RESOLUTIONS OF QUOTIENT SINGULARITIES AND THEIR COX RINGS

EXTENDED ABSTRACT

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The resolution of singularities allows one to modify an algebraic variety to obtain a nonsingular variety sharing many properties with the original, singular one. The study of nonsingular varieties is an easier task, due to their good geometric and topological properties. By the theorem of Hironaka [29] it is always possible to resolve singularities of a complex algebraic variety.

The dissertation is concerned with resolutions of quotient singularities over the field $\mathbb{C}$ of complex numbers. Quotient singularities are the singularities occuring in quotients of algebraic varieties by finite group actions. A good local model of such a singularity is a quotient $\mathbb{C}^n/G$ of an affine space by a linear action of a finite group. The study of quotient singularities dates back to the work of Hilbert and Noether [28], [40], who proved finite generation of the ring of invariants of a linear finite group action on a ring of polynomials. Other important classical result is due to Chevalley, Shephard and Todd [10], [46], who characterized groups G that yield non-singular quotients. Nowadays, quotient singularities form an important class of singularities, since they are in some sense typical well-behaved (*log terminal*) singularities in dimensions two and three (see [35, Sect. 3.2]). Methods of construction of algebraic varieties via quotients, such as the classical Kummer construction, are being developed and used in various contexts, including higher dimensions, see [1] and [20] for a recent example of application.

In the well-studied case of surfaces every quotient singularity admits a unique minimal resolution. The geometry of such a resolution can be described by the incidence diagram of the exceptional curves together with their self-intersection numbers. Among surface quotient singularities one distinguishes the class of Du Val singularities (see e.g. [22]) associated with Dynkin diagrams, which appear also in representation theory.

The modern notion of a crepant resolution generalizes the notion of the minimal resolution of a Du Val singularity. A remarkable difference is that in higher dimension a crepant resolution is rarely unique and it may even not exist at all. Nevertheless, the minimality property makes crepant resolutions highly desired objects. Moreover, the problem of non-existence can be avoided by introducing a further generalization, a minimal model, which is a partial resolution admitting a certain minimality property and which for a quotient singularity always exists. The study of crepant resolutions and minimal models of quotient singularities shows an interesting interplay of many techniques from various fields. Among them are geometry, representation theory, topology and even mathematical physics (see [15, 14]). One may observe such connections in the research on the McKay correspondence, which describes the relation between the structure of a group and the geometry of minimal models of a quotient (see [42], [44] for a survey). This research field appeared at the end of the previous century and is still developing.

The aim of this work is to develop general methods to study crepant resolutions of a quotient singularity via their Cox ring and apply them to three- and four-dimensional examples exhibiting interesting phenomena.

The Cox ring of a normal complex algebraic variety X with a finitely generated class group $\mathrm{Cl}(X)$ is a $\mathrm{Cl}(X)$-graded ring:

$$\mathcal{R}(X) = \bigoplus_{D \in \mathrm{Cl}(X)} H^0(X, D),$$

where multiplication is defined via identification of the space of sections $H^0(X, D)$ with a subspace of the field of rational functions on X. According to ideas presented in the seminal paper by Hu and Keel [30] if $\mathcal{R}(X)$ is a finitely generated $\mathbb{C}$-algebra it is a powerful tool to study the geometry of X and its small modifications. In particular one may recover X and all its codimension two modifications as quotients of open subsets of $\operatorname{Spec} \mathcal{R}(X)$ by the action of the characteristic quasitorus $\mathbb{T} = \mathrm{Hom}(\mathrm{Cl}(X), \mathbb{C}^*)$. This observation is the central theme of our work, the basis of all our research that contributes to this dissertation. The finite generation assumption is satisfied when X is a minimal model of a quotient singularity. In this case one may recover all the minimal models of the singularity together with birational relations between them (flops, see [43]) by using the geometric invariant theory (*GIT*) to construct quotients corresponding to chambers in a certain subdivision of the movable cone of X. Knowing the Cox ring one may find the movable cone of X, its subdivision and GIT quotients corresponding to chambers.

1. Motivation and state of the art

The existence and geometry of crepant resolutions of quotient singularities is an active field of research. Here we outline problems that motivated our work and summarize what was known before and what developed while we were conducting the research presented in the thesis.

Existence and construction of symplectic resolutions. The open problem that originally motivated our work is the problem of existence and construction of crepant resolutions of symplectic quotient singularities. A crepant resolution of such a singularity preserves the symplectic structure and thus is often called a symplectic resolution (the converse also holds – a resolution preserving the symplectic structure is crepant). Such resolutions may find applications in constructing Hyperkähler manifolds via the generalized Kummer construction [1]. Finding new Hyperkähler manifolds is an important and difficult problem in complex geometry.

