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Adam Śpiewak<br>Geometric properties of measures in finite-dimensional dynamical systems<br>PhD dissertation

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## Author's declaration:

I hereby declare that this dissertation is my own work.

January 16, 2020
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Supervisors's declaration:
This dissertation is ready to be reviewed.

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#### Abstract

This dissertation consists of two parts, both studying geometric properties of measures occuring in finite-dimensional dynamical systems, mainly from the point of view of the dimension theory.

The first part concerns probabilistic aspects of the Takens embedding theorem, dealing with the problem of reconstructing a dynamical system from a sequence of measurements performed via a one-dimensional observable. Classical results of that type state that for a typical observable, every initial state of the system is uniquely determined by a sequence of measurements as long as the number of measurements is greater than twice the dimension of the phase space. The main result of this part of the dissertation states that in the probabilistic setting the number of measurements can be reduced by half, i.e. almost every initial state of the system can be uniquely determined provided that the number of measurements is greater than the Hausdorff dimension of the phase space. This result partially proves a conjecture of Shroer, Sauer, Ott and Yorke from 1998. We provide also a non-dynamical probabilistic embedding theorem and several examples.

In the second part of the dissertation we consider a family of stationary probability measures for certain random dynamical systems on the unit interval and study their geometric properties. The measures we are interested in can be seen as stationary measures for Markov processes on the unit interval, which arise from random iterations of two piecewise-affine homeomorphisms of the interval. We call such random systems Alsedà-Misiurewicz systems (or AM-systems), as they were introduced and studied by Alsedà and Misiurewicz, who conjectured in 2014 that typically measures of that type should be singular with respect to the Lebesgue measure. We work towards characterization of parameters exhibiting this property. Our main result is establishing singularity of the corresponding stationary measures for certain sets of parameters, hence confirming the conjecture on these sets. We present two different approaches to proving singularity - one based on constructing invariant minimal Cantor sets and one based on estimating the expected return time to a suitably chosen interval. In the first case we calculate the Hausdorff dimension of the measure for certain parameters. We present also several auxiliary results concerning $A M$-systems.


Keywords: Takens delay embedding theorem, probabilistic embedding, Hausdorff dimension, box-counting dimension, random system, stationary measure, semigroup of interval homeomorphisms, minimal set

AMS Subject Classification: 37C45, 28A78, 28A80, 37E05, 37H10

## Streszczenie

Poniższa rozprawa składa się z dwóch części. Obie z nich badają geometryczne własności miar występujących w skończenie wymiarowych układach dynamicznych, głównie z punktu widzenia teorii wymiaru.

Część pierwsza dotyczy probabilistycznych aspektów twierdzenia Takensa o zanurzaniu, zajmującego się zagadnieniem rekonstrukcji układu dynamicznego z ciągu pomiarów wykonanych za pomocą jednowymiarowej obserwabli. Klasycznego wyniki z tej dziedziny orzekają, że dla typowej obserwabli, dowolny stan początkowy układu jest jednoznacznie wyznaczony przez ciąg pomiarów, o ile ich ilość przekracza dwukrotnie wymiar przestrzeni fazowej. Główny wynik tej części rozprawy stwierdza, że w kontekście probabilistycznym liczba pomiarów może być dwukrotnie zmniejszona, tzn. prawie każdy stan początkowy układu jest wyznaczony jednoznacznie, o ile ilość pomiarów jest większa od wymiaru Hausdorffa przestrzeni fazowej. Powyższy wynik dowodzi częściowo hipotezy Shroera, Sauera, Otta oraz Yorka z 1998 roku. Przedstawiamy także niedynamiczną wersję probabilistycznego twierdzenia o zanurzaniu oraz szereg przykładów.

W drugiej części rozprawy rozważamy rodzinę stacjonarnych miar probabilistycznych dla pewnych losowych układów dynamicznych na odcinku jednostkowym oraz badamy ich własności geometryczne. Rozważane miary mogą być traktowane jako miary stacjonarne dla procesu Markowa na odcinku, otrzymanego przez losowe iterowanie dwóch kawałkami afinicznych homeomorfizmów odcinka. Układy tej postaci nazywamy układami Alsedy-Misiurewicza (albo $A M$-układami), gdyż badania nad nimi rozpoczęli Alsedà oraz Misiurewicz, którzy postawili w 2014 roku hipotezę, że typowe miary stacjonarne dla takich układów są singularne względem miary Lebesgue'a. Głównym celem naszej pracy jest scharakteryzowanie parametrów posiadających tę własność. Naszym głównym wynikiem jest znalezienie pewnych zbiorów parametrów dla których odpowiednie miary są singularne, co dowodzi powyższą hipotezę dla tych zbiorów. Przedstawiamy dwa różne podejścia do dowodzenia singularności - jedno oparte na znajdowaniu minimalnych niezmienniczych zbiorów Cantora oraz drugie, wykorzystujące szacowanie oczekiwanego czasu powrotu do odpowiednio dobranego przedziału. W pierwszym przypadku wyliczamy wymiar Hausdorffa miary stacjonarnej dla pewnych parametrów. Przedstawiamy również kilka dodatkowych wyników dotyczących $A M$-układów.

Słowa kluczowe: twierdzenie Takensa o zanurzaniu, zanurzenie probabilistyczne, wymiar Hausdorffa, wymiar pudełkowy, układ losowy, miara stacjonarna, półgrupa homeomorfizmów odcinka, zbiór minimalny

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## Chapter 1

## Introduction

This dissertation consists of two parts, both studying geometric properties of measures in occuring in finite-dimensional dynamical systems, mainly from the point of view of the dimension theory. Let us begin by describing briefly the main topics of both parts, with more detailed description of problems and results given in subsequent sections.

The first part, contained in Chapter 3. concerns probabilistic aspects of the Takens embedding theorem, dealing with the problem of reconstructing a dynamical system from a sequence of measurements performed via a one-dimensional observable. More precisely, let $T: X \rightarrow X$ be a transformation on a phase space $X$. Fix $k \in \mathbb{N}$ and consider an observable $h: X \rightarrow \mathbb{R}$ together with the corresponding delay-coordinate map $X \ni x \mapsto \phi_{h}^{T}(x)=$ $\left(h(x), \ldots, h\left(T^{k-1} x\right)\right) \in \mathbb{R}^{k}$. Takens-type theorems, which originate from the original work of Takens 94 and were obtained in several categories (see Section 1.1 for more on their history), state that $\phi_{h}^{T}$ is an embedding (i.e. it is injective) for a typical (in a suitable sense) observable $h$ provided $k \in \mathbb{N}$ is large enough. Such theorems serve as a justification of the validity of procedures actually used by experimentalists and have been proved in several categories. A striking common feature of these results is that the number $k$ of measurements sufficient for the lossless reconstruction of the system is $k \approx 2 \operatorname{dim} X$, where $\operatorname{dim} X$ is the dimension of the set $X$. Moreover, this threshold is known to be optimal and goes in line with results on nondynamical embeddings (e.g. Whitney and Menger-Nöbeling theorems). The main result of this part of the dissertation is a probabilistic version of the Takens theorem. It corresponds to a scenario in which the accessible initial states of the system are coming from a given probability distribution $\mu$ on $X$ and one is interested in reconstructing the system almost surely subject to $\mu$, i.e. we want the map $\phi_{h}^{T}$ to be injective on a set of full measure $\mu$. The key conclusion is that in such a setting, it suffices to take $k \approx \operatorname{dim}_{H}(\mu)$ measurements, hence their number can be reduced by half compared to the non-probabilistic case (here $\operatorname{dim}_{H}$ is the Hausdorff dimension). The possibility of reducing by half the number of required measurements in the probabilistic setting was conjectured in a physical literature by Shroer, Sauer, Ott and Yorke [85]. As a corollary of our results, we prove their [85, Conjecture 1] for ergodic measures. As a by-product of our work, we prove also a non-dynamical probabilistic embedding theorem for linear embeddings in terms of the Hausdorff dimension, which strengthens previously obtained results in this direction. We also present several examples. All the results are based on a joint work with Krzysztof Barański and Yonatan Gutman. The contents of Chapter 3, except for the Theorem 1.3 and Section 3.4 can be found in a joint preprint [8]. Theorem 1.3 and Section 3.4, dealing with the conjectures of Shroer, Sauer, Ott and Yorke [85], are part of a work in progress with K. Barański and Y. Gutman.

The second part of the dissertation, contained in Chapter 4 is of a different flavour. We consider a specific family of probability measures on the unit interval and study their geometric properties. The measures we are interested in are stationary measures for Markov processes on the unit interval, which arise from random iterations of two piecewise-affine homeomorphisms of the unit interval. We call such random systems Alsedà-Misiurewicz systems (or

AM-systems), as they were introduced and studied by Alsedà and Misiurewicz in [2]. In the symmetric case, this gives a family of systems (and corresponding measures) parametrized by two real parameters. It is well known that such stationary measures are always either singular or absolutely continuous with respect to the Lebesgue measure. Alsedà and Misiurewicz conjectured in [2] that typically measures of that type should be singular. We work towards characterization of parameters exhibiting this property. Let us emphasize that this kind of questions are being asked for various types of random dynamical systems (iterated function systems) and even the case of contractive similarities on the unit interval (so called Bernoulli convolutions) is an active area of research, with full characterization of singular parameters still being unknown (see Section 1.2 for more details). Our main result is finding sets of parameters for which the corresponding measure is singular. We take two different approaches to obtain that goal. The first one enables us to give a detailed description of the stationary measure for parameters satisfying certain algebraic hypotheses, while the second one (based on bounding the Lyapunov exponent of the corresponding measure) allows us to find an open set of parameters, for which the corresponding measure is singular. Moreover, in certain cases we are able to calculate, or at least bound, the Hausdorff dimension of the stationary measure. All the results of Chapter 4, except for Section 4.8, can be found in a publication [9], joint with Krzysztof Barański. Section 4.8 is a part of a work in progress with K. Barański.

### 1.1 A probabilistic Takens theorem

Consider an experimentalist observing a physical system modeled by a discrete time dynamical system $(X, T)$, where $T: X \rightarrow X$ is the evolution rule during time unit and the phase space $X$ is a subset of the Euclidean space $\mathbb{R}^{N}$. It often happens that, for a given point $x \in X$, instead of an actual sequence of $k$ states $x, T x, \ldots, T^{k-1} x$, the observer's access is limited to the values of $k$ measurements $h(x), h(T x), \ldots, h\left(T^{k-1} x\right)$, for a real-valued observable $h: X \rightarrow \mathbb{R}$. Therefore, it is natural to ask, to what extent the original system can be reconstructed from such sequences of measurements and what is the minimal number $k$, referred to as the number of delay-coordinates, required for a reliable reconstruction. These questions have emerged in the physical literature (see e.g. [74, 85]) and inspired a number of mathematical results, known as Takens-type delay embedding theorems, stating that the reconstruction of $(X, T)$ is possible for certain observables $h$, as long as the measurements $h(x), h(T x), \ldots, h\left(T^{k-1} x\right)$ are known for all $x \in X$ and large enough $k$. Mathematically, this means that the delay-coordinate map $X \ni x \mapsto\left(h(x), h(T x), \ldots, h\left(T^{k-1} x\right)\right) \in \mathbb{R}^{k}$ is injective.

Takens-type theorems are considered as theoretical results which justify the validity of actual procedures used by experimentalists (see e.g. [46, 57, 79, 91). Note that one cannot expect a reliable reconstruction of the system based on the measurements of an a priori given observable $h$, as it may fail to distinguish the states of the system (e.g. if $h$ is a constant function). It is therefore necessary (and rather realistic) to assume that the experimentalists are able to perturb the given observable. The first result obtained in this area is the celebrated Takens delay embedding theorem for smooth systems on manifolds [93, Theorem 1]. It states that for given finite-dimensional $C^{2}$ manifold $M$ and a generic pair of $C^{2}$-diffeomorphism $T: M \rightarrow M$ and $C^{2}$-function $h: M \rightarrow \mathbb{R}$, the corresponding delay-coordinate map $\phi:$ $M \rightarrow \mathbb{R}^{k}, \phi(x)=\left(h(x), h(T x), \ldots, h\left(T^{k-1} x\right)\right)$ is a $C^{2}$-embedding (an injective immersion) as long as $k>2 \operatorname{dim} M$. Due to its strong connections with actually performed reconstruction procedures, Takens theorem has been met with interest among mathematical physicists (see e.g. [42, 84, 85, 96]). Let us recall its extension due to Sauer, Yorke and Casdagli [84]. In this setting, the number $k$ of the delay-coordinates should be two times larger than the upper box-counting dimension of the phase space $X$ (denoted by $\overline{\operatorname{dim}}_{B} X$; see Section 2.1 for the definition), and the perturbation is a polynomial of degree $2 k$. The formulation of the result follows 81].

Theorem 1.1 ([81, Theorem 14.5]). Let $X \subset \mathbb{R}^{N}$ be a compact set and let $T: X \rightarrow X$ be Lipschitz, injective and aperiodic (i.e. without periodic points). Let $k \in \mathbb{N}$ be such that $k>2 \overline{\operatorname{dim}}_{B} X$. Let $h: \mathbb{R}^{N} \rightarrow \mathbb{R}$ be a Lipschitz function and $h_{1}, \ldots, h_{m}: \mathbb{R}^{N} \rightarrow \mathbb{R}$ a basis of the space of real polynomials of $N$-variables of degree at most $2 k$. For $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{R}^{m}$ denote by $h_{\alpha}: \mathbb{R}^{N} \rightarrow \mathbb{R}$ the map

$$
h_{\alpha}(x)=h(x)+\sum_{j=1}^{m} \alpha_{j} h_{j}(x) .
$$

Then for Lebesgue almost every $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{R}^{m}$, the transformation

$$
\phi_{\alpha}^{T}: X \rightarrow \mathbb{R}^{k}, \quad \phi_{\alpha}^{T}(x)=\left(h_{\alpha}(x), h_{\alpha}(T x), \ldots, h_{\alpha}\left(T^{k-1} x\right)\right)
$$

is injective on $X$.
Note that Theorem 1.1 applies to any compact set $X \subset \mathbb{R}^{N}$, not necessarily a manifold. This is a useful feature, as it allows to consider sets with a complicated geometrical structure, such as fractal sets arising as attractors in chaotic dynamical systems, see e.g. [24]. Moreover, the upper box-counting dimension of $X$ can be smaller than the dimension of any smooth manifold containing $X$, so Theorem 1.1 may require fewer delay-coordinates than its smooth counterpart in [93]. These results were extended later by Robinson to finite-dimensional subsets of infinite-dimensional Banach spaces [80] (see also [81, Section 14.3]). Refer to [68] for a version of Takens' theorem with a fixed observable and perturbation performed on the dynamics. Takens' theorem involving Lebesgue covering dimension on compact metric spaces and a continuous observable was given in [40] (see [41] for a detailed proof). See also [17, 90] for Takens theorem for deterministically driven smooth systems and [88, 89] for stochastically driven smooth systems.

Usually, an experimentalist may perform only a finite number of observations $h\left(x_{j}\right), \ldots, h\left(T^{k-1} x_{j}\right)$ for some points $x_{j} \in X, j=1, \ldots, l$. We believe it is realistic to assume that there is an (explicit or implicit) random process determining which initial states $x_{j}$ are accessible to the experimentalist. In this work we are interested in the question of reconstruction of the system in a presence of such process, subject to it. Mathematically speaking, this corresponds to fixing a probability measure $\mu$ on $X$ and asking whether the delay-coordinate $\operatorname{map} \phi_{\alpha}^{T}$ is injective almost surely with respect to $\mu$ (i.e. whether there exists a subset of $X$ with full measure $\mu$, such that $\phi_{\alpha}^{T}$ is injective after restricting to this subset). Since in this setting we are allowed to neglect sets of probability zero, it is reasonable to ask whether the minimal number of delay-coordinates sufficient for the reconstruction of the system can be smaller than $2 \operatorname{dim} X$. Our main result states that this is indeed the case, and the number of delay-coordinates can be reduced by half for any (Borel) probability measure. The following theorem is a simplified version of our main result. See Section 2.2 for the definition of the Hausdorff dimension of a measure. See Theorem 3.15 for the full version and Section 3.3 for its proof.
Theorem 1.2 (Probabilistic Takens delay embedding theorem). Let $X \subset \mathbb{R}^{N}$ be $a$ Borel set, $\mu$ a Borel probability measure on $X$ and $T: X \rightarrow X$ an injective, Lipschitz and aperiodic map. Take $k \in \mathbb{N}$ such that $k>\operatorname{dim}_{H}(\mu)$. Let $h: \mathbb{R}^{N} \rightarrow \mathbb{R}$ be a locally Lipschitz function and $h_{1}, \ldots, h_{m}: \mathbb{R}^{N} \rightarrow \mathbb{R}$ a basis of the space of real polynomials of $N$ variables of degree at most $2 k-1$. For $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{R}^{m}$ denote by $h_{\alpha}: \mathbb{R}^{N} \rightarrow \mathbb{R}$ the map

$$
h_{\alpha}(x)=h(x)+\sum_{j=1}^{m} \alpha_{j} h_{j}(x)
$$

Then for Lebesgue almost every $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{R}^{m}$, there exists a Borel set $X_{\alpha} \subset X$ of full measure $\mu$, such that the delay-coordinate map

$$
\phi_{\alpha}^{T}: X \rightarrow \mathbb{R}^{k}, \quad \phi_{\alpha}^{T}(x)=\left(h_{\alpha}(x), h_{\alpha}(T x), \ldots, h_{\alpha}\left(T^{k-1} x\right)\right)
$$

is injective on $X_{\alpha}$. If $\mu$ is additionally $T$-invariant, then the sets $X_{\alpha}$ can be taken to be $T$-invariant, i.e. satisfy $T\left(X_{\alpha}\right) \subset X_{\alpha}$.

Recall that for any Borel set $X$ and a Borel probability measure $\mu$ on $X$ one has

$$
\begin{equation*}
\operatorname{dim}_{H}(\mu) \leqslant \operatorname{dim}_{H} X \leqslant \operatorname{dim}_{B} X \leqslant \overline{\operatorname{dim}}_{B} X \tag{1.1}
\end{equation*}
$$

(see (2.1) and the definition of $\operatorname{dim}_{H}(\mu)$ ). Since the inequalities in (1.1) may be strict, using the Hausdorff dimension instead of the box-counting one(s) may reduce further the required number of delay-coordinates. In particular there are compact sets $X \subset \mathbb{R}^{N}$ with $\operatorname{dim}_{H} X=$ 0 and $\overline{\operatorname{dim}}_{B} X=N$, hence Theorem 1.2 can reduce significantly the number of required delay-coordinates compared to Theorem 1.1 (in a probabilistic setting), as the box-counting dimension cannot be replaced by Hausdorff dimension in Theorem 1.1 (see Remark 3.10 for a detailed discussion).

Let us comment on consequences of $T$-invariance of $X_{\alpha}$ in the case when $\mu$ is $T$-invariant. Having injectivity of $\phi_{\alpha}^{T}$ in Theorem 1.2, it is natural to consider a model of the dynamical system $(X, T)$ embedded in $\mathbb{R}^{k}$, i.e. the dynamical system with phase space $\phi_{\alpha}^{T}\left(X_{\alpha}\right)$ and dynamics $\phi_{\alpha}^{T} \circ T \circ\left(\phi_{\alpha}^{T}\right)^{-1}$ on it. However, to have $\phi_{\alpha}^{T} \circ T \circ\left(\phi_{\alpha}^{T}\right)^{-1}$ well-defined, the set $X_{\alpha}$ should be $T$-invariant. This does not have to be the case in general, yet it holds if the measure $\mu$ is $T$-invariant.

An extended version of Theorem 1.2 is presented and proved in Section 3.3 as Theorem 3.15. It shows that the assumption $k>\operatorname{dim}(\mu)$ can be slightly weakened to $\mu \perp \mathcal{H}^{k}$ (here $\mathcal{H}^{k}$ denotes the $k$-dimensional Hausdorff measure), and in addition to Lipschitz observables $h$, one can consider functions which are $\beta$-Hölder on bounded sets for suitable $\beta \in(0,1]$. Moreover, the theorem holds for any Borel $\sigma$-finite measure $\mu$ on $X$. The assumption of aperiodicity of $T$ is not essential in Theorems 1.1 and 1.2 - it is enough to assume that sets of periodic points have dimension small enough. For details, see Theorem 3.15.

The problem of determining the minimal number of delay-coordinates required for reconstruction has been considered in the physical literature. In [74], the authors analyzed an algorithm which may by interpreted as an attempt to determine this number in a probabilistic setting. Our work provides rigorous results in this direction. Furthermore, the possibility of reducing twice the number of required measurements in the probabilistic scenario was conjectured by Shroer, Sauer, Ott and Yorke in 855 (together with a conjecture on the decay rate of the error probability). These conjectures are being invoked as justifications for reducing the number of measurements required for a reliable reconstruction of the system (see e.g. [58, 66, 71), including applications (see e.g. 779] studying neural activity for epileptic patients). However in the same spirit, our main result (Theorem 1.2 and its extension Theorem 3.15) formally does not settle these conjectures, as [85] considers a different setting. In Section 3.4 we use our results, together with the theory of topological conditional measures (see 87]), to prove [85, Conjecture 1] for ergodic invariant measures. Let us postpone formulating precisely conjectures of 85 to Section 3.4 (see Conjectures 3.20 and 3.21 ). Instead, we present here a strengthening of Theorem 1.2, which asserts additional properties of the almost surely defined inverse $\left(\phi_{h}^{T}\right)^{-1}$ and allows us to prove [85, Conjecture 1] for ergodic measures. See Theorem 3.24 for the full version and Section 3.4 for the proof.
Theorem 1.3. Assume that $X, \mu, T: X \rightarrow X, h: X \rightarrow \mathbb{R}$ and $k \in \mathbb{N}$ satisfy assumptions of Theorem 1.2. Assume additionally that $X$ is compact. Then for Lebesgue almost every $\alpha \in \mathbb{R}^{m}$ there exists a set $X_{\alpha} \subset X$ of full measure $\mu$ such that $\phi=\phi_{\alpha}^{T}$ is injective on $X_{\alpha}$ and for every $x \in X_{\alpha}$, the sequence of conditional measures

$$
\frac{1}{\mu\left(\phi^{-1}(B(\phi(x), \varepsilon))\right)} \mu \upharpoonright \phi^{-1}(B(\phi(x), \varepsilon))
$$

(on sets $\phi^{-1}\left(B(\phi(x), \varepsilon)=\{y \in X:\|\phi(x)-\phi(y)\|<\varepsilon\}\right.$ ) converges to $\delta_{x}$ in the weak* topology as $\varepsilon \searrow 0$. As a consequence, [85, Conjecture 1] holds for ergodic invariant measures.

Takens-type delay embedding theorems can be seen as dynamical versions of embedding theorems, which specify when a finite-dimensional set can be embedded (i.e. mapped injectively) into a Euclidean space. Indeed, under the assumptions of Theorem 1.1, the delaycoordinate map $\phi_{\alpha}^{T}$ is an embedding of $X$ into $\mathbb{R}^{k}$ for typical $\alpha$. Embedding theorems in various categories have been extensively studied in a number of papers (see Section 3.2 for a more detailed discussion). Recently, Alberti, Bölcskei, De Lellis, Koliander and Riegler [1] proved a probabilistic embedding theorem involving the modified lower box-counting dimension of the measure (see Theorem 3.8). We are able to improve this result by considering the Hausdorff dimension. Below we present a simplified version of our theorem, which can be seen as a non-dynamical counterpart of Theorem 1.2 and a probabilistic version of Mañé's linear embedding theorem [59]. Its extended version is formulated and proved in Section 3.2 as Theorem 3.5.

Theorem 1.4 (Probabilistic embedding theorem). Let $X \subset \mathbb{R}^{N}$ be a Borel set and let $\mu$ be a Borel probability measure on $X$. Take $k \in \mathbb{N}$ such that $k>\operatorname{dim}_{H}(\mu)$ and let $\phi: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$ be a Lipschitz function. Then for Lebesgue almost every linear transformation $L: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$ there exists a Borel set $X_{L} \subset X$ of full measure $\mu$, such that $\phi_{L}=\phi+L$ is injective on $X_{L}$.

We obtain also the following geometric corollary (see Section 3.2 for details).
Corollary 1.5 (Probabilistic injective projection theorem). Let $X \subset \mathbb{R}^{N}$ be a Borel set and let $\mu$ be a Borel probability measure on $X$. Then for every $k>\operatorname{dim}_{H}(\mu)$ and almost every $k$-dimensional linear subspace $S \subset \mathbb{R}^{N}$, the orthogonal projection of $X$ into $S$ is injective on a $\mu$-full measure subset of $X$.

This accompanies the classical Marstrand-Mattila projection theorem (see 61, 62]), stating that if $X \subset \mathbb{R}^{N}$ is Borel and $k \geqslant \operatorname{dim}_{H} X$, then for almost all $k$-dimensional linear subspaces $S \subset \mathbb{R}^{N}$, the Hausdorff dimension of the image of $X$ under the orthogonal projection into $S$ has Hausdorff dimension equal to $\operatorname{dim}_{H} X$.

We also provide several examples. Section 3.5 provides a probability measure with $\operatorname{dim}_{H} \mu<\underline{\operatorname{dim}}_{M B} \mu$, showing that Theorem 1.4 strengthens a previous result from [1]. Example 3.32 shows that in general the condition $k>\operatorname{dim}_{H}(\mu)$ in Theorem 1.4 cannot be replaced by $k \geqslant \operatorname{dim}_{H}(\mu)$. Example 3.34 shows that linear perturbations of the observable are not sufficient for Takens theorem. Kan's example from the Appendix to [84], shows that condition $k>2 \operatorname{dim}_{H} X$ is not sufficient for existence of a linear transformation into $\mathbb{R}^{k}$ which is injective on $X$, hence the Hausdorff dimension is not well suited for the deterministic embedding theorem. As in the probabilistic setting one can work with the Hausdorff dimension, we consider a set $X \subset \mathbb{R}^{2}$ similar to the one provided by Kan, which cannot be embedded linearly into $\mathbb{R}$, but when endowed with a natural probability measure, almost every linear transformation $L: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is injective on a set of full measure. We find these transformations and sets of injectivity explicitly.

### 1.2 Singular stationary measures for random piecewise affine interval homeomorphisms

In the second part of the disseration, we study a family of random dynamical systems consisting of pairs of piecewise affine interval homeomorphisms, which we call Alsedà-Misiurewicz systems, or $A M$-systems, as systems of this type were introduced and studied in [2] by Alsedà and Misiurewicz. An AM-system is the system $\left\{f_{-}, f_{+}\right\}$of increasing homeomorphisms of the interval $[0,1]$ of the form

$$
f_{-}(x)=\left\{\begin{array}{ll}
a_{-} x & \text { for } x \in\left[0, x_{-}\right] \\
1-b_{-}(1-x) & \text { for } x \in\left(x_{-}, 1\right]
\end{array}, \quad f_{+}(x)= \begin{cases}b_{+} x & \text { for } x \in\left[0, x_{+}\right] \\
1-a_{+}(1-x) & \text { for } x \in\left(x_{+}, 1\right]\end{cases}\right.
$$

where $0<a_{-}<1<b_{-}, 0<a_{+}<1<b_{+}$and

$$
x_{-}=\frac{b_{-}-1}{b_{-}-a_{-}}, \quad x_{+}=\frac{1-a_{+}}{b_{+}-a_{+}}
$$

See Figure 1.1 below.


Figure 1.1: An example of an $A M$-system.
We consider $\left\{f_{-}, f_{+}\right\}$as a random system with given probabilities $p_{-}, p_{+}$, where $p_{ \pm}>0$ and $p_{-}+p_{+}=1$. Formally, it means that $\left\{f_{-}, f_{+}\right\}$defines a step skew product

$$
\mathcal{F}^{+}: \Sigma_{2}^{+} \times[0,1] \rightarrow \Sigma_{2}^{+} \times[0,1], \quad \mathcal{F}^{+}(\underline{i}, x)=\left(\sigma(\underline{i}), f_{i_{1}}(x)\right)
$$

where $\underline{i}=\left(i_{n}\right)_{n \in \mathbb{N}}$ and $\sigma$ is shift on the space $\Sigma_{2}^{+}$of infinite one-sided sequences of two symbols $\{-,+\}$, with the Bernoulli probability distribution given by $\left(p_{-}, p_{+}\right)$(see Section 4.2). However, we are mainly interested in the behaviour of the system in the phase space $[0,1]$, studying distribution of trajectories of points $x \in[0,1]$ under $\left\{f_{-}, f_{+}\right\}$, i.e. $\left\{f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{n=0}^{\infty}$ for $i_{1}, i_{2}, \ldots \in\{-,+\}$.

Note that on the intervals $\left(0, \min \left(x_{-}, x_{+}\right)\right)$and $\left(\max \left(x_{-}, x_{+}\right), 1\right)$ the system $\left\{f_{-}, f_{+}\right\}$is equivalent (after a logarithmic change of coordinates), respectively, to two (typically different and non-symmetric) one-dimensional random walks, which are glued in a continuous way. This makes such systems interesting from a probabilistic point of view and we believe that they can serve as models for many stochastic phenomena which appear in random one-dimensional dynamics.

The behaviour of an $A M$-system depends on the values of the endpoint Lyapunov exponents, i.e.

$$
\Lambda(0)=p_{-} \ln f_{-}^{\prime}(0)+p_{+} \ln f_{+}^{\prime}(0), \quad \Lambda(1)=p_{-} \ln f_{-}^{\prime}(1)+p_{+} \ln f_{+}^{\prime}(1)
$$

For instance, if $\Lambda(0), \Lambda(1)$ are negative, then the endpoints of the interval are attracting in average, so a typical trajectory converges to one of them, which can give rise to two intermingled basins for the step skew product $\mathcal{F}^{+}$(see e.g. [15, 35, 52]). In this work we assume that the Lyapunov exponents $\Lambda(0), \Lambda(1)$ are positive. Then for almost all paths $\underline{i}=$ $\left(i_{n}\right)_{n} \in \Sigma_{2}^{+}$, any two trajectories defined by $\underline{i}$ converge to each other, i.e. $\mid f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)-f_{i_{n}} \circ$ $\cdots \circ f_{i_{1}}(y) \mid \rightarrow 0$ as $n \rightarrow \infty$ for $x, y \in(0,1)$. This phenomenon is called synchronization (see
e.g. [3, 11, [56, 78]). The main object of our study are stationary measures for such systems, i.e. Borel probability measures $\vartheta$ on $[0,1]$ satisfying

$$
\vartheta(A)=p_{-} \vartheta\left(f_{-}^{-1}(A)\right)+p_{+} \vartheta\left(f_{+}^{-1}(A)\right) \text { for every Borel set } A \subset[0,1]
$$

(see also Definition 2.9 and Section 2.3). In probabilistic language, these are stationary distributions for the Markov process on the unit interval, arising from iterating $f_{-}$and $f_{+}$ randomly according to a probability vector $\left(p_{-}, p_{+}\right)$in an i.i.d fahsion. Clearly, atomic measures $\delta_{0}$ and $\delta_{1}$ at the common fixed points 0 and 1 are stationary measures. It turns out, that if $\Lambda(0)$ and $\Lambda(1)$ are both strictly positive, then there exists a unique stationary measure $\mu$ on $[0,1]$ such that $\mu(\{0,1\})=0$ (see Theorem4.22). In the remainder of this section, as well as in Chapter 4, $\mu$ will always denote this unique stationary measure without atoms at the endpoints (note that we suppress in the notation the dependence of $\mu$ on the parameters). In this work we study the properties of the measure $\mu$, which we call the stationary measure for the $A M$-system. Measure $\mu$ has to be either singular or absolutely continuous with respect to the Lebesgue measure (Proposition 4.26) and our main goal is to understand which possibility occurs depending on the parameters $a_{-}, a_{+}, b_{-}, b_{+}, p_{-}, p_{+}$.

In [2], Alsedà and Misiurewicz were using properties of the measure $\mu$ to study the corresponding step skew product. In the course of their work, they showed that for some parameters, the stationary measure $\mu$ of an $A M$-system is equal to the Lebesgue measure and conjectured that $\mu$ should be singular for typical parameters. We provide a precise condition under which the stationary measure is equal to the Lebesgue measure (Theorem4.4). Our main result is verifying the conjecture on singularity for some sets of parameters. We focus on the symmetric case, i.e. $a_{-}=a_{+}$and $b_{-}=b_{+}$. We use two different approaches to prove singularity. First one, which is the main content of the article [9] joint with Krzysztof Barański, concentrates on systems exhibiting a resonance, i.e. with $\ln f_{+}^{\prime}(0) / \ln f_{-}^{\prime}(0)=\ln f_{+}^{\prime}(1) / \ln f_{-}^{\prime}(1) \in \mathbb{Q}$. We prove that for some resonant parameters, the measure $\mu$ is indeed singular and supported on an exceptional minimal set, which is a Cantor set of dimension smaller than 1. More precisely, we prove the following result.

Theorem 1.6. Let $\left\{f_{-}, f_{+}\right\}$be an AM-system. Assume that

1. $\Lambda(0)>0$ and $\Lambda(1)>0$,
2. $\left\{f_{-}, f_{+}\right\}$is symmetric, i.e. $a_{-}=a_{+}=a \in(0,1)$ and $b_{-}=b_{+}=b \in(1, \infty)$,
3. $\left\{f_{-}, f_{+}\right\}$exhibits a $(k: l)$-resonance for some relatively prime $k, l \in \mathbb{N}, k>l$, i.e.

$$
\ln f_{+}^{\prime}(0) / \ln f_{-}^{\prime}(0)=\ln f_{+}^{\prime}(1) / \ln f_{-}^{\prime}(1)=-\frac{k}{l}
$$

4. Inequality $\rho<\eta$ holds, where

$$
\rho=\left(f_{-}^{\prime}(0)\right)^{1 / l}=\left(f_{+}^{\prime}(0)\right)^{-1 / k}=\left(f_{+}^{\prime}(1)\right)^{1 / l}=\left(f_{-}^{\prime}(1)\right)^{-1 / k}
$$

and $\eta \in(1 / 2,1)$ is the unique solution of the equation $\eta^{k+l}-2 \eta^{k+1}+2 \eta-1=0$.
Then the unique stationary measure $\mu$ (without atoms at 0,1 ) is singular with

$$
\operatorname{dim}_{H}(\operatorname{supp} \mu)=\frac{\log \eta}{\log \rho}<1
$$

where $\operatorname{supp} \mu$ denotes the topological support of $\mu$. Moreover, $\operatorname{supp} \mu$ is a nowhere dense perfect set.

We give also a dynamical characterization of the set $\operatorname{supp}(\mu)$ and describe its geometrical structure as a countable union of disjoint self-similar sets (see Theorem 4.10 for more detailed version of Theorem 1.6). It turns out that there are two cases to be considered separately, depending of the resonance parameter $\ln f_{+}^{\prime}(0) / \ln f_{-}^{\prime}(0)$. In the case $\ln f_{+}^{\prime}(0) / \ln f_{-}^{\prime}(0) \in \mathbb{Z}$ (i.e. $l=1$ in Theorem 1.6), the self-similar sets forming $\operatorname{supp}(\mu)$ are attractors of finite iterated function systems, while in the case $\ln f_{+}^{\prime}(0) / \ln f_{-}^{\prime}(0) \notin \mathbb{Z}$ (i.e. $l>1$ ), these self-similar sets are attractors of infinite iterated function systems and the dynamics on $\operatorname{supp}(\mu)$ is much more complicated. In both cases, the proof is based on constructing explicitly minimal sets for the dynamics and studying their combinatorics. In the case $\ln f_{+}^{\prime}(0) / \ln f_{-}^{\prime}(0) \in \mathbb{Z}$ we are able to determine the value of the Hausdorff dimension of $\mu$ (see Theorem 4.12). Furthermore, we present an interesting example of an $A M$-system exhibiting a resonance, with a singular stationary measure of full support $[0,1]$ (Theorem 4.16 ). Finally, we show that the considered systems with the same resonance are topologically conjugate (Theorem4.15). See Section 4.1 for a detailed discussion of these results and Sections 4.2 - 4.7 for proofs.

The second approach allows us to consider also non-resonant parameters. The result presented below is a part of a work in progress with Krzysztof Barański. See Theorem 4.64 for more detailed version and Section 4.8 for the proof.

Theorem 1.7. There exists a non-empty and open set of parameters $(a, b) \in(0,1) \times(1, \infty)$ such that the stationary measure $\mu$ for the symmetric AM-system with $a_{-}=a_{+}=a$ and $b_{-}=b_{+}=b$ with probability vector $\left(p_{-}, p_{+}\right)=\left(\frac{1}{2}, \frac{1}{2}\right)$ is singular with $\operatorname{dim}_{H}(\mu)<1$.

The proof is based on inequality (see [51, Theorem 1])

$$
\operatorname{dim}_{H}(\mu) \leqslant-\frac{H\left(\left(p_{-}, p_{+}\right)\right)}{\chi(\mu)},
$$

where

$$
H\left(\left(p_{-}, p_{+}\right)\right):=-p_{-} \log p_{-}-p_{+} \log p_{+}
$$

is the entropy of the probability vector $\left(p_{-}, p_{+}\right)$and

$$
\chi(\mu):=\int_{[0,1]}\left(p_{-} \log f_{-}^{\prime}(x)+p_{+} \log f_{+}^{\prime}(x)\right) d \mu(x)
$$

is the Lyapunov exponent of the stationary measure $\mu$. In order to prove $-\frac{H\left(\left(p_{-}, p_{+}\right)\right)}{\chi(\mu)}<1$, we bound the Lyapunov exponent $\chi(\mu)$ by estimating the expected return time to the interval $\left[x_{-}, x_{+}\right]$and applying Kac's Lemma (see e.g. [77, Theorem 4.6]). Let us emphasize that the approach used to prove Theorem 1.6 cannot be used to produce singular stationary measure in the non-resonant case, as in such case $\operatorname{supp}(\mu)=[0,1]$ (see Corollary 4.33).

In order to put our research in a wider perspective, let us give now a brief historical account on the study of stationary measures for random dynamical systems. For the last forty years there has been an intensive interest in the study of non-autonomous real onedimensional dynamical systems, especially in the context of the theory of groups of smooth diffeomorphisms acting on the unit circle (see e.g. [36, 69] and the references therein). In a probabilistic approach, such a system equipped with an appropriate probability distribution generates in a natural way a Markov process on the circle (see e.g. [5, 22, 26, 54] as general references on random dynamical systems and iterated function systems). Recently, a continuously growing interest in random dynamics has led to an intensive study of random systems given by groups or semigroups of one-dimensional non-smooth maps, for instance interval or circle homeomorphisms (see e.g. [2, 18, 33, 34, 35, 60, 92]).

Let $f_{1}, \ldots, f_{m}, m \geqslant 2$, be homeomorphisms of a 1 -dimensional compact manifold $X$ (the closed interval or the unit circle). Such a system of maps generates a semigroup consisting of iterates $f_{i_{n}} \circ \cdots \circ f_{i_{1}}$ for $i_{1}, \ldots, i_{n} \in\{1, \ldots, m\}, n \in\{0,1,2, \ldots\}$. Let $\left(p_{1}, \ldots, p_{m}\right)$ be
a probability vector. The Krylov-Bogolyubov Theorem shows that a stationary probability measure (according to Definition 2.9) always exists (but is non-necessarily unique). However, in most cases little is known about its properties. Assuming some regularity of the system (e.g. forward and backward non-singularity of the transformations) and the uniqueness of the stationary measure, which occur for a wide class of systems (see e.g. [19]), we know that the stationary measure is either absolutely continuous or singular with respect to the Lebesgue measure - this is the case also for $A M$-systems with positive endpoint Lyapunov exponents (if we consider only measures without atoms at the endpoints - see Proposition 4.32). Determining which of the two possibilities occur is a well-known problem, especially in the context of groups of smooth diffeomorphisms acting on the circle (see e.g. [70, Question 18]). Up to now, an answer has been given only in some particular cases. For instance, a conjecture by Y. Guivarc'h, V. Kaimanovich and F. Ledrappier (see [20, Conjecture 1.21] states that for a finitely generated subgroup of $\operatorname{PSL}(2, \mathbb{R})$ acting smoothly on the circle, the stationary measure is singular. The conjecture was proved by Y. Guivarc'h and Y. Le Jan in [39] for non-cocompact subgroups and by B. Deroin, V. Kleptsyn and A. Navas in [20] for some minimal actions of the Thompson group and subgroups of $\operatorname{PSL}(2, \mathbb{R})$ by $C^{2}$-diffeomorphisms. On the other hand, the absolute continuity of the stationary measure was proved to hold for a number of random systems of non-homeomorphic maps of the interval (usually expanding at least at average), see e.g. [4, 16, 75].

Let us note that the question of determining singularity or absolute continuity of the stationary measure is non-trivial even in the apparently simple case of two contracting similarities $f_{1}, f_{2}$ of the unit interval $[0,1]$, given by $f_{1}(x)=\lambda x, f_{2}(x)=\lambda x+1-\lambda$ for $\lambda \in(0,1)$. Then the unique stationary measure $\nu_{\lambda}$ for the probability vector $(1 / 2,1 / 2)$ is called the symmetric Bernoulli convolution and is always either singular or absolutely continuous. It is known (see [86]) that the set of parameters $\lambda>1 / 2$ for which $\nu_{\lambda}$ is singular has Hausdorff dimension zero, and the only known values of "singular" parameters are the reciprocals of the Pisot numbers, as proved in [28]. It is a long-standing open question whether these are the only examples of singular Bernoulli convolutions. Despite many results in this direction, a complete answer is still unknown and stimulates an active research. See e.g. [76, 95] for comprehensive surveys on the subject.

Another approach to proving typical singularity of stationary measures for systems on the interval with positive endpoint Lyapunov exponents was taken recently in [18]. The authors consider much more general family of systems, namely all the systems $\left\{f_{1}, \ldots, f_{m}\right\}$ (together with probability vectors $\left(p_{1}, \ldots, p_{m}\right)$ ) consisting of orientation preserving homeomorphisms of the unit interval which are $C^{1}$ in the neighbourhoods of the endpoints, satisfying $\Lambda(0), \Lambda(1)>$ 0 and such that for every $x \in(0,1)$ there exist $i, j \in\{1, \ldots, m\}$ such that $f_{i}(x)<x<$ $f_{j}(x)$. They prove that for a topologically generic system in such a family, the corresponding stationary measure is singular. This however does not imply generic singularity of the stationy measure for $A M$-systems, as they form a meager subset of this family.

To our knowledge, Theorems 1.6 and 1.7 are the first explicit examples of non-atomic singular stationary measures for non-expanding random systems generated by semigroups of piecewise affine homeomorphisms of the circle of that type (note that an $A M$-system can be considered as a pair of homeomorphisms of the circle with a unique common fixed point). The fact that the maps are piecewise affine is especially interesting, since such systems are studied intensively and often serve as models for smooth systems (see e.g. [70, Questions 12 and 16]).

Notice that in the resonant case mentioned above, the stationary measure is supported on an exceptional minimal set (i.e. invariant Cantor sets where the systems is minimal), while in the non-resonant one, its support is equal to the entire interval [0, 1] (see Proposition 4.6). It should be noted that the properties of exceptional minimal sets are a well-known subject of interest, especially in the context of the groups of diffeomorphisms. For instance, a conjecture of Ghys and Sullivan says that exceptional minimal sets for groups of $C^{2}$-diffeomorphisms have

Lebesgue measure zero. The hypothesis has been recently verified by B. Deroin, V. Kleptsyn and A. Navas [21] for real-analytic diffeomorphisms, while the question remains open in the smooth case. Our work contributes to the study of such sets for piecewise affine systems.

### 1.3 Organization of the dissertation

The dissertation is organized as follows.
Chapter 2 contains preliminary material on the dimension theory of sets and measures together with basic facts on invariant and stationary measures.

Chapter 3 contains results on probabilistic embedding theorems (Section 3.2), probabilistic Takens theorem (Section 3.3) and conjectures of Schroer, Sauer, Ott and Yorke (Section 3.4). It also contains several examples (Sections 3.5 and 3.6).

Chapter 4 contains results on singular stationary measures for $A M$-systems. The proof of Theorem 1.6 is divided into two parts: Section 4.4 (case $l=1$ ) and Section 4.5 (case $l>1$ ). Proof of Theorem 1.7 can be found in Section 4.8.

## Chapter 2

## Preliminaries

In this chapter we present basic notions which will be used throughout the dissertation. We focus on the dimension theory of sets and measures, as well as the notions of invariant and stationary measures. It will be sufficient for us to consider only subsets of the Euclidean space $\mathbb{R}^{N}$, however some parts of the presented theory extend to the more general setting of metric spaces. We fill concentrate on notions and properties which will be used throughout this dissertation. For more information on dimension theory in Euclidean spaces see e.g. [30, 63, 81 .

Let us begin by fixing some notation. Consider the Euclidean space $\mathbb{R}^{N}$ for $N \in \mathbb{N}$, with the standard inner product $\langle\cdot, \cdot\rangle$ and the corresponding norm $\|\cdot\|$. The open $\delta$-ball around a point $x \in \mathbb{R}^{N}$ is denoted by $B_{N}(x, \delta)$. We will often denote it by $B(x, \delta)$, if the dimension is clear from the context. By $|X|$ we denote the diameter of a set $X \subset \mathbb{R}^{N}$. By $\bar{X}$ we will denote the closure of a set $X$ and by $\operatorname{Int}(X)$ its interior. A measurable space is a pair $(X, \mathcal{A})$ consisting of a set $X$ and a $\sigma$-algebra $\mathcal{A}$ of subsets of $X$. Elements of $\mathcal{A}$ are called measurable sets. By a measure on a measurable space $(X, \mathcal{A})$ we will understand a $\sigma$-additive function $\mu: \mathcal{A} \rightarrow[0, \infty]$ satisfying $\mu(\emptyset)=0$. We say that a measurable set $Y \subset X$ is of full measure $\mu$, if $\mu(X \backslash Y)=0$. A measure $\mu$ on a measurable space $(X, \mathcal{A})$ is called probabilistic if $\mu(X)=1$, finite if $\mu(X)<\infty$, and $\sigma$-finite if there exists a countable collection of measurable sets $A_{n}, n \in \mathbb{N}$ such that $\mu\left(A_{n}\right)<\infty$ for each $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_{n}=X$. For a measurable set $A$, we denote by $\left.\mu\right|_{A}$ or $\mu \upharpoonright A$ the restriction of $\mu$ to the set $A$. Measures $\mu$ and $\nu$ on a measurable space $(X, \mathcal{A})$ are called (mutually) singular if there exists a measurable set $A$ such that $\mu(A)=\nu(X \backslash A)=0$. We denote this fact by $\mu \perp \nu$. The Euclidean space $\mathbb{R}^{N}$ will be by default considered with the Borel $\sigma$-algebra (i.e. the $\sigma$-algebra generated by open sets). A Borel measure is a measure $\mu$ defined on the $\sigma$-algebra of Borel sets. The support of a Borel measure $\mu$ on $\mathbb{R}^{N}$ is the closed set $\operatorname{supp}(\mu)=\left\{x \in \mathbb{R}^{N}: \underset{\delta>0}{\forall} \mu(B(x, \delta))>0\right\}$. For a measure $\mu$ on a measurable space $(X, \mathcal{A})$ and a measurable function $f: X \rightarrow \mathbb{R}$, the essential supremum and essential infimum of $f$ with respect to $\mu$ are defined as

$$
\underset{x \sim \mu}{\operatorname{ess} \sup } f(x):=\inf \left\{\sup _{x \in A} f(x): A \in \mathcal{A} \text { and } \mu(X \backslash A)=0\right\}
$$

and

$$
\underset{x \sim \mu}{\operatorname{ess} \inf } f(x):=\sup \left\{\inf _{x \in A} f(x): A \in \mathcal{A} \text { and } \mu(X \backslash A)=0\right\}
$$

respectively. By Leb we will denote the Lebesgue measure. We take all the logarithms in the base 2. Note however that most of the considered notions (in particular: dimensions) do not depend on the choice of the base of the logarithms.

### 2.1 Dimensions of sets

For a bounded set $X \subset \mathbb{R}^{N}$ and $\delta>0$, let $N(X, \delta)$ denote the minimal number of balls of diameter at most $\delta$ required to cover $X$. The lower and upper box-counting (Minkowski) dimensions of $X$ are defined as

$$
\underline{\operatorname{dim}}_{B} X=\liminf _{\delta \rightarrow 0} \frac{\log N(X, \delta)}{-\log \delta} \quad \text { and } \quad \overline{\operatorname{dim}}_{B} X=\limsup _{\delta \rightarrow 0} \frac{\log N(X, \delta)}{-\log \delta}
$$

If $\underline{\operatorname{dim}}_{B} X=\overline{\operatorname{dim}}_{B} X$, then we denote their common value as $\operatorname{dim}_{B} X$ and call it the boxcounting (Minowski) dimension of $X$. The lower (resp. upper) box-counting dimension of an unbounded set is defined as the supremum of the lower (resp. upper) box-counting dimensions of its bounded subsets. The lower and upper modified box-counting (Minkowski) dimensions of $X \subset \mathbb{R}^{N}$ are defined as

$$
\begin{aligned}
& \underline{\operatorname{dim}}_{M B} X=\inf \left\{\sup _{i \in \mathbb{N}} \underline{\operatorname{dim}}_{B} K_{i}: X \subset \bigcup_{i=1}^{\infty} K_{i}, K_{i} \text { compact }\right\} \\
& \overline{\operatorname{dim}}_{M B} X=\inf \left\{\sup _{i \in \mathbb{N}} \overline{\operatorname{dim}}_{B} K_{i}: X \subset \bigcup_{i=1}^{\infty} K_{i}, K_{i} \text { compact }\right\}
\end{aligned}
$$

If $\underline{\operatorname{dim}}_{M B} X=\overline{\operatorname{dim}}_{M B} X$, then we denote their common value by $\operatorname{dim}_{M B} X$ and call it the modified box-counting (Minowski) dimension of $X$. For $s>0$, the $s$-dimensional (outer) Hausdorff measure of a set $X \subset \mathbb{R}^{N}$ is defined as

$$
\mathcal{H}^{s}(X)=\lim _{\delta \rightarrow 0} \inf \left\{\sum_{i=1}^{\infty}\left|U_{i}\right|^{s}: X \subset \bigcup_{i=1}^{\infty} U_{i},\left|U_{i}\right| \leqslant \delta\right\}
$$

The outer measure $\mathcal{H}^{s}$ becomes a measure after restricting to Borel sets. The Hausdorff dimension of $X$ is given as

$$
\operatorname{dim}_{H} X=\inf \left\{s>0: \mathcal{H}^{s}(X)=0\right\}
$$

or, equivalently, as

$$
\operatorname{dim}_{H} X=\sup \left\{s>0: \mathcal{H}^{s}(X)=\infty\right\}
$$

See [30, Chapter 3] for more details. With this notation, the following inequalities hold for $X \subset \mathbb{R}^{N}$ :

$$
\begin{align*}
& \operatorname{dim}_{H} X \leqslant \underline{\operatorname{dim}}_{M B} X \leqslant \overline{\operatorname{dim}}_{M B} X \leqslant \overline{\operatorname{dim}}_{B} X \leqslant N  \tag{2.1}\\
& \operatorname{dim}_{H} X \leqslant \underline{\operatorname{dim}}_{M B} X \leqslant \underline{\operatorname{dim}}_{B} X \leqslant \overline{\operatorname{dim}}_{B} X \leqslant N
\end{align*}
$$

The following proposition states basic properties of dimensions, which will be used throughout this work.

Proposition 2.1. For any sets $X, X_{1}, X_{2}, \ldots \subset \mathbb{R}^{N}$ the following holds
(1) if $X_{1} \subset X_{2}$, then $\operatorname{dim} X_{1} \leqslant \operatorname{dim} X_{2}$ for any notion of the dimension defined above,
(2) $\underline{\operatorname{dim}}_{B} X=\underline{\operatorname{dim}}_{B} \bar{X}$ and $\overline{\operatorname{dim}}_{B} X=\overline{\operatorname{dim}}_{B} \bar{X}$,
(3) $\overline{\operatorname{dim}}_{B}\left(X_{1} \cup X_{2}\right)=\max \left\{\overline{\operatorname{dim}}_{B} X_{1}, \overline{\operatorname{dim}}_{B} X_{2}\right\}$,
(4) $\overline{\operatorname{dim}}_{B}\left(X_{1} \times X_{2}\right) \leqslant \overline{\operatorname{dim}}_{B}\left(X_{1}\right)+\overline{\operatorname{dim}}_{B}\left(X_{2}\right)$
(5) $\operatorname{dim}\left(\bigcup_{j=1}^{\infty} X_{j}\right)=\sup \left\{\operatorname{dim} X_{j}: j=1,2, \ldots\right\}$, where $\operatorname{dim}$ can denote any of $\operatorname{dim}_{H}, \underline{\operatorname{dim}_{M B}}$ $\underline{j=1}$ and $\overline{\operatorname{dim}}_{M B}$,
(6) let $f: X \rightarrow \mathbb{R}^{k}$ be a Lipschitz map. Then $\operatorname{dim}(f(X)) \leqslant \operatorname{dim}(X)$ for any notion of the dimension defined above.

