Obliczenia symboliczne i algorytmy kombinatoryczne w spektralnej klasyfikacji skończonych zbiorów częściowo uporządkowanych

Symbolic computations and combinatorial algorithms in spectral classification of finite partially ordered sets

Summary

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In the dissertation we study the Coxeter spectral classification of finite partially ordered sets (*posets*) introduced in [17, 21]. Our research is motivated by certain results known in the representation theory of finite groups and finite-dimensional algebras, and on the other hand, is inspired by spectral graph theory [4]. Due to the significant use of algorithmic tools and calculation methods, these research can be viewed as a part of *Scientific Computing*, which is an interdisciplinary field of scientific research, in which the possibility of using computational results to solve complex theoretical problems is examined.

Motivation

In the spectral graph theory, certain graph properties are studied by means of algebraic tools. More precisely, with any graph there is uniquely associated matrix (e.g. adjacency, Laplace or Seidl, see [5]) and based on the spectrum of this matrix, various structural characterizations of a given graph are obtained. For example, as the following theorem shows, the spectrum of the adjacency matrix of the graph encodes information of its regularity.

Theorem 1. [3, Theorem 1.3.13] *Let G be a simple graph and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of the adjacency matrix $\mathrm{Ad}_G \in \mathbb{M}_n(\mathbb{Z})$. The graph G is regular if and only if $\sum_{i=1}^{n} \lambda_i^2 = n \cdot \lambda_1$.*

One of the inspirations to use algebraic methods in graph theory was an attempt to find such an invariant of the graph that determines it *uniquely*, up to the isomorphism [3, 4]. Assume that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are simple graphs, where $V_1 = V_2 = \{1, \ldots, n\}$. The graphs G_1, G_2 are isomorphic if and only if, there exists a permutation matrix $B \in \mathbb{M}_n(\mathbb{Z})$, such that

$$\mathrm{Ad}_{G_1} = B^{tr} \cdot \mathrm{Ad}_{G_2} \cdot B. \qquad (*)$$

From the equality ($*$) it follows that the spectra of the adjacency matrices of isomorphic graphs coincide. The converse implication does not hold in general: a counterexample can be found in among the graphs with 5 vertices [4, 5]. Hence, the other (additional) invariants that characterise uniquely (up to isomorphism) broad classes of graphs are studied [3].

In the dissertation we study the classification problem of finite partially ordered sets (*posets*) up to the two equivalences: $\sim_{\mathbb{Z}}$ and $\approx_{\mathbb{Z}}$, introduced in [17, 19] that are analogous to ($*$). With every poset $I = (\{1, \ldots, n\}, \preceq_I)$ we associate the incidence matrix $C_I = [c_{ij}] \in \mathbb{M}_n(\mathbb{Z})$, where $c_{ij} = 1$ if $i \preceq_I j$ and $c_{ij} = 0$ otherwise (see [17, 21]), and the symmetric Gram matrix $G_I := \frac{1}{2}\left(C_I + C_I^{tr}\right) \in \mathbb{M}_n(\frac{1}{2}\mathbb{Z})$. The posets I and J are called:

- *quadratic $\mathbb{Z}$-equivalent* ($I \sim_{\mathbb{Z}} J$), if $G_I = B^{tr} \cdot G_J \cdot B$,
- *bilinear $\mathbb{Z}$-equivalent* ($I \approx_{\mathbb{Z}} J$), if $C_I = B^{tr} \cdot C_J \cdot B$,

where $B \in \mathbb{M}_n(\mathbb{Z})$ is such a matrix that $\det B = \pm 1$.

Moreover, following [17], with every finite poset I we associate:

- **the Coxeter matrix** $\mathrm{Cox}_I := -C_I \cdot \left(C_I^{tr}\right)^{-1} \in \mathbb{M}_n(\mathbb{Z})$,
- **the Coxeter spectrum** $\mathbf{specc}_I := \{t \in \mathbb{C};\ \det(t \cdot E - \mathrm{Cox}_I) = 0\} \subseteq \mathbb{C}$.

