–(3.6) are instances of (3.8)–(3.11), respectively, we need to prove only (3.8)–(3.11). It is sufficient to show that, for every $x, x' \in \Delta^\mathcal{I}$, every basic role R of $\mathcal{L}_\Phi$ and every block $Y \in \mathbb{P}$, if $deg_R(x, Y) \neq deg_R(x', Y)$ then x and x' belong to different blocks of $\mathbb{P}$. This is clear for the case $deg_R(x) \neq deg_R(x')$. So, assume that $deg_R(x) = deg_R(x')$. Let Y' be the smallest block appeared during the execution of the algorithm such that Y' is a superset of Y and $deg_R(x, Y') = deg_R(x', Y')$ (the biggest one is $\Delta^\mathcal{I}$). Let $Y_1, \ldots, Y_k$ be the blocks obtained from the splitting of Y'. There exist $1 \leq i, j \leq k$ such that $i \neq j$, $deg_R(x, Y_i) \neq deg_R(x', Y_i)$ and $deg_R(x, Y_j) \neq deg_R(x', Y_j)$. Hence, $\langle Y_i, R\rangle$ or $\langle Y_j, R\rangle$ is inserted into $\mathbb{L}$ when Y' is split. It follows that, at some step, a pair $\langle Y'', R\rangle$ such that $deg_R(x, Y'') \neq deg_R(x', Y'')$ is extracted from $\mathbb{L}$ (Y'' is some subset of Y_i or Y_j). This pair separates x and x'. ■

Lemma 3.22. *Consider an execution of Algorithm 2. If $x, x' \in \Delta^{\mathcal{I}}$ are separated (i.e., belong to different blocks of the partition $\mathbb{P}$) then $x \not\equiv_{\Phi,\mathcal{I}} x'$.*

Proof. Assume that $x, x' \in \Delta^{\mathcal{I}}$ are separated. We prove that $x \not\equiv_{\Phi,\mathcal{I}} x'$ by induction on the iteration k of the main loop at which x and x' are separated.

Consider the base case $k = 0$ when x and x' belong to different equivalence classes of the equivalence relation $ECond_\Phi$. There are the following subcases:

- there exists $A \in \Sigma_C$ such that $x \in A^{\mathcal{I}}$ and $x' \notin A^{\mathcal{I}}$ or vice versa (i.e., $x \notin A^{\mathcal{I}}$ and $x' \in A^{\mathcal{I}}$);
- there exists a basic role R of $\mathcal{L}_\Phi$ such that $deg_R(x) \neq deg_R(x')$; without loss of generality, assume that $deg_R(x) = l > deg_R(x')$;
- $O \in \Phi$ and there exists $a \in \Sigma_I$ such that $x = a^{\mathcal{I}}$ and $x' \neq a^{\mathcal{I}}$ or vice versa (i.e., $x \neq a^{\mathcal{I}}$ and $x' = a^{\mathcal{I}}$);
- $\mathtt{Self} \in \Phi$ and there exists $r \in \Sigma_R$ such that $\langle x, x\rangle \in r^{\mathcal{I}}$ and $\langle x', x'\rangle \notin r^{\mathcal{I}}$ or vice versa (i.e., $\langle x, x\rangle \notin r^{\mathcal{I}}$ and $\langle x', x'\rangle \in r^{\mathcal{I}}$).

The concept A, $\geq l\, R.\top$, $\{a\}$ or $\exists r.\mathtt{Self}$ of $\mathcal{L}_\Phi$, respectively for these subcases, distinguishes x and x'. Hence $x \not\equiv_{\Phi,\mathcal{I}} x'$.

Now consider the induction step and assume that x and x' are separated by a pair $\langle Y, R\rangle$ at the iteration $k+1$ of the main loop. Thus, $deg_R(x, Y) \neq deg_R(x', Y)$. Without loss of generality, assume that $deg_R(x, Y) > deg_R(x', Y)$ and let $h = deg_R(x, Y)$. Let the partition $\mathbb{P}$ before the iteration $k+1$ be $\{Y_0, \ldots, Y_s\}$ with $Y_0 = Y$ and let $Y_i = \{y_{i,1}, \ldots, y_{i,t_i}\}$ for $0 \leq i \leq s$. By the induction assumption, for each $1 \leq i \leq s$, $1 \leq j \leq t_0$ and $1 \leq j' \leq t_i$, there exists a concept $C_{i,j,j'}$ such that $y_{0,j} \in C^{\mathcal{I}}_{i,j,j'}$ and $y_{i,j'} \notin C^{\mathcal{I}}_{i,j,j'}$. For $1 \leq i \leq s$ and $1 \leq j \leq t_0$, let $C_{i,j} = C_{i,j,1} \sqcap \ldots \sqcap C_{i,j,t_i}$, then $y_{0,j} \in C^{\mathcal{I}}_{i,j}$ and $y_{i,j'} \notin C^{\mathcal{I}}_{i,j}$ for all $1 \leq j' \leq t_i$. For $1 \leq i \leq s$, let $C_i = C_{i,1} \sqcup \ldots \sqcup C_{i,t_0}$, then $y_{0,j} \in C^{\mathcal{I}}_i$ for all $1 \leq j \leq t_0$, and $y_{i,j'} \notin C^{\mathcal{I}}_i$ for all $1 \leq j' \leq t_i$. Let $C = C_1 \sqcap \ldots \sqcap C_s$. Thus, $Y_0 \subseteq C^{\mathcal{I}}$ and $Y_i \cap C^{\mathcal{I}} = \emptyset$ for all $1 \leq i \leq s$, which means that $Y_0 = C^{\mathcal{I}}$. Therefore, $x \in (\geq h\, R.C)^{\mathcal{I}}$ and $x' \notin (\geq h\, R.C)^{\mathcal{I}}$, which implies $x \not\equiv_{\Phi,\mathcal{I}} x'$. ■

Proposition 3.23. *Algorithm 2 is correct and can be implemented to have time complexity $O(|\Sigma|(m+n)\log n)$, where $m = \sum_{r\in\Sigma_R} |r^{\mathcal{I}}|$ and $n = |\Delta^{\mathcal{I}}|$. A tighter bound for the complexity is $O(|\Sigma_I| + |\Sigma_C|n + |\Sigma_R|(m+n)\log n)$.*

Proof. (Sketch) The contrapositive of Lemma 3.22 states that if $x \equiv_{\Phi,\mathcal{I}} x'$ then x and x' are not separated. That is, $\equiv_{\Phi,\mathcal{I}}$ is a subset of the equivalence relation corresponding to the partition $\mathbb{P}$. As $\equiv_{\Phi,\mathcal{I}}$ and $\sim_{\Phi,\mathcal{I}}$ coincide (by Proposition 3.14), it follows that $\sim_{\Phi,\mathcal{I}}$ is a subset of the equivalence relation corresponding to the partition $\mathbb{P}$. By Lemma 3.21, the latter is also an $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$, hence it is the same as $\sim_{\Phi,\mathcal{I}}$ (the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$). That is, Algorithm 2 is correct.

To estimate complexity, notice that the steps 4-17 of Algorithm 2 are essentially the same as the skeleton of Hopcroft's automaton minimization algorithm [28, 51] used for refining the partition. The only difference is the way of splitting X by a pair $\langle Y, R\rangle$. The technique for this has been mentioned earlier. The complexity analysis of [51] can be applied to the steps 4-17 of Algorithm 2. The first change is that instead of the occurrences of $|\{p \in Q : \delta(p,a) \in Y\}|$ we have $|\{\langle x,y\rangle \in R^{\mathcal{I}} : y \in Y\}|$. The second change is that, as Hopcroft's automaton minimization algorithm is for deterministic automata but here we have nondeterminism (i.e., for each $R \in \Sigma_R^{\dagger}$, $R^{\mathcal{I}}$ is a binary relation but not a function), the last two lines of [51] are modified so that $O(n)$ is replaced by $O(m)$ and $O(|\Sigma| n \log n)$ is replaced by $O(|\Sigma_R| m \log n)$. Thus, we can conclude that the steps 4-17 can be implemented to have time complexity $O(|\Sigma_R^{\dagger}|(m+n)\log n)$, which is the same as $O(|\Sigma_R|(m+n)\log n)$.

Consider complexity of the step 2 of Algorithm 2. To compute equivalence classes of the equivalence relation $ECond_{\Phi}$, we start from the partition $\{\Delta^{\mathcal{I}}\}$ and then:

1. Refine the current partition by using the condition that, when $O \in \Phi$, x and x' should be in the same block only if, for every $a \in \Sigma_I$, $x = a^{\mathcal{I}}$ iff $x' = a^{\mathcal{I}}$. This can be done in $O(|\Sigma_I|)$ steps.

2. Refine the current partition by using the condition that x and x' should be in the same block only if, for every $A \in \Sigma_C$, $x \in A^{\mathcal{I}}$ iff $x' \in A^{\mathcal{I}}$. This can be done in $O(|\Sigma_C| n)$ steps.

3. Refine the current partition by using the condition that, when $\mathtt{Self} \in \Phi$, x and x' should be in the same block only if, for every $r \in \Sigma_R$, $\langle x, x\rangle \in r^{\mathcal{I}}$ iff $\langle x', x'\rangle \in r^{\mathcal{I}}$. This can be done in $O(|\Sigma_R| n)$ steps.

4. Refine the current partition by using the condition that x and x' should be in the same block only when $deg_R(x) = deg_R(x')$ (since when $Q \in \Phi$). This can be done in $O(|\Sigma_R| n)$ steps.

Summing up, the time complexity of the step 2 of Algorithm 2 is of rank $O(|\Sigma_I| + |\Sigma_C| n + |\Sigma_R| n)$. Therefore, Algorithm 2 can be implemented to have time complexity $O(|\Sigma_I| + |\Sigma_C| n + |\Sigma_R|(m+n)\log n)$. ∎

3.4.2 The Case $Q \notin \Phi$

Computing the partition corresponding to $\sim_{\Phi,\mathcal{I}}$ for the case $Q \notin \Phi$ differs from the relational coarsest partition problem studied in [49], among others, in that the "edges" are labeled by basic roles of $\mathcal{L}_{\Phi}$. Algorithm 3 (on page 44) is our adaptation of the Paige-Tarjan algorithm [49] for computing the partition corresponding to $\sim_{\Phi,\mathcal{I}}$ for the case $Q \notin \Phi$.[1] It is formulated in a way to reflect the traditional presentation of Hopcroft's automaton minimization algorithm.

Roughly speaking, the main problem is that the relation $R^{\mathcal{I}}$ for a basic role R of $\mathcal{L}_{\Phi}$ need not to be a function. Elements x and x' of a block X should be separated by a pair $\langle V, R\rangle$, where V is a block, not only when $\deg_R(x, V) > 0$ and $\deg_R(x', V) = 0$

[1] It is a correction for [19].

or vice versa (i.e., $\deg_R(x, V) = 0$ and $\deg_R(x', V) > 0$), but also when $\deg_R(x, V') > 0$ and $\deg_R(x', V') = 0$ or vice versa (i.e., $\deg_R(x, V') = 0$ and $\deg_R(x', V') > 0$), where V' is the complement of V w.r.t. an appropriate block Y containing V. Such a Y is the union of some blocks of the current partition $\mathbb{P}$. In [49], it is called a compound block.

Suppose that a block X cannot be split by a pair $\langle Y, R\rangle$ (in the sense that for every $x, x' \in X$, $\deg_R(x, Y) > 0$ iff $\deg_R(x', Y) > 0$). Let $V \subset Y$ and $V' = Y \setminus V$. Then splitting X by a tuple $\langle V, V', R\rangle$ is done as follows:

- Split X by $\langle V, R\rangle$ to obtain $X_1 = \{x \in X \mid \deg_R(x, V) > 0\}$ and $X_2 = X \setminus X_1$.
- Split X_1 by $\langle V', R\rangle$ to obtain $X_{1,1} = \{x \in X_1 \mid \deg_R(x, V') > 0\}$ and $X_{1,2} = X_1 \setminus X_{1,1}$.
- Then return the set $\{X_{1,1}, X_{1,2}, X_2\}$ after deleting empty sets.

If the result contains more than one block then we say that X is split by $\langle V, V', R\rangle$. Note that X_2 cannot be split by $\langle V', R\rangle$. Denote $R^{-1}(U) = \{x \in \Delta^{\mathcal{I}} \mid \deg_R(x, U) > 0\}$. Then, also observe that:

- $X_1 = X \cap R^{-1}(V)$ and $X_2 = X \setminus X_1$,
- $X_{1,2} = X_1 \cap (R^{-1}(V) \setminus R^{-1}(V'))$ and $X_{1,1} = X_1 \setminus X_{1,2}$.

This gives a good way for computing the split of X via $R^{-1}(V)$ and $R^{-1}(V) \setminus R^{-1}(V')$, as the computation can "start" from V and "look back" via R. This is a crucial observation of [49].

Regarding the idea "process the smaller half", observe that if Y is the union of at least two blocks from the current partition $\mathbb{P}$ and V is a minimal block of $\mathbb{P}$ such that $V \subset Y$ then $\#V \leq \#Y/2$.

Lemma 3.24. *Consider an execution of Algorithm 3. The resulting partition $\mathbb{P}$ corresponds to an $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$.*

Proof. Let Z be the equivalence relation corresponding to the partition $\mathbb{P}$. Consider the conditions (3.1)–(3.7) and (3.12)–(3.14) with $\mathcal{I}' = \mathcal{I}$. Clearly, (3.1), (3.2), (3.7), (3.12)–(3.14) hold. We need to prove only (3.3)–(3.6). It is sufficient to show that, for every $x, x' \in \Delta^{\mathcal{I}}$, every basic role R of $\mathcal{L}_\Phi$ and every block $V \in \mathbb{P}$, if ($deg_R(x, V) > 0$ and $deg_R(x', V) = 0$) or ($deg_R(x, V) = 0$ and $deg_R(x', V) > 0$) then x and x' belong to different blocks of $\mathbb{P}$. This is clear for the case when ($deg_R(x) > 0$ and $deg_R(x') = 0$) or ($deg_R(x) = 0$ and $deg_R(x') > 0$). So, assume that either ($deg_R(x) > 0$ and $deg_R(x') > 0$) or ($deg_R(x) = 0$ and $deg_R(x') = 0$). Let Y be the smallest block such that $V \subset Y$, $\langle Y, R\rangle$ appeared in $\mathbb{L}$ at some step and either ($deg_R(x, Y) > 0$ and $deg_R(x', Y) > 0$) or ($deg_R(x, Y) = 0$ and $deg_R(x', Y) = 0$). Such a set exists due to the candidate $\Delta^{\mathcal{I}}$.

Consider the moment when $\langle Y, R\rangle$ is extracted from $\mathbb{L}$. We have $Y = V_1 \cup \ldots \cup V_k$, where $k \geq 2$ and $V_1, \ldots, V_k$ are blocks of $\mathbb{P}$. Let V_1 be the minimal block among $V_1, \ldots, V_k$ that is taken for processing $\langle Y, R\rangle$ and let $V_1' = Y \setminus V_1$. If x and x' are still in the same block of $\mathbb{P}$ and they are not separated when splitting that block using $\langle V_1, V_1', R\rangle$ then the following conditions hold:

- ($deg_R(x, V_1) > 0$ and $deg_R(x', V_1) > 0$) or ($deg_R(x, V_1) = 0$ and $deg_R(x', V_1) = 0$),

Algorithm 3: computing the partition corresponding to $\sim_{\Phi,\mathcal{I}}$ for the case $Q \notin \Phi$

input : a set Φ of DL-features without Q, and a finite interpretation $\mathcal{I}$
output: the partition $\mathbb{P}$ corresponding to the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$

if $I \notin \Phi$ **then** let $\Sigma_R^\dagger = \Sigma_R$ **else** let $\Sigma_R^\dagger = \Sigma_R^\pm$;
set $\mathbb{P}$ to the partition corresponding to the equivalence relation $ECond_\Phi$;
$\mathbb{L} := \{\langle \Delta^\mathcal{I}, R\rangle \mid R \in \Sigma_R^\dagger\}$;
while $\mathbb{L} \neq \emptyset$ **do**
  extract a pair $\langle Y, R\rangle$ from $\mathbb{L}$;
  let V be a minimal block of $\mathbb{P}$ such that $V \subset Y$;
  $V' := Y \setminus V$;
  if *more than one block of* $\mathbb{P}$ *is a subset of* V' **then**
    add $\langle V', R\rangle$ to $\mathbb{L}$
  foreach $X \in \mathbb{P}$ *split by* $\langle V, V', R\rangle$ **do**
    split X by $\langle V, V', R\rangle$ into a set $\mathbb{X}$ of blocks;
    replace X in $\mathbb{P}$ by all the blocks of $\mathbb{X}$;
    foreach $S \in \Sigma_R^\dagger$ **do**
      if $\mathbb{L}$ *does not contain any pair* $\langle U, S\rangle$ *such that* $X \subset U$ **then**
        add $\langle X, S\rangle$ to $\mathbb{L}$

- $(deg_R(x, V_1') > 0$ and $deg_R(x', V_1') > 0)$ or $(deg_R(x, V_1') = 0$ and $deg_R(x', V_1') = 0)$,
- $k = 2$ and either $V \subset V_1$ or $V \subset V_1'$.

This implies that V_1 or V_1' is split at some step and that operation adds $\langle V_1, R\rangle$ or $\langle V_1', R\rangle$ to $\mathbb{L}$. This contradicts the minimality of Y. Therefore, x are x' must be separated by using $\langle V_1, V_1', R\rangle$. ∎

Lemma 3.25. *Consider an execution of Algorithm 3. If $x, x' \in \Delta^\mathcal{I}$ are separated (i.e., belong to different blocks of the partition $\mathbb{P}$) then $x \not\sim_{\Phi,\mathcal{I}} x'$.*

Proof. Assume that $x, x' \in \Delta^\mathcal{I}$ are separated. We prove that $x \not\sim_{\Phi,\mathcal{I}} x'$ by induction on the iteration k of the main loop at which x and x' are separated.

Consider the base case $k = 0$ when x and x' belong to different equivalence classes of the equivalence relation $ECond_\Phi$. There are the following subcases:

- there exists $A \in \Sigma_C$ such that $x \in A^\mathcal{I}$ and $x' \notin A^\mathcal{I}$ or vice versa (i.e., $x \notin A^\mathcal{I}$ and $x' \in A^\mathcal{I}$);
- there exists a basic role R of $\mathcal{L}_\Phi$ such that either $deg_R(x) > 0$ and $deg_R(x') = 0$ or $deg_R(x) = 0$ and $deg_R(x') > 0$;
- $O \in \Phi$ and there exists $a \in \Sigma_I$ such that $x = a^\mathcal{I}$ and $x' \neq a^\mathcal{I}$ or vice versa (i.e., $x \neq a^\mathcal{I}$ and $x' = a^\mathcal{I}$);

- $\mathtt{Self} \in \Phi$ and there exists $r \in \Sigma_R$ such that $\langle x, x\rangle \in r^{\mathcal{I}}$ and $\langle x', x'\rangle \notin r^{\mathcal{I}}$ or vice versa (i.e., $\langle x, x\rangle \notin r^{\mathcal{I}}$ and $\langle x', x'\rangle \in r^{\mathcal{I}}$).

The concept A, $\exists R.\top$, $\{a\}$ or $\exists r.\mathtt{Self}$ of $\mathcal{L}_\Phi$, respectively for these subcases, distinguishes x and x'. Hence $x \not\equiv_{\Phi,\mathcal{I}} x'$.

Now consider the induction step and assume that x and x' are separated by a tuple $\langle V, V', R\rangle$ at the iteration $k+1$ of the main loop. There are the following cases:

1. $\deg_R(x, V) > 0$ and $\deg_R(x', V) = 0$;
2. $\deg_R(x, V) = 0$ and $\deg_R(x', V) > 0$;
3. $\deg_R(x, V) > 0$, $\deg_R(x', V) > 0$, $\deg_R(x, V') > 0$ and $\deg_R(x', V') = 0$;
4. $\deg_R(x, V) > 0$, $\deg_R(x', V) > 0$, $\deg_R(x, V') = 0$ and $\deg_R(x', V') > 0$.

Using the induction assumption, in a similar way as in the proof of Lemma 3.22, it can be shown that there exist concepts C and C' such that $V = C^{\mathcal{I}}$ and $V' = C'^{\mathcal{I}}$. It can be seen that either $\exists R.C$ or $\exists R.C'$ distinguishes x and x'. ■

Proposition 3.26. *Algorithm 3 is correct and can be implemented to have time complexity $O(|\Sigma|(m+n)\log n)$, where $m = \sum_{r\in\Sigma_R} |r^{\mathcal{I}}|$ and $n = |\Delta^{\mathcal{I}}|$. A tighter bound for the complexity is $O(|\Sigma_I| + |\Sigma_C|n + |\Sigma_R|(m+n)\log n)$.*

Proof. (Sketch) The contrapositive of Lemma 3.25 states that if $x \equiv_{\Phi,\mathcal{I}} x'$ then x and x' are not separated. That is, $\equiv_{\Phi,\mathcal{I}}$ is a subset of the equivalence relation corresponding to the partition $\mathbb{P}$. As $\equiv_{\Phi,\mathcal{I}}$ and $\sim_{\Phi,\mathcal{I}}$ coincide (by Proposition 3.14), it follows that $\sim_{\Phi,\mathcal{I}}$ is a subset of the equivalence relation corresponding to the partition $\mathbb{P}$. By Lemma 3.24, the latter is also an $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$, hence it is the same as $\sim_{\Phi,\mathcal{I}}$ (the largest $\mathcal{L}_\Phi$-auto-bisimulation of $\mathcal{I}$). That is, Algorithm 3 is correct. This algorithm can be implemented in a similar way as the Paige-Tarjan algorithm for the relational coarsest partition problem [49] and its complexity can be estimated analogously. ■

3.5 Minimizing Interpretations: Applications

Minimizing an interpretation in a DL is not the same as minimizing an ontology in that DL. From the logical point of view, an ontology is specified by a knowledge base, which may have zero or infinitely many models. It is possible that minimizing interpretations may have some effects or may form a starting point for the study on ontology minimization. However, this is a challenging topic of automated reasoning in DLs and is beyond the scope of this dissertation. In this section we only discuss applications of minimizing interpretations, which is useful when one is dealing with a specific interpretation, e.g., with the *unique intended model* of a rule-based knowledge base in a DL or with a counterexample of an instance checking problem in a DL.

Note that if a knowledge base KB has the unique intended model $\mathcal{I}$ then a problem of checking $KB \models \varphi$, where φ is a terminological axiom, a role inclusion axiom or an individual assertion of the form $C(a)$, $R(a, b)$ or $a \doteq b$, is usually defined to be equivalent to the problem of checking whether $\mathcal{I} \models \varphi$. In this case, it makes sense to reduce $\mathcal{I}$ to $\mathcal{I}' = \mathcal{I}/_{\sim_{\Phi,\mathcal{I}}}$ when $\Phi \subseteq \{I, O, U\}$, and to $\mathcal{I}' = \mathcal{I}/^{QS}_{\sim_{\Phi,\mathcal{I}}}$ in the other case. According to Theorems 3.17 and 3.20, $\mathcal{I}'$ is a minimal version of $\mathcal{I}$ w.r.t. essential aspects. Furthermore, by Theorems 3.16 and 3.19, when φ is a query of one of the mentioned forms, $\mathcal{I} \models \varphi$ iff $\mathcal{I}' \models \varphi$, and hence $KB \models \varphi$ iff $\mathcal{I}' \models \varphi$. Clearly, the reduction is useful as it can be used for answering many queries.

We present below exemplary types of rule-based knowledge bases in DLs that have the unique intended model:

- **Acyclic knowledge bases**: The notion of acyclic knowledge bases in DLs is widely used (see, e.g., [48] for a definition). Under the unique name assumption and the closed world assumption, an acyclic knowledge base KB has the *standard model* (see [48] for details). The *unique intended model* of such a KB can be defined to be its standard model. We refer the reader to [48] for an example.

- **OWL 2 RL$^+$**: OWL 2 RL is a profile of OWL 2 Full recommended by W3C. It hase PTime data complexity. Knowledge bases in OWL 2 RL may be unsatisfiable (i.e., inconsistent), since their translations into Datalog may also need negative clauses as constraints. In [9] Cao et al. introduced OWL 2 RL$_0$ as the logical formalism of OWL 2 RL that ignores the predefined data types. They then introduced OWL 2 RL$^+$ as a maximal fragment of OWL 2 RL$_0$ with the property that every knowledge base KB expressed in OWL 2 RL$^+$ can be translated to an equivalent Datalog program P without negative clauses. The *unique intended model* of such a KB is the least Herbrand model of that Datalog program P.

- **WORL** and **SWORL**: In [10] Cao et at. introduced a Web ontology rule language called WORL, which combines a variant of OWL 2 RL with eDatalog$^\neg$. Similarly to the work on OWL 2 RL$^+$ [9], they disallowed those features of OWL 2 RL that play the role of constraints[2], allowed unary external checkable predicates, additional features like negation and the constructor $\geq n\, R.C$ to occur at the left hand side of $\sqsubseteq$ in concept inclusion axioms. They adopted some restrictions for the additional features to guarantee a translation of WORL programs into eDatalog$^\neg$. They also defined the rule language SWORL (stratified WORL) and developed the well-founded semantics for WORL and the standard semantics for SWORL via translation into eDatalog$^\neg$. Both WORL with respect to the well-founded semantics and SWORL with respect to the standard semantics have PTime data complexity. The *unique intended model* of a WORL knowledge base KB can be defined to be the well-founded model of KB, and the *unique intended model* of a SWORL knowledge base KB can be defined to be the standard model of KB.

[2] I.e., the ones that are translated to negative clauses of the form $\varphi \to \bot$.

Chapter 4

Bisimulation-Based Comparisons for Interpretations

In this chapter, we study bisimulation-based comparisons between interpretations in the DLs introduced in Chapter 2. The studied problems are: preservation of semi-positive concepts with respect to comparisons, the Hennessy-Milner property for comparisons, characterizing bisimulation for tidy interpretations by semi-positive concepts, and minimization of interpretations that preserves semi-positive concepts. The class of semi-positive concepts differs from the class of positive concepts in that, in the recursive definition, it also allows $\bot$. This involves non-seriality.

As mentioned in the introduction, "bisimulation-based comparison" is a synonym of "directed simulation". This latter term was introduced by Kurtonina and de Rijke in [36]. In that work, they first formulated directed simulation for a monomodal logic (denoted by $\mathcal{L}^{-}_{\Diamond,\Box}$, which is the monomodal logic K without negation) and then, as examples, they extended it to the DL $\mathcal{FLEUC}^{-}$, temporal logic, feature logics and languages with non-Boolean negation. They used directed simulation to obtain preservation (of negation-free formulas), safety and definability results. They also proved the corresponding Hennessy-Milner property for the considered monomodal logic.

In [42, 44], Nguyen studied the problem of constructing a least Kripke model for a positive modal logic progam in serial modal logics. He compared Kripke models w.r.t. positive consequences using relations that are in fact bisimulation-based comparisons. Other works by Nguyen on Horn fragments of modal and description logics also use bisimulation-based comparisons.

In [25], bisimulation-based comparisons are studied at an abstract level for coalgebraic modal logics under the name Λ-simulation, and the term "positive formula" is used instead of "semi-positive formula". As mentioned before, the term "simulation" traditionally has another meaning, and in our opinion $\bot$ should not be referred to as "positive". At an abstract level, the work [25] does not have a result like a Hennessy-Milner property.

In this chapter, to guarantee a Hennessy-Milner property, roles in semi-positive concepts have a specific syntax due to the presence of the test operator. The definition of semi-positive concepts itself in this chapter is not trivial (e.g., we have that if C is a semi-positive concept then $\leq n\, r.\neg C$ is also a semi-positive concept).

Our results on preservation of semi-positive concepts and the Hennessy-Milner property w.r.t. comparisons may overlap to a certain degree with the known ones.[1] However, our results on "characterizing bisimulation for tidy interpretations by semi-positive concepts" and "minimization that preserves semi-positive concepts" are novel.

This chapter is structured as follows. In Section 4.1 we introduce positive and semi-positive concepts. In Section 4.2 we define bisimulation-based comparisons for interpretations and present results on preservation of semi-positive concepts. In Section 4.3 we present results on the Hennessy-Milner property with respect to semi-positive concepts. In Section 4.4 we characterize bisimulation for tidy interpretations by semi-positive concepts. Section 4.5 is devoted to minimization of interpretations that preserves semi-positive concepts.