A theorem of Verbitsky [48] gives a necessary (but insufficient) criterion for the existence of a crepant resolution of a quotient $\mathbb{C}^{2n}/G$, where G is a finite group of linear transformations preserving the symplectic form on $\mathbb{C}^{2n}$. The condition says that the group G has to be generated by symplectic reflections, i.e. elements which fix a subspace of codimension 2.

Finite groups generated by symplectic reflections were classified by Cohen in [11]. The problem of existence of symplectic resolutions was investigated in [23, 4, 6, 7, 50]. According to [4] and [7, 4.1], for the following groups it is known that a symplectic resolution exists:

(1) S_{n+1} acting on $\mathbb{C}^{2n}$ via direct sum of two copies of the standard n-dimensional representation of S_{n+1}. Here a resolution can be constructed via the Hilbert scheme $\mathrm{Hilb}^{n+1}(\mathbb{C}^2)$. We study the example of S_3 which is the only four-dimensional representative of his family.
(2) $H \wr S_n = H^n \rtimes S_n$, where H is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$, so that $\mathbb{C}^2/H$ is a Du Val singularity, and S_n acts on H^n by permutations of coordinates. The natural product representation of H^n on $\mathbb{C}^{2n}$ extends naturally to the action of $H \wr S_n$. One of the resolutions is $\mathrm{Hilb}^n(S)$ where S is the minimal resolution of the Du Val

singularity $\mathbb{C}^2/H$. We study the smallest nontrivial four-dimensional example of this family: $\mathbb{Z}_2 \wr S_2$. Note that for $n = 1$ one obtains Du Val singularities and their minimal resolutions.

(3) A certain four-dimensional representation of the binary tetrahedral group. The existence of crepant resolutions was first proved by Bellamy in [4]; they were constructed later by Lehn and Sorger, see [36]. We study this example in the dissertation.

(4) A group of order 32 acting on $\mathbb{C}^4$, for which the existence of crepant resolutions was proved in [6], and their construction and the Cox ring were investigated in [21] and [25].

Apart from this list, [4], [7] and [50] give also negative results. The non-existence of crepant resolutions is shown for certain infinite families of representations from the Cohen's classification [11]. Notably, the results of the paper [50] that appeard when this work was in progress rely on the techniques developed in [21] and [17] (the second paper contributed partially to this thesis), using Cox rings. On the other hand, even in dimension 4 there still are infinitely many groups on the Cohen's list for which the question of existence of a crepant resolution remains unanswered.

There are also other recent results on the birational geometry of symplectic quotient singularities, related to our work. For example in [5], Bellamy used the work of Namikawa [39] to find the number of minimal models of a given symplectic quotient singularity. And as we were finishing this thesis, there appeared paper [3] by Bellamy and Craw in which they study an interpretation of resolutions for wreath products $H \wr S_n$ as in (2) above as certain moduli spaces, constructing them with the use of GIT.

Geometry of three-dimensional crepant resolutions. Three-dimensional quotient singularities $\mathbb{C}^3/G$ for $G \subset \mathrm{SL}_3(\mathbb{C})$ have also been studied extensively. In this case it is known, see [31, 32, 37, 45], that a crepant resolution exists. It is usually non-unique, but each two crepant resolutions differ by a modification in codimension two, and the set of all such resolutions for a given quotient singularity is finite. This set together with birational relations between its elements form a natural object of study.

As shown in [9], a crepant resolution of $\mathbb{C}^3/G$ can be constructed as the equivariant Hilbert scheme G-Hilb, which was conjectured by Nakamura in [38] (see also [13] for the case of G abelian). In [12] all small $\mathbb{Q}$-factorial modifications of G-Hilb for G abelian are analysed and constructed as certain moduli spaces. It was also conjectured that this might be the case for arbitrary finite subgroup of $\mathrm{SL}_3(\mathbb{C})$ – this is the celebrated Craw-Ishii conjecture. Another successful approach to investigating such resolutions is based on homological methods and noncommutative algebra [8, 49, 41]. However, up to now, significant results have been obtained only for groups not containing any elements of age 2. Geometrically, this condition is equivalent to saying that the resolution does not contract a divisor to a point (see [33]), which apparently makes these resolutions easier to deal with. See e.g. [41] for an application of this method to finite subgroups $G \subset \mathrm{SO}_3$, in particular to representations of dihedral groups, which we study in the dissertation.