Note that the equivalences $\approx_{\mathbb{Z}}$ and $\sim_{\mathbb{Z}}$ are more general than the isomorphism, because the posets isomorphism $I \simeq J$ implies the equivalences $I \approx_{\mathbb{Z}} J$ and $I \sim_{\mathbb{Z}} J$, but the converse implication does not hold in general.

It is shown in [18, 19], that the equivalence $I \approx_{\mathbb{Z}} J$ implies the equality of Coxeter spectra $\mathbf{specc}_I = \mathbf{specc}_J$ and the equality $I \sim_{\mathbb{Z}} J$. The main problem considered in the Coxeter spectral analysis of posets is to describe a broad class of connected nonnegative posets I that are determined by the Coxeter spectrum $\mathbf{specc}_I \subseteq \mathbb{C}$ uniquely, up to the relation $\approx_{\mathbb{Z}}$.

The Coxeter spectral classification of finite partially ordered sets can be viewed as a special case of edge-bipartite graphs classification [19–21] and it often uses the methods developed there. On the other hand, it is inspired by the representation theory of finite-dimensional algebras [1] and matrix representations of finite posets [7, 16]. Therefore, the main applications of the results presented in the dissertation are in these areas and are discussed in the articles [17–19], see also [7, Remark 5.12], and [14] oraz [13].

The aim and the main results

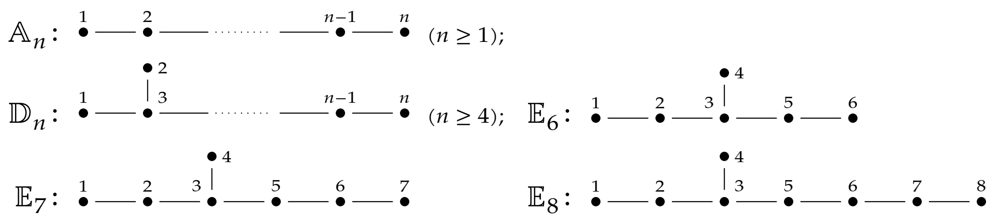
The aim of the research presented in the dissertation is the classification of the finite partially ordered sets with the symmetric Gram matrix positive semidefinite (i.e. *nonnegative posets*), up to the equivalences $\sim_{\mathbb{Z}}$ and $\approx_{\mathbb{Z}}$. The dissertation contains partial solutions to the following four problems formulated in the articles [17, 19, 20].

- Classify all finite posets up to the relation $\approx_{\mathbb{Z}}$.
- Classify all finite posets up to the relation $\sim_{\mathbb{Z}}$.
- Define a *minimal* set of spectral invariants that determines a broad class of partially ordered sets uniquely, up to the relation $\approx_{\mathbb{Z}}$.
- Construct efficient algorithms for spectral analysis of non-negative posets. In particular, the algorithms that:

– determine a corank of a finite poset,
– generate all, up to the isomorphism, nonnegative posets of a given corank,
– compute the set of roots,
– determine the Dynkin type of a non-negative poset,
– construct an $\mathbb{Z}$-invertible matrix $B \in \mathbb{M}_n(\mathbb{Z})$ that defines the $\approx_{\mathbb{Z}}$ relation between connected nonnegative posets I and J.

In the dissertation we consider finite *nonnegative posets I of corank* $\mathbf{crk}_I \in \{0, 1, 2\}$ of n elements, i.e. with the symmetric Gram matrix $G_I \in \mathbb{M}_n(\frac{1}{2}\mathbb{Z})$ positive semidefined of rank $\mathbf{rz}\, G_I = n - \mathbf{crk}_I \in \{n, n-1, n-2\}$.

