Singularities of minimizing harmonic maps into closed manifolds

Extended abstract

Michał Miśkiewicz

1 Motivation

In the greatest generality, our object of study are maps $u\colon \mathcal{M} \to \mathcal{N}$ between two Riemannian manifolds that minimize (or are critical points of) the Dirichlet energy

$$E(u) := \int_{\mathcal{M}} |\nabla u|^2,$$

called *harmonic maps*. In this dissertation we investigate the regularity properties of such maps, with special emphasis on the case $\mathcal{N} = \mathbb{S}^2$ and $\dim \mathcal{M} \geqslant 3$. As we shall see, harmonic maps are in general not regular, but their singularities are by now well understood. The study of their *singular sets* is the main objective of this work.

Harmonic maps appear naturally in various geometric problems. Let us mention several examples:

- if $\mathcal{M}$ is 1-dimensional, harmonic maps are geodesics on $\mathcal{N}$;
- if $\mathcal{N} = \mathbb{R}$, harmonic maps are simply harmonic functions on $\mathcal{M}$;
- if $\mathcal{M}$ is 2-dimensional, *conformal* harmonic maps are parametrizations of minimal surfaces (i.e., critical points of the area functional).

A detailed discussion of the classical theory of (smooth) harmonic maps can be found in two survey articles by Eells and Lemaire [6, 7].

2 Analytic difficulties

To better illustrate the difficulties connected to the geometry of $\mathcal{N}$, let us first consider the simplest case $\mathcal{M} = \mathbb{R}^n$ and $\mathcal{N} = \mathbb{R}$. Then, it is enough to assume that $u\colon \mathbb{R}^n \to \mathbb{R}$ is a *critical point* of E, i.e., $\frac{d}{dt}|_{t=0} E(u + t\varphi) = 0$ for each perturbation $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$. It follows that u satisfies the Euler-Lagrange equation $\Delta u = 0$ in the weak sense, and Weyl's lemma implies smoothness of u.

In the general case, we may assume that the target manifold $\mathcal{N} \subseteq \mathbb{R}^N$ is isometrically embedded into some (possibly high-dimensional) Euclidean space. Now each critical point

$u\colon \mathcal{M} \to \mathcal{N}$ satisfies

$$\frac{d}{dt}\Big|_{t=0} E(\pi_{\mathcal{N}}(u + t\varphi)) = 0 \quad \text{for every } \varphi \in C_c^\infty(\mathcal{M}, \mathbb{R}^N).$$

Note that $u+t\varphi$ is no longer a valid competitor as it takes values outside of $\mathcal{N}$, and thus it needs to be projected back onto $\mathcal{N}$ by the nearest-point projection $\pi_{\mathcal{N}}$. The resulting Euler-Lagrange equation can be rewritten as

$$-\Delta_{\mathcal{M}} u = A_u^{\mathcal{N}}(\nabla u, \nabla u), \tag{2.1}$$

where $\Delta_{\mathcal{M}}$ is the Laplace-Beltrami operator on $\mathcal{M}$ and $A_u^{\mathcal{N}}$ denotes the second fundamental form of the submanifold $\mathcal{N} \subseteq \mathbb{R}^N$ evaluated at u. To be precise, the right-hand side is to be understood as a sum $\sum_\alpha A_u^{\mathcal{N}}(\partial_\alpha u, \partial_\alpha u)$ over some orthonormal basis ∂_α of $T\mathcal{M}$.

This dissertation is focused on the special case when $\mathcal{M} \subseteq \mathbb{R}^n$ is a bounded flat domain and $\mathcal{N}$ is the standard sphere $\mathbb{S}^2$. Then, the equation (2.1) takes the simple form

$$-\Delta u = |\nabla u|^2 u.$$

In all possible cases of non-flat target manifolds, the quadratic non-linearity is troublesome and most standard techniques of regularity theory cannot be directly applied. Indeed, in higher dimensions the solutions may be singular:

- if $u \in W^{1,2}$ is a weak solution of (2.1) and $\dim \mathcal{M} \leqslant 2$, then u is a smooth classical solution (Hélein [11]);
- however, if $\dim \mathcal{M} \geqslant 3$, then u may be discontinuous everywhere (Rivière [22]).