4.1 Positive and Semi-Positive Concepts

Definition 4.1 (Positive Concept).
Let $\mathcal{L}_\Phi^{pos}$ be the smallest set of concepts and $\mathcal{L}_{\Phi,\exists}^{pos}$, $\mathcal{L}_{\Phi,\forall}^{pos}$ be the smallest sets of roles defined recursively as follows:

- if $r \in \Sigma_R$ then r is a role of $\mathcal{L}_{\Phi,\exists}^{pos}$ and $\mathcal{L}_{\Phi,\forall}^{pos}$,
- if $I \in \Phi$ and $r \in \Sigma_R$ then r^- is a role of $\mathcal{L}_{\Phi,\exists}^{pos}$ and $\mathcal{L}_{\Phi,\forall}^{pos}$,
- if R and S are roles of $\mathcal{L}_{\Phi,\exists}^{pos}$ and C is a concept of $\mathcal{L}_\Phi^{pos}$ then ε, $R \circ S$, $R \sqcup S$, R^* and $C?$ are roles of $\mathcal{L}_{\Phi,\exists}^{pos}$,
- if R and S are roles of $\mathcal{L}_{\Phi,\forall}^{pos}$ and C is a concept of $\mathcal{L}_\Phi^{pos}$ then ε, $R \circ S$, $R \sqcup S$, R^* and $(\neg C)?$ are roles of $\mathcal{L}_{\Phi,\forall}^{pos}$,
- if $A \in \Sigma_C$ then A is a concept of $\mathcal{L}_\Phi^{pos}$,
- if $O \in \Phi$ and $a \in \Sigma_I$ then $\{a\}$ is a concept of $\mathcal{L}_\Phi^{pos}$,
- if $\mathtt{Self} \in \Phi$ and $r \in \Sigma_R$ then $\exists r.\mathtt{Self}$ is a concept of $\mathcal{L}_\Phi^{pos}$,
- if C is a concept of $\mathcal{L}_\Phi^{pos}$, R is a role of $\mathcal{L}_{\Phi,\exists}^{pos}$ and S is a role of $\mathcal{L}_{\Phi,\forall}^{pos}$ then
  - $\top$, $C \sqcup D$, $C \sqcap D$, $\exists R.C$ and $\forall S.C$ are concepts of $\mathcal{L}_\Phi^{pos}$,
  - if $Q \in \Phi$, $r \in \Sigma_R$ and n is a natural number then $\geq n\, r.C$ and $\leq n\, r.(\neg C)$ are concepts of $\mathcal{L}_\Phi^{pos}$,
  - if $\{Q, I\} \subseteq \Phi$, $r \in \Sigma_R$ and n is a natural number then $\geq n\, r^-.C$ and $\leq n\, r^-.(\neg C)$ are concepts of $\mathcal{L}_\Phi^{pos}$,
  - if $U \in \Phi$ then $\forall U.C$ and $\exists U.C$ are concepts of $\mathcal{L}_\Phi^{pos}$.

A concept of $\mathcal{L}_\Phi^{pos}$ is called a *positive concept* of $\mathcal{L}_\Phi$. ■

[1] We are aware of only the mentioned papers [36, 25] as closely related works of other authors.

We introduce both $\mathcal{L}^{pos}_{\Phi,\forall}$ and $\mathcal{L}^{pos}_{\Phi,\exists}$ due to the test constructor of roles.

Example 4.2. The concepts $\exists(A?).B$ and $\forall((\neg A)?).B$ are positive concepts. They are equivalent to $A \sqcap B$ and $A \sqcup B$, respectively. That the concept $\leq n\, R.(\neg A)$ is positive should not be a surprise, as $\forall R.A$ is equivalent to $\leq 0\, R.(\neg A)$. ■

Definition 4.3 (Semi-Positive Concept).
Let $\mathcal{L}^{sp}_{\Phi}$ be the smallest set of concepts and $\mathcal{L}^{sp}_{\Phi,\exists}$, $\mathcal{L}^{sp}_{\Phi,\forall}$ be the smallest sets of roles defined analogously to the case of $\mathcal{L}^{pos}_{\Phi}$, $\mathcal{L}^{pos}_{\Phi,\exists}$, $\mathcal{L}^{pos}_{\Phi,\forall}$ except that $\perp$ is also allowed as a concept of $\mathcal{L}^{sp}_{\Phi}$. We call concepts of $\mathcal{L}^{sp}_{\Phi}$ *semi-positive concepts* of $\mathcal{L}_{\Phi}$. ■

4.2 Bisimulation-Based Comparisons

The following definition of $\mathcal{L}_{\Phi}$-comparison differs from the definition of $\mathcal{L}_{\Phi}$-bisimulation only in that, in the conditions (4.2), (4.7) and (4.14), the second implication ($\Rightarrow$) is used instead of equivalence ($\Leftrightarrow$). Technically, the conditions (4.1)–(4.14) differ from the conditions (3.1)–(3.14) only in that the three occurrences of $\Leftrightarrow$ in the latter are replaced by $\Rightarrow$.

Definition 4.4 ($\mathcal{L}_{\Phi}$-Comparison).
Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations. A non-empty binary relation $Z \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}$ is called an *$\mathcal{L}_{\Phi}$-comparison* between $\mathcal{I}$ and $\mathcal{I}'$ if the following conditions hold for every $a \in \Sigma_I$, $A \in \Sigma_C$, $r \in \Sigma_R$, $x, y \in \Delta^{\mathcal{I}}$, $x', y' \in \Delta^{\mathcal{I}'}$:

$$Z(a^{\mathcal{I}}, a^{\mathcal{I}'}) \tag{4.1}$$

$$Z(x, x') \Rightarrow [A^{\mathcal{I}}(x) \Rightarrow A^{\mathcal{I}'}(x')] \tag{4.2}$$

$$[Z(x, x') \wedge r^{\mathcal{I}}(x, y)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y, y') \wedge r^{\mathcal{I}'}(x', y')] \tag{4.3}$$

$$[Z(x, x') \wedge r^{\mathcal{I}'}(x', y')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z(y, y') \wedge r^{\mathcal{I}}(x, y)], \tag{4.4}$$

if $I \in \Phi$ then

$$[Z(x, x') \wedge r^{\mathcal{I}}(y, x)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y, y') \wedge r^{\mathcal{I}'}(y', x')] \tag{4.5}$$

$$[Z(x, x') \wedge r^{\mathcal{I}'}(y', x')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z(y, y') \wedge r^{\mathcal{I}}(y, x)], \tag{4.6}$$

if $O \in \Phi$ then

$$Z(x, x') \Rightarrow [x = a^{\mathcal{I}} \Rightarrow x' = a^{\mathcal{I}'}], \tag{4.7}$$

if $Q \in \Phi$ then

if $Z(x, x')$ holds and $y_1, \ldots, y_n$ ($n \geq 1$) are pairwise different elements of $\Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y_i)$ holds for every $1 \leq i \leq n$ then there exist pairwise different elements $y'_1, \ldots, y'_n$ of $\Delta^{\mathcal{I}'}$ such that $r^{\mathcal{I}'}(x', y'_i)$ and $Z(y_i, y'_i)$ hold for every $1 \leq i \leq n$ (4.8)

if $Z(x, x')$ holds and $y'_1, \ldots, y'_n$ ($n \geq 1$) are pairwise different elements of $\Delta^{\mathcal{I}'}$ such that $r^{\mathcal{I}'}(x', y'_i)$ holds for every $1 \leq i \leq n$ then there exist pairwise different elements $y_1, \ldots, y_n$ of $\Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y_i)$ and $Z(y_i, y'_i)$ hold for every $1 \leq i \leq n$, (4.9) if $\{Q, I\} \subseteq \Phi$ then (additionally)

$$\text{if } Z(x,x') \text{ holds and } y_1,\ldots,y_n \ (n \geq 1) \text{ are pairwise different elements of } \Delta^{\mathcal{I}} \text{ such that } r^{\mathcal{I}}(y_i,x) \text{ holds for every } 1 \leq i \leq n \text{ then there exist pairwise different elements } y'_1,\ldots,y'_n \text{ of } \Delta^{\mathcal{I}'} \text{ such that } r^{\mathcal{I}'}(y'_i,x') \text{ and } Z(y_i,y'_i) \text{ hold for every } 1 \leq i \leq n \tag{4.10}$$

$$\text{if } Z(x,x') \text{ holds and } y'_1,\ldots,y'_n \ (n \geq 1) \text{ are pairwise different elements of } \Delta^{\mathcal{I}'} \text{ such that } r^{\mathcal{I}'}(y'_i,x') \text{ holds for every } 1 \leq i \leq n \text{ then there exist pairwise different elements } y_1,\ldots,y_n \text{ of } \Delta^{\mathcal{I}} \text{ such that } r^{\mathcal{I}}(y_i,x) \text{ and } Z(y_i,y'_i) \text{ hold for every } 1 \leq i \leq n, \tag{4.11}$$

if $U \in \Phi$ then

$$\forall x \in \Delta^{\mathcal{I}}\, \exists x' \in \Delta^{\mathcal{I}'}\, Z(x,x') \tag{4.12}$$

$$\forall x' \in \Delta^{\mathcal{I}'}\, \exists x \in \Delta^{\mathcal{I}}\, Z(x,x'), \tag{4.13}$$

if $\mathtt{Self} \in \Phi$ then

$$Z(x,x') \Rightarrow [r^{\mathcal{I}}(x,x) \Rightarrow r^{\mathcal{I}'}(x',x')]. \tag{4.14}$$

For example, if $\Phi = \{I, Q\}$ then only the conditions (4.1)-(4.6) and (4.8)-(4.11) are essential. ■

The following lemma is similar to Lemma 3.1 and can easily be proved.

Lemma 4.1.

1. *The relation $\{\langle x,x\rangle \mid x \in \Delta^{\mathcal{I}}\}$ is an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}$.*

2. *If Z_1 is an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}_0$ and $\mathcal{I}_1$, and Z_2 is an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}_1$ and $\mathcal{I}_2$, then $Z_1 \circ Z_2$ is an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}_0$ and $\mathcal{I}_2$.*

3. *If $\mathcal{Z}$ is a set of $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$ then $\bigcup \mathcal{Z}$ is also an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$.*

Definition 4.5. We write $\mathcal{I} \lesssim_\Phi \mathcal{I}'$ to denote that there exists an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$. For $x \in \Delta^{\mathcal{I}}$ and $x' \in \Delta^{\mathcal{I}'}$, we write $x \lesssim_\Phi x'$ to denote that there exists an $\mathcal{L}_\Phi$-comparison Z between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z(x,x')$ holds. ■

By Lemma 4.1, the relation $\lesssim_\Phi$ between interpretations (resp. between elements of interpretations' domains) is a preorder.

To check whether there exists an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$, one can use Algorithm 4 (on page 51), which is very similar to Algorithm 1. It is straightforward to prove the following proposition.

Proposition 4.2. *Algorithm 4 is correct. Furthermore, if it returns Z (but not "false") then Z is a maximal (w.r.t. $\subseteq$) $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$.*

Algorithm 4: computing an $\mathcal{L}_\Phi$-comparison between two finite interpretations

input : a set Φ of DL-features and finite interpretations $\mathcal{I}$, $\mathcal{I}'$
output: an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$ if it exists, or *false* otherwise.

$Z := \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}$;
repeat
  foreach $x \in \Delta^{\mathcal{I}}$ *and* $x' \in \Delta^{\mathcal{I}'}$ **do**
    if *some condition among (4.2)-(4.11), (4.14) is related to* Φ *but not satisfied for some* A, r, y, y', a **then** delete the pair $\langle x, x' \rangle$ from Z;
until Z *was not modified during the last iteration*;
if *the condition (4.1) is not satisfied for some* $a \in \Sigma_I$ **then return** *false*;
if $U \in \Phi$ *and the condition (4.12) or (4.13) is not satisfied* **then return** *false*;
return Z;

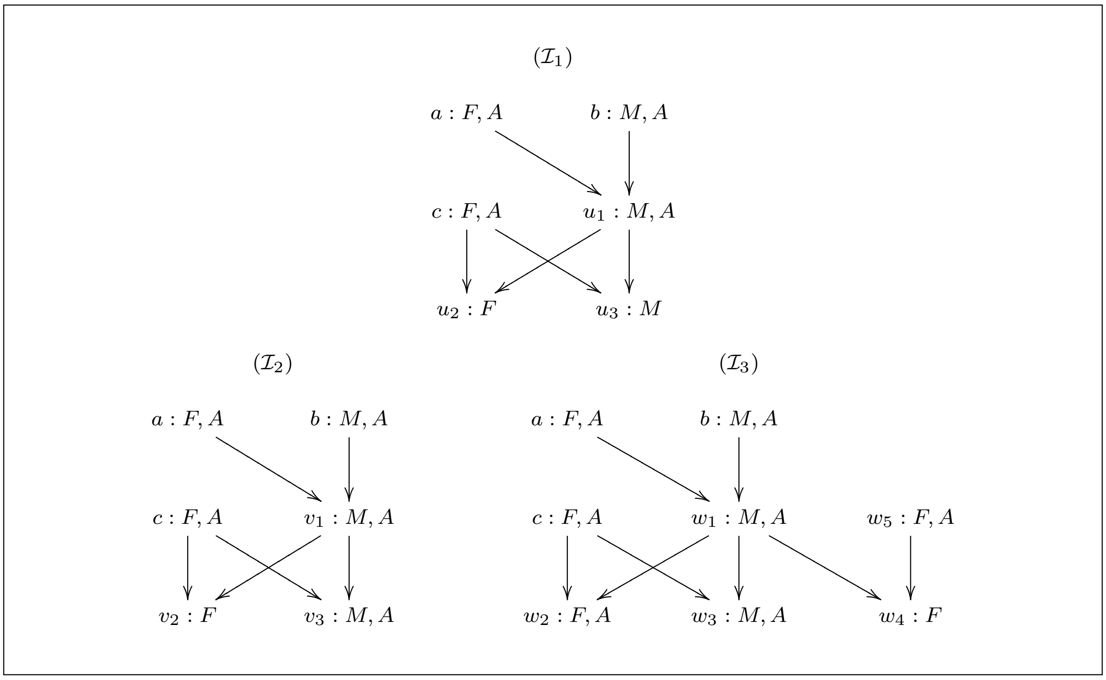


Figure 4.1: Interpretations used in Example 4.6.

Example 4.6. Let $\Sigma_I = \{a, b, c\}$, $\Sigma_C = \{F, M, A\}$ and $\Sigma_R = \{r\}$. The symbols have the meanings as in Example 3.4, except that A stands for *Adult*. In Figure 4.1 we present three interpretations $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{I}_3$ in a similar way as for Example 3.4. These interpretations are not $\mathcal{L}$-bisimilar to each other, but we have that:

- $\mathcal{I}_1 \lesssim_\Phi \mathcal{I}_2$ for any Φ,
- $\mathcal{I}_1 \lesssim_\Phi \mathcal{I}_3$ and $\mathcal{I}_2 \lesssim_\Phi \mathcal{I}_3$ (only) when $\Phi \subseteq \{O, \texttt{Self}\}$.

We do not have $\mathcal{I}_2 \lesssim_\Phi \mathcal{I}_1$ nor $\mathcal{I}_3 \lesssim_\Phi \mathcal{I}_2$ nor $\mathcal{I}_3 \lesssim_\Phi \mathcal{I}_1$ for any Φ. These assertions can be checked by using Algorithm 4. ■

Lemma 4.3. *Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations and Z be an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$. Then the following properties hold for every concept C of $\mathcal{L}^{sp}_\Phi$, every role R of $\mathcal{L}^{sp}_{\Phi,\exists}$, every role S of $\mathcal{L}^{sp}_{\Phi,\forall}$, every $x, y \in \Delta^\mathcal{I}$, every $x', y' \in \Delta^{\mathcal{I}'}$, and every $a \in \Sigma_I$:*

$$Z(x, x') \Rightarrow [C^\mathcal{I}(x) \Rightarrow C^{\mathcal{I}'}(x')] \tag{4.15}$$

$$[Z(x, x') \wedge R^\mathcal{I}(x, y)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z(y, y') \wedge R^{\mathcal{I}'}(x', y')] \tag{4.16}$$

$$[Z(x, x') \wedge S^{\mathcal{I}'}(x', y')] \Rightarrow \exists y \in \Delta^\mathcal{I}[Z(y, y') \wedge S^\mathcal{I}(x, y)]. \tag{4.17}$$

Proof. We prove this lemma by induction on the structures of C, R and S.

Consider the assertion (4.16). Suppose $Z(x, x')$ and $R^\mathcal{I}(x, y)$ hold. By induction on the structure of R we prove that there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y, y')$ and $R^{\mathcal{I}'}(x', y')$ hold. The base case occurs when R is a role name and the assertion for it follows from (4.3). The induction steps are given below.

- Case $R = \varepsilon$ is trivial.
- Case $R = R_1 \circ R_2$, where R_1 and R_2 are roles of $\mathcal{L}^{sp}_{\Phi,\exists}$: We have that $(R_1 \circ R_2)^\mathcal{I}(x, y)$ holds. Hence, there exists $z \in \Delta^\mathcal{I}$ such that $R_1^\mathcal{I}(x, z)$ and $R_2^\mathcal{I}(z, y)$ hold. By the inductive assumption of (4.16), there exists $z' \in \Delta^{\mathcal{I}'}$ such that $Z(z, z')$ and $R_1^{\mathcal{I}'}(x', z')$ hold, and there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y, y')$ and $R_2^{\mathcal{I}'}(z', y')$ hold. Since $R_1^{\mathcal{I}'}(x', z')$ and $R_2^{\mathcal{I}'}(z', y')$ hold, we have that $(R_1 \circ R_2)^{\mathcal{I}'}(x', y')$ holds, i.e. $R^{\mathcal{I}'}(x', y')$ holds.
- Case $R = R_1 \sqcup R_2$, where R_1 and R_2 are roles of $\mathcal{L}^{sp}_{\Phi,\exists}$, is trivial.
- Case $R = R_1^*$, where R_1 is a role of $\mathcal{L}^{sp}_{\Phi,\exists}$: Since $R^\mathcal{I}(x, y)$ holds, there exists $x_0, \ldots, x_k \in \Delta^\mathcal{I}$ such that $x_0 = x$, $x_k = y$ and, for $1 \leq i \leq k$, $R_1^\mathcal{I}(x_{i-1}, x_i)$ holds. Let $x'_0 = x'$. For each $1 \leq i \leq k$, since $Z(x_{i-1}, x'_{i-1})$ and $R_1^\mathcal{I}(x_{i-1}, x_i)$ hold, by the inductive assumption of (4.16), there exists $x'_i \in \Delta^\mathcal{I}$ such that $Z(x_i, x'_i)$ and $R_1^{\mathcal{I}'}(x'_{i-1}, x'_i)$ hold. Hence, $Z(x_k, x'_k)$ and $(R_1^*)^{\mathcal{I}'}(x'_0, x'_k)$ hold. Let $y' = x'_k$. Thus, $Z(y, y')$ and $R^{\mathcal{I}'}(x', y')$ hold.
- Case $R = (D?)$, where D is a concept of $\mathcal{L}^{sp}_\Phi$: By the definition of $(D?)^\mathcal{I}$, we have that $D^\mathcal{I}(x)$ holds and $x = y$. By the inductive assumption of (4.15), $D^{\mathcal{I}'}(x')$ holds, and therefore $R^{\mathcal{I}'}(x', x')$ holds. By choosing $y' = x'$, we have that $Z(y, y')$ and $R^{\mathcal{I}'}(x', y')$ hold.
- Case $I \in \Phi$ and $R = r^-$: The assertion for this case follows from (4.5).

The assertion (4.17) can be proved analogously as for (4.16) except for the case $S = (\neg C)?$, where C is a concept of $\mathcal{L}^{sp}_\Phi$. The proof for this case is as follows. Suppose $Z(x, x')$ and $S^{\mathcal{I}'}(x', y')$ hold. Thus, $(\neg C)^{\mathcal{I}'}(x')$ holds and $x' = y'$. By the contrapositive of the inductive assumption of (4.15), it follows that $(\neg C)^\mathcal{I}(x)$ holds. By choosing $y = x$, $Z(y, y')$ and $S^\mathcal{I}(x, y)$ hold.

Consider the assertion (4.15). Suppose $Z(x, x')$ and $C^{\mathcal{I}}(x)$ hold, where C is a concept of $\mathcal{L}_{\Phi}^{sp}$. We show that $C^{\mathcal{I}'}(x')$ holds. The cases when C is of the form $\top$, $\bot$, A, $D \sqcup D'$ or $D \sqcap D'$ are trivial.

- Case $C = \exists R.D$, where R is a role of $\mathcal{L}_{\Phi,\exists}^{sp}$ and D is a concept of $\mathcal{L}_{\Phi}^{sp}$: Since $(\exists R.D)^{\mathcal{I}}(x)$ holds, there exists $y \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y)$ and $D^{\mathcal{I}}(y)$ hold. By the inductive assumption of (4.16) (proved earlier), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y, y')$ and $R^{\mathcal{I}'}(x', y')$ hold. By the inductive assumption of (4.15), $D^{\mathcal{I}'}(y')$ holds. Therefore, $C^{\mathcal{I}'}(x')$ holds.

- Case $C = \forall S.D$, where S is a role of $\mathcal{L}_{\Phi,\forall}^{sp}$ and D is a concept of $\mathcal{L}_{\Phi}^{sp}$: Let y' be an arbitrary element of $\Delta^{\mathcal{I}'}$ such that $S^{\mathcal{I}'}(x', y')$ holds. We show that $D^{\mathcal{I}'}(y')$ holds. By the inductive assumption of (4.17) (proved earlier), there exists $y \in \Delta^{\mathcal{I}}$ such that $Z(y, y')$ and $S^{\mathcal{I}}(x, y)$ hold. Since $(\forall S.D)^{\mathcal{I}}(y)$ holds, it follows that $D^{\mathcal{I}}(y)$ holds. Therefore, by the inductive assumption of (4.15), it follows that $D^{\mathcal{I}'}(y')$ holds.

- Case $O \in \Phi$ and $C = \{a\}$: Since $\{a\}^{\mathcal{I}}(x)$ holds, we have that $x = a^{\mathcal{I}}$. By the condition (4.7), it follows that $x' = a^{\mathcal{I}'}$. Hence $C^{\mathcal{I}'}(x')$ holds.

- Case $\mathtt{Self} \in \Phi$ and $C = \exists r.\mathtt{Self}$: Since $(\exists r.\mathtt{Self})^{\mathcal{I}}(x)$ holds, we have that $r^{\mathcal{I}}(x, x)$ holds. By the condition (4.14), it follows that $r^{\mathcal{I}'}(x', x')$ holds. Hence $C^{\mathcal{I}'}(x')$ holds.

- Case $Q \in \Phi$ and $C = (\geq n\, r.D)$, where D is a concept of $\mathcal{L}_{\Phi}^{sp}$: Since $C^{\mathcal{I}}(x)$ holds, there exist pairwise different $y_1, \ldots, y_n \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y_i)$ and $D^{\mathcal{I}}(y_i)$ hold for all $1 \leq i \leq n$. Since $Z(x, x')$ holds, by the condition (4.8), there exist pairwise different $y'_1, \ldots, y'_n \in \Delta^{\mathcal{I}'}$ such that $r^{\mathcal{I}'}(x', y'_i)$ and $Z(y_i, y'_i)$ hold for all $1 \leq i \leq n$. Since $Z(y_i, y'_i)$ and $D^{\mathcal{I}}(y_i)$ hold, by the inductive assumption of (4.15), it follows that $D^{\mathcal{I}'}(y'_i)$ holds. Since $r^{\mathcal{I}'}(x', y'_i)$ and $D^{\mathcal{I}'}(y'_i)$ hold for all $1 \leq i \leq n$, it follows that $C^{\mathcal{I}'}(x')$ holds.

- Case $\{Q, I\} \subseteq \Phi$ and $C = (\geq n\, r^{-1}.D)$, where D is a concept of $\mathcal{L}_{\Phi}^{sp}$, can be proved analogously to the above case.

- Case $Q \in \Phi$ and $C = (\leq n\, r.(\neg D))$, where D is a concept of $\mathcal{L}_{\Phi}^{sp}$: For the sake of contradiction, suppose $C^{\mathcal{I}'}(x')$ does not hold. Thus, $(\neg C)^{\mathcal{I}'}(x')$ holds, which means $(\geq (n+1)\, r.(\neg D))^{\mathcal{I}'}(x')$ holds. Hence, there exist pairwise different $y'_1, \ldots, y'_{n+1} \in \Delta^{\mathcal{I}'}$ such that $r^{\mathcal{I}'}(x', y'_i)$ and $(\neg D)^{\mathcal{I}'}(y'_i)$ hold for all $1 \leq i \leq n+1$. By the condition (4.9), there exist pairwise different $y_1, \ldots, y_{n+1} \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y_i)$ and $Z(y_i, y'_i)$ hold for all $1 \leq i \leq n+1$. For each $1 \leq i \leq n+1$, since $Z(y_i, y'_i)$ holds and $D^{\mathcal{I}'}(y'_i)$ does not hold, by the inductive assumption of (4.15), $D^{\mathcal{I}}(y_i)$ does not hold, which means $(\neg D)^{\mathcal{I}}(y_i)$ holds. It follows that $(\neg C)^{\mathcal{I}}(x)$ holds, which contradicts the assumption that $C^{\mathcal{I}}(x)$ holds.

- Case $\{Q, I\} \subseteq \Phi$ and $C = (\leq n\, r^{-1}.(\neg D))$, where D is a concept of $\mathcal{L}_{\Phi}^{sp}$, can be proved analogously to the above case.

- Case $U \in \Phi$ and $C = \forall U.D$, where D is a concept of $\mathcal{L}_\Phi^{sp}$: Let $y' \in \Delta^{\mathcal{I}'}$. By the condition (4.13), there exists $y \in \Delta^{\mathcal{I}}$ such that $Z(y, y')$ holds. Since $C^{\mathcal{I}}(x)$ holds, it follows that $D^{\mathcal{I}}(y)$ holds. By the inductive assumption of (4.15), it follows that $D^{\mathcal{I}'}(y')$ holds. Hence $C^{\mathcal{I}'}(x')$ holds.

- Case $U \in \Phi$ and $C = \exists U.D$, where D is a concept of $\mathcal{L}_\Phi^{sp}$: Since $C^{\mathcal{I}}(x)$ holds, there exists $y \in \Delta^{\mathcal{I}}$ such that $D^{\mathcal{I}}(y)$ holds. By the condition (4.12), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z(y, y')$ holds. By the inductive assumption of (4.15), it follows that $D^{\mathcal{I}'}(y')$ holds. Hence $C^{\mathcal{I}'}(x')$ holds. ∎

Definition 4.7 (Preservation by $\mathcal{L}_\Phi$-Comparisons).
A concept C of $\mathcal{L}_\Phi$ is said to be *preserved by $\mathcal{L}_\Phi$-comparisons* if, for any interpretations $\mathcal{I}$, $\mathcal{I}'$ and any $\mathcal{L}_\Phi$-comparison Z between $\mathcal{I}$ and $\mathcal{I}'$, if $Z(x, x')$ holds and $x \in C^{\mathcal{I}}$ then $x' \in C^{\mathcal{I}'}$. ∎

The following theorem follows immediately from the assertion (4.15) of Lemma 4.3.

Theorem 4.4. *All concepts of $\mathcal{L}_\Phi^{sp}$ are preserved by $\mathcal{L}_\Phi$-comparisons.*

Corollary 4.5. *All concepts of $\mathcal{L}_\Phi^{pos}$ are preserved by $\mathcal{L}_\Phi$-comparisons.*

4.3 The Hennessy-Milner Property with Respect to Semi-Positive Concepts

In this section, we present theorems similar to the Hennessy-Milner property that are related to $\mathcal{L}_\Phi$-comparisons and semi-positive/positive concepts.

Definition 4.8. Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations, $x \in \Delta^{\mathcal{I}}$ and $x' \in \Delta^{\mathcal{I}'}$. Define that:

- x is *equivalent to* x' w.r.t. (concepts of) $\mathcal{L}_\Phi$, denoted by $x \equiv_\Phi x'$, if, for every concept C of $\mathcal{L}_\Phi$, $x \in C^{\mathcal{I}}$ iff $x' \in C^{\mathcal{I}'}$;

- x is *less than or equal to* x' w.r.t. concepts of $\mathcal{L}_\Phi^{sp}$ (resp. $\mathcal{L}_\Phi^{pos}$), denoted by $x \leq_\Phi^{sp} x'$ (resp. $x \leq_\Phi^{pos} x'$), if, for every concept C of $\mathcal{L}_\Phi^{sp}$ (resp. $\mathcal{L}_\Phi^{pos}$), $x \in C^{\mathcal{I}}$ implies $x' \in C^{\mathcal{I}'}$;

- x is *equivalent to* x' w.r.t. concepts of $\mathcal{L}_\Phi^{sp}$, denoted by $x \equiv_\Phi^{sp} x'$, if $x \leq_\Phi^{sp} x'$ and $x' \leq_\Phi^{sp} x$. ∎

We need the following lemma, which allows us to check the conditions (4.8)-(4.11) in another way.

Lemma 4.6. *Let $Z \subseteq \mathbf{S} \times \mathbf{S}'$ be a binary relation such that, for any natural number n and any pairwise different $x_1, \ldots, x_n \in \mathbf{S}$, there exist pairwise different $x'_1, \ldots, x'_n \in \mathbf{S}'$ with the property that, for any $1 \leq j \leq n$, there exists $1 \leq i \leq n$ such that $\langle x_i, x'_j \rangle \in Z$. Then, for any natural number n and any pairwise different $x_1, \ldots, x_n \in \mathbf{S}$, there exist pairwise different $x'_1, \ldots, x'_n \in \mathbf{S}'$ such that $\langle x_i, x'_i \rangle \in Z$ for all $1 \leq i \leq n$.*

Proof. We prove this lemma by induction on n. The base case $n = 0$ is trivial. Assuming that the induction hypothesis holds for some n, we show that it also holds for $n + 1$ (in the place of n). Suppose $X \cup \{x_0\} \subseteq \mathbf{S}$, $\#X = n$ and $x_0 \notin X$. We prove that there exists an injection $g : X \cup \{x_0\} \to \mathbf{S}'$ such that, for every $x \in X \cup \{x_0\}$, $\langle x, g(x)\rangle \in Z$.