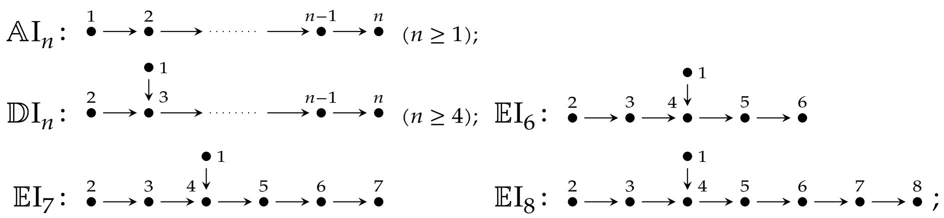
In addition to the Coxeter spectrum $\mathbf{specc}_I \subseteq \mathbb{C}$, in considered cases, an effective classification tool is the Dynkin type, i.e. unlabelled Dynkin diagram $\mathbf{Dyn}_I \in \{\mathcal{A}_n, \mathcal{D}_n, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8\}$. We recall from [2], that the simply laced Dynkin diagrams are the following simple graphs.



The most important theoretical results presented in the dissertation are as follows.

(a) Classification of connected posets I of corank $\mathbf{crk}_I = 0$, that are *one-peak* (i.e. have exactly one maximal element) or $|I| \leq 14$:

- up to the relations $\sim_{\mathbb{Z}}$ and $\approx_{\mathbb{Z}}$, I is one of the Dynkin posets $\mathbb{A}\mathrm{I}_n$, $\mathbb{D}\mathrm{I}_n$, $\mathbb{E}\mathrm{I}_6$, $\mathbb{E}\mathrm{I}_7$, $\mathbb{E}\mathrm{I}_8$:



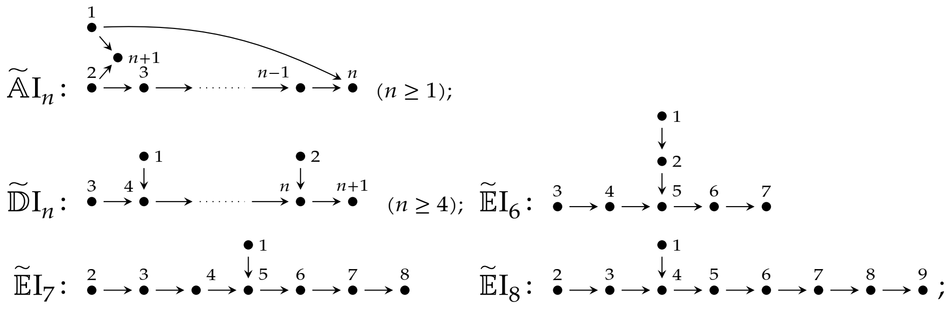
- the following equivalences hold:

$$I \approx_{\mathbb{Z}} J \;\Leftrightarrow\; I \sim_{\mathbb{Z}} J \;\Leftrightarrow\; \mathbf{specc}_I = \mathbf{specc}_J \;\Leftrightarrow\; \mathbf{Dyn}_I = \mathbf{Dyn}_J;$$

- all one-peak posets I, up to the isomorphism are described by four infinite series and 193 Hasse quvers with 6, 7, 8 vertices.

(b) Classification of connected posets I with $\mathbf{crk}_I = 1$, that are *one-peak* or $|I| \leq 15$:

- up to the relation $\sim_{\mathbb{Z}}$ (and $\approx_{\mathbb{Z}}$, if I is a one-peak poset) I is one of the Euclidean posets $\widetilde{\mathbb{A}}\mathrm{I}_n$, $\widetilde{\mathbb{D}}\mathrm{I}_n$, $\widetilde{\mathbb{E}}\mathrm{I}_6$, $\widetilde{\mathbb{E}}\mathrm{I}_7$, $\widetilde{\mathbb{E}}\mathrm{I}_8$:



- the following equivalence holds:

