The abstract of the dissertation
Forcing-theoretic framework for the Fraïssé theory

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Subject of the dissertation

The subject of the dissertation is the study of possible generalizations of the Fraïssé theory, using the method of forcing. In 1954 Roland Fraïssé discovered that many classes of finite models, like graphs or linear orders, can be canonically assigned certain infinite models. These infinite models are *universal* – they contain isomorphic copies of all finite models from the class – and *homogeneous* – each isomorphism between finite substructures can be extended to an automorphism of the whole structure. An infinite structure with these two properties is called a *Fraïssé limit*. This correspondence is reversible – given a countable, homogeneous model, one can recover the class of finite models from which it was built – it is exactly the class of its finite substructures. The Fraïssé theory studies this correspondence.

The Fraïssé limit of a class $\mathcal{K}$ has a natural connection with the forcing

$$\mathrm{Fn}\,(\omega, \mathcal{K}, \omega) = \{A \in \mathcal{K} |\ F(A) \in [\omega]^{<\omega}\},$$

where $F(A)$ denotes the universe of a structure A, and the ordering is given by the reversed inclusion of substructures. If $G \subseteq \mathrm{Fn}\,(\omega, \mathcal{K}, \omega)$ is a filter intersecting sufficiently many dense sets, then $\bigcup G$ is a structure isomorphic to the Fraïssé limit of $\mathcal{K}$.

This inspires a natural question about structures added in a similar way by the forcings

$$\mathrm{Fn}\,(S, \mathcal{K}, \omega) = \{A \in \mathcal{K} |\ F(A) \in [S]^{<\omega}\},$$

for uncountable sets S, in particular $S = \omega_1$. The study of such structures is the topic of Chapters 4 and 5.

It should be emphasized, that despite the obvious model-theoretic aspect, this is a dissertation about the set theory. The apparatus of model theory is very basic. On the other hand, the set-theoretic machinery is rather sophisticated, and this refers particularly to forcing-theoretic arguments. Although almost all forcing notions appearing in the dissertation are c.c.c. (and indeed most of them resemble the Cohen forcing) arguments sometimes get quite technical and involved.

Chapter 1

The chapter is a brief survey of the development of the Fraïssé theory, its relatives studied in the past, and their applications.

Chapter 2

The chapter is an introduction to the classical Fraïssé theory, with examples. WE describe the Fraïssé limits of the following classes: linear orders, graphs, Boolean algebras, partial orders, and metric spaces with rational distances.

Chapter 3

We introduce the Fraïssé-Jónsson theory, which is a modification of the classical Fraïssé theory, where we do not assume that the models from $\mathcal{K}$ are finite. Examples of the uncountable Fraïssé limits we can obtain this way, are countably saturated models of size 2^ω, in case

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Introduction

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This inspires a natural question about structures added in a similar way by the forcings

$$\mathrm{Fn}\,(S, \mathcal{K}, \omega) = \{A \in \mathcal{K} |\ F(A) \in [S]^{<\omega}\},$$

or more generally

$$\mathrm{Fn}\,(S, \mathcal{K}, \kappa) = \{A \in \mathcal{K} |\ F(A) \in [S]^{<\kappa}\},$$

for uncountable sets S, in particular $S = \omega_1$.

It should be emphasized, that despite the obvious model-theoretic aspect, this is a dissertation about the set theory. The apparatus of model theory is very basic. On the other hand, the set-theoretic machinery is rather sophisticated, and this refers particularly to forcing-theoretic arguments. Although almost all forcing notions appearing in the dissertation are c.c.c. (and indeed most of them resemble the Cohen forcing) arguments sometimes get quite technical and involved.

In the first chapter I sketch the history of the Fraïssé theory, and its variants that were studied in the past. The second chapter is an introduction to the classical Fraïssé theory, and the third – to the Fraïssé-Jónsson theory, which is a generalization of the classical Fraïssé theory, in which we work with classes of infinite models. The first three chapters are mostly of introductory nature. The only original results of these chapters are: part of Theorem 3.2.13 (adding the last condition), Theorem 3.2.15, Corollary 3.2.16 and Lemma 3.2.17. The essential part of the dissertation are results of Chapters 4 and 5.