By the inductive assumption, there exists a bijection $f : X \to X'$ such that $X' \subseteq \mathbf{S}'$ and, for every $x \in X$, $\langle x, f(x)\rangle \in Z$. Consider the following procedure:

---

1 $i := 0,\ \ X_0 := \emptyset,\ \ X'_0 := \emptyset$;
2 **while** *true* **do**
3 $i := i + 1$;
4   set X'_i to be any subset of $\mathbf{S}'$ with cardinality i such that, for any $x' \in X'_i$,
       there exists $x \in X_{i-1} \cup \{x_0\}$ such that $\langle x, x'\rangle \in Z$;
5   set x'_i to an arbitrary element from $X'_i \setminus f(X_{i-1})$;
6   **if** $x'_i \in X'$ **then**
7 $x_i := f^{-1}(x'_i)$;
8 $X_i := X_{i-1} \cup \{x_i\}$;
9   **else break**;

Note that, at any step of the execution of the above procedure, $\#X_{i-1} = i - 1$, $X_{i-1} \subseteq X$ and $\#X'_i = i$. Hence, at the step 5, x'_i can be set properly. Observe that the loop terminates after some iteration with $i \leq n + 1$. The reason is that: if $i = n + 1$ then $f(X_{i-1}) = f(X) = X'$ and $x'_i \notin X'$, which terminates the loop. Let k be the final value of i (when the loop terminates).

We prove by an inner induction on j from 1 to k that there exists a bijection $g_j : X_{j-1} \cup \{x_0\} \to f(X_{j-1}) \cup \{x'_j\}$ such that $\langle x, g_j(x)\rangle \in Z$ for every $x \in X_{j-1} \cup \{x_0\}$. The base case $j = 1$ is trivial. Assuming that the hypothesis of the inner induction holds for some $j < k$ and any natural number less than j, we show that it also holds for $j + 1$ (in the place of j).

Let $g_{j+1} : X_j \cup \{x_0\} \to f(X_j) \cup \{x'_{j+1}\}$ be specified as follows:

- if $\langle x_j, x'_{j+1}\rangle \in Z$ then $g_{j+1}(x_j) = x'_{j+1}$ and $g_{j+1}(x) = g_j(x)$ for $x \in X_{j-1} \cup \{x_0\}$,
- else let j' be a natural number such that $0 \leq j' < j$ and $\langle x_{j'}, x'_{j+1}\rangle \in Z$ (such j' exists due to the definition of x'_{j+1}) and define
  - $g_{j+1}(x_h) = g_{j'}(x_h)$ for $0 \leq h < j'$,
  - $g_{j+1}(x_{j'}) = x'_{j+1}$,
  - $g_{j+1}(x_h) = f(x_h)$ for $j' < h \leq j$.

By the inductive assumption of the inner induction for j and j', it is easy to see that g_{j+1} satisfies the induction hypothesis for $j + 1$.

We define the intended injection $g : X \cup \{x_0\} \to \mathbf{S}'$ as follows:

- $g(x_i) = g_k(x_i)$ for $0 \leq i < k$,
- $g(x) = f(x)$ for $x \in X \setminus (X_{k-1} \cup \{x_0\})$.

Since f and g_k are bijections, $g_k(X_{k-1} \cup \{x_0\}) = f(X_{k-1}) \cup \{x'_k\}$ and $x'_k \notin X'$, it is easy to see that g is an injection. Due to the properties of f and g_k, we also have that $\langle x, g(x)\rangle \in Z$ for all $x \in X \cup \{x_0\}$. ■

Theorem 4.7. *Let $\mathcal{I}$ and $\mathcal{I}'$ be modally saturated interpretations (w.r.t. $\mathcal{L}_\Phi$) such that, for every $a \in \Sigma_I$, $a^\mathcal{I} \leq^{sp}_\Phi a^{\mathcal{I}'}$. Suppose that if $U \in \Phi$ then: either both $\mathcal{I}$ and $\mathcal{I}'$ are unreachable-objects-free, or both of them are not unreachable-objects-free, or both of them are finite. Then, for every $x \in \Delta^\mathcal{I}$ and $x' \in \Delta^{\mathcal{I}'}$, $x \leq^{sp}_\Phi x'$ iff $x \lesssim_\Phi x'$. In particular, the relation $\{\langle x, x'\rangle \in \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'} \mid x \leq^{sp}_\Phi x'\}$ is an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$ when it is not empty.*

Proof. First, suppose Z is an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z(x, x')$ holds. We show that $x \leq^{sp}_\Phi x'$. Let C be an arbitrary concept of $\mathcal{L}^{sp}_\Phi$ such that $C^\mathcal{I}(x)$ holds. Thus, by the assertion (4.15) of Lemma 4.3, $C^{\mathcal{I}'}(x')$ holds. Therefore, $x \leq^{sp}_\Phi x'$.

Conversely, let $Z = \{\langle x, x'\rangle \in \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'} \mid x \leq^{sp}_\Phi x'\}$ and assume that Z is not empty. We show that Z is an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$.

- The condition (4.1) immediately follows from the assumption of the theorem.
- Consider the condition (4.2). If $Z(x, x')$ and $A^\mathcal{I}(x)$ hold, then by the definition of Z, $A^{\mathcal{I}'}(x')$ holds.
- Consider the condition (4.3). Suppose $Z(x, x')$ and $r^\mathcal{I}(x, y)$ hold. Let $\mathbf{S} = \{y' \in \Delta^{\mathcal{I}'} \mid r^{\mathcal{I}'}(x', y')\}$. We show that there exists $y' \in \mathbf{S}$ such that $Z(y, y')$ holds. For the sake of contradiction, suppose that, for every $y' \in \mathbf{S}$, $Z(y, y')$ does not hold, which means that $y \not\leq^{sp}_\Phi y'$. Thus, for every $y' \in \mathbf{S}$, there exists a concept $C_{y'}$ of $\mathcal{L}^{sp}_\Phi$ such that $y \in C^\mathcal{I}_{y'}$ but $y' \notin C^{\mathcal{I}'}_{y'}$. Let $\Gamma = \{C_{y'} \mid y' \in \mathbf{S}\}$. Thus, no $y' \in \mathbf{S}$ satisfies Γ (i.e. $\mathbf{S} \cap \Gamma^{\mathcal{I}'} = \emptyset$). Since $\mathcal{I}'$ is modally saturated, it follows that there exists a finite subset Λ of Γ such that, for every $y' \in \mathbf{S}$, $y' \notin \Lambda^{\mathcal{I}'}$. Consider the concept $C = \exists r.\bigsqcap\Lambda$ of $\mathcal{L}^{sp}_\Phi$. $C^\mathcal{I}(x)$ holds, but $C^{\mathcal{I}'}(x')$ does not. This contradicts $x \leq^{sp}_\Phi x'$.
- Consider the condition (4.4). Suppose $Z(x, x')$ and $r^{\mathcal{I}'}(x', y')$ hold. Let $\mathbf{S} = \{y \in \Delta^\mathcal{I} \mid r^\mathcal{I}(x, y)\}$. We show that there exists $y \in \mathbf{S}$ such that $Z(y, y')$ holds. For the sake of contradiction, suppose that, for every $y \in \mathbf{S}$, $Z(y, y')$ does not hold, i.e. $y \not\leq^{sp}_\Phi y'$. Thus, for every $y \in \mathbf{S}$, there exists a concept C_y of $\mathcal{L}^{sp}_\Phi$ such that $y \in C^\mathcal{I}_y$ but $y' \notin C^{\mathcal{I}'}_y$. Let $\Gamma = \{\neg C_y \mid y \in \mathbf{S}\}$. Thus, no $y \in \mathbf{S}$ satisfies Γ (i.e. $\mathbf{S} \cap \Gamma^\mathcal{I} = \emptyset$). Since $\mathcal{I}$ is modally saturated, it follows that there exists a finite subset Λ of Γ such that, for every $y \in \mathbf{S}$, $y \notin \Lambda^\mathcal{I}$. Thus, $x \in (\forall r.\neg\bigsqcap\Lambda)^\mathcal{I}$. Let $\Lambda = \{\neg C_{y_1}, \ldots, \neg C_{y_n}\}$ and $C = \forall r.(C_{y_1} \sqcup \ldots \sqcup C_{y_n})$ (we have $C = \forall r.\bot$ when $n = 0$). The concept C belongs to $\mathcal{L}^{sp}_\Phi$ and is equivalent to $\forall r.\neg\bigsqcap\Lambda$. Hence, $C^\mathcal{I}(x)$ holds, but $C^{\mathcal{I}'}(x')$ does not. This contradicts $x \leq^{sp}_\Phi x'$.
- The conditions (4.5) and (4.6) can be proved analogously as for the conditions (4.3) and (4.4), respectively.
- Consider the condition (4.7) and the case $O \in \Phi$. Suppose $Z(x, x')$ holds and $x = a^\mathcal{I}$. Since $\{a\}^\mathcal{I}(x)$ holds and $x \leq^{sp}_\Phi x'$, it follows that $\{a\}^{\mathcal{I}'}(x')$ holds. Therefore, $x' = a^{\mathcal{I}'}$.

- Consider the condition (4.8) and the case $Q \in \Phi$. Suppose $Z(x, x')$ holds, i.e., $x \leq_{\Phi}^{sp} x'$. Let $\mathbf{S} = \{y \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x, y)\}$ and $\mathbf{S}' = \{y' \in \Delta^{\mathcal{I}'} \mid r^{\mathcal{I}'}(x', y')\}$. Let $y_1, \ldots, y_n$ be pairwise different elements of $\mathbf{S}$. Let $\mathbf{S}'' = \{y' \in \mathbf{S}' \mid$ there exists $1 \leq i \leq n$ such that $y_i \leq_{\Phi}^{sp} y'\}$. To prove the condition (4.8), by Lemma 4.6, it is sufficient to prove that $\#\mathbf{S}'' \geq n$. For each $y' \in \mathbf{S}' \setminus \mathbf{S}''$, there exist concepts $D_{y',1}, \ldots, D_{y',n}$ of $\mathcal{L}_{\Phi}^{sp}$ such that $y_i \in D_{y',i}^{\mathcal{I}}$ and $y' \notin D_{y',i}^{\mathcal{I}'}$ for every $1 \leq i \leq n$. For each $y' \in \mathbf{S}' \setminus \mathbf{S}''$, let $C_{y'} = D_{y',1} \sqcup \ldots \sqcup D_{y',n}$, then we have that $y_i \in C_{y'}^{\mathcal{I}}$ for all $1 \leq i \leq n$, but $y' \notin C_{y'}^{\mathcal{I}'}$. Let $\Gamma = \{C_{y'} \mid y' \in \mathbf{S}' \setminus \mathbf{S}''\}$. Note that $\Gamma^{\mathcal{I}'} \cap (\mathbf{S}' \setminus \mathbf{S}'') = \emptyset$. For every finite subset Λ of Γ, since $y_1, \ldots, y_n \in \Lambda^{\mathcal{I}}$, we have $x \in (\geq n\, r.\bigsqcap \Lambda)^{\mathcal{I}}$, and since $x \leq_{\Phi}^{sp} x'$, we also have that $x' \in (\geq n\, r.\bigsqcap \Lambda)^{\mathcal{I}'}$, which means there are at least n pairwise different $y'_1, \ldots, y'_n \in \mathbf{S}'$ that belong to $\Lambda^{\mathcal{I}'}$. Since $\mathcal{I}'$ is modally saturated, it follows that there are at least n pairwise different $y'_1, \ldots, y'_n \in \mathbf{S}'$ that belong to $\Gamma^{\mathcal{I}'}$. Since $\Gamma^{\mathcal{I}'} \cap (\mathbf{S}' \setminus \mathbf{S}'') = \emptyset$, it follows that $\#\mathbf{S}'' \geq n$.

- Consider the condition (4.9) and the case $Q \in \Phi$. Suppose $Z(x, x')$ holds, i.e., $x \leq_{\Phi}^{sp} x'$. Let $\mathbf{S} = \{y \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x, y)\}$ and $\mathbf{S}' = \{y' \in \Delta^{\mathcal{I}'} \mid r^{\mathcal{I}'}(x', y')\}$. Let $y'_1, \ldots, y'_n$ be pairwise different elements of $\mathbf{S}'$. Let $\mathbf{S}'' = \{y \in \mathbf{S} \mid$ there exists $1 \leq i \leq n$ such that $y \leq_{\Phi}^{sp} y'_i\}$. To prove the condition (4.9), by Lemma 4.6, it is sufficient to prove that $\#\mathbf{S}'' \geq n$. For each $y \in \mathbf{S} \setminus \mathbf{S}''$, there exist concepts $D_{y,1}, \ldots, D_{y,n}$ of $\mathcal{L}_{\Phi}^{sp}$ such that $y \in D_{y,i}^{\mathcal{I}}$ and $y'_i \notin D_{y,i}^{\mathcal{I}'}$ for every $1 \leq i \leq n$. For each $y \in \mathbf{S} \setminus \mathbf{S}''$, let $C_y = D_{y,1} \sqcap \ldots \sqcap D_{y,n}$, then we have that $y \in C_y^{\mathcal{I}}$, but $y'_i \notin C_y^{\mathcal{I}'}$ for all $1 \leq i \leq n$. Let $\Gamma = \{\neg C_y \mid y \in \mathbf{S} \setminus \mathbf{S}''\}$. Note that $\Gamma^{\mathcal{I}} \cap (\mathbf{S} \setminus \mathbf{S}'') = \emptyset$. Consider any finite subset Λ of Γ and let $C = {\geq n\, r.\bigsqcap \Lambda}$. Since $y'_1, \ldots, y'_n \in \Lambda^{\mathcal{I}}$, we have that $x' \in C^{\mathcal{I}'}$. Since $\neg C$ is equivalent to a concept of $\mathcal{L}_{\Phi}^{sp}$, $x \leq_{\Phi}^{sp} x'$ and $x' \notin (\neg C)^{\mathcal{I}'}$, we must have that $x \notin (\neg C)^{\mathcal{I}}$, which means $x \in C^{\mathcal{I}}$. Hence, there are at least n pairwise different $y_1, \ldots, y_n \in \mathbf{S}$ that belong to $\Lambda^{\mathcal{I}}$. Since $\mathcal{I}$ is modally saturated, it follows that there are at least n pairwise different $y_1, \ldots, y_n \in \mathbf{S}$ that belong to $\Gamma^{\mathcal{I}}$. Since $\Gamma^{\mathcal{I}} \cap (\mathbf{S} \setminus \mathbf{S}'') = \emptyset$, it follows that $\#\mathbf{S}'' \geq n$.

- The conditions (4.10) and (4.11) can be proved analogously as for the conditions (4.8) and (4.9).

- Consider the assertion (4.12) and the case $U \in \Phi$. If $\mathcal{I}$ is unreachable-objects-free then the assertion (4.12) follows from the assertions (4.1), (4.3) and (4.5).

  Consider the case when $\mathcal{I}$ is not unreachable-objects-free. Thus, $\mathcal{I}'$ is also not unreachable-objects-free. Since Z is not empty, there exists $\langle y, y' \rangle \in Z$. We have $y \leq_{\Phi}^{sp} y'$. Let $x \in \Delta^{\mathcal{I}}$. For the sake of contradiction, suppose there is no $x' \in \Delta^{\mathcal{I}'}$ such that $x \leq_{\Phi}^{sp} x'$. Thus, for every $x' \in \Delta^{\mathcal{I}'}$, there exists a concept $C_{x'}$ of $\mathcal{L}_{\Phi}^{sp}$ such that $x \in C_{x'}^{\mathcal{I}}$ but $x' \notin C_{x'}^{\mathcal{I}'}$. Let $\Gamma = \{C_{x'} \mid x' \in \Delta^{\mathcal{I}'}\}$. For any finite subset Λ of Γ, since $x \in \Lambda^{\mathcal{I}}$, we have that $y \in (\exists U.\bigsqcap \Lambda)^{\mathcal{I}}$, which implies that $y' \in (\exists U.\bigsqcap \Lambda)^{\mathcal{I}'}$ (since $y \leq_{\Phi}^{sp} y'$), which means Λ is satisfied in $\mathcal{I}'$. Since $\mathcal{I}'$ is modally saturated and not unreachable-objects-free, it follows that Γ is satisfied in $\mathcal{I}'$, which is a contradiction.

  The case when $\mathcal{I}$ is finite can be proved analogously as for the above case.

- Consider the assertion (4.13) and the case $U \in \Phi$. If $\mathcal{I}'$ is unreachable-objects-free then the assertion (4.13) follows from the assertions (4.1), (4.4) and (4.6).

  Consider the case when $\mathcal{I}'$ is not unreachable-objects-free. Thus, $\mathcal{I}$ is also not unreachable-objects-free. Since Z is not empty, there exists $\langle y, y' \rangle \in Z$. We have $y \leq_{\Phi}^{sp} y'$. Let $x' \in \Delta^{\mathcal{I}'}$. For the sake of contradiction, suppose there is no $x \in \Delta^{\mathcal{I}}$ such that $x \leq_{\Phi}^{sp} x'$. Thus, for every $x \in \Delta^{\mathcal{I}}$, there exists a concept C_x of $\mathcal{L}_{\Phi}^{sp}$ such that $x \in C_x^{\mathcal{I}}$ but $x' \notin C_x^{\mathcal{I}'}$. Let $\Gamma = \{\neg C_x \mid x \in \Delta^{\mathcal{I}}\}$. Consider any finite subset Λ of Γ and let $C = \exists U.\bigsqcap \Lambda$. Since $x' \in \Lambda^{\mathcal{I}'}$, we have that $y' \in C^{\mathcal{I}'}$, and hence $y' \notin (\neg C)^{\mathcal{I}'}$. Since $\neg C$ is equivalent to a concept of $\mathcal{L}_{\Phi}^{sp}$ and $y \leq_{\Phi}^{sp} y'$, it follows that $y \notin (\neg C)^{\mathcal{I}}$, and hence $y \in C^{\mathcal{I}}$. This means Λ is satisfied in $\mathcal{I}$. Since $\mathcal{I}$ is modally saturated and not unreachable-objects-free, it follows that Γ is satisfied in $\mathcal{I}$, which is a contradiction.

  The case when $\mathcal{I}'$ is finite can be proved analogously as for the above case.

- Consider the condition (4.14) and the case $\mathtt{Self} \in \Phi$. Suppose $Z(x, x')$ and $r^{\mathcal{I}}(x, x)$ hold. Since $(\exists r.\mathtt{Self})^{\mathcal{I}}(x)$ holds and $x \leq_{\Phi}^{sp} x'$, it follows that $(\exists r.\mathtt{Self})^{\mathcal{I}'}(x')$ holds. Hence, $r^{\mathcal{I}'}(x', x')$ holds. ■

Let us analyze where $\perp$ is really used in the proof of Theorem 4.7. Observe that the notion of being modally saturated remains the same if in its definition only non-empty finite subsets of Γ are considered. We can modify the proof of Theorem 4.7 by changing every phrase "finite subset Λ" to "non-empty finite subset Λ" if the set **S** in the proof of the assertion (4.4) is not empty. If $\mathcal{I}$ is a serial interpretation then that **S** is always non-empty. It can be seen that $\perp$ is only used to guarantee that the set **S** in the proof of the assertion (4.4) can be empty. Therefore, we also have the following theorem.

Theorem 4.8. *Let $\mathcal{I}$ and $\mathcal{I}'$ be modally saturated interpretations (w.r.t. $\mathcal{L}_{\Phi}$) such that $\mathcal{I}$ is serial and, for every $a \in \Sigma_I$, $a^{\mathcal{I}} \leq_{\Phi}^{pos} a^{\mathcal{I}'}$. Suppose that if $U \in \Phi$ then: either both $\mathcal{I}$ and $\mathcal{I}'$ are unreachable-objects-free, or both of them are not unreachable-objects-free, or both of them are finite. Then, for every $x \in \Delta^{\mathcal{I}}$ and $x' \in \Delta^{\mathcal{I}'}$, $x \leq_{\Phi}^{pos} x'$ iff $x \lesssim_{\Phi} x'$. In particular, the relation $\{\langle x, x' \rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'} \mid x \leq_{\Phi}^{pos} x'\}$ is an $\mathcal{L}_{\Phi}$-comparison between $\mathcal{I}$ and $\mathcal{I}'$ when it is not empty.*

4.4 Characterizing Bisimulation for Tidy Interpretations by Semi-Positive Concepts

Before introducing tidy interpretations, let us consider the following example.

Example 4.9. Let $\Phi = \emptyset$, $\Sigma_I = \{a\}$, $\Sigma_C = \{A, B\}$, $\Sigma_R = \{r\}$ and let $\mathcal{I}$, $\mathcal{I}'$ be the interpretations specified and illustrated as follows:

- $\Delta^{\mathcal{I}} = \{u, v_0, v_1, v_2\}$, $a^{\mathcal{I}} = u$, $r^{\mathcal{I}} = \{\langle u, v_0 \rangle, \langle u, v_1 \rangle, \langle u, v_2 \rangle\}$,
  $A^{\mathcal{I}} = \{v_1, v_2\}$, $B^{\mathcal{I}} = \{v_2\}$,
- $\Delta^{\mathcal{I}'} = \{u, v_0, v_2\}$, $a^{\mathcal{I}'} = u$, $r^{\mathcal{I}'} = \{\langle u, v_0 \rangle, \langle u, v_2 \rangle\}$ and $A^{\mathcal{I}'} = B^{\mathcal{I}'} = \{v_2\}$.

$$
(\mathcal{I}) \qquad
\begin{tikzcd}
 & u = a^{\mathcal{I}} \arrow[dl] \arrow[d] \arrow[dr] & \\
v_0 & v_1 : A & v_2 : A, B
\end{tikzcd}
\qquad\qquad
(\mathcal{I}') \qquad
\begin{tikzcd}
 & u = a^{\mathcal{I}'} \arrow[dl] \arrow[dr] & \\
v_0 & & v_2 : A, B
\end{tikzcd}
$$

Notice that $\mathcal{I}'$ is obtained from $\mathcal{I}$ by deleting v_1. Observe that there are $\mathcal{L}_\Phi$-comparisons between $\mathcal{I}$ and $\mathcal{I}'$ as well as between $\mathcal{I}'$ and $\mathcal{I}$, but there is no $\mathcal{L}_\Phi$-bisimulations between $\mathcal{I}$ and $\mathcal{I}'$. In particular, $a^{\mathcal{I}} \equiv_\Phi^{sp} a^{\mathcal{I}'}$, but $a^{\mathcal{I}} \not\equiv_\Phi a^{\mathcal{I}'}$. Also observe that, for $C = \forall r.((A \sqcap B) \sqcup (\neg A \sqcap \neg B))$, we have $\mathcal{I}' \models C(a)$, but $\mathcal{I} \not\models C(a)$. This implies that the concept C cannot be expressed in $\mathcal{L}_\Phi^{sp}$. ∎

The point of the above example is that, when $Q \notin \Phi$, if v_0, v_1, v_2 are pairwise different r-successors of u, $v_0 \leq_\Phi^{sp} v_1$ and $v_1 \leq_\Phi^{sp} v_2$ then the edge $\langle u, v_1 \rangle \in r^{\mathcal{I}}$ is not essential for the semantics of semi-positive concepts at u. Also note that, when $Q \notin \Phi$, if v and v' are different r-successors of u such that $v \equiv_\Phi^{sp} v'$ then the edge $\langle u, v' \rangle \in r^{\mathcal{I}}$ is not essential for the semantics of semi-positive concepts at u.

Definition 4.10 ($\mathcal{L}_\Phi^{sp}$-Tidiness – for the Case $Q \notin \Phi$).
Suppose $Q \notin \Phi$. We say that an interpretation $\mathcal{I}$ is $\mathcal{L}_\Phi^{sp}$*-tidy* if it is unreachable-objects-free when $U \in \Phi$, and for every $x, y, y', y'' \in \Delta^{\mathcal{I}}$ and every basic role R of $\mathcal{L}_\Phi$,

- if $\{\langle x, y \rangle, \langle x, y' \rangle\} \subseteq R^{\mathcal{I}}$ and $y \equiv_\Phi^{sp} y'$ then $y = y'$,
- if $\{\langle x, y \rangle, \langle x, y' \rangle, \langle x, y'' \rangle\} \subseteq R^{\mathcal{I}}$, $y \leq_\Phi^{sp} y'$ and $y' \leq_\Phi^{sp} y''$ then $y = y'$ or $y' = y''$ or ($\texttt{Self} \in \Phi$ and $y' = x$). ∎

Example 4.11. Reconsider Example 4.9. Observe that $\mathcal{I}'$ is $\mathcal{L}_\Phi^{sp}$-tidy, but $\mathcal{I}$ is not. ∎

Theorem 4.9. *Suppose $Q \notin \Phi$. Let $\mathcal{I}$ and $\mathcal{I}'$ be modally saturated and $\mathcal{L}_\Phi^{sp}$-tidy interpretations such that, for every $a \in \Sigma_I$, $a^{\mathcal{I}} \equiv_\Phi^{sp} a^{\mathcal{I}'}$. Then, for every $x \in \Delta^{\mathcal{I}}$ and $x' \in \Delta^{\mathcal{I}'}$, $x \equiv_\Phi^{sp} x'$ iff $x \sim_\Phi x'$. In particular, the relation $\{\langle x, x' \rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'} \mid x \equiv_\Phi^{sp} x'\}$ is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ when it is not empty.*

Proof. If Z is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z(x, x')$ holds then, by Theorem 3.12, $x \equiv_\Phi x'$, and hence $x \equiv_\Phi^{sp} x'$. For the remaining assertions of the current theorem, let $Z = \{\langle x, x' \rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'} \mid x \equiv_\Phi^{sp} x'\}$ and assume that it is not empty. We show that it is an $\mathcal{L}_\Phi$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.

- The condition (3.1) immediately follows from the assumption of the theorem.
- Consider the condition (3.2). Suppose $Z(x, x')$ holds. By the definition of Z, $A^{\mathcal{I}}(x)$ holds iff $A^{\mathcal{I}'}(x')$ holds.
- Consider the condition (3.3). Suppose $Z(x, x') \wedge r^{\mathcal{I}}(x, y)$ holds. We show that there exists y' such that $Z(y, y') \wedge r^{\mathcal{I}'}(x', y')$ holds. For the case when $\texttt{Self} \in \Phi$ and $y = x$ we can just take $y' = x'$. So, suppose $\texttt{Self} \notin \Phi$ or $y \neq x$. Analogously to the proof of Theorem 4.7, it can be shown that there exists $y'_2 \in \Delta^{\mathcal{I}'}$ such that

$r^{\mathcal{I}'}(x', y'_2)$ holds and $y \leq^{sp}_{\Phi} y'_2$. Dually, there exists $y'_1 \in \Delta^{\mathcal{I}'}$ such that $r^{\mathcal{I}'}(x', y'_1)$ holds and $y'_1 \leq^{sp}_{\Phi} y$. Similarly, there exist $y_1, y_2 \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y_1)$ and $r^{\mathcal{I}}(x, y_2)$ hold, $y_1 \leq^{sp}_{\Phi} y'_1$ and $y'_2 \leq^{sp}_{\Phi} y_2$. Hence $y_1 \leq^{sp}_{\Phi} y \leq^{sp}_{\Phi} y_2$. Since $\mathcal{I}$ is $\mathcal{L}^{sp}_{\Phi}$-tidy, either $y = y_1$ or $y = y_2$. Since $y_1 \leq^{sp}_{\Phi} y'_1 \leq^{sp}_{\Phi} y$ and $y \leq^{sp}_{\Phi} y'_2 \leq^{sp}_{\Phi} y_2$, it follows that $y \equiv^{sp}_{\Phi} y'_1$ or $y \equiv^{sp}_{\Phi} y'_2$. If $y \equiv^{sp}_{\Phi} y'_1$ then choose $y' = y'_1$, else choose $y' = y'_2$. Thus, $Z(y, y') \wedge r^{\mathcal{I}'}(x', y')$ holds.

- The condition (3.4) as well as the conditions (3.5) and (3.6) for the case $I \in \Phi$ can be proved analogously.

- Consider the condition (3.7) and the case $O \in \Phi$. Suppose $Z(x, x')$ holds. Thus, $\{a\}^{\mathcal{I}}(x)$ holds iff $\{a\}^{\mathcal{I}'}(x')$ holds. That is, $x = a^{\mathcal{I}}$ iff $x' = a^{\mathcal{I}'}$.

- Consider the conditions (3.12) and (3.13) and the case $U \in \Phi$. By assumption, both $\mathcal{I}$ and $\mathcal{I}'$ are unreachable-objects-free. The condition (3.12) follows from the conditions (3.1), (3.3) and (3.4). Analogously, the condition (3.13) also holds.

- Consider the condition (3.14) and the case $\mathtt{Self} \in \Phi$. Suppose $Z(x, x')$ holds. Thus, $(\exists r.\mathtt{Self})^{\mathcal{I}}(x)$ holds iff $(\exists r.\mathtt{Self})^{\mathcal{I}'}(x')$ holds. That is, $r^{\mathcal{I}}(x, x)$ holds iff $r^{\mathcal{I}'}(x', x')$ holds. ■

Definition 4.12 ($\mathcal{L}^{sp}_{\Phi}$-Tidiness – for the Case $Q \in \Phi$).
Suppose $Q \in \Phi$. We say that an interpretation $\mathcal{I}$ is $\mathcal{L}^{sp}_{\Phi}$*-tidy* if it is unreachable-objects-free when $U \in \Phi$, and for every $x \in \Delta^{\mathcal{I}}$, every basic role R of $\mathcal{L}_{\Phi}$ and every R-successor y of x, the set $\{y' \in \Delta^{\mathcal{I}} \mid \langle x, y' \rangle \in R^{\mathcal{I}} \wedge y \leq^{sp}_{\Phi} y'\}$ is finite. ■

Clearly, if $Q \in \Phi$ and $U \notin \Phi$ then every finitely branching interpretation is $\mathcal{L}^{sp}_{\Phi}$-tidy.

