

# Stochastic Games and their Complexities

Extended Abstract of the PhD Thesis

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# Abstract

We study a class of games introduced by Mio to capture the *probabilistic  $\mu$ -calculi* called *branching games*. They are a subclass of stochastic two-player zero-sum turn-based infinite-time games of imperfect information. Branching games extend *Gale-Stewart* games by allowing players to split the execution of a play into new concurrent sub-games that continue their execution independently. In consequence, the play of a branching game has a tree-like structure, as opposed to linearly structured plays of Gale-Stewart games.

In this thesis, we focus our attention on *regular branching games*. Those are the branching games whose pay-off functions are the indicator functions of *regular sets* of infinite trees, i.e. the sets recognisable by *finite tree automata*. We study the problems of *determinacy*, *game value computability*, and the related problem of computing a *measure* of a regular set of infinite trees.

Determinacy is a property of a game that guarantees that none of the players gains or loses an advantage by revealing their strategy at the start of the game. In general, branching games are not determined: not even under *mixed strategies* nor when the winning sets are *topologically simple*. On the positive side, we show that regular branching games with *open* winning sets are determined under *mixed strategies*. Moreover, we show that *game automata* definable winning sets guarantee a stronger version of determinacy – the determinacy under *pure strategies*. Both results are accompanied by examples showing the limits of used techniques.

We give an answer to the problem of computing a value of a regular branching game. We show that a *mixed value* of a non-stochastic branching game is uncomputable and that a *pure value* of a stochastic branching game is also uncomputable. On the other hand, we provide an algorithm that computes all pure values of a given non-stochastic regular branching game.

We make a step towards a solution of the problem of computing measures of regular sets of trees. We provide an algorithm that computes the *uniform* measure of a regular winning set in two cases. Either when it is defined by a *first-order* formula with no descendant relation or when it is defined by a Boolean combination of *conjunctive queries*.

Finally, we use real-life data to show how to incorporate game-theoretic techniques in practice. We propose a general procedure that given a *time series* of data extracts a *reactive model* that can be used to predict the evolution of the system and advise on the strategies to achieve predefined goals. We use the procedure to create a game based on *Markov decision processes* that is used to predict and control level of pest in a tropical fruit farm.

# Introduction

From its very beginning *game theory* has been used to discover, understand, model, and predict the behaviour of naturally occurring systems. Game theory is especially useful when the systems in question are defined by an interaction of a number of agents that, not necessarily in cooperation, try to achieve their individual goals, e.g. a group of processes in an operating system competing for resources, a group of investment bankers trading shares, or a pack of predators hunting prey. Systems like those can be found in almost every branch of modern computer science, economy, or natural sciences. In computer science, games are used in semantics, verification, logic, and automata theory, to name a few, where they are used to define and formalise the notions of *interaction*. In economics, game theory is often associated with the *rational choice* in which we assume that the agents behave rationally. Lastly, in natural sciences games are often used to model complex events and ecosystems, where a number of competing parties try to achieve the best possible outcome, e.g. predator-prey equilibria.

**Games** The games considered in this thesis are an extension of the so-called *games on graphs*. Games on graphs are played on possibly infinite graphs with vertices distributed between the players. The players move a token, initially placed in one of the vertices, along the edges of the graph and in accordance to the ownership of the vertices. If a vertex is owned by a single player, then this player decides where to move the token. If a vertex is shared, then the players simultaneously and independently choose an action each; the chosen tuple of actions indicates the next placement of the token. The outcome of the game, called a *play*, is the trace of the token. After the game is played, every player achieves a score defined by a specific to the player *pay-off function*.

Games on graphs are often enhanced with probability. Such games, called *stochastic games*, introduce the uncertainty with a new type of vertices, called *random vertices*, in which the next position of the token is not decided by the players, but by the value of an associated random variable. The addition of random vertices is often encoded as an additional, fictitious, player that chooses its moves at random. This player is often referred to as *Nature*. In the case of stochastic games the score is the expected value of the pay-off function over the set of possible outcomes. Games with no random vertices are called *non-stochastic* or *pure* games.