For the proof see [30, Chapters 2 and 3].
In some cases, it will be convenient for us to consider covers by dyadic cubes rather than Euclidean balls. It turns out, that it gives rise to equivalent notions of dimension. Let us confine ourselves to subsets of $[0,1]^{N}$. For $n \in \mathbb{N}$ and $x_{1}, \ldots, x_{n} \in\{0,1\}$, let $\left[x_{1}, \ldots, x_{n}\right]$ denote the dyadic interval of length $2^{-n}$ corresponding to the sequence $\left(x_{1}, \ldots, x_{n}\right)$, i.e.

$$
\left[x_{1}, \ldots, x_{n}\right]= \begin{cases}{\left[\sum_{j=1}^{n} \frac{x_{j}}{2^{j}}, \sum_{j=1}^{n} \frac{x_{j}}{2^{j}}+\frac{1}{2^{n}}\right)} & \text { if } \sum_{j=1}^{n} \frac{x_{j}}{2^{j}}+\frac{1}{2^{n}}<1 \\ {\left[1-\frac{1}{2^{n}}, 1\right]} & \text { otherwise. }\end{cases}
$$

A dyadic cube of sidelength $2^{-n}$ in $[0,1]^{N}$ is an $N$-fold product of dyadic intervals of length $2^{-n}$. For $n \in \mathbb{N}$ and $x \in[0,1]^{N}$ let $D_{n}(x)$ be the unique dyadic cube of sidelength $2^{-n}$ containing $x$. Let $N^{\prime}\left(X, 2^{-n}\right)$ be the number of dyadic cubes of sidelength $2^{-n}$ intersecting $X$. Then (see e.g. [30, Section 2.1])

$$
\begin{equation*}
\underline{\operatorname{dim}}_{B} X=\liminf _{n \rightarrow \infty} \frac{\log N^{\prime}\left(X, 2^{-n}\right)}{n \log 2} \text { and } \overline{\operatorname{dim}}_{B} X=\limsup _{n \rightarrow \infty} \frac{\log N^{\prime}\left(X, 2^{-n}\right)}{n \log 2} \tag{2.2}
\end{equation*}
$$

### 2.2 Dimensions of measures

We define the Hausdorff dimension of a $\sigma$-finite Borel measure $\mu$ on $\mathbb{R}^{N}$ as

$$
\operatorname{dim}_{H}(\mu)=\inf \left\{\operatorname{dim}_{H} X: X \subset \mathbb{R}^{N} \text { is a Borel set of full measure } \mu\right\}
$$

and the upper and lower modified box-counting dimensions of $\mu$ as

$$
\begin{aligned}
& \underline{\operatorname{dim}}_{M B}(\mu)=\inf \left\{\underline{\operatorname{dim}}_{M B} X: X \subset \mathbb{R}^{N} \text { is a Borel set of full measure } \mu\right\}, \\
& \overline{\operatorname{dim}}_{M B}(\mu)=\inf \left\{\overline{\operatorname{dim}}_{M B} X: X \subset \mathbb{R}^{N} \text { is a Borel set of full measure } \mu\right\} .
\end{aligned}
$$

Note that a similar definition for the box-counting dimensions would give simply the dimension of the topological support $\operatorname{supp}(\mu)$ of the measure $\mu$, as box counting dimensions are stable under taking closure (Prop. 2.1.(2)). We will consider also the lower Hausdorff dimension of a $\sigma$-finite Borel measure $\mu$ on $\mathbb{R}^{N}$ defined as

$$
\underline{\operatorname{dim}}_{H}(\mu)=\inf \left\{\operatorname{dim}_{H} X: X \subset \mathbb{R}^{N} \text { is a Borel set of positive measure } \mu\right\}
$$

The inequalities (2.1) give

$$
\begin{equation*}
\underline{\operatorname{dim}}_{H}(\mu) \leqslant \operatorname{dim}_{H}(\mu) \leqslant \underline{\operatorname{dim}}_{M B}(\mu) \leqslant \overline{\operatorname{dim}}_{M B}(\mu) \tag{2.3}
\end{equation*}
$$

For a Borel probability measure $\mu$ on $\mathbb{R}^{N}$ with compact support define lower and upper information dimensions of $\mu$ as

$$
\underline{\mathrm{ID}}(\mu)=\liminf _{r \rightarrow 0} \int_{\operatorname{supp}(\mu)} \frac{\log \mu(B(x, r))}{\log r} d \mu(x) \text { and } \overline{\mathrm{ID}}(\mu)=\limsup _{r \rightarrow 0} \int_{\operatorname{supp}(\mu)} \frac{\log \mu(B(x, r))}{\log r} d \mu(x) .
$$

If $\underline{\mathrm{ID}}(\mu)=\overline{\mathrm{ID}}(\mu)$, then we denote their common value as $\operatorname{ID}(\mu)$ and call it the information dimension of $\mu$.

Remark 2.2 Information dimensions are often defined as

$$
\begin{equation*}
\underline{\mathrm{ID}}(\mu)=\liminf _{\varepsilon \rightarrow 0} \frac{1}{\log \varepsilon} \sum_{C \in \mathcal{C}_{\varepsilon}} \mu(C) \log \mu(C) \text { and } \overline{\mathrm{ID}}(\mu)=\limsup _{\varepsilon \rightarrow 0} \frac{1}{\log \varepsilon} \sum_{C \in \mathcal{C}_{\varepsilon}} \mu(C) \log \mu(C) \tag{2.4}
\end{equation*}
$$

where $\mathcal{C}_{\varepsilon}$ is the partition of $\mathbb{R}^{N}$ into cubes with side length $\varepsilon$ and vertices on the lattice $(\varepsilon \mathbb{Z})^{N}$. These definitions are equivalent with the previous ones (see e.g. [98, Appendix I]).

Definition 2.3 Let $\mu$ be a finite Borel measure on $\mathbb{R}^{N}$. The lower and upper local dimensions of a point $x \in \mathbb{R}^{N}$ are defined as

$$
\underline{d}(\mu, x)=\liminf _{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \text { and } \bar{d}(\mu, x)=\limsup _{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r}
$$

(we set $\underline{d}(\mu, x)=\bar{d}(\mu, x)$ for $x \notin \operatorname{supp}(\mu))$. We say that $\mu$ is lower (resp. upper) exactdimensional if $\underline{d}(\mu, x)=$ const $\mu$-almost surely (resp. $\bar{d}(\mu, x)=$ const $\mu$-almost surely). We say that $\mu$ is exact-dimensional if $\underline{d}(\mu, x)=\bar{d}(\mu, x)=$ const $\mu$-almost surely.

Proposition 2.4. Let $\mu$ be a Borel probability measure on $\mathbb{R}^{N}$ with compact support. Then
(1) $\operatorname{dim}_{H}(\mu)=\underset{x \sim \mu}{\operatorname{ess} \sup } \underline{d}(\mu, x)$,
(2) $\underline{\operatorname{dim}}_{H}(\mu)=\underset{x \sim \mu}{\operatorname{ess} \inf } \underline{d}(\mu, x)$,
(3) $\overline{\operatorname{dim}}_{M B}(\mu)=\underset{x \sim \mu}{\operatorname{ess} \sup } \bar{d}(\mu, x)$
(4) $\underline{\operatorname{dim}}_{H}(\mu) \leqslant \underline{\operatorname{ID}}(\mu)$,
(5) if $\mu$ is lower exact-dimensional, then $\underline{\operatorname{dim}}_{H}(\mu)=\operatorname{dim}_{H}(\mu) \leqslant \underline{\mathrm{ID}}(\mu)$,
(6) if $\mu$ is exact-dimensional, then $\underline{\operatorname{dim}}_{H}(\mu)=\operatorname{dim}_{H}(\mu)=\operatorname{ID}(\mu)=\underline{\operatorname{dim}}_{M B}(\mu)=\overline{\operatorname{dim}}_{M B}(\mu)$,

Proof. For (1) - (3) see [29, Propositions 10.2 and 10.3] together with [30, Proposition 3.9]. For (4) see [31, Thm 1.3] (see also [10, Thm. 2.1]). The statement (5) follows from (1), (2) and (4), while (6) follows from the previous points and [31, Thm 1.3].

It turns out that the Hausdorff dimension and the information dimension behave differently under taking convex combinations of measures.

Proposition 2.5. (1) Let $\mu_{1}, \mu_{2}, \ldots$ be Borel probability measures on $\mathbb{R}^{N}$ and let $\left(p_{1}, p_{2}, \ldots\right)$ be a probability vector with positive entries. Then

$$
\operatorname{dim}_{H}\left(\sum_{j=1}^{\infty} p_{j} \mu_{j}\right)=\sup \left\{\operatorname{dim}_{H}\left(\mu_{j}\right): j=1,2, \ldots\right\}
$$

(2) Let $\mu$ and $\nu$ be Borel probability measures with compact support on $\mathbb{R}^{N}$ and let $p \in(0,1)$. Then

$$
\overline{\mathrm{ID}}(p \mu+(1-p) \nu) \leqslant p \overline{\mathrm{ID}}(\mu)+(1-p) \overline{\mathrm{ID}}(\nu)
$$

and

$$
\underline{\mathrm{ID}}(p \mu+(1-p) \nu) \geqslant p \underline{\mathrm{ID}}(\mu)+(1-p) \underline{\mathrm{ID}}(\nu)
$$

Consequently

$$
\operatorname{ID}(p \mu+(1-p) \nu)=p \operatorname{ID}(\mu)+(1-p) \operatorname{ID}(\nu)
$$

provided that $\operatorname{ID}(\mu)$ and $\operatorname{ID}(\nu)$ exist.

Proof. For (1) see [44, Cor. 3.18], while (2) follows by applying [38, Lemma 3.4] to (2.4).

Similarly as for sets, one can equivalently define Hausdorff dimensions of measures in terms of local dimensions corresponding to dyadic cubes. More precisely, for a finite Borel measure $\mu$ on $\mathbb{R}^{N}$ and $x \in \mathbb{R}^{N}$ define the lower and upper dyadic local dimension of $\mu$ at $x$ as

$$
\underline{d}^{\prime}(\mu, x)=\liminf _{n \rightarrow \infty} \frac{-\log \mu\left(D_{n}(x)\right)}{n \log 2}, \quad \bar{d}^{\prime}(\mu, x)=\limsup _{n \rightarrow \infty} \frac{-\log \mu\left(D_{n}(x)\right)}{n \log 2} .
$$

It turns out that switching from Euclidean balls to dyadic cubes can alter the local dimensions only on a set of measure zero:

Proposition 2.6 ([44, Proposition 3.20]). Let $\mu$ be a finite Borel measure on $\mathbb{R}^{N}$. Then

$$
\underline{d}^{\prime}(\mu, x)=\underline{d}(\mu, x) \text { and } \bar{d}^{\prime}(\mu, x)=\bar{d}(\mu, x) \text { for } \mu \text {-a.e. } x \in \mathbb{R}^{N} .
$$

Consequently, we can replace $\underline{d}(\mu, x)$ with $\underline{d}^{\prime}(\mu, x)$ and $\bar{d}(\mu, x)$ with $\bar{d}^{\prime}(\mu, x)$ in Proposition 2.4 .
Note that the above result is proved in [44 only for the lower local dimensions. The upper case is analogous.

In this dissertation we will focus mainly on $\operatorname{dim}_{H}$, as our embedding theorems of Chapter 3 will be formulated in its terms. The stationary measures from Chapter 4 will turn out to be exact dimensional, hence all the above notions of dimension will coincide for them. The modified box-counting dimensions of a measure will appear only in Sections 3.2 and 3.5 , where we will compare probabilistic embedding theorem for $\operatorname{dim}_{H}$ with results obtained previously in 【1 and give an explicit example of a measure $\mu$ with $\operatorname{dim}_{H}(\mu)<\operatorname{dim}_{M B}(\mu)$. Information dimension will play a role in Section 3.4, as the conjectures of Shroer, Sauer, Ott and Yorke [85] are stated in its terms. We will however observe that neither the information dimension nor the lower Hausdorff dimension are well suited for almost sure embedding problems for arbitrary finite measures (see Example 3.33).

### 2.3 Invariant and stationary measures

Let us now introduce the notions of an invariant measure (for a deterministic dynamical system) and a stationary measure (for a random dynamical system).

Definition 2.7 Let $(X, \mathcal{A})$ be a measurable space and let $T: X \rightarrow X$ be a measurable map. For a measure $\mu$ on $(X, \mathcal{A})$, we denote by $T_{*} \mu$ the transport of $\mu$ by $T$, i.e. the measure defined as

$$
T_{*} \mu(A)=\mu\left(T^{-1} A\right) \text { for } A \in \mathcal{A} .
$$

Measure $\mu$ is called $T$-invariant if $T_{*} \mu=\mu$. Measure $\mu$ is called ergodic if any $T$-invariant set is either of full or zero measure, i.e. for any $A \in \mathcal{A}$, the condition $A=T^{-1}(A)$ implies $\mu(A)=0$ or $\mu(X \backslash A)=0$.

Ergodic measures for Lipschitz transformations are always lower (and upper) exact dimensional:

Proposition 2.8 ([29, Prop 10.6]). Let $X \subset \mathbb{R}^{N}$ be closed, let $T: X \rightarrow X$ be a Lipschitz map and let $\mu$ be a T-invariant and ergodic Borel probability measure. Then $\mu$ is lower (and upper) exact-dimensional and consequently $\operatorname{dim}_{H}(\mu)=\operatorname{dim}_{H}(\mu) \leqslant \underline{\operatorname{ID}}(\mu)$.

Let us now consider the case of random dynamics arising from iterated function systems.
Definition 2.9 Let $f_{1}, f_{2}, \ldots, f_{m}: X \rightarrow X$ be a collection of measurable maps on a measurable space $(X, \mathcal{A})$ and let $\left(p_{1}, \ldots, p_{m}\right)$ be a probabilistic vector. A measure $\mu$ on $(X, \mathcal{A})$ is called stationary if

$$
\mu=\sum_{j=1}^{m} p_{j}\left(f_{j}\right)_{*} \mu
$$

Let $\mathcal{M}$ denote the set of probability measures on $(X, \mathcal{A})$. The Markov operator $\mathcal{T}: \mathcal{M} \rightarrow \mathcal{M}$ is defined as

$$
\mathcal{T} \mu=\sum_{j=1}^{m} p_{j}\left(f_{j}\right)_{*} \mu
$$

Note that the set of stationary probability measures coincide with the set of fixed points of the Markov operator $\mathcal{T}$.

Consider now $X=[0,1]$ equipped with the Borel $\sigma$-algebra and assume that all $f_{1}, f_{2}, \ldots, f_{m}$ are piecewise differentiable homeomorphisms with $f_{j}^{\prime}>0$ (this will be the case in Chapter (4).

Definition 2.10 The Perron-Frobenius (or transfer) operator $T: L^{1}([0,1]$, Leb) $\rightarrow$ $L^{1}([0,1], \mathrm{Leb})$ is defined as

$$
T g=\sum_{j=1}^{m} p_{j}\left(f_{j}^{-1}\right)^{\prime} g \circ f_{j}^{-1}
$$

Note that the Perron-Frobenius operator transforms a density (Radon-Nikodym derivative) of a probability measure to the density of its image under the Markov operator (i.e. $\left.\frac{d P \mu}{d \mathrm{Leb}}=T\left(\frac{d \mu}{d \mathrm{Leb}}\right)\right)$.

Similarly as for the Markov operator, the stationary densities (densities of stationary measures with respect to the Lebesgue measure) are the fixed points of the Perron-Frobenius operator $T$.

One of the basic cases of calculation of the dimension of sets and stationary measures are the self-similar sets and measures. Let us begin by the classical result guaranteeing the existence of these objects.

Proposition 2.11 ([30, Thm 9.1] and [26, Prop. 3.3.15]). Let $f_{j}: \mathbb{R} \rightarrow \mathbb{R}, j=1, \ldots, m$ be strictly contracting, i.e. there exists $\lambda \in(0,1)$ such that $\left|f_{j}(x)-f_{j}(y)\right| \leqslant \lambda|x-y|$ for every $x, y \in \mathbb{R}$ and $j \in\{1, \ldots, m\}$. Then, there exists a unique non-empty compact set $X \subset \mathbb{R}$, called the attractor of the system $\left\{f_{1}, \ldots, f_{m}\right\}$, satisfying

$$
X=\bigcup_{j=1}^{m} f_{j}(X)
$$

Moreover, for any non-empty compact set $A \subset \mathbb{R}$, the following equality holds

$$
X=\bigcap_{n=1}^{\infty} \bigcup_{i_{1}, \ldots, i_{n}=1}^{m} f_{i_{1}} \circ \ldots \circ f_{i_{n}}(A)
$$

Similarly, let $\left(p_{1}, \ldots, p_{m}\right)$ be a probability vector. There exists a unique stationary Borel probability measure $\mu$ on $\mathbb{R}$ for the system $\left\{f_{1}, \ldots, f_{m}\right\}$ with probabilities $\left(p_{1}, \ldots, p_{m}\right)$.

Definition 2.12 Let $f_{j}: \mathbb{R} \rightarrow \mathbb{R}, j=1, \ldots, m$ be as in Proposition 2.11. The system $\left\{f_{1}, \ldots, f_{m}\right\}$ satisfies the Strong Separation Condition if $f_{i}(X) \cap f_{j}(X)=\bar{\emptyset}$ for $i \neq j$, where $X$ is the attractor of the system.

The attractor of a contracting iterated function system, as well as its stationary measure, can be described in terms of the natural projection from the symbolic space. We will make use of this description in Chapter 4.

Proposition 2.13. Let $f_{j}: \mathbb{R} \rightarrow \mathbb{R}, j=1, \ldots, m$ be as in Proposition 2.11. Fix $x \in \mathbb{R}$. Then the natural projection $\pi:\{1, \ldots, m\}^{\mathbb{N}} \rightarrow \mathbb{R}$ given by

$$
\pi\left(i_{1}, i_{2}, \ldots\right)=\lim _{n \rightarrow \infty} f_{i_{1}} \circ f_{i_{2}} \circ \ldots \circ f_{i_{n}}(x)
$$

is well defined and does not depend on $x$. Moreover, the attractor $X$ of the system $\left\{f_{1}, \ldots, f_{m}\right\}$ satisfies $X=\pi\left(\{1, \ldots, m\}^{\mathbb{N}}\right)$. Similarly, for a fixed probability vector $\left(p_{1}, \ldots, p_{m}\right)$, the corresponding stationary probability measure $\mu$ satisfies equality $\mu=\pi_{*}\left(\left(p_{1}, \ldots, p_{m}\right)^{\otimes \mathbb{N}}\right)$, where $\left(p_{1}, \ldots, p_{m}\right)^{\otimes \mathbb{N}}$ is the Bernoulli measure corresponding to the probability vector $\left(p_{1}, \ldots, p_{m}\right)$. If the system $\left\{f_{1}, \ldots, f_{m}\right\}$ satisfies the Strong Separation Condition, then $\pi$ is injective (hence it is a homeomorphism between $\{1, \ldots, m\}^{\mathbb{N}}$ and $X$ ).
See [49] for details.
Definition 2.14 Let $f_{j}: \mathbb{R} \rightarrow \mathbb{R}, j=1, \ldots, m$ be contracting similarities, i.e. maps of the form $f_{j}(x)=\lambda_{j} x+t_{j}$ for some $\lambda_{j}, t_{j} \in \mathbb{R}$ satisfying $0<\left|\lambda_{j}\right|<1$. Then the attractor of the system $\left\{f_{1}, \ldots, f_{m}\right\}$ is called a self-similar set and any stationary measure for this system is called a self-similar measure.

It turns out that it is easy to calculate dimension of self-similar sets and measures for systems satisfying the Strong Separation Condition. We will apply this formulas in Chapter 4 , where the stationary measures for $A M$-systems will turn out to be infinite convex combinations of self-similar measures.
Theorem 2.15 ([30, Thm 9.3] and [26, Thm 5.2.5]). Let $f_{j}: \mathbb{R} \rightarrow \mathbb{R}, j=1, \ldots, m$ be of the form $f_{j}(x)=\lambda_{j} x+t_{j}$, where $\lambda_{j}, t_{j} \in \mathbb{R}$ and $0<\left|\lambda_{j}\right|<1$ for each $j=1, \ldots, m$. Assume additionally that $f_{i}(X) \cap f_{j}(X)=\emptyset$ for $i \neq j$, where $X$ is the attractor of the system $\left\{f_{1}, \ldots, f_{m}\right\}$. Then the box-counting dimension of $X$ exists and $\operatorname{dim}_{H} X=\operatorname{dim}_{B} X=d$, where $d$ is the unique number $d \in[0,1]$ satisfying

$$
\sum_{j=1}^{m}\left|\lambda_{j}\right|^{d}=1 .
$$

Let $\left(p_{1}, \ldots, p_{m}\right)$ be a probability vector. Then the corresponding self-similar measure is exact dimensional and

$$
\operatorname{dim}_{H} \mu=\frac{\sum_{j=1}^{m} p_{j} \log p_{j}}{\sum_{j=1}^{m} p_{j} \log \left|\lambda_{j}\right|}
$$

Moreover, if all $p_{j}$ are strictly positive, then $\operatorname{supp}(\mu)=X$.
Remark 2.16 It is enough to assume weaker separation condition than $f_{i}(X) \cap f_{j}(X)=\emptyset$ for $i \neq j$. Namely, it is enough to assume the Open Set Condition, i.e. the existence of an open bounded non-empty set $U \subset \mathbb{R}$ such that $\bigcup_{j=1}^{m} f_{j}(U) \subset U$ with the sum being disjoint. However, the natural projection $\pi$ does not have to be injective in that case.
The formula for the dimension of a self-similar measure is given in terms of the entropy $H(p)$ of the probability vector $p=\left(p_{1}, \ldots, p_{m}\right)$, defined as

$$
H(p)=-\sum_{j=1}^{m} p_{j} \log p_{j},
$$

and the Lyapunov exponent $\chi(p)$ of the system, defined as

$$
\chi(p)=\sum_{j=1}^{m} p_{j} \log \left|\lambda_{j}\right| .
$$

In this notation, the above formula yields $\operatorname{dim}_{H} \mu=\frac{H(p)}{-\chi(p)}$. In the overlapping case the equality might no longer be valid, yet the inequality $\operatorname{dim}_{H} \mu \leqslant \frac{H(p)}{-\chi(p)}$ still holds. Actually, this type of inequality holds in much greater generality (see [51]) and we will use it in Section 4.8.

In Chapter 4 we will also make use of an extension of Theorem 2.15 to infinite collections of similarities on the interval (following [64]; see the proof of Proposition 4.62 for the details).

## Chapter 3

## A probabilistic Takens theorem

In this chapter we prove probabilistic Takens theorem and several accompanying results. The main results of this chapter are: Probabilistic embedding theorem 3.5, Probabilistic Takens theorem 3.15) and its extension 3.24. The latter allows us to prove [85, Conjecture 1] for ergodic measures (see Theorem 3.22). These are full versions of results formulated in Section 1.1. We also give an example of a measure with Hausdorff dimension strictly smaller than lower modified box dimension (see Theorem 3.29) and several further examples. All the results of this chapter, except for Section 3.4 (including Theorems 3.22 and 3.24 ), are taken from [8].

The chapter is organized as follows. In Section 3.1 we introduce notation, definitions and preliminary results used in the proofs of main theorems. Section 3.2 contains the formulation and proof of the extended version of the probabilistic embedding theorem (Theorem 3.5), while Section 3.3 is devoted to the proof of the extended version of the probabilistic Takens delay embedding theorem (Theorem 3.15). Both of these sections contain also historical remarks. In Section 3.4 we prove conjecture of Sauer, Shroer, Ott and Yorke 85 for ergodic invariant measures. Section 3.5 contains an example of a measure with Hausdorff dimension smaller than lower modified box dimension. In Section 3.6 we present several further examples. Apart from results and their proofs, Sections 3.2 and 3.3 contain historical remarks on embedding and Takens theorems.

### 3.1 Preliminaries

We say that function $\phi: X \rightarrow \mathbb{R}^{k}, X \subset \mathbb{R}^{N}$ is locally $\beta$-Hölder for $\beta>0$ if for every $x \in X$ there exists an open set $U \subset \mathbb{R}^{N}$ containing $x$ such that $\phi$ is $\beta$-Hölder on $U \cap X$, i.e.

$$
\begin{equation*}
\underset{C>0}{\exists} \underset{x, y \in U \cap X}{\forall}\|\phi(x)-\phi(y)\| \leqslant C\|x-y\|^{\beta} . \tag{3.1}
\end{equation*}
$$

Similarly, we say the $\phi$ is $\beta$-Hölder on bounded sets if for every bounded set $U \subset \mathbb{R}^{N}, \phi$ is $\beta$-Hölder on $U \cap X$ (i.e. (3.1) holds). We say that $\phi$ is Lipschitz (locally/on bounded sets) if it is 1-Hölder (locally/on bounded sets). Note that if $\phi: X \rightarrow \mathbb{R}^{k}$ is $\beta$-Hölder on bounded sets then it is also locally $\beta$-Hölder. The converse holds if $X$ is closed (but not for arbitrary $X \subset \mathbb{R}^{N}$ ). For $k \leqslant N$ we write $\operatorname{Gr}(k, N)$ for the $(k, N)$-Grassmannian, i.e. the space of all $k$-dimensional linear subspaces of $\mathbb{R}^{N}$, equipped with the standard rotation-invariant (Haar) measure (see [63, Section 3.9]) . By $\eta_{N}$ we denote the normalized Lebesgue measure on the unit ball $B_{N}(0,1)$, i.e.

$$
\eta_{N}=\left.\frac{1}{\kappa_{N}} \operatorname{Leb}\right|_{B_{N}(0,1)},
$$

where Leb is the Lebesgue measure on $\mathbb{R}^{N}$ and $\kappa_{N}=\operatorname{Leb}\left(B_{N}(0,1)\right)$.
For $N, k \in \mathbb{N}$ let $\operatorname{Lin}\left(\mathbb{R}^{N} ; \mathbb{R}^{k}\right)$ be the space of all linear transformations $L: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$. Such transformations are given by

$$
\begin{equation*}
L x=\left(\left\langle l_{1}, x\right\rangle, \ldots,\left\langle l_{k}, x\right\rangle\right), \tag{3.2}
\end{equation*}
$$

where $l_{1}, \ldots, l_{k} \in \mathbb{R}^{N}$. Thus, the space $\operatorname{Lin}\left(\mathbb{R}^{N} ; \mathbb{R}^{k}\right)$ can be identified with $\left(\mathbb{R}^{N}\right)^{k}$, and the Lebesgue measure on $\operatorname{Lin}\left(\mathbb{R}^{N} ; \mathbb{R}^{k}\right)$ is understood as $\bigotimes_{j=1}^{k}$ Leb, where Leb is the Lebesgue measure in $\mathbb{R}^{N}$. Within the space $\operatorname{Lin}\left(\mathbb{R}^{N} ; \mathbb{R}^{k}\right)$ we consider the space $E_{k}^{N}$ consisting of all linear transformations $L: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$ of the form (3.2), for which $l_{1}, \ldots, l_{k} \in B_{N}(0,1)$. Note that by the Cauchy-Schwarz inequality,

$$
\begin{equation*}
\|L x\| \leqslant \sqrt{N}\|x\| \tag{3.3}
\end{equation*}
$$

for every $L \in E_{k}^{N}$ and $x \in \mathbb{R}^{N}$.
By $\eta_{N, k}$ we denote the normalized Lebesgue measure on $E_{k}^{N}$, i.e. the probability measure on $E_{k}^{N}$ given by

$$
\eta_{N, k}=\left.\bigotimes_{j=1}^{k} \frac{1}{\kappa_{N}} \operatorname{Leb}\right|_{B_{N}(0,1)}
$$

where $\kappa_{N}=\operatorname{Leb}\left(B_{N}(0,1)\right)$. The following geometrical inequality is the key ingredient of the proof of Theorem 3.5.

Lemma 3.1 ([81, Lemma 4.1]). Let $L: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$ be a linear transformation. Then for every $x \in \mathbb{R}^{N} \backslash\{0\}, z \in \mathbb{R}^{k}$ and $\varepsilon>0$,

$$
\eta_{N, k}\left(\left\{L \in E_{k}^{N}:\|L x+z\| \leqslant \varepsilon\right\}\right) \leqslant C N^{k / 2} \frac{\varepsilon^{k}}{\|x\|^{k}}
$$

where $C>0$ is an absolute constant.
For $L \in \operatorname{Lin}\left(\mathbb{R}^{m} ; \mathbb{R}^{k}\right)$, where $m, k \in \mathbb{N}$, denote by $\sigma_{p}(L), p \in\{1, \ldots, k\}$, the $p$-th largest singular value of the matrix $L$, i.e. the $p$-th largest square root of an eigenvalue of the matrix $L^{*} L$. In the proof of Theorem 3.15, instead of Lemma 3.1 we will use the following lemma.

Lemma 3.2 ([81, Lemma 14.3]). Let $L: \mathbb{R}^{m} \rightarrow \mathbb{R}^{k}$ be a linear transformation. Assume that $\sigma_{p}(L)>0$ for some $p \in\{1, \ldots, k\}$. Then for every $z \in \mathbb{R}^{k}$ and $\rho, \varepsilon>0$,

$$
\frac{\operatorname{Leb}\left(\left\{\alpha \in B_{m}(0, \rho):\|L \alpha+z\| \leqslant \varepsilon\right\}\right)}{\operatorname{Leb}\left(B_{m}(0, \rho)\right)} \leqslant C_{m, k}\left(\frac{\varepsilon}{\sigma_{p}(L) \rho}\right)^{p}
$$

where $C_{m, k}>0$ is a constant depending only on $m, k$ and Leb is the Lebesgue measure on $\mathbb{R}^{m}$.

To verify the measurability of the sets occurring in subsequent proofs, we will use the two following elementary lemmas. Recall that a $\sigma$-compact set is a countable union of compact sets.

Lemma 3.3. Let $X \subset \mathbb{R}^{N}$ be a Borel set and let $\mu$ be a Borel $\sigma$-finite measure on $X$. Then there exists a $\sigma$-compact set $K \subset X$ of full measure $\mu$.

Proof. The proof follows from the fact that a finite Borel measure in a Euclidean space is regular (see e.g. [14, Theorem 1.1]). More precisely, as $\mu$ is $\sigma$-finite, there exists a sequence $K_{n}, n \in \mathbb{N}$ of Borel sets such that $X=\bigcup_{n=1}^{\infty} K_{n}$ and $\mu\left(K_{n}\right)<\infty$. By regularity of a finite Borel measure on $\mathbb{R}^{N}$, there exist compact sets $F_{n, k}, n, k \in \mathbb{N}$ such that $F_{n, k} \subset K_{n}$ and $\mu\left(K_{n} \backslash F_{n, k}\right) \leqslant \frac{1}{k}$ for all $n, k \in \mathbb{N}$. The desired $\sigma$-compact subset of $X$ of full measure is $K=\bigcup_{n, k \in \mathbb{N}} F_{n, k}$.

Lemma 3.4. Let $\mathcal{X}, \mathcal{Z}$ be metric spaces. Then the following hold.
(a) If $K \subset \mathcal{X} \times \mathcal{Z}$ is $\sigma$-compact, then so is $\pi_{\mathcal{X}}(K)$, where $\pi_{\mathcal{X}}: \mathcal{X} \times \mathcal{Z} \rightarrow \mathcal{X}$ is the projection given by $\pi_{\mathcal{X}}(x, z)=x$. In particular, $\pi_{\mathcal{X}}(K)$ is Borel.
(b) If $\mathcal{X}, \mathcal{Z}$ are $\sigma$-compact, $F: \mathcal{X} \times \mathcal{Z} \rightarrow \mathbb{R}^{k}, k \in \mathbb{N}$, is continuous and $K \subset \mathcal{X}$ is $\sigma$-compact, then the set

$$
\{(x, z) \in \mathcal{X} \times \mathcal{Z}: F(x, z)=F(y, z) \text { for some } y \in K \backslash\{x\}\}
$$

is $\sigma$-compact and hence Borel.

Proof. The statement (a) follows from the fact that $\pi_{\mathcal{X}}$ is continuous, and a continuous image of a compact set is also compact. To check (b), let $\pi_{\mathcal{X} \times \mathcal{Z}}: \mathcal{X} \times K \times \mathcal{Z} \rightarrow \mathcal{X} \times \mathcal{Z}$ be the projection $\pi_{\mathcal{X} \times \mathcal{Z}}(x, y, z)=(x, z)$. Then

$$
\begin{aligned}
& \{(x, z) \in \mathcal{X} \times \mathcal{Z}: F(x, z)=F(y, z) \text { for some } y \in K \backslash\{x\}\} \\
& =\pi_{\mathcal{X} \times \mathcal{Z}}(\{(x, y, z) \in \mathcal{X} \times K \times \mathcal{Z}: F(x, z)=F(y, z), d(x, y) \neq 0\}) \\
& =\bigcup_{n=1}^{\infty} \pi_{\mathcal{X} \times \mathcal{Z}}\left(\left\{(x, y, z) \in \mathcal{X} \times K \times \mathcal{Z}: F(x, z)=F(y, z), d(x, y) \geqslant \frac{1}{n}\right\}\right)
\end{aligned}
$$

where $d$ is the metric in $\mathcal{X}$. Since $d$ and $F$ are continuous, the set $(\{(x, y, z) \in \mathcal{X} \times K \times \mathcal{Z}$ : $\left.\left.F(x, z)=F(y, z), d(x, y) \geqslant \frac{1}{n}\right\}\right)$ is closed and consequently $\sigma$-compact, since $\mathcal{X}$ is $\sigma$-compact. Applying (a) ends the proof.

### 3.2 Probabilistic embedding theorem

In this section we prove an extended version of the Probabilistic embedding theorem, formulated below. Recall that $\mathcal{H}^{s}$ denotes the $s$-dimensional Hausdorff measure.

Theorem 3.5 (Probabilistic embedding theorem - extended version). Let $X \subset \mathbb{R}^{N}$ be a Borel set and $\mu$ be a Borel $\sigma$-finite measure on $X$. Fix $k \in \mathbb{N}$ and $\beta \in(0,1]$ such that $\mu \perp \mathcal{H}^{\beta k}$ and let $\phi: X \rightarrow \mathbb{R}^{k}$ be $\beta$-Hölder on bounded sets. Then for Lebesgue almost every linear transformation $L: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$ there exists a Borel set $X_{L} \subset X$ of full measure $\mu$, such that the map $\phi_{L}=\phi+L$ is injective on $X_{L}$.

Remark 3.6 Note that the assumption $\mu \perp \mathcal{H}^{\beta k}$ is fulfilled if $\operatorname{dim}_{H}(\mu)<\beta k$, so Theorem 3.5 is indeed an embedding theorem for the Hausdorff dimension. Moreover, it may happen that $\mu \perp \mathcal{H}^{\beta k}$ and $\operatorname{dim}_{H} X=\beta k$, hence the assumption $\mu \perp \mathcal{H}^{\beta k}$ is weaker than $\operatorname{dim}_{H}(\mu)<\beta k$. Note also that $\phi: X \rightarrow \mathbb{R}^{k}$ is $\beta$-Hölder on bounded sets provided that $\phi$ extends to a locally $\beta$-Hölder map on the closure $\bar{X}$. In particular, this assumption is fulfilled if $\phi$ is defined on $\mathbb{R}^{N}$ and locally $\beta$-Hölder. Consequently, Theorem 1.4 follows from Theorem 3.5. It is also straightforward to notice that if $\operatorname{dim}_{H} X=0$, then $\phi$ can be taken to be an arbitrary Hölder map.

Proof of Theorem 3.5. Note first that it sufficient to prove that the set $X_{L}$ exists for $\eta_{N, k^{-}}$ almost every $L \in E_{k}^{N}$. Indeed, if this is shown, then for a given $\beta$-Hölder map $\phi: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$ we can take sets $\mathcal{L}_{j} \subset E_{k}^{N}, j \in \mathbb{N}$, such that $\eta_{N, k}\left(\mathcal{L}_{j}\right)=1$ and for every $\tilde{L} \in \mathcal{L}_{j}$ the map $(\phi / j)_{\tilde{L}}=\phi / j+\tilde{L}$ is injective on a Borel set $X_{\tilde{L}}^{(j)} \subset X$ of full measure $\mu$. Then the set $\mathcal{L}=\bigcup_{j \in \mathbb{N}}\left\{j \tilde{L}: \tilde{L} \in \mathcal{L}_{j}\right\} \subset \operatorname{Lin}\left(\mathbb{R}^{N} ; \mathbb{R}^{k}\right)$ has full Lebesgue measure and for every $L \in \mathcal{L}$ there exists $j$ such that $L / j \in \mathcal{L}_{j}$, so $(\phi / j)_{L / j}=(\phi+L) / j$ is injective on $X_{L}=\bigcap_{j \in \mathbb{N}} X_{L / j}^{(j)}$ (and hence so is $\phi_{L}$ ), which has full measure $\mu$.

By the assumption $\mu \perp \mathcal{H}^{\beta k}$, there exists a Borel subset $\tilde{X}$ of $X$ of full measure $\mu$ and zero measure $\mathcal{H}^{\beta k}$. By Lemma 3.3 , we can assume that $\tilde{X}$ is $\sigma$-compact. Take $k \in \mathbb{N}, \beta \in(0,1]$ with $\mathcal{H}^{\beta k}(\tilde{X})=0$ and a $\beta$-Hölder map $\phi: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$. Set

$$
A=\left\{(x, L) \in \tilde{X} \times E_{k}^{N}: \phi_{L}(x)=\phi_{L}(y) \text { for some } y \in \tilde{X} \backslash\{x\}\right\}
$$

By Lemma 3.4, $A$ is Borel. For $x \in \tilde{X}$ and $L \in E_{k}^{N}$, denote by $A_{x}$ and $A^{L}$, respectively, the sections

$$
A_{x}=\left\{L \in E_{k}^{N}:(x, L) \in A\right\}, \quad A^{L}=\{x \in \tilde{X}:(x, L) \in A\}
$$

The sets $A_{x}$ and $A^{L}$ are Borel as sections of a Borel set. Observe first, that in order to prove the theorem it is enough to show $\eta_{N, k}\left(A_{x}\right)=0$ for every $x \in \tilde{X}$, since then by Fubini's theorem $([83$, Thm. 8.8$]),\left(\eta_{N, k} \otimes \mu\right)(A)=0$ and, consequently, $\mu\left(A^{L}\right)=0$ for $\eta_{N, k}$-almost every $L \in E_{k}^{N}$. Since $\phi_{L}$ is injective on $\tilde{X} \backslash A^{L}$, the assertion of the theorem is true.

Take a point $x \in \tilde{X}$. To show $\eta_{N, k}\left(A_{x}\right)=0$, it suffices to prove $\eta_{N, k}\left(A_{x, n}\right)=0$ for every $n \in \mathbb{N}$, where

$$
A_{x, n}=\left\{L \in E_{k}^{N}: \phi_{L}(x)=\phi_{L}(y) \text { for some } y \in K_{n}\right\}
$$

and

$$
K_{n}=\left\{y \in \tilde{X}: \frac{1}{n} \leqslant\|x-y\| \leqslant n\right\}
$$

Take $n \in \mathbb{N}$ and fix a small $\varepsilon>0$. Since $\mathcal{H}^{\beta k}\left(K_{n}\right) \leqslant \mathcal{H}^{\beta k}(\tilde{X})=0$, there exists a collection of balls $B_{N}\left(y_{i}, \varepsilon_{i}\right), i \in \mathbb{N}$, for some $y_{i} \in K_{n}, \varepsilon_{i}>0$, such that

$$
\begin{equation*}
K_{n} \subset \bigcup_{i \in \mathbb{N}} B_{N}\left(y_{i}, \varepsilon_{i}\right) \quad \text { and } \quad \sum_{i \in \mathbb{N}} \varepsilon_{i}^{\beta k} \leqslant \varepsilon \tag{3.4}
\end{equation*}
$$

Take $L \in A_{x, n}$ and $y \in K_{n}$ such that $\phi_{L}(x)=\phi_{L}(y)$. Then $y \in B_{N}\left(y_{i}, \varepsilon_{i}\right)$ for some $i \in \mathbb{N}$ and

$$
\begin{aligned}
\left\|L\left(y_{i}-x\right)+\phi\left(y_{i}\right)-\phi(x)\right\| & =\left\|\phi_{L}\left(y_{i}\right)-\phi_{L}(x)\right\| \\
& =\left\|\phi_{L}\left(y_{i}\right)-\phi_{L}(y)\right\| \\
& \leqslant\left\|\phi\left(y_{i}\right)-\phi(y)\right\|+\left\|L\left(y_{i}-y\right)\right\| \\
& \leqslant \tilde{M}_{n}\left\|y_{i}-y\right\|^{\beta}+\sqrt{N}\left\|y_{i}-y\right\| \\
& \leqslant M_{n} \varepsilon_{i}^{\beta}
\end{aligned}
$$

for some $\tilde{M}_{n}, M_{n}>0$, by (3.3) and the fact that $y, y_{i} \in B_{N}(x, n+\varepsilon)$ and $\phi$ is $\beta$-Hölder on bounded sets. This shows that

$$
A_{x, n} \subset \bigcup_{i \in \mathbb{N}}\left\{L \in E_{k}^{N}:\left\|L\left(y_{i}-x\right)+\phi\left(y_{i}\right)-\phi(x)\right\| \leqslant M_{n} \varepsilon_{i}^{\beta}\right\}
$$

By Lemma 3.1 and (3.4) we have

$$
\begin{aligned}
\eta_{N, k}\left(A_{x, n}\right) & \leqslant \sum_{i \in \mathbb{N}} \eta_{N, k}\left(\left\{L \in E_{k}^{N}:\left\|L\left(y_{i}-x\right)+\phi\left(y_{i}\right)-\phi(x)\right\| \leqslant M_{n} \varepsilon_{i}^{\beta}\right\}\right) \\
& \leqslant \frac{C N^{k / 2} M_{n}^{k}}{1 / n^{k}} \sum_{i \in \mathbb{N}} \varepsilon_{i}^{\beta k} \leqslant C N^{k / 2} M_{n}^{k} n^{k} \varepsilon
\end{aligned}
$$

Since $\varepsilon>0$ was arbitrary, we obtain $\eta_{N, k}\left(A_{x, n}\right)=0$, which ends the proof.
As a simple consequence of Theorem 3.5, we obtain the following corollary, formulated in a slightly simplified version in Chapter 1 as Corollary 1.5.

Corollary 3.7 (Probabilistic injective projection theorem - extended version). Let $X \subset \mathbb{R}^{N}$ be a Borel set and let $\mu$ be a Borel $\sigma$-finite measure on $X$. Then for every $k \in \mathbb{N}, k \leqslant N$ such that $\mu \perp \mathcal{H}^{k}$ and almost every $k$-dimensional linear subspace $S \subset \mathbb{R}^{N}$, the orthogonal projection of $X$ into $S$ is injective on a $\mu$-full measure subset of $X$ (depending on $S$ ).

Proof of Corollary 3.7. Apply Theorem 3.5 for the map $\phi \equiv 0$. Then we know that a linear map $L \in \operatorname{Lin}\left(\mathbb{R}^{N} ; \mathbb{R}^{k}\right)$ of the form (3.2) is injective on a set $X_{L} \subset X$ of full measure $\mu$ for Lebesgue almost every $\left(l_{1}, \ldots, l_{k}\right) \in\left(\mathbb{R}^{N}\right)^{k}$. We can assume that $l_{1}, \ldots, l_{k}$ are linearly independent for all such $L$, which also implies that the same holds for $L l_{1}, \ldots, L l_{k}$. Setting

$$
S_{L}=\operatorname{Span}\left(l_{1}, \ldots, l_{k}\right)
$$

and taking $V_{L} \in \operatorname{Lin}\left(\mathbb{R}^{k} ; \mathbb{R}^{N}\right)$ defined by $V_{L}\left(L l_{j}\right)=l_{j}$ for $j=1, \ldots, k$, we have

$$
V_{L} \circ L=\Pi_{S_{L}},
$$

where $\Pi_{S_{L}}$ is the orthogonal projection from $\mathbb{R}^{N}$ onto $S_{L}$ and $V_{L}$ is injective. It follows that $\Pi_{S_{L}}$ is injective on $X_{L}$ for almost every $\left(l_{1}, \ldots, l_{k}\right)$, so $\Pi_{S}$ is injective on a $\mu$-full measure subset of $X$ for almost every $k$-dimensional linear subspace $S \subset \mathbb{R}^{N}$.

Let us note that in general, the requirement $\mu \perp \mathcal{H}^{\beta k}$ in Theorem 3.5 cannot be replaced by a weaker condition $\operatorname{dim}_{H}(\mu) \leqslant \beta k$ - see Example 3.32 ,

Theorem 3.5 strengthens the following embedding theorem, proved recently by Alberti, Bölcskei, De Lellis, Koliander and Riegler in [1].

Theorem 3.8 ([1, Theorem II.1]). Let $\mu$ be a Borel probability measure in $\mathbb{R}^{N}$ and let $k \in \mathbb{N}$ be such that $k>{\underset{\operatorname{dim}}{M B}}^{\mu}$. Then for Lebesgue almost every linear transformation $L: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$ there exists a Borel set $X_{L} \subset \mathbb{R}^{N}$ such that $\mu\left(X_{L}\right)=1$ and $L$ is injective on $X_{L}$.

In fact, in [1] the authors introduced the notion of $\underline{\operatorname{dim}}_{M B} \mu$, denoting it by $K(\mu)$ and calling it the description complexity of the measure. In particular, Theorem 3.8 hold for measures $\mu$ supported on a Borel set $X \subset \mathbb{R}^{N}$ with $\underline{\operatorname{dim}}_{B} X<k$. By 2.1 , we have $\operatorname{dim}_{H} \mu \leqslant \underline{\operatorname{dim}}_{M B} \mu$, and in Section 3.5 we present an example (Theorem 3.29) showing that the inequality may be strict. Therefore, Theorem 3.5 actually strengthens Theorem 3.8.

Non-probabilistic embedding theorems were first obtained in topological and smooth categories. The well-known Menger-Nöbeling embedding theorem (see e.g. [48, Theorem V.2]) states that for a compact metric space $X$ with Lebesgue covering dimension at most $k$, a generic continuous transformation $\phi: X \rightarrow \mathbb{R}^{2 k+1}$ is injective (and hence defines a homeomorphism between $X$ and $\phi(X)$ ). Genericity means here that the set of injective transformations $\phi: X \rightarrow \mathbb{R}^{2 k+1}$ is a dense $G_{\delta}$ subset of $C\left(X ; \mathbb{R}^{2 k+1}\right)$ endowed with the supremum metric. The dimension $2 k+1$ is known to be optimal. The corresponding result in the category of smooth manifolds is the Whitney embedding theorem (see 97]). It states that for a given $k$-dimensional $C^{r}$-manifold $M$, a generic $C^{r}$-transformation from $M$ to $\mathbb{R}^{2 k+1}$ is a $C^{r}$-embedding (i.e. an injective immersion of class $C^{r}$ ).

Let us now compare Theorem 3.5t to non-probabilistic embedding theorems involving boxcounting dimension. One of the first results in this area was a theorem by Mañé 59, Lemma 1.1]. We present its formulation following [84, Theorem 4.6] and [81, Theorem 6.2] (originally, Mañé proved that topologically generic linear transformation is injective on $X$ ).

Theorem 3.9. Let $X \subset \mathbb{R}^{N}$ be a compact set. Let $k \in N$ be such that $k>2 \overline{\operatorname{dim}}_{B} X$ or $k>\operatorname{dim}_{H}(X-X)$. Then Lebesgue almost every linear transformation $L: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$ is injective on $X$.

Remark 3.10 As noticed by Mañé and communicated in [24, p. 627], his original statement in [59] is incorrect. Namely, he assumed $k>2 \operatorname{dim}_{H} X+1$ instead of $k>\operatorname{dim}_{H}(X-X)$. However, this is known to be insufficient for the existence of a linear embedding of $X$ into $\mathbb{R}^{k}$. In fact, in [84, Appendix A], Kan presented an example of a set $X \subset \mathbb{R}^{m}$ with $\operatorname{dim}_{H} X=0$, such that any linear transformation $L: \mathbb{R}^{m} \rightarrow \mathbb{R}^{m-1}$ fails to be injective on $X$. This proves that the assumption $k>2 \operatorname{dim}_{H} X$ is insufficient, while $k>2 \overline{\operatorname{dim}}_{B} X$ is sufficient. This stems from the fact that the proof of Theorem 3.9 actually requires the property $k>\operatorname{dim}_{H}(X-X)$, and applying (2.1), Proposition 2.16 and Proposition 2.14, we obtain

$$
\operatorname{dim}_{H}(X-X) \leqslant \operatorname{dim}_{H}(X \times X) \leqslant \overline{\operatorname{dim}}_{B}(X \times X) \leqslant 2 \overline{\operatorname{dim}}_{B} X,
$$

hence $k>2 \overline{\operatorname{dim}}_{B} X$ implies $k>\operatorname{dim}_{H}(X-X)$. On the other hand, (??) does not hold for the Hausdorff dimension (nor for the lower box-counting dimension), so $\operatorname{dim}_{H} X$ does not control $\operatorname{dim}_{H}(X-X)$. The fact that in Theorem 3.5 we can work with the Hausdorff dimension comes from the application of Fubini's theorem, which enables us to consider covers of the set $X$ itself, instead of $X-X$. In Section 3.6 we analyze Kan's example from the point of view of Theorem 3.5,
Theorem 3.9 is also true for subsets of an arbitrary Banach space $\mathfrak{B}$ for a prevalent set of linear transformations $L: \mathfrak{B} \rightarrow \mathbb{R}^{k}$ (see [81, Chapter 6] for details).

Note that the linear embedding from Theorem 3.5 need not preserve the dimension of $X$. Indeed, the Hausdorff and box-counting dimensions are invariants for bi-Lipschitz transformations, yet inverse of a linear map on a compact set does not have to be Lipschitz. Therefore, we only know that $\operatorname{dim} \phi_{L}(X) \leqslant \operatorname{dim} X$ (see [81, Proposition 2.8.iv and Lemma 3.3.iv]) and the inequality can be strict. For example, let $\phi \equiv 0$ and $X=\{(x, f(x)): x \in[0,1]\}$ be a graph of a (Hölder continuous) function $f:[0,1] \rightarrow \mathbb{R}$ with $\operatorname{dim}_{H} X>1$, e.g. the Weierstrass non-differentiable function. Then the linear projection $L: \mathbb{R}^{2} \rightarrow \mathbb{R}$ given by $L(x, y)=x$ satisfies $1=\operatorname{dim} L(X)<\operatorname{dim}_{H} X$. The following theorem shows that in the non-probabilistic setting, one can obtain $\beta$-Hölder continuity of the inverse map for small enough $\beta \in(0,1)$ (see [12, 25, 47] and [81, Chapter 4]).
Theorem 3.11. Let $X \subset \mathbb{R}^{N}$ be a compact set. Let $k \in \mathbb{N}$ be such that $k>2 \overline{\operatorname{dim}}_{B} X$ and let $\beta$ be such that $0<\beta<1-2 \overline{\operatorname{dim}}_{B} X / k$. Then Lebesgue almost every linear transformation $L: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$ is injective on $X$ with $\beta$-Hölder continuous inverse.
However, this is not true in the case of Theorem 3.5
Remark 3.12 In general, we cannot claim that the injective map $\left.\phi_{L}\right|_{X_{L}}$ from Theorem 3.5 has a Hölder continuous inverse. Indeed, it is well-known that for $n \in \mathbb{N}$ there are examples of compact sets $X \subset \mathbb{R}^{N}$ of Hausdorff and topological dimension equal to $n$, which do not embed topologically into $\mathbb{R}^{k}$ for $k \leqslant 2 n$ (showing the optimality of the bounds in the MengerNöbeling embedding theorem, see [48, Example V.3]). Consider a probability measure $\mu$ on $X$ with $\operatorname{supp} \mu=X$. Such measure exists for any compact set. If the map $\left.\phi_{L}\right|_{X_{L}}$ from Theorem 3.5 for $k=n+1$ had a Hölder continuous inverse $f=\phi_{L}^{-1}$, then we could extend $f$ from $\phi_{L}\left(X_{L}\right)$ to $\mathbb{R}^{n+1}$ preserving the Hölder continuity ([7, Theorem IV.7.5], see also [67]). Then $Y=\left\{x \in X: f \circ \phi_{L}(x)=x\right\}$ would be a closed subset of $X$ with $\mu(Y)=1$, hence $Y=X$, so $\phi_{L}$ would be a homeomorphism between $X$ and $\phi_{L}(X) \subset \mathbb{R}^{n+1}$, which would give a contradiction.