Cox rings of crepant resolutions. The case of surface quotient singularities was analyzed by Donten-Bury in [16]. In [21] Donten-Bury and Wiśniewski studied the example of the symplectic quotient singularity of dimension 4 by a certain group of order 32 (see (4) above, in the part concerning symplectic quotient singularities). These two papers lied down the ground for the study of crepant resolutions X of quotient singularity $\mathbb{C}^n/G$ via Cox ring $\mathcal{R}(X)$ by use of the natural embedding

$$\mathcal{R}(X) \subset \mathbb{C}[x_1, \dots, x_n]^{[G,G]}[\mathrm{Cl}(X)].$$

Here $[G,G] \subset G$ is the commutator subgroup and the rank of the finitely generated free abelian group $\mathrm{Cl}(X)$ may be computed via the McKay correspondence of Ito and Reid [33]. In other words, it was known form these works how to embed the Cox ring of a resolution of a quotient singularity given by G into the Laurent polynomial ring over the ring of invariants of the commutator subgroup of G. This embedding is the starting point for the research presented here.

Prior to our studies there were also known computational methods by Hausen, Keicher and Laface [26], [25] of finding Cox rings of birational modifications of a variety with known Cox ring. The main difficulty to apply them in our context is describing the crepant resolution in terms of the blow-up operation (but see [25] for an application to singularity studied in [21]).

There are also methods specific to the complexity one case, i.e. for varieties with the effective action of an algebraic torus of rank one smaller that the dimension of the variety [27]. They were generalized from complete to not necessarily complete case by Hausen and Wrobel in [34] when this work was in progress. As our ultimate goal is to understand the more general case we do not use these results.

As this work was in progress there appeared two more articles developing computational methods related to resolving singularities via Cox rings. First is the paper [19] by Donten-Bury and Keicher in which the methods of [26] were combined with tropical geometry to give an algorithm that in some cases computes the Cox ring of a resolution of a quotient singularity. This algorithm also relies heavily on the computational power, and we do not use it in this work. The second was the work of Yamagishi [50] providing an algorithm to check the slightly generalized valuative criterion given in [17] (in this work we present the criterion in even more general form, see theorem 4.1.13). We use a part of this algorithm, with some improvements, once, to study the geometry of crepant resolutions in the case of a three-dimensional quotient singularity given by an irreducible representation containing elements of age 2. The quick growth of computational complexity of this algorithm with the growth of the number of crepant divisors as well as the fact that it does not always produce the minimal set of generators suggests to seek for other methods.

As we mentioned before, there is an ongoing research on explicit minimal model program with use of homological methods by Wemyss and others [49], and the research on the symplectic varieties with use of noncommutative algebra by Bellamy, Namikawa, Schedler and others [4], [6, 39, 5]. We believe that as these studies develop – the one via Cox rings presented here, and the ones via homological methods and noncommutative algebra – each of them may benefit from the existence of the other ones.

2. Results of the dissertation

The dissertation contains general results giving methods to compute Cox ring of a crepant resolution without thorough knowledge of a geometry of such a resolution (and so allows one to construct such a resolution). These methods are then applied to study Cox rings and the geometry of resolutions for concrete (families of) quotient singularities. What is more, in the analysis of examples of symplectic quotient singularities we develop techniques using torus action. We plan to generalize this last set of methods and use in further studies. More detailed description of all these results can be found below.

Apart from that the work contains a few generalizations of results concerning the relation between the Cox ring of a variety and the birational geometry of this variety. Such results were present before in the literature under more restrictive assumptions. See in particular proposition 3.2.9 describing the cone of movable divisors and theorem 3.4.7

characterizing relative Mori Dream Spaces in terms of finite generation of their Cox ring. The latter result is then used to show that minimal models of quotient singularities defined by subgroups of $\mathrm{SL}_n(\mathbb{C})$ are relative Mori Dream Spaces (theorem 3.4.10). In other words, the Cox ring of such a minimal model allows one to recover all minimal models of the singularity and flops among them by means of GIT. For all these results proofs presented in the dissertation does not differ essentially from proofs of their special cases already present in the literature.

Finally, in part 6.4 of the work we introduce the notion of an equivariant Euler characteristic for a variety which is projective over an affine variety. We use this notion in the chapter dedicated to symplectic quotient singularities with a torus action. We define equivariant Euler characteristic in this setting as a certain formal Laurent series (definition 6.4.20) under an additional assumption on the weights of the torus action (assumption 6.4.15). It turns out, that just as in the projective case one may express equivariant Euler characteristic as an expansion of a certain rational function depending on weights of the torus action (corollary 6.4.22). The proof, as in the projective case, relies on the localization formula in the equivariant K-theory [47].