The abundance of possible applications and areas of relevance of game theory gave birth to many classes of games which are often defined by some of their properties and require

different tools to be analysed efficiently. The properties defining those classes include, but are not limited to, the duration of the game, the progress of time, the number of the players, the shape of the arena, the presence of uncertainties, players knowledge, and the form of the objectives. Considering duration of the game we can distinguish *one-shot games*, e.g. the matrix games, matching pennies, rock-paper-scissors; *finite time games*, e.g. chess, tic-tac-toe; and (potentially) *infinite time games*, like *reachability games*, *system-user* interaction, or *Gale-Stewart* games [6]. Note that from the technical point of view, one-shot games can be seen as (in)finite time games, and (in)finite time games can be seen as one-shot games. Indeed, we can either add some inconsequential moves or demand that players declare all their future decisions at the start of the game.

The progress of time leads to distinction between turn based games, which are played in rounds, and continuous time games, see e.g. [1]. In discrete time setting, we have concurrent games, where some vertices can be shared, e.g. Blackwell games [10], and turn-based games, where every vertex has at most one owner, e.g. Gale-Stewart games [6], or *parity games* [13, 4]. The *objectives* of games are usually given by families of pay-off functions, one for each player. An important class of games are *zero-sum* games, where the pay-off functions are chosen so that the sum of individual scores is zero. A game has a *winning set* if the possible scores are binary: *win* or *lose*. We say that a game is *regular* if it has a regular winning set, i.e. the inverse image of win is a *regular set*. By regular set we understand a set recognised by an alternating automaton on finite or infinite words or trees, see e.g. [21] for details.

**Determinacy** One of the most important notions in game theory is *determinacy*. Intuitively, a game is *determined* if no player gains an advantage knowing the *strategies* of the other players.

The exact definition of determinacy depends on the type of the game and the class of allowed strategies, e.g. in concurrent games or in matrix games with real valued pay-off functions the determinacy is defined in terms of equilibria, while in zero-sum turn-based games with winning sets, like Gale-Stewart games, in terms of *winning strategies*.

The celebrated result of Martin [9] states that Gale-Stewart games with Borel winning sets are determined under *pure* strategies. On the other hand, since the seminal work of Gale and Stewart [6], we know that not every game is determined under pure strategies. Therefore, broader classes of strategies are considered.

Nash theorem [14] states that in one-shot games with finitely many strategies, there exists at least one point of equilibrium of *mixed strategies*. A *mixed strategy* is a probability distri-

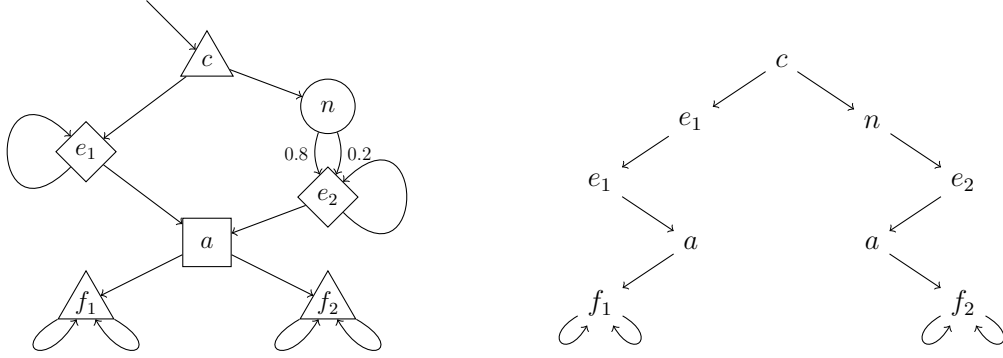


Figure 1: An example of a branching board and a play on this board. We denote Eve’s, Adam’s, *Nature*’s, and branching vertices by diamonds, squares, circles, and triangles respectively. *Nature*’s vertices are equipped with a probability distribution over the successors. The initial vertex is the only vertex with an arrow not having a source vertex. The successors  $\mathbb{R}$  and  $\mathbb{L}$  are drawn in the clockwise order, i.e.  $\mathbb{R}$  moves to the right and is drawn first in the clockwise order.

butions on the set of pure strategies. Nash theorem is an extension of the works of von Neumann and Morgenstern [22], who have shown the determinacy of two-payer zero-sum games, and was later improved by Glicksberg [7] who showed that in two-player zero-sum games the equilibrium exists if the space of possibly infinite number of strategies is compact and the pay-off function is semi-continuous. A similar result by Martin [10] states that Blackwell games, a class of infinite-time turn-based concurrent games with a finite number of possible actions per turn, are determined under mixed strategies. Note that both results by Martin hold in the stochastic set-up, see [10] for details.