An example of a singular harmonic map is

$$\mathbb{R}^n \ni x \longmapsto \frac{x}{|x|} \in \mathbb{S}^{n-1} \qquad (n \geqslant 3).$$

Let us stress that this map is not only a critical point, but also a minimizer in the sense of Definition 1 [13].

3 Minimizing harmonic maps

Since harmonic maps in dimensions 1 and 2 are smooth and very well understood, we focus on higher dimensional domains. Moreover, the discussion from the last section motivates the study of minimizers instead of merely critical points of the energy:

Definition 1 (minimizing harmonic maps)**.** Let $\mathcal{N} \subseteq \mathbb{R}^N$ be a smooth closed (i.e., compact, without boundary) submanifold, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain. We define the class of Sobolev maps $W^{1,2}(\Omega, \mathcal{N})$ by

$$W^{1,2}(\Omega, \mathcal{N}) = \left\{ u \in W^{1,2}(\Omega, \mathbb{R}^N) : u(x) \in \mathcal{N} \text{ for a.e. } x \in \Omega \right\}.$$

A map $u \in W^{1,2}(\Omega, \mathcal{N})$ is called a *minimizing harmonic map* in Ω if $E(u) \leqslant E(v)$ for each $v \in W^{1,2}(\Omega, \mathcal{N})$ satisfying $v = u$ on $\partial\Omega$.

As mentioned before, even minimizing harmonic maps $u\colon \Omega \to \mathcal{N}$ may be non-smooth. We shall call $x \in \Omega$ a *regular point* if u has a representative continuous at x, otherwise x is a *singular point*. We denote the set of all singular points by $\operatorname{sing} u$.

One can notice that we assumed the domain Ω to be flat in Definition 1 – this is motivated by the fact that the singular behavior of u is a local phenomenon. It is enough to study small balls around each singular point, and these balls are almost flat after rescaling to unit size. Indeed, most results in this dissertation can be generalized to a general domain $\mathcal{M}$ (see [20] and [26, Sec. 8] for detailed explanations).

One can prove (see Theorem 2) that u is actually smooth around each regular point. This dichotomy – discontinuity on the *singular set* and smoothness elsewhere – in some sense reduces the usual regularity problems to the study of the singular set. For this reason, from now on we will focus on estimating its dimension (which in typical situations is $n-3$) and size (i.e., Hausdorff measure), studying its manifold structure and its dependence on the boundary data.

4 Partial regularity

Some of the main available results for the singular set are summarized in the following two theorems.

Theorem 2 (1982-83, Schoen-Uhlenbeck [23, 24, 25])**.** Let $u\colon \Omega \to \mathcal{N}$ be a minimizing harmonic map in a bounded domain $\Omega \subseteq \mathbb{R}^n$. Then $\operatorname{sing} u \subseteq \Omega$ is a closed subset of Hausdorff dimension at most $n-3$, and u is smooth in $\Omega \setminus \operatorname{sing} u$. Moreover,

- if both the boundary $\partial\Omega$ and the boundary map $u|_{\partial\Omega}$ are sufficiently smooth ($C^{1,\alpha}$ is sufficient), then u is smooth on some neighborhood of $\partial\Omega$;
- in case $n = 3$, $\operatorname{sing} u$ is discrete.

Theorem 3 (2017, Naber-Valtorta [20])**.** If $u\colon \mathbf{B}_{2r}(p) \to \mathcal{N}$ is a locally minimizing harmonic map, then $\operatorname{sing} u$ is a rectifiable $(n-3)$-dimensional set. Moreover, its measure in a smaller ball $\mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_r(p))$ is bounded by a constant dependent only on n, $\mathcal{N}$ and the energy $r^{2-n}\int_{\mathbf{B}_{2r}} |\nabla u|^2$.

It follows from Theorem 2 that if both $\partial\Omega$ and $u|_{\partial\Omega}$ are sufficiently smooth and $n = 3$, then the singular set $\operatorname{sing} u$ consists of finitely many points. The higher-dimensional counterpart of this statement – that $\mathcal{H}^{n-3}(\operatorname{sing} u) < \infty$ – has been proved only recently by Naber and Valtorta [20]. This recent breakthrough and the new methods behind it were one of the reasons I have chosen this topic for my doctoral dissertation.