Theorem 4.10. *Suppose $Q \in \Phi$. Let $\mathcal{I}$ and $\mathcal{I}'$ be modally saturated and $\mathcal{L}^{sp}_{\Phi}$-tidy interpretations such that, for every $a \in \Sigma_I$, $a^{\mathcal{I}} \equiv^{sp}_{\Phi} a^{\mathcal{I}'}$. Then, for every $x \in \Delta^{\mathcal{I}}$ and $x' \in \Delta^{\mathcal{I}'}$, $x \equiv^{sp}_{\Phi} x'$ iff $x \sim_{\Phi} x'$. In particular, the relation $\{\langle x, x' \rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'} \mid x \equiv^{sp}_{\Phi} x'\}$ is an $\mathcal{L}_{\Phi}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ when it is not empty.*

Proof. Let $Z = \{\langle x, x' \rangle \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'} \mid x \equiv^{sp}_{\Phi} x'\}$ and assume that it is not empty. Analyzing the proof of Theorem 4.9, it can be seen that we only need to prove the conditions (3.3)-(3.6) and (3.8)-(3.11).

Suppose $Z(x, x')$ holds. Let R be a basic role of $\mathcal{L}^{sp}_{\Phi}$ and y_0 be an arbitrary R-successor of x. Let $\mathbf{S} = \{y \in \Delta^{\mathcal{I}} \mid \langle x, y \rangle \in R^{\mathcal{I}} \wedge y_0 \leq^{sp}_{\Phi} y\}$ and $\mathbf{S}' = \{y' \in \Delta^{\mathcal{I}'} \mid \langle x', y' \rangle \in R^{\mathcal{I}'} \wedge y_0 \leq^{sp}_{\Phi} y'\}$. Since $\mathcal{I}$ is $\mathcal{L}^{sp}_{\Phi}$-tidy, $\mathbf{S}$ is finite. By Theorem 4.7, $\leq^{sp}_{\Phi}$ is an $\mathcal{L}_{\Phi}$-comparison between $\mathcal{I}$ and $\mathcal{I}'$ as well as between $\mathcal{I}'$ and $\mathcal{I}$. Since $x \leq^{sp}_{\Phi} x'$ and $\mathbf{S}$ is finite, by (4.8) and (4.10) for $\leq^{sp}_{\Phi}$ (i.e., with Z replaced by $\leq^{sp}_{\Phi}$), there exists an injection $g : \mathbf{S} \to \mathbf{S}'$ such that $y \leq^{sp}_{\Phi} g(y)$ for every $y \in \mathbf{S}$. Since $x' \leq^{sp}_{\Phi} x$ and $\mathbf{S}'$ is finite, by (4.8) and (4.10) for $\leq^{sp}_{\Phi}$ (i.e., with Z replaced by $\leq^{sp}_{\Phi}$), there exists an injection $g' : \mathbf{S}' \to \mathbf{S}$ such that $y' \leq^{sp}_{\Phi} g'(y')$ for every $y' \in \mathbf{S}'$. Therefore, there exists a bijection $f : \mathbf{S} \to \mathbf{S}'$ such that $y \equiv^{sp}_{\Phi} f(y)$ for every $y \in \mathbf{S}$. Observe that this property implies the condition (3.8). The condition (3.9) as well as the conditions (3.10) and (3.11) can be proved analogously. The conditions (3.3)-(3.6) follow from the conditions (3.8)-(3.11), respectively. ■

Corollary 4.11. *Let $\mathcal{I}$ and $\mathcal{I}'$ be finitely branching interpretations (w.r.t. $\mathcal{L}_\Phi$) such that, for every $a \in \Sigma_I$, $a^\mathcal{I} \equiv^{sp}_\Phi a^{\mathcal{I}'}$. Suppose $Q \in \Phi$ and if $U \in \Phi$ then both $\mathcal{I}$ and $\mathcal{I}'$ are unreachable-objects-free. Then, for every $x \in \Delta^\mathcal{I}$ and $x' \in \Delta^{\mathcal{I}'}$, $x \equiv^{sp}_\Phi x'$ iff $x \equiv_\Phi x'$.*

This corollary follows from Theorems 4.10, 3.12 and the definition of $\mathcal{L}^{sp}_\Phi$-tidiness. As a consequence of this corollary, if $Q \in \Phi$ then $\mathcal{L}_\Phi$-bisimulation between interpretations that are finitely branching and unreachable-objects-free can be characterized by semi-positive concepts.

4.5 Minimization That Preserves Semi-Positive Concepts

In this section, we study the problem of minimizing an interpretation $\mathcal{I}$ w.r.t. the cardinality of the domain so that the resulting interpretation satisfies the same set of concept assertions in $\mathcal{L}^{sp}_\Phi$ as $\mathcal{I}$. When possible, we also consider infinite interpretations, however, the main objective is to develop methods that work for any finite interpretation and cover all the cases of Φ.

Definition 4.13 ($\mathcal{L}_\Phi$-Auto-Comparison).
An $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and itself is called an $\mathcal{L}_\Phi$-*auto-comparison of* $\mathcal{I}$. An $\mathcal{L}_\Phi$-auto-comparison of $\mathcal{I}$ is said to be the *largest* if it is larger than or equal to ($\supseteq$) any other $\mathcal{L}_\Phi$-auto-comparison of $\mathcal{I}$. ■

Proposition 4.12. *The largest $\mathcal{L}_\Phi$-auto-comparison of an interpretation $\mathcal{I}$ always exists and is a preorder.*

This proposition follows from Lemma 4.1. By Proposition 4.2, the largest $\mathcal{L}_\Phi$-auto-comparison of an interpretation $\mathcal{I}$ can be computed by applying Algorithm 4 to the interpretations $\mathcal{I}$ and $\mathcal{I}' = \mathcal{I}$.

Definition 4.14. Given an interpretation $\mathcal{I}$, by $\lesssim_{\Phi,\mathcal{I}}$ we denote the largest $\mathcal{L}_\Phi$-auto-comparison of $\mathcal{I}$. We define $\simeq_{\Phi,\mathcal{I}}$ to be $\lesssim_{\Phi,\mathcal{I}} \cap (\lesssim_{\Phi,\mathcal{I}})^{-1}$. ■

Definition 4.15. Let $\mathcal{I}$ be an interpretation or a QS-interpretation. By $\leq^{sp}_{\Phi,\mathcal{I}}$ we denote the binary relation on $\Delta^\mathcal{I}$ such that $x \leq^{sp}_{\Phi,\mathcal{I}} y$ iff, for every concept C of $\mathcal{L}^{sp}_\Phi$, if $x \in C^\mathcal{I}$ then $y \in C^\mathcal{I}$. By $\equiv^{sp}_{\Phi,\mathcal{I}}$ we denote the binary relation on $\Delta^\mathcal{I}$ such that $x \equiv^{sp}_{\Phi,\mathcal{I}} y$ iff, for every concept C of $\mathcal{L}^{sp}_\Phi$, $x \in C^\mathcal{I}$ iff $y \in C^\mathcal{I}$. ■

Proposition 4.13. *For every modally saturated interpretation $\mathcal{I}$, the relations $\leq^{sp}_{\Phi,\mathcal{I}}$ and $\lesssim_{\Phi,\mathcal{I}}$ coincide (i.e. $\leq^{sp}_{\Phi,\mathcal{I}}$ is the largest $\mathcal{L}_\Phi$-auto-comparison of $\mathcal{I}$) and, as a consequence, the relation $\simeq_{\Phi,\mathcal{I}}$ coincides with the equivalence relation $\equiv^{sp}_{\Phi,\mathcal{I}}$.*

This proposition follows from Theorem 4.7.

Lemma 4.14. *Let $\mathcal{I}$ be a finitely branching interpretation such that it is finite when $U \in \Phi$, and let $\mathcal{I}' = \mathcal{I}/_{\simeq_{\Phi,\mathcal{I}}}$ when $\{Q, \mathtt{Self}\} \cap \Phi = \emptyset$, and $\mathcal{I}' = \mathcal{I}/^{QS}_{\simeq_{\Phi,\mathcal{I}}}$ otherwise. Then:*

1. *for every $x \in \Delta^\mathcal{I}$ and every concept C of $\mathcal{L}^{sp}_\Phi$, $x \in C^\mathcal{I}$ iff $[x]_{\simeq_{\Phi,\mathcal{I}}} \in C^{\mathcal{I}'}$; consequently, $\mathcal{I}'$ satisfies the same set of concept assertions in $\mathcal{L}^{sp}_\Phi$ as $\mathcal{I}$;*

2. *for every* $x'_1, x'_2 \in \Delta^{\mathcal{I}'}$*, if* $x'_1 \neq x'_2$ *then* $x'_1 \not\equiv^{sp}_{\Phi,\mathcal{I}'} x'_2$*;*

3. *for every* $x_1, x_2 \in \Delta^{\mathcal{I}}$*,* $x_1 \lesssim_{\Phi,\mathcal{I}} x_2$ *iff* $[x_1]_{\simeq_{\Phi,\mathcal{I}}} \leq^{sp}_{\Phi,\mathcal{I}'} [x_2]_{\simeq_{\Phi,\mathcal{I}}}$*.*

Proof. Consider the first assertion for the case $\{Q, \texttt{Self}\} \cap \Phi = \emptyset$. It is straightforward to check that $Z = \{\langle x, [x']_{\simeq_{\Phi,\mathcal{I}}}\rangle \mid x \lesssim_{\Phi,\mathcal{I}} x'\}$ satisfies the conditions (4.1)-(4.7), (4.12), (4.13) and is thus an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}$ and $\mathcal{I}'$. Analogously, $Z' = \{\langle [x']_{\simeq_{\Phi,\mathcal{I}}}, x\rangle \mid x' \lesssim_{\Phi,\mathcal{I}} x\}$ is an $\mathcal{L}_\Phi$-comparison between $\mathcal{I}'$ and $\mathcal{I}$. By Theorem 4.4, it follows that, for every $x \in \Delta^{\mathcal{I}}$ and every concept C of $\mathcal{L}^{sp}_\Phi$, $x \in C^{\mathcal{I}}$ iff $[x]_{\simeq_{\Phi,\mathcal{I}}} \in C^{\mathcal{I}'}$.

Consider the first assertion for the case $\{Q, \texttt{Self}\} \cap \Phi \neq \emptyset$. It is straightforward to check that $Z = \{\langle x, [x']_{\simeq_{\Phi,\mathcal{I}}}\rangle \mid x \lesssim_{\Phi,\mathcal{I}} x'\}$ satisfies the properties (4.1)-(4.7), (4.12), (4.13). Additionally, the properties (4.15)-(4.17) can be proved analogously as in Lemma 4.3 except that the cases when ($Q \in \Phi$ and $C = \geq n\, R.D$) or ($Q \in \Phi$ and $C = \leq n\, R.\neg D$) or ($\texttt{Self} \in \Phi$ and $C = \exists r.\texttt{Self}$) in the proof of the assertion (4.15) are changed to the following:

- Case $Q \in \Phi$ and $C = (\geq n\, R.D)$, where R is a basic role and D is a concept of $\mathcal{L}^{sp}_\Phi$: Since $C^{\mathcal{I}}(x)$ holds, there exist pairwise different $y_1, \ldots, y_n \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y_i)$ and $D^{\mathcal{I}}(y_i)$ hold for all $1 \leq i \leq n$. Since $x \lesssim_{\Phi,\mathcal{I}} x'$, by the conditions (4.8) and (4.10), there exist pairwise different $y'_1, \ldots, y'_n \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x', y'_i)$ and $y_i \lesssim_{\Phi,\mathcal{I}} y'_i$ for all $1 \leq i \leq n$. Let the partition of $\{y'_1, \ldots, y'_n\}$ that corresponds to the equivalence relation $\simeq_{\Phi,\mathcal{I}}$ consist of pairwise different blocks $Y_{i_1}, \ldots, Y_{i_k}$, where $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ and $y'_{i_j} \in Y_{i_j}$ for all $1 \leq j \leq k$. For every $1 \leq j \leq k$, since $D^{\mathcal{I}}(y_{i_j})$ and $y_{i_j} \lesssim_{\Phi,\mathcal{I}} y'_{i_j}$ hold, by the inductive assumption of (4.15), $D^{\mathcal{I}'}([y'_{i_j}]_{\simeq_{\Phi,\mathcal{I}}})$ holds. By the definition of $\mathcal{I}'$, $\mathtt{Q}^{\mathcal{I}'}(R)([x']_{\simeq_{\Phi,\mathcal{I}}}, [y'_{i_j}]_{\simeq_{\Phi,\mathcal{I}}}) \geq \#Y_{i_j}$ for all $1 \leq j \leq k$. Hence $C^{\mathcal{I}'}([x']_{\simeq_{\Phi,\mathcal{I}}})$ holds.

- Case $Q \in \Phi$ and $C = (\leq n\, R.(\neg D))$, where R is a basic role and D is a concept of $\mathcal{L}^{sp}_\Phi$: For the sake of contradiction, suppose $[x']_{\simeq_{\Phi,\mathcal{I}}} \notin C^{\mathcal{I}'}$. Thus, $[x']_{\simeq_{\Phi,\mathcal{I}}} \in (\neg C)^{\mathcal{I}'}$, which means $[x']_{\simeq_{\Phi,\mathcal{I}}} \in (\geq (n+1)\, R.(\neg D))^{\mathcal{I}'}$. Hence, there exist pairwise different $[y'_1]_{\simeq_{\Phi,\mathcal{I}}}, \ldots, [y'_k]_{\simeq_{\Phi,\mathcal{I}}} \in (\neg D)^{\mathcal{I}'}$ such that $\Sigma_{1 \leq i \leq k} \mathtt{Q}^{\mathcal{I}'}(R)([x']_{\simeq_{\Phi,\mathcal{I}}}, [y'_i]_{\simeq_{\Phi,\mathcal{I}}}) \geq n+1$. For each $1 \leq i \leq k$, let $y'_{i,1}, \ldots, y'_{i,j_i}$ be all pairwise different elements of $[y'_i]_{\simeq_{\Phi,\mathcal{I}}}$ such that $R^{\mathcal{I}'}(x', y'_{i,j})$ holds for all $1 \leq j \leq j_i$. We have that $j_1 + \ldots + j_k \geq n+1$. For $1 \leq i \leq k$ and $1 \leq j \leq j_i$, since $[y'_i]_{\simeq_{\Phi,\mathcal{I}}} \notin D^{\mathcal{I}'}$ and $y'_{i,j} \lesssim_{\Phi,\mathcal{I}} y'_i$, by the contrapositive of the inductive assumption of (4.15), $y'_{i,j} \notin D^{\mathcal{I}'}$. Therefore, $x' \in (\geq (n+1)\, R.(\neg D))^{\mathcal{I}}$, which means $x' \notin C^{\mathcal{I}}$. Since $x \lesssim_{\Phi,\mathcal{I}} x'$, it follows that $x \notin C^{\mathcal{I}}$, which contradicts the assumption that $C^{\mathcal{I}}(x)$ holds.

- Case $\texttt{Self} \in \Phi$ and $C = \exists r.\texttt{Self}$: Since $(\exists r.\texttt{Self})^{\mathcal{I}}(x)$ holds, we have that $r^{\mathcal{I}}(x, x)$ holds. Since $x \lesssim_{\Phi,\mathcal{I}} x'$, by (4.14), it follows that $r^{\mathcal{I}'}(x', x')$ holds. Hence $C^{\mathcal{I}'}(x')$ holds.

Dually, it can be proved that the properties (4.1)-(4.7), (4.12), (4.13) and (4.15)-(4.17) are satisfied for $Z' = \{\langle [x]_{\simeq_{\Phi,\mathcal{I}}}, x'\rangle \mid x \lesssim_{\Phi,\mathcal{I}} x'\}$ in the place of Z. By the property (4.15) for Z and Z', for every $x \in \Delta^{\mathcal{I}}$ and every concept C of $\mathcal{L}_\Phi^{sp}$, $x \in C^{\mathcal{I}}$ iff $[x]_{\simeq_{\Phi,\mathcal{I}}} \in C^{\mathcal{I}'}$.

The second assertion follows from Proposition 4.13 and the definition of $\mathcal{I}/_{\simeq_{\Phi,\mathcal{I}}}$.

Consider the third assertion. By Proposition 4.13, $x_1 \lesssim_{\Phi,\mathcal{I}} x_2$ iff $x_1 \leq_{\Phi,\mathcal{I}}^{sp} x_2$. By the first assertion of this lemma, $x_1 \leq_{\Phi,\mathcal{I}}^{sp} x_2$ iff $[x_1]_{\simeq_{\Phi,\mathcal{I}}} \leq_{\Phi,\mathcal{I}'}^{sp} [x_2]_{\simeq_{\Phi,\mathcal{I}}}$. Hence, $x_1 \lesssim_{\Phi,\mathcal{I}} x_2$ iff $[x_1]_{\simeq_{\Phi,\mathcal{I}}} \leq_{\Phi,\mathcal{I}'}^{sp} [x_2]_{\simeq_{\Phi,\mathcal{I}}}$. ∎

Definition 4.16 (**$\mathcal{L}_\Phi^{sp}$-Extremal Individual**).
Let $\mathcal{I}$ be an interpretation or a QS-interpretation. We say that $x \in \Delta^{\mathcal{I}}$ is an $\mathcal{L}_\Phi^{sp}$-*maximal individual* if, for every $x' \in \Delta^{\mathcal{I}}$, if $x \leq_{\Phi,\mathcal{I}}^{sp} x'$ then $x \equiv_{\Phi,\mathcal{I}}^{sp} x'$. The notion of $\mathcal{L}_\Phi^{sp}$*-minimal individual* is defined dually. An $\mathcal{L}_\Phi^{sp}$*-extremal individual* is either an $\mathcal{L}_\Phi^{sp}$-maximal individual or an $\mathcal{L}_\Phi^{sp}$-minimal individual. ∎

Definition 4.17 (**$\mathcal{L}_\Phi^{sp}$-Extremal R-Successor**).
Let $\mathcal{I}$ be an interpretation or a QS-interpretation, R be a basic role of $\mathcal{L}_\Phi$ and let $x \in \Delta^{\mathcal{I}}$. We say that $y \in \Delta^{\mathcal{I}}$ is an $\mathcal{L}_\Phi^{sp}$*-maximal R-successor* of x if y is an R-successor of x and, for every R-successor y' of x, if $y \leq_{\Phi,\mathcal{I}}^{sp} y'$ then $y \equiv_{\Phi,\mathcal{I}}^{sp} y'$. The notion of $\mathcal{L}_\Phi^{sp}$*-minimal R-successor* is defined dually. An $\mathcal{L}_\Phi^{sp}$*-extremal* R-successor of x is either an $\mathcal{L}_\Phi^{sp}$-maximal R-successor or an $\mathcal{L}_\Phi^{sp}$-minimal R-successor of x. ∎

Definition 4.18 (**$\mathcal{L}_\Phi^{sp}$-Essential Individual**).
Let $\mathcal{I}$ be an interpretation or a QS-interpretation. The set of $\mathcal{L}_\Phi^{sp}$*-essential individuals* of $\mathcal{I}$ is defined to be the smallest subset of $\Delta^{\mathcal{I}}$ such that:

1. for every $a \in \Sigma_I$, $a^{\mathcal{I}}$ is $\mathcal{L}_\Phi^{sp}$-essential,
2. if $U \in \Phi$ and $x \in \Delta^{\mathcal{I}}$ is an $\mathcal{L}_\Phi^{sp}$-extremal individual then x is $\mathcal{L}_\Phi^{sp}$-essential.
3. if $Q \notin \Phi$, $x \in \Delta^{\mathcal{I}}$ is $\mathcal{L}_\Phi^{sp}$-essential, R is a basic role of $\mathcal{L}_\Phi$ and y is an $\mathcal{L}_\Phi^{sp}$-extremal R-successor of x, then y is $\mathcal{L}_\Phi^{sp}$-essential,
4. if $Q \in \Phi$, $x \in \Delta^{\mathcal{I}}$ is $\mathcal{L}_\Phi^{sp}$-essential and R is a basic role of $\mathcal{L}_\Phi$, then every R-successor of x is $\mathcal{L}_\Phi^{sp}$-essential, ∎

If $\mathcal{I}$ is a finitely branching interpretation/QS-interpretation then a property can be proved for $\mathcal{L}_\Phi^{sp}$-essential individuals of $\mathcal{I}$ by induction on the timestamp at which an individual is marked as $\mathcal{L}_\Phi^{sp}$-essential, assuming that individuals are marked as $\mathcal{L}_\Phi^{sp}$-essential by the items 1 and 2 of Definition 4.18 at the timestamp 0.

Theorem 4.15. *Suppose $\Sigma_I \neq \emptyset$, $\{Q, \mathtt{Self}\} \cap \Phi = \emptyset$ and $\mathcal{I}$ is a finitely branching interpretation such that it is finite when $U \in \Phi$. Then the interpretation $\mathcal{I}''$ obtained from $\mathcal{I}' = \mathcal{I}/_{\simeq_{\Phi,\mathcal{I}}}$ by deleting from the domain all non-$\mathcal{L}_\Phi^{sp}$-essential individuals and modifying the interpretation function accordingly is a minimal interpretation that satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$.*

Proof. By Lemma 4.14, $\mathcal{I}'$ satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$. It is easy to check that the restriction to $\Delta^{\mathcal{I}'} \times \Delta^{\mathcal{I}''}$ of the largest $\mathcal{L}_\Phi$-auto-comparison of $\mathcal{I}'$ is an $\mathcal{L}_\Phi^{sp}$-comparison between $\mathcal{I}'$ and $\mathcal{I}''$. Similarly, the restriction to $\Delta^{\mathcal{I}''} \times \Delta^{\mathcal{I}'}$

of the largest $\mathcal{L}_\Phi$-auto-comparison of $\mathcal{I}'$ is an $\mathcal{L}_\Phi^{sp}$-comparison between $\mathcal{I}''$ and $\mathcal{I}'$. By Theorem 4.4, $\mathcal{I}''$ satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}'$ and hence also as $\mathcal{I}$. Also observe that, for every $x \in \Delta^{\mathcal{I}''}$ and every concept C of $\mathcal{L}_\Phi^{sp}$, $x \in C^{\mathcal{I}'}$ iff $x \in C^{\mathcal{I}''}$. Due to this reason, when considering only concepts of $\mathcal{L}_\Phi^{sp}$, it does not matter whether an element $x \in \Delta^{\mathcal{I}''}$ is considered in the context $\mathcal{I}''$ or $\mathcal{I}'$.

Let $\mathcal{I}_2$ be any interpretation that satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$ and $\mathcal{I}''$. We need to show that $\#\Delta^{\mathcal{I}''} \leq \#\Delta^{\mathcal{I}_2}$. Observe that the domain of $\mathcal{I}''$ is countable. Thus, we assume that $\mathcal{I}_2$ is finitely branching. If $U \in \Phi$ then, by assumption, $\mathcal{I}$ is finite, and hence $\mathcal{I}''$ is also finite. So, for the case $U \in \Phi$ we assume that $\mathcal{I}_2$ is finite. By Theorem 4.7, there exist $\mathcal{L}_\Phi^{sp}$-comparisons between $\mathcal{I}''$ and $\mathcal{I}_2$ and between $\mathcal{I}_2$ and $\mathcal{I}''$.

By Lemma 4.14, for every $x_1, x_2 \in \Delta^{\mathcal{I}''}$, if $x_1 \neq x_2$ then $x_1 \not\equiv_\Phi^{sp} x_2$. Hence, it is sufficient to prove that, for every $x \in \Delta^{\mathcal{I}''}$, there exists $x' \in \Delta^{\mathcal{I}_2}$ such that $x \equiv_\Phi^{sp} x'$. We prove this by induction on the timestamp at which an element $x \in \Delta^{\mathcal{I}''}$ is marked as an $\mathcal{L}_\Phi^{sp}$-essential individual of $\mathcal{I}'$. There are the following base cases:

- Case $x = a^{\mathcal{I}'}$ for some $a \in \Sigma_I$: Take $x' = a^{\mathcal{I}_2}$. Since $\mathcal{I}''$ and $\mathcal{I}_2$ satisfy the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$, we have that $x \equiv_\Phi^{sp} x'$.
- Case $U \in \Phi$ and x is an $\mathcal{L}_\Phi^{sp}$-maximal individual of $\mathcal{I}'$: Because there are $\mathcal{L}_\Phi^{sp}$-comparisons between $\mathcal{I}''$ and $\mathcal{I}_2$ and between $\mathcal{I}_2$ and $\mathcal{I}''$, by (4.12) and Theorem 4.4, there exist $x' \in \Delta^{\mathcal{I}_2}$ and $x'' \in \Delta^{\mathcal{I}''}$ such that $x \leq_\Phi^{sp} x' \leq_\Phi^{sp} x''$. As x is an $\mathcal{L}_\Phi^{sp}$-maximal individual of $\mathcal{I}'$, it follows that $x \equiv_\Phi^{sp} x' \equiv_\Phi^{sp} x''$.
- The case when $U \in \Phi$ and x is an $\mathcal{L}_\Phi^{sp}$-minimal individual of $\mathcal{I}'$ is similar to the above case.

For the induction step, assume that x is an $\mathcal{L}_\Phi^{sp}$-extremal R-successor of x_0, where R is a basic role of $\mathcal{L}_\Phi$ and x_0 was marked as $\mathcal{L}_\Phi^{sp}$-essential earlier than x. By the inductive assumption, there exists $x'_0 \in \Delta^{\mathcal{I}_2}$ such that $x_0 \equiv_\Phi^{sp} x'_0$. Analogously to the base case with $U \in \Phi$, it can be seen that there exists an R-successor x' of x'_0 such that $x \equiv_\Phi^{sp} x'$. ∎

Theorem 4.16. *Suppose $\Sigma_I \neq \emptyset$, $\{Q, \mathtt{Self}\} \cap \Phi \neq \emptyset$ and $\mathcal{I}$ is a finitely branching interpretation such that it is finite when $U \in \Phi$. Then the QS-interpretation $\mathcal{I}''$ obtained from $\mathcal{I}' = \mathcal{I}/_{\simeq_{\Phi,\mathcal{I}}}^{QS}$ by deleting from the domain all non-$\mathcal{L}_\Phi^{sp}$-essential individuals and modifying the interpretation function accordingly satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$ and the cardinality of its domain is not bigger than the cardinality of the domain of any interpretation that satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$.*

Proof. By Lemma 4.14, $\mathcal{I}'$ satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$. By induction on the structure of C, it can be proved that, if C is a concept of $\mathcal{L}_\Phi^{sp}$ then, for every $x \in \Delta^{\mathcal{I}''}$, $x \in C^{\mathcal{I}''}$ iff $x \in C^{\mathcal{I}'}$. Due to this reason, when considering only concepts of $\mathcal{L}_\Phi^{sp}$, it does not matter whether an element $x \in \Delta^{\mathcal{I}''}$ is considered in the context $\mathcal{I}''$ or $\mathcal{I}'$. Also note that, as a consequence, $\mathcal{I}''$ satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}'$ and hence also as $\mathcal{I}$.

Let $\mathcal{I}_2$ be any interpretation that satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$. We need to show that $\#\Delta^{\mathcal{I}''} \leq \#\Delta^{\mathcal{I}_2}$. Observe that the domain of $\mathcal{I}''$ is countable. Thus, we assume that $\mathcal{I}_2$ is finitely branching. If $U \in \Phi$ then, by assumption, $\mathcal{I}$ is finite and hence $\mathcal{I}''$ is also finite. So, for the case $U \in \Phi$ we assume that $\mathcal{I}_2$ is finite. By Theorem 4.7, there exist $\mathcal{L}_\Phi^{sp}$-comparisons between $\mathcal{I}$ and $\mathcal{I}_2$ and between $\mathcal{I}_2$ and $\mathcal{I}$.

By Lemma 4.14, for every $x_1, x_2 \in \Delta^{\mathcal{I}''}$, if $x_1 \neq x_2$ then $x_1 \not\equiv_{\Phi,\mathcal{I}''}^{sp} x_2$. Hence, it is sufficient to prove that, for every $x \in \Delta^{\mathcal{I}''}$, there exists $x' \in \Delta^{\mathcal{I}_2}$ such that $x \equiv_\Phi^{sp} x'$. We prove this by induction on the timestamp at which an element $x \in \Delta^{\mathcal{I}''}$ is marked as an $\mathcal{L}_\Phi^{sp}$-essential individual of $\mathcal{I}'$. There are the following base cases:

- Case $x = a^{\mathcal{I}'}$ for some $a \in \Sigma_I$: Take $x' = a^{\mathcal{I}_2}$. Since $\mathcal{I}''$ and $\mathcal{I}_2$ satisfy the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$, we have that $x \equiv_\Phi^{sp} x'$.

- Case $U \in \Phi$ and x is an $\mathcal{L}_\Phi^{sp}$-maximal individual of $\mathcal{I}'$: Let x_0 be any element of $\Delta^{\mathcal{I}}$ such that $x = [x_0]_{\simeq_{\Phi,\mathcal{I}}}$. By Lemma 4.14, $x \equiv_\Phi^{sp} x_0$. Because there are $\mathcal{L}_\Phi^{sp}$-comparisons between $\mathcal{I}$ and $\mathcal{I}_2$ and between $\mathcal{I}_2$ and $\mathcal{I}$, by (4.12) and Theorem 4.4, there exist $x' \in \Delta^{\mathcal{I}_2}$ and $x'' \in \Delta^{\mathcal{I}}$ such that $x_0 \leq_\Phi^{sp} x' \leq_\Phi^{sp} x''$. By Lemma 4.14, $x'' \equiv_\Phi^{sp} [x'']_{\sim_{\Phi,\mathcal{I}}}$. It follows that $x \leq_\Phi^{sp} x' \leq_\Phi^{sp} [x'']_{\sim_{\Phi,\mathcal{I}}}$. As x is an $\mathcal{L}_\Phi^{sp}$-maximal individual of $\mathcal{I}'$, it follows that $x \equiv_\Phi^{sp} x' \equiv_\Phi^{sp} [x'']_{\sim_{\Phi,\mathcal{I}}}$.