**Branching games** In this thesis we study a special extension of stochastic two-player zero-sum turn-based games on graphs called *branching games* [11]. The novel addition of branching games [11] is yet another kind of vertices, as opposed to players’ vertices and random vertices, called *branching vertices*. A token placed in one of those vertices is split into a number of indistinguishable new copies of the token. The copies are placed in the successor vertices of the node, one in each, and moved with no information on whereabouts of the other copies. This new type of vertices can be seen as a delegation process, where the players delegate the resolution of the rest of the game to independent parties that cannot communicate. Note that branching games are games of imperfect information: we assume that when players decide where to move a copy of the token, they are unaware of the positions of the other copies. An example of a branching board is presented in Figure 1.

**Complexities of games** The main theoretical focus of this thesis is placed on the *computational complexity* of computing the values of the *regular branching games*. This can be seen as a natural extension of the work of Mio, who introduced branching games [12] and studied some of their properties [11].

We are interested in this family of games for two reasons:

- regular sets are a robust class with strongly developed theory and many good properties, e.g. closure properties, effective representations, and many decision procedures;
- regular sets are complex enough to not trivialise the problems and showcase interesting properties of branching games, e.g. lack of perfect information or perfect recall; for the definition of perfect recall see e.g. [8].

Considering the scope and the theme of the theoretical part of this thesis, we continue the work of Mio by considering branching games with regular winning sets and studying their computational complexity. In a grater scope, this research inscribes itself into a rich literature describing the complexity of  $\omega$ -regular games, for a survey see e.g. [2].

**Applications** As we mentioned at the start, using games to model complex ecosystems has always been an important motivation in the development of game theory. We contribute to this part of the research by creating a framework that allows an easy incorporation of game-theoretic methods. We propose a general procedure that given a time series of data, extracts a reactive model that can be used to predict the evolution of the system and advise on the strategies to achieve the predefined goals.

This is a case study, in which we were presented a data set to work with. Due to the nature of the data, we have decided to use *Markov decision processes* as our models of choice and *Baum-Welch* procedure to teach our models. Nevertheless, the described procedure is general and, if the data would allow, both the model and the teaching procedure can be replaced effortlessly.

## Organisation of the thesis and main results

The main theoretical work of this thesis consist in the studies of *branching games* with regular winning objectives. The secondary achievement of this thesis shows how game theory in conjunction with machine learning can be used in real-life applications in modern agriculture.

Before we proceed, we need to fix some notation. Let  $\mathcal{B}$  denote the branching vertices and the symbols  $E, A, \mathcal{N}$  denote the players Eve, Adam, *Nature*, respectively. If for the player's vertices we use the symbols of the players, then for a set  $S \subseteq \{E, A, \mathcal{N}, \mathcal{B}\}$  we say that game is  $S$ -branching if the symbols of vertices present in its board belong to the set  $S$ . By  $val_G^{XP}$  we denote the value that player  $P \in \{E, A\}$  can enforce in game  $G$  using only pure ( $\varepsilon$ ), behavioural ( $B$ ), or mixed ( $M$ )<sup>1</sup>. For instance,  $val_G^E$  is the value that Eve can enforce using only pure strategies and  $val_G^{MA}$  is the value that Adam can enforce using mixed strategies.

## Pure branching games

*Pure branching games* are the family of branching games with no stochastic elements, i.e. games with no random vertices. Moreover, when considering pure branching games we allow *pure strategies* only.

For those games the notion of a *winning strategy* can be defined. This allows to associate the *pure values* of the game and the determinacy with the existence of a winning strategy: the values belong to a binary set and the game is determined if and only if one of the players has a winning strategy.

Pure branching games are not necessarily determined under pure strategies [11]. Thus, we discuss the complexity of computing the values of a game and deciding the determinacy.

We start the discussion with the case of single-player games, which are necessarily determined under pure strategies.