The Fraïssé Theory

By a *structure*, we always understand a model of some first order theory. By an *embedding*, we always understand a homomorphism of structures, that is an isomorphism into its image.

Definition 1. We will say about a class of structures $\mathcal{K}$ that:

The Fraïssé-Jónsson Theory

The Fraïssé-Jónsson theory is a variant of the classical Fraïssé theory, where we do not assume that the models from $\mathcal{K}$ are finite. Examples of the uncountable Fraïssé limits we can obtain this way, are countably saturated models of size 2^ω, in case when $2^\omega = \omega_1$. Some of such models admit natural representations, like the Boolean algebra $\mathcal{P}(\omega)/\mathrm{Fin}$.

The Fraïssé-Jónsson theory essentially uses assumptions on the cardinal arithmetic, so it is natural to look for some canonical, saturated models, whose existence is not dependent on additional axioms of set theory.

Definition 4. A linear order $(L, \leq)$ is *countable saturated* if for all pairs of countable subsets $A, B \subseteq L$ such that $A < B$, there exists a point in L strictly between A and B (we permit that A or B is empty).

One can check, that the above definition is equivalent to the usual, model-theoretic, definition of an ω_1-saturated linear order. An example of such order is the set

$$\mathbb{L}^{\omega_1} = \{x \in [-1, 1]^{\omega_1} \,|\, |\{\alpha < \omega_1 : x(\alpha) \neq 0\}| \leq \omega\},$$

with the lexicographic ordering, i.e. $x < y$ if and only if $x(\alpha) < y(\alpha)$ for the least coordinate α, on which $x(\alpha) \neq y(\alpha)$. Under Continuum Hypothesis, this is a unique countably saturated linear order of size 2^ω, and this was known already to Hausdorff ([6]).

Definition 5. A linear order L is *prime countably saturated* if it is countably saturated, and each countably saturated linear order contains an isomorphic copy of L.

Without any assumptions on the cardinal arithmetic, the following is true.

Theorem 2 (Harzheim [5]). *Assume that $(L, \leq)$ is a countably saturated linear order. The following conditions are equivalent:*

- *L is prime countably saturated;*
- *L can be represented as an increasing sum $\bigcup_{\alpha<\omega_1} L_\alpha$, where L_α doesn't contain any copy of ω_1 or ω_1^*, for any index $\alpha < \omega_1$ (ω_1^* is the reversed ordering of ω_1).*

Theorem 3 (Harzheim, [5]). *All prime countably saturated linear orders are isomorphic.*

The proof of this result relies on the fact, that a certain class of linear orders has properties similar to a Fraïssé class. Indeed, $\mathbb{L}^{\omega_1}$ is an example of a Fraïssé class in some more general sense, described in [7]. The order $\mathbb{L}^{\omega_1}$ has also a certain homogeneity property.

Theorem 4 (Kostana). *Each isomorphism between countable or Dedekind complete subsets of $\mathbb{L}^{\omega_1}$ can be extended to an automorphism of $\mathbb{L}^{\omega_1}$.*

Generic Structures

Denote by $\mathcal{LO}$ the class of all linear orders. Let us look at the forcing

$$\mathbb{P} = \mathrm{Fn}(\kappa, \mathcal{LO}, \omega) = \{(A, \leq) \,|\, A \in [\kappa]^{<\omega},\ (A, \leq) \text{ is a linear order.}\},$$

where κ is any cardinal number, and the ordering is given by the relation

$$(A, \leq_A) \leq (B, \leq_B) \iff A \supseteq B, \quad \leq_A \restriction B \times B = \leq_B .$$

It is easy to check that the following sets are dense in $\mathbb{P}$:

- $D_\alpha = \{(A, \leq) \,|\, \alpha \in A\}$,
- $D_{\alpha,\beta} = \{(A, \leq) \,|\, \exists n < \omega \quad n \text{ is between } \alpha \text{ and } \beta\}$.

If $G \subseteq \mathbb{P}$ is therefore a suitable generic filter, then $\bigcup G$ is a separable linear order. In particular, it is isomorphic to $(\mathbb{Q}, \leq)$ if $\kappa = \omega$.