2.1. **General results.** Two general results are concentrated around the construction using the natural embedding

$$\Theta\colon \mathcal{R}(X) \to \mathbb{C}[x_1, \dots, x_n]^{[G,G]} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{Cl}(X)]$$

of the Cox ring of a crepant resolution $X \to \mathbb{C}^n/G$ for a finite subgroup $G \subset \mathrm{SL}_n(\mathbb{C})$ into the ring of Laurent polynomials over the ring of invariants of the commutator group $[G, G] \subset G$. This embedding was described in [16] and [21]. A natural idea is to describe generators of $\mathcal{R}(X)$ in terms of embedding Θ, since the structure of this bigger ring is easier to describe. In particular the finitely generated free abelian group $\mathrm{Cl}(X)$ is described by McKay correspondence in the version of Ito and Reid [33]. In chapter 4 we consider in a greater generality the construction, which for a resolution of a quotient singularity as above yields a finite set of elements of $\Theta(\mathcal{R}(X))$ for a given set of generators of $\mathbb{C}[x_1, \dots, x_n]^{[G,G]}$. In the following part we give two conditions equivalent to the fact that the "candidate set" for generators constructed in this way actually generates $\Theta(\mathcal{R}(X))$. The first condition, the *valuation compatibility criterion* (see theorem 4.1.13) is expressed in terms of valuations of exceptional divisors (computed by the McKay correspondence, see proposition 4.1.22) and lifts of this valuations via the map from the polynomial ring given by the choice of the "candidate set". The second condition (theorem 4.2.3) is based on the characterization theorem for Cox rings [2, Thm. 1.6.4.3].

The third important general result, or actually the combination of two such results (propositions 4.3.4 and 4.3.5) gives a bound for degrees of generators of the Cox ring with use of the multigraded Castelnuovo-Mumford regularity and the Kawamata-Viehweg vanishing.

2.2. **Geometry of concrete resolutions of singularities in dimension 3.** In chapter 5 we study Cox rings and geometry of resolutions of the singularities of the form $\mathbb{C}^3/G$ for $G \subset \mathrm{SL}_3(\mathbb{C})$. In particular, we prove that for G nonabelian and giving reducible representation on $\mathbb{C}^3$, the Cox ring of a crepant resolution is generated by $m + 4$ elements related by a single trinomial relation, where m is the number of conjugacy classes of elements of age 1 in G. The proof uses the valuation compatibility criterion. We also give the precise description of these $m + 4$ generators in terms of generators of $\mathbb{C}[x_1, x_2, x_3]^{[G,G]}$ and the recipe to obtain the trinomial relation. See theorem 5.2.3 and remark 5.2.7.

In the further part of the chapter we study in detail the case when G is a faithful representation of a dihedral group D_{2n}. In this case we also give an alternative proof of the above theorem, using the criterion based on the characterization theorem for Cox rings. Then we find the movable cone and its subdivision into GIT chambers corresponding to resolutions and flops between them. We describe how flops change the central fibre, i.e the fibre over $[0] \in \mathbb{C}^3/G$.

We also study the geometry of resolutions of simplest resolutions contracting divisor to a point, i.e. resolutions for smallest subgroups of $\mathrm{SL}_3(\mathbb{C})$ containing elements of age 2. One of the studied examples belongs to the class of irreducible representations, which not satisfy assumptions of theorem 5.2.3. In each case we find the Cox ring, the cone of movable divisors and its subdivision. In one case we also describe flops and how they change the central fibre.

2.3. **Symplectic case with a torus action.** In the final chapter we exemplify the techniques for bounding degrees of generators of the Cox ring in the study of three symplectic quotient singularities in dimension four. In each case we find the Cox ring and describe the central fibre as well as flops. The arguments here are based on the presence of a two-dimensional algebraic torus action, which comes from the fact that each of the three groups gives a reducible representation on $\mathbb{C}^4$. The line of the argument here is more subtle than in previous chapters as we are simultaneously studying the geometry of a resolution and moving toward the proof that given 'candidate' is an actual set of generators of the Cox ring. The final step of the proof is based on properties of equivariant Euler characteristic. Along the way we use computer calculations several times. First we compute components of the central fibre of a candidate for a resolution. Then, we use the division by a Gröbner basis to prove that the GIT quotient of the ring generated by the appropriate 'candidate set' is a crepant resolution. Finally, we use a computer algebra system to find the Hilbert series of this ring and compare it with the results obtained by properties of equivariant Euler characteristic to prove that this ring is the whole Cox ring.

We summarize this chapter presenting the strategy to generalize the arguments. We plan to apply this strategy in the study of other examples.

3. Additional Remarks

The dissertation is a substantially extended version of works of the author: [17, 18] (joint papers with Marią Donten-Bury) and [24].

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