**Theorem 1.** *Let  $G = \langle \mathcal{B}, L \rangle$  be a finitary branching game. If the game is  $\{E, \mathcal{B}\}$ -branching then deciding whether Eve has a winning strategy*

- *is in  $UP \cap co-UP$ , if  $L$  is given by a non-deterministic automaton,*
- *is  $EXP$ -complete, if  $L$  is given by an alternating automaton.*

*If the game is  $\{A, \mathcal{B}\}$ -branching then deciding whether Eve has a winning strategy*

- *can be done in  $UP \cap co-UP$ , if  $L$  is given by a game automaton,*
- *is  $EXP$ -complete, if  $L$  is given by a non-deterministic or an alternating automaton.*

In the case of finite two-player games, we show that the sets of pure winning strategies are regular sets of trees. From this we conclude that there is an algorithm that computes pure partial values.

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<sup>1</sup>If the game is clear from context, we often drop it from the subscript.

**Theorem 2.** *Let  $G$  be a  $\{A, E, \mathcal{B}\}$ -branching game with winning set given by an alternating automaton on trees and  $val \in \{val^E, val^A\}$ . Then, the value  $val$  problem is 2-EXP-complete.*

In consequence, we can decide whether a given game is determined under pure strategies.

**Corollary 3.** *Let  $G$  be a  $\{A, E, \mathcal{B}\}$ -branching game with winning set given by an alternating tree automaton. Then, there is an algorithm that in doubly exponential time decides whether game  $G$  is determined.*

We also provide an alternative proof of the existence of an undetermined branching game. The proof is based purely on computational complexity and is an immediate consequence of the following theorem.

**Theorem 4.** *Let  $G$  be an  $\{A, E, \mathcal{B}\}$ -branching game with a winning set given by a non-deterministic automaton. Then, the value  $val^E$  problem is 2-EXP-complete, and the value  $val^A$  problem is EXP-complete.*

Finally, we present an interesting properties of the branching games: the *dealternation* of the winning set.

**Lemma 5.** *There exists a polynomial time reduction that inputs a  $\{A, E, \mathcal{B}\}$ -branching game  $G$  with the winning condition given as an alternating tree automaton and constructs a  $\{A, E, \mathcal{B}\}$ -branching game  $G'$  with the winning condition given by a non-deterministic tree automaton, such that  $val_G^E = val_{G'}^E$ .*

Because of complexity reasons, such an operation is impossible without branching elements of the arena, recall that in  $\omega$ -regular games on graphs an exponential blow-up is unavoidable.

The above results were published in [17, 19].

## Stochastic branching games

We now lift the restrictions on the types of vertices and the type of strategies. Here we study the branching games with stochastic elements, i.e. we allow random vertices, and both behavioural and mixed strategies.

We show that branching games with regular objectives are not necessarily determined even under mixed strategies.

**Theorem 6.** *There is an  $\{A, E, \mathcal{B}\}$ -branching game with the winning set being a difference of two open sets that is not determined under mixed strategies.*



On the other hand, we show that if the winning objective is topologically, relatively, simple, i.e. is an open (or a closed) set, then the game is determined under mixed strategies.

**Theorem 7.** *Branching games with closed (resp. open) regular objectives are determined under mixed strategies.*

Contrary to the pure branching games, the computational complexity of deciding the value of a branching game with a regular winning set is intractable. We show that even in the single-player case there is no algorithm that can compute a value of an arbitrary branching game. In particular, we show that deciding whether a value of a branching game with effectively encoded regular winning set is strictly greater than a certain threshold is undecidable.

**Theorem 8.** *The value  $V$  problem of a regular branching game  $G = \langle \mathbf{B}, L \rangle$  is undecidable for every partial value  $V \in \{val^A, val^{BA}, val^{MA}, val^{ME}, val^{BE}, val^E\}$ . The problem is undecidable even for a fixed single player board.*

Finally, we present another interesting property of the branching games: the *derandomisation* property. We show that we can modify in polynomial space both the board and the winning set so that the mixed values remain unchanged and the new game is regular and has no random vertices.