- The case when $U \in \Phi$ and x is an $\mathcal{L}_\Phi^{sp}$-minimal individual of $\mathcal{I}'$ is similar to the above case.

For the induction step, assume that x is an $\mathcal{L}_\Phi^{sp}$-extremal R-successor of u, where R is a basic role of $\mathcal{L}_\Phi$ and u was marked as $\mathcal{L}_\Phi^{sp}$-essential earlier than x. By the inductive assumption, there exists $u' \in \Delta^{\mathcal{I}_2}$ such that $u \equiv_\Phi^{sp} u'$. Analogously to the base case with $U \in \Phi$, it can be seen that there exists an R-successor x' of u' such that $x \equiv_\Phi^{sp} x'$. ∎

Corollary 4.17. *Suppose $\Sigma_I \neq \emptyset$, $\Phi \subseteq \{I, O\}$ and $\mathcal{I}$ is a finitely branching interpretation without unreachable objects. Then $\mathcal{I}/^{QS}_{\simeq_{\Phi,\mathcal{I}}}$ satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$ and the cardinality of its domain is not bigger than the cardinality of the domain of any interpretation that satisfies the same set of concept assertions in $\mathcal{L}_\Phi^{sp}$ as $\mathcal{I}$.*

By using Algorithm 4 to compute the relation $\lesssim_{\Phi,\mathcal{I}}$ and then the relation $\simeq_{\Phi,\mathcal{I}}$, the "minimal" interpretation/QS-interpretation $\mathcal{I}''$ mentioned in Theorems 4.15 and 4.16 can be constructed in polynomial time in the size of $\mathcal{I}$, provided that $\mathcal{I}$ is finite. How to construct that interpretation/QS-interpretation efficiently in the spirit of Hopcroft's automaton minimization algorithm [28] and the Paige-Tarjan algorithm [49] remains, however, as an open problem.

Chapter 5

Comparing the Expressiveness of Description Logics

Expressiveness (expressive power) is a topic studied in the fields of formal languages, databases and logics. The Chomsky hierarchy provides fundamental results on the expressiveness of formal languages. In the field of databases, the works by Fagin [21, 22], Immerman [31, 32], Abiteboul and Vianu [1] provide important results on the expressiveness of query languages. Many results on the expressiveness of logics have also been obtained, e.g. in [24, 31, 61, 33, 54, 35, 55].

The expressiveness of description logics (DLs) has been studied in a number of works [4, 7, 8, 37, 40]. In [4] Baader proposed a formal definition of the expressive power of DLs. His definition is liberal in that it allows the compared logics to have different vocabularies. His work provides separation results for some early DLs. In [7] Borgida showed that certain DLs have the same expressiveness as the two or three variable fragment of first-order logic. The class of DLs considered in [7] is large, but the results only concern DLs without the reflexive and transitive closure of roles. In [8] Cadoli et al. considered the expressiveness of hybrid knowledge bases that combine a DL knowledge base with Horn rules. The used DL is $\mathcal{ALCNR}$. The work [37] by Kurtonina and de Rijke is a comprehensive work on the expressiveness of DLs that are sublogics of $\mathcal{ALCNR}$. It is based on bisimulation and provides many interesting results. In [40] Lutz et al. characterized the expressiveness and rewritability of DL TBoxes for the DLs that are sublogics of $\mathcal{ALCQIO}$. They used semantic notions such as bisimulation, equisimulation, disjoint union and direct product.

This chapter studies the expressiveness of the DLs introduced in Chapter 2. We compare the expressiveness of these DLs w.r.t. concepts, positive concepts, TBoxes and ABoxes. Our results about separating the expressiveness of DLs are based on bisimulations and bisimulation-based comparisons. They are naturally extended to the case when instead of $\mathcal{ALC}_{reg}$ we have any sublogic of $\mathcal{ALC}_{reg}$ that extends $\mathcal{ALC}$.

Our study differs significantly from all of [4, 7, 8, 37, 40], as the class of considered DLs is much larger than the ones considered in those works (we allow PDL-like role constructors as well as the universal role and the concept constructor $\exists r.\mathtt{Self}$) and our results about separating the expressiveness of DLs are obtained not only w.r.t. concepts and TBoxes but also w.r.t. positive concepts and ABoxes.

Definition 5.1 (**Equivalence between Concepts, TBoxes or ABoxes).**
Two concepts C and D are *equivalent* if, for every interpretation $\mathcal{I}$, $C^\mathcal{I} = D^\mathcal{I}$. Two TBoxes $\mathcal{T}_1$ and $\mathcal{T}_2$ are *equivalent* if, for every interpretation $\mathcal{I}$, $\mathcal{I}$ is a model of $\mathcal{T}_1$ iff $\mathcal{I}$ is a model of $\mathcal{T}_2$. Two ABoxes $\mathcal{A}_1$ and $\mathcal{A}_2$ are *equivalent* if, for every interpretation $\mathcal{I}$, $\mathcal{I}$ is a model of $\mathcal{A}_1$ iff $\mathcal{I}$ is a model of $\mathcal{A}_2$. ■

Definition 5.2 (**Comparing Description Logics).**
We say that a logic $\mathcal{L}_1$ is *at most as expressive as* a logic $\mathcal{L}_2$ w.r.t. concepts (resp. positive concepts, TBoxes, ABoxes), denoted by $\mathcal{L}_1 \leq_C \mathcal{L}_2$ (resp. $\mathcal{L}_1 \leq_{PC} \mathcal{L}_2$, $\mathcal{L}_1 \leq_T \mathcal{L}_2$, $\mathcal{L}_1 \leq_A \mathcal{L}_2$), if every concept (resp. positive concept, TBox, ABox) in $\mathcal{L}_1$ has an equivalent concept (resp. positive concept, TBox, ABox) in $\mathcal{L}_2$.

We say that a logic $\mathcal{L}_2$ is *more expressive* than a logic $\mathcal{L}_1$ (or $\mathcal{L}_1$ is *less expressive* than $\mathcal{L}_2$) w.r.t. concepts (resp. positive concepts, TBoxes, ABoxes), denoted by $\mathcal{L}_1 <_C \mathcal{L}_2$ (resp. $\mathcal{L}_1 <_{PC} \mathcal{L}_2$, $\mathcal{L}_1 <_T \mathcal{L}_2$, $\mathcal{L}_1 <_A \mathcal{L}_2$), if $\mathcal{L}_1 \leq_C \mathcal{L}_2$ (resp. $\mathcal{L}_1 \leq_{PC} \mathcal{L}_2$, $\mathcal{L}_1 \leq_T \mathcal{L}_2$, $\mathcal{L}_1 \leq_A \mathcal{L}_2$) and $\mathcal{L}_2 \not\leq_C \mathcal{L}_1$ (resp. $\mathcal{L}_2 \not\leq_{PC} \mathcal{L}_1$, $\mathcal{L}_2 \not\leq_T \mathcal{L}_1$, $\mathcal{L}_2 \not\leq_A \mathcal{L}_1$). ■

The following proposition clearly holds.

Proposition 5.1. *If a logic $\mathcal{L}_1$ is at most as expressive as a logic $\mathcal{L}_2$ w.r.t. concepts (resp. positive concepts, TBoxes, ABoxes) and a logic $\mathcal{L}_2$ is at most as expressive as $\mathcal{L}_3$ w.r.t. concepts (resp. positive concepts, TBoxes, ABoxes) then $\mathcal{L}_1$ is at most as expressive as $\mathcal{L}_3$ w.r.t. concepts (resp. positive concepts, TBoxes, ABoxes).*

Lemma 5.2. *Let Φ_1 and Φ_2 be sets of DL-features such that $\Phi_1 \subseteq \Phi_2$. Denote $\mathcal{L}_1 = \mathcal{L}_{\Phi_1}$ and $\mathcal{L}_2 = \mathcal{L}_{\Phi_2}$. Let $\mathcal{I}$, $\mathcal{I}'$ be interpretations and Z an $\mathcal{L}_1$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.*

1. *If $\mathcal{L}_1 \leq_C \mathcal{L}_2$, $x \in \Delta^\mathcal{I}, x' \in \Delta^{\mathcal{I}'}, Z(x, x')$ holds, and there exists a concept C of $\mathcal{L}_2$ such that $x \in C^\mathcal{I}$ but $x' \notin C^{\mathcal{I}'}$, then $\mathcal{L}_1 <_C \mathcal{L}_2$.*

2. *Suppose that $U \in \Phi_1$ or both $\mathcal{I}$ and $\mathcal{I}'$ are unreachable-objects-free. If $\mathcal{L}_1 \leq_T \mathcal{L}_2$ and there exists a TBox $\mathcal{T}$ in $\mathcal{L}_2$ such that $\mathcal{I}$ is a model of $\mathcal{T}$ but $\mathcal{I}'$ is not, then $\mathcal{L}_1 <_T \mathcal{L}_2$.*

3. *Suppose $O \in \Phi_1$. If $\mathcal{L}_1 \leq_A \mathcal{L}_2$ and there exists an ABox $\mathcal{A}$ in $\mathcal{L}_2$ such that $\mathcal{I}$ is a model of $\mathcal{A}$ but $\mathcal{I}'$ is not, then $\mathcal{L}_1 <_A \mathcal{L}_2$.*

Proof. Consider the first assertion. Suppose $\mathcal{L}_1 \leq_C \mathcal{L}_2$, $x \in \Delta^\mathcal{I}$, $x' \in \Delta^{\mathcal{I}'}$, $Z(x, x')$ holds and there exists a concept C of $\mathcal{L}_2$ such that $x \in C^\mathcal{I}$ but $x' \notin C^{\mathcal{I}'}$. We prove that $\mathcal{L}_2 \not\leq_C \mathcal{L}_1$. For the sake of contradiction, suppose $\mathcal{L}_2 \leq_C \mathcal{L}_1$. It follows that there exists a concept C' of $\mathcal{L}_1$ that is equivalent to C. Thus, $x \in C'^\mathcal{I}$ but $x' \notin C'^{\mathcal{I}'}$. Hence, C' is not invariant for Z, which contradicts Theorem 3.4. Therefore, $\mathcal{L}_1 <_C \mathcal{L}_2$.

Consider the second assertion. Suppose $\mathcal{L}_1 \leq_T \mathcal{L}_2$ and there exists a TBox $\mathcal{T}$ in $\mathcal{L}_2$ such that $\mathcal{I}$ is a model of $\mathcal{T}$ but $\mathcal{I}'$ is not. We prove that $\mathcal{L}_2 \not\leq_T \mathcal{L}_1$. For the sake of contradiction, suppose $\mathcal{L}_2 \leq_T \mathcal{L}_1$. It follows that there exists a TBox $\mathcal{T}'$ in $\mathcal{L}_1$ that is equivalent to $\mathcal{T}$. Thus, $\mathcal{I}$ is a model of $\mathcal{T}'$ but $\mathcal{I}'$ is not, which contradicts Corollary 3.5 or Theorem 3.6. Therefore, $\mathcal{L}_1 <_T \mathcal{L}_2$.

Consider the third assertion. Suppose $\mathcal{L}_1 \leq_A \mathcal{L}_2$ and there exists an ABox $\mathcal{A}$ in $\mathcal{L}_2$ such that $\mathcal{I}$ is a model of $\mathcal{A}$ but $\mathcal{I}'$ is not. We prove that $\mathcal{L}_2 \not\leq_A \mathcal{L}_1$. For the sake of contradiction, suppose $\mathcal{L}_2 \leq_A \mathcal{L}_1$. It follows that there exists an ABox $\mathcal{A}'$ in $\mathcal{L}_1$ that is equivalent to $\mathcal{A}$. Thus, $\mathcal{I}$ is a model of $\mathcal{A}'$ but $\mathcal{I}'$ is not, which contradicts Theorem 3.7. Therefore, $\mathcal{L}_1 <_A \mathcal{L}_2$. ■

Lemma 5.3. *Let Φ_1 and Φ_2 be sets of DL-features such that $\Phi_1 \subseteq \Phi_2$. Denote $\mathcal{L}_1 = \mathcal{L}_{\Phi_1}$ and $\mathcal{L}_2 = \mathcal{L}_{\Phi_2}$. Let $\mathcal{I}$, $\mathcal{I}'$ be interpretations and Z an $\mathcal{L}_1$-comparison between $\mathcal{I}$ and $\mathcal{I}'$. If $\mathcal{L}_1 \leq_{PC} \mathcal{L}_2$, $x \in \Delta^{\mathcal{I}}, x' \in \Delta^{\mathcal{I}'}, Z(x, x')$ holds, and there exists a positive concept C of $\mathcal{L}_2$ such that $x \in C^{\mathcal{I}}$ but $x' \notin C^{\mathcal{I}'}$, then $\mathcal{L}_1 <_{PC} \mathcal{L}_2$.*

Proof. Suppose $\mathcal{L}_1 \leq_{PC} \mathcal{L}_2$, $x \in \Delta^{\mathcal{I}}$, $x' \in \Delta^{\mathcal{I}'}$, $Z(x, x')$ holds and there exists a positive concept C of $\mathcal{L}_2$ such that $x \in C^{\mathcal{I}}$ but $x' \notin C^{\mathcal{I}'}$. We prove that $\mathcal{L}_2 \not\leq_{PC} \mathcal{L}_1$. For the sake of contradiction, suppose $\mathcal{L}_2 \leq_{PC} \mathcal{L}_1$. It follows that there exists a positive concept C' of $\mathcal{L}_1$ that is equivalent to C. Thus, $x \in C'^{\mathcal{I}}$ but $x' \notin C'^{\mathcal{I}'}$. It follows that C' is not preserved by Z, which contradicts Corollary 4.5. Hence, $\mathcal{L}_1 <_{PC} \mathcal{L}_2$. ■

From now on, we assume that Σ_C and Σ_R are not empty and Σ_I contains at least two individual names. Let $\{a, b\} \subseteq \Sigma_I$, $A \in \Sigma_C$ and $r \in \Sigma_R$.

Lemma 5.4.

1. *For any pair $\langle \mathcal{L}_1, \mathcal{L}_2 \rangle$ among $\langle \mathcal{L}_I, \mathcal{L}_{OQU\mathtt{Self}} \rangle, \langle \mathcal{L}_Q, \mathcal{L}_{IOU\mathtt{Self}} \rangle, \langle \mathcal{L}_{\mathtt{Self}}, \mathcal{L}_{IOQU} \rangle$, we have that: $\mathcal{L}_1 \not\leq_C \mathcal{L}_2$, $\mathcal{L}_1 \not\leq_{PC} \mathcal{L}_2$, $\mathcal{L}_1 \not\leq_T \mathcal{L}_2$, $\mathcal{L}_1 \not\leq_A \mathcal{L}_2$.*
2. *$\mathcal{L}_O \not\leq_C \mathcal{L}_{IQU\mathtt{Self}}$, $\mathcal{L}_O \not\leq_{PC} \mathcal{L}_{IQU\mathtt{Self}}$, $\mathcal{L}_O \not\leq_T \mathcal{L}_{IQU\mathtt{Self}}$.*
3. *$\mathcal{L}_U \not\leq_C \mathcal{L}_{IOQ\mathtt{Self}}$, $\mathcal{L}_U \not\leq_{PC} \mathcal{L}_{IOQ\mathtt{Self}}$, $\mathcal{L}_U \not\leq_A \mathcal{L}_{IOQ\mathtt{Self}}$.*

Proof. Let us compare $\mathcal{L}_I$ with $\mathcal{L}_{OQU\mathtt{Self}}$. Consider the interpretations $\mathcal{I}$, $\mathcal{I}'$ and the relation Z shown in the first part of Figure 5.1 (on page 70). The arrows denote the instances of r in $\mathcal{I}$ and $\mathcal{I}'$. The instances of A in $\mathcal{I}$ and $\mathcal{I}'$ are explicitly indicated in the figure. Let $B^{\mathcal{I}} = B^{\mathcal{I}'} = \emptyset$ for all $B \in \Sigma_C \setminus \{A\}$, $s^{\mathcal{I}} = s^{\mathcal{I}'} = \emptyset$ for all $s \in \Sigma_R \setminus \{r\}$, and $c^{\mathcal{I}} = a^{\mathcal{I}}$, $c^{\mathcal{I}'} = a^{\mathcal{I}'}$ for all $c \in \Sigma_I \setminus \{a, b\}$. The dotted lines in the figure indicate the instances of a binary relation $Z \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}$. It can be checked that Z is an $\mathcal{L}_{OQU\mathtt{Self}}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$. Consider the positive concept $C = \forall r \forall r^{-1}.A$ of $\mathcal{L}_I$. Clearly, $a^{\mathcal{I}} \in C^{\mathcal{I}}$ but $a^{\mathcal{I}'} \notin C^{\mathcal{I}'}$. By Theorem 3.4, C does not have any equivalent concept in $\mathcal{L}_{OQU\mathtt{Self}}$. Hence, $\mathcal{L}_I \not\leq_C \mathcal{L}_{OQU\mathtt{Self}}$. As Z is also an $\mathcal{L}_{OQU\mathtt{Self}}$-comparison between $\mathcal{I}$ and $\mathcal{I}'$, by Corollary 4.5, C does not have any equivalent positive concept in $\mathcal{L}_{OQU\mathtt{Self}}$ either. Hence, $\mathcal{L}_I \not\leq_{PC} \mathcal{L}_{OQU\mathtt{Self}}$. Consider the TBox $\mathcal{T} = \{A \sqsubseteq C\}$. Since $\mathcal{I} \models \mathcal{T}$ but $\mathcal{I}' \not\models \mathcal{T}$, by Theorem 3.6, $\mathcal{T}$ does not have any equivalent TBox in $\mathcal{L}_{OQU\mathtt{Self}}$. Hence $\mathcal{L}_I \not\leq_T \mathcal{L}_{OQU\mathtt{Self}}$. Consider the ABox $\mathcal{A} = \{C(a)\}$. Since $\mathcal{I} \models \mathcal{A}$ but $\mathcal{I}' \not\models \mathcal{A}$, by Theorem 3.7, $\mathcal{A}$ does not have any equivalent ABox in $\mathcal{L}_{OQU\mathtt{Self}}$. Hence $\mathcal{L}_I \not\leq_A \mathcal{L}_{OQU\mathtt{Self}}$.

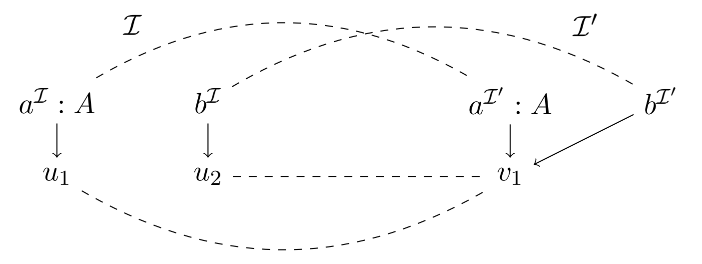
The proofs for the other pairs of logics can be done similarly, using $\mathcal{I}$, $\mathcal{I}'$, C specified in the next parts of Figure 5.1 (on page 70). For the parts without the presence of b, let $b^{\mathcal{I}} = a^{\mathcal{I}}$ and $b^{\mathcal{I}'} = a^{\mathcal{I}'}$. ■

<table>
<tr><td>

$\mathcal{L}_I$ vs.
$\mathcal{L}_{OQU\texttt{Self}}$

$C = \forall r \forall r^{-1}.A$

</td><td>

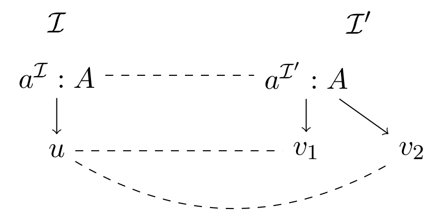


</td></tr>
<tr><td>

$\mathcal{L}_Q$ vs.
$\mathcal{L}_{IOU\texttt{Self}}$

$C = (\leq 1 r.\neg A)$

</td><td>

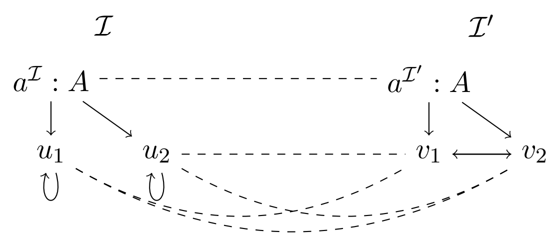


</td></tr>
<tr><td>

$\mathcal{L}_{\texttt{Self}}$ vs.
$\mathcal{L}_{IOQU}$

$C = \exists r \exists r.\texttt{Self}$

</td><td>

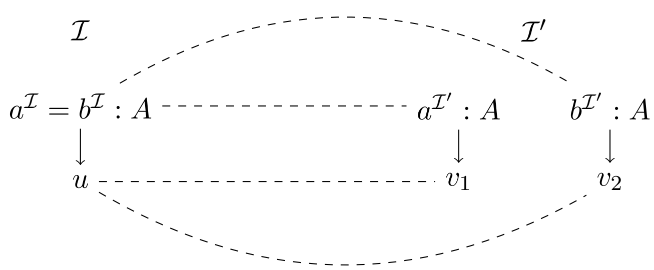


</td></tr>
<tr><td>

$\mathcal{L}_O$ vs.
$\mathcal{L}_{IQU\texttt{Self}}$

$C = \{b\}$

</td><td>

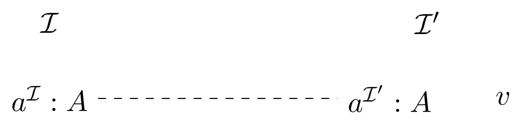


</td></tr>
<tr><td>

$\mathcal{L}_U$ vs.
$\mathcal{L}_{IOQ\texttt{Self}}$

$C = \forall U.A$

</td><td>



</td></tr>
</table>

Figure 5.1: An illustration for the proof of Lemma 5.4.

Theorem 5.5. *Let Φ and Φ' be subsets of $\{I, O, Q, U, \mathtt{Self}\}$.*

1. *If $\Phi \subset \Phi'$ then $\mathcal{L}_\Phi <_C \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi <_{PC} \mathcal{L}_{\Phi'}$.*
2. *If $\Phi \not\subseteq \Phi'$ then $\mathcal{L}_\Phi \not\leq_C \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi \not\leq_{PC} \mathcal{L}_{\Phi'}$.*
3. *If $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{U\}$ then $\mathcal{L}_\Phi <_T \mathcal{L}_{\Phi'}$.*
4. *If $\Phi \not\subseteq \Phi'$ and $\Phi \setminus \Phi' \neq \{U\}$ then $\mathcal{L}_\Phi \not\leq_T \mathcal{L}_{\Phi'}$.*
5. *If $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{O\}$ then $\mathcal{L}_\Phi <_A \mathcal{L}_{\Phi'}$.*
6. *If $\Phi \not\subseteq \Phi'$ and $\Phi \setminus \Phi' \neq \{O\}$ then $\mathcal{L}_\Phi \not\leq_A \mathcal{L}_{\Phi'}$.*

Proof. Consider the first assertion and suppose $\Phi \subset \Phi'$. Since every concept (resp. positive concept) of $\mathcal{L}_\Phi$ is also a concept (resp. positive concept) of $\mathcal{L}_{\Phi'}$, we have that $\mathcal{L}_\Phi \leq_C \mathcal{L}_{\Phi'}$ (resp. $\mathcal{L}_\Phi \leq_{PC} \mathcal{L}_{\Phi'}$). Since $\Phi' \setminus \Phi \neq \emptyset$, at least one feature among I, O, Q, U, $\mathtt{Self}$ belongs to $\Phi' \setminus \Phi$. Consider the case $I \in \Phi' \setminus \Phi$. The cases of other features are similar and omitted. For the sake of contradiction, suppose $\mathcal{L}_{\Phi'} \leq_C \mathcal{L}_\Phi$ (resp. $\mathcal{L}_{\Phi'} \leq_{PC} \mathcal{L}_\Phi$). Since $\mathcal{L}_I \leq_C \mathcal{L}_{\Phi'}$ (resp. $\mathcal{L}_I \leq_{PC} \mathcal{L}_{\Phi'}$) and $\mathcal{L}_\Phi \leq_C \mathcal{L}_{OQU\mathtt{Self}}$ (resp. $\mathcal{L}_\Phi \leq_{PC} \mathcal{L}_{OQU\mathtt{Self}}$), it follows that $\mathcal{L}_I \leq_C \mathcal{L}_{OQU\mathtt{Self}}$ (resp. $\mathcal{L}_I \leq_{PC} \mathcal{L}_{OQU\mathtt{Self}}$), which contradicts Lemma 5.4. Therefore, $\mathcal{L}_\Phi <_C \mathcal{L}_{\Phi'}$ (resp. $\mathcal{L}_\Phi <_{PC} \mathcal{L}_{\Phi'}$).

Consider the second assertion and suppose $\Phi \not\subseteq \Phi'$. Since $\Phi \setminus \Phi' \neq \emptyset$, at least one feature among I, O, Q, U, $\mathtt{Self}$ belongs to $\Phi \setminus \Phi'$. Consider the case $I \in \Phi \setminus \Phi'$. The cases of other features are similar and omitted. For the sake of contradiction, suppose $\mathcal{L}_\Phi \leq_C \mathcal{L}_{\Phi'}$ (resp. $\mathcal{L}_\Phi \leq_{PC} \mathcal{L}_{\Phi'}$). Since $\mathcal{L}_I \leq_C \mathcal{L}_\Phi$ (resp. $\mathcal{L}_I \leq_{PC} \mathcal{L}_\Phi$) and $\mathcal{L}_{\Phi'} \leq_C \mathcal{L}_{OQU\mathtt{Self}}$ (resp. $\mathcal{L}_{\Phi'} \leq_{PC} \mathcal{L}_{OQU\mathtt{Self}}$), it follows that $\mathcal{L}_I \leq_C \mathcal{L}_{OQU\mathtt{Self}}$ (resp. $\mathcal{L}_I \leq_{PC} \mathcal{L}_{OQU\mathtt{Self}}$), which contradicts Lemma 5.4. Therefore, $\mathcal{L}_\Phi \not\leq_C \mathcal{L}_{\Phi'}$ (resp. $\mathcal{L}_\Phi \not\leq_{PC} \mathcal{L}_{\Phi'}$).

Consider the third assertion and suppose $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{U\}$. At least one feature among I, O, Q, $\mathtt{Self}$ belongs to $\Phi' \setminus \Phi$. Consider the case $I \in \Phi' \setminus \Phi$. The cases of other features are similar and omitted. Since $\Phi \subset \Phi'$, $\mathcal{L}_\Phi \leq_T \mathcal{L}_{\Phi'}$. For the sake of contradiction, suppose $\mathcal{L}_{\Phi'} \leq_T \mathcal{L}_\Phi$. Since $\mathcal{L}_I \leq_T \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi \leq_T \mathcal{L}_{OQU\mathtt{Self}}$, it follows that $\mathcal{L}_I \leq_T \mathcal{L}_{OQU\mathtt{Self}}$, which contradicts Lemma 5.4. Therefore, $\mathcal{L}_\Phi <_T \mathcal{L}_{\Phi'}$.

Consider the fourth assertion and suppose $\Phi \not\subseteq \Phi'$ and $\Phi \setminus \Phi' \neq \{U\}$. At least one feature among I, O, Q, $\mathtt{Self}$ belongs to $\Phi \setminus \Phi'$. Consider the case $I \in \Phi \setminus \Phi'$. The cases of other features are similar and omitted. For the sake of contradiction, suppose $\mathcal{L}_\Phi \leq_T \mathcal{L}_{\Phi'}$. Since $\mathcal{L}_I \leq_T \mathcal{L}_\Phi$ and $\mathcal{L}_{\Phi'} \leq_T \mathcal{L}_{OQU\mathtt{Self}}$, it follows that $\mathcal{L}_I \leq_T \mathcal{L}_{OQU\mathtt{Self}}$, which contradicts Lemma 5.4. Therefore, $\mathcal{L}_\Phi \not\leq_T \mathcal{L}_{\Phi'}$.

Consider the fifth assertion and suppose $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{O\}$. At least one feature among I, Q, U, $\mathtt{Self}$ belongs to $\Phi' \setminus \Phi$. Consider the case $I \in \Phi' \setminus \Phi$. The cases of other features are similar and omitted. Since $\Phi \subset \Phi'$, $\mathcal{L}_\Phi \leq_A \mathcal{L}_{\Phi'}$. For the sake of contradiction, suppose $\mathcal{L}_{\Phi'} \leq_A \mathcal{L}_\Phi$. Since $\mathcal{L}_I \leq_A \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi \leq_A \mathcal{L}_{OQU\mathtt{Self}}$, it follows that $\mathcal{L}_I \leq_A \mathcal{L}_{OQU\mathtt{Self}}$, which contradicts Lemma 5.4. Therefore, $\mathcal{L}_\Phi <_A \mathcal{L}_{\Phi'}$.