### 3.3 Probabilistic Takens delay embedding theorem

In this section we present the proof of the extended probabilistic Takens delay embedding theorem. It turns out that linear perturbations are insufficient for Takens-type theorems, see Example 3.34. As observed in [84], it is enough to take perturbations from the space of polynomials of degree $2 k$. This can be easily extended to more general families of functions.

Definition 3.13 Let $X$ be a subset of $\mathbb{R}^{N}$. A family of transformations $h_{1}, \ldots, h_{m}: X \rightarrow \mathbb{R}$ is called a $k$-interpolating family on set $X$, if for every collection of distinct points $x_{1}, \ldots, x_{k} \in X$ and every $\xi=\left(\xi_{1}, \ldots, \xi_{k}\right) \in \mathbb{R}^{k}$ there exists $\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{R}^{m}$ such that $\sum_{j=1}^{m} \alpha_{j} h_{j}\left(x_{i}\right)=\xi_{i}$ for each $i=1, \ldots, k$. In other words, the matrix

$$
\left[\begin{array}{ccc}
h_{1}\left(x_{1}\right) & \ldots & h_{m}\left(x_{1}\right) \\
\vdots & \ddots & \vdots \\
h_{1}\left(x_{k}\right) & \ldots & h_{m}\left(x_{k}\right)
\end{array}\right]
$$

has full row rank as a transformation from $\mathbb{R}^{m}$ to $\mathbb{R}^{k}$. Note that the same is true for any collection of $l$ distinct points with $l \leqslant k$.

Example 3.14 It is known that any linear basis $h_{1}, \ldots, h_{m}$ of the space of real polynomials of $N$ variables of degree at most $k-1$ is a $k$-interpolating family (see e.g. [32, Sec. 1.2, eq. (1.9)]).

For a transformation $T: X \rightarrow X$ and $p \in \mathbb{N}$ denote by $\operatorname{Per}_{p}(T)$ the set of periodic points of minimal period $p$, i.e.

$$
\operatorname{Per}_{p}(T)=\left\{x \in X: T^{p} x=x \text { and } T^{j} x \neq x \text { for } j=1, \ldots, p-1\right\} .
$$

Theorem 3.15 (Probabilistic Takens delay embedding theorem - extended version). Let $X \subset \mathbb{R}^{N}$ be a Borel set, $\mu$ be a Borel $\sigma$-finite measure on $X$ and $T: X \rightarrow X$ an injective map, which is Lipschitz on bounded sets. Fix $k \in \mathbb{N}$ and $\beta \in(0,1]$ such that $\mu \perp \mathcal{H}^{\beta k}$ and assume $\left.\mu\right|_{\operatorname{Per}_{p}(T)} \perp \mathcal{H}^{\beta p}$ for every $p=1, \ldots, k-1$. Let $h: X \rightarrow \mathbb{R}$ be $\beta$-Hölder on bounded sets and $h_{1}, \ldots, h_{m}: X \rightarrow \mathbb{R}$ a $2 k$-interpolating family on $X$ consisting of transformations which are $\beta$-Hölder on bounded sets. For $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{R}^{m}$ denote by $h_{\alpha}: X \rightarrow \mathbb{R}$ the transformation

$$
h_{\alpha}(x)=h(x)+\sum_{j=1}^{m} \alpha_{j} h_{j}(x) .
$$

Then for Lebesgue almost every $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{R}^{m}$, there exists a Borel set $X_{\alpha} \subset X$ of full measure $\mu$, such that the delay-coordinate map

$$
\phi_{\alpha}^{T}: X \rightarrow \mathbb{R}^{k}, \quad \phi_{\alpha}^{T}(x)=\left(h_{\alpha}(x), h_{\alpha}(T x), \ldots, h_{\alpha}\left(T^{k-1} x\right)\right)
$$

is injective on $X_{\alpha}$. If $\mu$ is additionally $T$-invariant, then the sets $X_{\alpha}$ can be taken to be $T$-invariant, i.e. satisfy $T\left(X_{\alpha}\right) \subset X_{\alpha}$.

Remark 3.16 By Example 3.14 and observations of Remark 3.6. Theorem 1.2 follows from Theorem 3.15,

Under the notation of Theorem 3.15, we first show a preliminary lemma. For $x \in X$ define its full orbit $\operatorname{Orb}(x)$ as

$$
\operatorname{Orb}(x)=\left\{T^{n} x: n \geqslant 0\right\} \cup\left\{y \in X: T^{n} y=x \text { for some } n \in \mathbb{N}\right\}=\bigcup_{n \in \mathbb{Z}} T^{n}(\{x\})
$$

Note that since $T$ is injective, all full orbits are at most countable, and any two full orbits $\operatorname{Orb}(x)$ and $\operatorname{Orb}(y)$ are either equal or disjoint. For $x, y \in X$ let $D_{x, y}$ be the $k \times m$ matrix defined by

$$
D_{x, y}=\left[\begin{array}{ccc}
h_{1}(x)-h_{1}(y) & \ldots & h_{m}(x)-h_{m}(y) \\
h_{1}(T x)-h_{1}(T y) & \ldots & h_{m}(T x)-h_{m}(T y) \\
\vdots & \ddots & \vdots \\
h_{1}\left(T^{k-1} x\right)-h_{1}\left(T^{k-1} y\right) & \ldots & h_{m}\left(T^{k-1} x\right)-h_{m}\left(T^{k-1} y\right)
\end{array}\right]
$$

Lemma 3.17. For $x, y \in X$, the following statements hold.
(i) If $y \neq x$, then $\operatorname{rank} D_{x, y} \geqslant 1$.
(ii) If $y \notin \operatorname{Orb}(x)$ and $y \in \operatorname{Per}_{p}(T)$ for some $p \in\{1, \ldots, k-1\}$, then $\operatorname{rank} D_{x, y} \geqslant p$.
(iii) If $y \notin \operatorname{Orb}(x)$ and $y \notin \bigcup_{p=1}^{k-1} \operatorname{Per}_{p}(T)$, then $\operatorname{rank} D_{x, y}=k$.

Proof. For (i), it suffices to observe that the first row of $D_{x, y}$ is non-zero as long as $x \neq y$ and therefore $\operatorname{rank}\left(D_{x, y}\right) \geqslant 1$. Indeed, otherwise we would have $h_{j}(x)=h_{j}(y)$ for $j=1, \ldots, m$ which contradicts the fact that $h_{1}, \ldots, h_{m}$ is an interpolating family.

Assume now $y \notin \operatorname{Orb}(x)$, which implies $\operatorname{Orb}(y) \cap \operatorname{Orb}(x)=\emptyset$. Let $q$ (resp. $r$ ) be a maximal number from $\{1, \ldots, k\}$ such that the points $x, T x, \ldots, T^{q-1} x$ (resp. $y, T y, \ldots, T^{r-1} y$ ) are distinct. Notice that if $y \in \operatorname{Per}_{p}(T)$ for some $p \in\{1, \ldots, k-1\}$, then $r=p$, and if $y \notin$ $\bigcup_{p=1}^{k-1} \operatorname{Per}_{p}(T)$, then $r=k$. Thus, the assertions (ii)-(iii) of the lemma can be written simply as one condition

$$
\begin{equation*}
\operatorname{rank} D_{x, y} \geqslant r \tag{3.5}
\end{equation*}
$$

To show that (3.5) holds, denote the points $x, T x, \ldots, T^{q-1} x, y, T y, \ldots, T^{r-1} y$, preserving the order, by $z_{1}, \ldots, z_{l}$, for $l=q+r$. By the definition of $q, r$, we have $1 \leqslant l \leqslant 2 k$ and the points $z_{1}, \ldots, z_{l}$ are distinct. Thus, the matrix $D_{x, y}$ can be written as the product

$$
D_{x, y}=J_{x, y} V_{x, y}
$$

where

$$
V_{x, y}=\left[\begin{array}{ccc}
h_{1}\left(z_{1}\right) & \ldots & h_{m}\left(z_{1}\right) \\
\vdots & \ddots & \vdots \\
h_{1}\left(z_{l}\right) & \ldots & h_{m}\left(z_{l}\right)
\end{array}\right]
$$

and $J_{x, y}$ is a $k \times l$ matrix with entries in $\{-1,0,1\}$ and block structure of the form

$$
J_{x, y}=\left[\begin{array}{c|c}
* & -\mathrm{Id}_{r \times r} \\
\hline * & *
\end{array}\right],
$$

where $\mathrm{Id}_{r \times r}$ is the $r \times r$ identity matrix. It follows that rank $J_{x, y} \geqslant r$. Moreover, since $z_{1}, \ldots, z_{l}$ are distinct and $h_{1}, \ldots, h_{m}$ is a $2 k$-interpolating family, the matrix $V_{x, y}$ is of full rank, hence $\operatorname{rank} D_{x, y}=\operatorname{rank} J_{x, y} \geqslant r$, which ends the proof.

Proof of Theorem 3.15. We proceed similarly as in the proof of Theorem 3.5, using Lemma 3.2 instead of Lemma 3.1, together with the suitable rank estimates coming from Lemma 3.17. In the same way as in the proof of Theorem 3.5, we show that it is enough to check that the suitable set $X_{\alpha}$ exists for $\eta_{m}$-almost every $\alpha \in B_{m}(0,1)$.

Applying Lemma 3.3 to the sets $\operatorname{Per}_{p}(T), p=1, \ldots, k-1$ and (possibly zero) measures $\left.\mu\right|_{\operatorname{Per}_{p}(T)}$, we find (possibly empty) disjoint $\sigma$-compact sets $X_{1}, \ldots, X_{k-1} \subset X$ such that

$$
X_{p} \subset \operatorname{Per}_{p}(T), \quad \mu\left(X_{p}\right)=\mu\left(\operatorname{Per}_{p}(T)\right), \quad \mathcal{H}^{\beta p}\left(X_{p}\right)=0 \quad \text { for } p=1, \ldots, k-1
$$

Similarly, there exists a $\sigma$-compact set $X_{k} \subset X \backslash \bigcup_{p=1}^{k-1} \operatorname{Per}_{p}(T)$ such that

$$
\mu\left(X_{k}\right)=\mu\left(X \backslash \bigcup_{p=1}^{k-1} \operatorname{Per}_{p}(T)\right) \quad \text { and } \quad \mathcal{H}^{\beta k}\left(X_{k}\right)=0
$$

Note that $X_{k}$ contains both aperiodic and periodic points (with period at least $k$ ). Let

$$
\tilde{X}=\bigcup_{p=1}^{k} X_{p}
$$

Then $\tilde{X} \subset X$ is a $\sigma$-compact set of full measure $\mu$. Define

$$
A=\left\{(x, \alpha) \in \tilde{X} \times B_{m}(0,1): \phi_{\alpha}^{T}(x)=\phi_{\alpha}^{T}(y) \text { for some } y \in \tilde{X} \backslash\{x\}\right\}
$$

The set $A$ is Borel by Lemma 3.4. For $x \in \tilde{X}$ and $\alpha \in B_{m}(0,1)$, denote, respectively, by $A_{x}$ and $A^{\alpha}$, the Borel sections

$$
A_{x}=\left\{\alpha \in B_{m}(0,1):(x, \alpha) \in A\right\}, \quad A^{\alpha}=\{x \in \tilde{X}:(x, \alpha) \in A\}
$$

Observe that it is enough to prove $\eta_{m}\left(A_{x}\right)=0$ for every $x \in \tilde{X}$, since then by Fubini's theorem $\left(\left[83\right.\right.$, Thm. 8.8]), $\left(\eta_{m} \otimes \mu\right)(A)=0$ and, consequently, $\mu\left(A^{\alpha}\right)=0$ for $\eta_{m}$-almost every $\alpha \in B_{m}(0,1)$. As $\phi_{\alpha}^{T}$ is injective on $\tilde{X} \backslash A^{\alpha}$ and $\tilde{X}$ has full measure $\mu$, the proof of the theorem is finished.

Fix $x \in \tilde{X}$. To show $\eta_{m}\left(A_{x}\right)=0$, note that for $y \in \tilde{X}$,

$$
\begin{equation*}
\phi_{\alpha}^{T}(x)-\phi_{\alpha}^{T}(y)=D_{x, y} \alpha+w_{x, y} \tag{3.6}
\end{equation*}
$$

for

$$
w_{x, y}=\left[\begin{array}{c}
h(x)-h(y) \\
h(T x)-h(T y) \\
\vdots \\
h\left(T^{k-1} x\right)-h\left(T^{k-1} y\right)
\end{array}\right]
$$

Write $A_{x}$ as

$$
A_{x}=A_{x}^{\mathrm{orb}} \cup \bigcup_{p=1}^{k} A_{x}^{p}
$$

where

$$
\begin{aligned}
A_{x}^{\text {orb }} & =\left\{\alpha \in B_{m}(0,1): \phi_{\alpha}^{T}(x)=\phi_{\alpha}^{T}(y) \text { for some } y \in \tilde{X} \cap \operatorname{Orb}(x) \backslash\{x\}\right\} \\
A_{x}^{p} & =\left\{\alpha \in B_{m}(0,1): \phi_{\alpha}^{T}(x)=\phi_{\alpha}^{T}(y) \text { for some } y \in X_{p} \backslash\{x\}\right\}, \quad p=1, \ldots, k
\end{aligned}
$$

The set $A_{x}^{\mathrm{orb}}$ is Borel as a countable union of closed sets of the form

$$
\begin{equation*}
\left\{\alpha \in B_{m}(0,1): \phi_{\alpha}^{T}(x)=\phi_{\alpha}^{T}(y)\right\}, \quad y \in \tilde{X} \cap \operatorname{Orb}(x) \backslash\{x\} \tag{3.7}
\end{equation*}
$$

while each set $A_{x}^{p}$ is Borel as a section of the set

$$
\left\{(x, \alpha) \in \tilde{X} \times B_{m}(0,1): \phi_{\alpha}^{T}(x)=\phi_{\alpha}^{T}(y) \text { for some } y \in X_{p} \backslash\{x\}\right\}
$$

which is Borel by Lemma 3.4. To end the proof, it is enough to show that the sets $A_{x}^{\text {orb }}$ and $A_{x}^{p}, p=1, \ldots, k$, have $\eta_{m}$ measure zero.

To prove $\eta_{m}\left(A_{x}^{\text {orb }}\right)=0$ it suffices to check that the sets of the form 3.7) have $\eta_{m}$ measure zero. By (3.6), we have

$$
\left\{\alpha \in B_{m}(0,1): \phi_{\alpha}^{T}(x)=\phi_{\alpha}^{T}(y)\right\}=\left\{\alpha \in B_{m}(0,1): D_{x, y} \alpha=-w_{x, y}\right\}
$$

and Lemma 3.17 gives rank $D_{x, y} \geqslant 1$ whenever $y \neq x$, so each set of the form 3.7 is contained in an affine subspace of $\mathbb{R}^{m}$ of codimension at least 1 . Consequently, it has $\eta_{m}$ measure zero.

To prove $\eta_{m}\left(A_{x}^{p}\right)=0$ for $p=1, \ldots, k$, fix $n \in \mathbb{N}$ and define

$$
\begin{aligned}
X_{x}^{p, n} & =\left\{y \in X_{p}: \sigma_{p}\left(D_{x, y}\right) \geqslant \frac{1}{n} \text { and }\|y-x\| \leqslant n\right\}, \\
A_{x}^{p, n} & =\left\{\alpha \in B_{m}(0,1): \phi_{\alpha}^{T}(x)=\phi_{\alpha}^{T}(y) \text { for some } y \in X_{x}^{p, n} \backslash\{x\}\right\},
\end{aligned}
$$

where $\sigma_{p}\left(D_{x, y}\right)$ is the $p$-th largest singular value. Note that singular values of given order depend continuously on the coefficients of the matrix, see e.g. [37, Corollary 8.6.2]. Hence, the set $X_{x}^{p, n}$ is $\sigma$-compact as a closed subset of $X_{p}$ and by Lemma 3.4, the set $A_{x}^{p, n}$ is Borel.

By Lemma 3.17, for every $y \in X_{p} \backslash \operatorname{Orb}(x)$ we have rank $D_{x, y} \geqslant p$, which implies $\sigma_{p}\left(D_{x, y}\right)>0$ (see e.g. [81, Lemma 14.2]). Hence,

$$
A_{x}^{p} \backslash A_{x}^{\mathrm{orb}}=\bigcup_{n=1}^{\infty} A_{x}^{p, n} \backslash A_{x}^{\mathrm{orb}} .
$$

Consequently, it is enough to prove $\eta_{m}\left(A_{x}^{p, n} \backslash A_{x}^{\text {orb }}\right)=0$ for every $n \in \mathbb{N}$.
Fix $\varepsilon>0$. Since $\mathcal{H}^{\beta p}\left(X_{x}^{p, n} \backslash \operatorname{Orb}(x)\right) \leqslant \mathcal{H}^{\beta p}\left(X_{p}\right)=0$, there exists a collection of balls $B_{N}\left(y_{i}, \varepsilon_{i}\right)$, for $y_{i} \in X_{x}^{p, n} \backslash \operatorname{Orb}(x)$ and $0<\varepsilon_{i}<\varepsilon, i \in \mathbb{N}$, such that

$$
\begin{equation*}
X_{x}^{p, n} \backslash \operatorname{Orb}(x) \subset \bigcup_{i \in \mathbb{N}} B_{N}\left(y_{i}, \varepsilon_{i}\right) \quad \text { and } \quad \sum_{i \in \mathbb{N}} \varepsilon_{i}^{\beta p} \leqslant \varepsilon . \tag{3.8}
\end{equation*}
$$

Take $\alpha \in A_{x}^{p, n} \backslash A_{x}^{\text {orb }}$ and let $y \in X_{x}^{p, n} \backslash \operatorname{Orb}(x)$ be such that $\phi_{\alpha}^{T}(x)=\phi_{\alpha}^{T}(y)$. Then for $y_{i}$ with $y \in B\left(y_{i}, \varepsilon_{i}\right)$ we have

$$
\begin{align*}
\left\|D_{x, y_{i}} \alpha+w_{x, y_{i}}\right\| & =\left\|\phi_{\alpha}^{T}(x)-\phi_{\alpha}^{T}\left(y_{i}\right)\right\|=\left\|\phi_{\alpha}^{T}(y)-\phi_{\alpha}^{T}\left(y_{i}\right)\right\| \\
& \leqslant \sqrt{\sum_{s=0}^{k-1}\left(\left\|h\left(T^{s} y\right)-h\left(T^{s} y_{i}\right)\right\|+\sum_{j=1}^{m} \alpha_{j}\left\|h_{j}\left(T^{s} y\right)-h_{j}\left(T^{s} y_{i}\right)\right\|\right)^{2}}  \tag{3.9}\\
& \leqslant M_{n} \varepsilon_{i}^{\beta}
\end{align*}
$$

for some $M_{n}>0$ (depending also on $\left.m, k\right)$, since $\left\|y-y_{i}\right\| \leqslant \varepsilon_{i},\left\|\alpha_{j}\right\| \leqslant 1, y, y_{i} \in B_{N}(x, n+\varepsilon)$ and $T, h$ and $h_{j}$ are $\beta$-Hölder on bounded sets on $X$. By (3.9),

$$
A_{x}^{p, n} \backslash A_{x}^{\text {orb }} \subset \bigcup_{i \in \mathbb{N}}\left\{\alpha \in B_{m}(0,1):\left\|D_{x, y_{i}} \alpha+w_{x, y_{i}}\right\| \leqslant M_{n} \varepsilon_{i}^{\beta}\right\} .
$$

Since for every $i \in \mathbb{N}$ we have $\sigma_{p}\left(D_{x, y_{i}}\right) \geqslant 1 / n$, we can apply Lemma 3.2 and (3.8) to obtain

$$
\eta_{m}\left(A_{x}^{p, n} \backslash A_{x}^{\mathrm{orb}}\right) \leqslant \sum_{i \in \mathbb{N}} C_{m, k} \frac{M_{n}^{p} \varepsilon_{i}^{\beta p}}{1 / n^{p}} \leqslant C_{m, k} M_{n}^{p} n^{p} \varepsilon .
$$

Since $\varepsilon>0$ was arbitrary, we conclude that $\eta_{m}\left(A_{x}^{p, n} \backslash A_{x}^{\text {orb }}\right)=0$, so in fact $\eta_{m}\left(A_{x}^{p, n}\right)=0$.
Let us end the proof by showing that if $\mu$ is $T$-invariant, then $X_{\alpha}$ can be taken to be $T$-invariant. This follows from the fact that every Borel set $Y \subset X$ of full measure has a $T$-invariant subset of full measure. Indeed, let $B=\bigcup_{n=0}^{\infty} T^{-n}(X \backslash Y)$. Then $\mu(B)=0$ and it is easy to see that $Y \backslash B$ is $T$-invariant.

### 3.4 Conjectures of Shroer, Sauer, Ott and Yorke

The sufficiency of taking $k>\operatorname{dim}(X)$ measurements (instead of $k>2 \operatorname{dim}(X)$ ) for almost surely lossless reconstruction of the system via $k$ measurements was conjectured in a physical literature by Shroer, Sauer, Ott and Yorke [85, Conjectures 1 and 2]. They provided heurestic
arguments supporting this conjecture and numerical verification for certain systems, but no rigorous proofs. In this section we present these conjectures and explain their connections to the results from previous sections. We prove Conjecture 1 for general measures with Hausdorff dimension replacing the information dimension. As a corollary, we prove Conjecture 1 for ergodic measures. Let us introduce notions required to state the conjectures.

Definition 3.18 Let $X \subset \mathbb{R}^{N}$ be a Borel set and let $T: X \rightarrow X$ be a Borel transformation. Let $\mu$ be a Borel probability measure on $X$. Fix $k \in \mathbb{N}$, let $h: X \rightarrow \mathbb{R}$ be an observable and let $\phi(x)=\left(h(x), \ldots, h\left(T^{k-1} x\right)\right)$ be the corresponding delay-coordinate map. For $y_{0} \in \mathbb{R}^{k}$ and $\varepsilon>0$ such that $\phi_{*} \mu\left(B\left(y_{0}, \varepsilon\right)\right)>0$ define

$$
\chi_{\varepsilon}\left(y_{0}\right)=\frac{1}{\phi_{*} \mu\left(B\left(y_{0}, \varepsilon\right)\right)} \int_{\phi^{-1}\left(B\left(y_{0}, \varepsilon\right)\right)} \phi(T(x)) d \mu(x)
$$

and

$$
\sigma_{\varepsilon}\left(y_{0}\right)=\left(\frac{1}{\phi_{*} \mu\left(B\left(y_{0}, \varepsilon\right)\right)} \int_{\phi^{-1}\left(B\left(y_{0}, \varepsilon\right)\right)}\left\|\phi(T(x))-\chi_{\varepsilon}\left(y_{0}\right)\right\|^{2} d \mu(x)\right)^{\frac{1}{2}}
$$

In other words, $\chi_{\varepsilon}\left(y_{0}\right)$ is the conditional expectation of $\phi \circ T$ with respect to $\mu$ given $\phi \in$ $B\left(y_{0}, \varepsilon\right)$ and $\sigma_{\varepsilon}\left(y_{0}\right)$ is the corresponding conditional variance. For $y_{0}$ in the support of $\phi_{*} \mu$ define the prediction error at $y_{0}$ as

$$
\sigma\left(y_{0}\right)=\lim _{\varepsilon \rightarrow 0} \sigma_{\varepsilon}\left(y_{0}\right) .
$$

A point $y_{0}$ is said to be predictable if the above limit exists and $\sigma\left(y_{0}\right)=0$.
Note that the prediction error depends on the observable $h$. We simplify the notation by suppressing this dependence.

Definition 3.19 Let $X$ be a compact Riemannian manifold and $T: X \rightarrow X$ a smooth diffeomorphism. Let $\Lambda \subset X$ be a compact $T$-invariant set (an attractor) and let $B \subset M$ be the basin of attraction of $\Lambda$, i.e. a neighbourhood of $X$ characterized by $B=\{x \in X$ : $\left.\lim _{n \rightarrow \infty} \operatorname{dist}\left(T^{n} x, \Lambda\right)=0\right\}$. A $T$-invariant probability measure $\mu$ on $\Lambda$ is called a natural measure for $T$ if for almost every $x \in B$ (with respect to the volume measure on $X$ )

$$
\lim _{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{T^{k} x}=\mu
$$

where the limit is taken in the weak* topology.
Note that some authors use the name physical measure or SRB (Sinai-Ruelle-Bowen) measure for this or similar concepts (see e.g. [99]). The above notion of natural measure occurs is used in mathematical physics literature (see e.g. [72, 73]). We are ready now to state the conjectures of Shroer, Sauer, Ott and Yorke, following the formulations from [85]. Note that in [85] not all the details are precisely specified (e.g. the notion of genericity).

Conjecture 3.20 ([85, Conjecture 1]). If $\mu$ is a natural measure for a smooth diffeomorphism $T$ of a compact Riemannian manifold $X$, then $\phi_{*} \mu\left(\left\{x \in \mathbb{R}^{k}: x\right.\right.$ is predictable $\left.\}\right)=1$ holds for a generic observable if $k>\operatorname{ID}(\mu)$.

Conjecture 3.21 ([85, Conjecture 2]). Assume that $\mu$ is a natural measure for a smooth diffeomorphism $T$ of a compact Riemannian manifold $X$. Let $\delta>0$. Then the measure $\phi_{*} \mu\left(\sigma_{\varepsilon}>\delta\right)$ of points with finite prediction error $\sigma_{\varepsilon}>\delta$ scales for a generic observable in the following way (as $\varepsilon \searrow 0$ ):
(i) When $\operatorname{ID}(\mu)>k$, then $\mu\left(\sigma_{\varepsilon}>\delta\right) \sim O(1)$.
(ii) When $\frac{k}{2}<\operatorname{ID}(\mu)<k$, then $\mu\left(\sigma_{\varepsilon}>\delta\right) \sim \varepsilon^{k-\operatorname{ID}(\mu)}$ if self-intersections occur (the delay coordinate map is not injective). If self-intersections are absent, $\mu\left(\sigma_{\varepsilon}>\delta\right)=0$.
(iii) If $\operatorname{ID}(\mu)<\frac{k}{2}$ then $\mu\left(\sigma_{\varepsilon}>\delta\right)=0$.

If we understand the notion of generic observable as in Theorem 3.15, we can prove the following:

Theorem 3.22. Conjecture 3.20 holds with $\operatorname{dim}_{H}(\mu)$ replacing $\operatorname{ID}(\mu)$. Consequently, Conjecture 3.20 holds for ergodic measures.

Actually, we are able to prove a more general result - see Theorem 3.24 below. Indeed, the first assertion of Theorem 3.22 follows from Theorem 3.24 (see Lemma 3.25). The second assertion follows from the first one by Proposition 2.8. Before proceeding further, let us introduce additional notation.

Definition 3.23 Let $X \subset \mathbb{R}^{N}$ be a compact set, $T: X \rightarrow X$ be a continuous map and $\mu$ a Borel probability measure $\mu$ on $X$. For $x \in \operatorname{supp}(\mu)$ and $\varepsilon>0$, let $\mu_{x, \varepsilon}$ be the conditional distribution of $\mu$ on $\phi^{-1}(B(\phi(x), \varepsilon)$ ) (i.e. we condition on the event $\phi \in B(\phi(x), \varepsilon))$ :

$$
\mu_{x, \varepsilon}=\frac{1}{\mu\left(\phi^{-1}(B(\phi(x), \varepsilon))\right)} \mu \upharpoonright \phi^{-1}(B(\phi(x), \varepsilon)) .
$$

Theorem 3.24. Let $X \subset \mathbb{R}^{N}$ be a compact set, $\mu$ be a Borel probability measure on $X$ and $T: X \rightarrow X$ an injective Lipschitz map. Fix $k \in \mathbb{N}$ and $\beta \in(0,1]$ such that $\mu \perp \mathcal{H}^{\beta k}$ and assume $\left.\mu\right|_{\operatorname{Per}_{p}(T)} \perp \mathcal{H}^{\beta p}$ for every $p=1, \ldots, k-1$. Let $h: X \rightarrow \mathbb{R}$ be $\beta$-Hölder and $h_{1}, \ldots, h_{m}: X \rightarrow \mathbb{R}$ a $2 k$-interpolating family on $X$ consisting of $\beta$-Hölder transformations. For $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{R}^{m}$ denote by $h_{\alpha}: X \rightarrow \mathbb{R}$ the transformation

$$
h_{\alpha}(x)=h(x)+\sum_{j=1}^{m} \alpha_{j} h_{j}(x) .
$$

Then for Lebesgue almost every $\alpha=\left(\alpha_{1}, \ldots, \alpha_{m}\right) \in \mathbb{R}^{m}$, there exists a set $X_{\alpha} \subset X$ of full measure $\mu$, such that the delay-coordinate map

$$
\phi_{\alpha}^{T}: X \rightarrow \mathbb{R}^{k}, \quad \phi_{\alpha}^{T}(x)=\left(h_{\alpha}(x), h_{\alpha}(T x), \ldots, h_{\alpha}\left(T^{k-1} x\right)\right)
$$

is injective on $X_{\alpha}$. Moreover, $\lim _{\varepsilon \rightarrow 0} \mu_{x, \varepsilon}=\delta_{x}$ in the weak ${ }^{*}$ topology for every $x \in X_{\alpha}$ and $\phi_{\alpha}^{T}(x)$ is predictable for every $x \in X_{\alpha}$. If $\mu$ is additionally $T$-invariant, then the sets $X_{\alpha}$ can be taken to be $T$-invariant, i.e. satisfy $T\left(X_{\alpha}\right) \subset X_{\alpha}$.

Note that assumptions of the above theorem are the same as assumptions of Theorem 3.15, hence this is its strengthening, which asserts additional properties of $\phi_{\alpha}^{T}\left(\lim _{\varepsilon \rightarrow 0} \mu_{x, \varepsilon}=\delta_{x}\right.$ almost surely). As the next lemma shows, in order to establish almost sure predictability, it is enough to prove the convergence $\lim _{\varepsilon \rightarrow 0} \mu_{x, \varepsilon}=\delta_{x}$ for almost every $x \in X$.

Lemma 3.25. Let $X \subset \mathbb{R}^{N}$ be a compact set, $T: X \rightarrow X$ be a continuous map and $\mu$ a Borel probability measure $\mu$ on $X$. Let $\phi$ be the delay-coordinate map corresponding to a continuous observable $h: X \rightarrow \mathbb{R}$. Fix $x_{0} \in X$ and assume that $\lim _{\varepsilon \rightarrow 0} \mu_{x_{0}, \varepsilon}=\delta_{x_{0}}$, where the limit is in the weak* topology. Then $\phi\left(x_{0}\right)$ is predictable

Proof. Observe first that if $\mu_{x_{0}, \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \delta_{x_{0}}$, then

$$
\begin{equation*}
\chi_{\varepsilon}\left(\phi\left(x_{0}\right)\right)=\int_{X} \phi \circ T d \mu_{x_{0}, \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \phi \circ T\left(x_{0}\right), \tag{3.10}
\end{equation*}
$$

as $\phi \circ T$ is continuous. Moreover

$$
\begin{gathered}
\sigma_{\varepsilon}^{2}\left(\phi\left(x_{0}\right)\right)=\int_{X}\left\|\phi \circ T-\chi_{\varepsilon}\left(\phi\left(x_{0}\right)\right)\right\|^{2} d \mu_{x_{0}, \varepsilon} \leqslant \int_{X}\left\|\phi \circ T-\phi \circ T\left(x_{0}\right)\right\|^{2} d \mu_{x_{0}, \varepsilon}+ \\
+\int_{X}\left\|\chi_{\varepsilon}\left(\phi\left(x_{0}\right)\right)-\phi \circ T\left(x_{0}\right)\right\|^{2} d \mu_{x_{0}, \varepsilon}+2 \int_{X}\left\|\chi_{\varepsilon}\left(\phi\left(x_{0}\right)\right)-\phi \circ T\left(x_{0}\right)\right\|\left\|\phi \circ T-\phi \circ T\left(x_{0}\right)\right\| d \mu_{x_{0}, \varepsilon} \leqslant \\
\leqslant \int_{X}\left\|\phi \circ T-\phi \circ T\left(x_{0}\right)\right\|^{2} d \mu_{x_{0}, \varepsilon}+\left\|\chi_{\varepsilon}\left(\phi\left(x_{0}\right)\right)-\phi \circ T\left(x_{0}\right)\right\|^{2}+2 M\left\|\chi_{\varepsilon}\left(\phi\left(x_{0}\right)\right)-\phi \circ T\left(x_{0}\right)\right\|,
\end{gathered}
$$

where $M=\sup _{x \in X}\left\|\phi \circ T(x)-\phi \circ T\left(x_{0}\right)\right\|$. The latter integral converges to 0 as $\varepsilon \rightarrow 0$, since $\mu_{x_{0}, \varepsilon} \xrightarrow{\varepsilon \rightarrow 0} \delta_{x_{0}}$ and $\left\|\phi \circ T-\phi \circ T\left(x_{0}\right)\right\|^{2}$ is continuous. Applying 3.10 finishes the proof.

Remark 3.26 Note that in the proof of the above lemma, we did not use any special properties of the delay-coordinate map. In fact, Lemma 3.25 holds for any continuous function $\phi: X \rightarrow \mathbb{R}^{k}$.

In order to deduce Theorem 3.24 from Theorem 3.15, we will need the following version of the Rokhlin Disintegration Theorem in compact metric spaces. We will use it to deduce $\lim _{\varepsilon \rightarrow 0} \mu_{x, \varepsilon}=\delta_{x}$ for $\mu$-a.e. $x \in X$ from almost sure injectivity of the delay coordinate map. The existence and uniqueness of a system of conditional measures is a classical result, known as the Rokhlin disintegration theorem (see e.g. [82]). The crucial fact for us is that in the topological setting, the conditional measures can be defined as limits of conditional measures on preimages of shrinking balls and the convergence is ensured almost surely, which was proved by D. Simmons in 87].

Theorem 3.27 ([87, Theorems 2.1 and 2.2]). Let $X$ be a compact metric space and let $\mu$ be a Borel probability measure on $X$. Let $Y$ be a separable Riemannian manifold and let $\pi: X \rightarrow Y$ be a measurable map. Then for $\pi_{*} \mu$-almost every $y \in Y$, the sequence of probability measures $\frac{1}{\mu\left(\pi^{-1}(B(y, \varepsilon))\right)} \mu \upharpoonright \pi^{-1}(B(y, \varepsilon))$ converges weakly* to a probability measure $\mu_{y}$ as $\varepsilon \searrow 0$. Moreover, the collection of measures $\left\{\mu_{y}: y \in Y\right\}$ (we set here $\mu_{y}=0$ if the convergence does not hold) is a system of conditional measures of $\mu$ with respect to $\pi$, i.e.

1. for each $y \in Y, \mu_{y}$ is a measure on $\pi^{-1}(\{y\})$,
2. $\mu_{y}$ is a probability measure for $\pi_{*} \mu$-almost every $y \in Y$,
3. for every Borel set $B \subset X$, the transformation $Y \ni y \mapsto \mu_{y}(B) \in \mathbb{R}$ is measurable (with respect to the completion of the Borel $\sigma$-algebra according to the measure $\pi_{*} \mu$ on $Y$ ) and

$$
\mu(B)=\int_{Y} \mu_{y}(B) d \pi_{*} \mu(y) .
$$

Moreover, the system of conditional measures is unique, i.e. if $\left\{\nu_{y}: y \in Y\right\}$ is a collection of measures satisfying 1, - 圂, then $\nu_{y}=\mu_{y}$ for $\pi_{*} \mu$ almost every $y \in Y$.

The idea of the proof of Theorem 3.24 is the following. Theorem 3.15 guarantees that for almost every $\alpha \in \mathbb{R}^{m}$, the corresponding delay-coordinate map $\phi_{\alpha}^{T}: X \rightarrow \mathbb{R}^{k}$ is injective on a Borel set $X_{\alpha}$ of full measure. On the other hand, Theorem 3.27 guarantees that the measures $\mu_{x, \varepsilon}$ are almost surely convergent as $\varepsilon \searrow 0$, and the limits form a system of conditional measures of $\mu$ with respect to $\phi_{\alpha}^{T}$. Almost sure injectivity implies that the conditional measures are almost surely Dirac's deltas, hence indeed $\lim _{\varepsilon \rightarrow 0} \mu_{x, \varepsilon}=\delta_{x}$. Below we present a formal proof, were we check the details.

Proof of Theorem 3.24. By Theorem 3.15, for almost every $\alpha \in \mathbb{R}^{m}$ there exists a Borel set $X_{\alpha}$ of full measure $\mu$, such that the corresponding delay-coordinate map $\phi:=\phi_{\alpha}^{T}: X \rightarrow \mathbb{R}^{k}$ is injective on $X_{\alpha}$. Let us fix such $\alpha$. We shall prove $\lim _{\varepsilon \rightarrow 0} \mu_{x, \varepsilon}=\delta_{x}$ for $\mu$-almost every $x \in X$. By applying Theorem 3.27 with $\pi=\phi$ and $Y=\mathbb{R}^{k}$, we obtain that $\lim _{\varepsilon \rightarrow 0} \mu_{x, \varepsilon}=: \mu_{\phi(x)}$ exists for $x$ in a set $A$ of full measure $\mu$. Moreover, the system of measures $\left\{\mu_{y}: y \in Y\right\}$, where $\mu_{y}=\mu_{\phi(x)}$ if $y=\phi(x)$ for some $x \in A$ (note that the measure $\mu_{y}$ does not depend on the choice of the preimage $x$ of $y$ ) and $\mu_{y}=0$ otherwise, is a system of conditional measures of $\mu$ with respect to $\phi$. It remains to show that $\mu_{\phi(x)}=\delta_{x}$ for $\mu$-almost every $x \in X$. We claim that this follows from injectivity of $\phi$ on $X_{\alpha}$. For $y \in \phi\left(X_{\alpha}\right)$, denote by $\psi(y)$ the unique element $x \in X_{\alpha}$ such that $\phi(x)=y$. By the uniqueness in Theorem 3.27, it is enough to show that the collection of measures $\left\{\nu_{y}: y \in Y\right\}$ defined as $\nu_{y}=\delta_{\psi(y)}$ if $y \in \phi\left(X_{\alpha}\right)$ and $\nu_{y}=0$ otherwise, is also a system of conditional measures of $\mu$ with respect to $\phi$. Let us check first that for every Borel $B \subset X$, the map $f: Y \rightarrow \mathbb{R}$ defined as $f(y)=\nu_{y}(B)$ is measurable with respect to the completion of the Borel $\sigma$-algebra on $Y$ according to the measure $\phi_{*} \mu$. It is enough to show that $f$ is equal $\phi_{*} \mu$-almost surely to a Borel map. Note that $\phi\left(X_{\alpha}\right)$ is of full $\phi_{*} \mu$ measure and for $y \in \phi\left(X_{\alpha}\right)$ we have

$$
\begin{equation*}
f(y)=\delta_{\psi(y)}(B)=\mathbb{1}_{B}(\psi(y))=\mathbb{1}_{\phi\left(B \cap X_{\alpha}\right)}(y) . \tag{3.11}
\end{equation*}
$$

Therefore $f=\mathbb{1}_{\phi\left(B \cap X_{\alpha}\right)}$ holds $\phi_{*} \mu$-almost surely, hence it is enough to show that $\phi\left(B \cap X_{\alpha}\right)$ is Borel. This follows from [53, Theorem 15.1], as $\phi$ is continuous and injective on a Borel set $B \cap X_{\alpha}$. Moreover, by (3.11)

$$
\int_{Y} \nu_{y}(B) d \pi_{*} \mu(y)=\int_{Y} \mathbb{1}_{\phi\left(B \cap X_{\alpha}\right)}(y) \pi_{*} \mu(y)=\mu\left(\phi^{-1}\left(\phi\left(B \cap X_{\alpha}\right)\right) \cap X_{\alpha}\right)=\mu\left(B \cap X_{\alpha}\right)=\mu(B),
$$

hence point 3. of the definition of a system of conditional measures is fulfilled. As each $\nu_{y}$ is clearly a measure on $\phi^{-1}(\{y\})$, we conclude that $\left\{\nu_{y}: y \in Y\right\}$ is indeed a system of conditional measures of $\mu$ with respect to $\phi$. The invariance of $X_{\alpha}$ in the case of $T$-invariant measure follows by the same lines as in the proof of Theorem 3.15.

Note that the above proof does not use any specific properties of $\phi_{\alpha}^{T}$ other than its continuity. Therefore, we can similarly use Theorem 3.27 to prove analogous extension of Theorem 3.5.

Theorem 3.28. Let $X \subset \mathbb{R}^{N}$ be a compact set and $\mu$ a Borel probability measure on $X$. Fix $k \in \mathbb{N}$ and $\beta \in(0,1]$ such that $\mu \mathcal{H}^{\beta k}$ and let $\phi: X \rightarrow \mathbb{R}^{k}$ be $\beta$-Hölder on bounded sets. Then for Lebesgue almost every linear transformation $L: \mathbb{R}^{N} \rightarrow \mathbb{R}^{k}$ there exists a set $X_{L} \subset X$ of full measure $\mu$, such that the map $\phi_{L}=\phi+L$ is injective on $X_{L}$ and for every $x \in X_{L}$, the sequence of measures

$$
\mu_{x, \varepsilon}=\frac{1}{\mu\left(\phi^{-1}(B(\phi(x), \varepsilon))\right)} \mu \upharpoonright \phi_{L}^{-1}\left(B\left(\phi_{L}(x), \varepsilon\right)\right)
$$

converges weakly* to $\delta_{x}$ as $\varepsilon \searrow 0$.

### 3.5 Measure with Hausdorff dimension smaller than lower modified box dimension

To show that Theorem 3.5 is an actual strengthening of Theorem 3.8, we present an example of a measure $\mu$, for which $\operatorname{dim}_{H} \mu<\underline{\operatorname{dim}}_{M B} \mu$. As we were unable to find an explicit example in the literature, we give a suitable construction. More precisely, we show the following.

Theorem 3.29. There exists a Borel probability measure $\mu$ on $[0,1]^{2}$, such that $\operatorname{dim}_{H} \mu=1$ and $\underline{\operatorname{dim}}_{M B} \mu=2$.

Informally speaking, the idea is to construct a measure of exact lower local dimension 1 and exact upper dimension 2, but such that the lower local dimensions are attained along an uncountable collection of sequences of scales. In other words, there is no countable collection of positive numbers $\left\{r_{n}^{i}\right\}_{n, i=1}^{\infty}$ such that for every $i \in \mathbb{N}$, the sequence $r_{n}^{i}$ decreases to 0 as $n \rightarrow \infty$ and for $\mu$-a.e. $x$ there exists $i \in \mathbb{N}$ such that $\underline{d}(\mu, x)=\lim _{n \rightarrow 0} \frac{\log \mu\left(B\left(x, r_{n}^{i}\right)\right)}{\log r_{n}^{i}}=1$. If such countable collection would exist, then indeed we would have $\operatorname{dim}_{H}(\mu)=\underline{\operatorname{dim}}_{M B}(\mu)=1$. In the construction, we consider a collection of measures $\left\{\mu_{y}: y \in[0,1]\right\}$ on $[0,1]$, each with exact lower local dimension 0 and exact upper local dimension 1 , but with the sequence of scales $r_{n}(y)$, along which $\underline{\operatorname{dim}}_{H}\left(\mu_{y}\right)$ is attained, depending on the binary expansion of $y$. The desired measure is then $\mu$ on $[0,1]^{2}$ having $\mu_{y}$ as conditional measures on fibers $[0,1] \times\{y\}$. For technical reasons, it is easier to work with the dyadic local dimension. Let us proceed now with the formal proof.

For $x \in[0,1]$ we will write

$$
x=0 . x_{1} x_{2} \ldots
$$

where $0 . x_{1} x_{2} \ldots$ is the binary expansion of $x$, i.e.

$$
x=\sum_{j=1}^{\infty} \frac{x_{j}}{2^{j}}, \quad x_{1}, x_{2}, \ldots \in\{0,1\}
$$

For a dyadic rational we agree to choose its eventually terminating expansion, i.e. the one with $x_{j}=0$ for $j$ large enough, with an exception of the number 1 , for which we choose the expansion $0.111 \ldots$ Let $\pi:\{0,1\}^{\mathbb{N}} \rightarrow[0,1]$ be the coding map

$$
\pi\left(x_{1}, x_{2}, \ldots\right)=\sum_{j=1}^{\infty} \frac{x_{j}}{2^{j}}
$$

To begin the construction of $\mu$, fix an increasing sequence of positive integers $N_{k}, k \in \mathbb{N}$, such that $N_{k} \nearrow \infty$ with $\frac{S_{k}}{S_{k+1}} \leqslant \frac{1}{k+1}$, where $S_{k}=\sum_{j=1}^{k} N_{j}$. Consider the probability distributions $\mathbf{p}_{0}, \mathbf{p}_{1}$ on $\{0,1\}$ given by

$$
\mathbf{p}_{0}(\{0\})=0, \mathbf{p}_{0}(\{1\})=1, \quad \mathbf{p}_{1}(\{0\})=\mathbf{p}_{1}(\{1\})=\frac{1}{2}
$$

For $y=0 . y_{1} y_{2} \ldots \in[0,1]$, define the probability measure $\nu_{y}$ on $\{0,1\}^{\mathbb{N}}$ as the infinite product

$$
\nu_{y}=\bigotimes_{j=1}^{\infty} \bigotimes_{i=1}^{N_{j}} \mathbf{p}_{y_{j}}
$$

Further, let $\mu_{y}$ be the Borel probability measure on $[0,1]$ given by

$$
\mu_{y}=\pi_{*} \nu_{y}
$$

Finally, let $\mu$ be the Borel probability measure on $[0,1]^{2}$ defined as

$$
\mu(A)=\int_{[0,1]} \mu_{y}\left(A^{y}\right) d \operatorname{Leb}(y) \quad \text { for a Borel set } A \subset[0,1]^{2},
$$

where $A^{y}=\{x \in[0,1]:(x, y) \in A\}$. It is easy to see that $\mu$ is well-defined, as the function $y \mapsto \mu_{y}\left(A^{y}\right)$ is measurable for every Borel set $A \subset[0,1]^{2}$.

The proof of Theorem 3.29 is based on the analysis of the local dimension of $\mu$, defined in terms of dyadic squares (rather then balls). The following lemma gives estimates on the measure of dyadic squares at suitable scales (recall Chapter 2 for notation on dyadic cubes and local dimensions).

Lemma 3.30. Let $x=0 . x_{1} x_{2} \ldots, \in[0,1], y=0 . y_{1} y_{2} \ldots \in[0,1], n \in \mathbb{N}$ and $D=D_{n}(x, y)=$ $\left[x_{1}, \ldots, x_{n}\right] \times\left[y_{1}, \ldots, y_{n}\right]$. Let $k \in \mathbb{N}$ be such that $S_{k}<n \leqslant S_{k+1}$. Then the following hold.
(a) If $y_{k}=y_{k+1}=1$, then $\mu(D) \leqslant 2^{-\left(2-\frac{1}{k}\right) n}$.
(b) If $n=S_{k+1}$ and $y_{k+1}=0$, then either $\mu(D)=0$ or $\mu(D) \geqslant 2^{-\left(1+\frac{1}{k+1}\right) n}$.

Proof. Note that for $y^{\prime}=0 . y_{1}^{\prime} y_{2}^{\prime} \ldots \in[0,1]$ such that $\left(y_{1}^{\prime}, \ldots, y_{n}^{\prime}\right)=\left(y_{1}, \ldots, y_{n}\right)$ we have

$$
\begin{align*}
\mu_{y^{\prime}}\left(D^{y^{\prime}}\right)=\mu_{y^{\prime}}\left(\left[x_{1}, \ldots, x_{n}\right]\right)= & \mathbf{p}_{y_{1}^{\prime}}\left(\left\{x_{1}\right\}\right) \cdots \mathbf{p}_{y_{1}^{\prime}}\left(\left\{x_{S_{1}}\right\}\right) \mathbf{p}_{y_{2}^{\prime}}\left(\left\{x_{S_{1}+1}\right\}\right) \cdots \mathbf{p}_{y_{2}^{\prime}}\left(\left\{x_{S_{2}}\right\}\right)  \tag{3.12}\\
& \cdots \mathbf{p}_{y_{k+1}^{\prime}}\left(\left\{x_{S_{k}+1}\right\}\right) \cdots \mathbf{p}_{y_{k+1}^{\prime}}\left(\left\{x_{n}\right\}\right) .
\end{align*}
$$

Moreover, as $k<n$, the value of $\mu_{y^{\prime}}\left(D^{y^{\prime}}\right)$ depends only on $\left(y_{1}, \ldots, y_{n}\right)$ and $\left(x_{1}, \ldots, x_{n}\right)$. Using (3.12), we can prove both assertions of the lemma, as follows.