In general the $(n-3)$-dimensional bounds on the singular set cannot be improved. A typical singularity of a minimizing map into $\mathbb{S}^2$ looks like

$$\mathbb{R}^3 \times \mathbb{R}^{n-3} \ni (x, y) \mapsto x/|x| \in \mathbb{S}^2,$$

so it has an $(n-3)$-dimensional plane as its singular set.

5 Results of the dissertation

A number of other interesting properties were shown for special classes of target manifolds $\mathcal{N}$, especially for the standard sphere $\mathbb{S}^2$.

In addition to its intrinsic mathematical interest, this special case also appears in some physical models. The molecules of liquid crystals are small but relatively long, and their configuration minimizes an energy that penalizes changes of direction. Taking the averaged direction of molecules at each point, we obtain a map $u \colon \Omega \to \mathbb{R}P^2$ that minimizes a functional closely resembling the Dirichlet energy $E(u)$. Singularities of harmonic maps are related to *defects* of liquid crystals, i.e., points where the direction of molecules changes in a discontinuous way. Replacing $\mathbb{R}P^2$ (the space of directions) by $\mathbb{S}^2$ and simplifying the functional to E, we can still capture the main phenomena. An interested reader can be referred to [1] and [8].

Hölder regularity of the singular set

In the special case of maps $u \colon \mathbf{B}^4 \to \mathbb{S}^2$, Hardt and Lin [10] obtained the following remarkable structure result.

Theorem 4 (1980, Hardt-Lin [10])**.** The singular set of an energy minimizer $u \colon \mathbf{B}^4 \to \mathbb{S}^2$ is locally a union of a finite set and a finite family of Hölder continuous closed curves with a finite number of crossings.

In this thesis, we generalize this theorem to higher-dimensional domains. The main result – based on the author's work [16] – is the following (see [17, Cor. 3.1.5]). For any minimizing map $u \colon \Omega \to \mathbb{S}^2$ defined on $\Omega \subseteq \mathbb{R}^n$, one can distinguish the top-dimensional part of the singular set $\operatorname{sing}_* u \subseteq \operatorname{sing} u$, which is a subset of full $\mathcal{H}^{n-3}$-measure. Then, $\operatorname{sing}_* u$ is proved to be an open subset and a topological $(n-3)$-dimensional manifold of Hölder class $C^{0,\gamma}$ for every $\gamma \in (0,1)$.

In order to extract the topological obstruction responsible for preventing gaps in the singular set of maps into $\mathbb{S}^2$, we also show a more general conditional statement for minimizing maps into any target manifold $\mathcal{N}$.

Stability of singularities

In dimension $n = 3$, when minimizing harmonic maps have only isolated singularities, further refinements of Schoen and Uhlenbeck's results were available in the literature. Hardt and Lin [9] showed that the singularities are stable under Lipschitz perturbations of the boundary map.

Theorem 5 (1989, Hardt-Lin [9])**.** Let $\Omega \subseteq \mathbb{R}^3$ be a bounded smooth domain and $u \colon \Omega \to \mathbb{S}^2$ be a minimizing harmonic map with Lipschitz continuous boundary data $\varphi := u|_{\partial\Omega}$. If u_k is a sequence of minimizers with corresponding boundary maps φ_k and

$$\varphi_k \to \varphi \text{ in } \operatorname{Lip}(\partial\Omega, \mathbb{S}^2), \qquad u_k \to u \text{ in } W^{1,2}(\Omega, \mathbb{S}^2),$$

then for large k, u_k has the same number of singularities as u, and $\operatorname{sing} u_k$ converges to $\operatorname{sing} u$ (say, with respect to Hausdorff distance).

Even more, there exist bi-Lipschitz transformations η_k of Ω mapping $\operatorname{sing} u$ to $\operatorname{sing} u_k$ and such that $\|\eta_k - \operatorname{id}\|_{\mathrm{Lip}} \to 0$ and $\|u - u_k \circ \eta_k\|_{C^\beta} \to 0$ for some small $\beta > 0$.