Consider the last assertion and suppose $\Phi \not\subseteq \Phi'$ and $\Phi \setminus \Phi' \neq \{O\}$. At least one feature among I, Q, U, $\mathtt{Self}$ belongs to $\Phi \setminus \Phi'$. Consider the case $I \in \Phi \setminus \Phi'$. The cases of other features are similar and omitted. For the sake of contradiction, suppose $\mathcal{L}_\Phi \leq_A \mathcal{L}_{\Phi'}$. Since $\mathcal{L}_I \leq_A \mathcal{L}_\Phi$ and $\mathcal{L}_{\Phi'} \leq_A \mathcal{L}_{OQU\mathtt{Self}}$, it follows that $\mathcal{L}_I \leq_A \mathcal{L}_{OQU\mathtt{Self}}$, which contradicts Lemma 5.4. Therefore, $\mathcal{L}_\Phi \not\leq_A \mathcal{L}_{\Phi'}$. ∎

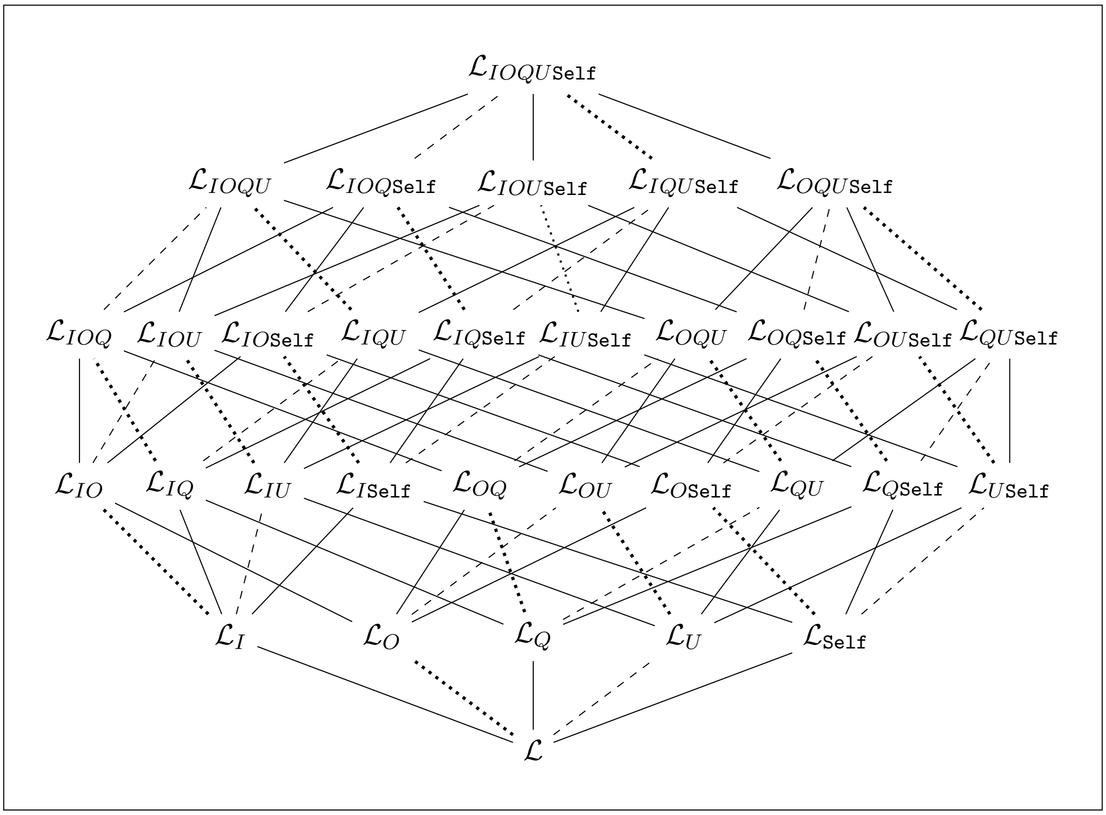


Figure 5.2: Comparing the expressiveness of description logics, where $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$. If there is a path from a logic $\mathcal{L}_2$ down to a logic $\mathcal{L}_1$ that contains either a normal edge or at least two edges then $\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$ w.r.t. concepts, positive concepts, TBoxes and ABoxes. If the path is a dotted edge then $\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$ w.r.t. concepts, positive concepts and TBoxes. If the path is a dashed edge then $\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$ w.r.t. concepts, positive concepts and ABoxes.

Definition 5.3. We define $\mathcal{ALC}$ to be the sublogic of $\mathcal{ALC}_{reg}$ such that the role constructors ε, $R \circ S$, $R \sqcup S$, R^* and $C?$ are disallowed. We say that $\mathcal{L}$ is a sublogic of $\mathcal{ALC}_{reg}$ that extends $\mathcal{ALC}$, denoted $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$, if it extends $\mathcal{ALC}$ with some of those role constructors. For $\Phi \subseteq \{I, O, Q, U, \mathtt{Self}\}$ and $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$, let $\mathcal{L}_\Phi$ and $\mathcal{L}_\Phi^{pos}$ be defined as usual in the spirit of Definitions 2.1 and 4.1. ■

Corollary 5.6. *Let $\mathcal{L}$ be any sublogic of $\mathcal{ALC}_{reg}$ that extends $\mathcal{ALC}$ and let Φ and Φ' be subsets of $\{I, O, Q, U, \mathtt{Self}\}$.*

1. *If $\Phi \subset \Phi'$ then $\mathcal{L}_\Phi <_C \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi <_{PC} \mathcal{L}_{\Phi'}$.*

2. *If $\Phi \not\subseteq \Phi'$ then $\mathcal{L}_\Phi \not\leq_C \mathcal{L}_{\Phi'}$ and $\mathcal{L}_\Phi \not\leq_{PC} \mathcal{L}_{\Phi'}$.*

3. *If $\Phi \subset \Phi'$ and $\Phi' \setminus \Phi \neq \{U\}$ then $\mathcal{L}_\Phi <_T \mathcal{L}_{\Phi'}$.*

4. If $\Phi \not\subseteq \Phi'$ *and* $\Phi \setminus \Phi' \neq \{U\}$ *then* $\mathcal{L}_\Phi \not\leq_T \mathcal{L}_{\Phi'}$.

5. If $\Phi \subset \Phi'$ *and* $\Phi' \setminus \Phi \neq \{O\}$ *then* $\mathcal{L}_\Phi <_A \mathcal{L}_{\Phi'}$.

6. If $\Phi \not\subseteq \Phi'$ *and* $\Phi \setminus \Phi' \neq \{O\}$ *then* $\mathcal{L}_\Phi \not\leq_A \mathcal{L}_{\Phi'}$.

Proof. Just observe that the concepts C listed in Figure 5.1 (on page 70) do not use any of the role constructors ε, $R \circ S$, $R \sqcup S$, R^*, $C?$. All the lemmas and theorems given in this chapter hold for the case when $\mathcal{L}$ is a sublogic of $\mathcal{ALC}_{reg}$ that extends $\mathcal{ALC}$. Their proofs do not require any change. ■

Figure 5.2 (on page 72) illustrates the relationship between the expressiveness of all the DLs that extend $\mathcal{L}$, where $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$, with any non-empty combination of the features I, O, Q, U, `Self`. Note that the problems whether $\mathcal{L}_\Phi <_T \mathcal{L}_{\Phi'}$ when $\Phi' \setminus \Phi = \{U\}$ and whether $\mathcal{L}_\Phi <_A \mathcal{L}_{\Phi'}$ when $\Phi' \setminus \Phi = \{O\}$ remain open.

Chapter 6

Concept Learning in Description Logics

Concept learning in DLs is useful for making decision rules as in traditional binary classification. It is also useful in ontology engineering, e.g., for suggesting definitions of important concepts. The major settings of concept learning in DLs are as follows:

Setting 1. Given a knowledge base KB and sets E^+, E^- of named individuals, learn a concept C in a DL L such that: (a) $KB \models C(a)$ for all $a \in E^+$, and (b) $KB \models \neg C(a)$ for all $a \in E^-$. The set E^+ (resp. E^-) contains positive (resp. negative) examples of C.

Setting 2. This setting differs from Setting 1 only in that the condition (b) is replaced by the weaker one: $KB \not\models C(a)$ for all $a \in E^-$.

Setting 3. Given an interpretation $\mathcal{I}$ and sets E^+, E^- of named individuals, learn a concept C in L such that: (a) $\mathcal{I} \models C(a)$ for all $a \in E^+$, and (b) $\mathcal{I} \models \neg C(a)$ for all $a \in E^-$. Note that $\mathcal{I} \not\models C(a)$ is the same as $\mathcal{I} \models \neg C(a)$.

In [11] Cohen and Hirsh studied PAC-learnability of an early DL formalism called CLASSIC. They proposed a concept learning algorithm based on "least common subsumers". In [38] Lambrix and Larocchia proposed a simple concept learning algorithm based on concept normalization. Badea and Nienhuys-Cheng [5], Iannone et al. [30], Fanizzi et al. [23], Lehmann and Hitzler [39] studied concept learning in DLs by using refinement operators as in inductive logic programming. The works [5, 30] use Setting 1, while the works [23, 39] use Setting 2.

Bisimulations in DLs have been used for concept learning in DLs in a number of papers [48, 57, 26, 56, 58]. In Section 6.1 we present a survey on these works. In Section 6.2 we present our results on possibility of correct learning in DLs using Setting 3. In Section 6.3 we also generalize common types of queries for DLs, introduce interpretation queries and present some consequences.

6.1 Bisimulation-Based Concept Learning

6.1.1 Using Setting 3

In [48] Nguyen and Szałas generalized our notion of bisimulations in DLs and some of our results [16] to model indiscernibility of objects. Their work is pioneering in using bisimulation for concept learning in DLs. It also concerns concept approximation by using bisimulation and Pawlak's rough set theory [52, 53]. The generalization deals with the following: the main language is $\mathcal{L}_\Phi$ using the signature Σ, but the concept to be learned may be restricted to a language $\mathcal{L}_{\Phi^\dagger}$ with a signature $\Sigma^\dagger$, where $\Phi^\dagger \subseteq \Phi$ and $\Sigma^\dagger \subseteq \Sigma$. For that they introduced the language $\mathcal{L}_{\Sigma,\Phi}$ and $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$-bisimulation between two interpretations $\mathcal{I}$ and $\mathcal{I}'$ (see [48] for details).

An $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$-bisimulation between $\mathcal{I}$ and itself is called an *$\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$-auto-bisimulation of $\mathcal{I}$*. An $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$-auto-bisimulation of $\mathcal{I}$ is said to be the *largest* if it is larger than or equal to ($\supseteq$) any other $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$-auto-bisimulation of $\mathcal{I}$.

An *information system in $\mathcal{L}_{\Sigma,\Phi}$* is a finite interpretation in $\mathcal{L}_{\Sigma,\Phi}$. It can be given explicitly or specified somehow, e.g., by a knowledge base in the Web ontology rule language OWL 2 RL$^+$ [9] (using the standard semantics) or WORL [10] (using the well-founded semantics) or SWORL [10] (using the stratified semantics).

Given an interpretation $\mathcal{I}$ in $\mathcal{L}_{\Sigma,\Phi}$, by $\sim_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$ we denote the largest $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$-auto-bisimulation of $\mathcal{I}$, and by $\equiv_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$ we denote the binary relation on $\Delta^\mathcal{I}$ with the property that $x \equiv_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}} x'$ iff x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$-equivalent to x'.

The following theorem correspond to Propositions 3.13 and 3.14.

Theorem 6.1. *[48, Theorem 19.3] Let Σ and $\Sigma^\dagger$ be DL-signatures such that $\Sigma^\dagger \subseteq \Sigma$, Φ and $\Phi^\dagger$ be sets of DL-features such that $\Phi^\dagger \subseteq \Phi$, and $\mathcal{I}$ be an interpretation in $\mathcal{L}_{\Sigma,\Phi}$. Then:*

- *the largest $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$-auto-bisimulation of $\mathcal{I}$ exists and is an equivalence relation,*
- *if $\mathcal{I}$ is finitely branching w.r.t. $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ then the relation $\equiv_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$ is the largest $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$-auto-bisimulation of $\mathcal{I}$ (i.e. the relations $\equiv_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$ and $\sim_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$ coincide).*

Definition 6.1. We say that a set Y is *split* by a set X if $Y \setminus X \neq \emptyset$ and $Y \cap X \neq \emptyset$. Thus, Y is not split by X if either $Y \subseteq X$ or $Y \cap X = \emptyset$. A partition $P = \{Y_1, \ldots, Y_n\}$ is *consistent* with a set X if, for every $1 \leq i \leq n$, Y_i is not split by X. ∎

Theorem 6.2. *[48, Theorem 19.4] Let $\mathcal{I}$ be an interpretation in $\mathcal{L}_{\Sigma,\Phi}$, and let $X \subseteq \Delta^\mathcal{I}$, $\Sigma^\dagger \subseteq \Sigma$ and $\Phi^\dagger \subseteq \Phi$. Then:*

1. *if there exists a concept C of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ such that $X = C^\mathcal{I}$ then the partition of $\Delta^\mathcal{I}$ by $\sim_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$ is consistent with X,*
2. *if the partition of $\Delta^\mathcal{I}$ by $\sim_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$ is consistent with X then there exists a concept C of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ such that $C^\mathcal{I} = X$.*

Let $\mathcal{I}$ be an information system in $\mathcal{L}_{\Sigma,\Phi}$ and let $A_d \in \Sigma_C$ be a concept name standing for the "decision attribute". Suppose that A_d can be expressed by a concept

C in $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$, for some specific $\Sigma^\dagger \subseteq \Sigma \setminus \{A_d\}$ and $\Phi^\dagger \subseteq \Phi$. How can we learn that concept C on the basis of $\mathcal{I}$? That is, how can we learn a definition of A_d in $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ on the basis of $\mathcal{I}$?

The idea of [48] for this task is based on the following observation:

> if A_d is definable in $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ then, by the first assertion of Theorem 6.2, $A_d^\mathcal{I}$ must be the union of some equivalence classes of $\Delta^\mathcal{I}$ w.r.t. $\sim_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$.

Nguyen and Szałas [48] proposed the following method:

1. Starting from the partition $\{\Delta^\mathcal{I}\}$, make subsequent granulations to reach the partition corresponding to $\sim_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$.
   - The granulation process can be stopped as soon as the current partition is consistent with $A_d^\mathcal{I}$ (or when some criteria are met).
   - In the granulation process, we denote the blocks created so far in all steps by $Y_1, \ldots, Y_n$, where the current partition $\{Y_{i_1}, \ldots, Y_{i_k}\}$ may consist of only some of them. We do not use the same subscript to denote blocks of different contents (i.e., we always use new subscripts obtained by increasing n for new blocks). We take care that, for each $1 \leq i \leq n$:
     - Y_i is characterized by an appropriate concept C_i (such that $Y_i = C_i^\mathcal{I}$),
     - we keep information about whether Y_i is split by $A_d^\mathcal{I}$,
     - if $Y_i \subseteq A_d^\mathcal{I}$ then $\mathit{LargestContainer}[i] := j$, where $1 \leq j \leq n$ is the subscript of the largest block Y_j such that $Y_i \subseteq Y_j \subseteq A_d^\mathcal{I}$.
2. At the end, let $j_1, \ldots, j_h$ be all the indices from $\{i_1, \ldots, i_k\}$ such that $Y_{j_t} \subseteq A_d^\mathcal{I}$ for $1 \leq t \leq h$, and let $\{l_1, \ldots, l_p\} = \{\mathit{LargestContainer}[j_t] \mid 1 \leq t \leq h\}$. Let C be a simplified form of $C_{l_1} \sqcup \ldots \sqcup C_{l_p}$. Return C as the result.

In [57] Tran et al. generalized and extended the concept learning method of [48] for DL-based information systems. They took attributes as basic elements of the language. Each attribute may be discrete or numeric. A Boolean attribute is treated as a concept name. They also allowed data roles and the features F (functionality) and N (unqualified number restriction). If σ is a data role and d belongs to the range of σ then $\exists\sigma.\{d\}$ is a concept. Concepts $\geq n\, R$ and $\leq n\, R$, where R is a basic role, mean $\geq n\, R.\top$ and $\leq n\, R.\top$, respectively, and can be used when the feature N is allowed. The concept $\leq 1\, r$ (resp. $\leq 1\, r^-$) is used to express functionality (resp. inverse functionality) of r. The Hennessy-Milner property (Theorem 3.12) is reformulated in a straightforward way for the extended language $\mathcal{L}_{\Sigma,\Phi}$ [57].

Reconsider the process of granulating $\{\Delta^\mathcal{I}\}$ for computing the partition corresponding to $\sim_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$. The works [48, 57] use the concepts listed in Figure 6.1 (on page 78) as *basic selectors* for the granulation process. Let the current partition of $\Delta^\mathcal{I}$ be $\{Y_{i_1}, \ldots, Y_{i_k}\}$. If a block Y_{i_j} $(1 \leq j \leq k)$ is split by $D^\mathcal{I}$, where D is a selector, then splitting Y_{i_j} by D is done as follows:

- $s := n + 1,\;\; t := n + 2,\;\; n := n + 2,$

- A, where $A \in \Sigma_C^\dagger$
- $A = d$, where $A \in \Sigma_A^\dagger \setminus \Sigma_C^\dagger$ and $d \in \mathit{range}(A)$
- $A \leq d$ and $A < d$, where $A \in \Sigma_{nA}^\dagger$, $d \in \mathit{range}(A)$ and d is not a minimal element of $\mathit{range}(A)$
- $A \geq d$ and $A > d$, where $A \in \Sigma_{nA}^\dagger$, $d \in \mathit{range}(A)$ and d is not a maximal element of $\mathit{range}(A)$
- $\exists\sigma.\{d\}$, where $\sigma \in \Sigma_{dR}^\dagger$ and $d \in \mathit{range}(\sigma)$
- $\exists r.C_i$, $\exists r.\top$ and $\forall r.C_i$, where $r \in \Sigma_{oR}^\dagger$ and $1 \leq i \leq n$
- $\exists r^-.C_i$, $\exists r^-.\top$ and $\forall r^-.C_i$, if $I \in \Phi^\dagger$, $r \in \Sigma_{oR}^\dagger$ and $1 \leq i \leq n$
- $\{a\}$, if $O \in \Phi^\dagger$ and $a \in \Sigma_I^\dagger$
- $\leq 1\, r$, if $F \in \Phi^\dagger$ and $r \in \Sigma_{oR}^\dagger$
- $\leq 1\, r^-$, if $\{F, I\} \subseteq \Phi^\dagger$ and $r \in \Sigma_{oR}^\dagger$
- $\geq l\, r$ and $\leq m\, r$, if $N \in \Phi^\dagger$, $r \in \Sigma_{oR}^\dagger$, $0 < l \leq \sharp\Delta^\mathcal{I}$ and $0 \leq m < \sharp\Delta^\mathcal{I}$
- $\geq l\, r^-$ and $\leq m\, r^-$, if $\{N, I\} \subseteq \Phi^\dagger$, $r \in \Sigma_{oR}^\dagger$, $0 < l \leq \sharp\Delta^\mathcal{I}$ and $0 \leq m < \sharp\Delta^\mathcal{I}$
- $\geq l\, r.C_i$ and $\leq m\, r.C_i$, if $Q \in \Phi^\dagger$, $r \in \Sigma_{oR}^\dagger$, $1 \leq i \leq n$, $0 < l \leq \sharp C_i$ and $0 \leq m < \sharp C_i$
- $\geq l\, r^-.C_i$ and $\leq m\, r^-.C_i$, if $\{Q, I\} \subseteq \Phi^\dagger$, $r \in \Sigma_{oR}^\dagger$, $1 \leq i \leq n$, $0 < l \leq \sharp C_i$ and $0 \leq m < \sharp C_i$
- $\exists r.\mathtt{Self}$, if $\mathtt{Self} \in \Phi^\dagger$ and $r \in \Sigma_{oR}^\dagger$

Figure 6.1: Basic selectors. Here, $\Sigma_A^\dagger$ denotes the set of attributes of $\Sigma^\dagger$, $\mathit{range}(A)$ denotes the range of the attribute A, $\Sigma_{nA}^\dagger$ denotes the set of numeric attributes of $\Sigma^\dagger$, $\Sigma_{dR}^\dagger$ denotes the set of data roles of $\Sigma^\dagger$, $\mathit{range}(\sigma)$ denotes the range of the data role σ, $\Sigma_{oR}^\dagger$ denotes the set of (object) role names of $\Sigma^\dagger$, n is the number of blocks created so far when granulating $\Delta^\mathcal{I}$, and C_i is the concept characterizing the block Y_i.

- $Y_s := Y_{i_j} \cap D^\mathcal{I}$, $C_s := C_{i_j} \sqcap D$,
- $Y_t := Y_{i_j} \cap (\neg D)^\mathcal{I}$, $C_t := C_{i_j} \sqcap \neg D$,
- the new partition of $\Delta^\mathcal{I}$ becomes $\{Y_{i_1}, \ldots, Y_{i_k}\} \setminus \{Y_{i_j}\} \cup \{Y_s, Y_t\}$.

It was proved in [57] that using the basic selectors listed in Figure 6.1 is sufficient to granulate $\Delta^{\mathcal{I}}$ to obtain the partition corresponding to $\sim_{\Sigma^\dagger,\Phi^\dagger,\mathcal{I}}$. In practice, we prefer as simple as possible definitions for the learned concept. Therefore, it is worth using also other selectors [48, 57, 58] (despite that they are expressible by the basic selectors over $\mathcal{I}$).

In [58] Tran et al. implemented the bisimulation-based concept learning method of [48, 57] (for most of the DLs considered in [48, 57]). They presented a domain partitioning method that use information gain and both basic selectors and extended selectors. The evaluation results of [58] show that the concept learning method of [48, 57] (for Setting 3) is valuable and extended selectors support it significantly.

We refer the reader to [48, 57, 58] for examples that illustrate concept learning for DL-based information systems.

6.1.2 Using Settings 1 and 2

In [26] Ha et al. developed bisimulation-based methods, called BBCL and dual-BBCL, for concept learning in DLs using Setting 1. Their method uses models of KB and bisimulations in those models to guide the search for the concept to be learned.

In [56] Tran et al. developed a bisimulation-based method, called BBCL2, for concept learning in DLs using Setting 2. Their method is based on the dual-BBCL method [26]. They made appropriate changes for dealing with the condition "$KB \not\models C(a)$ for all $a \in E^-$" instead of "$KB \models \neg C(a)$ for all $a \in E^-$".

The concept learning methods BBCL, dual-BBCL and BBCL2 are formulated for the DLs $\mathcal{L}_\Phi$ with $\mathcal{L} = \mathcal{ALC}$, $\Phi \subseteq \{F, I, N, O, Q, U, \mathtt{Self}\}$, (discrete and numeric) attributes and data roles. These DLs disallow the PDL-like role constructors, but it still covers a large class of DLs and well-known DLs like $\mathcal{ALC}$, $\mathcal{SHIQ}$, $\mathcal{SHOIQ}$, $\mathcal{SROIQ}$. We refer the reader to [26, 56] for illustrative examples about BBCL, dual-BBCL and BBCL2.

6.2 C-Learnability in Description Logics

In this section, we prove that any concept in any description logic that extends the basic DL $\mathcal{ALC}$ with some features amongst I (inverse), Q_k (qualified number restrictions with numbers bounded by a constant k), $\mathtt{Self}$ (local reflexivity of a role) can be learned if the training information system (specified as an interpretation) is good enough. That is, there exists a learning algorithm such that, for every concept C of those logics, there exists a training information system consistent with C such that applying the learning algorithm to the system results in a concept equivalent to C. We call this property *C-learnability* (possibility of correct learning). Our work uses Setting 3.

Our investigation uses bounded bisimulation in DLs and a new version of the algorithms proposed in the works [48, 57] that minimizes modal depths of resulting concepts. It shows a good property of the bisimulation-based concept learning methods proposed in [48, 57, 26, 56].

In this section, we only consider the DL-features I, Q_k and $\mathtt{Self}$.

Definition 6.2 (**The $\mathcal{L}_{\Sigma,\Phi}$ Language**).
Let Σ be a DL-signature and Φ be a set of DL-features. Let $\mathcal{L}$ stand for $\mathcal{ALC}$, which

is the name of a basic DL. The DL language $\mathcal{L}_{\Sigma,\Phi}$ allows *roles* and *concepts* defined recursively as follows:

- if $r \in \Sigma_R$ then r is role of $\mathcal{L}_{\Sigma,\Phi}$
- if $I \in \Phi$ then r^- is a role of $\mathcal{L}_{\Sigma,\Phi}$
- if $A \in \Sigma_C$ then A is concept of $\mathcal{L}_{\Sigma,\Phi}$
- if C and D are concepts of $\mathcal{L}_{\Sigma,\Phi}$, R is a role of $\mathcal{L}_{\Sigma,\Phi}$, $r \in \Sigma_R$, and h, k are natural numbers then
  - $\top$, $\bot$, $\neg C$, $C \sqcap D$, $C \sqcup D$, $\forall R.C$ and $\exists R.C$ are concepts of $\mathcal{L}_{\Sigma,\Phi}$
  - if $Q_k \in \Phi$ and $h \leq k$ then $\geq h\, R.C$ and $< h\, R.C$ are concepts of $\mathcal{L}_{\Sigma,\Phi}$ (we use $< h\, R.C$ instead of $\leq h\, R.C$ because it is more "dual" to $\geq h\, R.C$)
  - if $\mathtt{Self} \in \Phi$ then $\exists r.\mathtt{Self}$ is a concept of $\mathcal{L}_{\Sigma,\Phi}$. ■

Definition 6.3. An *information system* over Σ is defined to be a finite interpretation over Σ. ■

Definition 6.4 (Modal Depth).
The *modal depth* of a concept C, denoted by $\mathsf{mdepth}(C)$, is defined to be:

- 0 if C is of the form $\top$, $\bot$, A or $\exists r.\mathtt{Self}$,
- $\mathsf{mdepth}(D)$ if C is of the form $\neg D$,
- $\max(\mathsf{mdepth}(D), \mathsf{mdepth}(D'))$ if C is of the form $D \sqcap D'$ or $D \sqcup D'$,
- $\mathsf{mdepth}(D) + 1$ if C is of the form $\forall R.D$, $\exists R.D$, $\geq h\, R.C$ or $< h\, R.C$. ■

For example, $\mathsf{mdepth}(\exists r.(\forall s^-.(A \sqcup \exists r.\mathtt{Self}) \sqcap \exists s.(\neg A))) = 2$.

Definition 6.5 (The $\mathcal{L}_{\Sigma,\Phi,d}$ Language).
Let d denote a natural number. By $\mathcal{L}_{\Sigma,\Phi,d}$ we denote the sublanguage of $\mathcal{L}_{\Sigma,\Phi}$ that consists of concepts with modal depth not greater than d. ■

6.2.1 Concept Normalization

There are different normal forms for formulas or concepts (see, e.g., [45]). We provide below such a form. The aim is to introduce the notion of *universal interpretation* and a lemma about its existence. Our normal form uses the following normalization rules:

- Replace $\forall R.C$ by $\neg\exists R.\neg C$. Replace $< h\, R.C$ by $\neg \geq h\, R.C$.
- Replace $\geq 0\, R.C$ by $\top$ and replace $\geq 1\, R.C$ by $\exists R.C$.
- Push $\neg$ in depth through $\top$, $\bot$, $\neg$, $\sqcap$, $\sqcup$ according to De Morgan's laws.
- Represent $C_1 \sqcap \ldots \sqcap C_n$ as an "and"-set $\sqcap\{C_1, \ldots, C_n\}$ to make the order inessential and eliminate duplicates. Use a dual rule for $\sqcup$ and "or"-sets.

- Flatten an "and"-set $\sqcap\{\sqcap\{C_1, \ldots, C_i\}, C_{i+1}, \ldots, C_n\}$ to $\sqcap\{C_1, \ldots, C_n\}$. Replace $\sqcap\{C\}$ by C. Replace $\sqcap\{\top, C_1, \ldots, C_n\}$ by $\sqcap\{C_1, \ldots, C_n\}$. Replace $\sqcap\{\bot, C_1, \ldots, C_n\}$ by $\bot$. Use dual rules for "or"-sets.
- Replace $\exists R. \sqcup \{C_1, \ldots, C_n\}$ by $\sqcup\{\exists R.C_1, \ldots, \exists R.C_n\}$.
- Replace $\geq h\, R. \sqcup \{C_1, \ldots, C_n\}$ by the disjunction (using $\sqcup$) of all concepts of the form $\sqcap\{\geq h_1 R.C_1, \ldots, \geq h_n R.C_n\}$, where $h_1, \ldots, h_n$ are natural numbers such that $h_1 + \cdots + h_n = h$.
- Distribute $\sqcap$ over $\sqcup$.

Definition 6.6 (**DEG Normal Form).**
A concept is said to be in the *DEG normal form* (in short, *DEGNF*)[1] if it cannot be changed by any one of the above rules. ■

The following two lemmas can easily be proved.

Lemma 6.3. *Every concept can be translated to the DEG normal form. If C' is the DEG normal form of C then they are equivalent. A concept in the DEG normal form may contain $\sqcup$ only at the most outer level (i.e., either it does not contain $\sqcup$ or it must be of the form $\sqcup\{C_1, \ldots, C_n\}$, where $C_1, \ldots, C_n$ do not contain $\sqcup$).*

Lemma 6.4. *$\mathcal{L}_{\Sigma,\Phi,d}$ has only finitely many concepts in the DEG normal form. All of them can effectively be constructed.*

In the case $\Phi = \{I, Q_k, \mathtt{Self}\}$, $|\Sigma_C| = m$ and $|\Sigma_R| = n$, an upper bound $T(d)$ for the number of concepts in the DEG normal form of $\mathcal{L}_{\Sigma,\Phi,d}$ can be estimated as follows:

$$\begin{array}{rcl} T'(0) & = & 2^{2m+2n+2} \\ T'(l+1) & = & 2^{4k.n.T'(l)+2m+2n+2} \text{ for } l \geq 0 \\ T(d) & = & 2^{T'(d)}, \end{array}$$

where $T'(l)$ is an upper bound for the number of concepts in the DEG normal form of $\mathcal{L}_{\Sigma,\Phi,d}$ that do not use $\sqcup$ and have a modal depth not greater than l.