## Ad (a)

If $y_{k}=y_{k+1}=1$, then for $j \in\left\{S_{k-1}+1, \ldots, n\right\}$ we have $\mathbf{p}_{y_{l}}\left(x_{j}\right)=\frac{1}{2}$, where $l \in\{k, k+1\}$ is such that $S_{l-1}<j \leqslant S_{l}$. Therefore, in the product (3.12) there is at least $n-S_{k-1}$ terms equal to $\frac{1}{2}$. Consequently,

$$
\mu_{y^{\prime}}\left(D^{y^{\prime}}\right) \leqslant 2^{-\left(n-S_{k-1}\right)}=2^{-\left(1-\frac{S_{k-1}}{n}\right) n} \leqslant 2^{-\left(1-\frac{S_{k-1}}{S_{k}}\right) n} \leqslant 2^{-\left(1-\frac{1}{k}\right) n},
$$

hence

$$
\mu(D)=\int_{\left[y_{1}, \ldots, y_{n}\right]} \mu_{y^{\prime}}\left(D^{y^{\prime}}\right) d \operatorname{Leb}\left(y^{\prime}\right) \leqslant \operatorname{Leb}\left(\left[y_{1}, \ldots, y_{n}\right]\right) 2^{-n\left(1-\frac{1}{k}\right)}=2^{-n\left(2-\frac{1}{k}\right)} .
$$

## Ad (b).

Assume that $\mu(D) \neq 0$. Then all the terms in (3.12) have to be non-zero, so every term is equal to either $\frac{1}{2}$ or 1 . Moreover, as $y_{k+1}=0$ and $n=S_{k+1}$, we have

$$
\mathbf{p}_{y_{k+1}}\left(\left\{x_{S_{k}+1}\right\}\right) \cdots \mathbf{p}_{y_{k+1}}\left(\left\{x_{n}\right\}\right)=1
$$

and, consequently,

$$
\begin{aligned}
\mu(D)= & 2^{-n} \mathbf{p}_{y_{1}}\left(\left\{x_{1}\right\}\right) \cdots \mathbf{p}_{y_{1}}\left(\left\{x_{S_{1}}\right\}\right) \mathbf{p}_{y_{2}}\left(\left\{x_{S_{1}+1}\right\}\right) \cdots \mathbf{p}_{y_{2}}\left(\left\{x_{S_{2}}\right\}\right) \\
& \cdots \mathbf{p}_{y_{k}}\left(\left\{x_{S_{k-1}+1}\right\}\right) \cdots \mathbf{p}_{y_{k}}\left(\left\{x_{S_{k}}\right\}\right) \geqslant 2^{-n-S_{k}}=2^{-\left(1+\frac{S_{k}}{S_{k+1}}\right) n} \geqslant 2^{-\left(1+\frac{1}{k+1}\right) n} .
\end{aligned}
$$

Now we are ready to give the proof of Theorem 3.29,

Proof of Theorem 3.29. We begin by proving $\operatorname{dim}_{H} \mu=1$. Note that $\operatorname{dim}_{H} \mu \geqslant 1$, as $\mu$ projects under $[0,1]^{2} \ni(x, y) \mapsto y \in[0,1]$ to the Lebesgue measure, so it is sufficient to show $\operatorname{dim}_{H} \mu \leqslant 1$. By Propositions 2.41 and 2.6 , it is enough to prove that $\underline{d}^{\prime}(\mu,(x, y)) \leqslant 1$ for $\mu$-almost every $(x, y) \in[0,1]$. Note that for Lebesgue almost every $y=0 . y_{1} y_{2} \ldots \in$ $[0,1]$, the sequence $\left(y_{1}, y_{2}, \ldots\right)$ contains infinitely many zeros. Hence, it is sufficient to show $\underline{d}^{\prime}(\mu,(x, y)) \leqslant 1$ for $\mu_{y}$-almost every $x \in[0,1]$, assuming that $y=0 . y_{1} y_{2} \ldots \in[0,1]$ contains infinitely many zeros. Moreover, for $\mu_{y}$-almost every $x \in[0,1]$, we have $\mu\left(D_{n}(x, y)\right)>0$ for all $n \in \mathbb{N}$ (see 3.12 ). For such $x$, by Lemma 3.30 (b), we have

$$
\underline{d}^{\prime}(\mu,(x, y)) \leqslant \liminf _{k \rightarrow \infty} \frac{-\log \mu\left(D_{S_{n_{k}}}(x, y)\right)}{S_{n_{k}} \log 2} \leqslant \lim _{k \rightarrow \infty} \frac{\left(1+\frac{1}{n_{k}}\right) S_{n_{k}}}{S_{n_{k}}}=1
$$

Therefore, $\operatorname{dim}_{H} \mu \leqslant 1$, so in fact $\operatorname{dim}_{H} \mu=1$.
Let us prove now $\operatorname{dim}_{M B} \mu=2$. Since $\mu$ is supported on $[0,1]^{2}$, it suffices to show $\underline{\operatorname{dim}}_{M B} \mu \leqslant 2$. Let $A \subset[0,1]^{2}$ be a Borel set with $\mu(A)>0$. We show $\underline{\operatorname{dim}}_{B} A \geqslant 2$. Note that there exists $c>0$ such that the set

$$
\begin{equation*}
B=\left\{y \in[0,1]: \mu_{y}\left(A^{y}\right) \geqslant c\right\} \tag{3.13}
\end{equation*}
$$

satisfies $\operatorname{Leb}(B)>0$. Fix $\varepsilon \in\left(0, \frac{1}{4}\right)$. By the Lebesgue density theorem (see e.g. [44, Corollary $3.16]$ ), there exists a dyadic interval $I \subset[0,1]$ such that

$$
\begin{equation*}
\frac{\operatorname{Leb}(B \cap I)}{|I|} \geqslant 1-\varepsilon \tag{3.14}
\end{equation*}
$$

where $|I|=2^{-N}$ is the length of $I$. Fix $k \geqslant N+2$ and $n \in\left\{S_{k}+1, \ldots, S_{k+1}\right\}$. Consider the collection $\mathcal{C}_{n}$ of dyadic intervals of length $2^{-n}$ defined as

$$
\mathcal{C}_{n}=\left\{\left[y_{1}, \ldots, y_{n}\right]: y_{k}=y_{k+1}=1 \text { and }\left[y_{1}, \ldots, y_{n}\right] \cap B \cap I \neq \emptyset\right\}
$$

By (3.14), we have

$$
\begin{equation*}
\operatorname{Leb}\left(B \cap \bigcup \mathcal{C}_{n}\right) \geqslant\left(\frac{1}{4}-\varepsilon\right) 2^{-N} \tag{3.15}
\end{equation*}
$$

Let

$$
A_{n}=A \cap\left([0,1] \times\left(B \cap \bigcup \mathcal{C}_{n}\right)\right)
$$

Then $A_{n} \subset A$ and (3.13) together with (3.15) imply

$$
\begin{equation*}
\mu\left(A_{n}\right)=\int_{B \cap \bigcup \mathcal{C}_{n}} \mu_{y}\left(A^{y}\right) d \operatorname{Leb}(y) \geqslant c\left(\frac{1}{4}-\varepsilon\right) 2^{-N} \tag{3.16}
\end{equation*}
$$

Note that the above lower bound does not depend on $k$ and $n$. Let $N^{\prime}\left(A_{n}, 2^{-n}\right)$ be the number of dyadic squares of sidelength $2^{-n}$ intersecting $A_{n}$. If $D=I_{1} \times I_{2}$ is a dyadic square of sidelength $2^{-n}$ intersecting $A_{n}$, then $I_{2} \in \mathcal{C}_{n}$, hence by Lemma 3.30 (a) we have

$$
\mu(D) \leqslant 2^{-\left(2-\frac{1}{k}\right) n}
$$

As any two dyadic squares of the same sidelength are either equal or disjoint, 3.16 gives

$$
N^{\prime}\left(A, 2^{-n}\right) \geqslant N^{\prime}\left(A_{n}, 2^{-n}\right) \geqslant c\left(\frac{1}{4}-\varepsilon\right) 2^{-N+\left(2-\frac{1}{k}\right) n}
$$

Since $k$ and $n$ can be taken arbitrary large, invoking 2.2 gives $\operatorname{dim}_{B} A \geqslant 2$. Hence, $\underline{\operatorname{dim}}_{M B} \mu \geqslant 2$, so in fact $\underline{\operatorname{dim}}_{M B} \mu=2$.

Remark 3.31 Note that as (see Proposition 2.4)

$$
\underset{x \sim \mu}{\operatorname{ess} \sup } \underline{d}(\mu, x)=\operatorname{dim}_{H} \mu \leqslant \underline{\operatorname{dim}}_{M B} \mu \leqslant \overline{\operatorname{dim}}_{M B} \mu=\underset{x \sim \mu}{\operatorname{ess} \sup } \bar{d}(\mu, x),
$$

the equality $\operatorname{dim}_{H} \mu=\underline{\operatorname{dim}}_{M B} \mu$ holds for all exact dimensional measures.

### 3.6 Examples

In this section we present several examples which illustrate the usage of Theorem 3.5. Let us begin with an example showing that the requirement $\mu \perp \mathcal{H}^{\beta k}(X)$ in Theorem 3.5 cannot be replaced by weaker condition $\operatorname{dim}_{H}(\mu) \leqslant \beta k$.
Example 3.32 Let $k=\beta=1, X=\mathbb{S}^{1} \subset \mathbb{R}^{2}$ be the unit circle and let $\mu$ be the normalized Lebesgue measure on $\mathbb{S}^{1}$. Then $\operatorname{dim}_{H}(\mu)=1$ but $\mu$ and $\mathcal{H}^{1}$ are not mutually singular. We shall prove that there is no Lipschitz transformation $\phi: \mathbb{S}^{1} \rightarrow \mathbb{R}$ which is injective on a set of full measure $\mu$. Let $\phi$ be such a transformation. Then $\phi\left(\mathbb{S}^{1}\right)=[a, b]$ for some compact interval. As $\phi$ is injective on a set of full measure, interval $[a, b]$ is non-degenerate, i.e. $a<b$. Fix points $x, y \in \mathbb{S}^{1}$ with $\phi(x)=a, \phi(y)=b$. As $x \neq y$, there are exactly two open arcs $I, J \subset \mathbb{S}^{1}$ of positive measure joining $x$ and $y$ such that $\bar{I} \cap \bar{J}=\{x, y\}$ and $\bar{I} \cup \bar{J}=\mathbb{S}^{1}$ (these are clockwise and counter-clockwise arcs from $x$ to $y$ ). Clearly $\phi(\bar{I})=\phi(\bar{J})=[a, b]$. Let $A \subset \mathbb{S}^{1}$ be a Borel set such that $\phi$ is injective on $A$ and $\mu(A)=1$. As Lipschitz maps transform sets of zero Lebesgue measure to sets of zero Lebesgue measure, we conclude that $\phi(I \cap A)$ and $\phi(J \cap A)$ are disjoint Lebesgue measurable subsets of $[a, b]$, both with Lebesgue measure equal to $b-a$. This contradiction finishes the proof.

By using the above example, we can show that the information dimension is not well suited for embedding theorems, i.e. assumption $k>\overline{\mathrm{ID}}(\mu)$ is not sufficient for an existence of Lipschitz almost surely injective map into $\mathbb{R}^{k}$. As $\operatorname{dim}_{H}(\mu)<\overline{\mathrm{ID}}(\mu)$ (see Proposition 2.4. (4) ), the same is the case for the lower Hausdorff dimension. The latter observation is not very surprising, as $\operatorname{dim}_{H}$ gives control of the dimension only over a set of positive measure, hence one cannot expect to conclude injectivity on a set of full measure from a bound on $\underline{\operatorname{dim}}_{H}$.
Example 3.33 Let $\mathbb{S}^{1} \subset \mathbb{R}^{2}$ be the unit circle and let $\mu$ be the normalized Lebesgue measure on $\mathbb{S}^{1}$. Choose $p \in(0,1)$, fix point $x \in \mathbb{R}^{2} \backslash \mathbb{S}^{1}$ and let $\nu=p \delta_{x}+(1-p) \mu$. Then $\operatorname{ID}(\nu)=$ $(1-p)<1$ (see Proposition 2.5.(22), yet there does not exist a Lipschitz map $\phi: \mathbb{S} \cup\{x\} \rightarrow \mathbb{R}$ which is injective on a set of full measure $\nu$, a such map would be injective also on a set full measure $\mu$. This is however impossible, as Example 3.32 shows. Note that by considering $T:\{x\} \cup \mathbb{S}^{1} \rightarrow\{x\} \cup \mathbb{S}^{1}$ given as the identity, we see that $\overline{\mathrm{ID}}(\mu)$ is not well suited for the Probabilistic Takens Theorem 3.15 as well.

The next example shows that linear perturbations are not sufficient for Theorems 1.1 and 3.15
Example 3.34 We will show that it may happen that $\phi_{L}=\left(\phi(x)+L x, \ldots, \phi\left(T^{k-1} x\right)+\right.$ $L T^{k-1} x$ ) is not (almost surely) injective for a generic linear map $L: \mathbb{R}^{N} \rightarrow \mathbb{R}$. As an example, let $X=B_{2}(0,1) \subset \mathbb{R}^{2}$, fix $a \in(0,1)$ and define $T: X \rightarrow X$ as

$$
T(x)=a x .
$$

Then $T$ is a Lipschitz injective transformation on the unit disc $X \subset \mathbb{R}^{2}$ with zero being the unique periodic point. Fix $\phi \equiv 0$. We claim that there is no linear observable $L: \mathbb{R}^{2} \rightarrow \mathbb{R}$ which makes the delay map injective, i.e. for every $k \in \mathbb{N}$ and every $v \in \mathbb{R}^{2}$ the transformation $x \mapsto \phi_{v}^{T}(x)=\left(\langle x, v\rangle,\langle T x, v\rangle, \ldots,\left\langle T^{k-1} x, v\right\rangle\right) \in \mathbb{R}^{k}$ is not injective on $X$. This follows from the fact that for each 1-dimensional linear subspace $W \subset \mathbb{R}^{2}$ the set $W \cap X$ is $T$-invariant, hence $\phi_{v}^{T}=0$ on an infinite set $\operatorname{Ker}(\langle\cdot, v\rangle) \cap X$. We have seen that $\phi_{v}^{T}$ is not injective for any $v \in \mathbb{R}^{2}$. No we will see that it also not almost surely injecitve for $\mu$ being the Lebesgue measure on $X$. Note that for $v \in \mathbb{R}^{2}$ and $c \in \mathbb{R}$, the segment $W_{c}=\{z \in X:\langle z, v\rangle=$ c\} satisfies $T\left(W_{c}\right) \subset W_{a c}$, hence all points on $W_{c}$ will have the same observation vector $\left(\langle x, v\rangle,\langle T x, v\rangle, \ldots,\left\langle T^{k-1} x, v\right\rangle\right)=\left(c, a c, a^{2} c, \ldots, a^{k-1} c\right)$. Therefore, a set $X_{v} \subset X$ on which $\phi_{v}^{T}$ is injective can only have one point on each of the parallel segments $W_{c}$ contained in $X$. However, such a set $X_{v}$ cannot be of full Lebesgue measure. Note that the above example can be easily modified to make $T$ a homeomorphism.

### 3.6.1 A modified Kan's example

In the Appendix to [84], Kan presented an example of a compact set $K \subset \mathbb{R}^{N}$ with $\operatorname{dim}_{H} K=$ 0 and such that every linear transformation $L: \mathbb{R}^{N} \rightarrow \mathbb{R}^{N-1}$ fails to be injective on $K$ (see also Remark 3.10). It follows from Theorem 3.5, that whenever we are given a $\sigma$-finite Borel measure $\mu$ on such a set, then almost every linear transformation $L: \mathbb{R}^{N} \rightarrow \mathbb{R}$ is injective on a set of full measure $\mu$. To illustrate this, we construct a $\sigma$-compact set $X \subset \mathbb{R}^{2}$ with $\operatorname{dim}_{H} X=0$, which is a slight modification of Kan's example, equipped with a natural $\sigma$ finite Borel measure $\mu$, such that no linear transformation $L: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is injective on $X$, while for almost every $L$ we explicitly show a set $X_{L} \subset X$ of full measure $\mu$, such that $L$ is injective on $X_{L}$.

Following [84, Appendix], we begin with constructing compact sets $A, B \subset[0,1]$ such that

$$
\begin{equation*}
\operatorname{dim}_{H} A=\underline{\operatorname{dim}}_{B} A=\operatorname{dim}_{H} B=\underline{\operatorname{dim}}_{B} B=0, \tag{3.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\overline{\operatorname{dim}}_{B} A=\overline{\operatorname{dim}}_{B} B=1, \quad \underline{\operatorname{dim}}_{B}(A \cup B)=\overline{\operatorname{dim}}_{B}(A \cup B)=1 . \tag{3.18}
\end{equation*}
$$

Similarly as in the previous section, for $x \in[0,2)$ we write

$$
x=x_{0} \cdot x_{1} x_{2} \ldots,
$$

where $x_{0} \cdot x_{1} x_{2} \ldots$ is the binary expansion of $x$, i.e.

$$
x=\sum_{j=0}^{\infty} \frac{x_{j}}{2^{j}}, \quad x_{0}, x_{1}, x_{2}, \ldots \in\{0,1\} .
$$

For a dyadic rational we agree to choose its eventually terminating expansion, i.e. the one with $x_{j}=0$ for $j$ large enough (here, unlike in Section 3.5, it is convenient for us to choose $1.000 \ldots$ as the expansion of 1 ). Recall that $\pi:\{0,1\}^{\mathbb{N}} \rightarrow[0,1]$ is the coding map

$$
\pi\left(x_{1}, x_{2}, \ldots\right)=\sum_{j=1}^{\infty} \frac{x_{j}}{2^{j}}
$$

Let $M_{k}, k \geqslant 0$, be an increasing sequence of positive integers such that $M_{0}=1$ and $M_{k} \nearrow \infty$ with $\lim _{k \rightarrow \infty} \frac{M_{k+1}}{M_{k}}=\infty$. Define

$$
\begin{aligned}
\widetilde{A} & =\left\{\left(x_{1}, x_{2}, \ldots\right) \in\{0,1\}^{\mathbb{N}}: \text { for every even } k,\right. \\
& x_{j}=0 \text { for all } j \in\left[M_{k}, M_{k+1}\right) \\
& \text { or } \left.x_{j}=1 \text { for all } j \in\left[M_{k}, M_{k+1}\right)\right\}, \\
\widetilde{B}=\left\{\left(x_{1}, x_{2}, \ldots\right) \in\{0,1\}^{\mathbb{N}}: \text { for every odd } k,\right. & x_{j}=0 \text { for all } j \in\left[M_{k}, M_{k+1}\right) \\
& \text { or } \left.x_{j}=1 \text { for all } j \in\left[M_{k}, M_{k+1}\right)\right\},
\end{aligned}
$$

and set

$$
A=\pi(\widetilde{A}), \quad B=\pi(\widetilde{B}) .
$$

It is a straightforward calculation to check that $A$ and $B$ satisfy (3.17) and (3.18) (see 84, Appendix], [30, Example 7.8] or [81, Section 6.1]). Define $X \subset \mathbb{R}^{2}$ as

$$
X=\left(\{0\} \times \bigcup_{n \in \mathbb{Z}}(A+n)\right) \cup\left(\{1\} \times \bigcup_{n \in \mathbb{Z}}(B+n)\right) .
$$

By (3.17) and Proposition 2.4 5, we have $\operatorname{dim}_{H} X=0$. The following two propositions describe the embedding properties of the set $X$.
Proposition 3.35. No linear transformation $L: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is injective on $X$.

Proof. The map $L$ has the form $L(x, y)=\alpha x+\beta y$ for $\alpha, \beta \in \mathbb{R}$. Obviously, we can assume $\beta \neq 0$. Note that the points

$$
u=(0, a+n), \quad v=(1, b+m), \quad \text { for } a \in A, b \in B, n, m \in \mathbb{Z}
$$

are in $X$ and

$$
\begin{equation*}
L(u)=L(v) \quad \text { if and only if } \quad b-a=z, \tag{3.19}
\end{equation*}
$$

where

$$
z=-\frac{\alpha}{\beta}+n-m
$$

For given $\alpha$ and $\beta$, choose $n, m \in \mathbb{Z}$ such that $z \in[0,1)$. Consider the binary expansion $z=0 . z_{1} z_{2} \ldots$ and define

$$
a=0 . a_{1} a_{2} \ldots \in A, \quad b=0 . b_{1} b_{2} \ldots \in B
$$

setting

$$
\begin{array}{lll}
a_{j}=0, & b_{j}=z_{j} & \text { for } j \in\left[M_{k}, M_{k+1}\right),  \tag{3.20}\\
a_{j}=1-z_{j}, & b_{j}=1 & \text { for } k \text { is even, } \\
j \in\left[M_{k}, M_{k+1}\right), & \text { if } k \text { is odd }
\end{array}
$$

(if all $b_{j}$ are equal to 1 , we set $b=1$ ). Then $z=b-a$ and (3.19) implies that $L$ is not injective on $X$.

Let us now define a natural Borel $\sigma$-finite measure $\mu$ on $X$, starting from a pair of probability measures $\nu_{1}, \nu_{2}$ on $\widetilde{A}$ and $\widetilde{B}$, respectively. Let

$$
\nu_{1}=\bigotimes_{k=0}^{\infty} \mathbf{p}_{k}, \quad \nu_{2}=\bigotimes_{k=0}^{\infty} \mathbf{q}_{k},
$$

where $\mathbf{p}_{k}$ and $\mathbf{q}_{k}$ are probability measures on $\{0,1\}^{M_{k+1}-M_{k}}$ given as

$$
\mathbf{p}_{k}=\left\{\begin{array}{ll}
\frac{1}{2} \delta_{(0, \ldots, 0)}+\frac{1}{2} \delta_{(1, \ldots, 1)} & \text { if } k \text { is even } \\
\left(\frac{1}{2} \delta_{0}+\frac{1}{2} \delta_{1}\right)^{\otimes\left(M_{k+1}-M_{k}\right)} & \text { if } k \text { is odd }
\end{array}, \quad \mathbf{q}_{k}= \begin{cases}\left(\frac{1}{2} \delta_{0}+\frac{1}{2} \delta_{1}\right)^{\otimes\left(M_{k+1}-M_{k}\right)} & \text { if } k \text { is even } \\
\frac{1}{2} \delta_{(0, \ldots, 0)}+\frac{1}{2} \delta_{(1, \ldots, 1)} & \text { if } k \text { is odd }\end{cases}\right.
$$

and the symbol $\delta_{a}$ denotes the Dirac measure at $a$. Then $\operatorname{supp} \nu_{1}=\widetilde{A}, \operatorname{supp} \nu_{2}=\widetilde{B}$, hence defining

$$
\mu_{1}=\pi_{*}\left(\nu_{1}\right), \quad \mu_{2}=\pi_{*}\left(\nu_{2}\right),
$$

we obtain probability measures on $A, B$, respectively, with $\operatorname{supp} \mu_{1}=A, \operatorname{supp} \mu_{2}=B$. Finally, let

$$
\mu=\sum_{n \in \mathbb{Z}} \delta_{0} \otimes\left(\tau_{n}\right)_{*} \mu_{1}+\sum_{n \in \mathbb{Z}} \delta_{1} \otimes\left(\tau_{n}\right)_{*} \mu_{2},
$$

where $\tau_{n}: \mathbb{R} \rightarrow \mathbb{R}, \tau_{n}(x)=x+n, n \in \mathbb{Z}$. Clearly, $\mu$ is a Borel $\sigma$-finite measure with $\operatorname{supp} \mu=X$, hence $\operatorname{dim}_{H}(\mu)=0$. It follows from Theorem 3.5 that almost every linear transformation $L: \mathbb{R}^{2} \rightarrow \mathbb{R}$ is injective on a set of full measure $\mu$ (compare with Proposition 3.35). We will find now such transformations and sets of injectivity explicitly.

For $a \in A, b \in B$ let

$$
\begin{aligned}
A_{a}=\{x \in A \backslash\{1\}: & x+a=z_{0} \cdot z_{1} z_{2} \ldots \text { such that the sequence }\left(z_{0}, z_{1}, \ldots\right) \\
& \text { is constant on } \left.\left[M_{k}, M_{k+1}\right) \cap \mathbb{N} \text { for every odd } k\right\}, \\
B_{b}=\{x \in B \backslash\{1\}: & x+b=z_{0} \cdot z_{1} z_{2} \ldots \text { such that the sequence }\left(z_{0}, z_{1}, \ldots\right) \\
& \text { is constant on } \left.\left[M_{k}, M_{k+1}\right) \cap \mathbb{N} \text { for every even } k\right\} .
\end{aligned}
$$

Lemma 3.36. For every $a \in A, b \in B$, we have $\mu_{1}\left(A_{a}\right)=\mu_{2}\left(B_{b}\right)=0$.

Proof. Fix $b=b_{0} \cdot b_{1} b_{2} \ldots \in B$. We will show $\mu_{2}\left(B_{b}\right)=0$ (the fact $\mu_{1}\left(A_{a}\right)=0$ can be proved analogously). The proof proceeds by showing that for each even $k$, the vector $\left(x_{M_{k}}, \ldots, x_{M_{k+1}-1}\right)$, where $x=x_{0} \cdot x_{1} x_{2} \ldots \in B_{b}$, can assume at most four values. This will imply $\mu_{2}\left(B_{b}\right) \leqslant 4 \cdot 2^{-\left(M_{k+1}-M_{k}\right)}$ for each even $k$ and, consequently, $\mu_{2}\left(B_{b}\right)=0$. To show the assertion, fix an even $k$ and let

$$
\xi=\sum_{j=M_{k+1}}^{\infty} \frac{x_{j}+b_{j}}{2^{j}} .
$$

Note that $\xi<2^{-\left(M_{k+1}-2\right)}$ (as $\xi<2$ and we exclude expansions with digits eventually equal to 1). Hence, $\xi=\xi_{0} \cdot \xi_{1} \xi_{2} \ldots$ with $\xi_{j}=0$ for $j \leqslant M_{k+1}-2$. Note that, since $b$ is fixed, the values of $\xi_{M_{k+1}-1} \in\{0,1\}$ and $\left(x_{M_{k}}+b_{M_{k}}, \ldots, x_{M_{k+1}-1}+b_{M_{k+1}-1}\right) \in\{(0, \ldots, 0),(1, \ldots, 1)\}$ determine uniquely the value of $\left(x_{M_{k}}, \ldots, x_{M_{k+1}-1}\right)$. Therefore, $\left(x_{M_{k}}, \ldots, x_{M_{k+1}-1}\right)$ can assume at most four values.

Now for Lebesgue almost every linear transformation $L: \mathbb{R}^{2} \rightarrow \mathbb{R}$ we will construct a set $X_{L} \subset X$ of full measure $\mu$, such that $L$ is injective on $X_{L}$. As previously, write $L(x, y)=$ $\alpha x+\beta y$ for $\alpha, \beta \in \mathbb{R}$. Neglecting a set of zero Lebesgue measure, we can assume $\beta \neq 0$. Let $l \in \mathbb{Z}$ be such that

$$
\begin{equation*}
z=-\frac{\alpha}{\beta}+l \text { belongs to }[0,1) \tag{3.21}
\end{equation*}
$$

Similarly as in (3.20), we can write

$$
\begin{equation*}
z=a^{\prime}-b^{\prime}, \quad z-1=a^{\prime \prime}-b^{\prime \prime} \quad \text { for some } a^{\prime}, a^{\prime \prime} \in A, b^{\prime}, b^{\prime \prime} \in B \tag{3.22}
\end{equation*}
$$

Let

$$
X_{L}=\left(\{0\} \times \bigcup_{n \in \mathbb{Z}}(A+n)\right) \cup\left(\{1\} \times \bigcup_{n \in \mathbb{Z}}\left(\left(B \backslash\left(B_{b^{\prime}} \cup B_{b^{\prime \prime}} \cup\{1\}\right)\right)+n\right)\right)
$$

Then $X_{L} \subset X$ and Lemma 3.36 implies that $X_{L}$ has full measure $\mu$.
Proposition 3.37. For every $\alpha \in \mathbb{R}, \beta \in \mathbb{R} \backslash\{0\}$, the linear transformation $L: \mathbb{R}^{2} \rightarrow \mathbb{R}$, $L(x, y)=\alpha x+\beta y$, is injective on $X_{L}$.

For the proof of the proposition we will need the following simple lemma. Its proof is left to the reader.

Lemma 3.38. Let $x=x_{0} \cdot x_{1} x_{2} \ldots \in[0,1], y=y_{0} . y_{1} y_{2} \ldots \in[0,1], M, N \in \mathbb{N}, M<N-1$, be such that $x+y<2$ and sequences $\left(x_{M}, \ldots, x_{N}\right)$ and $\left(y_{M}, \ldots, y_{N}\right)$ are constant. Then $x+y=z_{0} . z_{1} z_{2} \ldots$, where the sequence $\left(z_{M}, \ldots, z_{N-1}\right)$ is constant.

Proof of Proposition 3.37. Assume, on the contrary, that there exist points $u, v \in X_{L}$ such that $L(u)=L(v)$. As $\beta \neq 0$, we cannot have $u, v \in\{0\} \times \mathbb{R}$ or $u, v \in\{1\} \times \mathbb{R}$. Hence, we can assume $u \in\{0\} \times \mathbb{R}, v \in\{1\} \times \mathbb{R}$. Then, following the previous notation, we have $u=(0, a+n), v=(1, b+m)$ for $a \in A, b \in B \backslash\left(B_{b^{\prime}} \cup B_{b^{\prime \prime}} \cup\{1\}\right), n, m \in \mathbb{Z}$. Note that $b-a \in[-1,1)$, so by $(3.19)$, we have

$$
b-a=z \quad \text { or } \quad b-a=z-1
$$

for $z$ from (3.21), and (3.22 implies

$$
b-a=a^{\prime}-b^{\prime} \quad \text { or } \quad b-a=a^{\prime \prime}-b^{\prime \prime} .
$$

Hence,

$$
a+a^{\prime}=b+b^{\prime} \quad \text { or } \quad a+a^{\prime \prime}=b+b^{\prime \prime}
$$

This is a contradiction, as Lemma 3.38 implies that the binary expansion sequences of $a+a^{\prime}$ and $a+a^{\prime \prime}$ are constant on $\left[M_{k}, M_{k+1}-1\right) \cap \mathbb{N}$ for every even $k$, while by the condition $b \in B \backslash\left(B_{b^{\prime}} \cup B_{b^{\prime \prime}} \cup\{1\}\right)$, the binary expansion sequences of $b+b^{\prime}$ and $b+b^{\prime \prime}$ are not constant on $\left[M_{k}, M_{k+1}\right) \cap \mathbb{N}$ for some even $k$.

## Chapter 4

## Singular stationary measures for random piecewise affine interval homeomorphisms

In this chapter we study stationary measures for Alsedà-Misiurewicz systems and provide sets of parameters for which these measures are singular with respect to the Lebesgue measure. The main results of this chapter are Theorems 4.10 and 4.12 (which deal the resonant case and are extensions of Theorem 1.6), as well as Theorem 4.64 (which gives an open set of parameters with singular stationary measure and extends Theorem 1.7).

The plan of this chapter is as follows. In Section 4.1 we describe the $A M$-systems and state the results in a precise way. Section 4.2 contains preliminaries, while Section 4.3 is devoted to the proofs of the auxiliary results and Theorem 4.4 (characterization of $A M$-systems with the stationary measure equal to the Lebesgue measure). The proofs of Theorems 4.10 and 4.12 are split into Section 4.4 (case $l=1$ ) and Section 4.5 (case $l>1$ ). Sections 4.6 and 4.7 contain, respectively, the proofs of Theorems 4.15 (establishing topological conjugacy between some $A M$-systems with the same resonance type) and 4.16 (an example of $A M$-system with resonance, stationary measure of full topological support, yet Hausdorff dimension smaller than one). Section 4.8 is devoted to the proof of Theorem 4.64 .

### 4.1 Main results

We begin with a precise description of an Alsedà-Misiurewicz system.
Definition 4.1 An $A M$-system is the system $\left\{f_{-}, f_{+}\right\}$of increasing homeomorphisms of the interval $[0,1]$ of the form

$$
f_{-}(x)=\left\{\begin{array}{ll}
a_{-} x & \text { for } x \in\left[0, x_{-}\right] \\
1-b_{-}(1-x) & \text { for } x \in\left(x_{-}, 1\right]
\end{array}, \quad f_{+}(x)= \begin{cases}b_{+} x & \text { for } x \in\left[0, x_{+}\right] \\
1-a_{+}(1-x) & \text { for } x \in\left(x_{+}, 1\right]\end{cases}\right.
$$

where $0<a_{-}<1<b_{-}, 0<a_{+}<1<b_{+}$and

$$
x_{-}=\frac{b_{-}-1}{b_{-}-a_{-}}, \quad x_{+}=\frac{1-a_{+}}{b_{+}-a_{+}}
$$

See Figure 4.1.
We consider an $A M$-system as a random system with probabilities $p_{-}, p_{+}$, where $p_{-}, p_{+}>$ $0, p_{-}+p_{+}=1$.


Figure 4.1: An example of an $A M$-system.

Definition 4.2 The endpoint Lyapunov exponents of an $A M$-system $\left\{f_{-}, f_{+}\right\}$with probabilities $p_{-}, p_{+}$are defined as

$$
\Lambda(0)=p_{-} \ln f_{-}^{\prime}(0)+p_{+} \ln f_{+}^{\prime}(0), \quad \Lambda(1)=p_{-} \ln f_{-}^{\prime}(1)+p_{+} \ln f_{+}^{\prime}(1)
$$

It is known (see [2, 34, 35]) that if the Lyapunov exponents are positive, then there exists a unique stationary measure without atoms at the endpoints of $[0,1]$, i.e. a Borel probability measure $\mu$ on $[0,1]$, such that

$$
\mu=p_{-}\left(f_{-}\right)_{*} \mu+p_{+}\left(f_{+}\right)_{*} \mu
$$

with $\mu(\{0,1\})=0$ (recall Definition 2.9). For details, see Theorem 4.22. Throughout the chapter, by a stationary measure for an $A M$-system with positive Lyapunov exponents we will mean the measure $\mu$. It is known that the measure $\mu$ is non-atomic and is either absolutely continuous or singular with respect to the Lebesgue measure (see Propositions 4.26 and 4.27).

Definition 4.3 We say that an $A M$-system $\left\{f_{-}, f_{+}\right\}$is of:

- disjoint type, if the intervals $\left[0, f_{-}\left(x_{-}\right)\right],\left[f_{+}\left(x_{+}\right), 1\right]$ are disjoint, i.e. $f_{-}\left(x_{-}\right)<f_{+}\left(x_{+}\right)$,
- border type, if the intervals $\left[0, f_{-}\left(x_{-}\right)\right],\left[f_{+}\left(x_{+}\right), 1\right]$ touch each other, i.e. $f_{-}\left(x_{-}\right)=$ $f_{+}\left(x_{+}\right)$,
- overlapping type, if the intervals $\left[0, f_{-}\left(x_{-}\right)\right],\left[f_{+}\left(x_{+}\right), 1\right]$ overlap, i.e. $f_{-}\left(x_{-}\right)>f_{+}\left(x_{+}\right)$.

See Figure 4.2.
Note that in the case $x_{+}<x_{-}$(which will be assumed throughout most of the chapter, see Lemma 4.29, , the system is of

- disjoint type, if $f_{-}\left(\left[x_{+}, x_{-}\right]\right), f_{+}\left(\left[x_{+}, x_{-}\right]\right)$are disjoint,
- border type, if $f_{-}\left(\left[x_{+}, x_{-}\right]\right) \cap f_{+}\left(\left[x_{+}, x_{-}\right]\right)=\left\{f_{-}\left(x_{-}\right)\right\}=\left\{f_{+}\left(x_{+}\right)\right\}$,
- overlapping type, if $f_{-}\left(\left[x_{+}, x_{-}\right]\right), f_{+}\left(\left[x_{+}, x_{-}\right]\right)$overlap.


Figure 4.2: Three types of $A M$-systems: disjoint, border and overlapping.

In [2, Theorem 6.1] Alsedà and Misiurewicz showed that if $a_{-}=a_{+}=a, b_{-}=b_{+}=b$, $1 / a+1 / b=2, p_{-}=p_{-}=1 / 2$, then the measure $\mu$ is the Lebesgue measure on $[0,1]$. The first result of our work, presented below, gives an exact condition for an $A M$-system to have a stationary Lebesgue measure.

Theorem 4.4. Let $\left\{f_{-}, f_{+}\right\}$be an AM-system with probabilities $p_{-}, p_{+}$, such that the Lyapunov exponents $\Lambda(0), \Lambda(1)$ are positive. Then the unique stationary measure $\mu$ (without atoms at 0,1 ) is the Lebesgue measure on $[0,1]$ if and only if the system is of border type and

$$
\frac{p_{-}}{a_{-}}+\frac{p_{+}}{b_{+}}=1 .
$$

In this case we also have $\frac{p_{-}}{b_{-}}+\frac{p_{+}}{a_{+}}=1$.
In [2] the authors conjectured that the stationary measure $\mu$ for an $A M$-system with positive Lyapunov exponents is typically singular. The main result of this chapter verifies this conjecture for some set of the system parameters. First, we split the $A M$-systems into two kinds: resonant and non-resonant, which have different kinds of behaviour.

Definition 4.5 We say that that an $A M$-system $\left\{f_{-}, f_{+}\right\}$with probabilities $p_{-}, p_{+}$exhibits a resonance at the point 0 , if

$$
\frac{\ln f_{+}^{\prime}(0)}{\ln f_{-}^{\prime}(0)} \in \mathbb{Q}
$$

More precisely, a ( $k: l$ )-resonance at 0 occurs for $k, l \in \mathbb{N}$ if

$$
\left(f_{-}^{\prime}(0)\right)^{k}\left(f_{+}^{\prime}(0)\right)^{l}=a_{-}^{k} b_{+}^{l}=1
$$

which is equivalent to $a_{-}=f_{-}^{\prime}(0)=\rho^{l}, b_{+}=f_{+}^{\prime}(0)=\rho^{-k}$ for some $\rho \in(0,1)$ and also to $\frac{\ln f_{+}^{\prime}(0)}{\ln f_{-}^{\prime}(0)}=-\frac{k}{l}$.

Analogously, a ( $k: l$ )-resonance at 1 occurs if

$$
\left(f_{-}^{\prime}(1)\right)^{l}\left(f_{+}^{\prime}(1)\right)^{k}=a_{+}^{k} b_{-}^{l}=1
$$

Without loss of generality, we always assume that $k, l$ are relatively prime.
We will show that in the resonant case the (topological) support of the stationary measure $\mu$ for some parameters is a Cantor set in $[0,1]$ of Hausdorff dimension smaller than 1 (see Theorems 4.10 and 4.12). A different situation occurs in the non-resonant case, as shown in the following proposition (for the definition of minimality see Definition 4.19 and for the proof refer to Proposition 4.31 and Corollary 4.33).

Proposition 4.6. If an AM-system with positive Lyapunov exponents has no resonance at one of the endpoints 0,1 , then it is minimal in $(0,1)$ and the support of $\mu$ is equal to $[0,1]$.

Before stating the main results of this chapter, we need to present some definitions. Let

$$
\mathcal{I}:[0,1] \rightarrow[0,1], \quad \mathcal{I}(x)=\mathcal{I}^{-1}(x)=1-x
$$

be the symmetry of $[0,1]$ with respect to its center.
Definition 4.7 An $A M$-system $\left\{f_{-}, f_{+}\right\}$is called symmetric, if $\mathcal{I} \circ f_{-}=f_{+} \circ \mathcal{I}$.
Obviously, a system $\left\{f_{-}, f_{+}\right\}$is symmetric if and only if $a_{-}=a_{+}$and $b_{-}=b_{+}$. It is straightforward that for symmetric systems we have $x_{+}=\mathcal{I}\left(x_{-}\right)$and $f_{+}\left(x_{+}\right)=\mathcal{I}\left(f_{-}\left(x_{-}\right)\right)$. Moreover, for symmetric systems the existence of $(k: l)$-resonance at 0 is equivalent to the existence of $(k: l)$-resonance at 1 . Note also that if a symmetric systems exhibits $(k: l)$ resonance, then the condition $k>l$ is equivalent to the positivity of the exponents $\Lambda(0), \Lambda(1)$ for $p_{-}=p_{+}=1 / 2$ (see the proof of Lemma 4.29).

Definition 4.8 For an $A M$-system of disjoint type, we call the interval $\left(f_{-}\left(x_{-}\right), f_{+}\left(x_{+}\right)\right)$the central interval of the system $\left\{f_{-}, f_{+}\right\}$.

Definition 4.9 Let $x \in(0,1)$ and $i_{1}, i_{2}, \ldots \in\{-,+\}$. We say that a trajectory $\left\{f_{i_{n}} \circ \cdots \circ\right.$ $\left.f_{i_{1}}(x)\right\}_{n=0}^{\infty}$ jumps over the central interval at the time $s$, for $s \geqslant 0$, if $f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x)$ and $f_{i_{s+1}} \circ \cdots \circ f_{i_{1}}(x)$ are in different components of the complement of the central interval in $[0,1]$.

The main results of this chapter shows the singularity of the stationary measure $\mu$ for some symmetric $A M$-systems of disjoint type, which exhibit a resonance.

Theorem 4.10. Let $\left\{f_{-}, f_{+}\right\}$be a symmetric AM-system of disjoint type with positive Lyapunov exponents. If the system exhibits $(k: l)$-resonance for some relatively prime $k, l \in \mathbb{N}$, $k>l$, and satisfies $\rho<\eta$, where

$$
\rho=\left(f_{-}^{\prime}(0)\right)^{1 / l}=\left(f_{+}^{\prime}(0)\right)^{-1 / k}=\left(f_{+}^{\prime}(1)\right)^{1 / l}=\left(f_{-}^{\prime}(1)\right)^{-1 / k}
$$

and $\eta \in(1 / 2,1)$ is the unique solution of the equation $\eta^{k+l}-2 \eta^{k+1}+2 \eta-1=0$, then the unique stationary measure $\mu$ (without atoms at 0,1 ) is singular with

$$
\operatorname{dim}_{H}(\operatorname{supp} \mu)=\frac{\log \eta}{\log \rho}<1
$$

where $\operatorname{supp} \mu$ denotes the topological support of $\mu$. Moreover, $\operatorname{supp} \mu$ is a nowhere dense perfect set consisting of all limit points of trajectories of any point $x \in(0,1)$ under $\left\{f_{-}, f_{+}\right\}$, which jump over the central interval infinitely many times.

Remark 4.11 The condition $\rho<\eta$ is equivalent to $\rho x_{-}<\frac{1}{2}$ and implies that the system is of disjoint type. In the case $l=1$ it holds for all systems of disjoint type. See Sections 4.4 and 4.5 for details.

In the case $l=1$ we give a more precise description of the measure $\mu$.
Theorem 4.12. Let $\left\{f_{-}, f_{+}\right\}$be a symmetric AM-system of disjoint type with probabilities $p_{-}, p_{+}$, such that the Lyapunov exponents are positive. If the system exhibits $(k: 1)$-resonance for some $k \in\{2,3, \ldots\}$, then

$$
\operatorname{dim}_{H} \mu=\frac{\sum_{r=1}^{k} r\left(\frac{p_{+}}{p_{-}} \eta_{-}^{r} \log \eta_{-}+\frac{p_{-}}{p_{+}} \eta_{+}^{r} \log \eta_{+}\right)}{\sum_{r=1}^{k} r\left(\frac{p_{+}}{p_{-}} \eta_{-}^{r}+\frac{p_{-}}{p_{+}} \eta_{+}^{r}\right) \log \rho}
$$

where $\rho$ is defined as above and $\eta_{-}, \eta_{+} \in(0,1)$ are, respectively, the unique solutions of the equations

$$
p_{+} \eta_{-}^{k+1}-\eta_{-}+p_{-}=0, \quad p_{-} \eta_{+}^{k+1}-\eta_{+}+p_{+}=0
$$

In particular, if $p_{-}=p_{+}=1 / 2$, then

$$
\operatorname{dim}_{H} \mu=\operatorname{dim}_{H}(\operatorname{supp} \mu)=\frac{\log \eta}{\log \rho}<1
$$

Remark 4.13 Under the assumptions of Theorem 4.10, if $l=1$ or $l>1, p_{-}=p_{+}=1 / 2$, then the stationary measure $\mu$ is a countable sum of (geometrically) similar copies, with disjoint supports, of a self-similar measure of an iterated function system with the Strong Separation Condition (recall Definition 2.14). In the case $l=1$ this iterated function system consists of $k$ maps, while in the case $l>1, p_{-}=p_{+}=1 / 2$ it is infinite. See Propositions 4.48 and 4.63.

Remark 4.14 For every $k \in\{2,3, \ldots\}$ and $\rho \in(0, \eta)$ and probability vector $\left(p_{-}, p_{+}\right)$with $p_{-}, p_{+} \in(1 /(k+1), k /(k+1))$, the assumptions of Theorem 4.12 are fulfilled for some $A M-$ system with $\rho=f_{-}^{\prime}(0)=f_{+}^{\prime}(1)$ and probabilities $p_{-}, p_{+}$. In particular, the theorem gives examples of $A M$-systems with $\operatorname{dim}_{H} \mu=d$ for arbitrary $d \in(0,1)$.

The next result shows that the considered resonant systems are uniquely determined (up to topological conjugacy) by their resonance data.

Theorem 4.15. Let $\left\{f_{-}, f_{+}\right\},\left\{g_{-}, g_{+}\right\}$be symmetric AM-systems of disjoint type. If both system exhibit ( $k: l$ )-resonance for some relatively prime $k, l \in \mathbb{N}, k>l$, and satisfy $\rho<\eta$, with $\rho, \eta$ defined as in Theorem 4.10, then they are topologically conjugated, i.e. there exists an increasing homeomorphism $h:[0,1] \rightarrow[0,1]$ such that

$$
g_{-} \circ h=h \circ f_{-}, \quad g_{+} \circ h=h \circ f_{+} .
$$

The next result shows that there exist symmetric resonant $A M$-systems with singular stationary measure of full support.

Theorem 4.16. If a symmetric $A M$-system with probabilities $p_{-}=p_{+}=1 / 2$ and positive Lyapunov exponents exhibits (5 : 2)-resonance and satisfies $\rho=\eta$, with $\rho, \eta$ defined as in Theorem 4.10, then $\mu$ is singular with

$$
\operatorname{dim}_{H} \mu<1, \quad \operatorname{supp} \mu=[0,1]
$$

Note that in this case the condition $\rho=\eta$ is equivalent to

$$
\rho^{7}-2 \rho^{6}+2 \rho-1=0
$$

which gives $\rho \approx 0.513649$.
Remark 4.17 The resonance (5:2) was chosen because the proof is relatively short in this case. Similar arguments work also for some other values of the resonance $(k: l)$ with $l>1$.

Our last result gives and open set of parameters for which the corresponding stationary measures is singular. In particular, there exist non-resonant $A M$-systems with singular stationary measure. See Theorem 4.64 for a more detailed formulation.

Theorem 4.18. There exists a non-empty and open set of parameters $(a, b) \in(0,1) \times(1, \infty)$ such that the stationary measure $\mu$ for the symmetric $A M$ system with $a_{-}=a_{+}=a, b_{-}=$ $b_{+}=b$ and probability vector $\left(p_{-}, p_{+}\right)=\left(\frac{1}{2}, \frac{1}{2}\right)$ is singular with $\operatorname{dim}_{H}(\mu)<1$.

### 4.2 Preliminaries

Notation We write $\mathbb{Z}^{*}=\mathbb{Z} \backslash\{0\}$. For $j \in \mathbb{Z}^{*}$ we set

$$
\operatorname{sgn}(j)=\left\{\begin{array}{ll}
- & \text { for } j<0 \\
+ & \text { for } j>0
\end{array} .\right.
$$

For $x \in \mathbb{R}, A \subset \mathbb{R}$ we use the notation

$$
x A=\{x y: y \in A\} .
$$

The convex hull of a set $A$ is denoted by conv $A$. We write $|I|$ for the length of an interval $I$. The symbol Leb denotes the Lebesgue measure.

The existence and uniqueness of stationary measures with atoms at the endpoints holds for much more general family of systems than $A M$-systems. For the sake of the completeness, let us state the corresponding result in the general case. Throughout this section we assume that $f_{1}, \ldots, f_{m}, m \geqslant 2$, are piecewise $C^{1}$ increasing homeomorphisms of the interval $[0,1]$, such that $f_{i}(0)=0, f_{i}(1)=1$ and $f_{i}(x) \neq x$ for $x \in(0,1), i=1, \ldots, m$.

For a set $A \subset[0,1]$ we define

$$
f(A)=f_{1}(A) \cup \ldots \cup f_{m}(A), \quad f^{-1}(A)=f_{1}^{-1}(A) \cup \ldots \cup f_{m}^{-1}(A)
$$

and, inductively,

$$
f^{0}(A)=A, \quad f^{n}(A)=f\left(f^{n-1}(A)\right), \quad f^{-n}(A)=f^{-1}\left(f^{-(n-1)}(A)\right)
$$

for $n \in \mathbb{N}$.
Definition 4.19 Suppose $f(X) \subset X$ for some $X \subset[0,1]$. We say that the system $\left\{f_{1}, \ldots, f_{m}\right\}$ is (forward) minimal in $X$, if the union of forward trajectories under $\left\{f_{1}, \ldots, f_{m}\right\}$ of every point in $X$ is dense in $X$, i.e. for every $x \in X$ and every non-empty open subset $U$ of $X$ there exist $i_{1}, \ldots, i_{n} \in\{1, \ldots, m\}, n \geqslant 0$, such that $f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x) \in U$.

Let $\left(p_{1}, \ldots, p_{m}\right)$ be a probability vector, i.e. $p_{1}, \ldots, p_{m} \in(0,1)$ and $p_{1}+\cdots+p_{m}=1$. We consider the symbolic space

$$
\Sigma_{m}^{+}=\{1, \ldots, m\}^{\mathbb{N}}
$$

equipped with the Bernoulli measure

$$
\operatorname{Ber}_{p_{1}, \ldots, p_{m}}^{+}=\bigotimes_{\mathbb{N}} \mathbb{P}_{p_{1}, \ldots, p_{m}}
$$

where $\mathbb{P}_{p_{1}, \ldots, p_{m}}$ is the probability distribution on $\{1, \ldots, m\}$ given by $\mathbb{P}_{p_{1}, \ldots, p_{m}}(\{i\})=p_{i}$, $i=1, \ldots, m$.

We study $\left\{f_{1}, \ldots, f_{m}\right\}$ as the random systems of maps, given by the step skew product

$$
\mathcal{F}^{+}: \Sigma_{m}^{+} \times[0,1] \rightarrow \Sigma_{m}^{+} \times[0,1], \quad \mathcal{F}^{+}(\underline{i}, x)=\left(\sigma(\underline{i}), f_{i_{1}}(x)\right)
$$

where $\underline{i}=\left(i_{n}\right)_{n \in \mathbb{N}}$ and $\sigma: \Sigma_{m}^{+} \rightarrow \Sigma_{m}^{+}$is the left-side shift, i.e. $\sigma\left(\left(i_{n}\right)_{n \in \mathbb{N}}\right)=\left(i_{n+1}\right)_{n \in \mathbb{N}}$.
By $\mathcal{T}$ and $T$ be will denote the corresponding Markov operator on measures (Definition 2.9) and the Perron-Frobenius transfer operator on densities (Definition 2.10), respectively.

Recall that the stationary measures of the system $\left\{f_{1}, \ldots, f_{m}\right\}$ with probabilities $p_{1}, \ldots, p_{m}$ coincide with the fixed points of the transfer operator $\mathcal{T}$, while the stationary densities (densities of stationary measures with respect to the Lebesgue measure) are the fixed points of the transfer operator $T$.

Proposition 4.20. Suppose that $f(X) \subset X$ for some $X \subset[0,1]$ and the system $\left\{f_{1}, \ldots, f_{m}\right\}$ is minimal in $X$. If $\vartheta$ is a stationary measure for the system and $\operatorname{supp} \vartheta \subset X$, then $\operatorname{supp} \vartheta=$ $X$.

The proof of this proposition is standard and can found e.g. in [19, Lemme 5.1] or [33, Lemma $2]$.

Note that since the maps $f_{i}$ fix the endpoints of the interval, the Dirac measures at 0 and 1 are stationary for any probabilities $p_{i}$. If we assume that the endpoints are repelling in average, then there exists a stationary measure with no atoms at 0,1 . More precisely, we have the following.

Definition 4.21 Assuming $f_{i}^{\prime}(0), f_{i}^{\prime}(1)>0, i=1, \ldots, m$, the Lyapunov exponents of the system $\left\{f_{1}, \ldots, f_{m}\right\}$ with probabilities $p_{1}, \ldots, p_{m}$ are defined as

$$
\Lambda(0)=p_{1} \ln f_{1}^{\prime}(0)+\cdots+p_{m} \ln f_{m}^{\prime}(0), \quad \Lambda(1)=p_{1} \ln f_{1}^{\prime}(1)+\cdots+p_{m} \ln f_{m}^{\prime}(1)
$$

Theorem 4.22 ([34, Proposition 4.1], [35, Lemmas 3.2-3.4]). If $\Lambda(0), \Lambda(1)>0$, then there exists a unique probability stationary measure $\mu$ for the system $\left\{f_{1}, \ldots, f_{m}\right\}$ with probabilities $p_{1}, \ldots, p_{m}$, such that $\mu(\{0,1\})=0$. Moreover, there exist positive constants $c, \alpha_{0}, \delta_{0}$ such that for every $\alpha \in\left(0, \alpha_{0}\right), \delta \in\left(0, \delta_{0}\right)$ and for

$$
\mathcal{D}_{c, \alpha, \delta}=\left\{\nu \in \mathcal{M}: \nu([0, x]), \nu([1-x, 1])<c x^{\alpha} \text { for every } x \in(0, \delta)\right\},
$$

we have $\mathcal{T}\left(\mathcal{D}_{c, \alpha, \delta}\right) \subset \mathcal{D}_{c, \alpha, \delta}$ and $\mu \in \mathcal{D}_{c, \alpha, \delta}$.

Remark 4.23 Actually, in [34, 35] the theorem was proved for systems of $C^{1}$-diffeomorphisms, but the proof goes through if we only assume that the maps are smooth in some neighbourhoods of 0,1 .

Remark 4.24 The uniqueness of the stationary measure $\mu \in \mathcal{D}_{c, \alpha, \delta}$ implies

$$
\frac{1}{N} \sum_{n=0}^{N-1} \mathcal{T}^{n} \nu \rightarrow \mu \quad \text { as } N \rightarrow \infty \quad \text { in weak-* topology } \quad \text { for every } \nu \in \mathcal{D}_{c, \alpha, \delta}
$$

Remark 4.25 The measure $\operatorname{Ber}_{p_{1}, \ldots, p_{m}}^{+} \times \mu$ is an $\mathcal{F}^{+}$-invariant measure on $\Sigma_{m}^{+} \times[0,1]$. Moreover, there is a Borel probability measure on $\Sigma_{m} \times[0,1]$, where $\Sigma_{m}=\{1, \ldots, m\}^{\mathbb{Z}}$, invariant with respect to the (extended) step skew product, which is associated to $\mu$ in a unique way (see [5, [35]).