This is an instance of a much more general phenomenon. In order to ensure that the forcing does not collapse cardinals, we need to impose some conditions on the class of models.

Generic Structures and Martin's Axiom

Structures added by the forcings $\mathrm{Fn}\,(\omega_1, \mathcal{K}, \omega)$ are rigid in the corresponding generic extensions, but it seems that this property is more inherent to the specific universe of set theory, rather than the structures themselves. More specific to the latter ones, is perhaps another property, which in the case of linear orders appears already in the work [2].

Definition 7 (Avraham-Shelah, [2]). An uncountable set $A \subseteq \mathbb{R}$ is *increasing* if for each uncountable family of pairwise disjoint n-tuples

$$\{(x_1^\xi, \ldots, x_n^\xi) | \xi < \omega_1\} \subseteq A^n$$

there exist $\eta \neq \xi < \omega_1$ such that for all $i, j = 1, \ldots, n$

$$x_i^\xi \leq x_i^\eta \iff x_j^\xi \leq x_j^\eta.$$

Authors of [2] used increasing sets to prove that Martin's Axiom does not guarantee, that all ω_1-dense, separable linear orders are isomorphic. The consistency with ZFC of the latter statement was proved by Baumgartner in [3], and from his proof easily follows also the consistency with $ZFC + MA + \neg CH$.

Theorem 8 (Avraham-Shelah, [2]). *The following is consistent with $ZFC + MA_{\omega_1}$: There exists an ω_1-dense separable linear order A with the property that each uncountable partial function*

$$f \subseteq A \times A$$

is monotone on an uncountable set.

From this property of a set A, it follows that $A = (A, \leq)$ cannot be isomorphic to its reversed order $A^* = (A, \geq)$. Therefore the conclusion of the Baumgartner's Theorem does not hold. What increasing sets have to do with the forcings $\mathrm{Fn}(S, \mathcal{K}, \omega)$ is explained by the next proposition.

Proposition 4. $\mathrm{Fn}(\omega_1, \mathcal{LO}, \omega) \Vdash$ *"$(A, \leq)$ is an increasing order".*

The usefulness of this notion, as well as its generalizations that we present below, comes from the fact, that the existence of an increasing set is consistent with Martin's Axiom.

Definition 8. Let (X, d) be a metric space.

- We call a pair of tuples $\overline{x} = (x_1, \ldots, x_n), \overline{y} = (y_1, \ldots, y_n) \in X^n$ *alike* if they satisfy the following axioms:

  A1 $\forall\, i, j = 1, \ldots, n \; (d(x_i, y_i) = d(x_j, y_j))$
  A2 $\forall\, i, j = 1, \ldots, n \; (d(x_i, x_j) = d(y_i, y_j))$
  A3 $\forall\, i, j = 1, \ldots, n \; (x_i \neq x_j \implies d(x_i, x_j) = d(x_i, y_j))$

  We then write $\overline{x} \circledast \overline{y}$.

- We call (X, d) *rectangular* if it is uncountable, and for any sequence of pairwise disjoint tuples $\{(x_1^\xi, \ldots, x_n^\xi) | \xi < \omega_1\} \subseteq X^n$, there are $\xi \neq \eta < \omega_1$, such that $(x_1^\xi, \ldots, x_n^\xi) \circledast (x_1^\eta, \ldots, x_n^\eta)$.
  (by "disjoint tuples", we mean that

  $$\{x_1^\xi, \ldots, x_n^\xi\} \cap \{x_1^\eta, \ldots, x_n^\eta\} = \emptyset$$

  whenever $\xi \neq \eta$.)

Following the ideas of Avraham, Rubin, and Shelah from [2] and [1], we prove a version of Theorem 8 for metric spaces.

Theorem 9 (Kostana). *The following is consistent with $ZFC + MA_{\omega_1}$: There exists a separable metric space X of size ω_1 with the property that each uncountable partial 1-1 function*

$$f \subseteq X \times X$$

is an isometry on an uncountable set. Moreover, distances between points of the space X are rational, and X has a dense copy of the rational Urysohn space.

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