Definition 6.7 (**Universal Interpretation).**
We say that an interpretation $\mathcal{I}$ over Σ is *universal* w.r.t. a sublanguage of $\mathcal{L}_{\Sigma,\Phi}$ if, for every satisfiable concept C of that sublanguage, $C^{\mathcal{I}} \neq \emptyset$. ■

Lemma 6.5. *There exists a finite universal interpretation w.r.t. $\mathcal{L}_{\Sigma,\Phi,d}$, which can effectively be constructed.*

Proof. Let $C_1, \ldots, C_n$ be all the satisfiable concepts in the DEG normal form of $\mathcal{L}_{\Sigma,\Phi,d}$. (By Lemma 6.4, the number of such concepts is finite.) For each $1 \leq i \leq n$, let $\mathcal{I}_i$ be a finite model satisfying C_i, which can effectively be constructed using some tableau algorithm (e.g., [29, 47]).[2] Without loss of generality we assume that these

[1] DEGNF stands for disjunctive-existential-greater-or-equal normal form.
[2] As RBoxes and TBoxes are not considered, $\mathcal{L}_{\Sigma,\Phi,d}$ has the finite model property.

interpretations have pairwise disjoint domains. Let $\mathcal{I}$ be any interpretation such that: $\Delta^{\mathcal{I}} = \Delta^{\mathcal{I}_1} \cup \ldots \cup \Delta^{\mathcal{I}_n}$; for $A \in \Sigma_C$, $A^{\mathcal{I}} = A^{\mathcal{I}_1} \cup \ldots \cup A^{\mathcal{I}_n}$; for $r \in \Sigma_R$, $r^{\mathcal{I}} = r^{\mathcal{I}_1} \cup \ldots \cup r^{\mathcal{I}_n}$ (individual names can be interpreted arbitrarily). It is easy to see that $\mathcal{I}$ is finite and universal w.r.t. $\mathcal{L}_{\Sigma,\Phi,d}$. ■

6.2.2 Bounded Bisimulation for Description Logics

Let d be a natural number and let

- Σ and $\Sigma^\dagger$ be DL-signatures such that $\Sigma^\dagger \subseteq \Sigma$
- Φ and $\Phi^\dagger$ be sets of DL-features such that $\Phi^\dagger \subseteq \Phi$
- $\mathcal{I}$ and $\mathcal{I}'$ be interpretations over Σ.

Definition 6.8 (Bounded Bisimulation).
A binary relation $Z_d \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}$ is called an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$*-bisimulation* between $\mathcal{I}$ and $\mathcal{I}'$ if there exists a sequence of binary relations $Z_d \subseteq \cdots \subseteq Z_0 \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}'}$ such that the following conditions hold for every $0 \leq i \leq d$, $0 \leq j < d$, $a \in \Sigma_I^\dagger$, $A \in \Sigma_C^\dagger$, $x, y \in \Delta^{\mathcal{I}}$, $x', y' \in \Delta^{\mathcal{I}'}$ and every role R of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$:

$$Z_i(a^{\mathcal{I}}, a^{\mathcal{I}'}) \tag{6.1}$$
$$Z_0(x, x') \Rightarrow [A^{\mathcal{I}}(x) \Leftrightarrow A^{\mathcal{I}'}(x')] \tag{6.2}$$
$$[Z_{j+1}(x, x') \wedge R^{\mathcal{I}}(x, y)] \Rightarrow \exists y' \in \Delta^{\mathcal{I}'}[Z_j(y, y') \wedge R^{\mathcal{I}'}(x', y')] \tag{6.3}$$
$$[Z_{j+1}(x, x') \wedge R^{\mathcal{I}'}(x', y')] \Rightarrow \exists y \in \Delta^{\mathcal{I}}[Z_j(y, y') \wedge R^{\mathcal{I}}(x, y)], \tag{6.4}$$

if $Q_k \in \Phi^\dagger$ and $1 \leq h \leq k$ then

if $Z_{j+1}(x, x')$ holds and $y_1, \ldots, y_h$ are pairwise different elements of $\Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y_l)$ holds for every $1 \leq l \leq h$ then there exist pairwise different elements $y'_1, \ldots, y'_h$ of $\Delta^{\mathcal{I}'}$ such that $R^{\mathcal{I}'}(x', y'_l)$ and $Z_j(y_l, y'_l)$ hold for every $1 \leq l \leq h$ (6.5)

if $Z_{j+1}(x, x')$ holds and $y'_1, \ldots, y'_h$ are pairwise different elements of $\Delta^{\mathcal{I}'}$ such that $R^{\mathcal{I}'}(x', y'_l)$ holds for every $1 \leq l \leq h$ then there exist pairwise different elements $y_1, \ldots, y_h$ of $\Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y_l)$ and $Z_j(y_l, y'_l)$ hold for every $1 \leq l \leq h$, (6.6)

if `Self` $\in \Phi^\dagger$ then

$$Z_0(x, x') \Rightarrow [r^{\mathcal{I}}(x, x) \Leftrightarrow r^{\mathcal{I}'}(x', x')]. \tag{6.7}$$

■

Lemma 6.6. *Let $\mathcal{I}$, $\mathcal{I}'$ and $\mathcal{I}''$ be interpretations.*

1. *The relation $\{\langle x, x\rangle \mid x \in \Delta^{\mathcal{I}}\}$ is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}$.*

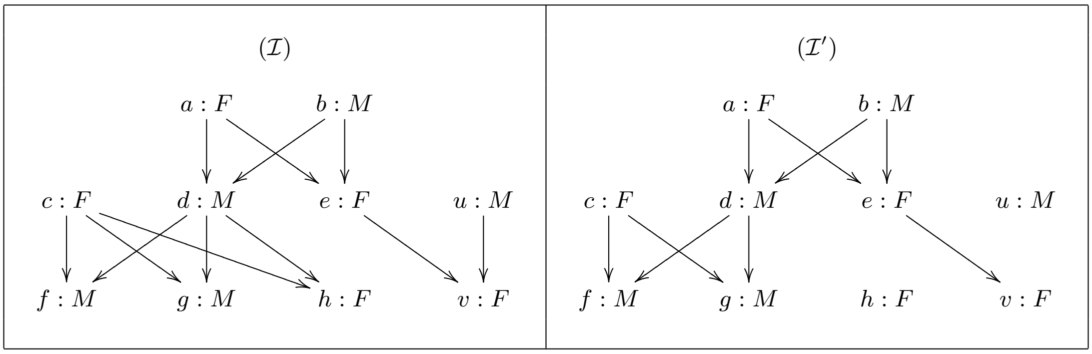


Figure 6.2: An illustration for Example 6.9.

2. *If Z is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ then Z^{-1} is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}'$ and $\mathcal{I}$.*

3. *If Z_1 is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$, and Z_2 is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}'$ and $\mathcal{I}''$, then $Z_1 \circ Z_2$ is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}''$.*

4. *If $\mathcal{Z}$ is a set of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulations between $\mathcal{I}$ and $\mathcal{I}'$ then $\bigcup \mathcal{Z}$ is also an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.*

The proof of this lemma is straightforward.

An interpretation $\mathcal{I}$ is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$*-bisimilar* to $\mathcal{I}'$ if there exists an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between them. By Lemma 6.6, this $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimilarity relation is an equivalence relation between interpretations. We say that $x \in \Delta^{\mathcal{I}}$ is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$*-bisimilar* to $x' \in \Delta^{\mathcal{I}'}$ if there exists an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation Z_d between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z_d(x, x')$ holds. This latter $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimilarity relation is also an equivalence relation (between elements of interpretations' domains).

Example 6.9. Let Σ be the signature and $\mathcal{I}$ be the interpretation specified in Example 2.9. This interpretation is illustrated in Figure 6.2 together with its modification $\mathcal{I}'$, which differs from $\mathcal{I}$ in that: $hasChild^{\mathcal{I}'}$ consists of only the elements illustrated by the edges shown at the right hand side of Figure 6.2 and $hasParent^{\mathcal{I}'}$, $Mother^{\mathcal{I}'}$, $Father^{\mathcal{I}'}$ are defined accordingly.

Let $\Phi = \{I, Q_2, Q_3, \mathtt{Self}\}$ (we add Q_2 to Φ just for convenience), $\Sigma_I^\dagger = \Sigma_I$, $\Sigma_C^\dagger = \{Male\}$ and $\Sigma_R^\dagger = \{hasChild\}$. Consider the following cases.

- The interpretations $\mathcal{I}$ and $\mathcal{I}'$ are $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,0}$-bisimilar (w.r.t. any $\Phi^\dagger \subseteq \Phi$).

- Case $\Phi^\dagger \subseteq \{Q_2, \mathtt{Self}\}$: $\mathcal{I}$ and $\mathcal{I}'$ are $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimilar (w.r.t. any d).

- Case $\Phi^\dagger \supseteq \{I\}$ and $d \geq 1$: $\mathcal{I}$ and $\mathcal{I}'$ are not $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimilar because $Helen^{\mathcal{I}}$ (h in $\mathcal{I}$) is not $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimilar to $Helen^{\mathcal{I}'}$ (h in $\mathcal{I}'$).

- Case $\Phi^\dagger \supseteq \{Q_3\}$ and $d \geq 1$: $\mathcal{I}$ and $\mathcal{I}'$ are not $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimilar because $Claudia^{\mathcal{I}}$ (c in $\mathcal{I}$) is not $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimilar to $Claudia^{\mathcal{I}'}$ (c in $\mathcal{I}'$).

If $\Sigma_I^\dagger = \{Alice, Bob\}$ and $\Phi^\dagger \subseteq \Phi$ then $\mathcal{I}$ and $\mathcal{I}'$ are $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,1}$-bisimilar. ■

Lemma 6.7. *Let Z_d be an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between interpretations $\mathcal{I}$ and $\mathcal{I}'$. Then, for every $x \in \Delta^{\mathcal{I}}$, every $x' \in \Delta^{\mathcal{I}'}$ and every concept C of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$, it holds that*

$$Z_d(x, x') \Rightarrow [C^{\mathcal{I}}(x) \Leftrightarrow C^{\mathcal{I}'}(x')].$$

Proof. We prove this lemma by induction on the structure of C. Let $x \in \Delta^{\mathcal{I}}$, $x' \in \Delta^{\mathcal{I}'}$, C be a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ and suppose $Z_d(x, x')$ and $C^{\mathcal{I}}(x)$ hold. We show that $C^{\mathcal{I}'}(x')$ also holds.

- The cases when C is of the form $\top$, $\bot$, A, $\neg D$, $D \sqcup D'$ or $D \sqcap D'$ are trivial.
- Case $C = \exists R.D$, where R is a role of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ and D is a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d-1}$: Since $C^{\mathcal{I}}(x)$ holds, there exists $y \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y)$ and $D^{\mathcal{I}}(y)$ hold. By the assertion (6.3), there exists $y' \in \Delta^{\mathcal{I}'}$ such that $Z_{d-1}(y, y')$ and $R^{\mathcal{I}'}(x', y')$ hold. By the induction assumption, it follows that $D^{\mathcal{I}'}(y')$ holds. Since $R^{\mathcal{I}'}(x', y')$ and $D^{\mathcal{I}'}(y')$ hold, it follows that $C^{\mathcal{I}'}(x')$ holds.
- Case $C = \forall R.D$, where R is a role of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ and D is a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d-1}$, is reduced to the above case, treating $\forall R.D$ as $\neg\exists R.\neg D$.
- Case $Q_k \in \Phi^\dagger$ and $C = (\geq h\, R.D)$, where $0 \leq h \leq k$, R is a role of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ and D is a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d-1}$: Since $C^{\mathcal{I}}(x)$ holds, there exist pairwise different $y_1, \ldots, y_h \in \Delta^{\mathcal{I}}$ such that $R^{\mathcal{I}}(x, y_i)$ and $D^{\mathcal{I}}(y_i)$ hold for all $1 \leq i \leq h$. Since $Z_d(x, x')$ holds, by the assertion (6.5), there exist pairwise different $y'_1, \ldots, y'_h \in \Delta^{\mathcal{I}'}$ such that $R^{\mathcal{I}'}(x', y'_i)$ and $Z_{d-1}(y_i, y'_i)$ hold for all $1 \leq i \leq h$. Since $Z_{d-1}(y_i, y'_i)$ and $D^{\mathcal{I}}(y_i)$ hold for every $1 \leq i \leq h$, by the induction assumption, it follows that $D^{\mathcal{I}'}(y'_i)$ holds for every $1 \leq i \leq h$. Therefore, $C^{\mathcal{I}'}(x')$ holds.
- Case $Q_k \in \Phi$ and $C = (< h\, R.D)$, where $0 \leq h \leq k$, R is a role of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ and D is a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d-1}$: This case is reduced to the above case, treating $< h\, R.D$ as $\neg(\geq h\, R.D)$.
- Case $\mathtt{Self} \in \Phi^\dagger$ and $C = \exists r.\mathtt{Self}$: Since $C^{\mathcal{I}}(x)$ holds, we have that $r^{\mathcal{I}}(x, x)$ holds. Since $Z_d(x, x')$ holds and $Z_d \subseteq Z_0$, it follows that $Z_0(x, x')$ holds. By the assertion (6.7), we have that $r^{\mathcal{I}'}(x', x')$ holds. Hence $C^{\mathcal{I}'}(x')$ holds. ■

Definition 6.10. An interpretation $\mathcal{I}$ over Σ is *finitely branching* (or *image-finite*) w.r.t. $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ and $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ if, for every $x \in \Delta^{\mathcal{I}}$ and every role R of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$, the set $\{y \in \Delta^{\mathcal{I}} \mid R^{\mathcal{I}}(x, y)\}$ is finite. ■

Definition 6.11. Let $x \in \Delta^{\mathcal{I}}$ and $x' \in \Delta^{\mathcal{I}'}$. We say that x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$*-equivalent* to x' if, for every concept C of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$, $x \in C^{\mathcal{I}}$ iff $x' \in C^{\mathcal{I}'}$. ■

Theorem 6.8 (The Hennessy-Milner Property)**.** *Let d be a natural number, Σ and $\Sigma^\dagger$ be DL-signatures such that $\Sigma^\dagger \subseteq \Sigma$, Φ and $\Phi^\dagger$ be sets of DL-features such that $\Phi^\dagger \subseteq \Phi$. Let $\mathcal{I}$ and $\mathcal{I}'$ be interpretations in $\mathcal{L}_{\Sigma,\Phi}$, finitely branching w.r.t. $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ and such that, for every $a \in \Sigma_I^\dagger$, $a^\mathcal{I}$ is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-equivalent to $a^{\mathcal{I}'}$. Then $x \in \Delta^\mathcal{I}$ is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-equivalent to $x' \in \Delta^{\mathcal{I}'}$ iff there exists an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation Z_d between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z_d(x,x')$ holds. In particular, the relation $\{\langle x,x'\rangle \in \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'} \mid x$ is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-equivalent to $x'\}$ is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.*

Proof. Consider the "$\Leftarrow$" direction. Suppose Z_d is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$ such that $Z_d(x,x')$ holds. By Lemma 6.7, for every concept C in $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$, $C^\mathcal{I}(x)$ holds iff $C^{\mathcal{I}'}(x')$ holds. Therefore, x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-equivalent to x'.

Now, consider the "$\Rightarrow$" direction. Define $Z_j = \{\langle x,x'\rangle \in \Delta^\mathcal{I} \times \Delta^{\mathcal{I}'} \mid x$ is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j}$-equivalent to $x'\}$ for every $1 \leq j \leq d$. We show that Z_d is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}$ and $\mathcal{I}'$.

- The condition (6.1) follows from the assumption of the theorem.
- Consider the condition (6.2) and suppose $Z_0(x,x')$ holds. By the definition of Z_0, it follows that x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,0}$-equivalent to x'. Therefore, for every concept name A, $A^\mathcal{I}(x)$ holds iff $A^{\mathcal{I}'}(x')$ holds.
- Consider the condition (6.3) and suppose $Z_{j+1}(x,x')$ and $R^\mathcal{I}(x,y)$ hold. Thus, x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j+1}$-equivalent to x'. Let $S = \{y' \in \Delta^{\mathcal{I}'} \mid R^{\mathcal{I}'}(x',y')\}$. We show that there exists $y' \in S$ such that $Z_j(y,y')$ holds. Since $x \in (\exists R.\top)^\mathcal{I}$ and x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j+1}$-equivalent to x', we have that $x' \in (\exists R.\top)^{\mathcal{I}'}$. Hence $S \neq \emptyset$. Since $\mathcal{I}'$ is finitely branching, S must be finite. Let the elements of S be $y'_1, \ldots, y'_n$. For the sake of contradiction, suppose that, for every $1 \leq i \leq n$, $Z_j(y,y'_i)$ does not hold, which means that y is not $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j}$-equivalent to y'_i. Thus, for every $1 \leq i \leq n$, there exists a concept C_i in $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j}$ such that $C_i^\mathcal{I}(y)$ holds, but $C_i^{\mathcal{I}'}(y'_i)$ does not. Let $C = \exists R.(C_1 \sqcap \ldots \sqcap C_n)$. Thus, C is a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j+1}$ and $C^\mathcal{I}(x)$ holds, but $C^{\mathcal{I}'}(x')$ does not, which contradicts the fact that x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j+1}$-equivalent to x'. Therefore, there exists $y'_i \in S$ such that $Z_j(y,y'_i)$ holds.
- The condition (6.4) can be proved analogously as for the condition (6.3).
- Consider the case $Q_k \in \Phi^\dagger$ and the conditions (6.5) and (6.6). Suppose $Z_{j+1}(x,x')$ holds. Thus, x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j+1}$-equivalent to x'. Let $S = \{y \in \Delta^\mathcal{I} \mid R^\mathcal{I}(x,y)\}$ and $S' = \{y' \in \Delta^{\mathcal{I}'} \mid R^{\mathcal{I}'}(x',y')\}$. Since $\mathcal{I}$ and $\mathcal{I}'$ are finitely branching, both S and S' are finite. Consider an arbitrary $y'' \in S \cup S'$ and let $y_1, \ldots, y_n \in S$ and $y'_1, \ldots, y'_{n'} \in S'$ be all the pairwise different elements that are $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j}$-equivalent to y''. To prove (6.5) and (6.6) it suffices to show that either $n = n'$ or ($n \geq k$ and $n' \geq k$). For the sake of contrary, assume that $n \neq n'$ and ($n < k$ or $n' < k$). Without loss of generality, also assume that $n < n'$. Thus, $n < k$ and $n+1 \leq k$. Let $\{t_1, \ldots, t_m\} = S \setminus \{y_1, \ldots, y_n\}$ and $\{t'_1, \ldots, t'_{m'}\} = S' \setminus \{y'_1, \ldots, y'_{n'}\}$. Let $\mathcal{I}'' = \mathcal{I}$ if $y'' \in S$, and let $\mathcal{I}'' = \mathcal{I}'$ otherwise. For each $1 \leq i \leq m$, there exists a concept D_i of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j}$ such that $y'' \in D_i^{\mathcal{I}''}$ but $t_i \notin D_i^\mathcal{I}$. Similarly, for each

$1 \le i \le m'$, there exists a concept D'_i of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j}$ such that $y'' \in (D'_i)^{\mathcal{I}''}$ but $t'_i \notin (D'_i)^{\mathcal{I}'}$. Let $D = (D_1 \sqcap \ldots \sqcap D_m \sqcap D'_1 \sqcap \ldots \sqcap D'_{m'})$. We have that $\{y_1, \ldots, y_n\} \subseteq D^{\mathcal{I}}$ (since $y'' \in D^{\mathcal{I}''}$) and $\{t_1, \ldots, t_m\} \cap D^{\mathcal{I}} = \emptyset$. Similarly, $\{y'_1, \ldots, y'_{n'}\} \subseteq D^{\mathcal{I}'}$ and $\{t'_1, \ldots, t'_{m'}\} \cap D^{\mathcal{I}'} = \emptyset$. Observe that $C = (\geq (n+1)\, R.D)$ is a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j+1}$, $C^{\mathcal{I}'}(x')$ holds, but $C^{\mathcal{I}}(x)$ does not. This contradicts the fact that x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,j+1}$-equivalent to x'.

- Consider the case $\mathtt{Self} \in \Phi^\dagger$ and the assertion (6.7). Suppose $Z_0(x, x')$ holds. Thus, x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,0}$-equivalent to x'. Let $C = \exists r.\mathtt{Self}$. Since $\mathsf{mdepth}(C) = 0$, it follows that $C^{\mathcal{I}}(x)$ holds iff $C^{\mathcal{I}'}(x')$ holds, which means $x \in (\exists r.\mathtt{Self})^{\mathcal{I}}$ iff $x' \in (\exists r.\mathtt{Self})^{\mathcal{I}'}$. Therefore, $r^{\mathcal{I}}(x, x)$ holds iff $r^{\mathcal{I}'}(x', x')$ holds. ■

Definition 6.12. An $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-bisimulation between $\mathcal{I}$ and itself is called an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-*auto-bisimulation of* $\mathcal{I}$. An $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-auto-bisimulation of $\mathcal{I}$ is said to be the *largest* if it is larger than or equal to ($\supseteq$) any other $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-auto-bisimulation of $\mathcal{I}$. ■

Definition 6.13. Given an interpretation $\mathcal{I}$ over Σ, by $\sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ we denote the largest $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-auto-bisimulation of $\mathcal{I}$, and by $\equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ we denote the binary relation on $\Delta^{\mathcal{I}}$ with the property that $x \equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}} x'$ iff x is $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-equivalent to x'. ■

Theorem 6.9. *Let d be a natural number, Σ and $\Sigma^\dagger$ be DL-signatures such that $\Sigma^\dagger \subseteq \Sigma$, Φ and $\Phi^\dagger$ be sets of DL-features such that $\Phi^\dagger \subseteq \Phi$, and $\mathcal{I}$ be an interpretation over Σ. Then the largest $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-auto-bisimulation of $\mathcal{I}$ exists and is an equivalence relation. Furthermore, if $\mathcal{I}$ is finitely branching w.r.t. $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ then the relation $\equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ is the largest $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-auto-bisimulation of $\mathcal{I}$ (i.e. the relations $\equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ and $\sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ coincide).*

Proof. It follows from Lemma 6.6 that the largest $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-auto-bisimulation of $\mathcal{I}$ exists and is an equivalence relation. Assume that $\mathcal{I}$ is finitely branching w.r.t. $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$. By Theorem 6.8, the relation $\equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-auto-bisimulation of $\mathcal{I}$. It remains to show that this $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-auto-bisimulation is the largest one. Suppose Z_d is another $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-auto-bisimulation of $\mathcal{I}$. If $Z_d(x, x')$ holds then, by Lemma 6.7, for every concept C of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$, $C^{\mathcal{I}}(x)$ holds iff $C^{\mathcal{I}'}(x')$ holds, and hence $x \equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}} x'$. Therefore, $Z_d \subseteq \equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$. ■

The following theorem differs from the ones given in [48, 57, 26] in that the considered languages are different.

Theorem 6.10. *Let d be a natural number, Σ and $\Sigma^\dagger$ be DL-signatures such that $\Sigma^\dagger \subseteq \Sigma$, Φ and $\Phi^\dagger$ be sets of DL-features such that $\Phi^\dagger \subseteq \Phi$, $\mathcal{I}$ be a finitely branching interpretation w.r.t. $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$, and let $X \subseteq \Delta^{\mathcal{I}}$. Then:*

1. *if there exists a concept C of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ such that $X = C^{\mathcal{I}}$ then the partition of $\Delta^{\mathcal{I}}$ by $\sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ is consistent with X*

2. *if the partition of $\Delta^{\mathcal{I}}$ by $\sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ is consistent with X then there exists a concept C of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ such that $C^{\mathcal{I}} = X$.*

Proof. By Theorem 6.9, $\sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ coincides with $\equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$.

Consider the first assertion and assume that $X = C^{\mathcal{I}}$ for some concept C of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$. Let Y be any element of the partition of $\Delta^{\mathcal{I}}$ by $\sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ such that $X \cap Y \neq \emptyset$. It suffices to show that $Y \subseteq X$. Let x be an arbitrary element of Y. Since $X \cap Y \neq \emptyset$, there exists $x' \in X \cap Y$. Since both x and x' belong to Y, $x' \sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}} x$. Since $\sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ coincides with $\equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$, we also have that $x' \equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}} x$. Since $x' \in X$ and $X = C^{\mathcal{I}}$, $C^{\mathcal{I}}(x')$ holds, which together with $x' \equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}} x$ implies that $C^{\mathcal{I}}(x)$ holds. Thus, $x \in X$ and we can conclude that $Y \subseteq X$.

Consider the second assertion and assume that the partition of $\Delta^{\mathcal{I}}$ by $\sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ is consistent with X. Let that partition be $\{Y_1, ..., Y_m, Y'_1, ..., Y'_n\}$, where $Y_i \subseteq X$ for all $1 \leq i \leq m$ and $Y'_j \cap X = \emptyset$ for all $1 \leq j \leq n$. We have that $X = Y_1 \cup \ldots \cup Y_m$. For each $1 \leq i \leq m$ and $1 \leq j \leq n$, since Y_i and Y'_j are different equivalence classes of $\equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ (the same as $\sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$), there exists a concept $C_{i,j}$ of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ such that $Y_i \subseteq C_{i,j}^{\mathcal{I}}$ and $Y'_j \cap C_{i,j}^{\mathcal{I}} = \emptyset$. For each $1 \leq i \leq m$, let $C_i = C_{i,1} \sqcap \ldots \sqcap C_{i,n}$. Thus, $Y_i \subseteq C_i^{\mathcal{I}}$ and $Y'_j \cap C_i^{\mathcal{I}} = \emptyset$ for all $1 \leq j \leq n$. Let $C = C_1 \sqcup \ldots \sqcup C_m$. Thus, $Y_i \subseteq C^{\mathcal{I}}$ for all $1 \leq i \leq m$ and $Y'_j \cap C^{\mathcal{I}} = \emptyset$ for all $1 \leq j \leq n$. Therefore $C^{\mathcal{I}} = X$. ∎

6.2.3 A Concept Learning Algorithm

Let $A_0 \in \Sigma_C$ be a concept name standing for the "decision attribute" and suppose that A_0 can be expressed by a concept C in $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$, where $\Sigma^\dagger \subseteq \Sigma \setminus \{A_0\}$ and $\Phi^\dagger \subseteq \Phi$. Let $\mathcal{I}$ be a training information system over Σ. How can we learn that concept C on the basis of $\mathcal{I}$? Nguyen and Szałas [48] gave a bisimulation-based method for this learning problem. In this section, by adopting a specific strategy we present a modified version of that method, called the *MiMoD* (minimizing-modal-depth) concept learning algorithm. This algorithm is used for analyzing C-learnability in the next section.

Our MiMoD algorithm is as follows:

1. Starting from the partition $\{\Delta^{\mathcal{I}}\}$, make subsequent granulations to reach a partition consistent with $A_0^{\mathcal{I}}$. In the granulation process, we denote the blocks created so far in all steps by $Y_1, \ldots, Y_n$, where the current partition may consist of only some of them. We do not use the same subscript to denote blocks of different contents (i.e. we always use new subscripts obtained by increasing n for new blocks). We take care that, for each $1 \leq i \leq n$, Y_i is characterized by a concept C_i such that $Y_i = C_i^{\mathcal{I}}$.

2. We use the following concepts as *selectors* for the granulation process, where $1 \leq i \leq n$:

   (a) A, where $A \in \Sigma_C^\dagger$

   (b) $\exists r.\mathtt{Self}$, if $\mathtt{Self} \in \Phi^\dagger$ and $r \in \Sigma_R^\dagger$

   (c) $\exists r.C_i$, where $r \in \Sigma_R^\dagger$

   (d) $\exists r^-.C_i$, if $I \in \Phi^\dagger$ and $r \in \Sigma_R^\dagger$

   (e) $\geq h\, r.C_i$, if $Q_k \in \Phi^\dagger$, $r \in \Sigma_R^\dagger$ and $1 \leq h \leq k$

(f) $\geq h\, r^{-}.C_i$, if $\{Q_k, I\} \subseteq \Phi^\dagger$, $r \in \Sigma_R^\dagger$ and $1 \leq h \leq k$.

A selector D has a *higher priority* than D' if $\mathsf{mdepth}(D) < \mathsf{mdepth}(D')$.

3. During the granulation process, if

   - a block Y_i of the current partition is split by $D^{\mathcal{I}}$, where D is a selector,
   - and there do not exist a block Y_j of the current partition and a selector D' with a higher priority than D such that Y_j is split by D'

   then split Y_i by D as follows:

   - $s := n+1,\ \ t := n+2,\ \ n := n+2$
   - $Y_s := Y_i \cap D^{\mathcal{I}},\ \ C_s := C_i \sqcap D$
   - $Y_t := Y_i \cap (\neg D)^{\mathcal{I}},\ \ C_t := C_i \sqcap \neg D$
   - replace Y_i in the current partition by Y_s and Y_t.

4. When the current partition becomes consistent with $A_0^{\mathcal{I}}$, return $C_{i_1} \sqcup \ldots \sqcup C_{i_j}$, where $i_1, \ldots, i_j$ are indices such that $Y_{i_1}, \ldots, Y_{i_j}$ are all the blocks of the current partition that are subsets of $A_0^{\mathcal{I}}$.

Observe that the above algorithm always terminates.