It is well-known (see e.g. [23, Theorem 2.5]) that whenever the operator $\mathcal{T}$ preserves absolute continuity and singularity of measures (with respect to the Lebesgue measure) and the stationary measure is unique, then it is of pure type (i.e. is either absolutely continuous or singular with respect to the Lebesgue measure). It is easy to see that the same holds for the measure $\mu$ from Theorem 4.22, as $f_{1}, \ldots, f_{m}$ are piecewise $C^{1}$ homeomorphisms. Hence, if $\Lambda(0), \Lambda(1)>0$, then the following two propositions hold.

Proposition 4.26. The stationary measure $\mu$ is either absolutely continuous or singular with respect to the Lebesgue measure.

Proposition 4.27. The stationary measure $\mu$ is non-atomic.

Proof. The proof follows [33, proof of Lemma 2] (see also [19, Lemme 5.1]). By Theorem 4.22, $\mu$ has no atoms at 0,1 . Suppose there exists an atom in $(0,1)$ and take $x \in(0,1)$ such that $\mu(\{x\})=\max \{\mu(\{y\}): y \in(0,1)\}$. Then, by the definition of stationary measure, $\mu\left(\left\{f_{i}^{-1}(x)\right\}\right)=\mu(\{x\})$ for every $i=1, \ldots, m$ and, consequently, $\mu\left(\left\{f_{i}^{-n}(x)\right\}\right)=\mu(\{x\})>0$ for every $n>0$. Since $f_{i}$ has no fixed points in $(0,1)$, the trajectory $\left\{f_{i}^{-n}(x)\right\}_{n=0}^{\infty}$ is strictly monotonic and thus infinite, which contradicts the finiteness of $\mu$.

The following lemma is useful in determining singularity of the measure $\mu$.
Lemma 4.28. If $X \subset(0,1)$ is non-empty, closed as a subset of $(0,1)$, and $f(X) \subset X$, then $\operatorname{supp} \mu \subset X \cup\{0,1\}$ and $\mu(X)=1$. Consequently, if there exists such a set $X$ of Lebesgue measure 0 , then $\mu$ is singular.

Proof. Take $x \in X$. Since $x \in(0,1)$, the Dirac measure $\delta_{x}$ at $x$ is in $\mathcal{D}_{c, \alpha, \delta}$ for sufficiently small $\delta>0$, so by Remark 4.24 we have

$$
\frac{1}{N} \sum_{n=0}^{N-1} \mathcal{T}^{n} \delta_{x} \rightarrow \mu \quad \text { as } N \rightarrow \infty \quad \text { in weak-* topology. }
$$

Since

$$
\mathcal{T}^{n} \delta_{x}=\sum_{i_{1}, \ldots, i_{n} \in\{1, \ldots, m\}} p_{i_{1}} \cdots p_{i_{n}} \delta_{f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)}
$$

 as $f(X) \subset X$. Since $\bar{X}=X \cup\{0,1\}$ and $\mu(\{0,1\})=0$, we have $\operatorname{supp} \mu \subset X \cup\{0,1\}$ and $\mu(X)=1$.

### 4.3 Preliminary results and proof of Theorem 4.4

From now on, we restrict our attention to $A M$-systems. In this section we prove Theorem 4.4 together with other preliminary results on the $A M$-systems. We begin with the following observation.

Lemma 4.29. Let $\left\{f_{-}, f_{+}\right\}$be an AM-system. If the Lyapunov exponents $\Lambda(0), \Lambda(1)$ are positive for the probabilities $p_{-}=p_{+}=1 / 2$, then $x_{+}<x_{-}$. In particular, $x_{+}<x_{-}$holds if the system is symmetric and exhibits a $(k: l)$-resonance for $k, l \in \mathbb{N}, k>l$.

Proof. The inequality $x_{+}<x_{-}$can be written as

$$
\frac{1-a_{+}}{b_{+}-a_{+}}<\frac{b_{-}-1}{b_{-}-a_{-}}
$$

which is equivalent to

$$
\begin{equation*}
\left(1-a_{-}\right)\left(1-a_{+}\right)<\left(b_{-}-1\right)\left(b_{+}-1\right) \tag{4.1}
\end{equation*}
$$

By the positivity of the Lyapunov exponents for $p_{-}=p_{+}=1 / 2$,

$$
b_{-}>\frac{1}{a_{+}}, \quad b_{+}>\frac{1}{a_{-}}
$$

so

$$
\left(b_{-}-1\right)\left(b_{+}-1\right)>\left(\frac{1}{a_{+}}-1\right)\left(\frac{1}{a_{-}}-1\right)=\frac{\left(1-a_{-}\right)\left(1-a_{+}\right)}{a_{-} a_{+}}>\left(1-a_{-}\right)\left(1-a_{+}\right)
$$

which gives 4.1. As already noted, if the system is symmetric and exhibits a $(k: l)$-resonance for $k>l$, then the assumption on the positivity of the Lyapunov exponents for $p_{-}=p_{+}=1 / 2$ is satisfied. Indeed, in this case we have $a_{-}=a_{+}=a \in(0,1)$ and $b_{-}=b_{+}=a^{-k / l}$, so

$$
\left.\frac{1}{2} \ln f_{-}^{\prime}(x)+\frac{1}{2} \ln f_{+}^{\prime}(x)\right)=\frac{1-k / l}{2} \ln a>0 .
$$

for $x=0,1$.
The following lemma is used in the proof of Theorem 4.4.
Lemma 4.30. If an AM-system $\left\{f_{-}, f_{+}\right\}$with probabilities $p_{-}, p_{+}$is of border type and $\frac{p_{-}}{a_{-}}+\frac{p_{+}}{b_{+}}=1$, then $\frac{p_{-}}{b_{-}}+\frac{p_{+}}{a_{+}}=1$. Conversely, if

$$
\frac{p_{-}}{a_{-}}+\frac{p_{+}}{b_{+}}=\frac{p_{-}}{b_{-}}+\frac{p_{+}}{a_{+}}=1,
$$

then the AM-system $\left\{f_{-}, f_{+}\right\}$with probabilities $p_{-}, p_{+}$is of border type.

Proof. An elementary calculation shows that the system is of border type if and only if

$$
\frac{a_{-}+a_{+}-1}{a_{-} a_{+}}=\frac{b_{-}+b_{+}-1}{b_{-} b_{+}},
$$

which is equivalent to

$$
\begin{equation*}
\frac{1-1 / b_{+}}{1 / a_{-}-1 / b_{+}}=\frac{1-1 / a_{+}}{1 / b_{-}-1 / a_{+}} . \tag{4.2}
\end{equation*}
$$

Suppose that the system is of border type and

$$
\frac{p_{-}}{a_{-}}+\frac{p_{+}}{b_{+}}=1 .
$$

Then

$$
p_{-}=\frac{1-1 / b_{+}}{1 / a_{-}-1 / b_{+}},
$$

so by (4.2),

$$
p_{-}=\frac{1-1 / a_{+}}{1 / b_{-}-1 / a_{+}},
$$

which gives

$$
\frac{p_{-}}{b_{-}}+\frac{p_{+}}{a_{+}}=1 .
$$

Conversely, suppose

$$
\frac{p_{-}}{a_{-}}+\frac{p_{+}}{b_{+}}=\frac{p_{-}}{b_{-}}+\frac{p_{+}}{a_{+}}=1 .
$$

Then

$$
p_{-}=\frac{1-1 / b_{+}}{1 / a_{-}-1 / b_{+}}=\frac{1-1 / a_{+}}{1 / b_{-}-1 / a_{+}},
$$

which gives 4.2).
The following proposition, which gives the first part of Proposition 4.6, is essentially proved in [50, Lemma 3] and [35, Proposition 2.1] (formally, in the case of diffeomorphisms). For completeness, we present the proof suited to our setup.

Proposition 4.31. If an $A M$-system $\left\{f_{-}, f_{+}\right\}$has no resonance at one of the endpoints 0,1 , then it is minimal in $(0,1)$.

Proof. To fix notation, assume that the system has no resonance at 0 (in the other case the proof is analogous). Choose $x_{0} \in(0,1)$. Since both families of intervals $\left[f_{-}^{n+1}\left(x_{0}\right), f_{-}^{n}\left(x_{0}\right)\right)$, $n \in \mathbb{Z}$, and $\left[f_{+}^{n}\left(x_{0}\right), f_{+}^{n+1}\left(x_{0}\right)\right), n \in \mathbb{Z}$, cover $(0,1)$, it is sufficient to prove that for every $x, y \in K$, where

$$
K=\left[f_{+}^{n_{0}}\left(x_{0}\right), f_{+}^{n_{0}+1}\left(x_{0}\right)\right)
$$

with some chosen $n_{0} \in \mathbb{Z}$ and every $\varepsilon>0$ there exist $n \in \mathbb{N}$ and $i_{1}, \ldots, i_{n} \in\{-,+\}$ such that

$$
\begin{equation*}
f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x) \in K \quad \text { and } \quad\left|f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)-y\right|<\varepsilon \tag{4.3}
\end{equation*}
$$

To show (4.3), we choose $n_{0}$ so that $K \subset\left(0, x_{+}\right)$and let

$$
\alpha=-\frac{\ln a_{-}}{\ln b_{+}}
$$

Since we assume that $\left\{f_{-}, f_{+}\right\}$has no resonance at 0 , we have $\alpha \in \mathbb{R}^{+} \backslash \mathbb{Q}$. Hence, for any $y \in K$ and $\delta>0$ we can find $k, l \in \mathbb{N}$ such that

$$
\begin{equation*}
0<k-\alpha l-\frac{\ln (y / x)}{\ln b_{+}}<\delta \tag{4.4}
\end{equation*}
$$

As

$$
b_{+}^{k} a_{-}^{l} x=e^{(k-\alpha l) \ln b_{+}+\ln x}=y b_{+}^{k-\alpha l-\ln (y / x) / \ln b_{+}},
$$

(4.4) implies

$$
y<b_{+}^{k} a_{-}^{l} x<y b_{+}^{\delta}=y+y\left(b_{+}^{\delta}-1\right)<y+b_{+}^{\delta}-1<y+\min (\varepsilon, \sup K-y)
$$

if $\delta$ is chosen sufficiently small. In particular,

$$
\begin{equation*}
b_{+}^{k} a_{-}^{l} x \in K \quad \text { and } \quad\left|b_{+}^{k} a_{-}^{l} x-y\right|<\varepsilon \tag{4.5}
\end{equation*}
$$

Since $x \in K \subset\left(0, x_{+}\right)$, we have $f_{-}^{l}(x)=a_{-}^{l} x$. Moreover, 4.5) implies $b_{+}^{j} a_{-}^{l} x \in\left(0, x_{+}\right)$for $j=0, \ldots, k$, which gives $f_{+}^{k}\left(f_{-}^{l}(x)\right)=b_{+}^{k} a_{-}^{l} x$. This together with 4.5) shows (4.3) and ends the proof.

Assume now that an $A M$-system $\left\{f_{-}, f_{+}\right\}$with probabilities $p_{-}, p_{+}$has positive Lyapunov exponents, which is equivalent to

$$
a_{-}^{p_{-}} b_{+}^{p_{+}}>1, \quad b_{-}^{p_{-}} a_{+}^{p_{+}}>1
$$

Then, by Theorem 4.22, there exists a unique probability stationary measure $\mu$ for the system, such that $\mu(\{0,1\})=0$. By Propositions 4.26 and 4.27 , we have the following.

Proposition 4.32. The stationary measure $\mu$ is non-atomic. Moreover, it is either absolutely continuous or singular with respect to the Lebesgue measure.

Propositions 4.20 and 4.31 imply the following corollary, which completes the proof of Proposition 4.6.

Corollary 4.33. If the system has no resonance at one of the endpoints 0,1 , then $\operatorname{supp} \mu=$ $[0,1]$.

We end the section by proving Theorem 4.4.

Proof of Theorem 4.4. Recall that he transfer operator $T$ on $L^{1}([0,1]$, Leb $)$ has the form (see Definition 2.10)

$$
T g=p_{-}\left(f_{-}^{-1}\right)^{\prime} g \circ f_{-}^{-1}+p_{+}\left(f_{+}^{-1}\right)^{\prime} g \circ f_{+}^{-1},
$$

The measure $\mu$ is the Lebesgue measure if and only if

$$
\begin{equation*}
T \mathbb{1}=\mathbb{1} \tag{4.6}
\end{equation*}
$$

for the constant unity function $\mathbb{1}$. If the system is of border type, then

$$
T \mathbb{1}(x)=\left\{\begin{array}{ll}
\frac{p_{-}}{a_{-}}+\frac{p_{+}}{b_{+}} & \text {for } x \leqslant f_{-}\left(x_{-}\right) \\
\frac{p_{-}}{b_{-}}+\frac{p_{+}}{a_{+}} & \text {for } x>f_{-}\left(x_{-}\right)
\end{array},\right.
$$

so (4.6) is equivalent to

$$
\begin{equation*}
\frac{p_{-}}{a_{-}}+\frac{p_{+}}{b_{+}}=\frac{p_{-}}{b_{-}}+\frac{p_{+}}{a_{+}}=1 . \tag{4.7}
\end{equation*}
$$

Conversely, if 4.6 holds, then applying it to points $x \in[0,1]$ close to the endpoints of $[0,1]$ we get (4.7). To end the proof, it is enough to use Lemma 4.30 .

Remark 4.34 As noted in the introduction, for the case $a_{-}=a_{+}=a, b_{-}=b_{+}=b$, $1 / a+1 / b=2, p_{-}=p_{-}=1 / 2$, Theorem 4.4 was proved in [2, Theorem 6.1].

### 4.4 Proofs of Theorems 4.10 (case $l=1$ ) and 4.12 .

In Theorems 4.10 and 4.12 we consider a symmetric $A M$-system of disjoint type $\left\{f_{-}, f_{+}\right\}$with probabilities $p_{-}, p_{+}$, positive Lyapunov exponents and a $(k: l)$-resonance for some relatively prime $k, l \in \mathbb{N}, k>l$. In this section we prove the results in the case $l=1$. The proof is divided into several parts concerning consecutive assertions of the theorems.

## Preliminaries

By assumption, $a_{-}=a_{+}=\rho, b_{-}=b_{+}=\rho^{-k}$, so the maps have the form

$$
f_{-}(x)=\left\{\begin{array}{ll}
\rho x & \text { for } x \in\left[0, x_{-}\right] \\
\mathcal{I}\left(\rho^{-k} \mathcal{I}(x)\right) & \text { for } x \in\left(x_{-}, 1\right]
\end{array}, \quad f_{+}(x)=\left\{\begin{array}{ll}
\rho^{-k} x & \text { for } x \in\left[0, x_{+}\right] \\
\mathcal{I}(\rho \mathcal{I}(x)) & \text { for } x \in\left[x_{+}, 1\right]
\end{array},\right.\right.
$$

where $\rho \in(0,1)$ and

$$
\begin{aligned}
x_{-} & =\frac{1-\rho^{k}}{1-\rho^{k+1}}, & x_{+} & =\mathcal{I}\left(x_{-}\right)=\frac{\rho^{k}-\rho^{k+1}}{1-\rho^{k+1}}, \\
f_{-}\left(x_{-}\right) & =\frac{\rho-\rho^{k+1}}{1-\rho^{k+1}}, & f_{+}\left(x_{+}\right) & =\mathcal{I}\left(f_{-}\left(x_{-}\right)\right)=\frac{1-\rho}{1-\rho^{k+1}} .
\end{aligned}
$$

Note that $x_{+}<x_{-}$(see Lemma 4.29) and $x_{+}<f_{-}\left(x_{-}\right)$. The assumption that the system is of disjoint type, i.e. the condition $f_{-}\left(x_{-}\right)<f_{+}\left(x_{+}\right)$, is equivalent to

$$
\begin{equation*}
\rho^{k+1}-2 \rho+1>0 \tag{4.8}
\end{equation*}
$$

and also to $\rho x_{-}<\frac{1}{2}$. For the function $h(\rho)=\rho^{k+1}-2 \rho+1, \rho \geqslant 0$ we have $h(1 / 2)>0$, $h(1)=0, h^{\prime}(\rho)<0$ for $\rho<\rho_{0}$ and $h^{\prime}(\rho)>0$ for $\rho>\rho_{0}$, where $\rho_{0}=(2 /(k+1))^{1 / k} \in(1 / 2,1)$. This implies that $h$ on $(0,1)$ has a unique zero $\eta \in(1 / 2,1)$, i.e.

$$
\eta^{k+1}-2 \eta+1=0
$$

and the condition $\rho<\eta$ is equivalent to (4.8) (this shows Remark 4.11 in the case $l=1$ ).
Since the system is symmetric, in fact we have

$$
\begin{equation*}
x_{+}<f_{-}\left(x_{-}\right)<\frac{1}{2}<f_{+}\left(x_{+}\right)<x_{+} . \tag{4.9}
\end{equation*}
$$

A simple computation shows that the condition of the positivity of the Lyapunov exponents is equivalent to

$$
\begin{equation*}
p_{-}, p_{+} \in\left(\frac{1}{k+1}, \frac{k}{k+1}\right) . \tag{4.10}
\end{equation*}
$$

Note that the above considerations prove Remark 4.14.

## Construction of the set $\Lambda$

Now we construct a set $\Lambda \subset(0,1)$ which will be shown later to be the support of the measure $\mu$ restricted to $(0,1)$. Our strategy is the following. First, we construct a family of disjoint closed intervals $I_{j}, j \in \mathbb{Z}^{*}$, with the union $I=\bigcup_{j \in \mathbb{Z}^{*}} I_{j}$ being forward-invariant under $\left\{f_{-}, f_{+}\right\}$. The disjointness of $I_{j}$ follows from the assumption that the system is of disjoint type. We check that the intervals $I_{-k}, \ldots, I_{-1}$ are mapped by $f_{+}$into $I_{1}$ with separation gaps, i.e. $f_{+}\left(I_{-k}\right), \ldots, f_{+}\left(I_{-1}\right)$ are disjoint subsets of $I_{1}$ (see Lemma 4.35 and Figure 4.3). Further iterates of these images and their similar copies generate an infinite collection of disjoint Cantor sets, whose union $\Lambda$ is fully invariant and minimal under the action of $\left\{f_{-}, f_{+}\right\}$(see Proposition 4.44). As we wish to calculate the dimension of $\Lambda$, it is convenient to describe $\Lambda$ as the union of the attractor $\Lambda_{-1}$ of a self-similar iterated function system $\left\{\phi_{r}\right\}_{r=1}^{k}$ on $I_{-1}$ and its similar copies. Moreover, as the successive levels of the Cantor set $\Lambda_{-1}$ are produced during jumps over the central interval $\left(x_{+}, x_{-}\right)$, we obtain a characterization of $\Lambda$ in terms of limit points of trajectories jumping over the central interval infinitely many times (see Proposition 4.43.

Let

$$
I_{-1}=\left[\rho f_{+}\left(x_{+}\right), \rho x_{-}\right]=\left[\rho f_{+}\left(x_{+}\right), f_{-}\left(x_{-}\right)\right]=\left[\rho \mathcal{I}\left(f_{-}\left(x_{-}\right)\right), f_{-}\left(x_{-}\right)\right]=\left[\frac{\rho-\rho^{2}}{1-\rho^{k+1}}, \frac{\rho-\rho^{k+1}}{1-\rho^{k+1}}\right]
$$

and for $j \in \mathbb{Z}^{*}$ define

$$
I_{j}=\left\{\begin{array}{ll}
\rho^{-j-1} I_{-1} & \text { for } j<0 \\
\mathcal{I}\left(\rho^{j-1} I_{-1}\right) & \text { for } j>0
\end{array} .\right.
$$

The following lemma is elementary and describes the combinatorics of the intervals $I_{j}, j \in \mathbb{Z}^{*}$.
Lemma 4.35. The following statements hold.
(a) $I_{-j}=\mathcal{I}\left(I_{j}\right)$ for $j \in \mathbb{Z}^{*}$.
(b) The sets $I_{j}, j \in \mathbb{Z}^{*}$ are pairwise disjoint and situated in $(0,1)$ in the increasing order with respect to $j$.
(c) $\inf I_{-k}=x_{+}, \sup I_{k}=x_{-}, \sup I_{-1}=f_{-}\left(x_{-}\right), \inf I_{1}=f_{+}\left(x_{+}\right)$. In particular,

$$
f_{-}(x)=\left\{\begin{array}{ll}
\rho x & \text { for } x \in \bigcup_{j=-\infty}^{k} I_{j} \\
\mathcal{I}\left(\rho^{-k} \mathcal{I}(x)\right) & \text { for } x \in \bigcup_{j=k+1}^{\infty} I_{j}
\end{array}, \quad f_{+}(x)=\left\{\begin{array}{ll}
\rho^{-k} x & \text { for } x \in \bigcup_{j=-\infty}^{-k-1} I_{j} \\
\mathcal{I}(\rho \mathcal{I}(x)) & \text { for } x \in \bigcup_{j=-k}^{\infty} I_{j}
\end{array} .\right.\right.
$$

(d) $f_{-}\left(I_{j}\right)=I_{j-1}$ for $j \leqslant-1, f_{-}\left(\operatorname{conv}\left(I_{1} \cup \cdots \cup I_{k}\right)\right)=I_{-1}, f_{-}\left(I_{j}\right)=I_{j-k}$ for $j \geqslant k+1$.
(e) $f_{+}\left(I_{j}\right)=I_{j+k}$ for $j \leqslant-k-1, f_{+}\left(\operatorname{conv}\left(I_{-k} \cup \cdots \cup I_{-1}\right)\right)=I_{1}, f_{+}\left(I_{j}\right)=I_{j+1}$ for $j \geqslant 1$.

See Figure 4.3 .


Figure 4.3: A schematic view of the action of $\left\{f_{-}, f_{+}\right\}$on the intervals $I_{j}$.

Proof. The assertion (a) follows directly from the definition of $I_{j}$. To show (b), we first check $\sup I_{-2}<\inf I_{-1}$. This is equivalent to

$$
\rho \frac{\rho-\rho^{k+1}}{1-\rho^{k+1}}<\frac{\rho-\rho^{2}}{1-\rho^{k+1}},
$$

which boils down to (4.8). By (4.9), $\sup I_{-1}<\inf I_{1}$. The rest of the assertion (b) follows directly from the above facts and the definition of $I_{j}$.

The assertions (c)-(e) are easy consequences of the definition of $I_{j}$, the symmetry of the system and the fact

$$
f_{-}^{-1}(x)=\rho^{-1} x, \quad f_{+}^{-1}(x)=\rho^{k} x \quad \text { for } x \in \bigcup_{j<0} I_{j},
$$

which follows from the definition of $f_{ \pm}$.

Let

$$
I=\bigcup_{j \in \mathbb{Z}} I_{j}, \quad I^{-}=\bigcup_{j<0} I_{j}, \quad I^{+}=\bigcup_{j>0} I_{j} .
$$

Note that Lemma 4.35 implies $f(I) \subset I$. More precisely, for every $i \in\{-,+\}$ and $j \in \mathbb{Z}^{*}$ we have

$$
f_{i}\left(I_{j}\right) \subset I_{j^{\prime}} \quad \text { for some } j^{\prime}=j^{\prime}(i, j) \in \mathbb{Z}^{*} .
$$

Lemma 4.36. For every $x \in(0,1)$ there exists $i_{1}, \ldots, i_{n} \in\{-,+\}, n \geqslant 0$, such that $f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x) \in I$.

Proof. Enumerate the components of $(0,1) \backslash I$ by $U_{j}, j \in \mathbb{Z}$, such that $U_{j}$ is the gap between $I_{j-1}$ and $I_{j}$ for $j<0, U_{0}$ is the gap between $I_{-1}$ and $I_{1}$, and $U_{j}$ is the gap between $I_{j}$ and $I_{j+1}$ for $j>0$. Take $x \in(0,1) \backslash I$. Since the system is symmetric, we can assume $x \in U_{j}, j \leqslant 0$. Then to prove the lemma it is enough to notice that by Lemma 4.35, we have $f_{-} \circ f_{+}^{\left\lfloor\frac{-j}{k}\right\rfloor+1}(x) \in I_{-1}$.

Consider the maps

$$
\phi_{r}: I_{-1} \rightarrow I_{-1}, \quad \phi_{r}(x)=\rho-\rho^{r} x, \quad r=1, \ldots, k .
$$

Note that

$$
\begin{equation*}
\phi_{r}(x)=\rho \mathcal{I}\left(\rho^{r-1} x\right)=f_{-}\left(\mathcal{I}\left(\rho^{r-1} x\right)\right)=\mathcal{I}\left(f_{+}\left(\rho^{r-1} x\right)\right) \tag{4.11}
\end{equation*}
$$

for $x \in I_{-1}$. Obviously, the maps $\phi_{r}$ are contracting similarities with $\left\|\phi_{r}^{\prime}\right\|=\rho^{r}$.
Let

$$
\Lambda_{-1}=\bigcap_{n=1}^{\infty} \bigcup_{r_{1}, \ldots, r_{n}=1}^{k} \phi_{r_{1}} \circ \cdots \circ \phi_{r_{n}}\left(I_{-1}\right)
$$

be the attractor of the iterated function system generated by $\left\{\phi_{r}\right\}_{r=1}^{k}$ on $I_{-1}$. By Theorem 2.11, it is the unique non-empty compact set in $I_{-1}$ satisfying

$$
\Lambda_{-1}=\bigcup_{r=1}^{k} \phi_{r}\left(\Lambda_{-1}\right)
$$

For $j \in \mathbb{Z}^{*}$ define

$$
\Lambda_{j}=\left\{\begin{array}{ll}
\rho^{-j-1} \Lambda_{-1} & \text { for } j<0 \\
\mathcal{I}\left(\rho^{j-1} \Lambda_{-1}\right) & \text { for } j>0
\end{array}, \quad \Lambda=\bigcup_{j \in \mathbb{Z}^{*}} \Lambda_{j}\right.
$$

Obviously, $\Lambda_{j}$ are pairwise disjoint compact sets and $\Lambda_{j} \subset I_{j}$. Furthermore, for $n \geqslant 0$, $r_{1}, \ldots, r_{n} \in\{1, \ldots, k\}$ let

$$
I_{j ; r_{1}, \ldots, r_{n}}= \begin{cases}\rho^{-j-1} \phi_{r_{1}} \circ \cdots \circ \phi_{r_{n}}\left(I_{-1}\right) & \text { for } j<0 \\ \mathcal{I}\left(\rho^{j-1} \phi_{r_{1}} \circ \cdots \circ \phi_{r_{n}}\left(I_{-1}\right)\right) & \text { for } j>0\end{cases}
$$

where for $n=0$ we set $I_{j ; r_{1}, \ldots, r_{n}}=I_{j}, \phi_{r_{1}} \circ \cdots \circ \phi_{r_{n}}=$ id. Since $\left|\phi_{r}^{\prime}\right|=\rho^{r}$, for every $j \in \mathbb{Z}^{*}$ and an infinite sequence $r_{1}, r_{2}, \ldots \in\{1, \ldots, k\}$ the segments $I_{j ; r_{1}, \ldots, r_{n}}, n \geqslant 0$, form a nested sequence of sets, such that

$$
\left|I_{j ; r_{1}, \ldots, r_{n}}\right|=\rho^{|j|-1+r_{1}+\cdots+r_{n}} \leqslant \rho^{n} \rightarrow 0 \text { as } n \rightarrow \infty
$$

so

$$
\bigcap_{n=1}^{\infty} I_{j ; r_{1}, \ldots, r_{n}}=\left\{x_{j ; r_{1}, r_{2}, \ldots .}\right\}
$$

for a point $x_{j ; r_{1}, r_{2}, \ldots} \in \Lambda$ and

$$
\Lambda=\bigcup_{j \in \mathbb{Z}^{*}} \bigcap_{n=1}^{\infty} \bigcup_{r_{1}, \ldots, r_{n}=1}^{k} I_{j ; r_{1}, \ldots, r_{n}}=\left\{x_{j ; r_{1}, r_{2}, \ldots}: j \in \mathbb{Z}^{*}, r_{1}, r_{2}, \ldots \in\{1, \ldots, k\}\right\}
$$

## Description of trajectories

Lemma 4.35 and 4.11 imply immediately the following.
Lemma 4.37. For $j \in \mathbb{Z}^{*}, r_{1}, r_{2}, \ldots \in\{1, \ldots, k\}, n \geqslant 0$,

$$
\begin{aligned}
f_{-}\left(I_{j ; r_{1}, \ldots, r_{n}}\right) & = \begin{cases}I_{j-1 ; r_{1}, \ldots, r_{n}} & \text { for } j<0 \\
I_{-1 ; j, r_{1}, \ldots, r_{n}} & \text { for } 1 \leqslant j \leqslant k, \\
I_{j-k ; r_{1}, \ldots, r_{n}} & \text { for } j>k\end{cases} \\
f_{+}\left(I_{j ; r_{1}, \ldots, r_{n}}\right) & = \begin{cases}I_{j+k ; r_{1}, \ldots, r_{n}} & \text { for } j<-k \\
I_{1 ;--j, r_{1}, \ldots, r_{n}} & \text { for }-k \leqslant j \leqslant-1 \\
I_{j+1 ; r_{1}, \ldots, r_{n}} & \text { for } j>0\end{cases}
\end{aligned}
$$

and

$$
\begin{aligned}
& f_{-}\left(x_{j ; r_{1}, r_{2}, \ldots}\right)= \begin{cases}x_{j-1 ; r_{1}, r_{2}, \ldots} & \text { for } j<0 \\
x_{-1 ; j, r_{1}, r_{2}, \ldots} & \text { for } 1 \leqslant j \leqslant k, \\
x_{j-k ; r_{1}, r_{2}, \ldots} & \text { for } j>k\end{cases} \\
& f_{+}\left(x_{j ; r_{1}, r_{2}, \ldots}\right)= \begin{cases}x_{j+k ; r_{1}, r_{2}, \ldots} & \text { for } j<-k \\
x_{1 ;-j, r_{1}, r_{2}, \ldots} & \text { for }-k \leqslant j \leqslant-1 . \\
x_{j+1 ; r_{1}, r_{2}, \ldots} & \text { for } j>0\end{cases}
\end{aligned}
$$

The following lemmas characterize trajectories jumping over the central interval. The first one follows directly from Lemma 4.35 .
Lemma 4.38. The following statements hold.
(a) If a trajectory $\left\{f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{n=0}^{\infty}$, for $x \in(0,1)$, jumps over the central interval at the time $s$, for $s \geqslant 0$, then $f_{i_{s+1}} \circ \cdots \circ f_{i_{1}}(x) \in I_{-1} \cup I_{1}$.
(b) A trajectory $\left\{f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{n=0}^{\infty}$, for $x \in I$, jumps over the central interval at the time $s$, for $s \geqslant 0$, if and only if

$$
\begin{aligned}
& f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x) \in \bigcup_{j=-k}^{-1} I_{j}, \quad i_{s+1}=+, \quad f_{i_{s+1}} \circ \cdots \circ f_{i_{1}}(x) \in I_{1} \\
& \text { or } \\
& f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x) \in \bigcup_{j=1}^{k} I_{j}, \quad i_{s+1}=-, \quad f_{i_{s+1}} \circ \cdots \circ f_{i_{1}}(x) \in I_{-1} .
\end{aligned}
$$

In particular, for given $j \in \mathbb{Z}^{*}$ and $i_{1}, i_{2}, \ldots \in\{-,+\}$, for all $x \in I_{j}$ the trajectories $\left\{f_{i_{n}} \circ\right.$ $\left.\cdots \circ f_{i_{1}}(x)\right\}_{n=0}^{\infty}$ jump over the central interval at the same times.

For $j, j^{\prime} \in \mathbb{Z}^{*}$ such that $\operatorname{sgn}(j)=\operatorname{sgn}\left(j^{\prime}\right)$, define

$$
F_{j, j^{\prime}}: I_{j} \underset{\text { onto }}{\longrightarrow} I_{j^{\prime}}, \quad F_{j, j^{\prime}}= \begin{cases}\left.f_{-}^{j-j^{\prime}}\right|_{I_{j}} & \text { for } j<0, j^{\prime} \leqslant j \\ \left.f_{+}^{\left\ulcorner\left(j^{\prime}-j\right) / k\right\rceil} \circ f_{-}^{j-j^{\prime}+k\left\lceil\left(j^{\prime}-j\right) / k\right\rceil}\right|_{I_{j}} & \text { for } j<0, j^{\prime}>j \\ f_{+}^{j^{\prime}-j} \mid I_{j} \\ \left.f_{-}^{\left\lceil\left(j-j^{\prime}\right) / k\right\rceil} \circ f_{+}^{j^{\prime}-j+k\left\lceil\left(j-j^{\prime}\right) / k\right\rceil}\right|_{I_{j}} & \text { for } j>0, j^{\prime} \geqslant j \\ f_{-}>0, j^{\prime}<j\end{cases}
$$

Note that $F_{j, j^{\prime}}=\left.f_{i_{n}} \circ \cdots \circ f_{i_{1}}\right|_{I_{j}}$ for some $i_{1}, \ldots, i_{n} \in\{-,+\}, n \geqslant 0$, and, by Lemma 4.35,

$$
F_{j, j^{\prime}}(x)= \begin{cases}\rho^{j-j^{\prime}} x & \text { for } j<0  \tag{4.12}\\ \mathcal{I}\left(\rho^{-j+j^{\prime}} \mathcal{I}(x)\right) & \text { for } j>0\end{cases}
$$

for $x \in I_{j}$. In particular, this implies

$$
F_{j, j}=\left.\operatorname{id}\right|_{I_{j}}, \quad F_{j^{\prime}, j^{\prime \prime}} \circ F_{j, j^{\prime}}=F_{j, j^{\prime \prime}}
$$

for $j, j^{\prime}, j^{\prime \prime} \in \mathbb{Z}^{*}$ such that $\operatorname{sgn}(j)=\operatorname{sgn}\left(j^{\prime}\right)=\operatorname{sgn}\left(j^{\prime \prime}\right)$. By Lemma 4.37,

$$
\begin{equation*}
F_{j, j^{\prime}}\left(I_{j ; r_{1}, \ldots, r_{n}}\right)=I_{j^{\prime} ; r_{1}, \ldots, r_{n}}, \quad F_{j, j^{\prime}}\left(x_{j ; r_{1}, r_{2}, \ldots}\right)=x_{j^{\prime} ; r_{1}, r_{2}, \ldots} \tag{4.13}
\end{equation*}
$$

for $r_{1}, r_{2}, \ldots \in\{1, \ldots, k\}, n \geqslant 0$.
Lemma 4.39. A trajectory $\left\{f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{n=0}^{\infty}$ of a point $x \in I_{j}, j \in \mathbb{Z}^{*}$, does not jump over the central interval at any time $0 \leqslant s<n$, for some $n \geqslant 0$, if and only if

$$
\left.f_{i_{n}} \circ \cdots \circ f_{i_{1}}\right|_{I_{j}}=F_{j, j^{\prime}}
$$

for $j^{\prime} \in \mathbb{Z}^{*}$ such that $f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x) \in I_{j^{\prime}}$ and $\operatorname{sgn}(j)=\operatorname{sgn}\left(j^{\prime}\right)$.
Proof. If a trajectory $\left\{f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{n=0}^{\infty}$ of $x \in I_{j}$ does not jump over the central interval at any time $0 \leqslant s<n$, then by Lemmas 4.35 and 4.38,

$$
f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)= \begin{cases}\rho^{j-j^{\prime}} x & \text { for } j<0 \\ \mathcal{I}\left(\rho^{-j+j^{\prime}} \mathcal{I}(x)\right) & \text { for } j>0\end{cases}
$$

for $j^{\prime} \in \mathbb{Z}^{*}$ such that $f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x) \in I_{j^{\prime}}$. Therefore, $\left.f_{i_{n}} \circ \cdots \circ f_{i_{1}}\right|_{I_{j}}=F_{j, j^{\prime}}$ by (4.12). The other implication follows directly from Lemmas 4.35 and 4.38 .

Define

$$
G_{r}^{-}: I_{1} \rightarrow I_{-1}, \quad G_{r}^{+}: I_{-1} \rightarrow I_{1}, \quad r \in\{1, \ldots, k\}
$$

setting

$$
G_{r}^{-}=f_{-} \circ F_{1, r}, \quad G_{r}^{+}=f_{+} \circ F_{-1,-r}
$$

We have $G_{r}^{ \pm}=\left.f_{i_{n}} \circ \cdots \circ f_{i_{1}}\right|_{I_{\mp 1}}$ for some $i_{1}, \ldots, i_{n} \in\{-,+\}, n \geqslant 0$. Moreover, by (4.11) and 4.12,

$$
\begin{equation*}
G_{r}^{-}=\left.\phi_{r} \circ \mathcal{I}\right|_{I_{1}}, \quad G_{r}^{+}=\mathcal{I} \circ \phi_{r} \tag{4.14}
\end{equation*}
$$

while Lemma 4.37 and 4.13 imply

$$
\begin{align*}
G_{r}^{-}\left(I_{1 ; r_{1}, \ldots, r_{n}}\right) & =I_{-1 ; r, r_{1}, \ldots, r_{n}}, & G_{r}^{-}\left(x_{1 ; r_{1}, r_{2}, \ldots}\right) & =x_{-1 ; r, r_{1}, r_{2}, \ldots}  \tag{4.15}\\
G_{r}^{+}\left(I_{-1 ; r_{1}, \ldots, r_{n}}\right) & =I_{1 ; r, r_{1}, \ldots, r_{n}}, & G_{r}^{+}\left(x_{-1 ; r_{1}, r_{2}, \ldots}\right) & =x_{1 ; r, r_{1}, r_{2}, \ldots}
\end{align*}
$$

for $r_{1}, r_{2}, \ldots \in\{1, \ldots, k\}, n \geqslant 0$.
Lemma 4.40. A trajectory $\left\{f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{n=0}^{\infty}$ of a point $x \in I$ jumps over the central interval at the time $s$, for some $s \geqslant 0$, if and only if

$$
f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x) \in I_{-r},\left.\quad f_{i_{s+1}}\right|_{I_{-r}}=G_{r}^{+} \circ F_{-r,-1}
$$

or

$$
f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x) \in I_{r},\left.\quad f_{i_{s+1}}\right|_{I_{r}}=G_{r}^{-} \circ F_{r, 1}
$$

for some $r \in\{1, \ldots, k\}$.
Proof. Follows directly from Lemmas $4.38,4.39$ and the definitions of the maps $F_{j, j^{\prime}}, G_{r}^{ \pm}$.
Lemma 4.41. A trajectory $\left\{f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{n=0}^{\infty}$ of a point $x \in I_{j}, j \in \mathbb{Z}^{*}$, jumps over the central interval (exactly) at the times $s_{1}, \ldots, s_{m}$, for some $0 \leqslant s_{1}<\cdots<s_{m}<n, 0 \leqslant m \leqslant n$, if and only if
$\left.f_{i_{n}} \circ \cdots \circ f_{i_{1}}\right|_{I_{j}}= \begin{cases}F_{-1, j^{\prime}} \circ G_{r_{1}}^{-} \circ G_{r_{2}}^{+} \circ \cdots \circ G_{r_{m-1}}^{-} \circ G_{r_{m}}^{+} \circ F_{j,-1} & \text { for } j<0, m \text { even } \\ F_{1, j^{\prime}} \circ G_{r_{1}}^{+} \circ G_{r_{2}}^{-} \circ \cdots \circ G_{r_{m-2}}^{+} \circ G_{r_{m-1}}^{-} \circ G_{r_{m}}^{+} \circ F_{j,-1} & \text { for } j<0, m \text { odd } \\ F_{1, j^{\prime}} \circ G_{r_{1}}^{+} \circ G_{r_{2}}^{-} \circ \cdots \circ G_{r_{m-1}}^{+} \circ G_{r_{m}}^{-} \circ F_{j, 1} & \text { for } j>0, m \text { even } \\ F_{-1, j^{\prime}} \circ G_{r_{1}}^{-} \circ G_{r_{2}}^{+} \circ \cdots \circ G_{r_{m-2}}^{-} \circ G_{r_{m-1}}^{+} \circ G_{r_{m}}^{-} \circ F_{j, 1} & \text { for } j>0, m \text { odd }\end{cases}$
for some $j^{\prime} \in \mathbb{Z}^{*}$ and $r_{1}, \ldots, r_{m} \in\{1, \ldots, k\}$, where $\operatorname{sgn}(j)=\operatorname{sgn}\left(j^{\prime}\right)$ when $m$ is even and $\operatorname{sgn}(j) \neq \operatorname{sgn}\left(j^{\prime}\right)$ when $m$ is odd. Moreover, in this case we have

$$
f_{i_{n}} \circ \cdots \circ f_{i_{1}}\left(I_{j}\right)=I_{j^{\prime} ; r_{1}, \ldots, r_{m}}
$$

and

$$
f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)= \begin{cases}\rho^{-j^{\prime}-1} \phi_{r_{1}} \circ \cdots \circ \phi_{r_{m}}\left(\rho^{j+1} x\right) & \text { for } j<0, m \text { even } \\ \mathcal{I}\left(\rho^{j^{\prime}-1} \phi_{r_{1}} \circ \cdots \circ \phi_{r_{m}}\left(\rho^{j+1} x\right)\right) & \text { for } j<0, m \text { odd } \\ \rho^{-j^{\prime}-1} \phi_{r_{1}} \circ \cdots \circ \phi_{r_{m}}\left(\rho^{-j+1} \mathcal{I}(x)\right) & \text { for } j>0, m \text { even } \\ \mathcal{I}\left(\rho^{j^{\prime}-1} \phi_{r_{1}} \circ \cdots \circ \phi_{r_{m}}\left(\rho^{-j+1} \mathcal{I}(x)\right)\right) & \text { for } j>0, m \text { odd }\end{cases}
$$

Proof. Follows directly from Lemmas 4.39 and 4.40, and 4.12, (4.13), 4.14, (4.15).
Definition 4.42 For $x \in(0,1)$ let $\omega_{\infty}(x)$ be the set of limit points of all trajectories of $x$ under $\left\{f_{-}, f_{+}\right\}$, which jump over the central interval infinitely many times, i.e.
$\omega_{\infty}(x)=\left\{\lim _{s \rightarrow \infty} f_{i_{n_{s}}} \circ \cdots \circ f_{i_{1}}(x): i_{1}, i_{2}, \ldots \in\{-,+\}, n_{s} \rightarrow \infty\right.$ as $s \rightarrow \infty$
and $\left\{f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{n=0}^{\infty}$ jumps over the central interval infinitely many times $\}$.

Proposition 4.43. For every $x \in(0,1)$,

$$
\omega_{\infty}(x)=\Lambda \cup\{0,1\}
$$

Proof. First, we prove $\omega_{\infty}(x) \subset \Lambda \cup\{0,1\}$ for $x \in(0,1)$. By Lemma 4.38(a), we can assume $x \in I$. Take $y \in \omega_{\infty}(x)$. Then $y=\lim _{s \rightarrow \infty} f_{i_{n_{s}}} \circ \cdots \circ f_{i_{1}}(x)$, where $n_{s} \rightarrow \infty$ and the trajectory $\left\{f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{n=0}^{\infty}$ jumps over the central interval infinitely many times. By Lemma 4.41, we have

$$
f_{i_{n_{s}}} \circ \cdots \circ f_{i_{1}}(x) \in I_{j(s) ; r_{1}(s), \ldots, r_{m(s)}(s)}
$$

for some $j(s) \in \mathbb{Z}^{*}, m(s) \geqslant 0, r_{1}(s), \ldots, r_{m(s)}(s) \in\{1, \ldots, k\}$, where $m(s) \rightarrow \infty$ as $s \rightarrow \infty$. Since $\left|I_{j(s) ; r_{1}(s), \ldots, r_{m(s)}(s)}\right| \leqslant \rho^{m(s)} \rightarrow 0$ as $s \rightarrow \infty, I_{j(s) ; r_{1}(s), \ldots, r_{m(s)}(s)} \cap \Lambda \neq 0$, we have $y \in \bar{\Lambda}=\Lambda \cup\{0,1\}$. In this way we have showed $\omega_{\infty}(x) \subset \Lambda \cup\{0,1\}$.

Now we prove $\Lambda \cup\{0,1\} \subset \omega_{\infty}(x)$ for $x \in(0,1)$. By Lemma 4.36, we can assume $x \in I_{j}$, $j \in \mathbb{Z}^{*}$. Since the system is symmetric, we can assume $j<0$. Take $y \in \Lambda$. Then $y=x_{j^{\prime} ; r_{1}, r_{2}, \ldots}$ for some $j^{\prime} \in \mathbb{Z}^{*}, r_{1}, r_{2}, \ldots \in\{1, \ldots, k\}$. Let

$$
F^{(0)}= \begin{cases}F_{j, j^{\prime}} & \text { if } j^{\prime}<0 \\ F_{1, j^{\prime}} \circ G_{1}^{+} \circ F_{j,-1} & \text { if } j^{\prime}>0\end{cases}
$$

and note that $F^{(0)}(x) \in I_{j^{\prime}}$. Define

$$
F^{(n)}= \begin{cases}F_{-1, j^{\prime}} \circ G_{r_{1}}^{-} \circ G_{r_{2}}^{+} \circ \cdots \circ G_{r_{n-1}}^{-} \circ G_{r_{n}}^{+} \circ F_{j^{\prime},-1} & \text { if } j^{\prime}<0 \\ F_{1, j^{\prime}} \circ G_{r_{1}}^{+} \circ G_{r_{2}}^{-} \circ \cdots \circ G_{r_{n-1}}^{+} \circ G_{r_{n}}^{-} \circ F_{j^{\prime}, 1} & \text { if } j^{\prime}>0\end{cases}
$$

for even $n>0$. Then $F^{(n)}$ is well-defined on $I_{j^{\prime}}$. Using (4.13) and 4.15 inductively, we see

$$
F^{(n)} \circ \cdots \circ F^{(2)} \circ F^{(0)}(x) \in I_{j^{\prime} ; r_{1}, \ldots, r_{n}}
$$

for every even $n>0$. Since $\left|I_{j^{\prime} ; r_{1}, \ldots, r_{n}}\right| \leqslant \rho^{n} \rightarrow 0$ as $n \rightarrow \infty$ and $\bigcap_{n \text { even }} I_{j^{\prime} ; r_{1}, \ldots, r_{n}}=\{y\}$, the trajectory defined by $\cdots \circ F^{(n)} \ldots \circ F^{(2)} \circ F^{(0)}(x)$ has $y$ as a limit point and, by Lemma 4.41, jumps over the central interval infinitely many times. This shows $\Lambda \subset \omega_{\infty}(x)$.

Take now $y \in\{0,1\}$ and define

$$
F^{(0)}= \begin{cases}F_{j,-1} & \text { if } y=0 \\ G_{1}^{+} \circ F_{j,-1} & \text { if } y=1\end{cases}
$$

and

$$
F^{(n)}= \begin{cases}F_{-1,-n-1} \circ G_{1}^{-} \circ G_{1}^{+} \circ \cdots \circ G_{1}^{-} \circ G_{1}^{+} \circ F_{-n+1,-1} & \text { if } y=0 \\ F_{1, n+1} \circ G_{1}^{+} \circ G_{1}^{-} \circ \cdots \circ G_{1}^{+} \circ G_{1}^{-} \circ F_{n-1,1} & \text { if } y=1\end{cases}
$$

for even $n>0$. Then, arguing as previously, we see that

$$
F^{(n)} \circ \cdots \circ F^{(2)} \circ F^{(0)}(x) \in \begin{cases}I_{-n-1} & \text { if } y=0 \\ I_{n+1} & \text { if } y=1\end{cases}
$$

for even $n>0$, the trajectory defined by $\cdots \circ F^{(n)} \cdots \circ F^{(2)} \circ F^{(0)}(x)$ has $y$ as its limit point and jumps over the central interval infinitely many times. This implies $\Lambda \cup\{0,1\} \subset \omega_{\infty}(x)$.

Proposition 4.44. We have

$$
\Lambda=f_{-}(\Lambda)=f_{+}(\Lambda)
$$

Moreover, the system $\left\{f_{-}, f_{+}\right\}$is minimal in $\Lambda$.

Proof. The first assertion follows directly from Lemma 4.37, while Proposition 4.43 implies minimality.

## Singularity of $\mu$

Proposition 4.45. We have

$$
\operatorname{supp} \mu=\Lambda \cup\{0,1\}, \quad \mu(\Lambda)=1
$$

Proof. By Proposition 4.44, we have $f(\Lambda)=\Lambda$. Moreover, $\Lambda$ is closed in $(0,1)$. Hence, Lemma 4.28 implies supp $\mu \subset \Lambda \cup\{0,1\}$ and $\mu(\Lambda)=1$. On the other hand, the system is minimal in $\Lambda$ by Proposition 4.44, so Proposition 4.20 gives $\operatorname{supp} \mu=\bar{\Lambda}=\Lambda \cup\{0,1\}$.

## Proposition 4.46.

$$
\operatorname{dim}_{H} \Lambda=\operatorname{dim}_{B} \Lambda=\frac{\log \eta}{\log \rho}<1
$$

where $\eta \in(1 / 2,1)$ is the unique solution of the equation $\eta^{k+1}-2 \eta+1=0$.

Proof. By definition, the maps $\phi_{r}: I_{-1} \rightarrow I_{-1}, r=1, \ldots, k$, are contractions and

$$
\phi_{r}\left(I_{-1}\right)=\left[\rho-\rho^{r} \frac{\rho-\rho^{k+1}}{1-\rho^{k+1}}, \rho-\rho^{r} \frac{\rho-\rho^{2}}{1-\rho^{k+1}}\right] .
$$

Using 4.8), we check that $\sup \phi_{r}\left(I_{-1}\right)<\inf \phi_{r+1}\left(I_{-1}\right)$ for $r=1, \ldots, k-1$. Consequently, $\left\{\phi_{r}\right\}_{r=1}^{k}$ is an iterated function system of contracting similarities with scales $\rho, \ldots, \rho^{k}$, respectively, satisfying the Strong Separation Condition (i.e. $\phi_{r}\left(I_{-1}\right)=\phi_{r}\left(I_{-1}\right), r=1, \ldots, k$, are pairwise disjoint). Therefore, its limit set $\Lambda_{-1}$ is a Cantor set with Hausdorff (and box) dimension equal to the unique positive number $d$ satisfying

$$
\rho^{d}+\cdots+\rho^{k d}=1
$$

(see Proposition 2.15). This equation is equivalent to $\eta^{k+1}-2 \eta+1=0$ for $\eta=\rho^{d}$. Hence,

$$
\operatorname{dim}_{H} \Lambda_{-1}=\operatorname{dim}_{B} \Lambda_{-1}=d=\frac{\log \eta}{\log \rho}
$$

Since $\Lambda_{j}, j \in \mathbb{Z}^{*}$, are disjoint similar copies of $\Lambda_{-1}$, we have $\operatorname{dim}_{H} \Lambda=\operatorname{dim}_{H} \Lambda_{-1}$ by Proposition 2.45. To see $\operatorname{dim}_{B} \Lambda=\operatorname{dim}_{B} \Lambda_{-1}$ note that $\Lambda=\psi\left(A \times \Lambda_{-1}\right) \cup \mathcal{I}\left(\psi\left(A \times \Lambda_{-1}\right)\right)$, where $A=\left\{\rho^{j}: j \geqslant 0\right\}$ and $\psi: A \times \Lambda_{-1} \rightarrow[0,1]$ is given as $\psi\left(\rho^{j}, x\right)=\rho^{j} x$. Since $\psi$ is Lipschitz and $\operatorname{dim}_{B} A=0$, applying points 1. 4 and 6, of Proposition 2.4 we obtain

$$
\operatorname{dim}_{B} \Lambda_{-1} \leqslant \operatorname{dim}_{B B} \Lambda \leqslant \overline{\operatorname{dim}}_{B B} \Lambda \leqslant \operatorname{dim}_{B} A+\operatorname{dim}_{B} \Lambda_{-1}=\operatorname{dim}_{B} \Lambda_{-1},
$$

hence $\operatorname{dim}_{B} \Lambda$ exists and equals $\operatorname{dim}_{B}\left(\Lambda_{-1}\right)=d$. The condition 4.8, equivalent to $\rho<\eta$, implies $\operatorname{dim}_{H} \Lambda_{-1}<1$.