**Lemma 9.** *There exists a polynomial space procedure that inputs a branching game  $G = \langle \mathbf{B}, L(\mathcal{A}) \rangle$  with a regular winning set given by an alternating tree automaton and outputs a non-stochastic  $\{A, E, \mathcal{B}\}$ -branching game  $G' = \langle \mathbf{B}', L(\mathcal{C}) \rangle$  with the winning set given by an alternating tree automaton  $\mathcal{C}$  such that for every partial value  $V$  from the set  $\{val^{BA}, val^{MA}, val^{ME}, val^{BE}\}$  we have that  $V_G = V_{G'}$ .*

In consequence, we obtain another undecidability result.

**Corollary 10.** *For every  $V \in \{val^{BA}, val^{MA}, val^{ME}, val^{BE}\}$ , the value  $V$  problem of a regular  $\{E, A, \mathcal{B}\}$ -branching game is undecidable.*

## Game automata winning sets

An interesting class of branching games are regular branching games with winning sets given by the so-called game automata. Game automata are a syntactic restriction of the alternating automata on trees, see [3] for details. We show that those games reduce in polynomial time to stochastic meta-parity games introduced by Mio [12].

**Theorem 11.** *There exists a logarithmic space procedure using an  $UP \cap co-UP$  oracle that inputs a finitary branching game  $G = \langle B, L(\mathcal{A}) \rangle$  with a regular winning set given by a game automaton  $\mathcal{A}$  and outputs a stochastic meta-parity game  $G'$  such that  $val_{G'}^E = val_G^E$  and  $val_{G'}^A = val_G^A$ . Moreover, if  $S \subseteq \{E, A, \mathcal{N}, \mathcal{B}\}$  and  $G$  is  $S$ -branching then  $G'$ , as a branching game, is also  $S$ -branching.*

Since stochastic meta-parity games are determined [11] and their value is computable (an unpublished result by Mio), branching games with winning conditions given by game automata are determined under pure strategies and their value is computable.

**Corollary 12.** *Let  $G = \langle B, L(\mathcal{A}) \rangle$  be a finitary branching game with a regular winning set given by a game automaton  $\mathcal{A}$ . Then, the game is determined under pure strategies and the value problem is decidable.*

Additionally, if the game has no stochastic vertices, then we can prove an even stronger reduction.

**Corollary 13.** *Let  $S \subseteq \{A, E, \mathcal{B}\}$ . Then, there exists a logarithmic space procedure using an  $UP \cap co-UP$  oracle that inputs a non-stochastic  $S$ -branching game  $G = \langle B, L(\mathcal{A}) \rangle$  with a regular winning set given by a game automaton  $\mathcal{A}$  and outputs a parity game  $G'$  such that  $val_{G'}^E = val_G^E$  and  $val_{G'}^A = val_G^A$ .*

## Measures

We also attack the problem of computing the uniform measure of a regular set of trees. This problem can be seen as a special case of computing a value of a given *half-player* game, i.e. a game with only branching and random vertices. It turns out that, in some restricted classes of first-order definable sets of trees, we can use *Gaifman locality* to show that the measure of a set of trees is rational and computable.

**Theorem 14.** *Let  $\varphi$  be a first-order sentence over the signature  $\Gamma \cup \{\text{root}, s_L, s_R, s\}$ . Then, the measure  $\mu^*(L(\varphi))$  is rational and computable in three-fold exponential space.*

**Theorem 15.** *Let  $q$  be a conjunctive query over the signature  $\Gamma \cup \{\varepsilon, s_L, s_R, s, \boxplus\}$ . Then, the measure of the language  $L(q)$  is rational and computable in exponential space.*

We leave the general problem unsolved, but we give an example, inspired by Potthoff's example [16], of a first-order definable set of trees with irrational, but algebraic, measure. Moreover, we conjecture that the measures of regular sets of trees are algebraic.

Those results have been partially published in [18].

## Plantation game

We show how game theory, and stochastic games in particular, can be used to support modern agriculture. We propose a *plantation game* framework, where we show how using *time series* of data describing a plantation one can create a tool that can *model and predict* the behaviour of the plantation and *advise* the owner on the best actions. This is a case study, where we take a time series describing a real fruit plantation and, using machine learning methods, create a model and, later, a game that can represent the interactions between the different elements of the fruit farm. Then, we show how the game can be used to predict the evolution of the system and how to use the game to create an artificial advisor, connecting the theory with real life applications.