Example 6.14. Consider the information system $\mathcal{I}$ given in Example 2.9. Let $\Sigma^\dagger = \{\mathit{Male}, \mathit{hasChild}\}$ and $\Phi^\dagger = \emptyset$. We want to apply the MiMoD algorithm to learn a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ that describes the concept *Father*. Recall that $\mathit{Father}^{\mathcal{I}} = \{b, d, u\}$. One of possible runs of the algorithm is as follows:

1. $Y_1 := \Delta^{\mathcal{I}}$, $C_1 = \top$, $\mathit{partition} := \{Y_1\}$,

2. splitting Y_1 by *Male*:

   - $Y_2 := \{b, d, f, g, u\}$, $C_2 := \mathit{Male}$,
   - $Y_3 := \{a, c, e, h, v\}$, $C_3 := \neg \mathit{Male}$,
   - $\mathit{partition} := \{Y_2, Y_3\}$,

3. splitting Y_2 by $\exists \mathit{hasChild}.\top$:

   - $Y_4 := \{b, d, u\}$, $C_4 := C_2 \sqcap \exists \mathit{hasChild}.\top$,
   - $Y_5 := \{f, g\}$, $C_5 := C_2 \sqcap \neg\exists \mathit{hasChild}.\top$,
   - $\mathit{partition} := \{Y_3, Y_4, Y_5\}$.

The obtained partition is consistent with $\mathit{Father}^{\mathcal{I}}$, having $Y_4 = \mathit{Father}^{\mathcal{I}}$ and Y_3, Y_5 disjoint with $\mathit{Father}^{\mathcal{I}}$. The returned concept is $C_4 = \mathit{Male} \sqcap \exists \mathit{hasChild}.\top$. ■

Example 6.15. Consider once again the information system $\mathcal{I}$ given in Example 2.9. Now let $\Sigma^\dagger = \{Male, hasChild\}$, $\Phi^\dagger = \{Q_3\}$ and let A_0 be a new concept name interpreted in $\mathcal{I}$ as $A_0^\mathcal{I} = \{c, d\}$. We want to apply the MiMoD algorithm to learn a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ that describes A_0. One of possible runs of the algorithm has first two steps as in Example 6.14 and then continues as follows:

3. splitting Y_2 by $\geq 3\, hasChild.\top$:
   - $Y_4 := \{d\}$, $C_4 := C_2 \sqcap (\geq 3\, hasChild.\top)$,
   - $Y_5 := \{b, f, g, u\}$, $C_5 := C_2 \sqcap \neg(\geq 3\, hasChild.\top)$,
   - *partition* $:= \{Y_3, Y_4, Y_5\}$.
4. splitting Y_3 by $\geq 3\, hasChild.\top$:
   - $Y_6 := \{c\}$, $C_6 := C_3 \sqcap (\geq 3\, hasChild.\top)$,
   - $Y_7 := \{a, e, h, v\}$, $C_7 := C_3 \sqcap \neg(\geq 3\, hasChild.\top)$,
   - *partition* $:= \{Y_4, Y_5, Y_6, Y_7\}$.

The obtained partition is consistent with $A_0^\mathcal{I}$, having $Y_4 \subset A_0^\mathcal{I}$, $Y_6 \subset A_0^\mathcal{I}$, and Y_5, Y_7 disjoint with $A_0^\mathcal{I}$. The returned concept is

$$C_4 \sqcup C_6 = [Male \sqcap (\geq 3\, hasChild.\top)] \sqcup [\neg Male \sqcap (\geq 3\, hasChild.\top)]$$

which is equivalent to $\geq 3\, hasChild.\top$. ■

Lemma 6.11. *Let Σ and $\Sigma^\dagger$ be DL-signatures such that $\Sigma^\dagger \subseteq \Sigma$, Φ and $\Phi^\dagger$ be sets of DL-features such that $\Phi^\dagger \subseteq \Phi$, and $\mathcal{I}$ be a finite interpretation over Σ. Suppose $A_0 \in \Sigma_C \setminus \Sigma_C^\dagger$ and C is a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ such that $A_0^\mathcal{I} = C^\mathcal{I}$. Let C' be a concept returned by the MiMoD algorithm for $\mathcal{I}$. Then C' is a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ such that $C'^\mathcal{I} = C^\mathcal{I}$ and $\mathsf{mdepth}(C') \leq \mathsf{mdepth}(C)$.*

Proof. Clearly, $C'^\mathcal{I} = A_0^\mathcal{I} = C^\mathcal{I}$. Consider the execution of the MiMoD algorithm on $\mathcal{I}$ that results in C'. By $\mathcal{P}_d$ we denote the partition of $\Delta^\mathcal{I}$ at the moment in that execution when $\max\{\mathsf{mdepth}(C_i) \mid Y_i \in \mathcal{P}_d\} = d$ and $\mathcal{P}_d$ cannot be granulated any more without using some selector with modal depth $d + 1$. Let d_{max} be the maximal value of such an index d (of some $\mathcal{P}_d$). Let Z_d be the equivalence relation corresponding to the partition $\mathcal{P}_d$, i.e. $Z_d = \{\langle x, x'\rangle \mid x, x' \in Y_i \text{ for some } Y_i \in \mathcal{P}_d\}$. It is straightforward to prove by induction on d that Z_d is an $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$-auto-bisimulation of $\mathcal{I}$. Hence, $Z_d \subseteq \sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$. Since each block of $\mathcal{P}_d$ is characterized by a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$, Z_d is a superset of $\equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$. Since $\equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ and $\sim_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$ coincide (Theorem 6.9), we have that $Z_d = \equiv_{\Sigma^\dagger,\Phi^\dagger,d,\mathcal{I}}$.

Since the algorithm terminates as soon as the current partition is consistent with $C^\mathcal{I}$, it follows that $d_{max} \leq \mathsf{mdepth}(C)$. Furthermore, if $d_{max} < \mathsf{mdepth}(C')$ then we also have $d_{max} < \mathsf{mdepth}(C)$. Since $\mathsf{mdepth}(C') \leq d_{max} + 1$, we conclude that $\mathsf{mdepth}(C') \leq \mathsf{mdepth}(C)$. ■

6.2.4 C-Learnability in Description Logics

Theorem 6.12. *Let d be a natural number, Σ and $\Sigma^\dagger$ be DL-signatures such that $\Sigma^\dagger \subseteq \Sigma$, Φ and $\Phi^\dagger$ be sets of DL-features such that $\Phi^\dagger \subseteq \Phi$, and $\mathcal{I}$ be a finite universal interpretation w.r.t. $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$. Suppose $A_0 \in \Sigma_C \setminus \Sigma_C^\dagger$ and C is a concept of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ such that $A_0^{\mathcal{I}} = C^{\mathcal{I}}$. Then any concept returned by the MiMoD algorithm for $\mathcal{I}$ is equivalent to C.*

Proof. Let C' be a concept returned by the MiMoD algorithm for $\mathcal{I}$. By Lemma 6.11, $C'^{\mathcal{I}} = C^{\mathcal{I}}$ and $\mathsf{mdepth}(C') \leq \mathsf{mdepth}(C)$. For the sake of contradiction, suppose C' is not equivalent to C. Thus, either $C \sqcap \neg C'$ or $C' \sqcap \neg C$ is satisfiable. Both of them belong to $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$. Since $\mathcal{I}$ is universal w.r.t. $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$, it follows that either $(C \sqcap \neg C')^{\mathcal{I}}$ or $(C' \sqcap \neg C)^{\mathcal{I}}$ is not empty, which contradicts the fact that $C'^{\mathcal{I}} = C^{\mathcal{I}}$. ∎

Theorem 6.13. *Any concept C in any description logic that extends $\mathcal{ALC}$ with some features amongst I, Q_k, `Self` can be learned if the training information system is good enough.*

Proof. Let the considered logic be $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ and let $d = \mathsf{mdepth}(C)$, $\Phi = \Phi^\dagger$ and $\Sigma = \Sigma^\dagger \cup \{A_0\}$, where $A_0 \notin \Sigma_C^\dagger$. By Lemma 6.5, there exists a finite universal interpretation $\mathcal{I}'$ w.r.t. $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$. Let $\mathcal{I}$ be the interpretation over Σ different from $\mathcal{I}'$ only in that $A_0^{\mathcal{I}}$ is defined to be $C^{\mathcal{I}'}$. Clearly, $\mathcal{I}$ is universal w.r.t. $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ and $A_0^{\mathcal{I}} = C^{\mathcal{I}}$. By Theorem 6.12, any concept returned by the MiMoD algorithm for $\mathcal{I}$ is equivalent to C. ∎

Assuming that the language $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger}$ is fixed, the MiMoD algorithm in the above two theorems for learning a concept C does not depend on C (nor the modal depth of C). Furthermore, the training information system $\mathcal{I}$ used for learning C depends on C only via its modal depth.

6.3 On Concept Learning Using Queries

Angluin [2] assumed that the learner has access to a fixed set of oracles that will answer specific kinds of queries about the concept to be learned. As mentioned earlier, she studied exact and probably exact learnability using different types of queries like membership, equivalence, subset, superset, disjointness and exhaustiveness. In this section, we generalize these types of queries for DLs, introduce interpretation queries and present some consequences. This mainly serves as a starting point for future work.

A type of queries is specified by a form of inputs and outputs for oracles. Let C denote the concept to be learned, which belongs to a language $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$. It is known to the oracles, but unknown to the learner. We assume that the learner knows $\Sigma^\dagger$ and whether $\Phi^\dagger$ contains I or `Self`, but it may not know d nor the (maximal) number k with $Q_k \in \Phi^\dagger$. Generalization of the types of queries studied by Angluin [2] is as follows.

Membership. The input is a pair of an interpretation $\mathcal{I}$ and an element $x \in \Delta^{\mathcal{I}}$, and the output is *yes* if $x \in C^{\mathcal{I}}$ and *no* otherwise.

Equivalence. The input is a concept D and the output is *yes* if $D \equiv C$ and *no* otherwise. If the answer is *no*, the oracle returns an interpretation $\mathcal{I}$ and an element $x \in D^{\mathcal{I}} \ominus C^{\mathcal{I}}$, where $\ominus$ denotes "symmetric difference".

Subset. The input is a concept D and the output is *yes* if $D \sqsubseteq C$ (i.e., $D^{\mathcal{I}} \subseteq C^{\mathcal{I}}$ for every interpretation $\mathcal{I}$) and *no* otherwise. If the answer is *no*, the oracle returns an interpretation $\mathcal{I}$ and an element $x \in D^{\mathcal{I}} - C^{\mathcal{I}}$.

Superset. The input is a concept D and the output is *yes* if $C \sqsubseteq D$ and *no* otherwise. If the answer is *no*, the oracle returns an interpretation $\mathcal{I}$ and an element $x \in C^{\mathcal{I}} - D^{\mathcal{I}}$.

Disjointness. The input is a concept D and the output is *yes* if $D \sqcap C$ is unsatisfiable and *no* otherwise. If the answer is *no*, the oracle returns an interpretation $\mathcal{I}$ and an element $x \in D^{\mathcal{I}} \cap C^{\mathcal{I}}$.

Disjointness. The input is a concept D and the output is *yes* if $D \sqcup C \equiv \top$ (i.e., $D^{\mathcal{I}} \cup C^{\mathcal{I}} = \Delta^{\mathcal{I}}$ for every interpretation $\mathcal{I}$) and *no* otherwise. If the answer is *no*, the oracle returns an interpretation $\mathcal{I}$ and an element $x \notin D^{\mathcal{I}} \cup C^{\mathcal{I}}$.

The input concept D is usually assumed to belong to the same language as C. In the *restricted* version, the above oracles return only *yes* or *no* without providing a counterexample x.

Valiant [59] studied concept learnability by using membership queries and oracles that generate positive examples. One can also consider oracles that generate negative examples. These oracles do not receive inputs, but only return examples. They are generalized for DLs as follows.

Positive Example. The output is a pair of an interpretation $\mathcal{I}$ and an element $x \in C^{\mathcal{I}}$.

Negative Example. The output is a pair of an interpretation $\mathcal{I}$ and an element $x \in \Delta^{\mathcal{I}} - C^{\mathcal{I}}$.

Our new type of queries is as follows, which generalizes membership queries.

Interpretation. The input is an interpretation $\mathcal{I}$ and the output is the set $C^{\mathcal{I}}$.

As a consequence of Theorem 6.12, we have the following corollary:

Corollary 6.14. *If $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ is known then each of its concepts can be learned using one interpretation query.*

We say that a concept C of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ is in the *h-DEG normal form* (in short, *h-DEGNF*) if it is in the DEG-normal form and

- every conjunction occurring in C has no more than h conjuncts,
- if C is a disjunction then it has no more than h disjuncts.

In the case $\Phi^\dagger = \{I, Q_k, \mathtt{Self}\}$, $|\Sigma_C^\dagger| = m$ and $|\Sigma_R^\dagger| = n$, an upper bound $S(d)$ for the number of concepts in the h-DEG normal form of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ can be estimated as follows:

$$\begin{array}{rcl} S'(0) & = & (2m+2n+2)^h \\ S'(l+1) & = & (4k.n.S'(l)+2m+2n+2)^h \text{ for } l \geq 0 \\ S(d) & = & (S'(d))^h, \end{array}$$

where $S'(l)$ is an upper bound for the number of concepts in the h-DEG normal form of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ that do not use $\sqcup$ and have a modal depth not greater than l.

Thus, $S(d) = (O(k.n.(m+n)^h))^{h^{d+1}}$. In the case h and d are constants, $S(d)$ is a polynomial (in k, m and n). We arrive at the following consequence, which is related to the learnability of bounded CNF boolean formulas in classical propositional calculus studied in [59, 2].

Proposition 6.15. *When h and d are fixed natural numbers, every concept C in the h-DEG normal form of $\mathcal{L}_{\Sigma^\dagger,\Phi^\dagger,d}$ can be learned using a polynomial number of equivalence queries.*

Chapter 7

Conclusions

We have studied bisimulations and bisimulation-based comparisons in a uniform way for a large class of DLs that extend $\mathcal{ALC}_{reg}$ with an arbitrary set of features among inverse roles, nominals, qualified number restrictions, the universal role, the concept constructor $\exists r.\mathtt{Self}$ as well as role axioms. This class contains useful DLs like $\mathcal{SROIQ}$, which is the logical base of the Web Ontology Language OWL 2. In comparison with the related works [37, 40], this class additionally allows the role constructors of PDL, the concept constructor $\exists r.\mathtt{Self}$ and the universal role as well as role axioms.

Our main contributions are the following:

Chapter 3:

- We proposed to treat named individuals as initial states and gave an appropriate condition for bisimulation. We introduced bisimulation conditions for the universal role and the concept constructor $\exists r.\mathtt{Self}$.
- We proved that all of the bisimulation conditions (3.1)-(3.14) can be combined together to guarantee invariance of concepts and the Hennessy-Milner property for the whole class of studied DLs.
- We addressed and gave results on invariance or preservation of ABoxes, RBoxes and knowledge bases in DLs. Independently with [40] we gave results on invariance of TBoxes. By examples, we showed that our results on invariance or preservation of TBoxes, ABoxes, RBoxes and knowledge bases in DLs are strong and cannot be extended in a straightforward way.
- We introduced a new notion called QS-interpretation, which is needed for dealing with minimizing interpretations in DLs with qualified number restrictions and/or the concept constructor $\exists r.\mathtt{Self}$.
- We formulated and proved results on minimality of quotient interpretations w.r.t. the largest auto-bisimulations.
- We adapted Hopcroft's automaton minimization algorithm [28] and the Paige-Tarjan algorithm [49] to give efficient algorithms for computing the partition corresponding to the largest auto-bisimulation of a finite interpretation in any DL of the considered family. The adaptation requires special

treatments for dealing with nondeterminism and the allowed constructors of the considered DLs.

Chapter 4:

- We proved that all of the conditions (4.1)-(4.14) can be combined together to guarantee preservation of semi-positive concepts and the Hennessy-Milner property w.r.t. semi-positive concepts for the whole class of studied DLs.
- We provided results on characterizing bisimulation for tidy interpretations by semi-positive concepts.
- We provided results on minimization of interpretations that preserves semi-positive concepts.

Chapter 5:

- We provided results about separating the expressiveness of the DLs that extend $\mathcal{L}$, where $\mathcal{ALC} \leq \mathcal{L} \leq \mathcal{ALC}_{reg}$, with any combination of the features I, O, Q, U, `Self`. Our separation results are w.r.t. concepts, positive concepts, TBoxes and ABoxes. Our work differs significantly from all of [4, 7, 8, 37, 40], as the class of considered DLs is much larger than the ones considered in those works and our results about separating the expressiveness of DLs are obtained not only w.r.t. concepts and TBoxes but also w.r.t. positive concepts and ABoxes.

Chapter 6:

- We proved that any concept in any description logic that extends $\mathcal{ALC}$ with some features amongst I, Q_k, `Self` can be learned if the training information system (specified as an interpretation) is good enough.
- For the above mentioned purpose, we introduced universal interpretations and bounded bisimulation in DLs and developed the MiMoD algorithm.

We also gave a survey on bisimulation-based concept learning in DLs and discussed applications of interpretation minimization as well as concept learning using queries.

This dissertation is a comprehensive work on bisimulations for DLs. Our results about separating expressiveness of DLs and C-learnability in DLs are interesting theoretical results. Our results on the largest auto-bisimulations found the logical basis for concept learning in DLs [48, 57, 26, 15, 56, 58]. That is, our results are useful for machine learning in the context of DLs.

Bibliography

[1] S. Abiteboul and V. Vianu. Expressive power of query languages. In *Theoretical Studies in Computer Science*, pages 207–251, 1992.

[2] D. Angluin. Queries and concept learning. *Machine Learning*, 2(4):319–342, 1987.

[3] C. Areces, P. Blackburn, and M. Marx. Hybrid logics: Characterization, interpolation and complexity. *J. Symb. Log.*, 66(3):977–1010, 2001.

[4] F. Baader. A formal definition for the expressive power of terminological knowledge representation languages. *J. Log. Comput.*, 6(1):33–54, 1996.

[5] L. Badea and S.-H. Nienhuys-Cheng. A refinement operator for description logics. In *Proceedings of ILP'2000*, volume 1866 of *LNCS*, pages 40–59. Springer, 2000.

[6] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Number 53 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 2001.

[7] A. Borgida. On the relative expressiveness of description logics and predicate logics. *Artif. Intell.*, 82(1-2):353–367, 1996.

[8] M. Cadoli, L. Palopoli, and M. Lenzerini. Datalog and description logics: Expressive power. In *Proceedings of APPIA-GULP-PRODE'1997*, pages 333–344, 1997.

[9] S.T. Cao, L.A. Nguyen, and A. Szałas. The web ontology rule language OWL 2 RL + and its extensions. *T. Computational Collective Intelligence*, 13:152–175, 2014.

[10] S.T. Cao, L.A. Nguyen, and A. Szałas. WORL: a nonmonotonic rule language for the semantic web. *Vietnam J. Computer Science*, 1(1):57–69, 2014.

[11] W.W. Cohen and H. Hirsh. Learning the Classic description logic: Theoretical and experimental results. In *Proceedings of KR'1994*, pages 121–133.

[12] W. Conradie. Definability and changing perspectives: The Beth property for three extensions of modal logic. Master's thesis, ILLC, University of Amsterdam, 2002.

[13] M. de Rijke. A note on graded modal logic. *Studia Logica*, 64(2):271–283, 2000.

[14] A.R. Divroodi. Comparing the expressiveness of description logics. In *Proceedings of Description Logics'2014*, CEUR Workshop Proceedings. CEUR-WS.org, 2014.

[15] A.R. Divroodi, Q.-T. Ha, L.A. Nguyen, and H.S. Nguyen. On C-learnability in description logics. In *Proceedings of ICCCI'2012 (1)*, volume 7653 of *LNCS*, pages 230–238. Springer, 2012.

[16] A.R. Divroodi and L.A. Nguyen. On bisimulations for description logics. In *Proceedings of CS&P'2011*, pages 99–110, 2011.

[17] A.R. Divroodi and L.A. Nguyen. On bisimulations for description logics. `http://arxiv.org/abs/1104.1964v1`, 2011-04-11.

[18] A.R. Divroodi and L.A. Nguyen. Bisimulation-based comparisons for interpretations in description logics. In *Proceedings of Description Logics'2013*, volume 1014 of *CEUR Workshop Proceedings*, pages 652–669. CEUR-WS.org, 2013.

[19] A.R. Divroodi and L.A. Nguyen. On bisimulations for description logics. *Information Sciences*, 295:465–493, 2015.

[20] B. Dunin-Kęplicz, L.A. Nguyen, and A. Szałas. Tractable approximate knowledge fusion using the Horn fragment of serial propositional dynamic logic. *Int. J. Approx. Reasoning*, 51(3):346–362, 2010.

[21] R. Fagin. Generalized first-order spectra and polynomial-time recognizable sets. In *SIAM-AMS Proceedings*, volume 7, 1974.

[22] R. Fagin. Monadic generalized spectra. *Zeitschr. f. math. Logik und Grundlagen d. Math.*, 21:89–96, 1975.

[23] N. Fanizzi, C. d'Amato, F. Esposito, and T. Lukasiewicz. Representing uncertain concepts in rough description logics via contextual indiscernibility relations. In *Proceedings of URSW'2008*, volume 423 of *CEUR Workshop Proceedings*, 2008.

[24] D.M. Gabbay. Expressive functional completeness in tense logic. In U. Monnich, editor, *Aspects of Philosophical Logic*, pages 91–117. Reidel, Dordrecht, 1981.

[25] Daniel Gorín and Lutz Schröder. Simulations and bisimulations for coalgebraic modal logics. *CoRR*, abs/1303.2467, 2013.

[26] Q.-T. Ha, T.-L.-G. Hoang, L.A. Nguyen, H.S. Nguyen, A. Szałas, and T.-L. Tran. A bisimulation-based method of concept learning for knowledge bases in description logics. In *Proceedings of SoICT'2012*, pages 241–249. ACM, 2012.

[27] M. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32(1):137–161, 1985.

[28] J. Hopcroft. An $n \log n$ algorithm for minimizing states in a finite automaton. Available at `ftp://reports.stanford.edu/pub/cstr/reports/cs/tr/71/190/CS-TR-71-190.pdf`, 1971.

[29] I. Horrocks, O. Kutz, and U. Sattler. The even more irresistible $\mathcal{SROIQ}$. In *Proc. of KR'2006*, pages 57–67. AAAI Press, 2006.

[30] L. Iannone, I. Palmisano, and N. Fanizzi. An algorithm based on counterfactuals for concept learning in the Semantic Web. *Appl. Intell.*, 26(2):139–159, 2007.

[31] N. Immerman. Upper and lower bounds for first order expressibility. *J. Comput. Syst. Sci.*, 25(1):76–98, 1982.

[32] N. Immerman. Relational queries computable in polynomial time. *Information and Control*, 68(1-3):86–104, 1986.

[33] N. Immerman and D. Kozen. Definability with bounded number of bound variables. In *Proceedings of LICS'1987*, pages 236–244. IEEE Computer Society, 1987.

[34] D. Janin and G. Lenzi. On the relationship between monadic and weak monadic second order logic on arbitrary trees, with applications to the mu-calculus. *Fundam. Inform.*, 61(3-4):247–265, 2004.

[35] D. Janin and I. Walukiewicz. On the expressive completeness of the propositional mu-calculus with respect to monadic second order logic. In *Proceedings of CONCUR'1996*, volume 1119 of *LNCS*, pages 263–277. Springer, 1996.

[36] N. Kurtonina and M. de Rijke. Simulating without negation. *J. Log. Comput.*, 7(4):501–522, 1997.

[37] N. Kurtonina and M. de Rijke. Expressiveness of concept expressions in first-order description logics. *Artif. Intell.*, 107(2):303–333, 1999.

[38] P. Lambrix and P. Larocchia. Learning composite concepts. In *Proc. of DL'1998*.

[39] J. Lehmann and P. Hitzler. Concept learning in description logics using refinement operators. *Machine Learning*, 78(1-2):203–250, 2010.

[40] C. Lutz, R. Piro, and F. Wolter. Description logic TBoxes: Model-theoretic characterizations and rewritability. In T. Walsh, editor, *Proceedings of IJCAI'2011*, pages 983–988, 2011.

[41] C. Lutz, R. Piro, and F. Wolter. Description logic tboxes: Model-theoretic characterizations and rewritability. *CoRR*, abs/1104.2844, 2011.

[42] L.A. Nguyen. Constructing the least models for positive modal logic programs. *Fundamenta Informaticae*, 42(1):29–60, 2000.

[43] L.A. Nguyen. A bottom-up method for the deterministic Horn fragment of the description logic $\mathcal{ALC}$. In *Proceedings of JELIA 2006, LNAI 4160*, pages 346–358. Springer, 2006.

[44] L.A. Nguyen. Constructing finite least Kripke models for positive logic programs in serial regular grammar logics. *Logic Journal of the IGPL*, 16(2):175–193, 2008.

[45] L.A. Nguyen. An efficient tableau prover using global caching for the description logic $\mathcal{ALC}$. *Fundamenta Informaticae*, 93(1-3):273–288, 2009.

[46] L.A. Nguyen. Horn knowledge bases in regular description logics with PTime data complexity. *Fundamenta Informaticae*, 104(4):349–384, 2010.

[47] L.A. Nguyen. ExpTime tableaux for the description logic SHIQ based on global state caching and integer linear feasibility checking. arXiv:1205.5838, 2012.

[48] L.A. Nguyen and A. Szałas. Logic-based roughification. In A. Skowron and Z. Suraj, editors, *Rough Sets and Intelligent Systems (To the Memory of Professor Zdzisław Pawlak), Vol. 1*, pages 529–556. Springer, 2012.

[49] R. Paige and R.E. Tarjan. Three partition refinement algorithms. *SIAM J. Comput.*, 16(6):973–989, 1987.

[50] D.M.R. Park. Concurrency and automata on infinite sequences. In Peter Deussen, editor, *Proceedings of the 5th GI-Conference*, volume 104 of *LNCS*, pages 167–183. Springer, 1981.

[51] P. Parys. Analiza algorytmu hopcrofta minimalizacji deterministycznego automatu skończonego. Available at `http://www.mimuw.edu.pl/~niwinski/Notatki/JAO/hopcroft.ps`, 2005.

[52] Z. Pawlak. *Rough Sets. Theoretical Aspects of Reasoning about Data.* Kluwer Academic Publishers, Dordrecht, 1991.

[53] Z. Pawlak and A. Skowron. Rudiments of rough sets. *Inf. Sci.*, 177(1):3–27, 2007.

[54] I.A. Stewart. Comparing the expressibility of languages formed using NP-complete operators. *J. Log. Comput.*, 1(3):305–330, 1991.

[55] D. Toman and D. Niwinski. First-order queries over temporal databases inexpressible in temporal logic. In *Proceedings of EDBT'1996*, volume 1057 of *LNCS*, pages 307–324. Springer, 1996.

[56] T.-L. Tran, Q.-T. Ha, T.-L.-G. Hoang, L.A. Nguyen, and H.S. Nguyen. Bisimulation-based concept learning in description logics. In *Proceedings of CS&P'2013*, volume 1032 of *CEUR Workshop Proceedings*, pages 421–433. CEUR-WS.org, 2013.

[57] T.-L. Tran, Q.-T. Ha, T.-L.-G. Hoang, L.A. Nguyen, H.S. Nguyen, and A. Szałas. Concept learning for description logic-based information systems. In *Proceedings of KSE'2012*, pages 65–73. IEEE Computer Society, 2012.

[58] T.-L. Tran, L.A. Nguyen, and T.-L.-G. Hoang. A domain partitioning method for bisimulation-based concept learning in description logics. In *Proceedings of ICCSAMA'2014*, volume 282 of *Advances in Intelligent Systems and Computing*, pages 297–312. Springer, 2014.

[59] L. G. Valiant. A theory of the learnable. *Commun. ACM*, 27(11):1134–1142, 1984.

[60] J. van Benthem. *Modal Correspondence Theory.* PhD thesis, Mathematisch Instituut & Instituut voor Grondslagenonderzoek, University of Amsterdam, 1976.

[61] J. van Benthem. *Modal Logic and Classical Logic.* Bibliopolis, Naples, 1983.

[62] J. van Benthem. Correspondence theory. In D. Gabbay and F. Guenther, editors, *Handbook of Philosophical Logic, Volume II*, pages 167–247. Reidel, Dordrecht, 1984.

Acknowledgements

Writing this part gives me a chance to look back over the past years and here I wish to express my gratitude to those who helped me during my education period, which I spent in learning and teaching.

Foremost, I would like to thank my supervisor, Dr. Hab. Linh Anh Nguyen, for his guidance, kindness, advice and endless support. His guidance, attention and patience in explaining problems increased my knowledge and skills in Logic and Computer Science, a part of which is presented in this PhD dissertation.

I would like to thank Prof. Damian Niwiński for endless support and guidance. I would also like to thank all the professors of MIMUW from whom I learnt some subjects, for creating a proper scientific environment and holding useful courses. I thank PhD students and researchers of MIMUW for answering my questions and creating good time when we were together, especially the members of the Automata Theory Group and my dear friend Son Thanh Cao. I also thank the staff of the faculty MIMUW. Their efforts helped me to facilitate my education and research.

I would like to express the most special thanks to my parents. My achievement at this level of education is due to their encouragement. During the seventeen years I worked as an instructor of Math and Computer Science at high school and university in Iran, I tried to get more knowledge and scientific skills every year so that I could improve my teaching for my students. This required a lot of redeployment inside Iran and to Poland. It could happen because I have a patient family that believed in me and supported me in any case. So, by all of my heart, I dedicate this PhD dissertation to my lovely wife, Leila, and my lovely son, Morteza.

Life in Warsaw was enjoyable and exciting during the course of my PhD study. Therefore, thanks to all of my Polish and Iranian friends in Warsaw, who enhanced my spirit and motivation by their words or behaviors, especially to Mr. Enayatolla Montazeri, Dr. Mostafa Bakhoda and Dr. Maryam Ghaffari. I wish the best to all of them.

During my PhD study I received financial support from the Faculty of Mathematics, Informatics and Mechanics of the University of Warsaw (MIMUW) and from the Polish National Science Center (NCN) under Grant No. 2011/01/B/ST6/02759.

Index