Propositions 4.45 and 4.46 imply the following.
Corollary 4.47. The measure $\mu$ is singular with $\operatorname{dim}_{B}(\operatorname{supp} \mu)=\operatorname{dim}_{H}(\operatorname{supp} \mu)<1$.

## Dimension of $\boldsymbol{\mu}$

To determine the exact form of $\mu$, consider the natural projection for the IFS $\left\{\phi_{r}\right\}_{r=1}^{k}$ on $I_{-1}$ given by

$$
\pi_{-1}: \Sigma_{k}^{+} \rightarrow \Lambda_{-1}, \quad \pi_{-1}\left(r_{1}, r_{2}, \ldots\right)=\lim _{n \rightarrow \infty} \phi_{r_{1}} \circ \phi_{r_{2}} \circ \cdots \circ \phi_{r_{n}}(x)=x_{-1 ; r_{1}, r_{2}, \ldots}
$$

where $\Sigma_{k}^{+}=\{1, \ldots, k\}^{\mathbb{N}}$ and $x$ is any point from $I_{-1}$. By Proposition $2.13, \pi_{-1}$ is well defined, does not depend on the choice of $x$ and it is a $\pi_{-1}$, since the IFS satisfies the Strong Separation Condition. It follows that $\Lambda$ is homeomorphic to $\mathbb{Z}^{*} \times \Sigma_{k}^{+}$with the topology defined as the product of the discrete topology on $\mathbb{Z}^{*}$ and the standard (product) topology on $\Sigma_{k}^{+}$. The homeomorphism is given by

$$
\pi: \mathbb{Z}^{*} \times \Sigma_{k}^{+} \rightarrow X, \quad \pi\left(j, r_{1}, r_{2}, \ldots\right)=x_{j ; r_{1}, r_{2}, \ldots}= \begin{cases}\rho^{-j-1} \pi_{-1}\left(r_{1}, r_{2}, \ldots\right) & \text { for } j<0 \\ \mathcal{I}\left(\rho^{j-1} \pi_{-1}\left(r_{1}, r_{2}, \ldots\right)\right) & \text { for } j>0\end{cases}
$$

Let $\tilde{f}_{-}, \tilde{f}_{+}: \mathbb{Z}^{*} \times \Sigma_{k}^{+} \rightarrow \mathbb{Z}^{*} \times \Sigma_{k}^{+}$be the lifts by $\pi$ of $\left.f_{-}\right|_{\Lambda},\left.f_{+}\right|_{\Lambda}$, respectively, i.e.

$$
\begin{equation*}
\pi \circ \tilde{f}_{i}=f_{i} \circ \pi, \quad i \in\{-,+\} . \tag{4.16}
\end{equation*}
$$

Lemma 4.37 implies

$$
\begin{align*}
& \tilde{f}_{-}\left(j, r_{1}, r_{2}, \ldots\right)= \begin{cases}\left(j-1, r_{1}, r_{2}, \ldots\right) & \text { for } j<0 \\
\left(-1, j, r_{1}, r_{2}, \ldots\right) & \text { for } 1 \leqslant j \leqslant k \\
\left(j-k, r_{1}, r_{2}, \ldots\right) & \text { for } j>k\end{cases} \\
& \tilde{f}_{+}\left(j, r_{1}, r_{2}, \ldots\right)= \begin{cases}\left(j+k, r_{1}, r_{2}, \ldots\right) & \text { for } j<-k \\
\left(1,-j, r_{1}, r_{2}, \ldots\right) & \text { for }-k \leqslant j \leqslant-1 . \\
\left(j+1, r_{1}, r_{2}, \ldots\right) & \text { for } j>0\end{cases} \tag{4.17}
\end{align*}
$$

Due to (4.16), there is a one-to-one correspondence between stationary probability measures for the system $\left\{f_{-}, f_{+}\right\}$on $\Lambda$ with probabilities $p_{-}, p_{+}$and for the system $\left\{\tilde{f}_{-}, \tilde{f}_{+}\right\}$with probabilities $p_{-}, p_{+}$, both considered on $\sigma$-algebras of Borel sets. Since there is a unique stationary probability measure $\mu$ for $\left\{f_{-}, f_{+}\right\}$on $\Lambda$, there is also a unique stationary probability measure $\tilde{\mu}$ for $\left\{\tilde{f}_{-}, \tilde{f}_{+}\right\}$. Moreover, $\mu=\pi_{*} \tilde{\mu}$.

Now we determine the structure of the measure $\tilde{\mu}$.
Proposition 4.48. There exist numbers $c_{-}, c_{+}>0$ and probabilistic vectors $\beta^{-}=$ $\left(\beta_{1}^{-}, \ldots, \beta_{k}^{-}\right), \beta^{+}=\left(\beta_{1}^{+}, \ldots, \beta_{k}^{+}\right)$, such that $c_{-} \sum_{j=1}^{\infty} \eta_{-}^{j}+c_{+} \sum_{j=1}^{\infty} \eta_{+}^{j}=1$, where $\eta_{-}, \eta_{+} \in(0,1)$ are the unique solutions of the equations

$$
p_{+} \eta_{-}^{k+1}-\eta_{-}+p_{-}=0, \quad p_{-} \eta_{+}^{k+1}-\eta_{+}+p_{+}=0,
$$

respectively, and

$$
\tilde{\mu}=\sum_{j \in \mathbb{Z}^{*}} \eta_{j} \delta_{j} \otimes \nu_{j},
$$

where

$$
\eta_{j}=\left\{\begin{array}{ll}
c_{-} \eta_{-}^{-j} & \text { for } j<0 \\
c_{+} \eta_{+}^{j} & \text { for } j>0
\end{array},\right.
$$

$\nu_{j}$ is a probability measure on $\Sigma_{k}^{+}$given by

$$
\nu_{j}=\left\{\begin{array}{l}
\mathbb{P}_{\beta^{-}} \otimes \mathbb{P}_{\beta^{+}} \otimes \mathbb{P}_{\beta^{-}} \otimes \mathbb{P}_{\beta^{+}} \otimes \cdots \\
\mathbb{P}_{\beta^{+}} \otimes \mathbb{P}_{\beta^{-}} \otimes \mathbb{P}_{\beta^{+}} \otimes \mathbb{P}_{\beta^{-}} \otimes \cdots \\
\text { for } j<0 \\
\text { for } j>0
\end{array}, \quad j \in \mathbb{Z}^{*},\right.
$$

and $\delta_{j}$ is the Dirac measure at $j$.

Proof. Let

$$
h^{-}(x)=p_{+} x^{k+1}-x+p_{-}, \quad h^{+}(x)=p_{-} x^{k+1}-x+p_{+} .
$$

Since $h^{ \pm}$are convex, $h^{ \pm}(0)>0, h^{ \pm}(1)=0$ and, by (4.10), $\left(h^{ \pm}\right)^{\prime}(1)>0$, the function $h^{ \pm}$has a unique zero in $(0,1)$, which determines the values of $\eta_{-}, \eta_{+}$. Suppose that $c_{ \pm}, \beta_{1}^{ \pm}, \ldots, \beta_{k}^{ \pm}>0$ satisfy

$$
\begin{equation*}
c_{-} \sum_{j=1}^{\infty} \eta_{-}^{j}+c_{+} \sum_{j=1}^{\infty} \eta_{+}^{j}=1, \quad \sum_{r=1}^{k} \beta_{r}^{-}=1, \quad \sum_{r=1}^{k} \beta_{r}^{+}=1 \tag{4.18}
\end{equation*}
$$

Then the measure

$$
\nu=\sum_{j \in \mathbb{Z}^{*}} \eta_{j} \delta_{j} \otimes \nu_{j}
$$

for $\eta_{j}, \nu_{j}$ as in the statement of the proposition is a probability measure on $\mathbb{Z}^{*} \times \Sigma_{k}^{+}$. Let

$$
\left[j, r_{1}, \ldots, r_{n}\right]=\left\{\left(j^{\prime}, r_{1}^{\prime}, r_{2}^{\prime}, \ldots\right) \in \mathbb{Z}^{*} \times \Sigma_{k}^{+}: j^{\prime}=j, r_{1}^{\prime}=r_{1}, \ldots, r_{n}^{\prime}=r_{n}\right\}
$$

for $j \in \mathbb{Z}^{*}, n \geqslant 0$ and $r_{1}, \ldots, r_{n} \in\{1, \ldots, k\}$ be the cylinders in $\mathbb{Z}^{*} \times \Sigma_{k}^{+}$. By definition,

$$
\nu\left(\left[j, r_{1}, \ldots, r_{n}\right]\right)=\left\{\begin{array}{ll}
c_{-} \eta_{-}^{-j} \beta_{r_{1}}^{-} \beta_{r_{2}}^{+} \cdots \beta_{r_{n-1}}^{-} \beta_{r_{n}}^{+} & \text {for } j<0, n \text { even }  \tag{4.19}\\
c_{-} \eta_{-}^{-j} \beta_{r_{1}}^{-} \beta_{r_{2}}^{+} \cdots \beta_{r_{n-2}}^{-} \beta_{r_{n-1}}^{+} \beta_{r_{n}}^{-} & \text {for } j<0, n \text { odd } \\
c_{+} \eta_{+}^{j} \beta_{r_{1}}^{+} \beta_{r_{2}}^{-} \cdots \beta_{r_{n-1}}^{+} \beta_{r_{n}} & \text { for } j>0, n \text { even } \\
c_{+} \eta_{+}^{j} \beta_{r_{1}}^{+} \beta_{r_{2}^{-}}^{-} \cdots \beta_{r_{n-2}}^{+} \beta_{r_{n-1}}^{-} \beta_{r_{n}}^{+} & \text {for } j>0, n \text { odd }
\end{array} .\right.
$$

Now we prove that for some choice of the constants $c_{ \pm}, \beta_{1}^{ \pm}, \ldots, \beta_{k}^{ \pm}>0$ satisfying (4.18) the measure $\nu$ is stationary for $\left\{\tilde{f}_{-}, \tilde{f}_{+}\right\}$with probabilities $p_{-}, p_{+}$. Note that to show that $\nu$ is stationary, it is enough to check

$$
\begin{equation*}
\nu\left(\left[j, r_{1}, \ldots, r_{n}\right]\right)=p_{-} \nu\left(\tilde{f}_{-}^{-1}\left(\left[j, r_{1}, \ldots, r_{n}\right]\right)\right)+p_{+} \nu\left(\tilde{f}_{+}^{-1}\left(\left[j, r_{1}, \ldots, r_{n}\right]\right)\right) \tag{4.20}
\end{equation*}
$$

for $j \in \mathbb{Z}^{*}$, even $n \in \mathbb{N}$ and $r_{1}, \ldots r_{n} \in\{1, \ldots, k\}$, because the corresponding cylinders $\left[j, r_{1}, \ldots, r_{n}\right]$ generate the $\sigma$-algebra of Borel sets in $\mathbb{Z}^{*} \times \Sigma_{k}^{+}$. By 4.17),

$$
\begin{aligned}
& \tilde{f}_{-}^{-1}\left(\left[j, r_{1}, \ldots, r_{n}\right]\right)= \begin{cases}{\left[j+1, r_{1}, \ldots, r_{n}\right]} & \text { for } j<-1 \\
{\left[r_{1}, r_{2}, \ldots, r_{n}\right]} & \text { for } j=-1, \\
{\left[j+k, r_{1}, \ldots, r_{n}\right]} & \text { for } j>0\end{cases} \\
& \tilde{f}_{+}^{-1}\left(\left[j, r_{1}, \ldots, r_{n}\right]\right)= \begin{cases}{\left[j-k, r_{1}, \ldots, r_{n}\right]} & \text { for } j<0 \\
{\left[-r_{1}, r_{2}, \ldots, r_{n}\right]} & \text { for } j=1 \\
{\left[j-1, r_{1}, \ldots, r_{n}\right]} & \text { for } j>1\end{cases}
\end{aligned}
$$

Using this together with (4.19), we check that 4.20 for even $n \in \mathbb{N}$ (split into four cases: $j<-1, j>1, j=-1, j=1$, respectively) is equivalent to the following system of equations:

$$
\left\{\begin{array}{l}
\eta_{-}=p_{-}+p_{+} \eta_{-}^{k+1}  \tag{4.21}\\
\eta_{+}=p_{+}+p_{-} \eta_{+}^{k+1} \\
c_{-} \eta_{-} \beta_{r}^{-}=p_{-} c_{+} \eta_{+}^{r}+p_{+} c_{-} \eta_{-}^{k+1} \beta_{r}^{-} \\
c_{+} \eta_{+} \beta_{r}^{+}=p_{+} c_{-} \eta_{-}^{r}+p_{-} c_{+} \eta_{+}^{k+1} \beta_{r}^{+}
\end{array} \text {for } r=1, \ldots, k, k, k, \ldots, k\right.
$$

(where we write $r$ instead of $r_{1}$ ).
Now we solve the system (4.21) together with 4.18). The first two equations of (4.21) agree with the definitions of $\eta_{-}, \eta_{+}$. Substituting them, respectively, into the third and fourth ones, we obtain

$$
\begin{equation*}
c_{-} \beta_{r}^{-}=c_{+} \eta_{+}^{r}, \quad c_{+} \beta_{r}^{+}=c_{-} \eta_{-}^{r} . \tag{4.22}
\end{equation*}
$$

Summing this over $r \in\{1, \ldots, k\}$ and using the second and third equation of (4.18), we have

$$
c_{-}=c_{+} \frac{\eta_{+}-\eta_{+}^{k+1}}{1-\eta_{+}}, c_{+}=c_{-} \frac{\eta_{-}-\eta_{-}^{k+1}}{1-\eta_{-}}
$$

and substituting the second and first equation of (4.21) respectively, we arrive at a single equation

$$
c_{-} p_{-}=c_{+} p_{+},
$$

which together with the first equation of (4.18) gives

$$
c_{-}=\frac{p_{+}}{p_{+} \eta_{-} /\left(1-\eta_{-}\right)+p_{-} \eta_{+} /\left(1-\eta_{+}\right)}, \quad c_{+}=\frac{p_{-}}{p_{+} \eta_{-} /\left(1-\eta_{-}\right)+p_{-} \eta_{+} /\left(1-\eta_{+}\right)} .
$$

Using (4.22), we finally obtain

$$
\beta_{r}^{-}=\frac{p_{-}}{p_{+}} \eta_{+}^{r}, \quad \beta_{r}^{+}=\frac{p_{+}}{p_{-}} \eta_{-}^{r}, \quad r=1, \ldots, k .
$$

The numbers $c_{ \pm}, \beta_{1}^{ \pm}, \ldots, \beta_{k}^{ \pm}$satisfy (4.21) and (4.18). In this way we showed that the system of equations (4.21) and 4.18) has a unique solution for which the measure $\nu$ is stationary. By the uniqueness of such a measure, we have $\nu=\tilde{\mu}$.

Finally, we determine the Hausdorff dimension of the measure $\mu$. Since by Proposition 4.48 $\left.\mu\right|_{I_{j}}=\pi_{*}\left(\eta_{j} \delta_{j} \otimes \nu_{j}\right)$ for $j \in \mathbb{Z}^{*}$, applying Proposition 2.5 gives

$$
\operatorname{dim}_{H} \mu=\left.\sup _{j \in \mathbb{Z}^{*}} \operatorname{dim}_{H} \mu\right|_{I_{j}}=\sup _{j \in \mathbb{Z}^{*}} \operatorname{dim}_{H} \pi_{*}\left(\eta_{j} \delta_{j} \otimes \nu_{j}\right) .
$$

Note that the measure $\pi_{*}\left(\eta_{j} \delta_{j} \otimes \nu_{j}\right)$, supported on the Cantor set $\Lambda_{j}$, is bi-Lipschitz isomorphic (after normalization) to the measure $\pi_{*}\left(\eta_{-1} \delta_{-1} \otimes \nu_{-1}\right)$, which (after normalization) is the selfsimilar measure for the iterated function system $\left\{\phi_{r} \circ \phi_{s}\right\}_{r, s=1}^{k}$ with probabilities $\left(\beta_{r}^{-} \beta_{s}^{+}\right)_{r, s=1}^{k}$. Therefore, Theorem 2.15 gives

$$
\begin{aligned}
\operatorname{dim}_{H} \pi_{*}\left(\eta_{j} \delta_{j} \otimes \nu_{j}\right)= & \frac{\sum_{r, s=1}^{k} \beta_{r}^{-} \beta_{s}^{+} \log \beta_{r}^{-} \beta_{s}^{+}}{\sum_{r, s=1}^{k} \beta_{r}^{-} \beta_{s}^{+} \log \rho^{r+s}}=\frac{\sum_{r=1}^{k}\left(\beta_{r}^{-} \log \beta_{r}^{-}+\beta_{r}^{+} \log \beta_{r}^{+}\right)}{\sum_{r=1}^{k}\left(\beta_{r}^{-}+\beta_{r}^{+}\right) \log \rho^{r}} \\
= & \frac{\sum_{r=1}^{k} r\left(\frac{p_{+}}{p_{-}} \eta_{-}^{r} \log \eta_{-}+\frac{p_{-}}{p_{+}} \eta_{+}^{r} \log \eta_{+}\right)}{\sum_{r=1}^{k} r\left(\frac{p_{+}}{p_{-}} \eta_{-}^{r}+\frac{p_{-}}{p_{+}} \eta_{+}^{r}\right) \log \rho} .
\end{aligned}
$$

### 4.5 Proof of Theorem 4.10. Case $l>1$

## Preliminaries

In Theorem 4.10 we consider a symmetric $A M$-system $\left\{f_{-}, f_{+}\right\}$of disjoint type with probabilities $p_{-}, p_{+}$, positive Lyapunov exponents and a ( $k: l$ )-resonance for some relatively prime $k, l \in \mathbb{N}, k>l$. In this section we deal with the case $l>1$. Our approach is similar to the case $l=1$, however the combinatorics of the obtained system of intervals is more complicated and produces Cantor sets which are attractors for infinite iterated function systems.

We have

$$
f_{-}(x)=\left\{\begin{array}{ll}
\rho^{l} x & \text { for } x \in\left[0, x_{-}\right] \\
\mathcal{I}\left(\rho^{-k} \mathcal{I}(x)\right) & \text { for } x \in\left(x_{-}, 1\right]
\end{array}, \quad f_{+}(x)=\left\{\begin{array}{ll}
\rho^{-k} x & \text { for } x \in\left[0, x_{+}\right] \\
\mathcal{I}\left(\rho^{\prime} \mathcal{I}(x)\right) & \text { for } x \in\left(x_{+}, 1\right]
\end{array},\right.\right.
$$

where $\rho \in(0,1), k, l \in \mathbb{N}, 1<l<k$ and

$$
\begin{aligned}
x_{-} & =\frac{1-\rho^{k}}{1-\rho^{k+l}}, & x_{+} & =\mathcal{I}\left(x_{-}\right)=\frac{\rho^{k}-\rho^{k+l}}{1-\rho^{k+l}}, \\
f_{-}\left(x_{-}\right) & =\frac{\rho^{l}-\rho^{k+l}}{1-\rho^{k+l}}, & f_{+}\left(x_{+}\right) & =\mathcal{I}\left(f_{-}\left(x_{-}\right)\right)=\frac{1-\rho^{l}}{1-\rho^{k+l}} .
\end{aligned}
$$

In particular, we have

$$
x_{+}<f_{-}\left(x_{-}\right) .
$$

A direct computation gives

$$
\begin{equation*}
\mathcal{I}\left(\rho^{k} \mathcal{I}\left(\rho^{l} x_{-}\right)\right)=x_{-} . \tag{4.23}
\end{equation*}
$$

We assume that the system is of disjoint type, which is equivalent to

$$
\rho^{k+l}-2 \rho^{l}+1>0
$$

and also (by symmetry) to

$$
f_{-}\left(x_{-}\right)<\frac{1}{2}
$$

Hence, since the system is symmetric, we have

$$
x_{+}<f_{-}\left(x_{-}\right)<\frac{1}{2}<f_{+}\left(x_{+}\right)<x_{-}
$$

Consider the function $h(\rho)=\rho^{k+l}-2 \rho^{k+1}+2 \rho-1, \rho \geqslant 0$. We have $h(0), h(1 / 2)<0$, $h(1)=0, h^{\prime}(1)<0$ and $h^{\prime \prime}$ has exactly one zero in $(0,+\infty)$. This implies that $h$ on $(0,1)$ has a unique zero $\eta \in(1 / 2,1)$, i.e.

$$
\eta^{k+l}-2 \eta^{k+1}+2 \eta-1=0
$$

and the assumption $\rho<\eta$ is equivalent to

$$
\begin{equation*}
\rho^{k+l}-2 \rho^{k+1}+2 \rho-1<0 \tag{4.24}
\end{equation*}
$$

and also to

$$
\begin{equation*}
\rho x_{-}<\frac{1}{2} . \tag{4.25}
\end{equation*}
$$

In particular, this shows that the condition $\rho<\eta$ implies that the system is of disjoint type, which proves Remark 4.11.

Finally, notice that the positivity of the Lyapunov exponents of the system is equivalent to

$$
p_{-}, p_{+} \in\left(\frac{l}{k+l}, \frac{k}{k+l}\right)
$$

## Construction of the set $\Lambda$

Let us define the basic intervals $I_{j} \in \mathbb{Z}^{*}$ in the same manner as in the case $l=1$, i.e.

$$
I_{-1}=\left[\rho f_{+}\left(x_{+}\right), \rho x_{-}\right]=\left[\rho f_{+}\left(x_{+}\right), \rho^{1-l} f_{-}\left(x_{-}\right)\right]=\left[\rho \mathcal{I}\left(\rho^{l} x_{-}\right), \rho x_{-}\right]=\left[\frac{\rho-\rho^{1+l}}{1-\rho^{k+l}}, \frac{\rho-\rho^{k+1}}{1-\rho^{k+l}}\right]
$$

and note that by 4.25,

$$
\begin{equation*}
\sup I_{-1}<\frac{1}{2} \tag{4.26}
\end{equation*}
$$

For $j \in \mathbb{Z}^{*}$ let

$$
I_{j}= \begin{cases}\rho^{-j-1} I_{-1} & \text { for } j<0 \\ \mathcal{I}\left(\rho^{j-1} I_{-1}\right) & \text { for } j>0\end{cases}
$$

Let us now explain briefly the differences compared to the case $l=1$. Unlike previously, the union $\bigcup_{j \in \mathbb{Z}^{*}} I_{j}$ is no longer forward-invariant under $\left\{f_{-}, f_{+}\right\}$. More precisely, $f_{+}\left(I_{-k} \cup \ldots \cup\right.$ $\left.I_{-l}\right) \subset I_{l}$, but $f_{+}\left(I_{-l+1} \cup \ldots \cup I_{-1}\right)$ is situated between $I_{l}$ and $I_{l+1}$, inside a larger interval $J_{l}$ (see Lemma 4.51 and Figure 4.5). Therefore, our first step is extending the family $\left\{I_{j}\right\}_{j \in \mathbb{Z}^{*}}$ to a larger family $\left\{I_{\mathbf{j}}\right\}_{\mathbf{j} \in \mathcal{J}}$ consisting of similar copies of intervals $f_{+}\left(I_{-l+1}\right), \ldots, f_{+}\left(I_{-1}\right)$ and their further iterates which are not contained in the intervals obtained in previous steps of the construction (see Figures 4.4 and 4.5). As a result, we obtain a forward-invariant family of intervals, which has infinitely many elements inside each of the (disjoint) intervals $J_{j}$. As before, we iterate the intervals from this family to produce a fully invariant and minimal union of disjoint Cantor sets. The corresponding iterated function system $\left\{\Phi_{\mathbf{r}}\right\}_{\mathbf{r} \in \mathcal{R}}$ on $I_{-1}$ is generated by the action of $f_{+}$on the interval $\left[x_{+}, f_{-}\left(x_{-}\right)\right]$, which maps some of the intervals $I_{\mathrm{j}}$ into $I_{l}$. This infinite IFS has a Cantor set attractor $\Lambda_{-1} \subset I_{-1}$, which is copied inside each of the intervals $I_{\mathbf{j}}$ to form a suitable invariant minimal set $\Lambda \subset(0,1)$.

Let

$$
J_{-1}=\left[\rho \mathcal{I}\left(\rho x_{-}\right), \rho x_{-}\right]
$$

and note that

$$
I_{-1} \subset J_{-1}, \quad \sup I_{-1}=\sup J_{-1}
$$

As previously, consider the maps

$$
\phi_{r}(x)=\rho \mathcal{I}\left(\rho^{r-1} x\right)=\rho-\rho^{r} x, \quad r=1, \ldots, k
$$

for $x \in J_{-1}$. Recall that $\phi_{r}$ are orientation-reversing contracting similarities with $\left|\phi_{r}^{\prime}\right|=\rho^{r}$ and $\phi_{1}<\cdots<\phi_{k}$.

Lemma 4.49. We have

$$
\phi_{r}\left(J_{-1}\right) \subset\left\{\begin{array}{ll}
J_{-1} \backslash I_{-1} & \text { for } r=1, \ldots, l-1 \\
I_{-1} & \text { for } r=l, \ldots, k-1
\end{array}, \quad \phi_{k}\left(I_{-1}\right) \subset I_{-1}\right.
$$

Moreover, $\phi_{r}\left(J_{-1}\right), r=1, \ldots, k$, are pairwise disjoint.

Proof. By definition,

$$
\phi_{r}\left(J_{-1}\right)=\left[\rho \mathcal{I}\left(\rho^{r} x_{-}\right), \rho \mathcal{I}\left(\rho^{r} \mathcal{I}\left(\rho x_{-}\right)\right)\right] \quad \phi_{r}\left(I_{-1}\right)=\left[\rho \mathcal{I}\left(\rho^{r} x_{-}\right), \rho \mathcal{I}\left(\rho^{r} \mathcal{I}\left(\rho^{l} x_{-}\right)\right)\right] .
$$

It is obvious that $\inf \phi_{r}\left(J_{-1}\right) \geqslant \inf J_{-1}$ for $r=1, \ldots, k$ and $\inf \phi_{r}\left(J_{-1}\right) \geqslant \inf I_{-1}$ for $r=$ $l, \ldots, k$. The inequality $\sup \phi_{r}\left(J_{-1}\right)<\inf I_{-1}$ for $r=1, \ldots, l-1$ boils down to (4.24), while $\sup \phi_{r}\left(J_{-1}\right) \leqslant \sup I_{-1}$ for $r=l, \ldots, k-1$ is equivalent to $\rho^{k+l}+\rho^{k-r}+\rho-\rho^{k+1}-\rho^{k+l-r}+1 \leqslant 0$. For $l \leqslant r \leqslant k-1$ it is enough to have $\rho^{k+l}+2 \rho-\rho^{k+1}-\rho^{k}-1 \leqslant 0$ (as $\rho^{k-r} \leqslant \rho$ and $\rho^{k+l-r} \geqslant \rho^{k}$ ). By (4.24) this can be reduced to $\rho^{k+1}-\rho^{k} \leqslant 0$ which is obviously true, since $\rho \in(0,1)$. This proves the first assertion. To show $\phi_{k}\left(I_{-1}\right) \subset I_{-1}$, it is enough to notice that $\sup \phi_{k}\left(I_{-1}\right)=\sup I_{-1}$ holds due to 4.23$)$. To check the disjointness of $\phi_{r}\left(J_{-1}\right)$, we notice that the inequality $\sup \phi_{r}\left(J_{-1}\right)<\inf \phi_{r+1}\left(J_{-1}\right), r=1, \ldots, k-1$, is equivalent to 4.24).

For $j \in \mathbb{Z}^{*}$ let

$$
\mathcal{J}_{j}=\{j\} \times\left(\{\emptyset\} \cup \bigcup_{n=1}^{\infty}\{1, \ldots, l-1\}^{n}\right), \quad \mathcal{J}=\bigcup_{j \in \mathbb{Z}^{*}} \mathcal{J}_{j} .
$$

We will denote the elements of $\mathcal{J}$ by $\mathbf{j}=\left(j, j_{1}, \ldots, j_{n}\right)$, where $j \in \mathbb{Z}^{*}, n \geqslant 0, j_{1}, \ldots, j_{n} \in$ $\{1, \ldots, l-1\}$, with the convention that $j_{1}, \ldots, j_{n}$ for $n=0$ is the empty sequence.

For $\mathbf{j}=\left(j, j_{1}, \ldots, j_{n}\right) \in \mathcal{J}$ define

$$
I_{\mathbf{j}}=I_{j, j_{1}, \ldots, j_{n}}=\left\{\begin{array}{ll}
\rho^{-j-1} \phi_{j_{1}} \circ \cdots \circ \phi_{j_{n}}\left(I_{-1}\right) & \text { for } j<0 \\
\mathcal{I}\left(\rho^{j-1} \phi_{j_{1}} \circ \cdots \circ \phi_{j_{n}}\left(I_{-1}\right)\right) & \text { for } j>0
\end{array}, \quad I=\bigcup_{\mathbf{j} \in \mathcal{J}} I_{\mathbf{j}} .\right.
$$

Note that this notation is compatible with our previous definition of $I_{j}$ for $j \in \mathbb{Z}^{*}$. Furthermore, for $j \in \mathbb{Z}^{*}$ let

$$
J_{j}=\left\{\begin{array}{ll}
\rho^{-j-1} J_{-1} & \text { for } j<0 \\
\mathcal{I}\left(\rho^{j-1} J_{-1}\right) & \text { for } j>0
\end{array}, \quad J=\bigcup_{j \in \mathbb{Z}} J_{j}\right.
$$

The following lemmas describe the combinatorics of the intervals $I_{\mathbf{j}}, \mathbf{j} \in \mathcal{J}$.
Lemma 4.50. The following statements hold.
(a) $I_{-j, j_{1}, \ldots, j_{n}}=\mathcal{I}\left(I_{j, j_{1}, \ldots, j_{n}}\right), J_{-j}=\mathcal{I}\left(J_{j}\right)$ for $j \in \mathbb{Z}^{*}, n \geqslant 0, j_{1}, \ldots, j_{n} \in\{1, \ldots, l-1\}$.
(b) The segments $J_{j}, j \in \mathbb{Z}^{*}$, are pairwise disjoint.
(c) For $j \in \mathbb{Z}^{*}$, the segments $I_{\mathbf{j}}, \mathbf{j} \in \mathcal{J}_{j}$, are pairwise disjoint subsets of $J_{j}$.
(d) For $j \in \mathbb{Z}^{*}$, we have $\inf J_{j}=\inf I_{j, 1}, \sup J_{j}=\sup I_{j}$ for $j<0$ and $\inf J_{j}=\inf I_{j}$, $\sup J_{j}=\sup I_{j, 1}$ for $j>0$. In particular,

$$
J_{j}=\operatorname{conv} \bigcup_{\mathbf{j} \in \mathcal{J}_{j}} I_{\mathbf{j}}
$$

(e) Let $j \in \mathbb{Z}^{*}$. Then for $j<0$ (resp. $j>0$ ), the segments $I_{j, j_{1}}, j_{1}=1, \ldots, l-1$, are situated in $J_{j}$ in the increasing (resp. decreasing) order with respect to $j_{1}$, to the left (resp. right) of $I_{j}$.
(f) Let $j \in \mathbb{Z}^{*}, j_{1}, \ldots, j_{n} \in\{1, \ldots, l-1\}$ for $n \geqslant 1$. Then for $j<0$ and even $n$ or $j>0$ and odd $n$ (resp. $j<0$ and odd $n$ or $j>0$ and even $n$ ), the segments $I_{j, j_{1}, \ldots, j_{n+1}}$, $j_{n+1}=1, \ldots, l-1$ are situated in $J_{j}$ in the increasing (resp. decreasing) order with respect to $j_{n+1}$, between $I_{j, j_{1}, \ldots, j_{n}}$ and $I_{j, j_{1}, \ldots, j_{n-1}, j_{n}+1}$ if $j_{n}<l-1$, and between $I_{j, j_{1}, \ldots, j_{n}}$ and $I_{j, j_{1}, \ldots, j_{n-1}}$ if $j_{n}=l-1$.
(g) $\inf I_{-k}=x_{+}, \sup I_{-l}=f_{-}\left(x_{-}\right), \inf I_{l}=f_{+}\left(x_{+}\right), \sup I_{k}=x_{-}$.

See Figures 4.4 and 4.5 .


Figure 4.4: A schematic view of the location of the intervals $I_{j, j_{1}, \ldots, j_{n}}$ within $J_{j}$ for $j<0$.

Proof. The assertion (a) is straightforward. To show (b), it is enough to use (4.26) and check $\sup J_{j-1}<\inf J_{j}$ for $j<0$ (and use the symmetry of the system). By a direct computation, the latter inequality is equivalent to (4.24). By symmetry and the definition of $I_{\mathrm{j}}$ and $J_{j}$, showing (c)-(f) we can assume $j=-1$. First, we prove (c). Since $I_{-1} \subset J_{-1}$, Lemma 4.49 implies $I_{\mathbf{j}} \subset J_{-1}$ for $\mathbf{j} \in \mathcal{J}_{-1}$. To show the disjointness of $I_{\mathbf{j}}$, suppose that $I_{-1, j_{1}, \ldots, j_{n}} \cap$ $I_{-1, j_{1}^{\prime}, \ldots, j_{n^{\prime}}^{\prime}} \neq \emptyset$ for some distinct $\left(-1, j_{1}, \ldots, j_{n}\right),\left(-1, j_{1}^{\prime}, \ldots, j_{n^{\prime}}^{\prime}\right) \in \mathcal{J}_{-1}$. We can assume $n^{\prime} \geqslant n$. Applying suitable sequence of inverses of maps $\phi_{r}$ to both segments, we can suppose
$j_{1} \neq j_{1}^{\prime}$ or $I_{-1, j_{1}^{\prime}, \ldots, j_{n^{\prime}}^{\prime}}=I_{-1}$. In the first case we have a contradiction with the last assertion of Lemma 4.49, while the second case contradicts with the first assertion of it. This proves (c). The first part of (d) is straightforward. Together with (c), it shows the second part. The assertion (e) follows from (c) and the fact $\phi_{1}<\cdots<\phi_{l-1}$. The first part of (f) holds by a direct checking. In turn, together with the fact that the maps $\phi_{r}$ reverse the orientation and $\phi_{1}<\cdots<\phi_{l-1}$, it proves the second part by induction. The assertion (g) is straightforward.

The following lemma is a direct consequence of the definition of the maps $f_{ \pm}$and Lemma 4.50. See Figure 4.5.

Lemma 4.51. We have

$$
\begin{aligned}
& f_{-}(x)=\left\{\begin{array}{ll}
\rho^{l} x & \text { for } x \in I_{k} \cup \bigcup_{j=-\infty}^{k-1} J_{j} \\
\mathcal{I}\left(\rho^{-k} \mathcal{I}(x)\right) & \text { for } x \in \bigcup_{j=k}^{\infty} J_{j} \backslash I_{k}
\end{array},\right. \\
& f_{+}(x)= \begin{cases}\rho^{-k} x & \text { for } x \in \bigcup_{j=-\infty}^{-k} J_{j} \backslash I_{-k} \\
\mathcal{I}\left(\rho^{l} \mathcal{I}(x)\right) & \text { for } x \in I_{-k} \cup \bigcup_{j=-k+1}^{\infty} J_{j}\end{cases}
\end{aligned}
$$

Moreover, for $\left(j, j_{1}, \ldots, j_{n}\right) \in \mathcal{J}$, we have:

$$
\begin{aligned}
f_{+}\left(I_{j, j_{1}, \ldots, j_{n}}\right) & =I_{j+k, j_{1}, \ldots, j_{n}} & & \text { for } j<-k, \\
f_{+}\left(I_{-k, j_{1}, \ldots, j_{n}}\right) & =I_{j_{1}, j_{2}, \ldots, j_{n}} & & \text { for } n>0, \\
f_{+}\left(\operatorname{conv}\left(I_{-k} \cup \bigcup_{j=-k+1}^{-l} J_{j}\right)\right) & =I_{l}, & & \\
f_{+}\left(I_{j, j_{1}, \ldots, j_{n}}\right) & =I_{l,-j, j_{1} \ldots, j_{n}} & & \text { for }-l+1 \leqslant j \leqslant-1, \\
f_{+}\left(I_{j, j_{1}, \ldots, j_{n}}\right) & =I_{j+l, j_{1}, \ldots, j_{n}} & & \text { for } j>0 .
\end{aligned}
$$

Analogously,

$$
\begin{aligned}
f_{-}\left(I_{j, j_{1}, \ldots, j_{n}}\right) & =I_{j-l, j_{1}, \ldots, j_{n}} & & \text { for } j<0, \\
f_{-}\left(I_{j, j_{1}, \ldots, j_{n}}\right) & =I_{-l, j, j_{1} \ldots, j_{n}} & & \text { for } 1 \leqslant j \leqslant l-1, \\
f_{-}\left(\operatorname{conv}\left(I_{k} \cup \bigcup_{j=l}^{k-1} J_{j}\right)\right) & =I_{-l}, & & \\
f_{-}\left(I_{k, j_{1}, \ldots, j_{n}}\right) & =I_{-j_{1}, j_{2}, \ldots, j_{n}} & & \text { for } n>0, \\
f_{-}\left(I_{j, j_{1}, \ldots, j_{n}}\right) & =I_{j-k, j_{1}, \ldots, j_{n}} & & \text { for } j>k .
\end{aligned}
$$

In particular, Lemma 4.51 implies $f(I) \subset I$. More precisely, for every $i \in\{-,+\}$ and $\mathbf{j} \in \mathcal{J}$,

$$
f_{i}\left(I_{\mathbf{j}}\right) \subset I_{\mathbf{j}^{\prime}} \quad \text { for some } \mathbf{j}^{\prime}=\mathbf{j}^{\prime}(i, \mathbf{j}) \in \mathcal{J}
$$

Let

$$
\mathcal{R}=\{l, \ldots, k\} \cup \bigcup_{n=1}^{\infty}\{l, \ldots, k-1\} \times\{1, \ldots, l-1\}^{n}
$$

We will denote the elements of $\mathcal{R}$ by $\mathbf{r}=\left(r, r_{1}, \ldots, r_{n}\right), n \geqslant 0$, where $r \in\{l, \ldots, k\}$ in the case $n=0, r \in\{l, \ldots, k-1\}$ in the case $n>0$ and $r_{1}, \ldots, r_{n} \in\{1, \ldots, l-1\}$, with the convention that $r_{1}, \ldots, r_{n}$ for $n=0$ is the empty sequence. Note that

$$
\mathcal{R} \subset \mathcal{J}
$$

For $\mathbf{r}=\left(r, r_{1}, \ldots, r_{n}\right) \in \mathcal{R}$ define the maps

$$
\Phi_{\mathbf{r}}= \begin{cases}\phi_{r} & \text { for } n=0 \\ \phi_{r} \circ \phi_{r_{1}} \circ \cdots \circ \phi_{r_{n}} & \text { for } n>0\end{cases}
$$



Figure 4.5: A schematic view of the action of $f_{+}$on the intervals $I_{\mathbf{j}}$.
on the interval $I_{-1}$. By Lemma 4.49,

$$
\Phi_{\mathbf{r}}: I_{-1} \rightarrow I_{-1}, \quad \mathbf{r} \in \mathcal{R},
$$

so the family $\left\{\Phi_{\mathbf{r}}\right\}_{\mathbf{r} \in \mathcal{R}}$ is a countable infinite iterated function system of contractions in $I_{-1}$ satisfying $\lim _{s \rightarrow \infty}\left|\phi_{\mathbf{r}_{s}}\left(I_{-1}\right)\right|=0$ for any sequence $\left(\mathbf{r}_{s}\right)_{s=1}^{\infty}$ of mutually distinct elements of $\mathcal{R}$. Moreover, the definition of $\Phi_{\mathbf{r}}$ implies

$$
\left\{\Phi_{\mathbf{r}}\left(I_{-1}\right)\right\}_{\mathbf{r} \in \mathcal{R}}=\left\{\phi_{r}\left(I_{\mathbf{j}}\right): r \in\{l, \ldots, k-1\}, \mathbf{j} \in \mathcal{J}_{-1}\right\} \cup\left\{\phi_{k}\left(I_{-1}\right)\right\} .
$$

This together with Lemma 4.49 implies that $\Phi_{\mathbf{r}}\left(I_{-1}\right), \mathbf{r} \in \mathcal{R}$, are pairwise disjoint. Similarly as before, we are interested in the limit set of this system. As the family $\left\{\Phi_{\mathbf{r}}\right\}_{\mathbf{r} \in \mathcal{R}}$ is infinite, there are two limit sets one can consider:

$$
\begin{equation*}
L=\bigcap_{m=1}^{\infty} \bigcup_{\mathbf{r}_{1}, \ldots, \mathbf{r}_{m} \in \mathcal{R}} \Phi_{\mathbf{r}_{1}} \circ \cdots \circ \Phi_{\mathbf{r}_{m}}\left(I_{-1}\right) \tag{4.27}
\end{equation*}
$$

and its closure

$$
\Lambda_{-1}=\bar{L}
$$

It is easy to see that they satisfy

$$
L=\bigcup_{\mathbf{r} \in \mathcal{R}} \Phi_{\mathbf{r}}(L), \quad \Lambda_{-1}=\overline{\bigcup_{\mathbf{r} \in \mathcal{R}} \Phi_{\mathbf{r}}\left(\Lambda_{-1}\right)}=\bigcap_{m=1}^{\infty} \overline{\bigcup_{\mathbf{r}_{1}, \ldots, \mathbf{r}_{m} \in \mathcal{R}} \Phi_{\mathbf{r}_{1}} \circ \cdots \circ \Phi_{\mathbf{r}_{m}}\left(I_{-1}\right)}
$$

(see e.g. [64, Section 2]). As our goal is to find the minimal attractor of the system $\left\{f_{-}, f_{+}\right\}$ (which equals also the support of $\mu$ ), we will focus on $\Lambda_{-1}$. However, we will use the set $L$ in the proof of Proposition 4.62, as it is better suited for calculating the Hausdorff dimension.

For $\mathbf{j}=\left(j, j_{1}, \ldots, j_{n}\right) \in \mathcal{J}$ let

$$
\Lambda_{\mathbf{j}}=\Lambda_{j, j_{1}, \ldots, j_{n}}=\left\{\begin{array}{ll}
\rho^{-j-1} \phi_{j_{1}, \ldots, j_{n}}\left(\Lambda_{-1}\right) & \text { for } j<0 \\
\mathcal{I}\left(\rho^{j-1} \phi_{j_{1}, \ldots, j_{n}}\left(\Lambda_{-1}\right)\right) & \text { for } j>0
\end{array}, \quad \Lambda=\overline{\bigcup_{\mathbf{j} \in \mathcal{J}} \Lambda_{\mathbf{j}}} \cap(0,1)\right.
$$

where we write

$$
\phi_{j_{1}, \ldots, j_{n}}=\phi_{j_{1}} \circ \cdots \circ \phi_{j_{n}} .
$$

Obviously, $\Lambda_{\mathbf{j}} \subset I_{\mathbf{j}}$ for $j \in \mathcal{J}$ and $\Lambda \subset \bigcup_{j \in \mathbb{Z} *} J_{j}$. Furthermore, for $m \geqslant 0$ and $\mathbf{r}_{1}, \ldots, \mathbf{r}_{m} \in \mathcal{R}$ let

$$
I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}= \begin{cases}\rho^{-j-1} \phi_{j_{1}, \ldots, j_{n}}\left(\Phi_{\mathbf{r}_{1}} \circ \cdots \circ \Phi_{\mathbf{r}_{m}}\left(I_{-1}\right)\right) & \text { for } j<0 \\ \mathcal{I}\left(\rho^{j-1} \phi_{j_{1}, \ldots, j_{n}}\left(\Phi_{\mathbf{r}_{1}} \circ \cdots \circ \Phi_{\mathbf{r}_{m}}\left(I_{-1}\right)\right)\right) & \text { for } j>0\end{cases}
$$

(for $m=0$ the set $I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}$ is equal to $I_{\mathbf{j}}$ ). As $\left|\Phi_{\mathbf{r}}^{\prime}\right| \leqslant \rho$, for $\mathbf{j} \in \mathcal{J}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots \in \mathcal{R}$ we have

$$
\left|I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right| \leqslant \rho^{m} \rightarrow 0 \text { as } m \rightarrow \infty
$$

SO

$$
\bigcap_{m=1}^{\infty} I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}=\left\{x_{\mathbf{j} ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right\}
$$

for a point $x_{\mathbf{j} ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} \in \Lambda$ and

$$
\Lambda=\overline{\bigcup_{\mathbf{j} \in \mathcal{J}} \bigcap_{m=1}^{\infty} \overline{\bigcup_{\mathbf{r}_{1}, \ldots, \mathbf{r}_{m} \in \mathcal{R}}} I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}} \cap(0,1)=\overline{\left\{x_{\mathbf{j} ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}: \mathbf{j} \in \mathcal{J}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots \in \mathcal{R}\right\}} \cap(0,1) .
$$

## Description of trajectories

Lemma 4.51 implies the following.
Lemma 4.52. For $\left(j, j_{1}, \ldots, j_{n}\right) \in \mathcal{J}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots, \in \mathcal{R}$ and $m \geqslant 0$, we have:

$$
\begin{gathered}
f_{-}\left(I_{\left(j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right)= \begin{cases}I_{\left(j-l, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j<0 \\
I_{\left(-l, j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } 1 \leqslant j \leqslant l-1 \\
I_{-l ;\left(j, j_{1}, \ldots, j_{n}\right), \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } l \leqslant j \leqslant k-1 \text { or } j=k, n=0, \\
I_{\left(-j_{1}, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j=k, n>0 \\
I_{\left(j-k, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j>k\end{cases} \\
f_{+}\left(I_{\left.\left(j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}\right)}= \begin{cases}I_{\left(j+k, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j<-k \\
I_{\left(j_{1}, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j=-k, n>0 \\
I_{l ;\left(-j, j_{1}, \ldots, j_{n}\right), \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j=-k, n=0 \text { or }-k+1 \leqslant j \leqslant-l, \\
I_{\left(l,-j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for }-l+1 \leqslant j \leqslant-1 \\
I_{\left(j+l, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j>0\end{cases} \right.
\end{gathered}
$$

and

$$
\begin{gathered}
f_{-}\left(x_{\left(j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right)= \begin{cases}x_{\left(j-l, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} & \text { for } j<0 \\
x_{\left(-l, j_{1}, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} & \text { for } 1 \leqslant j \leqslant l-1 \\
x_{-l ;\left(j, j_{1}, \ldots, j_{n}\right), \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} & \text { for } l \leqslant j \leqslant k-1 \text { or } j=k, n=0, \\
x_{\left(-j_{1}, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} & \text { for } j=k, n>0 \\
x_{\left(j-k, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} & \text { for } j>k\end{cases} \\
f_{+}\left(x_{\left(j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right)=\left\{\begin{array}{lll}
x_{\left(j+k, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}, & \text { for } j<-k \\
x_{\left(j_{1}, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} & \text { for } j=-k, n>0 \\
x_{l ;\left(-j, j_{1}, \ldots, j_{n}\right), \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} & \text { for } j=-k, n=0 \text { or }-k+1 \leqslant j \leqslant-l . \\
x_{\left(l,-j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} & \text { for }-l+1 \leqslant j \leqslant-1 \\
x_{\left(j+l, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} & \text { for } j>0
\end{array}\right.
\end{gathered}
$$

The next lemma follows directly from Lemma 4.51 .
Lemma 4.53. The following statements hold.
(a) If a trajectory $\left\{f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{N=0}^{\infty}$, for $x \in(0,1)$, jumps over the central interval at the time $s$, for $s \geqslant 0$, then $f_{i_{s+1}} \circ \cdots \circ f_{i_{1}}(x) \in I_{-l} \cup I_{l}$.
(b) A trajectory $\left\{f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{N=0}^{\infty}$, for $x \in J$, jumps over the central interval at the time $s$, for $s \geqslant 0$, if and only if

$$
\begin{aligned}
& f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x) \in I_{-k} \cup \bigcup_{j=-k+1}^{-l} J_{j}, \quad i_{s+1}=+, \quad f_{i_{s+1}} \circ \cdots \circ f_{i_{1}}(x) \in I_{l} \\
& \text { or } \\
& f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x) \in I_{k} \cup \bigcup_{j=l}^{k-1} J_{j}, \quad i_{s+1}=-, \quad f_{i_{s+1}} \circ \cdots \circ f_{i_{1}}(x) \in I_{-l} .
\end{aligned}
$$

In particular, for given $\mathbf{j} \in \mathcal{J}$ and $i_{1}, i_{2}, \ldots \in\{-,+\}$, for all $x \in I_{\mathbf{j}}$ the trajectories $\left\{f_{i_{N}} \circ\right.$ $\left.\cdots \circ f_{i_{1}}(x)\right\}_{N=0}^{\infty}$ jump over the central interval at the same times.

Since $k, l$ are relatively prime, there exist $N_{1}, N_{2}>0$ such that $N_{1} l-N_{2} k=1$. Let

$$
F_{-}=f_{+}^{N_{2}} \circ f_{-}^{N_{1}}, \quad F_{+}=f_{-}^{N_{2}} \circ f_{+}^{N_{1}} .
$$

Then

$$
\begin{array}{lll}
F_{-}\left(J_{j}\right)=J_{j-1}, & F_{-}(x)=\rho x & \text { for } x \in J_{j}, j<0, \\
F_{+}\left(J_{j}\right)=J_{j+1}, & F_{+}(x)=\mathcal{I} \rho \mathcal{I}(x) & \text { for } x \in J_{j}, j>0 .
\end{array}
$$

For $j, j^{\prime} \in \mathbb{Z}^{*}$ such that $\operatorname{sgn}(j)=\operatorname{sgn}\left(j^{\prime}\right)$, define

$$
F_{j, j^{\prime}}: J_{j} \xrightarrow[\text { onto }]{\longrightarrow} J_{j^{\prime}}, \quad F_{j, j^{\prime}}= \begin{cases}\left.F_{-}^{j-j^{\prime}}\right|_{J_{j}} & \text { for } j<0, j^{\prime} \leqslant j \\ \left.f_{+}^{\left\lceil\left(j^{\prime}-j\right) / k\right\rceil} \circ F_{-}^{j-j^{\prime}-k\left\lceil\left(j^{\prime}-j\right) / k\right\rceil}\right|_{J_{j}} & \text { for } j>0, j^{\prime}>j \\ \left.F_{+}^{j^{\prime}-j}\right|_{J_{j}} & \text { for } j<0, j^{\prime} \geqslant j \\ \left.f_{-}^{\left\lceil\left(j-j^{\prime}\right) / k\right]} \circ F_{+}^{j^{\prime}-j-k\left\lceil\left(j-j^{\prime}\right) / k\right\rceil}\right|_{J_{j}} & \text { for } j>0, j^{\prime}<j\end{cases}
$$

We have $F_{j, j^{\prime}}=\left.f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{J_{j}}$ for some $i_{1}, \ldots, i_{N} \in\{-,+\}, N \geqslant 0$, and, by Lemma 4.51,

$$
F_{j, j^{\prime}}(x)= \begin{cases}\rho^{j-j^{\prime}} x & \text { for } j<0  \tag{4.28}\\ \mathcal{I}\left(\rho^{-j+j^{\prime}} \mathcal{I}(x)\right) & \text { for } j>0\end{cases}
$$

for $x \in J_{j}$. In particular, this implies

$$
F_{j, j}=\left.\operatorname{id}\right|_{J_{j}}, \quad F_{j^{\prime}, j^{\prime \prime}} \circ F_{j, j^{\prime}}=F_{j, j^{\prime \prime}}
$$

for $j, j^{\prime}, j^{\prime \prime} \in \mathbb{Z}^{*}$ such that $\operatorname{sgn}(j)=\operatorname{sgn}\left(j^{\prime}\right)=\operatorname{sgn}\left(j^{\prime \prime}\right)$. By Lemma 4.52,

$$
\begin{align*}
F_{j, j^{\prime}}\left(I_{\left(j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right) & =I_{\left(j^{\prime}, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}},  \tag{4.29}\\
F_{j, j^{\prime}}\left(x_{\left(j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right) & =x_{\left(j^{\prime}, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}
\end{align*}
$$

for $j_{1}, \ldots, j_{n} \in\{1, \ldots, l-1\}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots \in \mathcal{R}, n, m \geqslant 0$.
Lemma 4.54. For every $x \in(0,1)$ there exists $i_{1}, \ldots, i_{n} \in\{-,+\}, n \geqslant 0$, such that $f_{i_{n}} \circ \cdots \circ f_{i_{1}}(x) \in I$.