## Conclusions and future work

In the theoretical part of the thesis we study the properties of regular branching games. We can differentiate three groups of results. The first one concerns the determinacy of branching games, the second one the computational complexity of computing game values, and the last one the computational complexity of computing the measures of regular sets of trees.

**Determinacy** In the case of determinacy we have shown that regular branching games with open (closed) sets are determined under mixed strategies, see Theorem 7. This is the limit in the terms of topological hierarchy of sets: we provide an example of a regular branching game that has a winning set that is a difference of two open sets, but which is not determined under mixed strategies.

In the case of both pure and behavioural strategies even clopen sets or sets of trees of bounded depth do not guarantee determinacy. This is showcased in an example that combines the classic game of “Matching Pennies” with the observation of Mio, see [11, Example 4.1.18]. This example stays contrary to Nash-like results in the perfect-information games with perfect recall, which state that finite duration games with finite set of actions are determined under behavioural strategies.

Still, those results do not characterise the classes of winning sets that guarantee determination. Indeed, we show that branching games with game automata definable winning sets are determined under pure strategies, see Corollary 12. The game automata recognisable sets can be of big topological complexity, for details see e.g. [5, 15]. Therefore, we think that it would be interesting to find new families of winning sets that guarantee determinacy.

**Computing game values** We have solved the general problem of computing values of regular branching games. In particular, we have shown that

- there is no algorithm that given a branching game with an arbitrary regular winning set computes any of the game's partial values, see Theorem 8;
- there is no algorithm that given a non-stochastic branching game with an arbitrary regular winning set computes any of the behavioural or mixed values, see Corollary 10;
- there is an algorithm that given a non-stochastic branching game with a regular winning set computes all of the pure values, see Theorem 2.

The exact computational complexity of the algorithm in the last bullet depends on the kinds of vertices on the board and the representation of the winning set.

The negative results are not necessarily surprising: endowing systems of imperfect information with probability often leads to undecidability, e.g. probabilistic automata [20].

While the above results answer the question of computability in the general case negatively, the positive results give hope that a smart restriction on the set of possible winning sets may yield a class of branching games with computable values. An example of such a class are branching games with winning sets defined by game automata, for which the values coincide and are computable, see Corollary 12. Another promising class of regular winning sets may arise not from syntactic restrictions on automata, but by putting restrictions on the expressive power of monadic second-order logic. The class of special interest are the Boolean combinations of conjunctive queries, for the reason mentioned in the following paragraph.

**Measures** We have tackled the problem of computing the coin-flipping measure of a given regular set of trees. We have shown that

- there is an algorithm that given a first-order formula  $\varphi$  not using the descendant relation computes the uniform measure of the set  $L(\varphi)$ , see Theorem 14;
- there is an algorithm that given a Boolean combination of conjunctive queries  $\varphi$  computes the uniform measure of the set  $L(\varphi)$ , see Theorem 15.

The involved techniques use the notion of locality and cannot be extended to the full power of monadic second-order logic. Even the full first order-logic is not captured in the scope of those results.

An obvious direction of future research is to try to find algorithms computing the measure of any arbitrary regular set or at least for a set definable by some logic subsumed by monadic

second-order logic, e.g.  $CTL^*$ , *weak monadic second-order logic*, or *alternation free  $\mu$ -calculus*. A less obvious direction of research would be to extend the techniques presented in this thesis to arbitrary measures generated by graphs.

**Applications** We have also presented a generic way of using machine learning methods and game theory to create a simple advisor taught on a time series describing a closed ecosystem. The presented approach is simple yet robust, allowing easy exchange of used tools and techniques.

In this particular case we have used Markov decision processes and the Baum-Welch procedure to create a stochastic two-player game that represents a fruit farm. This game can be used to predict the presence of pests, in the form of fruit flies and to plan chemical treatments that will help with the management of the population of the flies.

The provided data did not allow us to use the rich theory we have developed in the study of branching games. Still, we think that designing and developing tools which incorporate branching games would be an interesting direction of future research. In our opinion, the systems that would benefit the most from the branching games representation are those, where two adversaries oversee a number of independent, non-communicating agents: e.g. virus outbreak, or breeding bacteria.

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