Based on the joint work with Katarzyna Mazowiecka and Armin Schikorra [14], we show a higher-dimensional counterpart (see [17, Thm. 7.1.1]). With the same assumptions on Ω, u and φ as above, if u_k is a sequence of minimizers with boundary data φ_k and

$$u_k \to u \text{ in } W^{1,2}, \quad \varphi_k \to \varphi \text{ in } W^{1,n-1},$$

then

$$\mathcal{H}^{n-3} \llcorner \operatorname{sing} u_k \xrightarrow{d_W} \mathcal{H}^{n-3} \llcorner \operatorname{sing} u,$$

where d_W denotes the 1-Wasserstein distance between Hausdorff measures on singular sets of u_k and u. In particular, the total measure $\mathcal{H}^{n-3}(\operatorname{sing} u_k)$ tends to $\mathcal{H}^{n-3}(\operatorname{sing} u_k)$.

Note that this recovers most of Hardt and Lin's Theorem 5 in the case $n = 3$ (except for the diffeomorphism statement). Indeed, $\mathcal{H}^0$ is simply the counting measure, so Wasserstein convergence implies that $\# \operatorname{sing} u_k = \# \operatorname{sing} u$ for large k and that $\operatorname{sing} u_k$ converges to $\operatorname{sing} u$ with respect to Hausdorff distance. However, generalizing the diffeomorphism statement to higher dimensions is very hard – even the bi-Lipschitz regularity of $\operatorname{sing}_* u$ is an open problem for $n > 3$.

As in the original paper [9], the heart of the argument lies in the *local* case. If we restrict u to a small enough ball around a singularity, it is close to its tangent map, and after rescaling the problem reduces to the following (which is the content of Lemma [17, Lemma 7.4.1]). If $u\colon \mathbf{B}_{80} \to \mathbb{S}^2$ is close enough to its tangent map (δ-flat in the sense of [17, Def. 3.3.3]), then

$$(1-\varepsilon)\omega_{n-3} \leqslant \mathcal{H}^{n-3}(\operatorname{sing} u \cap \mathbf{B}_1) \leqslant (1+\varepsilon)\omega_{n-3}.$$

This means that the measure of $\operatorname{sing} u \cap \mathbf{B}_1$ is close to the measure of the singular set of its tangent map in $\mathbf{B}_1$, which is an $(n-3)$-dimensional disc.

Linear bound on the measure of singularities

Almgren and Lieb estimated the number of singularities in terms of the boundary map [1].

Theorem 6 (1988, Almgren-Lieb [1]**).** Let $\Omega \subseteq \mathbb{R}^3$ be a bounded smooth domain and $u\colon \Omega \to \mathbb{S}^2$ be a minimizing harmonic map with boundary data $\varphi \in W^{1,2}(\partial\Omega, \mathcal{N})$. Then

$$\# \operatorname{sing} u \leqslant C(\Omega) \int_{\partial\Omega} |\nabla\varphi(x)|^2 \, \mathrm{d}\mathcal{H}^2(x).$$

Again, together with Katarzyna Mazowiecka and Armin Schikorra [14], we were able to generalize this result to an arbitrary dimension (see [17, Thm. 6.1.1]). If $\Omega \subseteq \mathbb{R}^n$ is a bounded smooth domain and $u\colon \Omega \to \mathbb{S}^2$ is a minimizing map with boundary data $\varphi \in W^{1,n-1}(\partial\Omega, \mathbb{S}^2)$, then

$$\mathcal{H}^{n-3}(\operatorname{sing} u) \leqslant C(\Omega) \int_{\partial\Omega} |\nabla\varphi(x)|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1}(x).$$

As in the case $n = 3$, a non-linear estimate $\mathcal{H}^{n-3}(\operatorname{sing} u) \leqslant C(\Omega, \|\varphi\|_{\mathrm{Lip}})$ is much easier to obtain (see [17, Thm. 6.1.3]). Thus, the power of our result lies in the linear dependence on the energy, and in the use of $W^{1,n-1}$-norm of the boundary map, which again does not control the distance of singularities from the boundary.