Proof. If $x \in J_{j}$ for $j<0$ (resp. $j>0$ ), then it is enough to notice that by Lemma 4.51, we have $f_{+} \circ F_{j,-l}(x) \in I_{l}$ (resp. $f_{-} \circ F_{j, l}(x) \in I_{-l}$ ). Suppose $x \in(0,1) \backslash J$. Enumerate the components of $(0,1) \backslash J$ by $U_{j}, j \in \mathbb{Z}$, such that $U_{j}$ is the gap between $J_{j-1}$ and $J_{j}$ for $j<0$, $U_{0}$ is the gap between $J_{-1}$ and $J_{1}$, and $U_{j}$ is the gap between $J_{j}$ and $J_{j+1}$ for $j>0$. Since the system is symmetric, we can assume $x \in U_{j}, j \leqslant 0$. Then, by Lemma 4.51, we have $f_{-} \circ f_{+}^{\left\lfloor\frac{-j}{k}\right\rfloor+1}(x) \in I_{-l}$.

Define

$$
G_{j}^{-}: J_{1} \rightarrow J_{-1}, \quad G_{j}^{+}: J_{-1} \rightarrow J_{1}, \quad j \in\{1, \ldots, l-1\},
$$

by

$$
G_{j}^{-}=F_{-l,-1} \circ f_{-} \circ F_{1, j}, \quad G_{j}^{+}=F_{l, 1} \circ f_{+} \circ F_{-1,-j}
$$

Note that $G_{j}^{ \pm}=\left.f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{J_{\mp 1}}$ for some $i_{1}, \ldots, i_{N} \in\{-,+\}, N \geqslant 0$. By (4.28), we have

$$
\begin{equation*}
G_{j}^{-}=\left.\phi_{j} \circ \mathcal{I}\right|_{J_{1}}, \quad G_{j}^{+}=\mathcal{I} \circ \phi_{j} \tag{4.30}
\end{equation*}
$$

and by Lemma 4.52 and 4.29),

$$
\begin{align*}
G_{j}^{-}\left(I_{\left(1, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right) & =I_{\left(-1, j, j_{1}, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}, \\
G_{j}^{+}\left(I_{\left(-1, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right) & =I_{\left(1, j, j_{1}, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}},  \tag{4.31}\\
G_{j}^{-}\left(x_{\left(1, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right) & =x_{\left(-1, j, j_{1}, j_{2}, \ldots, \ldots j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}, \\
G_{j}^{+}\left(x_{\left(-1, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots},\right. & =x_{\left(1, j, j j_{1}, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots},
\end{align*}
$$

for $j_{1}, \ldots, j_{n} \in\{1, \ldots, l-1\}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots \in \mathcal{R}, n, m \geqslant 0$.
Define also

$$
H_{-}: \bigcup_{\left\{\left(1, j_{1}, \ldots, j_{n}\right) \in \mathcal{J}: n>0\right\}} I_{1, j_{1}, \ldots, j_{n}} \rightarrow J_{-1}, \quad H_{+}: \bigcup_{\left\{\left(-1, j_{1}, \ldots, j_{n}\right) \in \mathcal{J}: n>0\right\}} I_{-1, j_{1}, \ldots, j_{n}} \rightarrow J_{1}
$$

by
$\left.H_{-}\right|_{I_{1, j_{1}, \ldots, j_{n}}}=\left.F_{-j_{1},-1} \circ f_{-} \circ F_{1, k}\right|_{I_{1, j_{1}, \ldots, j_{n}}},\left.\quad H_{+}\right|_{I_{-1, j_{1}, \ldots, j_{n}}}=\left.F_{j_{1}, 1} \circ f_{+} \circ F_{-1,-k}\right|_{I_{-1, j_{1}, \ldots, j_{n}}}$.
Again, $H_{ \pm}=\left.f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{I_{\mp 1, j_{1}, \ldots, j_{n}}}$ for some $i_{1}, \ldots, i_{N} \in\{-,+\}, N \geqslant 0$. By 4.28) and (4.30),

$$
\begin{align*}
\left.H_{-}\right|_{I_{1, j_{1}, \ldots, j_{n}}} & =\left.\left(G_{j_{1}}^{+}\right)^{-1}\right|_{I_{1, j_{1}}, \ldots, j_{n}}=\left.\phi_{j_{1}}^{-1} \circ \mathcal{I}\right|_{I_{1, j_{1}, \ldots, j_{n}}},  \tag{4.32}\\
\left.H_{+}\right|_{I_{-1, j_{1}, \ldots, j_{n}}} & =\left.\left(G_{j_{1}}^{-}\right)^{-1}\right|_{I_{-1, j_{1}}, \ldots, j_{n}}=\left.\mathcal{I} \circ \phi_{j_{1}}^{-1}\right|_{I_{1, j_{1}, \ldots, j_{n}}}
\end{align*}
$$

while Lemma 4.52 and 4.29 give

$$
\begin{align*}
H_{-}\left(I_{\left(1, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right) & =I_{\left(-1, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} \\
H_{+}\left(I_{\left(-1, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right) & =I_{\left(1, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}  \tag{4.33}\\
H_{-}\left(x_{\left(1, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right) & =x_{\left(-1, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} \\
H_{+}\left(x_{\left(-1, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right) & =x_{\left(1, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}
\end{align*}
$$

for $j, j_{1}, \ldots, j_{n} \in\{1, \ldots, l-1\}, n>0, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots \in \mathcal{R}, m \geqslant 0$.
We introduce the following notation. For $\mathbf{j}=\left(j, j_{1}, \ldots, j_{n}\right) \in \mathcal{J}$ we set $\mathbf{j}<0$ (resp. $\mathbf{j}>0$ ) if $j<0$ (resp. $j>0)$. We also write $-\mathbf{j}=\left(-j, j_{1}, \ldots, j_{n}\right)$ and set $\operatorname{sgn}(\mathbf{j})=\operatorname{sgn}(j), n(\mathbf{j})=n$.

For $\mathbf{j}=\left(j, j_{1}, \ldots, j_{n}\right), \mathbf{j}^{\prime}=\left(j^{\prime}, j_{1}^{\prime}, \ldots, j_{n^{\prime}}^{\prime}\right) \in \mathcal{J}, \operatorname{such}$ that $\operatorname{sgn}(\mathbf{j})=\operatorname{sgn}\left(\mathbf{j}^{\prime}\right)$ and $n(\mathbf{j})-n\left(\mathbf{j}^{\prime}\right)$ is even, or $\operatorname{sgn}(\mathbf{j}) \neq \operatorname{sgn}\left(\mathbf{j}^{\prime}\right)$ and $n(\mathbf{j})-n\left(\mathbf{j}^{\prime}\right)$ is odd, define

$$
F_{\mathbf{j}, \mathbf{j}^{\prime}}: I_{\mathbf{j}} \xrightarrow[\text { onto }]{ } I_{\mathbf{j}^{\prime}}
$$

by

$$
F_{j, j j^{\prime}}=\left\{\begin{array}{l}
F_{-1, j^{\prime}} \circ G_{j_{1}^{\prime}}^{-} \circ G_{j_{2}^{\prime}}^{+} \circ \cdots \circ G_{j_{n^{\prime}}^{\prime}}^{-} \circ G_{j_{n^{\prime}}^{\prime}}^{+} \circ\left(H_{-} \circ H_{+}\right)^{n / 2} \circ F_{j,-1} \\
F_{1, j^{\prime}} \circ G_{j_{1}^{\prime}}^{+} \circ G_{j_{2}^{\prime}}^{-} \circ \cdots \circ G_{j_{n^{\prime}-2}}^{+} \circ G_{j_{n^{\prime}-1}^{\prime}}^{-} \circ G_{j_{n^{\prime}}^{\prime}}^{+} \circ\left(H_{-} \circ H_{+}\right)^{n / 2} \circ F_{j,-1} \\
F_{1, j^{\prime}} \circ G_{j_{1}^{\prime}}^{+} \circ G_{j_{2}^{\prime}}^{-} \circ \cdots \circ G_{j_{n^{\prime}-1}^{\prime}}^{+} \circ G_{j_{n^{\prime}}^{\prime}}^{-} \circ H_{+} \circ\left(H_{-} \circ H_{+}\right)^{\lfloor n / 2\rfloor} \circ F_{j,-1} \\
F_{-1, j^{\prime}} \circ G_{j_{1}^{\prime}}^{-} \circ G_{j_{2}^{\prime}}^{+} \circ \cdots \circ G_{j_{n^{\prime}-2}^{\prime}}^{-} \circ G_{j_{n^{\prime}-1}^{\prime}}^{+} \circ G_{j_{n^{\prime}}^{\prime}}^{-} \circ H_{+} \circ\left(H_{-} \circ H_{+}\right)^{\lfloor n / 2\rfloor} \circ F_{j,-1} \\
F_{1, j^{\prime}} \circ G_{j_{1}^{\prime}}^{+} \circ G_{j_{2}^{\prime}}^{-} \circ \cdots \circ G_{j_{n^{\prime}-1}^{\prime}}^{+} \circ G_{j_{n^{\prime}}^{\prime}}^{-} \circ\left(H_{+} \circ H_{-}\right)^{n / 2} \circ F_{j, 1} \\
F_{-1, j^{\prime}} \circ G_{j_{1}^{\prime}}^{-} \circ G_{j_{2}^{\prime}}^{+} \circ \cdots \circ G_{j_{n^{\prime}-2}^{\prime}}^{-} \circ G_{j_{n^{\prime}-1}^{\prime}}^{+} \circ G_{j_{n^{\prime}}^{\prime}}^{-} \circ\left(H_{+} \circ H_{-}\right)^{n / 2} \circ F_{j, 1} \\
F_{-1, j^{\prime}} \circ G_{j_{1}^{\prime}}^{-} \circ G_{j_{2}^{\prime}}^{+} \circ \cdots \circ G_{j_{n^{\prime}-1}^{\prime}}^{-} \circ G_{j_{n^{\prime}}^{\prime}}^{+} \circ H_{-} \circ\left(H_{+} \circ H_{-}\right)^{\lfloor n / 2\rfloor} \circ F_{j, 1} \\
F_{1, j^{\prime}} \circ G_{j_{1}^{\prime}}^{+} \circ G_{j_{2}^{\prime}}^{-} \circ \cdots \circ G_{j_{n^{\prime}-2}^{\prime}}^{+} \circ G_{j_{n^{\prime}-1}^{\prime}}^{-} \circ G_{j_{n^{\prime}}^{\prime}}^{+} \circ H_{-} \circ\left(H_{+} \circ H_{-}\right)^{\lfloor n / 2\rfloor} \circ F_{j, 1}
\end{array}\right.
$$

for

$$
\left\{\begin{array}{l}
\mathbf{j}<0, n(\mathbf{j}) \text { even, } n\left(\mathbf{j}^{\prime}\right) \text { even } \\
\mathbf{j}<0, n(\mathbf{j}) \text { even, } n\left(\mathbf{j}^{\prime}\right) \text { odd } \\
\mathbf{j}<0, n(\mathbf{j}) \text { odd, } n\left(\mathbf{j}^{\prime}\right) \text { even } \\
\mathbf{j}<0, n(\mathbf{j}) \text { odd, } n\left(\mathbf{j}^{\prime}\right) \text { odd } \\
\mathbf{j}>0, n(\mathbf{j}) \text { even, } n\left(\mathbf{j}^{\prime}\right) \text { even } \\
\mathbf{j}>0, n(\mathbf{j}) \text { even, } n\left(\mathbf{j}^{\prime}\right) \text { odd } \\
\mathbf{j}>0, n(\mathbf{j}) \text { odd, } n\left(\mathbf{j}^{\prime}\right) \text { even } \\
\mathbf{j}>0, n(\mathbf{j}) \text { odd, } n\left(\mathbf{j}^{\prime}\right) \text { odd }
\end{array}\right.
$$

respectively. Note that in the case $n=n^{\prime}=0$ the definition of $F_{\mathbf{j}, \mathbf{j}^{\prime}}=F_{j, j^{\prime}}$ agrees with the previous one. We have $F_{\mathbf{j}, \mathbf{j}^{\prime}}=\left.f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{I_{\mathbf{j}}}$ for some $i_{1}, \ldots, i_{N} \in\{-,+\}, N \geqslant 0$.

By 4.28, 4.30 and 4.32,

$$
F_{\mathbf{j}, \mathbf{j}^{\prime}}(x)= \begin{cases}\rho^{-j^{\prime}-1}\left(\phi_{j_{1}^{\prime}, \ldots, j_{n}^{\prime}} \circ \phi_{n_{1}, \ldots, j_{n}}^{-1}\left(\rho^{j+1} x\right)\right) & \text { if } \mathbf{j}<0, \mathbf{j}^{\prime}<0  \tag{4.34}\\ \mathcal{I}\left(\rho^{j^{\prime}-1}\left(\phi_{j_{1}^{\prime}, \ldots, j_{n}^{\prime}}^{\prime} \circ \phi_{j_{1}, \ldots, j_{n}}^{-1}\left(\rho^{j+1} x\right)\right)\right) & \text { if } \mathbf{j}<0, \mathbf{j}^{\prime}>0 \\ \rho^{-j^{\prime}-1}\left(\phi_{j_{1}^{\prime}, \ldots, j_{n^{\prime}}^{\prime}} \circ \phi_{j_{1}, \ldots, j_{n}}^{-1}\left(\rho^{-j+1} \mathcal{I}(x)\right)\right) & \text { if } \mathbf{j}>0, \mathbf{j}^{\prime}<0 \\ \mathcal{I}\left(\rho^{j^{\prime}-1}\left(\phi_{j_{1}^{\prime}, \ldots, j_{n^{\prime}}^{\prime}}^{\prime} \circ \phi_{j_{1}, \ldots, j_{n}}^{-1}\left(\rho^{-j+1} \mathcal{I}(x)\right)\right)\right) & \text { if } \mathbf{j}>0, \mathbf{j}^{\prime}>0\end{cases}
$$

for $x \in I_{\mathbf{j}}$. In particular, this gives

$$
F_{\mathbf{j}, \mathbf{j}}=\left.\operatorname{id}\right|_{I_{\mathbf{j}}}, \quad F_{\mathbf{j}^{\prime}, \mathbf{j}^{\prime \prime}} \circ F_{\mathbf{j}, \mathbf{j}^{\prime}}=F_{\mathbf{j}, \mathbf{j}^{\prime \prime}}
$$

for suitable $\mathbf{j}, \mathbf{j}^{\prime}, \mathbf{j}^{\prime \prime} \in \mathbb{Z}^{*}$. Moreover, (4.29), 4.31) and (4.33) imply

$$
\begin{equation*}
F_{\mathbf{j}, \mathbf{j}^{\prime}}\left(I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right)=I_{\mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}, \quad F_{\mathbf{j}, \mathbf{j}^{\prime}}\left(x_{\mathbf{j} ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right)=x_{\mathbf{j}^{\prime} ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} \tag{4.35}
\end{equation*}
$$

for $\mathbf{r}_{1}, \mathbf{r}_{2}, \ldots \in \mathcal{R}, m \geqslant 0$.
Lemma 4.55. A trajectory $\left\{f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{N=0}^{\infty}$ of a point $x \in I_{\mathbf{j}}, \mathbf{j} \in \mathcal{J}$, does not jump over the central interval at any time $0 \leqslant s<N$, for some $N \geqslant 0$, if and only if

$$
\left.f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{I_{\mathrm{j}}}=F_{\mathrm{j}, \mathrm{j}^{\prime}}
$$

for $\mathbf{j}^{\prime} \in \mathcal{J}$ such that $f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x) \in I_{\mathbf{j}^{\prime}}$, where $\operatorname{sgn}(\mathbf{j})=\operatorname{sgn}\left(\mathbf{j}^{\prime}\right)$ and $n(\mathbf{j})-n\left(\mathbf{j}^{\prime}\right)$ is even, or $\operatorname{sgn}(\mathbf{j}) \neq \operatorname{sgn}\left(\mathbf{j}^{\prime}\right)$ and $n(\mathbf{j})-n\left(\mathbf{j}^{\prime}\right)$ is odd.

Proof. If a trajectory $\left\{f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{N=0}^{\infty}$ of $x \in I_{\mathbf{j}}$ does not jump over the central interval at any time $0 \leqslant s<N$, then by Lemmas 4.51 and 4.53 ,

$$
f_{i_{N}} \circ \cdots \circ f_{i_{1}}\left(I_{\mathbf{j}}\right)=I_{\mathbf{j}^{\prime}}
$$

where $\mathbf{j}^{\prime} \in \mathcal{J}$ such that $\operatorname{sgn}(\mathbf{j})=\operatorname{sgn}\left(\mathbf{j}^{\prime}\right)$ and $n(\mathbf{j})-n\left(\mathbf{j}^{\prime}\right)$ is even, or $\operatorname{sgn}(\mathbf{j}) \neq \operatorname{sgn}\left(\mathbf{j}^{\prime}\right)$ and $n(\mathbf{j})-n\left(\mathbf{j}^{\prime}\right)$ is odd. Consequently, $F_{\mathbf{j}, \mathbf{j}^{\prime}}$ is defined on $I_{\mathbf{j}}$ and $\left.\left(F_{\mathbf{j}, \mathbf{j}^{\prime}}\right)^{-1} \circ f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{I_{\mathbf{j}}}$ is an increasing affine homeomorphism from $I_{\mathrm{j}}$ onto itself, so it is equal to the identity. Therefore, $\left.f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{I_{\mathrm{j}}}=F_{\mathrm{j}_{\mathrm{j}} \mathrm{j}^{\prime}}$. The other implication follows from Lemmas 4.51 and 4.53 and the definitions of the maps $F_{j, j^{\prime}}, G_{j}^{ \pm}, H_{ \pm}$.

Define, for $\mathbf{r} \in \mathcal{R}$,

$$
\begin{array}{lll}
G_{\mathbf{r}}^{+,-}: I_{1} \rightarrow I_{-1}, & G_{\mathbf{r}}^{-,+,}: I_{-1} \rightarrow I_{1} & \text { for } n(\mathbf{r}) \text { even, }, \\
G_{\mathbf{r}}^{-,-}: I_{-1} \rightarrow I_{-1}, & G_{\mathbf{r}}^{+,+}: I_{1} \rightarrow I_{1} & \text { for } n(\mathbf{r}) \text { odd, },
\end{array}
$$

setting

$$
\begin{array}{ll}
G_{\mathbf{r}}^{+,-}=F_{-l,-1} \circ f_{-} \circ F_{1, \mathbf{r}}, & G_{\mathbf{r}}^{-,+}=F_{l, 1} \circ f_{+} \circ F_{-1,-\mathbf{r}}, \\
G_{\mathbf{r}}^{-,-}=F_{-l,-1} \circ f_{-} \circ F_{-1, \mathbf{r}}, & G_{\mathbf{r}}^{+,+}=F_{l, 1} \circ f_{+} \circ F_{1,-\mathbf{r}} .
\end{array}
$$

Note that $G_{\mathbf{r}}^{ \pm, \pm}=\left.f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{I_{ \pm 1}}$ for some $i_{1}, \ldots, i_{N} \in\{-,+\}, N \geqslant 0$. By Lemma 4.51 and (4.34), we have

$$
\begin{array}{ll}
G_{\mathbf{r}}^{+,-}=\left.\Phi_{\mathbf{r}} \circ \mathcal{I}\right|_{I_{1}}, & G_{\mathbf{r}}^{-,+}=\mathcal{I} \circ \Phi_{\mathbf{r}},  \tag{4.36}\\
G_{\mathbf{r}}^{-,-}=\Phi_{\mathbf{r}}, & G_{\mathbf{r}}^{+,+}=\left.\mathcal{I} \circ \Phi_{\mathbf{r}} \circ \mathcal{I}\right|_{I_{1}},
\end{array}
$$

while by Lemma 4.52 and (4.35),

$$
\begin{align*}
G_{\mathbf{r}}^{+,-}\left(I_{1 ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right) & =I_{-1 ; \mathbf{r}, \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}, & G_{\mathbf{r}}^{+,-}\left(x_{1 ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right) & =x_{-1 ; \mathbf{r}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots} \\
G_{\mathbf{r}}^{-},+\left(I_{-1 ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right) & =I_{1 ; \mathbf{r}, \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}, & G_{\mathbf{r}}^{-,+}\left(x_{-1 ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right) & =x_{1 ; \mathbf{r}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}  \tag{4.37}\\
G_{\mathbf{r}}^{-,-}\left(I_{-1 ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right) & =I_{-1 ; \mathbf{r}, \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}, & G_{\mathbf{r}}^{-,-}\left(x_{-1 ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right) & =x_{-1 ; \mathbf{r}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}^{+,+}\left(I_{1 ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right)
\end{align*} I_{1 ; \mathbf{r}, \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}, \quad G_{\mathbf{r}}^{+,+}\left(x_{1 ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}, \ldots x_{1 ; \mathbf{r}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}\right.
$$

for $\mathbf{r}_{1}, \mathbf{r}_{2}, \ldots \in \mathcal{R}, m \geqslant 0$.
Lemma 4.56. A trajectory $\left\{f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{N=0}^{\infty}$ of a point $x \in I$ jumps over the central interval at the time $s$, for some $s \geqslant 0$, if and only if one of the four following possibilities:

$$
\begin{array}{lll}
f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x) \in I_{-\mathbf{r}}, & \left.f_{i_{s+1}}\right|_{I_{-\mathbf{r}}}=F_{1, l} \circ G_{\mathbf{r}}^{-,+} \circ F_{-\mathbf{r},-1}, & n(\mathbf{r}) \text { even, } \\
f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x) \in I_{-\mathbf{r}}, & f_{i_{s+1}} I_{-\mathbf{r}}=F_{1, l} \circ G_{\mathbf{r}}^{+,+} \circ F_{-\mathbf{r}, 1}, & n(\mathbf{r}) \text { odd }, \\
f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x) \in I_{\mathbf{r}}, & \left.f_{i_{s+1}}\right|_{I_{\mathbf{r}}}=F_{-1,-l} \circ G_{\mathbf{r}}^{+,-} \circ F_{\mathbf{r}, 1}, & n(\mathbf{r}) \text { even, } \\
f_{i_{s}} \circ \cdots \circ f_{i_{1}}(x) \in I_{\mathbf{r}}, & f_{i_{s+1}} I_{I_{\mathbf{r}}}=F_{-1,-l} \circ G_{\mathbf{r}}^{-,-} \circ F_{\mathbf{r},-1} & n(\mathbf{r}) \text { odd },
\end{array}
$$

holds for some $\mathbf{r} \in \mathcal{R}$.

Proof. Follows directly from Lemma 4.53 .

Lemma 4.57. A trajectory $\left\{f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{N=0}^{\infty}$ of a point $x \in I_{\mathbf{j}}, \mathbf{j} \in \mathcal{J}$, jumps over the central interval (exactly) at the times $s_{1}, \ldots, s_{m}$, for some $0 \leqslant s_{1}<\cdots<s_{m}<N$, $0 \leqslant m \leqslant N$, if and only if

$$
\left.f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{I_{\mathbf{j}}}=F_{t^{\prime}, \mathbf{j}^{\prime}} \circ G_{\mathbf{r}_{1}}^{\sigma_{1}, \sigma_{0}} \circ \cdots \circ G_{\mathbf{r}_{m}}^{\sigma_{m}, \sigma_{m-1}} \circ F_{\mathbf{j}, t}
$$

for some $\mathbf{r}_{1}, \ldots, \mathbf{r}_{m} \in \mathcal{R}$, where $\sigma_{s} \in\{-,+\}$, $s=0, \ldots, m, t, t^{\prime} \in\{-1,1\}$, are defined by backward induction as

$$
\begin{aligned}
\sigma_{m} & =\left\{\begin{array}{ll}
- & \text { if } \mathbf{j}<0, n(\mathbf{j}) \text { is even, or } \mathbf{j}>0, n(\mathbf{j}) \text { is odd } \\
+ & \text { if } \mathbf{j}<0, n(\mathbf{j}) \text { is odd, or } \mathbf{j}>0, n(\mathbf{j}) \text { is even }
\end{array}, \operatorname{sgn}(t)=\sigma_{m}\right. \\
\sigma_{s-1} & =\left\{\begin{array}{ll}
- & \text { if } \sigma_{s}=-, n\left(\mathbf{r}_{s}\right) \text { is odd, or } \sigma_{s}=+, n\left(\mathbf{r}_{s}\right) \text { is even } \\
+ & \text { if } \sigma_{s}=-, n\left(\mathbf{r}_{s}\right) \text { is even, or } \sigma_{s}=+, n\left(\mathbf{r}_{s}\right) \text { is odd }
\end{array} \quad \text { for } s=m, \ldots, 1,\right.
\end{aligned}
$$

and

$$
\operatorname{sgn}(t)=\sigma_{m}, \quad \operatorname{sgn}\left(t^{\prime}\right)=\sigma_{0}
$$

with $\mathbf{j}^{\prime} \in \mathcal{J}$ such that $\operatorname{sgn}\left(\mathbf{j}^{\prime}\right)=\sigma_{0}$ and $n\left(\mathbf{j}^{\prime}\right)$ is even, or $\operatorname{sgn}\left(\mathbf{j}^{\prime}\right) \neq \sigma_{0}$ and $n\left(\mathbf{j}^{\prime}\right)$ is odd. Moreover, in this case we have

$$
f_{i_{N}} \circ \cdots \circ f_{i_{1}}\left(I_{\mathbf{j}}\right)=I_{\mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}
$$

and

$$
\begin{aligned}
& f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x) \\
& = \begin{cases}\rho^{-j^{\prime}-1}\left(\phi_{j_{1}^{\prime}, \ldots, j_{n}^{\prime}} \circ \Phi_{r_{1}} \circ \cdots \circ \Phi_{r_{m}} \circ \phi_{j_{1}, \ldots, j_{n}}^{-1}\left(\rho^{j+1} x\right)\right) & \text { for } \mathbf{j}<0, \mathbf{j}^{\prime}<0 \\
\mathcal{I}\left(\rho^{j^{\prime}-1}\left(\phi_{j_{1}^{\prime}, \ldots, j_{n}^{\prime}} \circ \Phi_{r_{1}} \circ \cdots \circ \Phi_{r_{m}} \circ \phi_{j_{1}, \ldots, j_{n}}^{-1}\left(\rho^{j+1} x\right)\right)\right) & \text { for } \mathbf{j}<0, \mathbf{j}^{\prime}>0 \\
\rho^{-j^{\prime}-1}\left(\phi_{j_{1}^{\prime}, \ldots, j_{n}^{\prime}} \circ \Phi_{r_{1}} \circ \cdots \circ \Phi_{r_{m}} \circ \phi_{j_{1}, \ldots, j_{n}}^{-1}\left(\rho^{-j+1} \mathcal{I}(x)\right)\right) & \text { for } \mathbf{j}>0, \mathbf{j}^{\prime}<0 \\
\mathcal{I}\left(\rho^{j^{\prime}-1}\left(\phi_{j_{1}^{\prime}, \ldots, j_{n}^{\prime}} \circ \Phi_{r_{1}} \circ \cdots \circ \Phi_{r_{m}} \circ \phi_{j_{1}, \ldots, j_{n}}^{-1}\left(\rho^{-j+1} \mathcal{I}(x)\right)\right)\right) & \text { for } \mathbf{j}>0, \mathbf{j}^{\prime}>0\end{cases}
\end{aligned}
$$

where $\mathbf{j}=\left(j, j_{1}, \ldots, j_{n}\right), \mathbf{j}^{\prime}=\left(j^{\prime}, j_{1}^{\prime}, \ldots, j_{n^{\prime}}^{\prime}\right)$.

Proof. The definitions of $\sigma_{s}, t, t^{\prime}$ and the conditions for $\mathbf{j}^{\prime}$ imply that all the considered maps are well-defined. The assertions of the lemma follow directly from Lemmas 4.55 and 4.56, and (4.34), 4.35), 4.36), 4.37).

Lemma 4.58. For every $\mathbf{j}, \mathbf{j}^{\prime} \in \mathcal{J}, m \geqslant 0$ and $\mathbf{r}_{1}, \ldots, \mathbf{r}_{m} \in \mathcal{R}$, there exists a map

$$
\mathbf{F}_{\mathbf{j}_{\mathbf{j}} \mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}: I_{\mathbf{j}} \rightarrow I_{\mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}
$$

such that $\mathbf{F}_{\mathbf{j}, \mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}=\left.f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{I_{\mathbf{j}}}$ for some $i_{1}, \ldots, i_{N} \in\{-,+\}, N \geqslant 0$ and any trajectory of $x \in \mathcal{J}$ defined by $\cdots \circ f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x)$ jumps over the central interval at the times $s_{1}, \ldots, s_{m+1}$, for some $0 \leqslant s_{1}<\cdots<s_{m+1}<N$.

Proof. Since the system is symmetric, we can assume $\mathbf{j}<0$.
Let
$t=\left\{\begin{array}{ll}-1 & \text { if } n(\mathbf{j}) \text { is even } \\ 1 & \text { if } n(\mathbf{j}) \text { is odd }\end{array}, \quad t^{\prime}= \begin{cases}-1 & \text { if } \mathbf{j}^{\prime}<0, n\left(\mathbf{j}^{\prime}\right) \text { is even, or } \mathbf{j}^{\prime}>0, n\left(\mathbf{j}^{\prime}\right) \text { is odd } \\ 1 & \text { if } \mathbf{j}^{\prime}<0, n\left(\mathbf{j}^{\prime}\right) \text { is odd, or } \mathbf{j}^{\prime}>0, n\left(\mathbf{j}^{\prime}\right) \text { is even }\end{cases}\right.$
and

$$
p=\#\left\{s \in\{1, \ldots, m\}: n\left(\mathbf{r}_{s}\right) \text { is even }\right\}
$$

Define $\mathbf{r}_{m+1} \in \mathcal{R}$ by

$$
\mathbf{r}_{m+1}= \begin{cases}l & \text { if } t=t^{\prime}, p \text { is odd, or } t \neq t^{\prime}, p \text { is even } \\ (l, 1) & \text { if } t=t^{\prime}, p \text { is even, or } t \neq t^{\prime}, p \text { is odd }\end{cases}
$$

We have

$$
n\left(\mathbf{r}_{m+1}\right) \text { is } \begin{cases}\text { even } & \text { if } t=t^{\prime}, p \text { is odd, or } t \neq t^{\prime}, p \text { is even }  \tag{4.38}\\ \text { odd } & \text { if } t=t^{\prime}, p \text { is even, or } t \neq t^{\prime}, p \text { is odd }\end{cases}
$$

Furthermore, define $\sigma_{s} \in\{-,+\}, s=0, \ldots, m+1$ by

$$
\begin{aligned}
\sigma_{m+1} & =\operatorname{sgn}(t) \\
\sigma_{s-1} & =\left\{\begin{array}{ll}
- & \text { if } \sigma_{s}=-, n\left(\mathbf{r}_{s}\right) \text { is odd, or } \sigma_{s}=+, n\left(\mathbf{r}_{s}\right) \text { is even } \\
+ & \text { if } \sigma_{s}=-, n\left(\mathbf{r}_{s}\right) \text { is even, or } \sigma_{s}=+, n\left(\mathbf{r}_{s}\right) \text { is odd }
\end{array} \quad \text { for } s=m+1, \ldots, 1\right.
\end{aligned}
$$

By the definition of $t$, we have

$$
\sigma_{m+1}= \begin{cases}- & \text { if } n(\mathbf{j}) \text { is even } \\ + & \text { if } n(\mathbf{j}) \text { is odd }\end{cases}
$$

Note that $\sigma_{s-1} \neq \sigma_{s}$ if and only if $n\left(\mathbf{r}_{s}\right)$ is even. Therefore, as $\sigma_{m+1}=\operatorname{sgn}(t)$ we obtain

$$
\sigma_{0}= \begin{cases}\operatorname{sgn}(t) & \text { if } p \text { is even, } n\left(\mathbf{r}_{m+1}\right) \text { is odd, or } p \text { is odd, } n\left(\mathbf{r}_{m+1}\right) \text { is even } \\ -\operatorname{sgn}(t) & \text { if } p \text { is even, } n\left(\mathbf{r}_{m+1}\right) \text { is even, or } p \text { is odd, } n\left(\mathbf{r}_{m+1}\right) \text { is odd }\end{cases}
$$

where $-\operatorname{sgn}(t)=-($ resp. +$)$ if $\operatorname{sgn}(t)=+($ resp. -$)$. This together with 4.38) implies

$$
\sigma_{0}=\operatorname{sgn}\left(t^{\prime}\right)
$$

Moreover, by the definition of $t^{\prime}$, we have $\operatorname{sgn}\left(\mathbf{j}^{\prime}\right)=\sigma_{0}$ and $n\left(\mathbf{j}^{\prime}\right)$ is even, or $\operatorname{sgn}\left(\mathbf{j}^{\prime}\right) \neq \sigma_{0}$ and $n\left(\mathbf{j}^{\prime}\right)$ is odd. This implies that if we define

$$
\mathbf{F}_{\mathbf{j}, \mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}=F_{t^{\prime}, \mathbf{j}^{\prime}} \circ G_{\mathbf{r}_{1}}^{\sigma_{1}, \sigma_{0}} \circ \cdots \circ G_{\mathbf{r}_{m+1}}^{\sigma_{m+1}, \sigma_{m}} \circ F_{\mathbf{j}, t}
$$

then by Lemma 4.57 (with $m$ replaced by $m+1$ ), $\mathbf{F}_{\mathbf{j}, \mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}$ is well-defined on $I_{\mathbf{j}}$ and $\mathbf{F}_{\mathbf{j}, \mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}=\left.f_{i_{N}} \circ \cdots \circ f_{i_{1}}\right|_{I_{\mathbf{j}}}$ for some $i_{1}, \ldots, i_{N} \in\{-,+\}, N \geqslant 0$. Moreover, any trajectory of $x \in \mathcal{J}$ defined by $\cdots \circ f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x)$ jumps over the central interval at the times $s_{1}, \ldots, s_{m+1}$, for some $0 \leqslant s_{1}<\cdots<s_{m+1}<N$. By 4.35) and 4.37,

$$
\mathbf{F}_{\mathbf{j}, \mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\left(I_{\mathbf{j}}\right)=I_{\mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m+1}} \subset I_{\mathbf{j}^{\prime} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}
$$

Proposition 4.59. For every $x \in(0,1)$,

$$
\omega_{\infty}(x)=\bar{\Lambda}=\Lambda \cup\{0,1\}
$$

Proof. First, we prove $\omega_{\infty}(x) \subset \Lambda \cup\{0,1\}$ for $x \in(0,1)$. By Lemma 4.53(a), we can assume $x \in I$. Take $y \in \omega_{\infty}(x)$. We have $y=\lim _{s \rightarrow \infty} f_{i_{N_{s}}} \circ \cdots \circ f_{i_{1}}(x)$, where $N_{s} \rightarrow \infty$ and the trajectory $\left\{f_{i_{N}} \circ \cdots \circ f_{i_{1}}(x)\right\}_{N=0}^{\infty}$ jumps over the central interval infinitely many times. By Lemma 4.57,

$$
f_{i_{N_{s}}} \circ \cdots \circ f_{i_{1}}(x) \in I_{\mathbf{j}(s) ; \mathbf{r}_{1}(s), \ldots, \mathbf{r}_{m(s)}(s)}
$$

for some $\mathbf{j}(s) \in \mathcal{J}$ and $\mathbf{r}_{1}(s), \ldots, \mathbf{r}_{m(s)}(s) \in \mathcal{R}$, where $m(s) \rightarrow \infty$ as $s \rightarrow \infty$. Moreover, $\left|I_{\mathbf{j}(s) ; \mathbf{r}_{1}(s), \ldots, \mathbf{r}_{m(s)}(s)}\right| \leqslant \rho^{m(s)} \rightarrow 0$ as $s \rightarrow \infty$ and $I_{\mathbf{j}(s) ; \mathbf{r}_{1}(s), \ldots, \mathbf{r}_{m(s)}(s)} \cap \Lambda \neq 0$. Hence, $y \in \bar{\Lambda}=$ $\Lambda \cup\{0,1\}$, which shows $\omega_{\infty}(x) \subset \Lambda \cup\{0,1\}$.

Now we prove $\Lambda \cup\{0,1\} \subset \omega_{\infty}(x)$ for $x \in(0,1)$. By Lemma 4.54, we can assume $x \in I_{\mathrm{j}}$
 $\mathbf{r}_{1}(s), \mathbf{r}_{2}(s), \ldots \in \mathcal{R}, s \geqslant 0$. Using Lemma 4.58, define inductively

$$
\begin{aligned}
F^{(0)} & =\mathbf{F}_{\mathbf{j} \mathbf{j}^{\prime}(0)}, \\
F^{(s)} & =\mathbf{F}_{\mathbf{j}^{\prime}(s-1), \mathbf{j}^{\prime}(s) ; \mathbf{r}_{1}(s), \ldots, \mathbf{r}_{s}(s)} \quad \text { for } s>0 .
\end{aligned}
$$

By Lemma 4.58, the trajectory of $x$ under $\left\{f_{-}, f_{+}\right\}$defined by $\cdots \circ F^{(s)} \ldots \circ F^{(0)}(x)$ is well-defined and jumps over the central interval infinitely many times. Moreover,

$$
F^{(s)} \circ \ldots \circ F^{(0)}\left(I_{\mathbf{j}}\right) \subset I_{\mathbf{j}^{\prime}(s) ; \mathbf{r}_{1}(s), \ldots, \mathbf{r}_{s}(s)},
$$

so

$$
\left|F^{(s)} \circ \ldots \circ F^{(0)}(x)-y\right| \leqslant\left|I_{\mathbf{j}^{\prime}(s) ; \mathbf{r}_{1}(s), \ldots, \mathbf{r}_{s}(s)}\right|+\left|y-x_{\mathbf{j}^{\prime}(s) ; \mathbf{r}_{1}(s), \mathbf{r}_{2}(s), \ldots}\right| \rightarrow 0
$$

as $s \rightarrow \infty$, since $\left|I_{\mathbf{j}^{\prime}(s) ; \mathbf{r}_{1}(s), \ldots, \mathbf{r}_{s}(s)}\right| \leqslant \rho^{s} \rightarrow \infty$. Hence, $y$ is a limit point of this trajectory.
Take now $y \in\{0,1\}$. Then, by Lemma 4.58, we see

$$
\begin{array}{r}
\mathbf{F}_{2 s-1,-2 s} \circ \cdots \circ \mathbf{F}_{-2,3} \circ \mathbf{F}_{1,-2} \circ \mathbf{F}_{\mathbf{j}, 1}(x) \in I_{-2 s}, \\
\mathbf{F}_{-2 s, 2 s+1} \circ \mathbf{F}_{2 s-1,-2 s} \circ \cdots \circ \mathbf{F}_{-2,3} \circ \mathbf{F}_{1,-2} \circ \mathbf{F}_{\mathbf{j}, 1}(x) \in I_{2 s+1}
\end{array}
$$

for $s>0$, the trajectory defined by

$$
\cdots \circ \mathbf{F}_{-2 s, 2 s+1} \circ \mathbf{F}_{2 s-1,-2 s} \circ \cdots \circ \mathbf{F}_{-2,3} \circ \mathbf{F}_{1,-2} \circ \mathbf{F}_{\mathbf{j}, 1}(x)
$$

jumps over the central interval infinitely many times and has $y$ as its limit point. Hence, $\Lambda \cup\{0,1\} \subset \omega_{\infty}(x)$.

Proposition 4.60. We have

$$
\Lambda=f_{-}(\Lambda)=f_{+}(\Lambda)
$$

Moreover, the system $\left\{f_{-}, f_{+}\right\}$is minimal in $\Lambda$.

Proof. The first assertion follows directly from Lemma 4.52, while Proposition 4.59 implies minimality.

## Singularity of $\boldsymbol{\mu}$

Proposition 4.61. We have

$$
\operatorname{supp} \mu=\Lambda \cup\{0,1\}, \quad \mu(\Lambda)=1
$$

Proof. Similarly as for the case $l=1$, it is enough to use Proposition 4.60.

## Proposition 4.62.

$$
\operatorname{dim}_{H} \Lambda=\frac{\log \eta}{\log \rho}<1
$$

where $\eta \in(1 / 2,1)$ is the unique solution of the equation $\eta^{k+l}-2 \eta^{k+1}+2 \eta-1=0$.

Proof. Our first goal is to determine the dimension of $\Lambda_{-1}$. We begin with calculating the dimension of the $L$ defined in 4.27). Recall that $\left\{\Phi_{\mathbf{r}}\right\}_{\mathbf{r} \in \mathcal{R}}$ is an iterated function system of contracting similarities on $I_{-1}$, satisfying the Strong Separation Condition. It is well-known (see e.g. [64, Theorem 3.15]) that for such systems $\operatorname{dim}_{H} L$ is equal to the (unique) zero of the topological pressure function

$$
P(t)=\lim _{n \rightarrow \infty} \frac{1}{n} \log \sum_{\mathbf{r}_{1}, \ldots, \mathbf{r}_{n} \in \mathcal{R}}\left\|\left(\Phi_{\mathbf{r}_{1}} \circ \cdots \circ \Phi_{\mathbf{r}_{n}}\right)^{\prime}\right\|^{t}
$$

provided the system is regular (i.e. zero of the pressure function exists). Note that this is an analog of the formula from Theorem 2.15. Since $\Phi_{\mathbf{r}}$ are affine, we have

$$
\begin{align*}
P(t) & =\log \sum_{\mathbf{r} \in \mathcal{R}}\left\|\Phi_{\mathbf{r}}^{\prime}\right\|^{t} \\
& =\log \left(\sum_{r=l}^{k}\left|\phi_{r}^{\prime}\right|^{t}+\sum_{n=1}^{\infty} \sum_{r=l}^{k-1} \sum_{r_{1}, \ldots, r_{n}=1}^{l-1}\left|\left(\phi_{r} \circ \phi_{r_{1}} \circ \cdots \circ \phi_{r_{n}}\right)^{\prime}\right|^{t}\right) \\
& =\log \left(\left|\phi_{k}^{\prime}\right|^{t}+\sum_{r=l}^{k-1}\left|\phi_{r}^{\prime}\right|^{t} \sum_{n=0}^{\infty}\left(\sum_{r_{1}=1}^{l-1}\left|\phi_{r_{1}}^{\prime}\right|^{t}\right)^{n}\right)  \tag{4.39}\\
& =\log \left(\rho^{k t}+\sum_{r=l}^{k-1} \rho^{r t} \sum_{n=0}^{\infty}\left(\sum_{r_{1}=1}^{l-1} \rho^{r_{1} t}\right)^{n}\right) \\
& =\log \frac{\rho^{l t}-2 \rho^{(k+1) t}+\rho^{(k+l) t}}{1-2 \rho^{t}+\rho^{l t}}
\end{align*}
$$

provided $\rho^{t}+\cdots+\rho^{(l-1) t}<1$, which is equivalent to $\rho^{l t}-2 \rho^{t}+1>0$. Since by Lemma 4.49, $\phi_{r}\left(J_{-1}\right), r=1, \ldots, k$ are pairwise disjoint subset of $J_{-1}$, we have $\rho^{t}+\cdots+\rho^{(l-1) t}<1$ for $t=1$. It follows that $\rho^{l t}-2 \rho^{t}+1>0$ for $t \in\left(t_{0}, 1\right]$, where $t_{0}=\inf \{t>0: P(t)<\infty\} \in(0,1)$ is the unique solution of the equation $\rho^{l t_{0}}-2 \rho^{t_{0}}+1=0$. Moreover, the condition $P(1)<0$ is equivalent to

$$
\frac{\rho^{l}-2 \rho^{k+1}+\rho^{k+l}}{1-2 \rho+\rho^{l}}<1
$$

which is the same as 4.24 . Since $t \mapsto P(t)$ is strictly decreasing and continuous whenever it is finite (see [64]) and $\lim _{t \rightarrow t_{0}^{+}} P(t)=+\infty$, we see that there exists $d \in\left(t_{0}, 1\right)$ such that $P(d)=0$. By 4.39), we have $\eta=\rho^{d}$, so

$$
\operatorname{dim}_{H} L=d=\frac{\log \eta}{\log \rho}<1
$$

We will prove now that $\operatorname{dim}_{H} \Lambda_{-1}=\operatorname{dim}_{H} L$, i.e. taking the closure does not increase the Hausdorff dimension of $L$. To that end, let $L(\infty)$ be the "asymptotic boundary" of the system $\left\{\Phi_{\mathbf{r}}\right\}_{\mathbf{r} \in \mathcal{R}}$, i.e. the set of all limit points of sequences $\left(x_{s}\right)_{s=1}^{\infty}$, where $x_{s} \in \Phi_{\mathbf{r}_{s}}\left(I_{-1}\right)$ and $\left\{\mathbf{r}_{s}\right\}_{s=1}^{\infty}$ consists of mutually distinct elements of $\mathcal{R}$. It follows from [64, Lemma 2.1] that

$$
\Lambda_{-1}=\bar{L}=L \cup \bigcup_{m=0}^{\infty} \bigcup_{\mathbf{r}_{1}, \ldots, \mathbf{r}_{m} \in \mathcal{R}} \Phi_{\mathbf{r}_{1}} \circ \cdots \circ \Phi_{\mathbf{r}_{m}}(L(\infty))
$$

As the above sum is countable and the transformations $\Phi_{\mathbf{r}}$ are bi-Lipschitz, applying points 5 and 6 of Proposition 2.4 yields

$$
\operatorname{dim}_{H} \Lambda_{-1}=\max \left\{\operatorname{dim}_{H} L, \operatorname{dim}_{H} L(\infty)\right\}
$$

Using Lemmas 4.49 and 4.50 , it is easy to see that

$$
L(\infty)=\bigcup_{r=l}^{k-1} \phi_{r}(K)
$$

where $K$ is the limit set of the iterated function system $\left\{\phi_{r}\right\}_{r=1}^{l-1}$ on $J_{-1}$. By Lemma 4.49, this system satisfies the Strong Separation Condition, so its box and Hausdorff dimension are both equal to the unique solution $t_{0} \in(0,1)$ of the equation $\rho^{l t_{0}}-2 \rho^{t_{0}}+1=0$ (Theorem 2.15). As noted above, we have $t_{0}<d$, hence $\operatorname{dim}_{H} \Lambda_{-1}=d$. By Lemma 4.50, the sets $\Lambda_{\mathbf{j}}$, $\mathbf{j} \in \mathcal{J}$, are disjoint similar copies of $\Lambda_{-1}$, $\operatorname{so~}_{\operatorname{dim}}^{H} \bigcup_{\mathbf{j} \in \mathcal{J}} \Lambda_{\mathbf{j}}=\operatorname{dim}_{H} \Lambda_{-1}$. To end the proof, note that $\Lambda \backslash \bigcup_{\mathbf{j} \in \mathcal{J}} \Lambda_{\mathbf{j}}=\bigcup_{j>0}\left(\rho^{j} K \cup \mathcal{I}\left(\rho^{j} K\right)\right)$, hence, by the same argument as in the proof of Proposition 4.46 ,

$$
\operatorname{dim}_{H}\left(\Lambda \backslash \bigcup_{\mathbf{j} \in \mathcal{J}} \Lambda_{\mathbf{j}}\right)=t_{0}<d
$$

Finally, this implies $\operatorname{dim}_{H} \Lambda=d$.
The following proposition gives some information about the structure of the measure $\mu$ in the case of equal probabilities $p_{-}, p_{+}$. It states that $\mu$ restricted to $I_{\mathbf{j}}$ for $\mathbf{j} \in \mathcal{J}$ is the self-similar measure (with weight $m_{\mathbf{j}}$ ) for the system $\left\{\phi_{\mathbf{r}}\right\}_{\mathbf{r} \in \mathcal{R}}$ corresponding to some infinite probability vector $\left(\beta_{\mathbf{r}}\right)_{\mathbf{r} \in \mathcal{R}}$. We do not find explicit formulas for $m_{\mathbf{j}}$ and $\beta_{\mathbf{r}}$ in terms of the parameters of the original $A M$-system (the expression for $m_{\mathbf{j}}$ is a general form of a solution of a certain difference equation). Therefore, we are unable to give a formula for $\operatorname{dim}_{H}(\mu)$ in the case $l>1$. On the other hand, this description is sufficient for the proof of Theorem4.7.

Proposition 4.63. Suppose $p_{-}=p_{+}=1 / 2$. Then for $\mathbf{j}=\left(j, j_{1}, \ldots, j_{n}\right) \in \mathcal{J}, \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}$, $m \geqslant 0$, we have

$$
\mu\left(I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right)=m_{\mathbf{j}} \beta_{\mathbf{r}_{1}} \cdots \beta_{\mathbf{r}_{m}}
$$

for

$$
\begin{aligned}
m_{\mathbf{j}}=m_{j, j_{1}, \ldots, j_{n}} & =\mu\left(I_{\mathbf{j}}\right)=A_{1, j_{1}, \ldots, j_{n}} \lambda_{1}^{|j|}+\cdots+A_{p, j_{1}, \ldots, j_{n}} \lambda_{p}^{|j|} \\
& +A_{p+1, j_{1}, \ldots, j_{n}} \lambda_{p+1}^{|j|}+\overline{A_{p+1, j_{1}, \ldots, j_{n}}} \overline{\lambda_{p+1}}{ }^{|j|}+\cdots+A_{q, j_{1}, \ldots, j_{n}} \lambda_{q}^{|j|}+\overline{A_{q, j_{1}, \ldots, j_{n}}} \overline{\lambda_{q}}
\end{aligned}
$$

where $A_{1, j_{1}, \ldots, j_{n}}, \ldots, A_{p, j_{1}, \ldots, j_{n}} \in \mathbb{R}, A_{p+1, j_{1}, \ldots, j_{n}}, \ldots, A_{q, j_{1}, \ldots, j_{n}} \in \mathbb{C}$, moreover $\lambda_{1}, \ldots, \lambda_{p}$ (resp. $\lambda_{p+1}, \overline{\lambda_{p+1}}, \ldots, \lambda_{q}, \overline{\lambda_{q}}$ ) are real (resp. non-real) roots of the polynomial $x^{k+l}-2 x^{l}-1$ of moduli smaller than 1 and

$$
\beta_{\mathbf{r}}=\frac{m_{\mathbf{r}}}{2 m_{l}-m_{l+k}}
$$

Proof. Let $m_{\mathbf{j}}=\mu\left(I_{\mathbf{j}}\right)$ for $\mathbf{j}=\left(j, j_{1}, \ldots, j_{n}\right) \in \mathcal{J}$ and define $\beta_{\mathbf{r}}$ for $\mathbf{r} \in \mathcal{R}$ as in the proposition. Note that the assumption $p_{-}=p_{+}=1 / 2$ and the uniqueness of $\mu$ imply (recall that $-\mathbf{j}=$ $\left(-j, j_{1}, \ldots, j_{n}\right)$ for $\left.\mathbf{j}=\left(j, j_{1}, \ldots, j_{n}\right)\right)$

$$
\begin{equation*}
m_{-\mathbf{j}}=m_{\mathbf{j}} \tag{4.40}
\end{equation*}
$$

Furthermore, by Lemma 4.51 and the stationarity of $\mu$, for every fixed $j_{1}, \ldots, j_{n}$ we have

$$
m_{j+k, j_{1}, \ldots, j_{n}}=2 m_{j, j_{1}, \ldots, j_{n}}-m_{j-l, j_{1}, \ldots, j_{n}}
$$

for every $j \in \mathbb{N}, j \geqslant l+1$. This defines a linear difference equation with characteristic polynomial $x^{k+l}-2 x^{l}-1$. It is well-known (see e.g. [27]) that a solution of such an equation has the form

$$
\begin{aligned}
m_{j, j_{1}, \ldots, j_{n}} & =A_{1, j_{1}, \ldots, j_{n}} \lambda_{1}^{j}+\cdots+A_{p, j_{1}, \ldots, j_{n}} \lambda_{p}^{j} \\
& +A_{p+1, j_{1}, \ldots, j_{n}} \lambda_{p+1}^{j}+{\overline{A_{p+1, j_{1}, \ldots, j_{n}}}{\overline{\lambda_{p+1}}}^{j}+\cdots+A_{q, j_{1}, \ldots, j_{n}} \lambda_{q}^{j}+{\overline{A_{q, j_{1}, \ldots, j_{n}}}}_{\lambda_{q}}{ }^{j}}^{2}
\end{aligned}
$$

$j \in \mathbb{N}$, where $\lambda_{1}, \ldots, \lambda_{p}$ (resp. $\lambda_{p+1}, \overline{\lambda_{p+1}}, \ldots, \lambda_{q}, \overline{\lambda_{q}}$ ) are real (resp. non-real) roots of the characteristic polynomial and $A_{1, j_{1}, \ldots, j_{n}}, \ldots, A_{p, j_{1}, \ldots, j_{n}} \in \mathbb{R}, A_{p+1, j_{1}, \ldots, j_{n}}, \ldots, A_{q, j_{1}, \ldots, j_{n}} \in \mathbb{C}$. Since $\sum_{j=1}^{\infty} m_{j, j_{1}, \ldots, j_{n}} \leqslant \mu(I)=1$, in fact we take into account only the roots of moduli smaller than 1. This proves that $m_{\mathbf{j}}$ has the form described in the proposition.