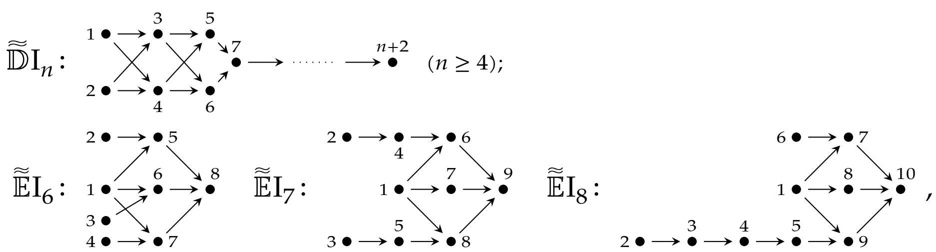
$$I \approx_{\mathbb{Z}} J \Leftrightarrow (\mathbf{specc}_I, \mathbf{Dyn}_I) = (\mathbf{specc}_J, \mathbf{Dyn}_J);$$

- all one-peak posets I, up to the isomorphism are described by seven infinite series and 422 Hasse quvers with $7, 8, 9$ vertices;
- if I is a one peak poset, then the following equivalences hold:

$$I \approx_{\mathbb{Z}} J \quad\Leftrightarrow\quad I \sim_{\mathbb{Z}} J \quad\Leftrightarrow\quad \mathbf{specc}_I = \mathbf{specc}_J \quad\Leftrightarrow\quad \mathbf{Dyn}_I = \mathbf{Dyn}_J.$$

(c) Classification of connected posets I of corank $\mathbf{crk}_I = 2$, that are *one-peak* or $|I| \leq 15$:

- up to the relation $\sim_{\mathbb{Z}}$, I is one of the Euclidean posets of corank two $\widetilde{\widetilde{\mathbb{D}}}\mathrm{I}_n$, $\widetilde{\widetilde{\mathbb{E}}}\mathrm{I}_6$, $\widetilde{\widetilde{\mathbb{E}}}\mathrm{I}_7$, $\widetilde{\widetilde{\mathbb{E}}}\mathrm{I}_8$:



- the following equivalence holds:

$$I \approx_{\mathbb{Z}} J \Leftrightarrow (\mathbf{specc}_I, \mathbf{Dyn}_I) = (\mathbf{specc}_J, \mathbf{Dyn}_J),$$

  in particular, if $|I| \notin \{9, 10\} \not\ni |J|$, then the following equivalences hold:

$$\mathbf{specc}_I = \mathbf{specc}_J \quad\Leftrightarrow\quad I \approx_{\mathbb{Z}} J \quad\Leftrightarrow\quad I \sim_{\mathbb{Z}} J \quad\Leftrightarrow\quad \mathbf{Dyn}_I = \mathbf{Dyn}_J;$$

- all one-peak posets I, up to the isomorphism are described by 14 series and 426 incidence matrices;
- up to the relation $\approx_{\mathbb{Z}}$ a one-peak poset I is one of the Euclidean poset of corank two $\widetilde{\widetilde{\mathbb{D}}}\mathrm{I}_n$, $\widetilde{\widetilde{\mathbb{E}}}\mathrm{I}_6$, $\widetilde{\widetilde{\mathbb{E}}}\mathrm{I}_7$, $\widetilde{\widetilde{\mathbb{E}}}\mathrm{I}_8$ and of Dynkin type $\mathbf{Dyn}_I \in \{\mathcal{D}_{|I|-2}, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8\}$;
- if I and J are one-peak, then the following equivalences hold:

$$I \approx_{\mathbb{Z}} J \quad\Leftrightarrow\quad I \sim_{\mathbb{Z}} J \quad\Leftrightarrow\quad \mathbf{specc}_I = \mathbf{specc}_J \quad\Leftrightarrow\quad \mathbf{Dyn}_I = \mathbf{Dyn}_J.$$

The most important combinatorial algorithms presented in the dissertation are the following two algorithms of exponential running time. They allow to construct an $\mathbb{Z}$-invertible matrix $B \in \mathbb{M}_n(\mathbb{Z})$ that defines the equivalence $\approx_{\mathbb{Z}}$ between a pair of finite posets I and J.