The strategy of the proof is close to the original, based on refined boundary regularity results of the following type: if a minimizer $u\colon \mathbf{B}_1^+ \to \mathbb{S}^2$ has a boundary map $\varphi\colon \mathbf{B}_1^{n-1} \to \mathbb{S}^2$ with small energy, then some region of $\mathbf{B}_1^+$ is free of singularities. However, the crucial ingredient here is the *hot spot lemma* due to Almgren and Lieb [1, Thm. 2.4], generalized to higher dimensions. It yields the same regularity conclusion with a weakened assumption – we only assume that the energy of φ on $\mathbf{B}_1^{n-1} \setminus \mathbf{B}_\varepsilon$ is small, while its behavior on the small ball $\mathbf{B}_\varepsilon$ (called the hot spot) can be arbitrarily wild.

The original paper of Almgren and Lieb [1] relies on the classification of singularities of maps into $\mathbb{S}^2$ to show a lower bound on the distance between two singularities. Replacing this bound by Naber and Valtorta's Theorem 3, we are able to obtain a similar result in an arbitrary dimension. A similar strategy was used in the context of minimal surfaces by Edelen in [3], where he combined interior measure bounds due to Naber and Valtorta [19] with boundary regularity results to obtain global bounds on the singular set.

Moreover, the only special property of $\mathbb{S}^2$ needed in course of the proof is the extension property [17, Thm. 2.5.1], which can be shown for a wider class of target manifolds. Thus, the final result holds for maps into any closed simply connected Riemannian manifold $\mathcal{N}$.

Discrete Reifenberg-type theorem

We also discuss Reifenberg-type theorems, which are at the heart of Naber and Valtorta's measure estimates. Theorems of this kind have wide applicability in the study of singular sets in various geometric problems [19, 12, 2], and in particular to singularities of minimizing harmonic maps [20]. They are also interesting in themselves as general results in geometric measure theory.

Various extensions of the discrete Reifenberg theorem from [20] were published in the author's paper [15]. The main result presented in the dissertation [17, Thm. 5.1.1] is phrased in terms of so called Jones' height excess numbers. Fixing a Radon measure μ on $\mathbb{R}^n$ and some dimension $0 < k < n$, the quantity

$$\beta^2_{\mu,2}(x,r) := \inf\left\{ r^{-(k+2)} \int_{\mathbf{B}_r(x)} d^2(y,V)\,\mathrm{d}\mu(y) : V \text{ is a } k\text{-dim affine plane} \right\}$$

measures how far $\mu \llcorner \mathbf{B}_r(x)$ is from being supported on some k-dimensional plane ($d(\cdot, V)$ denotes the distance to V). Our main result states that (under some technical assumptions on μ) the condition

$$r^{-k} \int_{\mathbf{B}_r(x)} \int_0^r \beta^2_{\mu,2}(y,s) \frac{\mathrm{d}s}{s}\,\mathrm{d}\mu(y) \leqslant J \quad \text{on each ball } \mathbf{B}_r(x)$$

implies the bound $\mu(\mathbf{B}_r(x)) \leqslant C(n)(1 + J^{1/2}) r^k$ on every ball.

These technical assumptions are automatically satisfied if μ is the Hausdorff measure on some k-dimensional set (i.e., $\mu = \mathcal{H}^k \llcorner S$) or if it is a discrete measure $\mu = \sum_j \omega_k r_j^k \delta_{x_j}$ associated to some family $\{\mathbf{B}_{r_j}(x_j)\}$ of disjoint balls. The name *discrete Reifenberg theorem* comes from the fact that the proof follows by a careful application of the classical Reifenberg construction [21], first in the case when μ is a discrete measure. Indeed, a version for more general measures follows easily from the discrete case. Simple modifications allow also for the use of $\beta_{\mu,q}$-numbers with $q \geqslant 2$ (which involve the distance $d(\cdot, V)$ to the power q), and for weakened assumptions. Let us remark here that Edelen, Naber and Valtorta [4, 5] later published even more general versions of the theorem (see also the lecture notes [18]).

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