To show $\mu\left(I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right)=m_{\mathbf{j}} \beta_{\mathbf{r}_{1}} \cdots \beta_{\mathbf{r}_{m}}$, note that by Lemma 4.51 and the stationarity of $\mu$,

$$
m_{l}=\frac{1}{2} \sum_{\mathbf{r} \in \mathcal{R}} m_{\mathbf{r}}+\frac{1}{2} m_{l+k}
$$

which together with Proposition 4.61 implies

$$
\beta_{\mathbf{r}}>0, \quad \sum_{\mathbf{r} \in \mathcal{R}} \beta_{\mathbf{r}}=1
$$

Let $\nu\left(I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right)=m_{\mathbf{j}} \beta_{\mathbf{r}_{1}} \cdots \beta_{\mathbf{r}_{m}}$ for $\mathbf{j} \in \mathcal{J}, \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}, m \geqslant 0$. Since the family of sets $I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}$ generates the $\sigma$-algebra of Borel sets in $\Lambda, \nu$ extends to a Borel probability measure on $\Lambda$. Therefore, by the uniqueness of the stationary measure, to prove the proposition it is sufficient to check that $\nu$ is stationary. It is enough to verify

$$
\begin{equation*}
\nu\left(I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right)=\frac{1}{2} \nu\left(f_{-}^{-1}\left(I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right)\right)+\frac{1}{2} \nu\left(f_{+}^{-1}\left(I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right)\right) \tag{4.41}
\end{equation*}
$$

By Lemma 4.52, for $\mathbf{j}=\left(j, j_{1}, \ldots, j_{n}\right) \in \mathcal{J}$, we have

$$
\begin{align*}
& f_{-}^{-1}\left(I_{\left(j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}\right)= \begin{cases}I_{\left(j+l, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j \leqslant-l-1 \\
I_{\left(j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j=-l, n>0 \\
I_{\mathbf{r}_{1} ; \mathbf{r}_{2}, \ldots, \mathbf{r}_{m}} & \text { for } j=-l, n=0 \\
I_{\left(k,-j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for }-l+1 \leqslant j \leqslant-1 \\
I_{\left(j+k, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j>0\end{cases}  \tag{4.42}\\
& f_{+}^{-1}\left(I_{\left.\left(j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}\right)}= \begin{cases}I_{\left(j-k, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j<0 \\
I_{\left(-k, j, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } 1 \leqslant j \leqslant l-1 \\
I_{-\mathbf{r}_{1} ; \mathbf{r}_{2}, \ldots, \mathbf{r}_{m}} & \text { for } j=l, n=0 \\
I_{\left(-j_{1}, j_{2}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j=l, n>0 \\
I_{\left(j-l, j_{1}, \ldots, j_{n}\right) ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}} & \text { for } j \geqslant l+1\end{cases} \right.
\end{align*}
$$

By (4.40) and 4.42 , the statement 4.41 is equivalent to the systems of equations

$$
\begin{cases}m_{j, j_{1}, j_{2}, \ldots, j_{n}}=\frac{1}{2} m_{k, j, j_{1}, j_{2}, \ldots, j_{n}}+\frac{1}{2} m_{j+k, j_{1}, j_{2}, \ldots, j_{n}} & \text { for } 1 \leqslant j \leqslant l-1 \\ m_{l} \beta_{\mathbf{r}}=\frac{1}{2} m_{\mathbf{r}}+\frac{1}{2} m_{l+k} \beta_{\mathbf{r}} & \\ m_{l, j_{1}, j_{2}, \ldots, j_{n}}=\frac{1}{2} m_{j_{1}, j_{2}, \ldots, j_{n}}+\frac{1}{2} m_{l+k, j_{1}, j_{2}, \ldots, j_{n}} & \text { for } n>0 \\ m_{j, j_{1}, j_{2}, \ldots, j_{n}}=\frac{1}{2} m_{j-l, j_{1}, j_{2}, \ldots, j_{n}}+\frac{1}{2} m_{j+k, j_{1}, j_{2}, \ldots, j_{n}} & \text { for } j \geqslant l+1\end{cases}
$$

where $\left(j, j_{1}, j_{2}, \ldots, j_{n}\right) \in \mathcal{J}$ and $\mathbf{r} \in \mathcal{R}$. The second equation is equivalent to the definition of $\beta_{\mathbf{r}}$ and the remaining ones hold due to $4.40,4.42$ for $m=0$, and the fact that $\mu$ is stationary.

### 4.6 Proof of Theorem 4.15

Let $\Lambda(f)$ and $\Lambda(g)$ be the sets constructed in Section 4.4 (in the case $l=1$ ) or Section 4.5 (in the case $l>1$ ) for the systems $\left\{f_{-}, f_{+}\right\}$and $\left\{g_{-}, g_{+}\right\}$, respectively. Following the notation
used in these sections, we have

$$
\begin{aligned}
& \Lambda(f)=\left\{x_{j ; r_{1}, r_{2}, \ldots}^{(f)}: j \in \mathbb{Z}^{*}, r_{1}, r_{2}, \ldots \in\{1, \ldots, k\}\right\} \\
& \Lambda(g)=\left\{x_{j ; r_{1}, r_{2}, \ldots}^{(g)}: j \in \mathbb{Z}^{*}, r_{1}, r_{2}, \ldots \in\{1, \ldots, k\}\right\}
\end{aligned}
$$

in the case $l=1$ and

$$
\begin{aligned}
& \Lambda(f)=\overline{\left\{x_{\mathbf{j} ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}^{(f)}: \mathbf{j} \in \mathcal{J}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots \in \mathcal{R}\right\}} \cap(0,1), \\
& \Lambda(g)=\overline{\left\{x_{\mathbf{j} ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}^{(g)}: \mathbf{j} \in \mathcal{J}, \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots \in \mathcal{R}\right\}} \cap(0,1)
\end{aligned}
$$

in the case $l>1$. We define the conjugating homeomorphism $h$ setting

$$
h\left(x_{j ; r_{1}, r_{2}, \ldots}^{(f)}\right)=x_{j ; r_{1}, r_{2}, \ldots}^{(g)}
$$

in the case $l=1$ and

$$
h\left(x_{\mathbf{j} ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}^{(f)}\right)=x_{\mathbf{j} ; \mathbf{r}_{1}, \mathbf{r}_{2}, \ldots}^{(g)}
$$

in the case $l>1$ (with a unique continuous extension to $\Lambda$ ). By the definition of $\Lambda(f), \Lambda(g)$, the map $h$ is an increasing homeomorphism between $\Lambda(f)$ and $\Lambda(g)$, while Lemmas 4.37 and 4.52 imply that it conjugates $\left.\left\{f_{-}, f_{+}\right\}\right|_{\Lambda(f)}$ to $\left.\left\{g_{-}, g_{+}\right\}\right|_{\Lambda(g)}$. It is easy to see that $h$ can be extended to an increasing homeomorphism of $[0,1]$ conjugating $\left\{f_{-}, f_{+}\right\}$to $\left\{g_{-}, g_{+}\right\}$, such that $h$ is affine on each component of $(0,1) \backslash \Lambda(f)$. For completeness, below we present a detailed construction for the case $l=1$, leaving the case $l>1$ to the reader.

From the considerations preceding Proposition 4.48, it follows that $\left.\left\{f_{-}, f_{+}\right\}\right|_{\Lambda(f)}$ and $\left.\left\{g_{-}, g_{+}\right\}\right|_{\Lambda(g)}$ are both conjugated to the system $\left\{\tilde{f}_{-}, \tilde{f}_{+}\right\}$acting on $\mathbb{Z}^{*} \times \Sigma_{k}$. Hence, there exists a homeomorphism $h: \Lambda(f) \rightarrow \Lambda(g)$ conjugating $\left\{f_{-}, f_{+}\right\}$on $\Lambda(f)$ to $\left\{g_{-}, g_{+}\right\}$on $\Lambda(g)$. We claim that $h$ can be extended in a continuous and equivariant manner to the interval $[0,1]$. To show this, we describe the structure of the complement of $\Lambda(f)$ in $[0,1]$.

Like in the proof of Lemma 4.36, let

$$
U_{0}=\left(f_{-}\left(x_{-}\right), f_{+}\left(x_{+}\right)\right)=\left(f_{-}\left(x_{-}\right), \mathcal{I}\left(f_{-}\left(x_{-}\right)\right)\right)=\left(\frac{\rho-\rho^{k+1}}{1-\rho^{k+1}}, \frac{1-\rho}{1-\rho^{k+1}}\right)
$$

and for $j \in \mathbb{Z}^{*}$ define

$$
U_{j}= \begin{cases}\rho^{-j} U_{0} & \text { for } j<0 \\ \mathcal{I}\left(\rho^{j} U_{0}\right) & \text { for } j>0\end{cases}
$$

By Lemma 4.35, the following statements hold.
(a) $U_{-j}=\mathcal{I}\left(U_{j}\right)$ for $j \in \mathbb{Z}$.
(b) The sets $U_{j}, j \in \mathbb{Z}$, are pairwise disjoint and together with $I_{j}, j \in \mathbb{Z}^{*}$, form a partition of $(0,1)$, where $U_{j}$ is the gap between $I_{j-1}$ and $I_{j}$ for $j<0, U_{0}$ is the gap between $I_{-1}$ and $I_{1}$, and $U_{j}$ is the gap between $I_{j}$ and $I_{j+1}$ for $j>0$.
(c) $f_{-}\left(U_{j}\right)=U_{j-1}$ for $j \leqslant 0, f_{-}\left(I_{1} \cup U_{1} \cup \cdots \cup I_{k-1} \cup U_{k-1} \cup I_{k}\right)=I_{-1}, f_{-}\left(U_{j}\right)=U_{j-k}$ for $j \geqslant k$.
(d) $f_{+}\left(U_{j}\right)=U_{j+k}$ for $j \leqslant-k, f_{+}\left(I_{-k} \cup U_{-k+1} \cup \cdots \cup I_{-2} \cup U_{-1} \cup I_{-1}\right)=I_{1}, f_{+}\left(U_{j}\right)=U_{j+1}$ for $j \geqslant 0$.

For $s=1, \ldots, k-1$, define

$$
U_{-1}^{s}=f_{-}\left(U_{s}\right)=\rho\left(U_{s}\right) \subset I_{-1}
$$

Note that $U_{-1}^{s}$ are the gaps between cylinders of the first order for the iterated function system $\left\{\phi_{1}, \ldots, \phi_{k}\right\}$ on $I_{-1}$. More precisely, $U_{-1}^{1}, \ldots, U_{-1}^{k-1}$ together with $I_{-1 ; 1}, \ldots, I_{-1 ; k}$ form a partition of $I_{-1}$, and are situated in the order

$$
I_{-1 ; 1}, U_{-1}^{1}, I_{-1 ; 2}, U_{-1}^{2}, \ldots, I_{-1 ; k-1}, U_{-1}^{k-1}, I_{-1 ; k}
$$

For $j \in \mathbb{Z}^{*}, s \in\{1, \ldots, k-1\}$ and $r_{1}, \ldots, r_{n} \in\{1, \ldots, k\}, n \geqslant 0$, define

$$
U_{j ; r_{1}, \ldots, r_{n}}^{s}= \begin{cases}\rho^{-j-1} \phi_{r_{1}} \circ \cdots \circ \phi_{r_{n}}\left(U_{-1}^{s}\right) & \text { for } j<0 \\ \mathcal{I}\left(\rho^{j-1} \phi_{r_{1}} \circ \cdots \circ \phi_{r_{n}}\left(U_{-1}^{s}\right)\right) & \text { for } j>0\end{cases}
$$

where $\phi_{r_{1}} \circ \ldots \circ \phi_{r_{n}}=\mathrm{id}, U_{j ; r_{1}, \ldots, r_{n}}^{s}=U_{j}^{s}$ for $n=0$, which agrees with the previous definition for $j=-1$. Note that for a fixed $j \in \mathbb{Z}^{*}$, the collection of disjoint intervals $\left\{U_{j ; r_{1}, \ldots, r_{n}}^{s}: 1 \leqslant\right.$ $\left.s \leqslant k-1, n \geqslant 0, r_{1}, \ldots, r_{n} \in\{1, \ldots, k\}\right\}$ forms the complement of the Cantor set $\Lambda_{j}$ and

$$
(0,1) \backslash \Lambda(f)=\bigcup_{j \in \mathbb{Z}} U_{j} \cup \bigcup_{j \in \mathbb{Z}^{*}} \bigcup_{s=1}^{k-1} \bigcup_{n=0}^{\infty} \bigcup_{r_{1}, \ldots, r_{n}=1}^{k} U_{j ; r_{1}, \ldots, r_{n}}^{s}
$$

with the union being disjoint. We can carry the same construction for the system $\left\{g_{-}, g_{+}\right\}$, yielding a decomposition

$$
(0,1) \backslash \Lambda(g)=\bigcup_{j \in \mathbb{Z}} U_{j} \cup \bigcup_{j \in \mathbb{Z}^{*}} \bigcup_{s=1}^{k-1} \bigcup_{n=0}^{\infty} \bigcup_{r_{1}, \ldots, r_{n}=1}^{k} V_{j ; r_{1}, \ldots, r_{n}}^{s}
$$

for analogously defined $V_{j}, V_{j ; r_{1}, \ldots, r_{n}}^{s}$. By Lemma 4.35, for $j \in \mathbb{Z}$,

$$
\begin{align*}
& f_{-}\left(U_{j}\right)= \begin{cases}U_{j-1} & \text { for } j \leqslant 0 \\
U_{-1}^{j} & \text { for } 1 \leqslant j \leqslant k-1, \\
U_{j-k} & \text { for } j \geqslant k\end{cases} \\
& g_{-}\left(V_{j}\right)= \begin{cases}V_{j-1} & \text { for } j \leqslant 0 \\
V_{-1}^{j} & \text { for } 1 \leqslant j \leqslant k-1, \\
V_{j-k} & \text { for } j \geqslant k\end{cases} \tag{4.43}
\end{align*}
$$

and for $j \in \mathbb{Z}^{*}, s \in\{1, \ldots, k-1\}, r_{1}, \ldots, r_{n} \in\{1, \ldots, k\}, n \geqslant 0$,

$$
\begin{align*}
& f_{-}\left(U_{j ; r_{1}, \ldots, r_{n}}^{s}\right)= \begin{cases}U_{j-1 ; r_{1}, \ldots, r_{n}}^{s} & \text { for } j \leqslant 0 \\
U_{-1 ; j, r_{1}, \ldots, r_{n}}^{s} & \text { for } 1 \leqslant j \leqslant k-1, \\
U_{j-k ; r_{1}, \ldots, r_{n}}^{s} & \text { for } j \geqslant k\end{cases} \\
& f_{+}\left(U_{j ; r_{1}, \ldots, r_{n}}^{s}\right)= \begin{cases}U_{j+k ; r_{1}, \ldots, r_{n}}^{s} & \text { for } j \leqslant-k \\
U_{1 ;-j, r_{1}, \ldots, r_{n}}^{s} & \text { for }-k+1 \leqslant j \leqslant-1, \\
U_{j+1 ; r_{1}, \ldots, r_{n}}^{s} & \text { for } j \geqslant 0\end{cases}  \tag{4.44}\\
& g_{-}\left(V_{j ; r_{1}, \ldots, r_{n}}^{s}\right)= \begin{cases}V_{j-1 ; r_{1}, \ldots, r_{n}}^{s} & \text { for } j \leqslant 0 \\
V_{-1 ; j, r_{1}, \ldots, r_{n}}^{s} & \text { for } 1 \leqslant j \leqslant k-1, \\
V_{j-k ; r_{1}, \ldots, r_{n}}^{s} & \text { for } j \geqslant k\end{cases} \\
& g_{+}\left(V_{j ; r_{1}, \ldots, r_{n}}^{s}\right)= \begin{cases}V_{j+k ; r_{1}, \ldots, r_{n}}^{s} & \text { for } j \leqslant-k \\
V_{1 ;-j, r_{1}, \ldots, r_{n}}^{s} & \text { for }-k+1 \leqslant j \leqslant-1 . \\
V_{j+1 ; r_{1}, \ldots, r_{n}}^{s} & \text { for } j \geqslant 0\end{cases}
\end{align*}
$$

We can now extend $h$ to an increasing homeomorphism of $[0,1]$ as follows: on $U_{j}, j \in \mathbb{Z}$, we define $h$ to be the unique affine increasing homeomorphism such that $h\left(U_{j}\right)=V_{j}$ and
on $U_{j ; r_{1}, \ldots, r_{n}}^{s}, j \in \mathbb{Z}^{*}, s \in\{1, \ldots, k-1\}, n \geqslant 0, r_{1}, \ldots, r_{n} \in\{1, \ldots, k\}$, we set $h$ to be the unique affine increasing homeomorphism such that $h\left(U_{j ; r_{1}, \ldots, r_{n}}^{s}\right)=V_{j ; r_{1}, \ldots, r_{n}}^{s}$. Finally, we set $h(0)=0, h(1)=1$. It is easy to see that $h$ is a homeomorphism of $[0,1]$. Using (4.43) and (4.44) we see that

$$
f_{ \pm}\left(U_{j}\right)=h^{-1} \circ g_{ \pm} \circ h\left(U_{j}\right) \quad \text { and } \quad f_{ \pm}\left(U_{j ; r_{1}, \ldots, r_{n}}^{s}\right)=h^{-1} \circ g_{ \pm} \circ h\left(U_{j ; r_{1}, \ldots, r_{n}}^{s}\right) .
$$

Since $f_{ \pm}$and $h^{-1} \circ g_{ \pm} \circ h$ are both affine and increasing on each of the above intervals, we have $f_{ \pm}=h^{-1} \circ g_{ \pm} \circ h$ on each of them.

### 4.7 Proof of Theorem 4.16

We consider a symmetric $A M$-system with probabilities $p_{-}=p_{+}=1 / 2$ and positive Lyapunov exponents, which exhibits (5:2)-resonance and satisfies $\rho=\eta$. The latter condition is equivalent to

$$
\begin{equation*}
\rho^{7}-2 \rho^{6}+2 \rho-1=0 \tag{4.45}
\end{equation*}
$$

and to $\rho x_{-}=1 / 2$. Note that this implies $f_{-}\left(x_{-}\right)=\rho^{2} x_{-}<1 / 2$, so the system is of disjoint type (see the beginning of the proof of Theorem 4.10 in the case $l>1$ ).

Define segments $J_{j}, j \in \mathbb{Z}^{*}$ as in the case $\rho<\eta$. We have

$$
J_{j}=\left\{\begin{array}{ll}
{\left[\rho^{-j} / 2, \rho^{-j+1} / 2\right]} & \text { for } j<0 \\
{\left[\mathcal{I}\left(\rho^{j} / 2\right), \mathcal{I}\left(\rho^{j-1} / 2\right)\right]} & \text { for } j>0
\end{array},\right.
$$

so the segments $J_{j}$ have pairwise disjoint interiors, each two consecutive intervals (according to the order in $\left.\mathbb{Z}^{*}\right)$ have a common endpoint and $\bigcup_{j \in \mathbb{Z}^{*}} J_{j}=(0,1)$. Similarly, defining maps $\phi_{r}$ and intervals $I_{\mathbf{j}}, \mathbf{j} \in \mathcal{J}$ as in the case $\rho<\eta$ and proceeding as in the proofs of Lemmas 4.49 and 4.50, we check that for each $j \in \mathbb{Z}^{*}$, the intervals $I_{\mathbf{j}}, \mathbf{j} \in \mathcal{J}_{j}$ are contained in $J_{j}$, have disjoint interiors and satisfy $\sum_{\mathbf{j} \in \mathcal{J}_{j}}\left|I_{\mathbf{j}}\right|=\left|J_{\mathbf{j}}\right|$. Analogously, we can define maps $\Phi_{\mathbf{r}}, \mathbf{r} \in \mathcal{R}$, intervals $I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}$ and sets $\Lambda_{\mathbf{j}}, \Lambda$ in the same way as in the case $\rho<\eta$. The maps $\Phi_{\mathbf{r}}$ form an iterated function system in $I_{-1}$, such that the intervals $\Phi_{\mathbf{r}}\left(I_{-1}\right)$ have disjoint interiors and $\sum_{\mathbf{r} \in \mathcal{R}}\left|\Phi_{\mathbf{r}}\left(I_{-1}\right)\right|=\sum_{\mathbf{r} \in \mathcal{R}}\left|I_{-1 ; \mathbf{r}}\right|=\left|I_{-1}\right|$. Hence, $\Lambda_{-1}=I_{-1}$ and the pressure (4.39) satisfies $P(1)=0$. The combinatorics of the intervals $I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}$ is the same as in the case $\rho<\eta$, so Lemmas 4.51 and 4.52 and Propositions 4.59 and 4.60 still hold. We have $\Lambda_{\mathbf{j}}=I_{\mathbf{j}}$ for $\mathbf{j} \in \mathcal{J}$ and $\Lambda=(0,1)$.

By Theorem 4.22, there exists a unique stationary measure $\mu$, and Proposition 4.60 implies $\operatorname{supp} \mu=\Lambda \cup\{0,1\}=[0,1]$. By Proposition 4.27, the measure $\mu$ is non-atomic. Hence, the measure of the endpoints of the intervals $I_{\mathbf{j} ; \mathbf{r}_{1}, \ldots, \mathbf{r}_{m}}$ is zero. In particular, Proposition 4.63 holds in this case with the same proof.

The above facts show that $\left\{\Phi_{\mathbf{r}}\right\}_{\mathbf{r} \in \mathcal{R}}$ is a countable iterated function system of contracting similarities on $I_{-1}$ satisfying the Open Set Condition, with the attractor $\Lambda_{-1}=I_{-1}$. By Proposition 4.63, the probability measure

$$
\mu_{-1}=\frac{\left.\mu\right|_{I_{-1}}}{\mu\left(I_{-1}\right)}
$$

is the self-similar measure for this system with probabilities $\beta_{\mathbf{r}}, \mathbf{r} \in \mathcal{R}$.
To prove Theorem 4.16, we show $\operatorname{dim}_{H} \mu<1$. Since by Proposition 4.63, the measure $\mu$ is a countable linear combination of $\mu_{-1}$ and its similar copies $\left.\mu\right|_{I_{\mathbf{j}}}, \mathbf{j} \in \mathcal{J}$, it is sufficient to show $\operatorname{dim}_{H} \mu_{-1}<1$. Let

$$
h\left(\mu_{-1}\right)=-\sum_{\mathbf{r} \in \mathcal{R}} \beta_{\mathbf{r}} \log \beta_{\mathbf{r}}
$$

be the entropy of $\mu_{-1}$. The proof splits into two cases depending whether $h\left(\mu_{-1}\right)$ is finite or infinite. To shorten the proof, we do not determine which case actually takes place, but we consider both possibilities.

Suppose first that $h\left(\mu_{-1}\right)$ is infinite. Then we have $\operatorname{dim}_{H} \mu_{-1} \leqslant t_{0}<1$, where $t_{0}=$ $\inf \{t>0: P(t)<\infty\}$ is the unique solution of the equation $\rho^{l t_{0}}-2 \rho^{t_{0}}+1=0$ (see the proof of Proposition 4.62). This fact follows from [6, Proposition 3.1], which is based on [55, Theorem 4.1]. Actually, the mentioned results in [6, 55] are formulated for a more specific class of iterated function systems, but the proofs are valid in the general case of self-similar systems on the interval.

Suppose now that $h\left(\mu_{-1}\right)$ is finite. Recall that the self-similar iterated function system $\left\{\Phi_{\mathrm{r}}\right\}_{\mathbf{r} \in \mathcal{R}}$ on $I_{-1}$ is regular with the attractor $\Lambda_{-1}=I_{-1}$. In particular, the normalized Lebesgue measure $\mathcal{L}=\left.\operatorname{Leb}\right|_{I_{-1} /}\left|I_{-1}\right|$ is the Gibbs and equilibrium state for the geometrical potential in dimension 1 and also the 1-conformal measure for this system on $I_{-1}$ (see [65, Section 4.4]). Moreover, the Lyapunov exponent

$$
\chi(\mathcal{L})=\sum_{\mathbf{r} \in \mathcal{R}}\left\|\Phi_{\mathbf{r}}^{\prime}\right\| \log \left\|\Phi_{\mathbf{r}}^{\prime}\right\|
$$

of the measure $\mathcal{L}$ is finite, since (similarly as in 4.39) by the definition of the set $\mathcal{R}$ in the considered case,

$$
\chi(\mathcal{L})=\sum_{r=2}^{4} \sum_{n=1}^{\infty} \rho^{r+n} \log \left(\rho^{r+n}\right)+\sum_{r=2}^{5} \rho^{r} \log \left(\rho^{r}\right)>-\infty .
$$

In such a situation [65, Theorem 4.4.7] (see also [43, Theorem 4.6]) asserts that either the self-similar measure $\mu_{-1}$ is equal to $\mathcal{L}$ or $\operatorname{dim}_{H} \mu_{-1}<\operatorname{dim}_{H} \Lambda_{-1}=1$. Therefore, to end the proof of the theorem, it is sufficient to show $\mu_{-1} \neq \mathcal{L}$.

Suppose $\mu_{-1}=\mathcal{L}$. Then

$$
\begin{equation*}
\frac{\mu\left(I_{-1 ; r}\right)}{\mu\left(I_{-1}\right)}=\frac{\left|I_{-1 ; r}\right|}{\left|I_{-1}\right|}=\rho^{r} \tag{4.46}
\end{equation*}
$$

for $r \in\{2,3,4,5\}$. Consider the characteristic polynomial $x^{k+l}-2 x^{l}+1$ from Proposition 4.63 . In the considered case it has the form

$$
h(x)=x^{7}-2 x^{2}+1=(x-1)\left(x^{3}+x^{2}-1\right)\left(x^{3}+x+1\right) .
$$

Computing the derivatives, we check that the polynomial $x^{3}+x^{2}-1$ has a unique real root $\alpha \in(0,1)$, while $x^{3}+x+1$ has a unique real root $\beta \in(-1,0)$. By Viete's formulas for these polynomials, the remaining non-real roots of $h$ have moduli greater than 1. Therefore, Proposition 4.63 implies that for $j \in \mathbb{Z}^{*}$ and $r \in\{2,3,4,5\}$,

$$
\begin{equation*}
\mu\left(I_{j}\right)=A \alpha^{|j|}+B \beta^{|j|}, \quad \mu\left(I_{-1 ; r}\right)=\frac{\mu\left(I_{-1}\right) \mu\left(I_{r}\right)}{2 \mu\left(I_{2}\right)-\mu\left(I_{7}\right)} \tag{4.47}
\end{equation*}
$$

for some $A, B \in \mathbb{R}$. Since $\mu\left(I_{j}\right)>0$, we have $(A, B) \neq(0,0)$.
By (4.46) and 4.47,

$$
A \alpha^{r}+B \beta^{r}=q \rho^{r}, \quad r=2,3,4,5,
$$

where $q=2 \mu\left(I_{2}\right)-\mu\left(I_{7}\right)>0$. This implies $A \alpha^{r+1}+B \beta^{r+1}=\rho\left(A \alpha^{r}+B \beta^{r}\right)$ for $r=2,3,4$, which gives

$$
\left(\frac{\alpha}{\beta}\right)^{r} A(\alpha-\rho)=B(\rho-\beta), \quad r=2,3,4 .
$$

We have $(A, B) \neq(0,0)$. Moreover, $\rho \neq \beta$ because $\rho>0, \beta<0$. If $\rho=\alpha$, then by 4.45) and the definition of $\alpha$,

$$
\rho^{6}-\rho^{5}-\rho^{4}-\rho=\frac{\rho^{7}-2 \rho^{6}+2 \rho-1}{\rho-1}+\rho^{3}+\rho^{2}-1=0
$$

which is impossible since $\rho \in(0,1)$. Hence, $A, B, \alpha-\rho, \rho-\beta \neq 0$ and we can write

$$
\left(\frac{\alpha}{\beta}\right)^{r}=\frac{B(\rho-\beta)}{A(\alpha-\rho)}, \quad r=2,3,4,
$$

which implies $\alpha=\beta$ and makes a contradiction. This ends the proof of Theorem 4.16,

### 4.8 Singularity by estimating return times

In this section we present another approach to proving singularity of stationary measures for $A M$-systems. Namely, rather than finding explicitly a closed invariant set of Lebesgue measure zero (note that this strategy has to fail in a non-resonant case, as then $\operatorname{supp}(\mu)=[0,1]$ - see Proposition 4.31), we use the well-known bound on the dimension of stationary measure in terms of its entropy and Lyapunov exponent (see 4.50) below). We find an open set of parameters for which the Lyapunov exponent is small enough (hence the average contraction is strong enough) to guarantee $\operatorname{dim}_{H}(\mu)<1$. The upper bound on the Lyapunov exponent is based on bounding the expected return time to the suitably chosen interval. In particular, we find non-resonant parameters for which the corresponding stationary measure is singular.

For this part of the work, it is convenient to consider a new parametrization of symmetric $A M$-systems. For $a \in(0,1)$ and $\gamma>0$, we set $a_{-}=a_{+}=a$ and $b_{-}=b_{+}=a^{-\gamma}$ in Definition 4.1. In other words, we consider now $A M$-systems as pairs $\left\{f_{-}, f_{+}\right\}$consisting of transformations

$$
f_{-}(x)=\left\{\begin{array}{ll}
a x & \text { for } x \in\left[0, x_{-}\right]  \tag{4.48}\\
\mathcal{I}\left(a^{-\gamma} \mathcal{I}(x)\right) & \text { for } x \in\left(x_{-}, 1\right]
\end{array}, \quad f_{+}(x)=\left\{\begin{array}{ll}
a^{-\gamma} x & \text { for } x \in\left[0, x_{+}\right] \\
\mathcal{I}(a \mathcal{I}(x)) & \text { for } x \in\left(x_{+}, 1\right]
\end{array},\right.\right.
$$

where

$$
x_{+}=\frac{1-a}{a^{-\gamma}-a}, x_{-}=\frac{a^{-\gamma}-1}{a^{-\gamma}-a} .
$$

Note that an $A M$-system of the above form exhibits resonance if and only if $\gamma \in \mathbb{Q}$. Let us fix the probability vector as $\left(p_{-}, p_{+}\right)=\left(\frac{1}{2}, \frac{1}{2}\right)$. Note that the endpoint Lyapunov exponents for the system of the above form are given by $\Lambda(0)=\Lambda(1)=\frac{1-\gamma}{2} \log a$. From now on, we will assume that $\gamma>1$, hence $\Lambda(0)$ and $\Lambda(1)$ are positive. Therefore, there exists a unique stationary probability measure $\mu$ such that $\mu(\{0,1\})=0$ (recall Theorem 4.22).

Theorem 4.64. There is a non-empty and open set of parameters $(a, \gamma) \in(0,1) \times(1, \infty)$ such that the corresponding stationary measure $\mu$ is singular with $\operatorname{dim}_{H}(\mu)<1$. More precisely, if $\gamma \in\left(1, \frac{3}{2}\right)$ and $x_{+}<f_{-}\left(\frac{1}{2}\right)$, then

$$
\operatorname{dim}_{H}(\mu) \leqslant \frac{(4 \gamma-1) \log 2}{(\gamma-1)\left(\gamma-\frac{3}{2}\right) \log a}
$$

Therefore, if additionally $a \in(0,1)$ is such that $\log a<\frac{(4 \gamma-1) \log 2}{(\gamma-1)\left(\gamma-\frac{3}{2}\right)}$, then $\operatorname{dim}_{H}(\mu)<1$. These three conditions hold simultaneously on an open and non-empty set of parameters $(a, \gamma) \in(0,1) \times(1, \infty)$.

Remark 4.65 Note that the change of parametrization $(0,1) \times(0, \infty) \ni(a, \gamma) \mapsto\left(a, a^{-\gamma}\right) \in$ $(0,1) \times(1, \infty)$ is a diffeomorphism, hence Theorem 4.64 indeed implies Theorem 1.7 (and Theorem 4.18).

Remark 4.66 All the systems with singular stationary measures covered by Theorem 4.64 are of disjoint type (recall Definition 4.3). To see that, observe first that any system of the form 4.48) with $a<\frac{1}{2}$ is of disjoint type, i.e. satisfies $f_{-}\left(x_{-}\right)<f_{+}\left(x_{+}\right)$, or, equivalently, $f_{-}\left(x_{-}\right)<\frac{1}{2}$. Indeed, the latter condition is equivalent to

$$
\begin{equation*}
2 a^{1-\gamma}-a-a^{-\gamma}<0 \tag{4.49}
\end{equation*}
$$

and if $a<\frac{1}{2}$ then

$$
2 a^{1-\gamma}-a-a^{-\gamma}<2 a^{-\gamma}(2 a-1)<0 .
$$

Therefore 4.49 is fulfilled provided $a<\frac{1}{2}$. On the other hand, conditions $\gamma \in\left(1, \frac{3}{2}\right)$ and $\log a<\frac{(4 \gamma-1) \log 2}{(\gamma-1)\left(\gamma-\frac{3}{2}\right)}$ imply that $a<2^{-48}<\frac{1}{2}$. Indeed, we have

$$
\log a<\frac{(4 \gamma-1) \log 2}{(\gamma-1)\left(\gamma-\frac{3}{2}\right)} \leqslant \frac{3 \log 2}{(\gamma-1)\left(\gamma-\frac{3}{2}\right)} \leqslant \frac{3 \log 2}{-\left(\frac{1}{4}\right)^{2}}=-48 \log 2
$$

hence $a<2^{-48}$. Therefore any system covered by Theorem 4.64 satisfies 4.49 , hence is of disjoint type.

The proof of Theorem 4.64 is based on an upper bound on the Hausdorff dimension of a stationary measure in terms of its entropy and Lyapunov exponent. To state it, let

$$
H\left(\left(p_{-}, p_{+}\right)\right):=-p_{-} \log p_{-}-p_{+} \log p_{+}
$$

be the entropy of the probability vector $\left(p_{-}, p_{+}\right)$and let

$$
\chi(\mu):=\int_{[0,1]}\left(p_{-} \log f_{-}^{\prime}(x)+p_{+} \log f_{+}^{\prime}(x)\right) d \mu(x)
$$

be the Lyapunov exponent of the stationary measure $\mu$. As measure $\mu$ is non-atomic (see Proposition 4.32) and $f_{-}, f_{+}$are differentiable everywhere except for points $x_{-}, x_{+}$, the Lyapunov exponent $\chi(\mu)$ is well defined. Moreover, measure $\mu$ is ergodic (cf. [35, Lemmas 3.2, 3.4]). It follows from [51, Theorem 1] that

$$
\begin{equation*}
\operatorname{dim}_{H}(\mu) \leqslant-\frac{H\left(\left(p_{-}, p_{+}\right)\right)}{\chi(\mu)} \tag{4.50}
\end{equation*}
$$

as long as $\chi(\mu)<0$. Since $f_{-}$and $f_{+}$are piecewise affine, we can easily express $\chi(\mu)$ in terms of parameters and the measure of the middle interval $M:=\left[x_{+}, x_{-}\right]$. Note that interval $M$ depends on parameters $a$ and $\gamma$, but we suppress that dependence from the notation (we will do the same for intervals defined later in the proof of Theorem 4.64). We have

$$
H\left(\left(p_{-}, p_{+}\right)\right)=H\left(\left(\frac{1}{2}, \frac{1}{2}\right)\right)=\log 2
$$

and

$$
\begin{equation*}
\chi(\mu)=(1-\mu(M)) \frac{1-\gamma}{2} \log a+\mu(M) \log a=\left(\frac{1-\gamma}{2}+\mu(M) \frac{1+\gamma}{2}\right) \log a . \tag{4.51}
\end{equation*}
$$

Clearly, in order to bound $\chi(\mu)$ from above, we have to bound $\mu(M)$ from below. The next lemma provides such an estimate.

Lemma 4.67. Let $a \in(0,1)$ and $\gamma>1$ be such that $x_{+}<f_{-}\left(\frac{1}{2}\right)$. Then

$$
\begin{equation*}
\mu(M) \geqslant \frac{\gamma-1}{2 \gamma-\frac{1}{2}} \tag{4.52}
\end{equation*}
$$

Moreover, the condition $x_{+}<f_{-}\left(\frac{1}{2}\right)$ is equivalent to

$$
\begin{equation*}
2 a^{-1}-2+a-a^{-\gamma}<0 \tag{4.53}
\end{equation*}
$$

and for given $\gamma>1$, it holds for all $a \in(0,1)$ small enough.
Before giving the proof of the above lemma, let us explain how it implies Theorem 4.64,

Proof of Theorem 4.64. Fix $a$ and $\gamma$ satisfying $x_{+}<f_{-}\left(\frac{1}{2}\right)$. By Lemma 4.67

$$
\frac{1-\gamma}{2}+\mu(M) \frac{1+\gamma}{2} \geqslant \frac{1-\gamma}{2}+\frac{\gamma-1}{2 \gamma-\frac{1}{2}} \frac{1+\gamma}{2}=\frac{(\gamma-1)\left(\frac{3}{2}-\gamma\right)}{4 \gamma-1}
$$

The above expression is positive for $\gamma<\frac{3}{2}$ and the bound does not depend on $a$. Hence, by 4.51,,$\chi(\mu)$ is negative for $\gamma \in\left(1, \frac{3}{2}\right)$ and $a \in(0,1)$ satisfying 4.53. Moreover, for such $\gamma$

$$
\lim _{a \rightarrow 0} \chi(\mu)=\lim _{a \rightarrow 0}\left(\frac{1-\gamma}{2}+\mu(M) \frac{1+\gamma}{2}\right) \log a=-\infty
$$

We can therefore apply 4.50 to conclude that $\operatorname{dim}_{H}(\mu)<1$ for $a$ small enough. More precisely, $\operatorname{dim}_{H}(\mu)<1$ provided that $\gamma \in\left(1, \frac{3}{2}\right)$ and $a \in(0,1)$ satisfies 4.53 together with

$$
\log a<\frac{(4 \gamma-1) \log 2}{(\gamma-1)\left(\gamma-\frac{3}{2}\right)}
$$

Proof of Lemma 4.67. Let us begin by proving the second assertion of the lemma. As $\frac{1}{2}<x_{-}$, we have $f_{-}\left(\frac{1}{2}\right)=\frac{a}{2}$, hence a direct computation yields that inequality $x_{+}<f_{-}\left(\frac{1}{2}\right)$ is equivalent to 4.53 . For a fixed $\gamma>1$, this condition is satisfied for $a>0$ small enough, as the left hand side of 4.53 converges to $-\infty$ as $a \searrow 0$.

The proof of 4.52 is based on the Kac's Lemma (see e.g. [77, Theorem 4.6]) and the observation that outside of $M$, the system $\left\{f_{-}, f_{+}\right\}$acts like a random walk with a drift (after a logarithmic change of coordinates). Note first that $\mu(M)>0$. Indeed, we have

$$
\begin{equation*}
f_{+}^{-1}\left(x_{-}\right)>x_{+} \tag{4.54}
\end{equation*}
$$

as it is straightforward to check that this inequality is equivalent to $a^{1-\gamma}>1$, which holds since $\gamma>1$ and $a \in(0,1)$. This means that sets $M$ and $f_{+}^{-1}(M)$ are not disjoint. By symmetry, $M$ and $f_{-}^{-1}(M)$ are also not disjoint. As $\lim _{n \rightarrow \infty} f_{+}^{-n}\left(x_{-}\right)=0$ and $\lim _{n \rightarrow \infty} f_{+}^{-n}\left(x_{+}\right)=1$, we see that $\bigcup_{n=0}^{\infty} f_{+}^{-n}(M) \cup f_{-}^{-n}(M)=(0,1)$ and hence $\mu(M)>0$, as $\mu$ is stationary and $\mu(\{0,1\})=0$.

We will apply Kac's Lemma to the skew product

$$
\mathcal{F}^{+}: \Sigma_{2}^{+} \times[0,1] \rightarrow \Sigma_{2}^{+} \times[0,1], \quad \mathcal{F}^{+}(\underline{i}, x)=\left(\sigma(\underline{i}), f_{i_{1}}(x)\right)
$$

where $\Sigma_{2}^{+}=\{-,+\}^{\mathbb{N}}, \underline{i}=\left(i_{1}, i_{2}, \ldots\right) \in \Sigma_{2}^{+}$and $\sigma: \Sigma_{2}^{+} \rightarrow \Sigma_{2}^{+}$is the left-side shift. Let $n_{\Sigma_{2}^{+} \times M}: \Sigma_{2}^{+} \times M \rightarrow \mathbb{N} \cup\{\infty\}$ be the first return time to $\Sigma_{2}^{+} \times M$, i.e.

$$
n_{\Sigma_{2}^{+} \times M}(\underline{i}, x):=\inf \left\{n \geqslant 1: \mathcal{F}^{+n}(\underline{i}, x) \in \Sigma_{2}^{+} \times M\right\} .
$$

Let $\mathbb{P}=\operatorname{Ber}_{\frac{1}{2}, \frac{1}{2}}^{+}$be the $\left(\frac{1}{2}, \frac{1}{2}\right)$-Bernoulli measure on $\Sigma_{2}^{+}$. Since $\mathbb{P} \otimes \mu$ is ergodic for $\mathcal{F}^{+}$(cf. [35, Lemmas 3.2 and A.2]) and $\mathbb{P} \otimes \mu\left(\Sigma_{2}^{+} \times M\right)=\mu(M)>0$, the Kac's Lemma ([77, Theorem 4.6]) implies that

$$
\begin{equation*}
\int_{\Sigma_{2}^{+} \times M} n_{\Sigma_{2}^{+} \times M}(\underline{i}, x) d \tilde{\mu}=\frac{1}{\mu(M)}, \tag{4.55}
\end{equation*}
$$

where $\tilde{\mu}=\left.\frac{1}{\mu(M)} \mathbb{P} \otimes \mu\right|_{\Sigma_{2}^{+} \times M}$. Let us define two more intervals:

$$
L=\left[x_{+}, f_{-}^{-1}\left(x_{+}\right)\right], \quad R=\mathcal{I}(L)
$$

Note that $L$ and $R$ are disjoint if and only if $f_{-}^{-1}\left(x_{+}\right)<\frac{1}{2}$, which is equivalent to our assumption $x_{+}<f_{-}\left(\frac{1}{2}\right)$. We will assume from now on that 4.53 is fulfilled, so $L \cap R=\emptyset$.

Let $C:=(\sup L, \inf R)$, i.e. $M=L \cup C \cup R$ with the union being disjoint. From the definitions of $L, C$ and $R$ we get that

$$
\begin{equation*}
f_{-}(L) \subset\left[0, x_{+}\right], f_{-}(C \cup R) \subset M, f_{+}(R) \subset\left[x_{-}, 1\right], f_{+}(L \cup C) \subset M . \tag{4.56}
\end{equation*}
$$

Let

$$
E=\left\{(\underline{i}, x) \in \Sigma \times M: f_{i_{1}}(x) \notin M\right\}=\left\{(\underline{i}, x) \in \Sigma \times M: n_{\Sigma_{2}^{+} \times M}>1\right\} .
$$

By (4.56) and disjointness of $L$ and $R$, we see that for a given $x \in M$, it cannot happen that $f_{-}(x)$ and $f_{+}(x)$ both belong to the complement of $M$. Therefore

$$
\begin{equation*}
\tilde{\mu}(E) \leqslant \frac{1}{2} . \tag{4.57}
\end{equation*}
$$

Moreover, (4.56) gives

$$
E=\left\{i_{1}=-\right\} \times L \cup\left\{i_{1}=+\right\} \times R,
$$

hence

$$
\begin{equation*}
\tilde{\mu}(E)=\frac{\mu(L)}{\mu(M)} \tag{4.58}
\end{equation*}
$$

Using the symmetry of the system $\left(f_{-} \circ \mathcal{I}=\mathcal{I} \circ f_{+}\right)$and the choice of equal probabilities, we obtain

$$
\begin{aligned}
\int_{\Sigma_{2}^{+} \times M} n_{\Sigma_{2}^{+} \times M}(\underline{i}, x) d \tilde{\mu}(\underline{i}, x)=1-\tilde{\mu}(E)+\int_{E} n_{\Sigma_{2}^{+} \times M}(\underline{i}, x) d \tilde{\mu}(\underline{i}, x)= \\
=1-\tilde{\mu}(E)+2 \int_{\left\{i_{1}=-\right\} \times L} n_{\Sigma_{2}^{+} \times M}(\underline{i}, x) \tilde{\mu}(\underline{i}, x)=(*) .
\end{aligned}
$$

Note that it follows from (4.54) that $f_{+}\left(x_{+}\right)<x_{-}$, hence a trajectory cannot jump from $\left[0, x_{+}\right)$to ( $\left.x_{-}, 1\right]$ without passing through $M$. Combining this observation with the fact that transformations $f_{-}$and $f_{+}$are increasing, we conclude that $n_{\Sigma_{2}^{+} \times M}(\underline{i}, x) \leqslant n_{\Sigma_{2}^{+} \times M}\left(\underline{i}, x_{+}\right)$ for $(i, x) \in\left\{i_{1}=-\right\} \times L$. We can therefore estimate further and apply (4.57) together with (4.58) to obtain

$$
\begin{align*}
(*) & \leqslant 1-\tilde{\mu}(E)+2 \int_{\left\{i_{1}=-\right\} \times L} n_{\Sigma_{2}^{+} \times M}\left(\underline{i}, x_{+}\right) d \tilde{\mu}(\underline{i}, x)= \\
& =1-\tilde{\mu}(E)+\frac{2}{\mu(M)} \int_{\left\{i_{1}=-\right\}}\left(\int_{L} n_{\Sigma_{2}^{+} \times M}\left(\underline{i}, x_{+}\right) d \mu(x)\right) d \mathbb{P}(\underline{i})= \\
& =1-\tilde{\mu}(E)+\frac{2 \mu(L)}{\mu(M)} \int_{\left\{i_{1}=-\right\}} n_{\Sigma_{2}^{+} \times M}\left(\underline{i}, x_{+}\right) d \mathbb{P}(\underline{i})=  \tag{4.59}\\
& =1-\tilde{\mu}(E)+\frac{\tilde{\mu}(E)}{\mathbb{P}\left(\left\{i_{1}=-\right\}\right)} \int_{\left\{i_{1}=-\right\}} n_{\Sigma_{2}^{+} \times M}\left(\underline{i}, x_{+}\right) d \mathbb{P}(\underline{i})= \\
& =1-\tilde{\mu}(E)+\tilde{\mu}(E) \tilde{\mathbb{E}} N=1+\tilde{\mu}(E)(\tilde{\mathbb{E}} N-1) \leqslant 1+\frac{\tilde{\mathbb{E}} N-1}{2},
\end{align*}
$$

where $N(\underline{i})=\inf \left\{n \geqslant 1: f_{i_{n}} \circ \ldots \circ f_{i_{1}}\left(x_{+}\right) \in M\right\}$ and the expectation $\tilde{\mathbb{E}}$ is taken with respect to the conditional measure $\tilde{\mathbb{P}}=\left.\frac{1}{\mathbb{P}\left(\left\{i_{1}=-\right\}\right)} \mathbb{P}\right|_{\left\{i_{1}=-\right\}}$. As $f_{i_{n}} \circ \ldots \circ f_{i_{1}}\left(x_{+}\right) \in\left[0, x_{+}\right)$provided $\underline{i} \in\left\{i_{1}=-\right\}$ and $n<N(\underline{i})$ (we use again the observation that a trajectory cannot jump from $\left[0, x_{+}\right)$to ( $\left.x_{-}, 1\right]$ without passing through $M$, which follows from (4.54)). Define random variables

$$
X_{j}: \Sigma_{2}^{+} \rightarrow \mathbb{R}, X_{j}(\underline{( })=\left\{\begin{array}{ll}
1 & \text { if } i_{j}=- \\
-\gamma & \text { if } i_{j}=+
\end{array}, j \in \mathbb{N} .\right.
$$

For $\underline{i} \in\left\{i_{1}=-\right\}$ we have

$$
N(\underline{i})=\inf \left\{n \geqslant 1: a^{1+X_{2}+\ldots+X_{n}} x_{+} \geqslant x_{+}\right\}=\inf \left\{n \geqslant 2: X_{2}+\ldots+X_{n} \leqslant-1\right\} .
$$

Note that $X_{2}, X_{3}, \ldots$ is an i.i.d. sequence with $\tilde{\mathbb{P}}\left(X_{j}=1\right)=\tilde{\mathbb{P}}\left(X_{j}=-\gamma\right)=\frac{1}{2}$ and $N$ is a stopping time for $\left\{X_{j}\right\}_{j=2}^{\infty}$ statisfying $\tilde{\mathbb{E}} N<\infty$. Indeed, Hoeffding's inequality [45, Theorem 2] gives

$$
\begin{gathered}
\tilde{\mathbb{P}}(N>n+1) \leqslant \tilde{\mathbb{P}}\left(\sum_{j=2}^{n+1} X_{j}>-1\right) \leqslant \tilde{\mathbb{P}}\left(\frac{1}{n} \sum_{j=2}^{n+1} X_{j}-\frac{1-\gamma}{2} \geqslant-\frac{1}{n}-\frac{1-\gamma}{2}\right) \leqslant \\
\quad \leqslant \exp \left(-\frac{2 n^{2}\left(\frac{1}{n}+\frac{1-\gamma}{2}\right)^{2}}{n(\gamma+1)^{2}}\right) \leqslant \exp (-c n)
\end{gathered}
$$

for some constant $c>0$ and $n \in \mathbb{N}$ large enough. As $\tilde{\mathbb{E}} N=\sum_{n=0}^{\infty} \tilde{\mathbb{P}}(N>n)$, the above inequality implies $\tilde{\mathbb{E}} N<\infty$. Let $S_{N}(\underline{i})=\sum_{n=2}^{N(i)} X_{n}(\underline{i})$. This random variable is well defined, since $2 \leqslant N<\infty$ holds $\tilde{\mathbb{P}}$-almost surely. As $\tilde{\mathbb{E}} N<\infty$, we can apply Wald's identity [13, Problem 22.8] to obtain

$$
\begin{equation*}
\tilde{\mathbb{E}} S_{N}=\tilde{\mathbb{E}} X_{2}(\tilde{\mathbb{E}} N-1)=\frac{1-\gamma}{2}(\tilde{\mathbb{E}} N-1) . \tag{4.60}
\end{equation*}
$$

In order to estimate $\tilde{\mathbb{E}} S_{N}$, we condition on $X_{2}$ and note that $-\gamma<-1$ and $S_{N} \geqslant-1-\gamma$ almost surely. This gives

$$
\tilde{\mathbb{E}} S_{N}=\frac{1}{2} \tilde{\mathbb{E}}\left(S_{N} \mid X_{2}=-\gamma\right)+\frac{1}{2} \tilde{\mathbb{E}}\left(S_{N} \mid X_{2}=1\right) \geqslant \frac{-\gamma}{2}+\frac{-1-\gamma}{2}=\frac{-1-2 \gamma}{2} .
$$

Combining this with (4.60) we get

$$
\tilde{\mathbb{E}} N-1 \leqslant \frac{1+2 \gamma}{\gamma-1} .
$$

Applying the above estimate to (4.59) we see that

$$
\int_{\Sigma_{2}^{+} \times M} n_{\Sigma_{2}^{+} \times M}(\underline{i}, x) d \tilde{\mu}(\underline{i}, x) \leqslant 1+\frac{1+2 \gamma}{2(\gamma-1)}=\frac{2 \gamma-\frac{1}{2}}{\gamma-1} .
$$

Invoking 4.55 finishes the proof.

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