(a) Exhaustive search algorithm for the case $\mathbf{crk}_I = 0 = \mathbf{crk}_J$. This algorithm guarantees to find a requested matrix, thus allows deterministic verification of the bilinear $\mathbb{Z}$-equivalence. In the dissertation we discuss a simple modification of the algorithm that guarantees the computation of *all* matrices defining the $\approx_{\mathbb{Z}}$ equivalence.

(b) Heuristic algorithm for the case $\mathbf{crk}_I, \mathbf{crk}_J \in \{1, 2\}$. This algorithm is sensitive to input data and does not guarantee the determination of a requested matrix.

In addition, in the dissertation we discuss our implementation of a package of algorithms for Coxeter spectral analysis of non-negative partially ordered sets, including the discussion of their computational complexity. These algorithms are a basic tool in proofs of the majority of theoretical results presented in the dissertation.

One of the applications of the Coxeter spectral classification of one-peak positive posets presented in the dissertation is the proof of the existence of only a finite number of *Tits-sincere* positive posets. We apply it in an alternative proof of the theorem on the existence of a finite number of *almost TP-critical* posets, significantly simpler than presented in [15].

Algorithmic and theoretical tools

Symbolic and combinatorial algorithms play a crucial rôle in the dissertation. They are a basis of the presented experimental results and are an integral part of the classification proofs. Due to the nature of the calculations, the emphasis in the design of algorithms was placed on the correctness of the obtained results. The issue of minimizing the running time of algorithms is of secondary priority, because the most time-consuming algorithms are required to be run only once.

In the dissertation we use not only newly developed algorithms, but also our original implementations of known numerical algorithms (eg Sylvester's algorithm), dedicated symbolic algorithms as well as publicly accessible programming libraries (for the needs of solving the graph isomorphism problems or integer linear programming).

One of the most important theoretical tools used in the dissertation are the *abstract root systems* in the sense of [2]. Using them, with any finite connected partially ordered set I of m elements of corank $\mathbf{crk}_I = r \in \{0, 1, 2\}$ we uniqly associate the simply laced Dynkin diagram $\mathbf{Dyn}_I \in \{\mathcal{A}_{m-r}, \mathcal{D}_{m-r}, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8\}$, which (in some cases) defines I uniqly, up to the equivalence $\approx_{\mathbb{Z}}$. The Dynkin diagram $\mathbf{Dyn}_I \in \{\mathcal{A}_{m-r}, \mathcal{D}_{m-r}, \mathcal{E}_6, \mathcal{E}_7, \mathcal{E}_8\}$ in fact is a Coxeter graph of the root system $\mathcal{R}_J = \{v \in \mathbb{Z}^{m-r};\, v \cdot G_J \cdot v^{tr} = 1\} \subseteq \mathbb{Z}^{m-r}$ determined by positive poset $J \subseteq I$ of $m - r$ elements.

A very important tool in the Coxeter spectral analysis of finite partially ordered sets is also the *Φ_I-mesh root system* $\Gamma(\mathcal{R}_I, \Phi_I)$ in the sense of [18].

The set $\mathcal{R}_I \subseteq \mathbb{Z}^n$ of *roots of unity* of the poset I of corank $\mathbf{crk}_I \in \{0, 1, 2\}$ plays a crucial rôle in the construction of algorithms presented in the dissertation. The main reason of that is the fact that the columns of any $\mathbb{Z}$-invertible matrix $B \in \mathbb{M}_n(\mathbb{Z})$ that defines the equivalence $I \approx_{\mathbb{Z}} J$ between posets of n-elements belong to the set $\mathcal{R}_J \subseteq \mathbb{Z}^n$.

Publications

Some of the results presented in the dissertation has been supported by NCN grant 2011/03/B/ST1/00824 and published in the following international journals:

- *Linear Algebra and its Applications* [7, 10, 11],
- *European Journal of Combinatorics* [9],
- *Fundamenta Informaticae* [12],
- *Colloquium Mathematicum* [6],
- *Algebra and Discrete Mathematics* [8].

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